Categories and Sheaves

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Introduction

These notes provide an abstract introduction to sheaf theory and sheaf cohomology. The goal is to understand Grothendieck's six operations, abstractly.[1] The abstract machinery divides the difficulty into multiple smaller steps, so the difficulty of the theorems is not felt. I have tried to provide intuitive motivation for the categorical definitions. The purpose of these notes is to prepare the reader for Kashiwara-Schapira's books.

The notion of a sheaf axiomatizes this 'local nature'. Given a topological space X, a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X.

Denote by CU the set of real-valued continuous functions on U. If $V \subset U$ then f restricted to V is a continuous map, $f|_{UV} : V \to \mathbb{R}$. The map, $f \mapsto f|_{UV}$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_{UV})|_{VW} = f|_{UW}.$$

So, this is functorial in nature, i.e., $|_{VW} \circ |_{UV} = |_{UW}$. This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\text{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\text{op}}$ is the opposite category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\text{op}}$ the functor assigns a morphism, the restriction morphism in $\mathbf{Sets}$, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_{UV}$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Sets}, \tag{pre-sheaf}$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps $U \subset V$ and the corresponding maps in $\mathbf{Sets}$ are called restriction maps, $|_{UV} : \mathcal{F}U \to \mathcal{F}V$.

Denote by $\mathrm{PSh}(X)$ the collection of all pre-sheaves over a topological space X. Each pre-sheaf is a functor from $\mathcal{O}(X)^{\text{op}}$ to $\mathbf{Sets}$ can be considered an object and the natural transformation between the two pre-sheaves as morphism between these objects. So, the category of pre-sheaves is the functor category,

$$\mathrm{PSh}(X) = \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}}$$

So, to understand pre-sheaves, we could use tools from studying functor categories. The goal of the first half of this document will be to develop the tools needed to study functor categories, and hence pre-sheaves.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions. Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function

[1] Fosco from mathoverflow commented once that 'If it's not possible to derive these statements from purely Kan-extensional arguments, then sheaves do belong to Algebraic Geometry'. This is the motivation for taking an abstract route.

$f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i\in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$CU \overset{e}{\dashrightarrow} \prod_i CU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

A sheaf of sets $\mathcal{F}$ on a topological space X is a functor,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i \in I} U_i$ of an open set U of X yields an equalizer diagram.

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j). \tag{collation property}$$

The category of sheaves, $\mathrm{Sh}(X)$, is a subcategory of the functor category,

$$\mathrm{Sh}(X) \rightarrowtail \mathrm{PSh}(X) = \widehat{\mathcal{O}(X)} = \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}},$$

which is the category consisting of all functors from $\mathcal{O}(X)^{\mathrm{op}}$ to $\mathbf{Sets}$, that satisfy the collation. The obstruction of going from local to global is carefully studied via cohomology. The goal of this document is to develop the tools needed to study cohomology of sheaves.

The starting point is the develop the necessary tools from category theory to study these objects. The main concepts of category theory is the representables-limits-adjoints triad. While Kashiwara-Schapira use a lot of abstract categorical tools, their approach still contains lots of difficult proofs. My goal is to introduce a bit more abstraction than Kashiwara-Schapira, but significantly reduce the difficulty of proofs.

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Part I

The Six Functors

1 | Representables, Adjoints & Limits

Category theory develops very general tools, which can be applied to many mathematical phenomena. This chapter is an abstract study of pre-sheaves. The use of category theory in sheaf theory was started by Grothendieck based on the following philosophical position,

Yoneda-Grothendieck. An object is determined by its relation to other objects.

The Yoneda lemma makes this precise and exploitable. Yoneda lemma embeds a given category inside the category of functors from the given category to the category of sets, via the functor, 'maps to the given object' or 'maps from the given object'. This allows us to utilize the nice properties of the target category in our case the category of sets.

1.1 | Category of Functors

A set is a collection of 'elements'. A category $\mathcal{A}$ is more sophisticated, it possesses 'objects' similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{A}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects. Studying objects by their relations to other objects could be called the Yoneda-Grothendieck philosophy.

A functor $\mathcal{F}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of a mapping of objects of $\mathcal{A}$ to objects of $\mathcal{B}$, $X \mapsto \mathcal{F}X$ together with a map of the set of homomorphisms,

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X,\mathcal{F}Y).$$

the image of $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ denoted by $\mathcal{F}(f)$. That takes identity to identity and respects composition[1] i.e.,

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ can be thought of as a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. A functor $\mathcal{F}$ is faithful if the map $\mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X,\mathcal{F}Y)$ is injective for all X, Y. It's full if the map is surjective. If it's a bijection the functor is called fully faithful.

A natural transformation κ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, denoted by,

$$\kappa : \mathcal{F} \Rightarrow \mathcal{G},$$

[1] The composition $f \circ g$ assumes they are composable.

is a collection of mappings κ_X for every $X \in \mathcal{A}$, such that for all $f : X \to Y$, the diagram,

$$\begin{array}{ccc} X & \mathcal{F}X \xrightarrow{\kappa_X} \mathcal{G}X \\ \downarrow f & \mathcal{F}(f)\downarrow \qquad \downarrow \mathcal{G}(f) \\ Y & \mathcal{F}Y \xrightarrow[\kappa_Y]{} \mathcal{G}Y \end{array} \tag{natural transformation}$$

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors $\mathcal{F}$ and $\mathcal{G}$ is denoted by,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

We say two functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $\mathcal{F} \cong \mathcal{G}$. The collections of all functors from $\mathcal{A}$ to $\mathcal{B}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{B}^{\mathcal{A}}$. The nice thing about functor category $\mathcal{B}^{\mathcal{A}}$ is that if $\mathcal{B}$ has some nice properties then $\mathcal{B}^{\mathcal{A}}$ inherits these useful properties.

Equivalence of two categories can be thought of as giving two complementary description of same mathematical object. We can compare two categories $\mathcal{A}$ and $\mathcal{B}$ via the functors between them. The starting point is the functor category $\mathcal{B}^{\mathcal{A}}$.

A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if there is a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that

$$\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}, \quad \text{and} \quad \mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}},$$

where the identity functor $\mathbb{1}_{\mathcal{A}}$ sends objects of $\mathcal{A}$ to the same objects, and morphisms to the same morphisms. $\mathcal{G}$ is called quasi-inverse functor. In such a case, $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent. Quotient categories can be defined when we have an equivalence relation on the collection of morphisms. The objects remain the same, and the hom-sets get quotiented.

1.2 | Representable Functors

Many of the definitions and properties of algebraic objects can be expressed in categorical language. Representable functor define new properties using functors we understand well. Definitions are simpler to study and they inherit many interesting properties from nicely behaved categories such as the category of sets.

Each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ tells us about all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{A}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{A}}(-, X)$. These are called hom functors.

$$\begin{aligned} h^X : \ & \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X). \end{aligned}$$

which maps each morphism $f : Y \to Z$ to a morphism of hom sets given by the composition,

$$Y \xrightarrow{f} Z \xrightarrow{g} X$$

We will denote this by,

$$\begin{aligned} h^X(f) : \ & h^X(Y) \to h^X(Z) \\ & g \mapsto g \circ f. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{A}}(Y,X)$s are all sets. Such categories are called locally small categories.

A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be $\mathcal{G} \cong h_X$. Where $\cong$ stands for natural isomorphism.

1.2.1 | Yoneda Embedding

Yoneda embedding and representable functors allow us to use the nice properties (ability to take limits) of the category of sets to study more complex categories that are not so nice. We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\mathrm{Hom}_{\mathcal{A}}(-,X)$ and $\mathrm{Hom}_{\mathcal{A}}(X,-)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{A}$ and the functor $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$.

Theorem 1.2.1. (Yoneda Lemma) *For a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{A}$, there is a natural bijection,*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \cong \mathcal{F}X \tag{Yoneda}$$

such that $\kappa \in \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \leftrightarrow \kappa_X(\mathbb{1}_X) \in \mathcal{F}X$.

Proof

In the natural transformation diagram, replace $\mathcal{F}$ by h^X, and $\mathcal{G}$ by $\mathcal{F}$. $\kappa_X : h^X X \to \mathcal{F}X$. Now, $h^X X = \mathrm{Hom}_{\mathcal{A}}(X,X)$, which contains $\mathbb{1}_X$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) &\to \mathcal{F}X \\ \kappa &\mapsto \kappa_X(\mathbb{1}_X). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $Y \in \mathcal{A}$. For any $f : Y \to X$, we have,

$$\begin{tikzcd} X & h^X X \arrow[r, "\kappa_X"] \arrow[d, "h^X(f)"'] & \mathcal{F}X \arrow[d, "\mathcal{F}(f)"] & \mathbb{1}_X \arrow[r, mapsto, "\kappa_X"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ Y \arrow[u, "f"] & h^X Y \arrow[r, "\kappa_Y"'] & \mathcal{F}Y & f \arrow[r, mapsto, "\kappa_Y"'] & \kappa_Y(f) \end{tikzcd}$$

Hence $\kappa_Y(f) = \mathcal{F}(f)(\mu(\kappa))$, or the action of κ_Y is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_Y(f) = \varphi_Y(f)$ for all $Y \in \mathcal{A}$, so it's injective.

For surjectivity we have to show that for all sets $x \in \mathcal{F}X$, there exists a natural transformation φ such that $\varphi_X(\mathbb{1}_X) = x$. For $x \in \mathcal{F}X$, and $f : Y \to X$, construct the map,

$$\begin{aligned} \varphi : h^X &\to \mathcal{F} \\ f &\mapsto \mathcal{F}(f)(x). \end{aligned}$$

this satisfies the requirement that $\varphi_X(\mathbb{1}_X) = x$, because clearly, $\mathbb{1}_X \mapsto \mathcal{F}(\mathbb{1}_X)(x) = \mathbb{1}_x(x) = x$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} Y & h^X Y \arrow[r, "\varphi_Y"] \arrow[d, "h^X(g)"'] & \mathcal{F}Y \arrow[d, "\mathcal{F}(g)"] \\ Z \arrow[u, "g"] & h^X Z \arrow[r, "\varphi_Z"'] & \mathcal{F}Z \end{tikzcd}$$

commutes for all $Y, Z \in \mathcal{A}$, $g \in \mathrm{Hom}_{\mathcal{A}}(Z, Y)$. For $f : Y \to X$, by definition of φ,

$$\mathcal{F}(g) \circ (\varphi_Y(f)) = \mathcal{F}(g) \circ \mathcal{F}(f)(x)$$

which by functoriality of $\mathcal{F}$ is $= \mathcal{F}(f \circ g)(x)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_Z \circ (h_X(g)(f)) = \varphi_Z(h_X(f \circ g))$$

which again by definition of φ is $= \mathcal{F}(f \circ g)(x)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \to \mathcal{F}X$ is a bijection. $\square$

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof covariant version is exactly the same, just have to reverse the arrows on the category $\mathcal{A}$. The Yoneda lemma gives us an embedding of the category $\mathcal{A}$ inside the functor category $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, given by,

$$X \mapsto h^X.$$

This embedding is called the Yoneda embedding $h^{(-)} : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, which sends an object $X \in \mathcal{A}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{A}}(-, X)$. These functors are fully faithful by Yoneda lemma, because by replacing the functor $\mathcal{F}$ by h^Y we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, h^Y) \cong h^Y(X) = \mathrm{Hom}_{\mathcal{A}}(X, Y). \tag{weak Yoneda}$$

Similarly for the covariant embedding, in which case this will be $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{A}}(Y, X)$.

Given a contravariant functor, $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$, the Yoneda tells us that we can think of the action of $\mathcal{F}$ on the element X as natural transformations to the hom functor h^X in the functor category. So, every functor $\mathcal{F}$ can extended and be thought of as a representable functor,

$$\begin{aligned} h^{\mathcal{F}} : (\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{G} &\mapsto \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(\mathcal{G}, \mathcal{F}) \end{aligned}$$

where elements $X \in \mathcal{A}$ are to be interpreted as the elements $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$. The following frequently used corollary of Yoneda lemma allows us to compare objects of locally small category by their hom-functors, and hence using the properties of the category of sets.

Lemma 1.2.2. (Yoneda Principle)

$$h^X \cong h^Y \Rightarrow X \cong Y. \tag{Yoneda principle}$$

$\square$

Note that, Yoneda associates to each set in $\mathcal{F}X$ a natural transformation between h^X and $\mathcal{F}$. If the functor $\mathcal{F}$ is representable, i.e., there exists $Y \in \mathcal{A}$ such that there exists a natural isomorphism,

$$\mathcal{F} \xrightarrow[\alpha]{\cong} h^Y$$

Let $\mu(\alpha)$ be the corresponding element in $\mathcal{F}Y = \mathrm{Hom}_{\mathcal{A}}(Y, Y)$. The pair $(Y, \mu(\alpha))$ is called a universal object for $\mathcal{F}$. It's such that for any other object $Z \in \mathcal{A}$, and each $g \in \mathcal{F}X = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ there exists a unique morphism $f : X \to Y$ such that,

$$\mathcal{F}(f)(\mu(\alpha)) = g.$$

1.2.2 | Representable Constructions

The Yoneda-Grothendieck philosophy now has a precise formulation; the properties of a category can be thought of as representability properties in functor category. An object existing can be stated as some functor being representable. A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be

$$\mathcal{G} \cong h_X$$

where $\cong$ stands for natural isomorphism. We will now see this in the following examples.

