# Enough Measure Theory

These notes should serve as a quick review of measure theory. We will skip the proofs involving limsups and liminfs and other set theoretic proofs. For important theorems proofs are sketched. A convention we will always assume is that $0 \cdot \infty = \infty \cdot 0 = 0$.

## 1 | Measure Spaces

In this section we gather together the structures needed to measure 'sizes' of sets. Aim of this section is to identify the minimum structure needed to define such a measure. We want the measure of sets to be a positive number. i.e., $\mu(A) \in [0, \infty]$. The empty set should have zero measure, $\mu(\varnothing) = 0$. For disjoint sets, the total measure should be sum of measures of the individual sets.

$$\mu(\coprod_{i \in F} A_i) = \textstyle\sum_{i \in F} \mu(A_i),$$

where $F$ is a finite set. If this holds for countable $F$ then $\mu$ is called $\sigma$-additive.

One way to measure sets is to use geometric notions such as length of an interval to measure the sets. This as we will see in this section is not possible for all sets as there are certain pathological sets. Let $\mathcal{I}_{\mathbb{R}}$ denote the set of all intervals of the form $(a, b] \subset \mathbb{R}$ where $a \leq b$. On this set, we can define the measure of the interval to be the length of the interval $\lambda : I_{\mathbb{R}} \to [0, \infty]$ defined by,

$$\lambda((a, b]) = b - a$$

It can be showed that the length of interval is $\sigma$-additive. Similarly we have the volume of cubes for higher dimensions. However the problem is that the length function cannot be defined for all subsets of $\mathbb{R}$ while also respecting translation invariance, i.e., $\mu(U + t) = \mu(U)$ for $t \in \mathbb{R}$. This was shown by Vitali, he constructed a set which if the measure satisfied the above rules would have both infinite measure and be a subset of a finite measure set.

**Theorem 1.1. (Vitali's theorem)** $\lambda : \mathcal{I}_{\mathbb{R}} \to [0, \infty]$ *cannot be extended to* $\mathcal{P}(\mathbb{R})$.

So we can't extend the length function to all subsets so that it remains a measure. The aim is then to define a notion of measure coming from the length function to a large subset $\mathcal{B}(\mathbb{R})$ of $\mathcal{P}(\mathbb{R})$.

### 1.1 | Semi-rings, $\sigma$-Algebras, Pre-Measures, and Measures

$\sigma$-algebras act as the family of subsets of a set $X$ that we can measure. So, what do we want to do? Our aim is to be able to define a measure on lots of subsets. So the definition of a $\sigma$-algebra should be such that it has the nice behavior and leave out pathologies.

Let $\Sigma(X) \subset \mathcal{P}(X)$ be a collection of subsets of $X$ that can be 'measured'. We expect the size of 'nothing' to be zero, so it should first be that the emptyset is measurable. Similarly the whole set $X$ should be measurable, could have infinite measure but it must be measurable.

$$\varnothing \in \Sigma(X), \;\; X \in \Sigma(X). \tag{1S}$$

If $A$ is measurable then we expect the size of $A^c$ to be the size of $X$ minus the size of $A$. So, the set $A^c$ must be measurable.

$$A \in \Sigma(X), \;\; \text{then } A^c \in \Sigma(X). \tag{2S}$$

If two sets $A \in \Sigma(X)$ and $B \in \Sigma(X)$ are measurable then we expect $A \cup B$ to be measurable and if they are disjoint the total size should be sum of the individual sizes,

$$A \cup B \in \Sigma(X) \tag{3S}$$

We can extend this to countable union as well while still having some nice behavior. Since our aim is to maximize the stuff we can measure. So, we allow countable union also to be measurable.

$$\{A_i\}_{i\in\mathbb{N}} \subset \Sigma(X), \;\; \text{then } \bigcup_{i\geq 1} A_i \in \Sigma(X). \tag{4S}$$

