Part IV
Spectral Theorem

In these notes we go through the basic theory of C^*-algebras and spectral theory. The topics in this document will include spectral mapping theorem, Gelfand-Naimark theory, and spectral theorem.

1 | Spectral mapping theorem

Our goal is now to abstract out the properties of operators on Hilbert spaces and study them. This will help us study more general quantum systems. Let $\mathcal{A}$ be a Banach space over $\mathbb{C}$ i.e., $\mathcal{A}$ is a vector space with a norm such that it's also complete under this norm. It's called a Banach algebra if it has a product structure such that,

$$\|AB\| \leq \|A\| \, \|B\|.$$

Since $\|A_1B_1 - A_2B_2\| \leq \|A_1\|\|B_1 - B_2\| + \|B_2\|\|A_1 - A_2\|$ the multiplication map is continuous. An involutive Banach algebra or a *-algebra is a Banach algebra with a $*$-operation,

$$A \mapsto A^*$$

such that, $(A^*)^* = A$, $(A+B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda} A^*$, $(AB)^* = B^*A^*$, and $\|A\| = \|A^*\|$. All these properties are imported from what we expect from the adjoint operation on operators on Hilbert spaces.

A *-algebra that satisfies,

$$\|A^*A\| = \|A^*\|\|A\| = \|A\|^2,$$

is called a C^*-algebra. It can be checked that the algebra of bounded operators on a Hilbert space $\mathcal{H}$ forms a C^*-algebra with respect to the adjoint operation. By embedding into the operators on the algebra every C^*-algebra can be made to contain the unit element. Hence we will assume every C^*-algebra to be unital in this document.

An element $A \in \mathcal{A}$ is said to be invertible if there exists a unique element A^{-1} such that $AA^{-1} = A^{-1}A = 1$. The set of all invertible elements forms a group and is called the general linear group of $\mathcal{A}$. Denoted by $\mathcal{G}(\mathcal{A})$. Our aim is to show that $\mathcal{G}(\mathcal{A})$ is open in $\mathcal{A}$.

Consider the unit ball around $1 \in \mathcal{A}$, i.e., $B_1(1) = \{A \mid \|A - 1\| \leq 1\}$. Since $\|A - 1\| \leq 1$, $\sum_{n\geq 0} \|A - 1\|^n < \infty$, so, $A' = \sum_{n \geq 0}(A-1)^n$ converges.

$$\begin{aligned} AA' = A'A = (1 - (1-A))A' = A' - (1-A)A' &= \textstyle\sum_{n\geq 0}(1-A)^n - (1-A)A' \\ &= \textstyle\sum_{n\geq 0}(1-A)^n - \sum_{n\geq 1}(1-A)^n = 1. \end{aligned}$$

So, every $A \in B_1(1)$ is invertible. As a corollary, if $\|A\| < |\lambda|$, then $(A - \lambda)$ is invertible with the inverse, $(A-\lambda)^{-1} = -\sum_{n\geq 0} A^n/\lambda^{n+1}$. Since left multiplication by an element $L_B(A) = BA$ is continuous, for $B \in \mathcal{G}(\mathcal{A})$, L_B is invertible with inverse $L_{B^{-1}}$.

Since the open unit ball around 1 is invertible 1 is in the interior of $\mathcal{G}(\mathcal{A})$. Using this we can obtain open balls around every element $B \in \mathcal{G}(\mathcal{A})$ using translations. $B \in \mathcal{G}(\mathcal{A})$, then the continuous map L_B takes the open ball around 1 to an open ball around B i.e., $L_B(B_1(1))$ is an open ball around B entirely contained in $\mathcal{G}(\mathcal{A})$. Hence $\mathcal{G}(\mathcal{A})$ is open.

2 | Gelfand-Naimark theory

The Gelfand-Naimark theorem gives a Hilbert nullstellensatz type relation between geometric objects and commutative C^*-algebras. All algebras in this section will be assumed unital and commutative.