1.2.2.1 | Products & Coproducts

Let $\mathcal{A}$ be a category and consider a family $\{X_i\}_{i \in I}$ of objects of $\mathcal{A}$ indexed by a set I, then we can consider the contravariant functor,

$$\mathcal{G} : Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $\mathcal{G}(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, P)$. This is called the product, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
 & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists ! h"'] \arrow[dr, "f_j"] & \\
X_i & \prod_{i \in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I. So, the property that the category $\mathcal{A}$ has products, is translated into a statement that certain functor is representable.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $\mathcal{F}(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i \in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd} X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i \in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\ & Y & \end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

1.2.2.2 | Kernel & Cokernel

For sets, the kernel of two maps s, t is defined as the set $\ker(s, t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{A}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"] \arrow[drr, bend left, "j"] & & \\ \ker(f,g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \end{tikzcd}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object Z is called a zero object if for any object A, there exists a unique morphism $Z \to A$ and a unique morphism $A \to Z$. It's unique upto isomorphism and denoted by 0. Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \to \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{A}}(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(Z, X)\big).$$

This can be visualized by the diagram,

$$\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f, g) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd}
Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

1.2.2.3 | Pullback or Fibered Product

Let $\mathcal{I}$ be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \to Z$ then for functors $\mathcal{F} : \mathcal{I} \to \mathcal{A}$, pullback $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$\mathcal{F}Y \xrightarrow{f} \mathcal{F}X \xleftarrow{g} \mathcal{F}Z$$

in a category $\mathcal{A}$ is the commutative square with vertex $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ such that any other commutative square factors through it, i.e.,

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_P"] \arrow[ddr, bend right, "g_P"'] & & \\
& \mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}Z \arrow[d, "g"] \\
& \mathcal{F}Y \arrow[r, "f"'] & \mathcal{F}X
\end{tikzcd}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ as above,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{F}Z \arrow[d, dashed] \arrow[ddr, bend left, "f_P"] & \\
\mathcal{F}Y \arrow[r, dashed] \arrow[drr, bend right, "g_P"'] & \mathcal{F}Y \coprod_{\mathcal{F}X} \mathcal{F}Z \arrow[dr, dashed, "\exists!"] & \\
& & C
\end{tikzcd}$$

1.2.2.4 | Exponentiation

The categorical notions of product and coproduct correspond to the arithematic operations such as multiplication and addition. We can similarly talk about exponentiation. In the category of sets, **Sets**, for $X, Z \in \mathcal{A}$, Z^X is the function set consisting of all functions $h : X \to Z$. Here we have the bijection,

$$\mathrm{Hom}(Y \times X, Z) \to \mathrm{Hom}(Y, Z^X).$$

for a function, $f : Y \times X \to Z$, this map sends each $y \in Y$ to the function $f(y, -) \in Z^X$. Conversely given a function $f' : Y \to Z^X$, we can define a map $f(y, x) = f'(y)(x)$. So,

$$\mathrm{Hom}(Y \times X, Z) \cong \mathrm{Hom}(Y, Z^X)$$

or equivalently, $(-)^X$ is the right adjoint of $(-) \times X$. By setting $Y = 1$, we obtain,

$$Z^X \cong \mathrm{Hom}(1, Z^X) \cong \mathrm{Hom}(X, Z).$$

Exponentiation can be representably defined in terms of products.

1.3 | Adjoint Situations

Categories are compared by means of functors, and functors themselves are compared via natural transformations. Equivalence of categories allows us to basically think of the two categories as the same thing. This is however too restrictive. The relaxation of the notion of equivalence gives us the notion of adjoint.

The philosophy of adjoint functors is the following; when we want to study an object in mathematics, belonging to some weird category, we can take it, via a functor to some well understood category. But now this new category will not have the same meaning to the objects as the original category. So we would like a functor to get back to the original category. This functor is the adjoint functor.

An adjuntion from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functors,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B},$$

such that there is a natural isomorphisms of bifunctors $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y)$ and $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \tag{adjoint}$$

for all $X \in \mathcal{A}, Y \in \mathcal{B}$. Denote by $\mathcal{F} \dashv \mathcal{G}$. Since composition of natural isomorphisms is also a natural isomorphism if $\mathcal{F}$ has two adjoints $\mathcal{G}$ and $\widehat{\mathcal{G}}$, then we have,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, \widehat{\mathcal{G}}Y).$$

So, by Yoneda principle, adjoints if they exist are unique upto isomorphism. Consider the following two adjoint situations,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\leftrightarrows}} \mathcal{B} \underset{\mathcal{K}}{\overset{\mathcal{H}}{\leftrightarrows}} \mathcal{C},$$

By definition, we have for all $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{H} \circ \mathcal{F}X, Y)$, hence,

$$\mathcal{F} \circ \mathcal{H} \dashv \mathcal{G} \circ \mathcal{K}$$

When we are working with locally small categories, we can exploit the properties of the category of sets. We can look at adjoints from a functor category perspective, and representable functors.

Given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, for each $X \in \mathcal{B}$, we have the composite functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) := h^X \circ \mathcal{F} : \mathcal{A} \to \mathcal{B} &\to \mathbf{Sets} \\ A &\mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

So, $\widehat{\mathcal{F}}$ is a functor to the functor category,

$$\widehat{\mathcal{F}} : \mathcal{B} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}},$$

which sends $X \mapsto \widehat{\mathcal{F}}(X)$. For each morphism $f : X \to Y$ in $\mathcal{B}$, the functor $\widehat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \to X$,

$$\widehat{\mathcal{F}}(f) : g \mapsto f \circ g$$

So, $\widehat{\mathcal{F}}(f \circ h) = \widehat{\mathcal{F}}(f) \circ \widehat{\mathcal{F}}(h)$, i.e., it's a covariant functor.

Lemma 1.3.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$ admits a left adjoint iff for all $X \in \mathcal{B}$,*

$$\widehat{\mathcal{F}}(X) : A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$$

is representable.

Proof

$\Leftarrow$ Suppose $\widehat{\mathcal{F}}(X)$ is representable for all $X \in \mathcal{B}$, then, $\exists\ \mathcal{G}X \in \mathcal{A}$ with, $\widehat{\mathcal{F}}(X) \cong h^{\mathcal{G}X}$, i.e.,

$$\widehat{\mathcal{F}}(X)(A) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X)$$

We have to make sure this is functorial, i.e., $X \mapsto \mathcal{G}X$ is a functor from $\mathcal{B}$ to $\mathcal{A}$. So, now we have to show that for each morphism $f : X \to Y$ there exists a morphism in the functor category, a natural transformation of functors, $\mathcal{G}f : \widehat{\mathcal{F}}(X) \to \widehat{\mathcal{F}}(Y)$, defined to be the maps that makes the following diagram commute.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X) \arrow[r] \arrow[d, "\mathcal{G}(f)\circ"'] & \widehat{F}(X)(A) \arrow[d, "\widehat{\mathcal{F}}(f)"] \\ \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}Y) \arrow[r] & \widehat{F}(Y)(A) \end{tikzcd}$$

This also satisfies the composition needs by construction. By Yoneda lemma, this determines the functor $\mathcal{G}$ uniquely upto isomorphism.

$\Rightarrow$ The other direction is obvious and follows directly from the definition of adjoint, i.e., if there exists a left adjoint $\mathcal{G} \dashv \mathcal{F}$ each functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ is representable with representative $\mathcal{G}X$. $\square$

1.3.1 | Adjoints as Reflections

The notion of reflection of a functor provides a bit more intuitive meaning of what adjoints are doing. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor. We want to associate to each object $B \in \mathcal{B}$ and object $R_B \in \mathcal{A}$ such that $\mathcal{F}R_B$ is the best estimate of B in $\mathcal{A}$. In categorical terms, the estimation is done with morphisms. So, the 'best estimate' is a morphism,

$$\kappa_B : B \to \mathcal{F}R_B$$

such that for any other $A \in \mathcal{A}$, with an estimation $\varkappa : B \to \mathcal{F}A$ factors uniquely through R_B. R_B together with the morphism κ_B is called the reflection of B along $\mathcal{F}$. Visualised by the diagram,

$$\begin{tikzcd} R_B \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_B \arrow[d, "\mathcal{F}(f)"'] & B \arrow[l, "\kappa_B"'] \arrow[dl, "\varkappa"] \\ A & \mathcal{F}A & \end{tikzcd} \tag{reflection}$$

that's to say there exists a unique morphism $f : R_B \to A$ such that,

$$\mathcal{F}(f) \circ \kappa_B = \varkappa.$$

Intuitively κ_B is a better estimate than $\varkappa$. We can't have two best estimates κ_B and κ'_B because in that case we have two maps $f : R_B \to R'_B$ and $f' : R'_B \to R_B$, such that,

$$\mathcal{F}(f) \circ \kappa_B = \kappa'_B, \quad \mathcal{F}(f') \circ \kappa'_B = \kappa_B$$

So we get,

$$\mathcal{F}(f \circ f') \circ \kappa'_B = \kappa'_B,$$

By uniqueness this means $f \circ f' = \mathbb{1}_{R'_B}$, so any two reflections are isomorphic.

Lemma 1.3.2. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, suppose reflection exists for each $B \in \mathcal{B}$, then there exists a unique functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{G}B = R_B$ and a natural transformation κ such that,*

$$\kappa_B : B \to \mathcal{F} \circ \mathcal{G}B.$$

Proof

The existence of reflection to each object in $\mathcal{B}$ gives us an associated object for each object in $\mathcal{B}$, we want to understand what happens to the morphisms.

Let $f : X \to Y$ be a morphism in $\mathcal{B}$. Then we have,

$$\begin{tikzcd} R_X \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_X \arrow[d, "\mathcal{F}(f)"'] & X \arrow[l, "\kappa_X"'] \arrow[d, "g"] \\ R_Y & \mathcal{F}R_Y & Y \arrow[l, "\kappa_Y"] \end{tikzcd}$$

In this diagram, $\kappa_Y \circ g : X \to \mathcal{F}R_Y$ is an estimate, and hence there must exist a morphism $f : R_X \to R_Y$ such that $\mathcal{F}(f) \circ \kappa_X = \kappa_Y \circ g$. So, we define,

$$\mathcal{G}(g) := f.$$

By construction this makes κ a natural transformation which is determined by the components κ_X. By exploiting uniqueness we can show that this is functorial, that is $\mathcal{G}(g \circ h) = \mathcal{G}(g) \circ \mathcal{G}(h)$. □

If the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a reflection, by definition, to each morphism $\varkappa : X \to \mathcal{F}A$ there exists a unique morphism $f : \mathcal{G}X \to A$. Conversely, any map f uniquely determines $\varkappa$ by,

$$\mathcal{F}(f) \circ \kappa_X = \varkappa.$$

Which means we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, A) \cong \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}A).$$

Since this holds for every X and A it is the adjoint condition. So, adjoint and reflection are the same functor. Note that in the construction of the 'reflection functor' we assumed every object in $\mathcal{B}$ has a reflection. The adjoint functor theorems try to simplify this condition, under additional constraints.

1.4 | Limits & Colimits

The notion of limits and colimits is very important as they allow us to construct new objects and functors. They are also very closely related to adjoint functors. To heuristacally motivate, limits is the categorical notion of 'closest' object to or from a system. Here, we intend to find an object that's nearest to a category. The notion of nearness comes from the morphisms, so the idea is to put the starting category in some other category, where the morphisms provide some sort of 'categorical distance', and then use this notion of distance of the target category to describe the 'limit'.