$\mathcal{A} \subset \mathcal{P}(X)$ is an algebra if it satisfies conditions 1S, 2S, and 3S. If an algebra $\Sigma(X) \subset \mathcal{P}(X)$ also satisfies the condition 4S it is called a $\sigma$-algebra. Clearly $\mathcal{P}(X)$ is a $\sigma$-algebra on $X$ and $I_{\mathbb{R}}$ is not a $\sigma$-algebra. Immediate consequence of the definition is that all standard operations of sets such as union, intersection, difference, symmetric difference are also in $\Sigma(X)$.

$$A, B \in \Sigma(X), \;\;\; A \cup B,\; A \cap B,\; A\backslash B,\; A\Delta B \in \Sigma(X)$$

More importantly lim sup and lim inf also are measurable. Suppose $\Sigma_i(X) \subset \mathcal{P}(X)$ be $\sigma$-algebras, then it can be easily checked that $\Sigma(X) = \cap_i \Sigma_i(X)$ is also a $\sigma$-algebra. For any collection of subsets $S \subset \mathcal{P}(X)$, the $\sigma$-algebra generated by $S$, denoted by $\sigma(S)$ is the intersection of all $\sigma$-algebras containing $S$.

$$\sigma(S) = \bigcap_{S \subset \Sigma(X)} \Sigma(X).$$

It is hence the smallest $\sigma$-algebra containing $S$. Given a collection of subsets $S$, we can construct $\sigma(S)$ as follows, include all the elements of $S$, add all complements of elements of $S$, add $X$ and $\varnothing$, add all countable unions and their complements. Since any $\sigma$-algebra containing $S$ will have these sets it's the desired intersection. If $\Sigma(X)$ is a $\sigma$-algebra of $X$ and $Y \subset X$ then $Y \cap \Sigma(X)$ is a $\sigma$-algebra of $Y$.

If $(X, \Gamma)$ is a topological space where $\Gamma$ is the collection of all open sets. The $\sigma$-algebra generated by the collection $\Gamma$ is the smallest $\sigma$-algebra generated by $\Gamma$ is called the Borel $\sigma$-algebra. Denoted by $\mathcal{B}(\Gamma)$, elements of $\mathcal{B}(\Gamma)$ are called Borel sets.

**Theorem 1.2.** *Let $f : Y \to X$ be a continuous map, then, $f^{-1}(\mathcal{B}(\Gamma)) = \mathcal{B}(f^{-1}(\Gamma))$.*

The proof of the theorem involves some basic set theory and we will not discuss it here. $\mathcal{B}(\mathcal{I}_{\mathbb{R}})$ will be the $\sigma$-algebra where we will define the Lebesgue measure. Let $(\mathbb{R}, \Gamma_{\mathbb{R}})$ be the standard topology on $\mathbb{R}$. If $I_k^x = (x - 1/k, x]$ we get that $\{x\} = \cap_{k\in\mathbb{N}} I_i^x$. So $\{x\} \in \mathcal{B}(\mathcal{I}_\mathbb{R})$. Hence we have,

$$\mathcal{B}(\mathcal{I}_\mathbb{R}) = \mathcal{B}(\Gamma_\mathbb{R}).$$

Though $\mathcal{I}_\mathbb{R}$ is not a $\sigma$-algebra, it has some nice properties. Open intervals, closed intervals, half closed intervals, also generate $\mathcal{B}(\mathbb{R})$. We are interested in abstracting out some properties of the length function on intervals.

The empty set can be taken to have zero length or the empty set belongs to $\mathcal{I} = \mathcal{I}_\mathbb{R}$.