Let $\mathcal{A}$ be a commutative Banach algebra, a multiplicative functional φ is a linear functional that's also an algebra homomorphism, $\varphi : \mathcal{A} \to \mathbb{C}$,

$$\varphi : AB \mapsto \varphi(A)\varphi(B).$$

The set of all multiplicative functionals will be called the spectrum of $\mathcal{A}$ denoted by, $\sigma(\mathcal{A})$. The reason for this name will soon become clear. Multiplicative linear functionals are also called characters in some books.

Let $\varphi \in \sigma(\mathcal{A})$, for any $A \in \mathcal{A}$, we have, $\varphi(A) = \varphi(1 \cdot A) = \varphi(1)\varphi(A)$, or $\varphi(1) = 1$. If A is invertible then $\varphi(A^{-1})\varphi(A) = \varphi(A^{-1}A) = 1$ or $\varphi(A)$ is non-zero. Suppose $|\varphi(A)| \nleq \|A\|$, then, $A - |\varphi(A)|$ is invertible.

$$\varphi(A - |\varphi(A)|) = \varphi(A) - |\varphi(A)|$$

adjusting the phase of A this term can be made zero. This is however a contradiction as φ is non-zero for invertible elements of $\mathcal{A}$. So for every $\varphi \in \sigma(\mathcal{A})$, we have $|\varphi(A)| \leq \|A\|$. Equipped with the weak* topology, $\sigma(\mathcal{A})$ is a closed subset of the closed unit ball B of $\mathcal{A}^*$.

$$\sigma(\mathcal{A}) \subset B, \quad \text{is closed}$$

By Alaoglu's theorem, A, $\sigma(\mathcal{A})$ is a compact Hausdorff space.

A left (or right, in our case it's irrelevant as we are dealing with commutative algebras) ideal of $\mathcal{A}$ is a subalgebra $\mathcal{I} \subset \mathcal{A}$ such that $AB \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and for all $B \in \mathcal{A}$. $\mathcal{I}$ is a proper ideal if $\mathcal{I} \neq \mathcal{A}$, and $\mathcal{I}$ is a maximal ideal if it's not contained in any proper ideal. If an ideal contains invertible an element, say A then $AA^{-1} = 1 \in \mathcal{I}$ which means that $B \in \mathcal{I}$ for all $B \in \mathcal{A}$, or $\mathcal{I} = \mathcal{A}$. If $A \in \mathcal{A}$ is not invertible then $\mathcal{I}_A = \{BA \mid B \in \mathcal{A}\}$ is an ideal containing A. Let $\overline{\mathcal{I}}$ be the closure of $\mathcal{I}$. Since the invertible elements of $\mathcal{A}$ form a group and is an open set in $\mathcal{A}$. $\overline{\mathcal{I}}$ cannot contain the identity of $\mathcal{A}$. $\overline{\mathcal{I}}$ is a proper ideal. Every ideal is contained in some maximal ideal, and since the closure of a proper ideal is also a proper ideal, the maximal ideals are closed. The collection of all maximal ideals of $\mathcal{A}$ will be denoted by $\mathcal{M}(\mathcal{A})$. Every non invertible element is contained in some maximal ideal.

Let $\varphi \in \sigma(\mathcal{A})$, for $A \in \ker(\varphi)$, and for all $B \in \mathcal{A}$,

$$\varphi(AB) = \varphi(A)\varphi(B) = 0,$$

so $AB \in \ker(\varphi)$. So it's an ideal. Since $\varphi(1) = 1 \notin \ker(\varphi)$ it's a proper ideal. Suppose $\ker(\varphi)$ is not a maximal ideal, and let $\ker(\varphi) \subsetneq \mathcal{I}$ with $\mathcal{I}$ a proper ideal.

Let $A \in \mathcal{I} \setminus \ker(\varphi)$, then we have, $A = (A - \varphi(A) \cdot 1) + \varphi(A) \cdot 1$. So, we can write $A = A' + \lambda \cdot 1$, for some $A' = A - \varphi(A) \cdot 1 \in \ker(\varphi)$ and $\lambda \in \mathbb{C}$. So, 1 is in the span of A and $\ker(\varphi)$. Equivalently, $\mathcal{I} = \mathcal{A}$ (!). $\ker(\varphi)$ is indeed a maximal ideal. Our goal is to relate the maximal ideals and multiplicative linear functionals.