Let $\mathcal{I}$ and $\mathcal{A}$ be two categories. An inductive system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{F} : \mathcal{I} \to \mathcal{A}.$$

Intuitively, the limit of a system is an object in $\mathcal{A}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $X \in \mathcal{A}$ the constant functor $\Delta X : \mathcal{I} \to \mathcal{A}$ that sends everything in $\mathcal{I}$ to X, and each morphism in $\mathcal{I}$ to the identity on X. A relation between an object X and the system $\mathcal{F}$ is a natural transformation between ΔX and $\mathcal{F}$. Such a natural transformation is called a cone. The collection of all such cones is the set of all natural transformations,

$$\begin{aligned} C_{\mathcal{F}} : \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}). \end{aligned}$$

It's a contravariant functor from $\mathcal{A}$ to **Sets**. If the functor $C_{\mathcal{F}}$ is representable, there exists an object $L_{\mathcal{F}} \in \mathcal{A}$ such that,

$$C_{\mathcal{F}} \cong h^{L_{\mathcal{F}}}.$$

So, in such case $C_{\mathcal{F}}(X) \cong \mathrm{Hom}_{\mathcal{A}}(X, L_{\mathcal{F}})$. The representative $L_{\mathcal{F}}$ if it exists is called the colimit of the system $\mathcal{F}$, and is denoted by $\varprojlim \mathcal{F} \coloneqq L_{\mathcal{F}}$ i.e.,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \tag{colimit}$$

and hence every cone must factor through $L_{\mathcal{F}}$. Intuitively the limit is the 'closest' object to the system. The notion of closeness must come from morphisms, so if there exists any other object with morphisms to the system, then it must be 'farther' than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

This means that for all objects $X \in \mathcal{A}$ and all family of morphisms $f_I : X \to \mathcal{F}I$, in $\mathcal{A}$ such that for all $f \in \mathrm{Hom}_{\mathcal{I}}(I, J)$, with $f_J = f_I \circ \mathcal{F}(f)$ factors uniquely through $\varprojlim \mathcal{F}$.

$$\begin{tikzcd}
& X \arrow[ddl, "f_I"'] \arrow[d, dashed] \arrow[ddr, "f_J"] & \\
& \varprojlim \mathcal{F} \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_J"'] & \\
\mathcal{F}I \arrow[rr, "\mathcal{F}(f)"'] & & \mathcal{F}J
\end{tikzcd}$$

A projective system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}.$$

Similar to the inductive system, for projective system $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^{\mathcal{G}} &: \mathcal{A} \to \mathbf{Sets} \\ & X \mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X). \end{aligned}$$

If it's representable with representative $L^{\mathcal{G}}$,

$$C^{\mathcal{G}} \cong h_{L^{\mathcal{G}}}.$$

Denote the representative by $\varinjlim \mathcal{G} \coloneqq L^{\mathcal{G}}$ is called the limit of the projective system. If the limit exists, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X) \cong \mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, X) \tag{limit}$$

Projective limits can be written in terms of universal property as,

$$\begin{tikzcd}
\mathcal{G}I \arrow[rr, "\mathcal{G}(f)"] \arrow[dr, "\varphi_I"] \arrow[ddr, "f_I"'] & & \mathcal{G}J \arrow[dl, "\varphi_J"'] \arrow[ddl, "f_J"] \\
& \varinjlim \mathcal{G} \arrow[d, dashed] & \\
& X &
\end{tikzcd}$$

Note that if $\mathcal{I}$ admits initial object 0, then the limit $\varprojlim \mathcal{F}$ corresponds to the object $\mathcal{F}(0)$. Similarly for colimit, with terminal object.

1.4.1 | Limit-Colimit Calculus

A category $\mathcal{A}$ is cocomplete with respect to $\mathcal{I}$ if for all inductive systems indexed by $\mathcal{I}$, the colimit exists, if $\mathcal{I}$ is not explicitly said, then it means that $\mathcal{A}$ is cocomplete with respect to all small categories. $\mathcal{A}$ is complete with respect to $\mathcal{I}$ if it has all limits for all projective systems indexed by $\mathcal{I}$. $\mathcal{A}$ is called bicomplete if it's both complete and cocomplete.

Theorem 1.4.1. *$\mathcal{A}$ is cocomplete $\Rightarrow \mathcal{A}^{\mathcal{K}}$ is cocomplete.*

Proof

Given an inductive system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$, the goal is to construct a new functor $\varinjlim \mathcal{F}$ in $\mathcal{A}^{\mathcal{I}}$ such that,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varinjlim \mathcal{F})$$

where $\Delta^{\mathcal{K}}$ is the constant functor in $\mathcal{A}^{\mathcal{K}}$ which sends $A \in \mathcal{A}$ to the constant functor $\Delta^{\mathcal{K}}A$. This is then by definition the colimit of the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$.

In order to do this we have to describe where $\varinjlim \mathcal{F}$ sends elements of $\mathcal{I}$. The construction involves argument wise assignment of objects in $\mathcal{A}$ for each object in $\mathcal{I}$, and then showing the functoriality, that is verify it respects composition of morphisms.

Construction. By definition of colimit, for every $I \in \mathcal{I}$, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}I, \mathcal{F}I) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}I, \varinjlim(\mathcal{F}I)) \qquad \text{(isomorphism)}$$

here $\mathcal{F}I : \mathcal{K} \to \mathcal{A}$ is a fixed functor, an inductive system and $\varinjlim(\mathcal{F}I)$ its inductive limit. So we define the associated object by,

$$\varinjlim \mathcal{F}(I) := \varinjlim(\mathcal{F}I).$$

Functoriality.

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}(\cdot), \mathcal{F}(\cdot)) : \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathbf{Sets}$$

is a bifunctor, so for each natural transformation $\kappa : \Delta^{\mathcal{K}}\mathcal{H} \Rightarrow \mathcal{F}$,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, bend left, "\mathcal{F}"] \arrow[dr, "\mathcal{H}"'] & \arrow[d, phantom, "\Uparrow \kappa"] & \mathcal{A}^{\mathcal{K}} \\ & \mathcal{A} \arrow[ur, "\Delta^{\mathcal{K}}"'] & \end{tikzcd}$$

and morphism $f : I \to J$, we have maps,

$$\begin{tikzcd} I \arrow[d, "f"'] & \Delta^{\mathcal{K}}\mathcal{H}I \arrow[r, "\kappa_I"] \arrow[d, "\Delta^{\mathcal{K}}\mathcal{H}(f)"'] & \mathcal{F}I \arrow[d, "\mathcal{F}(f)"] \\ J & \Delta^{\mathcal{K}}\mathcal{H}J \arrow[r, "\kappa_J"'] & \mathcal{F}J \end{tikzcd}$$

The commutative square gives us, $\kappa_J \circ \Delta^{\mathcal{K}}\mathcal{H}(f) = \mathcal{F}(f) \circ \kappa_I$, and the isomorphism of sets gives us a morphisms, $\widehat{\kappa}_I$ and $\widehat{\kappa}_J$ such that,

$$\widehat{\kappa}_I \circ \mathcal{H}(f) = \varinjlim(\mathcal{F}(f)) \circ \widehat{\kappa}_J.$$

So, $\widehat{\kappa} : \mathcal{H} \to \varprojlim \mathcal{F}(\cdot)$ is a natural transformation. So we get,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

Since this natural transformation is defined using the natural transformation κ it will satisfy the required compositions. So we have, $\varprojlim \mathcal{F}(f \circ g) = (\varprojlim \mathcal{F}(f)) \circ (\varprojlim \mathcal{F}(g))$. $\square$

This gives us an adjoint situation, where colimit is left-adjoint to the constant functor and the constant functor is left-adjoint to the limit,

$$\varinjlim \dashv \Delta \dashv \varprojlim \qquad \text{(limit-diagonal adjointness)}$$

Theorem 1.4.2.

$$\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}),$$

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, A) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, A).$$

Proof

The idea is to study an appropriate functor category, get hom-set isomorphisms and then apply Yoneda principle. So, we have to show for each set $X \in \mathbf{Sets}$, we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$$

Using the limit-diagonal adjointness, this reduces to showing $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$ and $\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$ are isomorphic. Δs are in the appropriate categories.

Let $\kappa : \Delta X \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$ be a natural transformation, then κ is determined by its components

$$\kappa_I : (\Delta X)I \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I).$$

Each $(\Delta X)I$ is a set, and hence the maps κ_I is itself determined by its action on the elements of the set X, So, for each $x \in X$, $\kappa_I(x)$ is a morphism in $\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I)$.

If we think of A as the constant functor, we can define using $\varphi_x(\cdot) := \kappa_{(\cdot)}(x)$ defines an element $\varphi \in \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$. This is a bijection and hence,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$$

Applying the limit-diagonal adjointness again to this and the Yoneda principle, we get that $\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$. The other isomorphism is also similar. $\square$

So, hom-functor takes limits to colimits in the first argument and takes limits to limits in the second argument. This can now be used to prove many things easily.

Intuitively, limits take us to the 'last object' in a diagram and colimits take us to the 'first object' in a diagram. Here 'most' is quantified in terms of morphisms, and first and last are quantified by the direction of the morphisms. So, if the indexing category is a product category, then we can think of limits as trying to find the bottom right corner of the rectangle diagram, and similarly the colimit is the top left corner. So, whether we go to right most first and then go to the bottom or whether go to the bottom first and then go to the right shouldn't change which object we reach after doing this. We now formalize this.

Theorem 1.4.3. (Fubini for limits)

$$\varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathcal{F} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F} \cong \varinjlim_{\mathcal{I}\times\mathcal{J}} \mathcal{F}$$

Proof

This can be proved using the relation between the constant functors. Let $\Delta^{\mathcal{I}\times\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}\times\mathcal{J}}$, $\Delta^{\mathcal{I}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ and $\Delta^{\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ be the constant functors in the appropriate functor category. Then we have, $\Delta^{\mathcal{I}\times\mathcal{J}} = \Delta^{\mathcal{I}}\Delta^{\mathcal{J}}$. This gives us,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{I}\times\mathcal{J}} \mathcal{F}) &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}\times\mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}}\Delta^{\mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F}). \end{aligned}$$

By Yoneda principle we have the required isomorphism. Since $\mathcal{I} \times \mathcal{J} \cong \mathcal{J} \times \mathcal{I}$, the other isomorphism also follows. □

Lemma 1.4.4. *Right adjoints preserve limits.*

Proof

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint say $\mathcal{G} : \mathcal{B} \to \mathcal{A}$, and suppose $\mathcal{H} : \mathcal{I} \to \mathcal{A}$ is an inductive system with a limit $\varprojlim \mathcal{H}$, we must prove that $\mathcal{F}(\varprojlim \mathcal{H})$ is the limit of the inductive system $\mathcal{F} \circ \mathcal{H}$.

$\mathcal{F}\mathcal{H} : \mathcal{I} \to \mathcal{B}$ is an inductive system in $\mathcal{B}$ indexed by $\mathcal{I}$. For all $X \in \mathcal{B}$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F} \varprojlim \mathcal{H}) &\cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \varprojlim \mathcal{H}) \\ &\cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \mathcal{H}) \\ &\cong \varprojlim \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}\mathcal{H}) \\ &\cong \mathrm{Hom}_{\mathcal{B}}(X, \varprojlim \mathcal{F}\mathcal{H}) \end{aligned}$$

By Yoneda principle, we have, $\mathcal{F}(\varprojlim \mathcal{H}) \cong \varprojlim \mathcal{F} \circ \mathcal{H}$. □

Right adjoints preserve limits, can be remembered by the acronym, RAPL. Under additional conditions on the category $\mathcal{A}$, the converse holds, these theorems are called adjoint functor theorems.

1.4.2 | End-Coend Calculus

Similar to how natural transformations relate functors between two categories, dinatural transformations relate bifunctors. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors. Bifunctors are functors when one of the arguments are fixed. We will denote $\mathcal{F}^X$ for the contravariant functor $\mathcal{F}(\cdot, X)$, and $\mathcal{F}_X$ for the covariant functor $\mathcal{F}(X, \cdot)$.

Each morphism $f : X \to Y$ gives rise to the following morphisms in $\mathcal{B}$,

$$\begin{tikzcd} X \arrow[dd, "f"] & & \mathcal{F}(X,X) \arrow[dl, "\mathcal{F}^X(f)"'] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\ & \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \arrow[dl, "\mathcal{G}^Y(f)"] \\ Y & & \mathcal{F}(Y,Y) & \mathcal{G}(Y,Y) & \end{tikzcd}$$

A dinatural transformation $\kappa : \mathcal{F} \to \mathcal{G}$ consists of a family of morphisms,

$$\kappa_X : \mathcal{F}(X, X) \to \mathcal{G}(X, X)$$

such that for any $f : X \to Y$ the following commutes,

$$\begin{tikzcd} X \arrow[dd, "f"'] \\ \\ J \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{F}(X,X) \arrow[r, "\kappa_X"] \arrow[dl, "\mathcal{F}^X(f)"'] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\ \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \\ & \mathcal{F}(Y,Y) \arrow[r, "\kappa_Y"] & \mathcal{G}(Y,Y) \arrow[ur, "\mathcal{G}^Y(f)"'] & \end{tikzcd}$$

Similar to the case of limits and cones, we can describe ‘doubly indexed limits’. Suppose we are given a system $\mathcal{F} : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{B}$, intuitively, the end of the system is an object in $\mathcal{B}$ that is ‘closest’ to the system.