$$\varnothing \in \mathcal{I} \tag{1R}$$

Intersection of two intervals is again an interval, so we have,

$$A, B \in I_\mathbb{R} \implies A \cap B \in \mathcal{I} \tag{2R}$$

An interval minus an interval is the disjoint union of two intervals. So in general we want,

$$A\backslash B = \coprod_{i=1}^{n} C_i, \;\; C_i \in \mathcal{I}. \tag{3R}$$

A subset $\mathcal{I} \subset \mathcal{P}(X)$ is said to be a semi-ring if it satisfies the above three conditions. We can define 'pre-measures' on these semi-rings. If $\mathcal{I} \subseteq \mathcal{P}(X)$ is a semi-ring, and $A, B_1, \dots B_m \in \mathcal{I}$, then by simple induction it can be showed that there exist $C_1, \dots C_n \in \mathcal{I}$ disjoint, such that

$$A\backslash(\cup_{j=1}^m B_j) = \coprod_{i=1}^{n} C_i.$$

If $\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{H} \subseteq \mathcal{P}(Y)$ are two semi-rings, then it can be showed that $\mathcal{I} \times \mathcal{H}$ is a semi-ring. Proof is some set theory manipulations. We can now define pre-measure on semirings that captures the intuition we have about the length function on $\mathcal{I}_\mathbb{R}$.

**Definition 1.1.** Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a semi-ring. A pre-measure on $\mathcal{I}$ is a map, $\mu : \mathcal{I} \to [0, \infty]$ with $\mu(\varnothing) = 0$ and $\mu$ is said to be $\sigma$-additive or a pre-measure if

$$\mu(\coprod_{i\geq 1} A_i) = \textstyle\sum_{i\geq 1} \mu(A_i).$$

$\mu$ is finitely additive or a content on $\mathcal{I}$, if, $\mu(\coprod_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. we say $\mu$ is a measure if $\mu$ is $\sigma$-additive and $\mathcal{I}$ is a $\sigma$-algebra.

A measure space is a triple $(X, \Sigma(X), \mu)$ where $\Sigma(X)$ is a $\sigma$-algebra of the set $X$ and $\mu$ is a measure on $\Sigma(X)$. $(X, \Sigma(X))$ is called a measurable space. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a semi-ring and $\mu$ is a pre-measure.

Suppose $A, B \in \mathcal{I}$ with $B \subseteq A$, then $A = B \coprod (A\backslash B)$, but $A\backslash B = \coprod_{i=1}^n C_i$. So, we have,

$$\mu(A) = \mu(B) + \textstyle\sum_{i=1}^n \mu(C_i) \geq \mu(B). \tag{monotonicity}$$

If $A \cup B \in \mathcal{I}$ then we have, $A = (A \cap B) \cup (A\backslash B) = (A \cap B) \cup \coprod_{i=1}^n C_i$ and similarly $B = (A \cap B) \cup \coprod_{j=1}^m D_j$. So we have, $A \cup B = (A \cap B) \cup \coprod_{i=1}^n C_i \cup \coprod_{j=1}^m D_j$. So we get,

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B). \tag{parallelogram}$$

Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{I}$ with $\cup_{i\geq 1} A_i \in \mathcal{I}$ then we can write $\cup_{i\in\mathbb{N}} A_i$ as a disjoint union by taking $A_i' = A_i \backslash (A_1 \cup \cdots \cup A_{i-1})$. Now each of these is a union of disjoint sets $A_i' = \coprod_{k=1}^{n} C_{k,i}$. So we have,

$$\mu(\bigcup_{i\geq 1} A_i) \leq \textstyle\sum_{i\geq 1} \mu(A_i). \qquad (\sigma\text{-subadditivity})$$

Let $A_n \in \mathcal{I}$ and $A_n \nearrow A$ then by taking $A_i' = A_i \backslash (A_1 \cup \cdots \cup A_{i-1}) = \coprod_{k=1}^{m_i} C_{k,i}$. Then $A$ can be written as a countable union,

$$A = \bigcup_{i\geq 1} A_i' = \bigcup_{i\geq 1} \coprod_{k=1}^{m_i} C_{k,i}$$

Let $A_N = \cup_{n=1}^{N} \cup_{k=1}^{m_i} C_{k,i}$. This is a disjoint union and hence we have,

$$\mu(A) = \textstyle\sum_{n\geq 1} \sum_{k=1}^{m_i} \mu(C_{k,i}) = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{k=1}^{m_i} \mu(C_{k,i}) = \lim_{N\to\infty} \mu(A_N)$$

$$\lim_{n\to\infty} \mu(A_n) = \mu(A) \qquad \text{(cont. from below)}$$

Similarly continuity from above is proved with some similar argument. First step in most of the proofs like this involve writing the union as a disjoint union.

