Financial Time Series Modelling Using Recurrent Fractal Interpolation Functions (RFIF)

Based on the work of Manousopoulos et al.
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Introduction

- Financial time series often exhibit non-smooth, irregular patterns.
- Standard interpolation methods (e.g., cubic splines) struggle with abrupt changes.
- **Recurrent Fractal Interpolation Functions (RFIF)** provide an alternative approach.

Why Use Fractal Interpolation?

- Financial data like cryptocurrency prices are volatile and exhibit fractal-like behavior.
- RFIFs allow modeling of local irregularities while capturing overall patterns.
- Applications: filling missing data, short-term forecasting, and high-frequency data modeling.

Theory of Fractal Interpolation Functions

Fractal Interpolation: Uses iterated function systems (IFS) to model complex, self-affine patterns.
Given data points $P = \{(u_m, v_m) \in I \times \mathbb{R} : m = 0, 1, \ldots, M\}$, chose a subset of these points for interpolation:
$Q = \{(x_i, y_i) : i = 0, 1, \ldots, N \leq M\}$, such that, $x_0 = u_0$ and $x_N = u_M$, where $I = [u_0, u_M] = [x_0, x_N]$ form the region of interest and $I_n = [x_{n-1}, x_n]$ forms the n-th interpolation interval.
Let $(\mathbb{R}^2, ||\cdot||_2)$ be the metric space. An IFS is defined as:

$$\{\mathbb{R}^2; w_n, n = 1, 2, \ldots, N\}$$

where $w_n : X \to X$ are contraction mappings with contractivity factors d_n such that $|d_n| < 1$ usually chosen beforehand.

Self-Affine Fractal Interpolation

For a chosen set of interpolation points Q, fractal interpolation generates an affine transformation:

$$w_n : [x_0, x_N] \to [x_{n-1}, x_n]$$

with,

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}$$

such that,

$$w_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \text{ and } w_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

where the parameters a_n, c_n, e_n, f_n are chosen based on interpolation constraints.

Recurrent Fractal Interpolation Functions (RFIF)

- Self-affine FIF transformations maps **entire graph** to its corresponding interpolation interval.
- RFIFs generalize fractal interpolation by allowing piecewise self-affinity.
- Each segment of data (**address interval**) is mapped to a corresponding interpolation interval.
- Allows RFIFs to model localized patterns effectively.

Constructing RFIF: Theory

For each interpolation interval $I_n = [x_{n-1}, x_n]$, we associate two address points $(x'_{n,1}, y'_{n,1})$ and $(x'_{n,2}, y'_{n,2}) \in P$ (from the entire data set) to form the address interval $I'_n = [x'_{n,1}, x'_{n,2}]$.
The affine transformation is then:

$$w_n : I'_n \to I_n$$

with,

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}$$

such that,

$$w_n \begin{pmatrix} x'_{n,1} \\ y'_{n,1} \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \text{ and } w_n \begin{pmatrix} x'_{n,2} \\ y'_{n,2} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Noting, for contractivity constraint, $(x'_{n,2} - x'_{n,1}) > (x_n - x_{n-1})$

Constructing RFIF: Choosing the Address Intervals

Define:

$$P_n = \{(u_m, v_m) \in P : x_{n-1} \leq u_m \leq x_n\}$$

be the set of data points appearing in the n-th interpolation interval, and,

$$A_n = \{(u_m, v_m) \in P : x'_{n,1} \leq u_m \leq x'_{n,2}\}$$

be the set of data points appearing the n-th address interval. With,

$$A(P, k) \equiv \{A \subseteq P : |A| = k\}$$

being the set of possible subsets of P with k consecutive points, choose,

$$A_n = \arg \min_{A \in A(P,k)} h(P_n, w_n(A))$$

which minimizes the Hausdorff distance $h(\cdot, \cdot)$ between n-th interpolation interval and itself.