This can be formalised using the functor category as follows; Attach to each object $E \in \mathcal{B}$ the constant bifunctor $\Delta E : \mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{B}$ that sends everything in $\mathcal{A}^{\text{op}} \times \mathcal{A}$ to E, and each morphism in $\mathcal{B}$ to the identity on E. A relation between an object X and the system $\mathcal{F}$ is a dinatural transformation δ between ΔE and $\mathcal{F}$. Represented by the diagram,

$$\begin{tikzcd} X \arrow[dd, "f"'] \\ \\ Y \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{F}(X,X) \arrow[dr, "\mathcal{F}^X(f)"] & \\ E \arrow[ur, "\delta^E_X"] \arrow[dr, "\delta^E_Y"'] & & \mathcal{F}(X,Y) \\ & \mathcal{F}(Y,Y) \arrow[ur, "\mathcal{F}_Y(f)"'] & \end{tikzcd}$$

Such a dinatural transformation is called a wedge. The collection of all such wedges is the set of all such dinatural transformations,

$$\begin{aligned} W_{\mathcal{F}} : \mathcal{B} &\to \textbf{Sets} \\ E &\mapsto \mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\text{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}). \end{aligned}$$

The end of the system $\mathcal{F}$ is intuitively the object which is closest to the system. A relation between an object $E \in \mathcal{B}$ and the system $\mathcal{F}$ consists of a dinatural transformation from the constant bifunctor ΔE to $\mathcal{F}$. So, the ‘closest’ would be such that any other dinatural transformation should factor through the ‘closest’ one.

The end of a system $\mathcal{F}$, is an object $\int_{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\omega : \Delta \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$ such that every other wedge factors through it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\text{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(E, \int_{\mathcal{A}} \mathcal{F}) \tag{end}$$

Also denoted by $\int_{A \in \mathcal{A}} \mathcal{F}A$. Expressed in a commutative diagram by,

$$\begin{tikzcd} E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\kappa_X"] \arrow[ddr, bend right, "\kappa_Y"'] & & \\ & \int_{\mathcal{A}} \mathcal{F} \arrow[r, dashed, "\omega_X"] \arrow[d, dashed, "\omega_Y"'] & \mathcal{F}(X,X) \arrow[d, "\mathcal{F}_X(f)"] \\ & \mathcal{F}(Y,Y) \arrow[r, "\mathcal{F}^Y(f)"'] & \mathcal{F}(X,Y) \end{tikzcd}$$

The integral notation corresponds to the intuition that the equalizer given in $\int_{\mathcal{A}}$ can be thought of as an averaging operation on the functor where we run through the objects of $\mathcal{A}$.

The coend of a system $\mathcal{F}$, is an object $\int^{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\sigma : \mathcal{F} \to \Delta \int^{\mathcal{A}} \mathcal{F}$, such that every other cowedge factors from it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\mathcal{F}, \Delta E) \cong \mathrm{Hom}_{\mathcal{B}}(\int^{\mathcal{A}} \mathcal{F}, E) \tag{coend}$$

Expressed in a commutative diagram by,

$$\begin{tikzcd}
\mathcal{F}(Y,X) \arrow[r, "\mathcal{F}_Y(f)"] \arrow[d, "\mathcal{F}^X(f)"'] & \mathcal{F}(Y,Y) \arrow[d, dashed] \arrow[rdd, bend left, "\kappa_Y"] & \\
\mathcal{F}(X,X) \arrow[r, dashed] \arrow[rrd, bend right, "\kappa_X"'] & \int^{\mathcal{A}} \mathcal{F} \arrow[rd, dashed, "\exists!"] & \\
 & & E
\end{tikzcd}$$

Functoriality. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors and let $\eta : \mathcal{F} \to \mathcal{G}$ be a dimatural transformation. Any connection from an object E to the system $\mathcal{F}$ gives rise to a connection from the object to the system $\mathcal{G}$, given by the composition.

In particular we have the composition for the connection $\omega : \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$. So, this connection must factor through $\int_{\mathcal{A}} \mathcal{G}$, and hence we have a map, whose components are given by the composition,

$$\begin{tikzcd}
\int_{\mathcal{A}} \mathcal{F} \arrow[rd, dashed, "\exists!"] \arrow[rrd, bend left, "\eta_X \circ \omega_X"] \arrow[rdd, bend right, "\eta_Y \circ \omega_Y"'] & & \\
 & \int_{\mathcal{A}} \mathcal{G} \arrow[r, dashed, "\varphi_X"] \arrow[d, dashed, "\varphi_Y"'] & \mathcal{G}(X,X) \arrow[d, "\mathcal{G}_X(f)"] \\
 & \mathcal{G}(Y,Y) \arrow[r, "\mathcal{G}^Y(f)"'] & \mathcal{G}(X,Y)
\end{tikzcd}$$

This is functorial since the $\int_{\mathcal{A}} \eta : \int_{\mathcal{A}} \mathcal{F} \to \int_{\mathcal{A}} \mathcal{G}$ are defined component wise and so the composition will be component wise. So $\int$ is functorial, that is, $\int_{\mathcal{A}}(\eta \circ \kappa) = \int_{\mathcal{A}} \eta \circ \int_{\mathcal{A}} \kappa$.

Fubini Rule. Given a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$, the end in the first two coordinates, gives rise to a functor,

$$\int_{\mathcal{A}} \mathcal{F} : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}.$$

The end of this functor is then,

$$\int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F} \in \mathcal{C}$$

With some tedious checking it turns out that,

$$\int_{\mathcal{A} \times \mathcal{B}} \mathcal{F} \cong \int_{\mathcal{A}} \int_{\mathcal{B}} \mathcal{F} \cong \int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F}. \tag{Fubini for ends}$$

1.4.2.1 | (Co)ends as (Co)limits

The motivation for the definition of ends/coends was very similar to that of limits/colimits. So we should expect the two concepts to be closely related. To intuitively motivate the relation between ends and limits, we have to intuitively understand how ends describe average of a system and limits describe closeness to the system. Limits only take into account the relation an object has 'with' the system, that is, a natural transformation from constant functor to the system. In case of ends, the relation 'between' the objects of the system is important, this is encoded in the dinatural transformation. So, the 'average' should also be expected to be some sort of limit taken over morphisms between objects of the indexing category. This is what makes it average over the relations between objects of the system.

We can think of ends and coends as limits and colimits. The first step is then to associate to each bifunctor system a functor, and turn the ends/coends of bifunctor systems into limits/colimits of systems. This involves estalishing an equivalence between the category of bifunctors and an appropriate category of functors. Consider a bifunctor

$$\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$$

attaches to pairs of objects in $\mathcal{A}$ objects in $\mathcal{B}$ such that the relation between the pair of objects is preserved. So, we could think of the functor as assigning to each morphism between pairs fo objects in $\mathcal{A}$ an object of $\mathcal{B}$. So, we can start with a new category where objects are morphisms of the category $\mathcal{A}$.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$, the bifunctor assigns to the pair X, Y the object $\mathcal{F}(X, Y)$, so instead we could assign to the morphism f, the object $\mathcal{F}(\mathrm{src}(f), \mathrm{tgt}(f))$. Where src is the source object of f, and $\mathrm{tgt}(f)$ is the target. To make sure the bifunctoriality transfers to functoriality of this new association we have to define the morphisms suitably. For two every morphism $f, g \in \mathrm{Hom}_{\mathcal{A}}$ define a morphism between them to be a pair of morphisms,

$$\begin{tikzcd} X \arrow[d, "f"'] & \widehat{X} \arrow[l] \arrow[d, "g"] \\ Y \arrow[r] & \widehat{Y}. \end{tikzcd} \tag{bimorph}$$

Note that the reverse direction of the connection from $\widehat{X}$ to X is necessary to make the association a functor. Because the bifunctor is contravariant in the first argument, with the second argument fixed. So with this, we have a new functor,

$$\widehat{\mathcal{F}} : \widehat{\mathcal{A}} \to \mathcal{B}$$

where $\widehat{\mathcal{A}}$ is the category consisting of morphisms of $\mathcal{A}$ as objects and the above rule for morphisms, bimorph. So the functor,

$$\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}} \to \mathcal{B}^{\widehat{\mathcal{A}}}, \quad \mathcal{F} \mapsto \widehat{\mathcal{F}},$$

is an equivalence of categories, and must respect initial/terminal objects. Since limits/colimits and ends/coends are initial/terminal objects in the respective categories we have,

$$\int_{\mathcal{A}} \mathcal{F} \cong \varprojlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}, \quad \int^{\mathcal{A}} \mathcal{F} \cong \varinjlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}. \tag{(co)ends as (co)limits}$$

For a detailed proof, see [?]. This immediately leads to the following observation that if $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ preserves all limits then it preserves all the ends that exist. If $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$ is a system, and $\mathcal{G}$ is a continuous functor then

$$\mathcal{G}(\int_{\mathcal{A}} \mathcal{F}) \cong \int_{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

If $\mathcal{G}$ is a contravariant functor and cocontinuous, then,

$$\mathcal{G}(\int_{\mathcal{A}} \mathcal{F}) \cong \int^{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

An immediate corollary is that Hom functors preserve ends in the second argument and maps ends to coends in the first argument.

Corollary 1.4.5. (Hom-(Co)End relations)

$$\mathrm{Hom}_{\mathcal{B}}(\int_{\mathcal{A}} \mathcal{F}, A) \cong \int^{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, A)$$

$$\mathrm{Hom}_{\mathcal{B}}(A, \int_{\mathcal{A}} \mathcal{F}) \cong \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(A, \mathcal{F}).$$

□

Theorem 1.4.6.

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A).$$

Proof

We have to show that $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$ satisfies the universal property of ends. That's, any collection of morphisms from an object $X \in$ **Sets** to the system $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-), \mathcal{G}(-))$ factors through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$.

Suppose we have a wedge $\tau : X \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-), \mathcal{G}(-))$, then for each $A \in \mathcal{A}$ we have a morphisms $\tau_A : X \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$. Since X is a set this is a set map, which maps each $x \in X$ to a morphism between the objects $\mathcal{F}A$ and $\mathcal{G}A$. That's to say $\tau_A(x) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$.

For a morphism $f : A \to B$, the wedge condition says that,

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\tau_A"] \arrow[ddr, bend right, "\tau_B"'] & & \\
& \int_{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \arrow[r, dashed, "\omega_A"] \arrow[d, dashed, "\omega_B"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \arrow[d, "\circ\mathcal{F}(f)"] \\
& \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}B, \mathcal{G}B) \arrow[r, "\mathcal{G}(f)\circ"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}B)
\end{tikzcd}$$

So, we have,

$$\mathcal{G}(f) \circ \tau_A(x) = \tau_B(x) \circ \mathcal{F}(f).$$

This means that $\tau_{(-)}(x)$ is a natural transformation,

$$\begin{array}{ccc} A & & \mathcal{F}A \xrightarrow{\tau_A(x)} \mathcal{G}A \\ \downarrow f & & \mathcal{F}(f)\downarrow \qquad\quad \downarrow \mathcal{G}(f) \\ B & & \mathcal{F}B \xrightarrow[\tau_B(x)]{} \mathcal{G}B \end{array}$$

So, it must factor through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$. This is precisely the universal property of ends. So,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A).$$

□

The collection of all natural transformations from $\mathcal{F}$ to $\mathcal{G}$ can be thought of as taking an average of elements of $\mathcal{A}$ as 'measured' by the functor $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-), \mathcal{G}(-))$.

1.4.3 | Weighted limits & Colimits

1.4.4 | Density theorem for Pre-Sheaves

We now have the necessary tools to prove the famous result about the category of pre-sheaves, $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Every pre-sheaf can be canonically presented as a colimit of representable functors. For the sake of familiarity we will assume that $\mathcal{C} = \mathcal{O}(X)$.

Theorem 1.4.7. (Co-Yoneda/Density Formula) $\mathcal{F} \in \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$,

$$\mathcal{F}V \cong \int^{U \in \mathcal{O}(X)} \mathcal{F}U \times \mathrm{Hom}_{\mathcal{O}(X)}(V, U) \cong \int_{U \in \mathcal{O}(X)} (\mathcal{F}U)^{\mathrm{Hom}_{\mathcal{O}(X)}(U, V)}.$$

Proof

By Yoneda applied to the functor

$$\mathcal{H} : V \mapsto \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W),$$

we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}}(h^V, \mathcal{H}) \cong \mathcal{H}V$$

The following chain of isomorphisms lets us sneak in Yoneda principle,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W) &\cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}}\left(h^V, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}(-), W)\right) \\ &\cong \int_{U \in \mathcal{O}(X)} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}U, W)\right) \\ &\cong \int_{U \in \mathcal{O}(X)} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U \times \mathcal{F}U, W\right) \\ &\cong \mathrm{Hom}_{\mathbf{Sets}}\left(\int^{U \in \mathcal{O}(X)} h^V U \times \mathcal{F}U, W\right) \end{aligned}$$

Here, in the second step we used 1.4.6, and in the third step we used hom-tensor adjointness in **Sets**. The last step follows from hom-sets taking coends to ends in the first argument. So, we have by Yoneda principle,

$$\mathcal{F}V \cong \int^{U \in \mathcal{O}(X)} h^V U \times \mathcal{F}U$$

This holds when the pre-sheaf is to any category that's cartesian closed, and the product are replaced with the correct tensor product in the target category. In case of modules this corresponds to the internal hom-tensor product adjointness. □

When $\mathcal{O}(X)^{\mathrm{op}}$ is the category of open sets, $\mathrm{Hom}_{\mathcal{O}(X)^{\mathrm{op}}}(U, V)$ is nonempty, only when $V \subseteq U$, in which case its a singleton set, with the inclusion map as the only map. If we consider continuous functions as the pre-sheaf, and $V \subset U$, any 'continuous function' belonging to U will give rise to a 'continuous function' in V. Intuitively, we want to include all such functions.