The length function on $\mathcal{I}_{\mathbb{R}}$ is a pre-measure. It's called the Lebesgue-Borel pre-measure. Our aim is to extend the length pre-measure on $\mathcal{I}_{\mathbb{R}}$ to $\mathcal{B}(\mathbb{R})$.

Let $\lambda : \mathcal{I}_{\mathbb{R}} \to [0, \infty]$ be a pre-measure. We can think of $\lambda((a, b])$ as a $\overline{\mathbb{R}}$-valued function dependent on two variables $a$ and $b$. The pre-measure is said to be locally finite or Lebesgue-Stieltjes if $\lambda((a, b]) < \infty$. Define a function $F_\lambda : \mathbb{R} \to \mathbb{R}$ by,

$$F_\lambda(t) = \begin{cases} \lambda((0, t]) & \text{if } t \geq 0 \\ -\lambda((0, t]) & \text{if } t < 0, \end{cases}$$

then $\lambda((a, b]) = F_\lambda(b) - F_\lambda(a)$ for $a < b \in \mathbb{R}$. A non-decreasing right continuous function $F : \mathbb{R} \to \mathbb{R}$ i.e., $(0, t_n] \searrow (0, t]$ implies $F(t_n) \searrow F(t)$ is called a distribution function.

## 1.2 | Outer Measures, Extension, and Uniqueness

The aim for this section is to develop tools needed to extend pre-measure on semi-ring to a measure on some appropriate, large enough $\sigma$-algebra. The approach is to extend the pre-measure $\mu$ to a set function $\mu^*$ on $\mathcal{P}(X)$. From this pick a large $\sigma$-algebra $\Sigma(\mu^*)$ on which $\mu^*$ is $\sigma$-additivity. We should also expect that $\Sigma(\mu^*)$ contains $\sigma(\mathcal{I})$.

Suppose we have a semi-ring $\mathcal{I}$ of subsets of $X$ and we are given a pre-measure,

$$\mu : \mathcal{I} \to [0, \infty],$$

we can approximate the size of subsets of $X$ using the subsets in $\mathcal{I}$. Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{I}$ be a cover of a subset $A \subset X$ i.e., $A \subseteq \cup_{i\geq 1} A_i$. Then the size of $A$ should be less than the sum of sizes of all subsets $A_i$. The quantity,

$$\mu^*(A) = \inf\{\textstyle\sum_{i\geq 1} \mu(A_i) \mid A \subseteq \bigcup_{i\geq 1} A_i,\ \ A_i \in \mathcal{I}\}$$

can be thought of as an approximation of the size of $A$ using the subsets in $\mathcal{I}$. This is called an outer measure generated by $\mu$. We can abstract out the nice properties of this quantity which can be later used to construct general measures.

The set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is such that

$$\mu^*(\varnothing) = 0 \tag{1O}$$

if $B \subseteq A$ then any cover of $A$ will also be a cover of $B$, so we have,

$$\mu^*(B) \leq \mu^*(A) \tag{2O}$$

we expect the sum of sizes of individual subsets should be greater than the size of the union. If $\{A_i\}_{i\in\mathbb{N}}$ is a sequence of sets in $\mathcal{P}(X)$ then, there exists a cover $\cup_{j\geq 1}\{O_{j,i}\}$ of $A_i$ such that $\mu^*(A_i) \geq \sum_j \mu(O_j) - \epsilon/2^i$ for every $\epsilon$. For example, any set from the collection of sets $\arg\inf\{\sum_j \mu(O_j)\}$ works. The union of $\{A_i\}_{i\geq 1}$ is contained in the union $\cup_{i,j} O_{i,j}$. Hence we have,

$$\mu^*(\bigcup_{i\geq 1} A_i) \leq \textstyle\sum_{i\geq 1}\sum_{j\geq 1}\mu(O_{i,j}) \leq \sum_{i\geq 1}\mu^*(A_i) + \epsilon.$$