Theorem 2.1.

$$\varphi \mapsto \ker(\varphi),$$

is a one-to-one correspondence between $\sigma(\mathcal{A})$ and $\mathcal{M}(\mathcal{A})$.

Sketch of Proof

Suppose $\ker(\varphi) = \ker(\varkappa)$, every $A \in \mathcal{A}$ can be written as, $A = \varphi(A) \cdot 1 + B$ for some $B \in \ker(\varphi)$. So we have, $\varkappa(A) = \varphi(A)\varkappa(1) + \varkappa(B)$. Since $\ker(\varphi) = \ker(\varkappa)$ we have $\varkappa(B) = 0$ and hence for all $A \in \mathcal{A}$,

$$\varphi(A) = \varkappa(A),$$

or $\varphi = \varkappa$. Hence the mapping $\varphi \mapsto \ker(\varphi)$ is injective.

Suppose $\mathcal{I}$ is a maximal ideal. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the quotient map. $\mathcal{A}/\mathcal{I}$ inherits algebra structure from $\mathcal{A}$ and also inherits a norm $\|A + \mathcal{I}\| = \inf\{\|A + I\| \mid I \in \mathcal{I}\}$ making it a Banach algebra.

$\mathcal{A}/\mathcal{I}$ has no non-trivial ideals, because otherwise if $\mathcal{I}'$ is an ideal of $\mathcal{A}/\mathcal{I}$ then, consider $\pi^{-1}(\mathcal{I}')$. For all $J \in \pi^{-1}(\mathcal{I}')$ and $A \in \mathcal{A}$ since $\pi(J) \in \mathcal{I}'$ we have,

$$\pi(JA) = \pi(J)\pi(A) \in \mathcal{I}'.$$

So, $JA \in \pi^{-1}(\mathcal{I}')$ and hence $\pi^{-1}(\mathcal{I}')$ is an ideal. Since $\mathcal{I} \subsetneq \pi^{-1}(\mathcal{I}')$ it cannot be a maximal ideal. This is a contradiction as we assumed it to be a maximal ideal. Hence every non-zero element of $\mathcal{A}/\mathcal{I}$ is invertible because otherwise we can construct an ideal containing the element. By Gelfand-Mazur theorem we have,

$$\mathcal{A}/\mathcal{I} \cong \mathbb{C} \cdot 1$$

Let the above isomorphism be φ. The composition, $\varphi \circ \pi$ is in $\sigma(\mathcal{A})$ with $\ker(\varphi \circ \pi) = \mathcal{I}$. The map $\varphi \mapsto \ker(\varphi)$ is surjective. □

This allows us to think of $\mathcal{M}(\mathcal{A})$ as a compact Hausdorff space. For every $A \in \mathcal{A}$ we have a map, $\widehat{A}(\varphi) = \varphi(A)$. With the weak* topology on $\sigma(\mathcal{A})$, $\widehat{A}$ is a continuous map on $\sigma(\mathcal{A})$. The map,

$$\Gamma : A \mapsto \widehat{A}$$

is called Gelfand tranformation on $\mathcal{A}$. It's a map from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$. Here $C(X)$ means continuous maps on X to $\mathbb{C}$. If $A, B \in \mathcal{A}$ then we have,

$$\widehat{AB}(\varphi) = \varphi(AB) = \varphi(A)\varphi(B) = \widehat{A}(\varphi)\widehat{B}(\varphi).$$

So, the Gelfand transformation is an algebra homomorphism, and $\widehat{1}(\varphi) = \varphi(1) = 1$, so $\widehat{1}$ is a constant function. If A is invertible then for all $\varphi \in \sigma(\mathcal{A})$ we have, $\varphi(AA^{-1}) = 1$ or $\varphi(A)$ is non vanishing. Conversely suppose $\widehat{A}$ is never vanishing, and suppose A is not invertible, then there exists a maximal ideal $\mathcal{I}_A$ containing A. Let the associate multiplicative functional be φ_A such that $\ker \varphi_A = \mathcal{I}_A$. So we have,

$$\varphi_A(A) = \widehat{A}(\varphi_A) = 0$$

this is a contradiction as we started with the assumption that $\widehat{A}$ is non-vanishing. Hence A is invertible if and only if $\widehat{A}$ is non-vanishing. A *-algebra $\mathcal{A}$ is said to be symmetric if

$$\Gamma(A^*) = \widehat{A^*} = \overline{\widehat{A}}.$$

Our goal is to show that for commutative C^*-algebras the Gelfand transform is an isometric isomorphism.