Digression: Enriched Categories

Enriched categories are categories, where the hom-sets have additional structure. The hom-sets in practice usually are richer than merely being sets, they come equipped with additional structure. This part makes sure that the construction we do using the structure of **Sets**, especially with products, can be extended to categories whose hom-sets have additional structure. Here will informally discuss the minimal necessary stuff from the theory of enriched categories so that the reader doesn't feel out of place.

An enriched category is a category in which the hom-sets come equipped with additional structure, that is, the hom-sets are objects in some *enriching* category, usually denoted by $\mathcal{V}$. $\mathcal{V}$ is called the base for enrichment. This already requires the category $\mathcal{V}$ to have some special properties.

Given any two composable morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, and $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$ in the category $\mathcal{A}$, we can consider the composition of the morphisms,

$$g \circ f \in \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

If $\mathcal{A}$ is a $\mathcal{V}$ enriched category, then $f, g, g \circ f \in \mathcal{V}$. So the notion of composition of morphisms in the category $\mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

should correspond to a notion of 'composition of objects' or a product in the enriching category,

$$\odot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

There should be unit object corresponding to the identity morphisms, denoted by $\mathbb{1}_{\mathcal{V}}$ such that multiplication by the unit object leaves every object unchanged. This can be formalized by saying there exist natural transformations such that,

$$\mathbb{1}_{\mathcal{V}} \odot V \cong V \odot \mathbb{1}_{\mathcal{V}} \cong V \tag{identity}$$

for all $V \in \mathcal{V}$. Since composition of morphisms is associative, we want $\odot$ to also be associative. This can be formalized by saying there exist natural transformations such that,

$$U \odot (V \odot W) \cong (U \odot V) \odot W \tag{associativity}$$

These properties can also be formalized in terms of a commutative diagram but we will skip that. A category $\mathcal{V}$ with a 'product' $\odot$ with identity, and associativity is called a monoidal category. $\odot$ is called the monoidal product. If in addition the product is such that $U \odot V \cong V \odot U$ it's called a symmetric monoidal category.

A category $\mathcal{A}$ is said to be enriched by a monoidal category $\mathcal{V}$ if the hom-sets belong to $\mathcal{V}$ and the composition corresponds to the monoidal product $\odot$. We will denote the hom-sets of a $\mathcal{V}$-enriched category $\mathcal{A}$ by,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X, Y)$$

for all $X, Y \in \mathcal{A}$. When there is no confusion we will drop the superscript $\mathcal{V}$, or sometimes $\mathcal{A}^{\mathcal{V}}(X, Y)$

A closed monoidal category is monoidal category $\mathcal{V}$ where the functors $- \odot V : \mathcal{V} \to \mathcal{V}$ admits a right adjoint denoted by $\mathcal{H}om^{\mathcal{V}}(V, -)$. The family of right adjoints assemble in a unique way to give a bifunctor,

$$\mathcal{H}om^{\mathcal{V}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}.$$

such that,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(U \times V, W) \cong \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(U, \mathcal{H}om^{\mathcal{V}}(V, W))$$

for all $U, V, W \in \mathcal{V}$. $\mathcal{H}om^{\mathcal{V}}$ is called the internal hom. The internal homs act as the product $\odot$, and hence, $\mathcal{V}$ together with $\mathcal{H}om^{\mathcal{V}}$ is an enriched category over itself.

A category is called cartesian closed if it's locally small, that is, it's enriched by the category of sets, and if $\odot$ is the cartesian product in **Sets**.

The small $\mathcal{V}$-categories themselves form a category. A $\mathcal{V}$-functor is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between $\mathcal{V}$-enriched categories such that the morphisms,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X, Y) \xrightarrow{\mathcal{F}_{X,Y}} \mathrm{Hom}^{\mathcal{V}}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$$

commute with the $\odot$ operation and the identity. Similarly a $\mathcal{V}$-natural transformation between a pair of $\mathcal{V}$-functors $\mathcal{F}$ and $\mathcal{G}$ consist of natural transformation between $\mathcal{F}$ and $\mathcal{G}$ such that it commutes with composition by the natural transformations in identity and associativity. The collection of all $\mathcal{V}$-natural transformations between $\mathcal{F}$ and $\mathcal{G}$ will be denoted by,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}^{\mathcal{B}}}(\mathcal{F}, \mathcal{G})$$

We will now state the enriched Yoneda lemma without proof.

Theorem 1.4.8. (Enriched Yoneda Lemma) *$\mathcal{A}$ be a small $\mathcal{V}$-category, then, for all $\mathcal{V}$-functors $\mathcal{F} : \mathcal{A} \to \mathcal{V}$,*

$$\mathcal{F}X \cong \mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}^{\mathcal{V}}}(\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X, -), \mathcal{F}).$$

□

The ideas of the proofs have the same ideas, but now with more verification to be done. Similarly we will have a notion of limits and colimits for the enriched categories, called weighted limits and colimits. It will have properties similar to limits and colimits, and using these we get the enriched density formula.

Theorem 1.4.9. (Enriched Density Formula) *Let $\mathcal{F} \in \mathcal{V}^{\mathcal{A}^{\text{op}}}$, then,*

$$\mathcal{F}V \cong \int^{U \in \mathcal{O}(X)} h^V U \odot \mathcal{F}U$$

$\square$

For proofs, and details, see [?],[?] and the references therein.

1.5 | Kan Extension of Functors

Given two functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$, the goal of Kan extension is to find a new functor $\mathcal{K}$ such that the manipulation done by composite functor $\mathcal{K} \circ \mathcal{F}$ is closest to the manipulation done by the functor $\mathcal{G}$. The obstruction comes from $\mathcal{F}$ losing information that $\mathcal{G}$ preserves.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd} & \mathcal{B} & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C} \end{tikzcd}$$

The left Kan extension of $\mathcal{G}$ along $\mathcal{F}$ is a functor is a functor $\operatorname{Lan}_{\mathcal{F}} \mathcal{G} : \mathcal{B} \to \mathcal{C}$ such that there exists a natural transformation $\mathcal{G} \Rightarrow (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F}$, such that any other extension $\mathcal{K}$ with natural transformation $\mathcal{G} \to \mathcal{K} \circ \mathcal{F}$ factors through $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$, i.e., $\mathcal{G} \Rightarrow (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F}$.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathcal{K}"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[rr, phantom, bend left=40, "\Uparrow"] & & \mathcal{C} \end{tikzcd}$$

So, in terms of a commutative diagram a Kan extension is the functor that makes the diagram as close to commutative as possible.

The left Kan extension $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$ can be visualised by the diagram,

$$\mathcal{G} \Longrightarrow (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{K} \circ \mathcal{F} \Longrightarrow \cdots$$

This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{K} \circ \mathcal{F}$ there exists a natural transformation from $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$ to $\mathcal{K}$, i.e.,

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{B}}}(\operatorname{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K})$$

Intuitively the left Kan extension is the left most functor to the functor in the above visualisation. Here 'most' is quantified in terms of natural transformations. The left Kan extension is the left most (as explained above) extension of the functor $\mathcal{G}$ with respect to $\mathcal{F}$. We should expect the construction of the 'left most' functor to be related to taking colimits in an appropriate category. If $\mathcal{C}$ is cocomplete, the functor category to $\mathcal{C}$ will also be cocomplete, and $\mathcal{A}$ has some 'nice properties' then we could think of these as a system in the functor category and intuitively, the left 'most' should exist.

The right Kan extension is defined similarly, and only the direction of the natural transformation is changed from $\mathcal{K} \circ \mathcal{F}$ to $\mathcal{G}$.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathcal{H}"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C} \end{tikzcd}$$

The right Kan extension is denoted by $\operatorname{Ran}_{\mathcal{F}} \mathcal{G}$. It's the right most functor in the following visualization,

$$\cdots \Longrightarrow \mathcal{H} \circ \mathcal{F} \Longrightarrow (\operatorname{Ran}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{G}$$

Similar to left Kan extensions, we should expect the construction of the 'right most' functor to be related to taking limits in an appropriate category. This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{H} \circ \mathcal{F}$ there exists a natural transformation from $\mathcal{H}$ to $\operatorname{Ran}_{\mathcal{F}} \mathcal{G}$, i.e.,

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{H}\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathcal{H}, \operatorname{Ran}_{\mathcal{F}} \mathcal{G}).$$

Kan extension are functors in the appropriate functor category.

Lemma 1.5.1. *$\forall \mathcal{G}$, $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$, $\operatorname{Ran}_{\mathcal{F}} \mathcal{G}$ exist, then,*

$$\operatorname{Lan}_{\mathcal{F}} \dashv \circ \mathcal{F} \dashv \operatorname{Ran}_{\mathcal{F}}.$$

The proof of this adjointness with precomposition has already been described, and directly follows from definition of Kan extensions, for this to make sense we only need the fact that $\operatorname{Lan}_{\mathcal{F}}$ and $\operatorname{Ran}_{\mathcal{F}}$ are functors, i.e., $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$ and $\operatorname{Ran}_{\mathcal{F}} \mathcal{G}$ exist $\forall \mathcal{G}$, and satisfy some composition rules, which it will due to the definition.

Lemma 1.5.2. *If $\mathcal{L}$ is a left adjoints, then,*

$$\operatorname{Lan}_{\mathcal{F}}(\mathcal{L} \circ \mathcal{G}) \cong \mathcal{L} \circ \operatorname{Lan}_{\mathcal{F}} \mathcal{G}.$$

Proof

What's happening is that left adjoints preserve colimits, and this guarantees the existence of the Kan extenstion for the composition. So, when the required colimits exist, the notion of closest makes sense. The closest functor to $\mathcal{L}$ is $\mathcal{L}$, so once we know the Kan extension exists, it must be the one described above.

Suppose $\mathcal{G}$ has a left Kan extension along $\mathcal{F}$,

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\operatorname{Lan}_{\mathcal{F}} \mathcal{G}"] \arrow[d, phantom, "\Uparrow"] \arrow[drr, bend left, "\mathcal{K}"] & & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C} \arrow[r, shift left, "\mathcal{L}"] & \mathcal{D} \arrow[l, shift left, "\mathcal{R}"] \end{tikzcd}$$

Let $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ be a functor that's left adjoint, i.e., there exists a functor $\mathcal{R} : \mathcal{D} \to \mathcal{C}$ to which $\mathcal{L}$ is a left adjoint, then we have,

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}^{\mathcal{B}}}(\mathcal{L} \operatorname{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K}) &\cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{B}}}(\operatorname{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{R}\mathcal{K}) \\ &\cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, (\mathcal{R}\mathcal{K})\mathcal{F}) \cong \operatorname{Hom}_{\mathcal{D}^{\mathcal{A}}}(\mathcal{L}\mathcal{G}, \mathcal{K}\mathcal{F}) \end{aligned}$$

So, by definition of Kan extension, we have that $\mathcal{L} \operatorname{Lan}_{\mathcal{F}} \mathcal{G} \cong \operatorname{Lan}_{\mathcal{F}}(\mathcal{L}\mathcal{G})$. $\square$

1.5.1 | Kan Extensions as Coends

Certain additional constraints on the starting categories guarantees the existence of Kan extensions. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd} & \mathcal{B} & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C} \end{tikzcd}$$

Theorem 1.5.3. (Existence/Coend Formula)

$$\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}B \cong \int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, B) \odot \mathcal{G}A$$

whenever $\mathcal{A}$ is small, $\mathcal{B}$ is locally small, $\mathcal{C}$ is cocomplete.

Proof

Note that the local smallness of $\mathcal{B}$, or being enriched by the category of sets is needed for the existence of tensor product $\odot$, which is needed for the existence of extensions, this condition maybe replaced with an appropriate enriched category with a tensor product. Once they exist, the cocompleteness is needed for the existence of the limit, and to take the limits, we need the starting category to be small. So the conditions we require are to be expected.