Since the choice of $\epsilon$ was arbitrary, we have,

$$\mu^*(\bigcup_{i\geq 1} A_i) \leq \textstyle\sum_{i\geq 1}\mu^*(A_i). \tag{3O}$$

A set map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is called an outer measure if it satisfies the conditions 1O, 2O, and 3O. The condition 3O is called subadditivity. We have to now recognize the sets for which the outer measure measures the right size. Our goal now is to choose a subset of $\mathcal{P}(X)$ for which the $\sigma$-additivity holds.

What we expect from such sets is that the outer measure is a good estimate of the size. The inner measure would be the outer estimate of the complement of the set. Suppose $A \subset X$ and $Q \subset X$, we expect both estimates to be equal, i.e., $\mu^*(Q \cap A) = \mu^*(Q) - \mu^*(Q \cap A^c)$.

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \cap A^c) \;\; \forall Q \subseteq X. \tag{4O}$$

In such a case $A$ is said to additively intersect all sets with respect to $\mu^*$. We use $Q$ here because $\mu^*(X)$ can be infinite.

A set $A \subseteq X$ is $\mu^*$-measurable if it satisfies (4O). Let $\Sigma(\mu^*)$ denote the set of all $\mu^*$-measurable sets. Our aim is to show that $\Sigma(\mu^*)$ is a $\sigma$-algebra and is the largest $\sigma$-algebra on which $\mu^*|_{\Sigma(\mu^*)}$ is a measure.

Clearly, $\varnothing \in \Sigma(\mu^*)$. For $A \in \Sigma(\mu^*)$, the complement $A^c \in \Sigma(\mu^*)$ due to the symmetry of 4O. We have to now show $\Sigma(\mu^*)$ also satisfies 3S and 4S. Let $A$, $B \in \Sigma(\mu^*)$. For all $Q \subseteq X$,

$$\begin{aligned}\mu^*(Q) &= \mu^*(Q \cap A) + \mu^*(Q \cap A^c) = \mu^*(Q \cap A) + \mu^*(Q \cap A^c \cap B) + \mu^*(Q \cap A^c \cap B^c)\\ &\geq \mu^*((Q \cap A) \cup (Q \cap A^c \cap B)) + \mu^*(Q \cap (A \cup B)^c).\end{aligned}$$

So, $\mu^*(Q) \geq \mu^*(Q \cap (A \cup B)) + \mu^*(Q \cap (A \cup B)^c)$. Together with $\mu^*(\cup_{i\leq 1} A_i) \leq \sum_{i\geq 1}\mu^*(A_i)$ it's an equality and hence $A \cup B \in \Sigma(\mu^*)$. By induction it holds for all finite unions and therefore for for countable union. Since $\mu^*$ is an outermeasure and is $\sigma$-additive on $\Sigma(\mu^*)$ it's a measure on $\Sigma(\mu^*)$.

**Theorem 1.3. (Carathéodory Theorem)** *$\Sigma(\mu^*)$ is a $\sigma$-algebra, $\mu^*|_{\Sigma(\mu^*)}$ a measure.*

We now have to show $\Sigma(\mu^*)$ is large enough, i.e., $\sigma(\mathcal{I}) \subseteq \Sigma(\mu^*)$ when $\mu^*$ is an outer measure generated by the premeasure $\mu$ on $\mathcal{I}$. Let $Q \subseteq X$ with $\mu^*(Q) < \infty$ and let $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{I}$ be a countable cover of $Q$. $A_i = (A_i \cap A) \cup (A_i \cap A^c)$. $A_i \cap A = B_i \in \mathcal{I}$ by 2R and $A_i \cap A^c = A_i \backslash A = \coprod_{j=1}^{n_i} C_{i,j}$ by 3R for $B_i, C_{i,j} \in \mathcal{I}$.

