Part I

Gelfand-Naimark Theory & Spectral Theorem

In these notes we go through the basic theory of C^*-algebras and spectral theory. The topics in this document will include spectral mapping theorem, Gelfand-Naimark theory.

1 | Gelfand-Naimark Theory

Our goal is now to abstract out the properties of operators on Hilbert spaces and study them. This will help us study more general quantum systems. Let $\mathcal{A}$ be a Banach space over $\mathbb{C}$ i.e., $\mathcal{A}$ is a vector space with a norm such that it's also complete under this norm. It's called a Banach algebra if it has a product structure such that,

$$\|AB\| \leq \|A\| \, \|B\|.$$

Since $\|A_1B_1 - A_2B_2\| \leq \|A_1\|\|B_1 - B_2\| + \|B_2\|\|A_1 - A_2\|$ the multiplication map is continuous. An involutive Banach algebra or a *-algebra is a Banach algebra with a $*$-operation,

$$A \mapsto A^*$$

such that, $(A^*)^* = A$, $(A + B)^* = A^* + B^*$, $(\lambda A)^* = \overline{\lambda} A^*$, $(AB)^* = B^*A^*$, and $\|A\| = \|A^*\|$. All these properties are imported from what we expect from the adjoint operation on operators on Hilbert spaces.

A *-algebra that satisfies,

$$\|A^*A\| = \|A^*\|\|A\| = \|A\|^2,$$

is called a C^*-algebra. It can be checked that the algebra of bounded operators on a Hilbert space $\mathcal{H}$ forms a C^*-algebra with respect to the adjoint operation. By embedding into the operators on the algebra every C^*-algebra can be made to contain the unit element. Hence we will assume every C^*-algebra to be unital in this document.

1.1 | Spectral Mapping Theorem

An element $A \in \mathcal{A}$ is said to be invertible if there exists a unique element A^{-1} such that $AA^{-1} = A^{-1}A = 1$. The set of all invertible elements forms a group and is called the general linear group of $\mathcal{A}$. Denoted by $\mathcal{G}(\mathcal{A})$. Our aim is to show that $\mathcal{G}(\mathcal{A})$ is open in $\mathcal{A}$.

Consider the unit ball around $1 \in \mathcal{A}$, i.e., $B_1(1) = \{A \mid \|A-1\| \leq 1\}$. Since $\|A-1\| \leq 1$, $\sum_{n\geq 0} \|A-1\|^n < \infty$, so, $A' = \sum_{n\geq 0}(A-1)^n$ converges.

$$\begin{aligned} AA' = A'A = (1-(1-A))A' = A' - (1-A)A' &= \textstyle\sum_{n\geq 0}(1-A)^n - (1-A)A' \\ &= \textstyle\sum_{n\geq 0}(1-A)^n - \sum_{n\geq 1}(1-A)^n = 1. \end{aligned}$$

So, every $A \in B_1(1)$ is invertible. As a corollary, if $\|A\| < |\lambda|$, then $(A-\lambda)$ is invertible with the inverse, $(A-\lambda)^{-1} = -\sum_{n\geq 0} A^n/\lambda^{n+1}$. Since left multiplication by an element $L_B(A) = BA$ is continuous, for $B \in \mathcal{G}(\mathcal{A})$, L_B is invertible with inverse $L_{B^{-1}}$.

Since the open unit ball around 1 is invertible 1 is in the interior of $\mathcal{G}(\mathcal{A})$. Using this we can obtain open balls around every element $B \in \mathcal{G}(\mathcal{A})$ using translations. $B \in \mathcal{G}(\mathcal{A})$, then the continuous map L_B takes the open ball around 1 to an open ball around B i.e., $L_B(B_1(1))$ is an open ball around B entirely contained in $\mathcal{G}(\mathcal{A})$. Hence $\mathcal{G}(\mathcal{A})$ is open.