The proof is again find a chain of isomorphisms so we can sneak in Yoneda principle. Firstly, by the end formula for natural transformations, we have,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}\mathcal{F}A\right)$$

Applying Yoneda to the functor $\mathcal{H} : X \mapsto \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}X\big)$, we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}\mathcal{F}A\big) = \mathcal{H}(\mathcal{F}A)$$

Applying the end formula for natural transformations again, we get,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}(h^{\mathcal{F}A}B, \mathcal{H}B)$$

This gives us the following double 'integral',

$$\int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}B\right)\right)$$

By using the hom-tensor adjointness we get,

$$\begin{aligned} \int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}B\right)\right) &\cong \int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\left(h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\right) \\ &\cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\left(\int_{A\in\mathcal{A}} h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\right) \\ &\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}\left(\int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, -) \odot \mathcal{G}A, \mathcal{K}\right) \end{aligned}$$

Here, we used the Fubini rule for ends, and hom-(co)end relations in the first step, and end formula for natural transformations in the next step. Now by Yoneda principle proves the theorem. ☐

Assume the conditions of coend formula are satisfied. For each functor $\mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, the left Kan extension along $\mathcal{F}$ is the functor, $\operatorname{Lan}_{\mathcal{F}}(-) : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{B}}$, $\mathcal{G} \mapsto \operatorname{Lan}_{\mathcal{F}} \mathcal{G}$. Intuitively we expect 'nearest' composed with 'nearest' to be the 'nearest'. So we will have,

$$\operatorname{Lan}_{\mathcal{F}\circ\mathcal{E}} \mathcal{G} = \operatorname{Lan}_{\mathcal{F}}(\operatorname{Lan}_{\mathcal{E}} \mathcal{G}).$$

The proof is a simple application of the coend formula and some end-coend calculus.

1.5.2 | All Concepts are Kan Extensions

Any concept that talks about some notion of nearest/closest/best should be expected to be related to Kan extensions. Kan extensions is a very general construction which unites limits, adjoints, and other constructions. Many of the categorical constructions involve limits or adjunctions. Limits correspond to nearest objects to/from a system. Adjoints when viewed as reflections also say that the associated objects are the nearest to the original object when transformed back. So both of these concepts should be expected to be related to Kan extensions.

1.5.2.1 | (Co)Limits as Kan Extensions

Limits are the nearest object to a system. We can now think of objects in a category $\mathcal{A}$ as functors from a terminal category to the category $\mathcal{A}$. Here the terminal category is a category with one object $*$ and one morphism, the identity morphism on the object $\mathbb{1}_*$. This category will be denoted by $\mathbb{1}$. Now each object A in $\mathcal{A}$ can be thought of as a functor $\widehat{A} : \mathbb{1} \to \mathcal{A}$. Since we want the 'nearest' object to a system, we want a nearest approximation of the system by a functor that corresponds to such an object. This is a Kan extension situation.

The terminal category $\mathbb{1}$ is the unique object in the category of categories, such that there exist only one functor from any category $\mathcal{I}$ to $\mathbb{1}$, which sends everything to the only object in the category, and every morphism to the only morphism in $\mathbb{1}$. Denote this functor by $\mathcal{T}$. Then we have the following Kan extension problem,

$$\begin{tikzcd}
 & \mathbb{1} \arrow[dr, dashed, "\mathcal{H}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{I} \arrow[ur, "\mathcal{T}"] \arrow[rr, "\mathcal{F}"'] & {} & \mathcal{A}
\end{tikzcd}$$

For each functor $\mathcal{H} : \mathbb{1} \to \mathcal{A}$ corresponds to an object H in the category $\mathcal{A}$, and $\mathcal{H} \circ \mathcal{T}$ represents the constant system which maps everything in the indexing category $\mathcal{I}$ to a constant object. The natural transformations from $\mathcal{H} \circ \mathcal{T}$ to $\mathcal{F}$ represent the cones, from the object H to the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$.

The limit of a system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ is the right Kan extension of $\mathcal{F}$ along $\mathcal{T}$,

$$\varprojlim \mathcal{F} = \operatorname{Ran}_{\mathcal{T}} \mathcal{F}(*).$$

Similarly the colimit of a system is the object that's nearest from the system. So, the colimit will be the left Kan extension of the system along $\mathcal{T}$.

$$\varinjlim \mathcal{F} \cong \operatorname{Lan}_{\mathcal{T}} \mathcal{F}(*).$$

1.5.2.2 | Adjoint Functor Theorem

Adjoint functors when viewed as reflections §1.3.1, bring with them some notion of 'nearness'. So adjoints should be expected to be related to Kan extensions. Consider an adjoint situation $\mathcal{F} \dashv \mathcal{G}$,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B},$$

If we think of objects in $\mathcal{A}$ as having some information, and after performing $\mathcal{F}$ we lose information in the category $\mathcal{A}$, then the functor $\mathcal{G}$ is such that $\mathcal{GF}$ is the 'best' recovery of the original information about $\mathcal{A}$. So, $\mathcal{GF}$ is the 'nearest' to the identity functor $\mathbb{1}_{\mathcal{A}}$, 'along' $\mathcal{F}$. This is a natural transformation,

$$\mathcal{GF} \Rightarrow \mathbb{1}_{\mathcal{A}}\,.$$

The nearest means that any other 'recovery' will have to factor through this that's to say, $\cdots \Rightarrow \mathcal{GF} \Rightarrow \mathbb{1}_{\mathcal{A}}$. Similarly, the left adjoint is the 'nearest' from the identity functor on $\mathcal{B}$, so we have,

$$\mathbb{1}_{\mathcal{A}} \Rightarrow \mathcal{FG}.$$

So, we get the following Kan extension way of thinking about adjoints.

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}}\,\mathbb{1}_{\mathcal{A}} \cong \mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A} \arrow[dr, dashed, "\mathrm{Ran}_{\mathcal{G}}\,\mathbb{1}_{\mathcal{B}} \cong \mathcal{F}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{B} \arrow[ur, "\mathcal{G}"] \arrow[rr, "\mathbb{1}_{\mathcal{B}}"'] & & \mathcal{B}
\end{tikzcd}$$

So, the right adjoint $\mathcal{G}$ is the left Kan extension of $\mathbb{1}_{\mathcal{A}}$ along $\mathcal{F}$. Similarly, the left adjoint $\mathcal{F}$ is the right Kan extension of $\mathbb{1}_{\mathcal{B}}$ along $\mathcal{G}$. An adjoint situation can be expressed as the following pair of Kan extensions,

$$\begin{tikzcd}
 & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[d, phantom, "\Uparrow"] \arrow[dr, dashed, shift left] & & \mathcal{B} \arrow[dl, "\mathcal{G}"] \arrow[dl, phantom, "\Uparrow", shift right=6] \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \arrow[ul, dashed, shift left] &
\end{tikzcd}$$

Theorem 1.5.4. (Adjoint Functor Theorem)

1.5.2.3 | Nerve Realization

2 | Abelian Categories

2.1 | Abelian Categories

2.2 | Diagram Chasing in Abelian Categories

2.2.1 | Salamander Lemma

2.2.2 | Corollaries of Salamander Lemma

3 | Derived Categories & Functors

The goal of cohomology is to measure the obstruction to exactness. So, if a morphism of complexes keeps the obstruction to exactness the same, then we have not lost the information about the obstruction by going in between such complexes.

A morphism of complexes,

$$\begin{array}{ccc} A^\bullet & & \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots \\ \downarrow f & & \quad\quad \downarrow f^{i-1} \quad\quad \downarrow f^i \quad\quad \downarrow f^{i+1} \\ B^\bullet & & \cdots \xrightarrow[\partial^B_{i-2}]{} B^{i-1} \xrightarrow[\partial^B_{i-1}]{} B^i \xrightarrow[\partial^B_i]{} B^{i+1} \xrightarrow[\partial^B_{i+1}]{} \cdots \end{array}$$

is called a quasi-isomorphism ($\cong_Q$) if the induced morphism at cohomology, $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all i. We want to treat quasi-isomorphic objects as isomorphic objects. A derived category is a category in which such quasi-isomorphic complexes are treated as the same object, i.e., quasi-isomorphisms are isomorphisms.

Derived functors are

The ideal use of these notes is to skim through the material to get an idea of what's happening before actually reading the material carefully from textbooks.

3.1 | Derived Categories

Given an additive category $\mathcal{A}$, a differential object in $\mathcal{A}$ is a sequence,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

where each $A^i \in \mathcal{A}$ and homomorphisms $\partial_i : A^i \to A^{i+1}$ are morphisms in the category $\mathcal{A}$. The morphisms between two such differential objects $A^\bullet$ and $B^\bullet$, $A^\bullet \xrightarrow{u} B^\bullet$ consists of morphisms $u^i : A^i \to B^i$ such that the following commutative diagram commutes,

$$\begin{array}{ccc} A^\bullet & & \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots \\ \downarrow u & & \quad\quad \downarrow u^{i-1} \quad\quad \downarrow u^i \quad\quad \downarrow u^{i+1} \\ B^\bullet & & \cdots \xrightarrow[\partial^B_{i-2}]{} B^{i-1} \xrightarrow[\partial^B_{i-1}]{} B^i \xrightarrow[\partial^B_i]{} B^{i+1} \xrightarrow[\partial^B_{i+1}]{} \cdots \end{array}$$

The set of morphisms $A^\bullet \to B^\bullet$ is denoted by $\text{Hom}(A^\bullet, B^\bullet)$. The category with differential objects as objects and $\text{Hom}(A^\bullet, B^\bullet)$ is the functor category $\mathcal{A}^{\mathbb{Z}}$.

3.1.1 | Complexes & Cohomology Functors

A sequence of morphisms in $\mathcal{A}$,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

is called complex if for all i,

$$\partial^A_i \circ \partial^A_{i-1} = 0,$$

The category of complexes over an abelian category $\mathcal{A}$ is an abelian subcategory of $\mathcal{A}^{\mathbb{Z}}$. Denoted by $\mathcal{C}(\mathcal{A})$. Since $\mathcal{A}$ is an abelian category we can split $\partial^{i-1} : A^{i-1} \to A^i$ as

$$\ker \partial^{i-1} \to A^{i-1} \to \Im \partial^{i-1} \to A^i \to \operatorname{coker} \partial^{i-1}.$$

Then using the definition of ker and coker, we get the following diagram,

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] & & \ker \partial^A_i \arrow[d] & & \ker \partial^A_{i+1} \arrow[d, dashed] & \\
\cdots \arrow[ur, dashed, tail] \arrow[r] & A^{i-1} \arrow[r] \arrow[d, dashed] & \Im \partial^A_{i-1} \arrow[r] \arrow[ur, dashed, tail, "f_i"] & A^i \arrow[r] \arrow[d] & \Im \partial^A_i \arrow[r] \arrow[ur, dashed, tail] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \operatorname{coker} \partial^A_{i-1} \arrow[ur, dashed] & & \operatorname{coker} \partial^A_{i-1} \arrow[ur, dashed, "c_i"'] & & \operatorname{coker} \partial^A_i . \arrow[ur, dashed] &
\end{tikzcd} \qquad \text{(split)}$$

The complex $A^\bullet$ is called exact if the morphisms are such that,

$$\text{Im } \partial^A_{i-1} \cong \ker \partial^A_i.$$

Given a complex $A^\bullet$, the cohomology is the 'objects' by which sequence fails to be exact. If the complex $A^\bullet$ is not exact at the node A_i, then the $\ker \partial^A_i$ is 'larger' than Im ∂^A_{i-1}. The extra stuff in $\ker \partial^A_i$ that's not in $\Im \partial^A_{i-1}$ is the cokernel of the morphism f_i. So, we have by universal property of kernels, there exists a morphism,

$$\text{Im } \partial^A_{i-1} \xrightarrow{f_i} \ker \partial^A_i \longrightarrow \operatorname{coker} f_i =: \overline{H}^i(A^\bullet).$$

The extra stuff in $\ker \partial^A_i$ that's not in the image of ∂^A_{i-1} is also the kernel of cokernel $\operatorname{coker} \partial^A_{i-1}$. By universal property of cokernels, there exists a morphism,

$$\underline{H}^i(A^\bullet) \coloneqq \ker c_i \longrightarrow \operatorname{coker} \partial^A_{i-1} \xrightarrow{c_i} \Im \partial^A_i.$$

This is the object we want and it's this part which is making the complex not exact.

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] & & \ker \partial^A_i \arrow[d] \arrow[r, dashed] & \overline{H}^i(A^\bullet) & \ker \partial^A_{i+1} \arrow[d, dashed] & \\
\cdots \arrow[ur, dashed, tail] \arrow[r] & A^{i-1} \arrow[r] \arrow[d, dashed] & \Im \partial^A_{i-1} \arrow[r] \arrow[ur, dashed, tail, "f_i"] & A^i \arrow[r] \arrow[d] & \Im \partial^A_i \arrow[r] \arrow[ur, dashed, tail] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \operatorname{coker} \partial^A_{i-1} \arrow[ur, dashed] & \underline{H}^i(A^\bullet) \arrow[r, dashed] & \operatorname{coker} \partial^A_{i-1} \arrow[ur, dashed, "c_i"'] & & \operatorname{coker} \partial^A_i . \arrow[ur, dashed] &
\end{tikzcd}$$

In this diagram, the sequences,

$$\ker \partial^A_i \to A^i \to \operatorname{coker} \partial^A_{i-1}$$

need not be exact. It's this non-exactness we are trying to capture. Let the composite morphism $\ker \partial_i^A \to A^i \to \operatorname{coker} \partial_{i-1}^A$ be h^i, the cohomology can be defined as the image of the morphism h^i,

$$\ker \partial_i^A \to H^i(A^\bullet) \coloneqq \Im h^i \to \operatorname{coker} \partial_{i-1}^A$$

So, $H^i(A^\bullet)$ is the replacement of A^i in split that makes the split into an exact sequence. The three definitions are equivalent, see [**?**] for the proof. So, $\Im h_i = \ker c_i = \operatorname{coker} f_i$.