$$\mu^*(A_i) = \mu^*(B_i) + \textstyle\sum_{j=1}^{n_i} \mu^*(C_{i,j}).$$

So, the sum is then given by, $\sum_{i\geq 1} \mu^*(A_i) = \sum_{i\geq 1} \mu^*(B_i) + \sum_{i\geq 1}\sum_{j=1}^{n_i} \mu^*(C_{i,j}) \geq \mu^*(Q \cap A) + \mu^*(Q \cap A^c)$. Since the choice of the cover $\{A_i\}_{i\in\mathbb{N}}$ was arbitrary we have,

$$\mu^*(Q) = \inf\{\textstyle\sum_{i\geq 1} \mu(A_i) \mid A \subseteq \cup_{i\geq 1} A_i\} \geq \mu^*(Q \cap A) + \mu^*(Q \cap A^c).$$

So, $A \in \Sigma(\mu^*)$.

**Theorem 1.4. (Carathéodory's Extension Theorem)** *If $\mu : \mathcal{I} \to [0, \infty]$ be a pre-measure, then, $\sigma(\mathcal{I}) \subseteq \Sigma(\mu^*)$ and $\mu^*|_{\Sigma(\mu^*)}$ is a measure.*

This extension is however not unique. For example, if we take $\mathcal{I} = \{\varnothing\}$ and $\mu(\varnothing) = 0$ has infinitely many extensions. The next question we could ask is if and when a measure $\mu : \Sigma(X) \to [0, \infty]$ is characterized by its values on some smaller family of subsets $\mathcal{E} \subseteq \Sigma(X)$. Let $\mu, \nu$ be two measures on $(X, \Sigma(X))$.

with $\mu(X) = \nu(X) < \infty$. Let $\mathcal{D}$ be the family of subsets of $X$ such that for each $A \in \mathcal{D}$, $\mu(A) = \nu(A)$. $\mathcal{D}$ has the following properties. By assumption

$$X \in \mathcal{D} \tag{1D}$$

If $A \in \mathcal{D}$, i.e., $\mu(A) = \nu(A)$, then so is $A^c$ because if $\mu(A) = \nu(A)$ then $\mu(A^c) = \mu(X) - \mu(A) = \nu(X) - \nu(A) = \nu(A^c)$.

$$A \in \mathcal{D} \implies A^c \in \mathcal{D} \tag{2D}$$

Let $\{A_i\}_{i\in\mathbb{N}}$ be a disjoint collection, $\mu(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \mu(A_i) = \sum_{i\geq 1} \nu(A_i) = \nu(\bigcup_{i\geq 1} A_i)$. So we have,

$$A_i \in \mathcal{D}, \implies \bigcup_{i\geq 1} A_i \in \mathcal{D} \tag{3D}$$

A family of subsets $\mathcal{D} \subseteq \mathcal{P}(X)$ is called a Dynkin system or $\lambda$-system if it fullfils 1D, 2D, and 3D. The difference between $\sigma$-algebra and a Dynkin system is that the union in Dynkin system should be disjoint. Clearly every $\sigma$-algebra is a Dynkin system.

A collection $\mathcal{E}$ is said to be $\cap$-stable or $\pi$-system if for every $A, B \in \mathcal{E}$, $A \cap B \in \mathcal{E}$. Suppose $\mathcal{D}$ is $\cap$-stable, then $B\backslash A = A^c \cap B$, and hence we can write $A \cup B = A \cap B \backslash A \in \mathcal{D}$. This can be extended to countable union similarly, and hence $\mathcal{D}$ is a $\sigma$-algebra iff it's $\cap$-stable.