Theorem 2.2.

$$\|\widehat{A}\|_{sup} \leq \|A\|.$$

Sketch of Proof

Let $\lambda \in \sigma(A)$, i.e., $A - \lambda$ is not invertible. There exists φ_A such that $\varphi_A(A-\lambda) = 0$. So, we have,

$$\varphi_A(A) = \lambda.$$

So, λ is in the range of $\widehat{A}$. Conversely, suppose μ is in the range of $\widehat{A}$, then there exists $\varphi \in \sigma(\mathcal{A})$ such that $\widehat{A}(\varphi) = \mu$, or $\varphi(A-\lambda) = 0$, which means that $A-\lambda$ is not invertible. So, range of $\widehat{A}$ is same as spectrum of $\sigma(A)$.

Now, $\|\widehat{A}\|_{sup} = \sup_{\varphi \in \sigma(\mathcal{A})}\{|\widehat{A}(\varphi)|\}$. So, $\|\widehat{A}\|_{sup} = \rho(A) \leq \|A\|$. □

Suppose $\mathcal{A}$ is symmetric, i.e., $\widehat{A^*} = \overline{\widehat{A}}$, then for all self-adjoint elements, $A = A^*$, $\widehat{A} = \overline{\widehat{A}}$. $\widehat{A}$ is a real valued function. Conversely, every element A can be written as a combination of self-adjoint operators, $A = A_1 + iA_2$, so we have, $A^* = A_1^* - iA_2^*$, and hence,

$$\widehat{A^*} = \widehat{A_1} - i\widehat{A_2} = \overline{\widehat{A}}.$$

So, $\mathcal{A}$ is symmetric if and only if $\widehat{A}$ is real valued function for self-adjoint A.

If $\mathcal{A}$ is a C^*-algebra then we have $\|B^*B\| = \|B\|^2$ for all $B \in \mathcal{A}$. Let $A \in \mathcal{A}$ be self-adjoint, consider $B = A + it$, then we have,

$$\|B\|^2 = \|B^*B\| = \|A\|^2 + t^2$$

Since, $\varphi(B)^2 \leq \|B\|^2 = \|A\|^2 + t^2$, we get,

$$\begin{aligned} \varphi(A+it)^2 &= (Re(\varphi(A)) + iIm(\varphi(A)) + it)^2 \\ &= Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t + t^2 \leq \|A\|^2 + t^2. \end{aligned}$$

Which means $Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t \leq \|A\|^2$ i.e., right side is independent of t, so on the left side $Im(\varphi(A))$ must be zero. Hence $\varphi(A)$ is real valued for all $\varphi \in \sigma(\mathcal{A})$ or equivalently $\widehat{A}$ is real valued for all $A = A^*$. Hence C^*-algebras are symmetric.

Theorem 2.3. *If $\mathcal{A}$ is symmetric then $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$.*

Sketch of Proof

The proof is an application of B. If $\mathcal{A}$ is symmetric then $\Gamma(\mathcal{A})$ is closed under conplex conjugation because,

$$\Gamma(A)^* = \Gamma(A^*).$$

So, $\Gamma(\mathcal{A})$ is a self-adjoint subalgebra. $\Gamma(1) = 1$, so $\Gamma(\mathcal{A})$ contains constant functions, and $\Gamma(\mathcal{A})$ separates the points on $\sigma(\mathcal{A})$, because if $\varphi, \varkappa \in \sigma(\mathcal{A})$ with $\varphi \neq \varkappa$ then there exists $A \in \mathcal{A}$ such that $\varphi(A) \neq \varkappa(A)$ i.e., $\Gamma(A)$ is such that $\Gamma(A)(\varphi) \neq \Gamma(A)(\varkappa)$.