Let $A \in \mathcal{A}$, the spectrum of A in $\mathcal{A}$ is defined as,

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid (A-\lambda) \text{ is not invertible}\}.$$

$\sigma(A)$ is a closed subset of the disk $\{\lambda \mid |\lambda| \leq \|A\|\}$. For any $\lambda \notin \sigma(A)$, the resolvent of A is defined as,

$$R_A(\lambda) = (\lambda - A)^{-1}$$

where $R_A : \mathbb{C}\backslash\sigma(A) \to \mathcal{A}$. If $\lambda, \mu \notin \sigma(A)$ then we have,

$$\begin{aligned} (\mu-\lambda)\mathbb{I} &= (\mu - A) - (\lambda - A) \\ &= (\lambda-A)(\lambda-A)^{-1}(\mu-A) - (\lambda-A)(\mu-A)(\mu-A)^{-1} \\ &= (\lambda-A)R_A(\lambda)(\mu-A) - (\lambda-A)R_A(\mu)(\mu-A)^{-1} \\ &= (\lambda-A)[R_A(\lambda) - R_A(\mu)](\mu-A) \end{aligned}$$

So we have,

$$R_A(\lambda)(\mu-\lambda)R_A(\mu) = R(\lambda)(\lambda-A)[R_A(\lambda)-R_A(\mu)](\mu-A)R_A(\mu)$$

$$\frac{R_A(\lambda)-R_A(\mu)}{\mu-\lambda} = R_A(\lambda)R_A(\mu)$$

So, as $\lambda \to \mu$, $R'_A(\lambda)$ exists and is equal to $-R_A(\lambda)^2$. $R_A(\lambda)$ is continuous in λ. $R_A(\lambda)$ is analytic $\mathcal{A}$ valued function on $\mathbb{C}\backslash\sigma(A)$, i.e., complex derivative $R'_A(\lambda)$ exists and is continuous.

Suppose $\sigma(A)$ is empty, then R_A is an analytic function on all of $\mathbb{C}$. As $\lambda \to \infty$ we have,

$$\|R_A(\lambda)\| = |\lambda|^{-1}\|(1-\lambda^{-1}A^{-1})^{-1}\|$$

Since $(1-\lambda^{-1}A^{-1}) \to 1$ as $\lambda \to \infty$ we have, $\|R_A(\lambda)\| \to 0$ as $\lambda \to \infty$. Since $\lim_{\lambda\to\mu}[\varphi(R_A(\lambda)) - \varphi(R_A(\mu))]/(\lambda-\mu) = \lim_{\lambda\to\mu}(\varphi(R_A(\lambda)-R_A(\mu)))/(\lambda-\mu)$. So, $\varphi \circ R_A$ is a bounded analytic function. Since bounded entire functions are constant by Liouville's theorem R_A is a constant function, equal to zero which is a contradiction. $\sigma(A)$ is also closed and bounded hence it's closed.

Lemma 1.1. *If $A \in \mathcal{A}$ then $\sigma(A) \subset \mathbb{C}$ is nonempty and compact.*

Suppose there exists $A \neq \lambda 1$, then $A - \lambda 1 \neq 0$, if every element of $\mathcal{A}$ is invertible we have, $(A-\lambda)$ is invertible for all $\lambda \in \mathbb{C}$ or $\sigma(A)$ is empty which cannot happen by previous lemma.

Theorem 1.2. (Gelfand-Mazur) *If $\mathcal{A}$ is a Banach algebra in which every non-zero element is invertible, then $\mathcal{A} \cong \mathbb{C}$.*

If $p(z)$ is a polynomial, then the map $p(z) \mapsto p(A)$ is a homomorphism from $\mathbb{C}[z]$ to the algebra generated by 1 and A denoted by $[1, A]$.

Theorem 1.3. (Spectral Mapping Theorem) $p(z) = \sum_{i=0}^{N} a_i z^i$. *Then,*

$$\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) \mid \lambda \in \sigma(A)\}$$

Proof

Fix $\lambda \in \mathbb{C}$, without loss of generality assume $a_N \neq 0$. Then we have by fundamental theorem of algebra, $p(z) - \lambda = a_N \prod_{n=1}^{N}(z - \lambda_i)$ since $p(z) \mapsto p(A)$ is an algebra homomorphism we have,

$$p(A) - \lambda = a_N \prod_{n=1}^{N}(A - \lambda_i)$$

So, $\lambda \notin \sigma(p(A))$ if and only if $\lambda_i \notin \sigma(A)$ and $\lambda \notin p(\sigma(A))$. $\square$