Evaluating performance: RMSE

- Let $\tilde{y}_m$ be the prediction to the m-th datapoint y_m, then,
- **Root Mean Square Error**:

$$RMSE(y_m, \tilde{y}_m) = \left(\frac{1}{M+1} \sum_{m=0}^{M} (\tilde{y}_m - y_m)^2 \right)^{\frac{1}{2}}$$

- Each transformation w maps larger interval of discrete data to smaller interval of discrete data.
- Handled by taking average of such predictions $\tilde{y}_k$ of points $(\tilde{x}_k, \tilde{y}_k)$ such that $\tilde{x}_k$ fall near data point u_m:

$$\tilde{y}_m \equiv \frac{1}{\text{no. of points in sum}} \sum_{k \mid \tilde{x}_k \in [u_m - 0.5, u_m + 0.5]} \tilde{y}_k$$

Application: Bitcoin Dataset

Dataset: Weekly Bitcoin prices from 23 Dec 2018 to 16 Dec 2020 (105 data points).
Method:

- Select every 4th data point as an interpolation point (total 27 points).
- Address intervals span 25 points.
- Minimize the Hausdorff distance to determine the optimal mapping for each interval.

RFIF for Bitcoin Data

The resulting RFIF captures the Bitcoin dataset's fluctuations effectively.

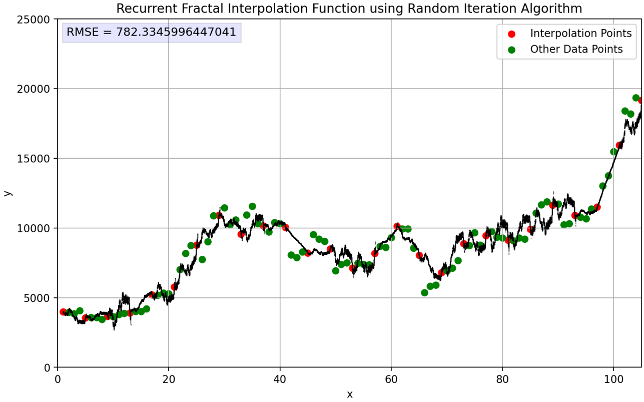

Figure: Bitcoin time series modeled by RFIF (red points are interpolation points).

Comparison to Traditional Models

- We compare RFIF to **ARIMA (1,1,0):** Autoregressive model suited for stationary data:
- If Y_t is our time series for each time point t, define the $d = 1$ order difference as:

$$y_t \equiv Y_{t+1} - Y_t$$

- Then, the ARIMA(1,1,0) model of Y_t is:

$$y_t = a_1 y_{t-1} + \varepsilon_t$$

ARIMA(1,1,0) with cubic spline on Bitcoin Data

The resulting ARIMA(1,1,0) with cubic spline interpolation not able to capture dataset's quick fluctuations effectively.

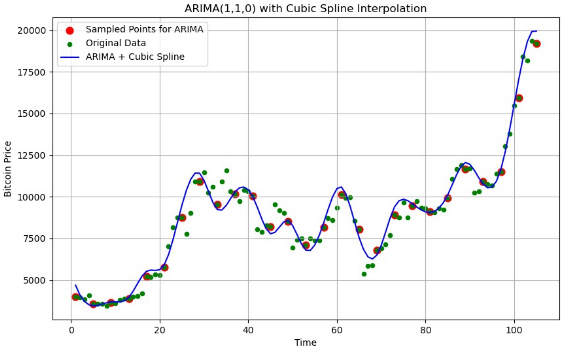

Figure: Bitcoin time series modeled by ARIMA(1,1,0) with cubic spline (red points are interpolation points).

Conclusion

- RFIF is effective for modeling non-smooth, fluctuating financial time series.
- Successfully applied to Bitcoin dataset, capturing volatility better than traditional models.

References

- Manousopoulos, P., Drakopoulos, V., Polyzos, E. *Financial Time Series Modelling Using Fractal Interpolation Functions*. AppliedMath 2023, 3, 510–524.
- D. S. Mazel and M. H. Hayes, *"Using iterated function systems to model discrete sequences,"* in IEEE Transactions on Signal Processing, vol. 40, no. 7, pp. 1724-1734, July 1992
- Barnsley, M. F. *Fractals Everywhere*. 3rd ed., Dover Publications, 2012.