So, $H^i(A^\bullet)$ is well defined and we get an exact sequence of morphisms,

$$\begin{tikzcd}
 & \ker \partial_i^A \arrow[dl, dashed, "f_i"'] \arrow[d, dashed] & \\
\mathrm{Im}\ \partial_{i-1}^A & H^i(A^\bullet) \arrow[d, dashed] & \Im \partial_i^A \\
 & \operatorname{coker} \partial_{i-1}^A \arrow[ur, dashed, "c_i"] &
\end{tikzcd}$$

The new exact sequence we have is,

$$\begin{tikzcd}
 & \ker \partial_i^A \arrow[d] & & \ker \partial_i^A \arrow[d] & & \ker \partial_{i+1}^A \arrow[d] & \\
\cdots \arrow[ur] & H^{i-1}(A^\bullet) \arrow[d] & \mathrm{Im}\ \partial_{i-1}^A \arrow[ur] & H^i(A^\bullet) \arrow[d] & \Im \partial_i^A \arrow[ur] & H^{i+1}(A^\bullet) \arrow[d] & \cdots \\
 & \operatorname{coker} \partial_{i-1}^A \arrow[ur] & & \operatorname{coker} \partial_{i-1}^A \arrow[ur] & & \operatorname{coker} \partial_i^A \arrow[ur] &
\end{tikzcd}$$

By replacing the A^is with $H^i(A^\bullet)$ we get an exact sequence. This gives us a collection of functors from $\mathcal{C}(\mathcal{A})$ to $\mathcal{A}$, called the cohomology functors,

$$H^i : C(\mathcal{A}) \to \mathcal{A},\ H^i(A^\bullet) = \ker \partial_i^A / \mathrm{Im}\ \partial_{i-1}^A.$$

A morphism of complexes,

$$\begin{tikzcd}
A^\bullet \arrow[d, dashed, "f"] & \cdots \arrow[r, "\partial_{i-2}^A"] & A^{i-1} \arrow[r, "\partial_{i-1}^A"] \arrow[d, dashed, "f^{i-1}"] & A^i \arrow[r, "\partial_i^A"] \arrow[d, dashed, "f^i"] & A^{i+1} \arrow[r, "\partial_{i+1}^A"] \arrow[d, dashed, "f^{i+1}"] & \cdots \\
B^\bullet & \cdots \arrow[r, "\partial_{i-2}^B"'] & B^{i-1} \arrow[r, "\partial_{i-1}^B"'] & B^i \arrow[r, "\partial_i^B"'] & B^{i+1} \arrow[r, "\partial_{i+1}^B"'] & \cdots
\end{tikzcd}$$

is called a quasi-isomorphism (qis) if the induced morphism at cohomology,

$$H^i(f) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$$

is an isomorphism for all i. Induced morphism makes sense because cohomology is a functor.

Quasi-isomorphisms manipulate the complexes so that the obstruction to exactness is preserved. So the collection of quasi-isomorphic complexes have same obstructions and could be treated to be the same. We want to treat quasi-isomorphic objects as isomorphic objects. For the sake of simplicity, we will stop writing the bullets for the remainder of this note.

3.1.2 | Localization of Categories

Localization is the process of adding all the formal inverses to a collection of morphisms. Note that the objects remain the same, but there will be more relation between these objects. Let $\mathcal{Q}$ denote a collection of morphisms in $\mathcal{C}$, the aim of localization is to construct a new category $\mathcal{C}[\mathcal{Q}^{-1}]$ and a functor,

$$\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$$

which sends morphisms belonging to $\mathcal{Q}$ to isomorphisms in $\mathcal{C}[\mathcal{Q}^{-1}]$. This construction being 'universal' i.e., for any other category $\mathcal{D}$ with a functor $\widehat{\mathcal{L}_{\mathcal{Q}}} : \mathcal{C} \to \mathcal{D}$, which sends morphisms in $\mathcal{Q}$ to isomorphisms gets factored through $\mathcal{L}_{\mathcal{Q}}$, that's to say there exists a functor

$$\mathcal{H} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$$

such that $\widehat{\mathcal{L}_{\mathcal{Q}}} = \mathcal{H} \circ \mathcal{L}_{\mathcal{Q}}$, so we have the following commutative diagram of functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widehat{\mathcal{L}_{\mathcal{Q}}}"] \arrow[d, "\mathcal{L}_{\mathcal{Q}}"'] & \mathcal{D} \\ \mathcal{C}[\mathcal{Q}^{-1}] \arrow[ur, "\mathcal{H}"'] & \end{tikzcd}$$

The extra things $\mathcal{D}$ has that's not already in $\mathcal{C}[\mathcal{Q}^{-1}]$ should not related to the localization process, i.e adding inverses to the morphisms in $\mathcal{Q}$. This can be formalized by saying precomposition of functors with $\mathcal{L}_{\mathcal{Q}}$ is an isomorphism of the respective natural transformations. If we have two functors $\mathcal{F}, \mathcal{G} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$, then,

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}[\mathcal{Q}^{-1}]}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{D}}}(\mathcal{F} \circ \mathcal{L}_{\mathcal{Q}}, \mathcal{G} \circ \mathcal{L}_{\mathcal{Q}}).$$

Or equivalently, the functor,

$$\circ \mathcal{L}_{\mathcal{Q}} : \mathcal{C}[\mathcal{Q}^{-1}]^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}.$$

is fully faithful. $\mathcal{C}[\mathcal{Q}^{-1}]$ together with the functor $\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$ is called the localization of $\mathcal{C}$ with $\mathcal{Q}$.

Localization, if it exists is unique upto equivalence of categories. The problem now is to understand what constraint on $\mathcal{Q}$ that guarantees the existence of localization of $\mathcal{C}$ with respect to $\mathcal{Q}$. Every category $\mathcal{D}$ that has the needed inverses to $\mathcal{Q}$ will be such that, there exists a functor from the localization $\mathcal{C}[\mathcal{Q}^{-1}]$ to it,

$$\mathcal{C}[\mathcal{Q}^{-1}] \xrightarrow{\mathcal{H}} \mathcal{D}$$

So the localization is the 'left most' category that satisfies certain properties, contains formal inverses. We should expect some colimit type thing happening here in the category of categories.

Sketch of Construction

The goal is to add the 'inverses' and turn it into a category. Let $\mathcal{Q}^{-1}$ be the set in $\mathcal{C}^{\mathrm{op}}$ corresponding to the collection of morphisms $\mathcal{Q}$. Note here that we are assuming the hom-sets are small sets. The new category should include these extra morphisms.

Consider first the graph with objects of $\mathcal{C}$ as vertices, and the arrows of the graph consists of morphisms in $\mathcal{C}$ together with the collection $\mathcal{Q}^{-1}$. So, the new collection of morphisms is given by, $\mathrm{Hom}_{\mathcal{C}} \coprod \mathcal{Q}^{-1}$. With concatenation as composition this is a category, denoted by

$\mathcal{FC}[\mathcal{Q}^{-1}]$. The identity morphisms given by the empty path from and to the same vertex. To turn this into a category we need we have to define equivalences that make the compositions $f \circ f^{-1}$ into identities for all $f \in \mathcal{Q}$.

So the localization of $\mathcal{C}$ with $\mathcal{Q}$ is,

$$\mathcal{C}[\mathcal{Q}^{-1}] \coloneqq \mathcal{FC}(\mathcal{Q}^{-1})/\sim$$

where the quotient consists of same objects and morphisms are quotiented by the above equivalence. This quotient is the colimit we needed. □

For $\mathcal{Q}$ with special properties more direct formulas for the hom-sets of $\mathcal{C}[\mathcal{Q}^{-1}]$ can be obtained. The localization of the category of complexes with the collection of quasi-isomorphisms is called the derived category.

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$$

where $\mathcal{Q}$ is the collection of quasi-isomorphisms. Our goal is to obtain a more direct formula for localization of the category of complexes with quasi-isomorphisms.

3.1.2.1 | Localization with Quasi-Isomorphisms

We want to get an explicit description of localization for the case of quasi-isomorphisms. This can be done with the special properties the collection of quasi-isomorphisms comes equipped with. Composition of morphisms that preserve cohomology also preserve cohomology, so composition of quasi-isomorphisms is also a quasi-isomorphism. Identity morphisms on the complexes preserve cohomology.

Let $f : A \to B$ be a morphism, and suppose $q : B \to C$ is a quasi-isomorphism, then it's invertible in the localized category. So, $\mathcal{L}(q)$ is an isomorphism, so we could just invert and get a map $A \to C$.

$$\begin{tikzcd} & \mathcal{L}(C) \arrow[d, "\cong"] \\ \mathcal{L}(A) \arrow[r] \arrow[ur, dashed] & \mathcal{L}(B). \end{tikzcd}$$

So, in the original category, there should exist a quasi-isomorphism, $A \dashrightarrow_{\cong_{\mathcal{Q}}} D$ such that the following diagram commutes,

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed, "\cong_{\mathcal{Q}}"] & C \arrow[d, "\cong_{\mathcal{Q}}"] \\ A \arrow[r] & B. \end{tikzcd} \tag{extension}$$

Similarly for the arrows reversed, for the same reason. Suppose we have two morphisms, $f, g : B \to C$ and a quasi-isomorphism $\tau : C \to D$ such that $\tau \circ f = \tau \circ g$ then it means that f and g manipulate the information about cohomology similarly. So, what τ did was rearrage the relevant information to make the two morphisms equal. It should also be possible to rearrange this information before manipulation by f and g. So, there should exist a quasi-isomorphism $\sigma : A \to B$ such that $f \circ \sigma = g \circ \sigma$.

$$\begin{tikzcd} A \arrow[r, dashed, "\cong_{\mathcal{Q}}"] & B \arrow[r, shift left] \arrow[r, shift right] & C \arrow[r, dashed, "\cong_{\mathcal{Q}}"] & D. \end{tikzcd} \tag{symmetry}$$

A collection of morphisms $\mathcal{Q} \subset \mathrm{Hom}_{\mathcal{C}}$ is called a right multiplicative system if it contains every isomorphism in $\mathrm{Hom}_{\mathcal{C}}$, closed under composition, and satisfies extension and symmetry.

The explicit construction of the localization of a right multiplicative system is as follows, $\mathcal{C}[\mathcal{Q}^{-1}]$ consists of the same objects as $\mathcal{C}$. For morphisms, we should think of objects that are quasi-isomorphic as the same object. So, the morphisms in the derived category will be represented by pairs of morphisms in the original category of complexes, visualised by the diagram,

$$\begin{tikzcd} \mathcal{L}(C) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd}$$

The important part is the transformation described by the map f. Consider two morphisms between $\mathcal{L}(A)$ and $\mathcal{L}(B)$,

$$\begin{tikzcd} \mathcal{L}(C) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(D) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd}$$

Qualitatively, these two morphisms represent the same morphism if they transform same quasi-isomorphic information. So the non-quasi-isomorphic maps are the important parts. This means that there exists a morphism representing this transformation of the quasi-isomorphic information to which the two morphisms above are related. This can be described by saying there exists a morphism, such that the following diagram commutes,

$$\begin{tikzcd} & C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[d, dashed] \arrow[dr] & \\ A & E \arrow[l, dashed, "\cong_{\mathcal{Q}}"'] \arrow[r, dashed] & B \\ & D \arrow[ul, "\cong_{\mathcal{Q}}"] \arrow[u, dashed] \arrow[ur] & \end{tikzcd} \tag{derived equivalence}$$

By the symmetry of the diagram this is clearly an equivalence relation.

For the composition of morphisms in the derived category, for two morphisms, $\mathcal{L}(A) \to \mathcal{L}(B)$ and $\mathcal{L}(B) \to \mathcal{L}(C)$ the composition of the morphisms corresponds to the following representative in the original category of complexes,

$$\begin{tikzcd} & & E \arrow[dl, dashed, "\cong_{\mathcal{Q}}"'] \arrow[dr, dashed] & & \\ & C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & & D \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & \\ A & & B & & C \end{tikzcd} \tag{derived composition}$$

Note that the above square is possible due to extension property. The category with same objects as the category of complexes, with morphisms described by the equivalence classes of morphisms in the category of complexes as described in derived equivalence, together with composition as described in derived composition is the derived category $\mathcal{D}(\mathcal{A})$.