Clearly, the intersection of Dynkin systems is again a Dynkin system. If $\mathcal{E} \subseteq \mathcal{P}(X)$ the Dynkin system generated by $\mathcal{E}$ is the intersection of all Dynkin systems containing $\mathcal{E}$.

$$\delta(\mathcal{E}) = \bigcap_{\mathcal{E}\subseteq\mathcal{D}} \mathcal{D}$$

It can be showed with some basic set theory proof that if $\mathcal{E}$ is $\cap$-stable then so is $\delta(\mathcal{E})$. This is equivalent to the following statement:

**Theorem 1.5. (Dynkin Lemma or $\pi$-$\lambda$-Theorem)** *If $\mathcal{E}$, $\cap$-stable then $\delta(\mathcal{E}) = \sigma(\mathcal{E})$.*

A set function $\mu : \mathcal{E} \to [0, \infty]$ is called $\sigma$-finite on $\mathcal{E}$ if there exists a sequence $\{A_i\}_{i\in\mathbb{N}}$ with $\mu(A_i) < \infty$ such that $X = \bigcup_{i\geq 1} A_i$. A measure space $(X, \Sigma(X), \mu)$ is called $\sigma$-finite if $\mu$ is $\sigma$-finite.

Let $\mu, \nu$ be two finite measures on $(X, \Sigma(X))$ with $\mu(X) = \nu(X)$, and $\mathcal{E}$ be a $\cap$-stable generator of $\Sigma(X)$ such that $\mu|_{\mathcal{E}} = \nu|_{\mathcal{E}}$. The collection,

$$\mathcal{D} = \{A \mid \mu(A) = \nu(A)\}$$

is a Dynkin system with $\mathcal{E} \subseteq \mathcal{D}$ and so, $\delta(\mathcal{E}) \subseteq \mathcal{D}$. Since $\mathcal{E}$ is $\cap$-stable we have, $\delta(\mathcal{E}) = \sigma(\mathcal{E}) = \Sigma(X)$. So we have, $\Sigma(X) \subseteq \mathcal{D}$ or $\mu = \nu$. If $\mathcal{E}$ is $\sigma$-finite then it is $\cap$-stable.

**Theorem 1.6. (Uniqueness)** *If $\mu : \mathcal{I} \to [0, \infty]$ is a $\sigma$-finite pre-measure with extension $\mu^* : \Sigma(\mu^*) \to [0, \infty]$ then the extension is uniquely determined on $\sigma(\mathcal{I})$.*

A measure space $(X, \Sigma(X), \mu)$ is called complete if for every $B \subseteq A \in \Sigma(X)$ with $\mu(A) = 0$, $B \in \Sigma(X)$ with $\mu(B) = 0$ i.e., every subset of the null set is measurable. It's easier to deal with complete measure spaces as forget about certain subsets once they are in the null set.

## 1.3 | Some important Measures

The first important measure is the Lebesgue measure. The semi-ring is $\mathcal{I}_{\mathbb{R}}$ and the pre-measure is the volume function, $\lambda^d : \mathcal{I}_{\mathbb{R}} \to [0, \infty]$ given by,

$$\lambda^d(\prod_{j=1}^{d}(a_j, b_j]) := \prod_{j=1}^{d}(b_j - a_j) \tag{L}$$

We have two $\sigma$-algebras of interest to us, $\sigma(\mathcal{I}_{\mathbb{R}}) = \mathcal{B}_{\mathbb{R}^d}$ and $\Sigma(\lambda^{d*}) = \mathcal{L}_{\mathbb{R}^d}$. $\Sigma(\lambda^{d*})$ is called the Lebesgue $\sigma$-algebra on $\mathbb{R}^d$. By Carathéodory extension theorem we have $\mathcal{B}_{\mathbb{R}^d} \subseteq \mathcal{L}_{\mathbb{R}^d}$.

Clearly $\lambda^d$ is a $\sigma$-finite pre-measure on $\mathcal{I}_{\mathbb{R}^d}$.

**Theorem 1.7. (Lebesgue Measure)** $\lambda^d$ *extends uniquely to a complete measure on* $\mathcal{L}_{\mathbb{R}^d} \supseteq \mathcal{B}(\mathbb{R}^d)$ *such that,*

$$\lambda^d(A) = \lambda^d(A + t) \tag{1L}$$

*where* $A \in \mathcal{L}_{\mathbb{R}^d}$, $A + t = \{a + t \mid a \in A\}$.

# 2 | Integration

## 2.1 | Convergence Theorems

## References