So by Stone-Weierstrass theorem $\Gamma(\mathcal{A})$ is a dense subset of $C(\sigma(\mathcal{A}))$. □

Suppose $A \in \mathcal{A}$, let $\sigma(A)$ be the spectrum of the operator, i.e., $\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not intertible}\}$. Suppose $\lambda \in \sigma(A)$ then $A - \lambda$ is not invertible, hence there exists some maximal ideal $\mathcal{I}_\lambda$ containing $A - \lambda$. Let $\varphi_\lambda \in \sigma(\mathcal{A})$ such that $\ker(\varphi_\lambda) = \mathcal{I}_\lambda$. Or equivalently, $\varphi_\lambda(A - \lambda) = 0$, or

$$\varphi_\lambda(A) = \lambda$$

So, to each $\lambda \in \sigma(A)$ we have a multiplicative functional φ_λ such that $\varphi_\lambda(A) = \lambda$.

If $\mathcal{A} = [A, 1]$, i.e., if $\mathcal{A}$ is generated by the identity and the operator A then $\varphi \in \sigma(\mathcal{A})$ is determined by its action on A. Since $\varphi(A^{-1}) = \varphi(A)^{-1}$ and $\varphi(A^*) = \overline{\varphi(A)}$ we have, $\widehat{A}(\varphi_1) = \widehat{A}(\varphi_2) \implies \varphi_1 = \varphi_2$. The map,

$$\widehat{A} : \sigma([A, 1]) \to \sigma(A)$$

is injective and surjective.

Theorem 2.4. (Gelfand-Naimark theorem) *If $\mathcal{A}$ is a unital commutative C^*-algebra then Γ is an isometric *-isomorphism of $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.*

Sketch of Proof

Suppose $\mathcal{A}$ is a commutative Banach algebra, we will show that $\|\widehat{A}\|_{\text{sup}} = \|A\|$ if and only if $\|A^{2^k}\| = \|A\|^{2^k}$ for $k \geq 1$. If $\|\widehat{A}\|_{\text{sup}} = \|A\|$ then,

$$\|A^{2^k}\| \leq \|A\|^{2^k} = \|\widehat{A}\|_{\text{sup}}^{2^k} = \|\widehat{A}^{2^k}\|_{\text{sup}} \leq \|A^{2^k}\|.$$

Here in the first step we used the product norm inequality, in the second step the assumption that $\|\widehat{A}\|_{\text{sup}} = \|A\|$, in the third step the definition of sup norm, and in the fourth step the fact that $\varphi(A) \leq \|A\|$ for all $\varphi \in \sigma(\mathcal{A})$. So,

$$\|\widehat{A}\|_{\text{sup}} = \|A\| \implies \|A^{2^k}\| = \|A\|^{2^k}.$$

Conversely, if $\|A^{2^k}\| = \|A\|^{2^k}$ for all $k \geq 1$, we have, $\|A^{2^k}\|^{1/2^k} = \|A\|$, but since $\lim_k \|A^{2^k}\|^{1/2^k} = \rho(A)$ and since $\|\widehat{A}\|_{sup} = \rho(A)$, we have,

$$\|A^{2^k}\| = \|A\|^{2^k} \implies \|\widehat{A}\|_{sup} = \|A\|.$$

Now for the case of commutative C^*-algebra $\mathcal{A}$, for any $B \in \mathcal{A}$, the element $A = B^*B$ is self-adjoint and hence,

$$\|A^{2^k}\| = \|(A^{2^k - 1})^*(A^{2^k - 1})\| = \|A^{2^k - 1}\|^2.$$

So, we have $\|A^{2^k}\| = \|A\|^{2^k}$ and hence $\|\widehat{A}\|_{sup} = \|A\|$. Since $\mathcal{A}$ is a C^*-algebra we also have, $\|B^*B\| = \|B^2\|$, so we have,

$$\|B\|^2 = \|A\| = \|\widehat{A}\|_{sup} = \|\widehat{|B|^2}\|_{sup} = \|\widehat{B}\|_{sup}^2.$$

Γ is an isometry with closed, dense and injective range. $\square$

3 | Spectral Theorem

4 | TLDR

A | Alaoglu's theorem

B | Stone-Weierstrass theorem

References

[1] V S Sunder, Functional Analysis: Spectral Theory, Birkhauser Advanced Texts, 1991