For proof that this is indeed the localization of $\mathcal{C}(\mathcal{A})$ with $\mathcal{Q}$ see, [**?**]. What needs to be done is show that any other functor which takes quasi-isomorphisms to isomorphisms must factor through this category.

3.1.3 | Structure of Derived Category

3.1.3.1 | Triangulated Categories

3.2 | Derived Functors

3.2.1 | Resolutions

3.2.2 | via Kan Extensions

3.2.3 | Spectral Sequences

4 | Grothendieck Topology

5 | Geometric Morphisms

In these notes we study certain adjoint pairs that don't need the language of derived categories. These can be called internal functors or external depending on whether we stay within that category of sheaves over a topological space or we leave the category. These are four of Grothendieck's 'six operations'.

5.1 | Direct and Inverse Image Sheaves

Continuous maps $X \to Y$ gives rise to an adjoint pair of functors $\mathrm{Sh}(X) \rightleftharpoons \mathrm{Sh}(Y)$ called direct image, and inverse image, these are external operations, i.e., we are moving to a different topological space. A topological space X determines a category $\mathrm{PSh}(X)$ of sheaves on X. A continuous map of spaces $f : X \to Y$ will induce functors in both directions, forward and backward, on the associated category of pre-sheaves $\mathrm{PSh}(X)$ and $\mathrm{PSh}(Y)$.

Let $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ be a sheaf on X. Using this sheaf we can construct a sheaf on Y as follows, the sheaf assigns to each open set $U \subset X$ a set $\mathcal{F}U$, we can associate the same set $\mathcal{F}U$ to the open set whose image under f was U. The continuous function f gives us a functor of categories of open sets,

$$f^{-1} : \mathcal{O}(Y)^{\mathrm{op}} \to \mathcal{O}(X)^{\mathrm{op}}.$$

This gives rise to a new sheaf, the induced sheaf $f_*\mathcal{F}$ on Y, defined as the composition of functors,

$$\mathcal{O}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}.$$

Defined for each open set V of Y, i.e., $V \in \mathcal{O}(Y)$ by,

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

This is defined because $f^{-1}V$ is an open set by definition of continuous functions. $f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$ under f. This gives us a functor,

$$f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y).$$

This respects the composition of functions, i.e.,

$$(fg)_* = f_*g_*$$

So, if we set $\mathrm{PSh}(f) = f_*$, then PSh is a functor from the category of topological spaces to the category of sheaves. The direct image functor f_* has a left exact, left adjoint f^*. The construction of this left adjoint is considerably more complex.

5.1.1 | Inverse Image Sheaves via Étale Space

To construct inverse image, we make use of pullbacks. Let $E \to Y$ be a bundle over Y, we can then pullback E along a function $f : X \to Y$ giving us the bundle, $f^*E \to X$,

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

f^* is a functor $f^* : \mathbf{Bund}\ Y \to \mathbf{Bund}\ X$.

Suppose E is an étale bundle over Y, i.e., around each point $e \in E$ has a neighborhood U_e that is homeomorphic to its image $\pi(U_e)$. By definition of pullback, the space f^*E consists of points which we can label using points of X and E by $\langle x, e\rangle$ such that $fx = \pi e$, i.e., the pullback is the universal equalizer of the two maps.

By definition of étale spaces, there is a neighborhood U_e of e that's mapped homeomorphically to its image. Using this image we can construct an open neighborhood of x, by taking $f^{-1}(\pi(U_e))$. This is possible because by definition of pullback we have $fx = \pi e$. So, $\langle f^{-1}(\pi(U_e)), U_e\rangle$ is an open neighborhood of $\langle x, r\rangle$ that is mapped homeomorphically onto $f^{-1}(\pi(U_e))$ of X. Hence f^*E is étale.

If we start with a sheaf $\mathcal{F}$ on Y, a point in the pullback $f^*\mathcal{F}$ is of the form, $\langle x, \operatorname{germ}_{f(x)} s\rangle$ where $s \in \mathcal{F}V$ is an element of the sheaf $\mathcal{F}$.

Lemma 5.1.1. *Pullback of an étale space under a continuous map is étale.* □

This gives us a map of sheaves,

$$\mathrm{Sh}(Y) \xrightarrow{\mathcal{E}} \textbf{Etale}\, Y \xrightarrow{f^*} \textbf{Etale}\, X \xrightarrow{\Gamma} \mathrm{Sh}(X).$$

Here the first map is the bundling of the stalks of a sheaf to an étale space, and the last map is taking the sheaf of sections of the bundle. The composition gives us a functor of sheaves, which we denote again by f^*,

$$\mathrm{Sh}(Y) \xrightarrow{f^*} \mathrm{Sh}(X).$$

Theorem 5.1.2.

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, shift left, "f_*"] & \mathrm{Sh}(Y) \arrow[l, shift left, "f^*"] \end{tikzcd}, \qquad f^* \dashv f_*.$$

Sketch of Proof

So, what we need to prove is that, $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*F, G) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(F, f_*G)$, which for the sake of brevity we will denote by,

$$\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Sh}(Y)(F, f_*G).$$

Since we have an equivalence between the category of étale spaces and category of sheaves over X, we have that $\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$, where $\mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$ is the collection of morphisms between étale spaces $\mathcal{E}f^*F$ and $\mathcal{E}G$.

Let $K(\mathcal{E}f^*F, \mathcal{E}G)$ be the set of functions $k : \mathcal{E}f^*F \to \mathcal{E}G$ over X.

5.1.2 | via Kan Extension

5.2 | Enriched Categories

5.3 | Category of Abelian Sheaves

Sheaves encountered often in geometry are abelian sheaves. Let X be a topological space, and consider continuous functions $f : X \to \mathbb{R}$ on the topological space X. It forms a sheaf,

$$C : \mathcal{O}(X)^{\text{op}} \to \mathbf{Sets}.$$

Continuous functions can be added, substracted, multiplied and scaled to obtain new continuous functions. For each open set $U \in \mathcal{O}(X)$, the collection of all continuous functions on it CU forms an $\mathbb{R}$-algebra. Each CU are $\mathbb{R}$-module objects in the category **Sets**.

Since we can encounter lot of sheaves which take values in an abelian category, it's justified to give them special attention.[1] Let $\mathcal{A}$ be an abelian category, an abelian pre-sheaf on a topological space X is a functor $\mathcal{F}$,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathcal{A}, \tag{pre-sheaf}$$

The category of all pre-sheaves on a topological space X with values in $\mathcal{A}$ is denoted by $\text{PSh}(X, \mathcal{A})$. The functor category,

$$\text{PSh}(X, \mathcal{A}) \coloneqq \mathcal{A}^{\mathcal{O}(X)^{\text{op}}}.$$

admits small limits since the abelian category $\mathcal{A}$ admits small limits , small colimits, and the small filtered limits are exact and limits in functor category are computed pointwise. Once we have small limits we can start doing all sorts of things like take pullbacks, pushforward, products, coproducts, etc.

A pre-sheaf is a sheaf if the local associations $\mathcal{F}U_i$ are restrictions of global association. So, given an open covering $U = \bigcup_{i \in I} U_i$, if $f_i \in \mathcal{F}U_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it should mean that there exists a section $f \in \mathcal{F}U$ such that $f_i = f|_{U_i}$. The maps $f_i \in \mathcal{F}U_i$ and $f_j \in \mathcal{F}U_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from $\mathcal{F}U$ to $\prod_i \mathcal{F}U_i$ such that $pe = qe$. $\mathcal{F}U \to \prod_i \mathcal{F}U_i$

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \prod_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{F}(U_i \cap U_j). \end{tikzcd} \tag{collation}$$

This is the collation property. For general sites, the intersection $\cap$ will be replaced by the fibered product $\prod_{i,j}$ and the open covers are replaced by covers on sites, injective maps replaced by monic maps and so on. We are interested in exploiting the target category now, i.e., $\mathcal{A}$.

[1]Abelian categories are categories where we can do homological algebra, i.e., kernels, cokernels, images, coimages, direct sums, products, etc exist. The important result we need about them is that if $\mathcal{A}$ is an abelian category then so is the functor category $\mathcal{A}^{\mathcal{C}}$ for any category $\mathcal{C}$. For a discussion on abelian categories see [**?**].

5.3.1 | Internal $\mathcal{H}om$

The operations of interest to us now are within a category of pre-sheaves $\mathrm{PSh}(X, \mathcal{A})$ or sheaves $\mathrm{Sh}(X, \mathcal{A})$ on a given topological space X or site. Since we are not going to use or do anything special with the underlying topological space or site, we will for the sake of simplicity assume it to be a topological space X.

Consider two pre-sheaves, $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(X, \mathcal{A})$, for any $U \subset X$, consider the new pre-sheaves, the restructions, $\mathcal{F}|_U, \mathcal{G}|_U \in \mathrm{PSh}(U, \mathcal{A})$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{tikzcd} U \arrow[d, "|_V"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_V)"'] & \mathcal{G}U \arrow[d, "G(|_V)"] \\ V & \mathcal{F}V \arrow[r, "\kappa_V"'] & \mathcal{G}V. \end{tikzcd}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom, denoted by,

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(X, \mathcal{A}).$$

Sometimes also written as $\mathcal{G}^{\mathcal{F}}$. We will now show that the collation property holds for the $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, and hence $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, i.e., for any cover $\{U_i\}$ of U, the following is an exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}om(\mathcal{F},\mathcal{G})U \arrow[r, dashed, "e"] & \prod_i \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i \prod_{i,j} U_j). \end{tikzcd}$$

To show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{F}, \mathcal{G})U$ and at $\prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 5.3.1. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then so is $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i\in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})U$, and $U_i \subset U$, we have,

$$\begin{tikzcd} U \arrow[d, "|_{U_i}"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_{U_i})"'] & \mathcal{G}U \arrow[d, "G(|_{U_i})"] \\ U_i & \mathcal{F}U_i \arrow[r, "\kappa_{U_i}"'] & \mathcal{G}U_i. \end{tikzcd}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{F}(U_i)$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \prod V} = 0.$$

Now, $\{W \prod U_i\}$ is a cover of W, and $\mathcal{G}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \prod_{i,j} U_j} = \kappa_j|_{U_i \prod_{i,j} U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{G}$ is a sheaf to patch these natural transformations.

Since $\mathcal{G}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{F}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

here the first map comes from the natural transformation, $\mathcal{F}V \ni f \mapsto \kappa_i(f|_{V \prod U_i})$. Since $\mathcal{G}$ is a sheaf, this must uniquely factor through $\mathcal{G}V$, by definition of equaliser. Hence, we have,

$$\mathcal{F}V \xrightarrow{\exists!} \mathcal{G}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ defines the patched up element that equalizes the diagram. $\square$

Note here that $\mathcal{F}$ need not be a sheaf, the above proposition also holds if $\mathcal{F}$ is a pre-sheaf, and $\mathcal{G}$ is a sheaf.

5.3.1.1 | Hom-Tensor Adjointness

Note here that the natural transformations κ respect the abelian structure. The exponentiation category $\mathcal{G}^{\mathcal{F}}$ has more structure than the standard exponentiation in **Sets**. Let $\mathcal{R}$ be a sheaf of commutative rings.

$$\begin{aligned} \mathcal{R} \,:\, \mathcal{O}(X)^{\text{op}} &\to \text{ CRings} \\ U &\mapsto \mathcal{R}U. \end{aligned}$$

Then we can consider the pre-sheaves which take values in the category ${}_{\mathcal{R}}\text{Mod}$ of $\mathcal{R}$-modules, i.e., U gets mapped to $\mathcal{R}U$-modules. We will denote such pre-sheaves by $\text{PSh}(X, \mathcal{R})$. It naturally inherits a tensor product from the module structure. So, for two pre-sheaves $\mathcal{F}$ and $\mathcal{G}$, we can construct the tensor product pre-sheaf,

$$\begin{aligned} \mathcal{F} \widehat{\otimes}_{\mathcal{R}} \mathcal{G} \,:\, \mathcal{O}(X)^{\text{op}} &\to \ {}_{\mathcal{R}}\text{Mod} \\ U &\mapsto \mathcal{F}U \otimes_{\mathcal{R}U} \mathcal{G}U \end{aligned}$$

We will denote the sheafification of this pre-sheaf by,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} \in \text{Sh}(X, \mathcal{R}).$$

This is a bifunctor,

$$\cdot \otimes_{\mathcal{R}} \cdot : \text{ Sh}(X, \mathcal{R}) \times \text{Sh}(X, \mathcal{R}) \to \text{Sh}(X, \mathcal{R}).$$

5.3.2 | Tensor Product

6 | Verdier Duality

Part II

Differential Geometry & Microlocalization

[1] W SCHLAG, A Course in Complex Analysis and Riemann Surfaces, AMS, 2010

[2] R NARASIMHAN, Complex Analysis in One Variable, Second Edition Springer, 2000