The spectrum depends on the ambient algebra. If $A - \lambda$ is invertible in $\mathcal{A}$ with inverse $(A - \lambda)^{-1}$ but $(A - \lambda)^{-1}$ might not be in $\mathcal{B} \subsetneq \mathcal{A}$. So we have,

$$\sigma_{\mathcal{B}}(A) \supset \sigma_{\mathcal{A}}(A).$$

where $\sigma_{\mathcal{B}}(A)$ is the spectrum with respect to $\mathcal{B}$. The spectral radius is defined as follows,

$$\rho(A) = \sup_{\lambda \in \sigma(A)} \{|\lambda|\}$$

Clearly $\rho(A) \leq \|A\|$ because otherwise there exists some $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ or $(A - \lambda)$ is invertible. The spectral radius is given by the following formula (hard proof which I will skip here),

Theorem 1.4. (Spectral Radius Formula)

$$\rho(A) = \lim_{n \to \infty} \|A^n\|^{1/n}.$$

Proof

We will show that $\limsup \|A^n\|^{1/n} \leq \rho(A) \leq \liminf \|A^n\|^{1/n}$. We can write $\lambda^n 1 - A^n = (\lambda - A) \sum_{i=0}^{n-1} \lambda^i A^{n-1-i}$, which means that if $\lambda^n - A^n$ is invertible then so is $\lambda - A$. So, $\lambda \in \sigma(A)$, then $\lambda^n \in \Sigma(A^n)$. So, $|\lambda|^n \leq \|A^n\|$. So, $\rho(A) \leq \liminf \|A^n\|^{1/n}$.

For the other inequality, we have to do a little bit of complex analysis, the idea is to use bounded linear functionals on $\mathcal{A}$ to construct analytic functions, and use their properties to infer properties of A. Let $\varphi \in \mathcal{A}^*$, then

$$\varphi \circ R_A(\lambda)$$

is analytic for all $|\lambda| > \rho(A)$. It's Laurent series about infinity is,

$$\textstyle\sum_{i=0}^{\infty} \varphi(A^n)/\lambda^{n+1}.$$

This series converges for $|\lambda| > \rho(A)$. So, for any such λ, $|\varphi(A^n)/\lambda^{n+1}| \leq C_\varphi$, for all n. So, there exists some C such that

$$\|A^n\|/|\lambda|^n \leq C$$

and hence $\|A^n\|^{1/n} \leq C^{1/n}|\lambda|$. Hence we have, $\limsup \|A^n\|^{1/n} \leq \rho(A)$. □

So, for self-adjoint and normal elements we have $\|A^2\| = \|A\|^2$. By applying spectral radius formula we get that, for normal elements,

$$\|A\| = \rho(A).$$

1.2 | Maximal Ideals & Spectrum

The Gelfand-Naimark theorem gives a Hilbert nullstellensatz type relation between geometric objects and commutative C^*-algebras. All algebras in this section will be assumed unital and commutative.

Let $\mathcal{A}$ be a commutative Banach algebra, a multiplicative functional φ is a linear functional that's also an algebra homomorphism, $\varphi : \mathcal{A} \to \mathbb{C}$,

$$\varphi : AB \mapsto \varphi(A)\varphi(B).$$

The set of all multiplicative functionals will be called the spectrum of $\mathcal{A}$ denoted by, $\sigma(\mathcal{A})$. The reason for this name will soon become clear. Multiplicative linear functionals are also called characters in some books.

Let $\varphi \in \sigma(\mathcal{A})$, for any $A \in \mathcal{A}$, we have, $\varphi(A) = \varphi(1 \cdot A) = \varphi(1)\varphi(A)$, or $\varphi(1) = 1$. If A is invertible then $\varphi(A^{-1})\varphi(A) = \varphi(A^{-1}A) = 1$ or $\varphi(A)$ is non-zero. Suppose $|\varphi(A)| \nleq \|A\|$, then, $A - |\varphi(A)|$ is invertible.

$$\varphi(A - |\varphi(A)|) = \varphi(A) - |\varphi(A)|$$

adjusting the phase of A this term can be made zero. This is however a contradiction as φ is non-zero for invertible elements of $\mathcal{A}$. So for every $\varphi \in \sigma(\mathcal{A})$, we have $|\varphi(A)| \leq \|A\|$. Equipped with the weak* topology, $\sigma(\mathcal{A})$ is a closed subset of the closed unit ball B of $\mathcal{A}^*$.

$$\sigma(\mathcal{A}) \subset B, \quad \text{is closed}$$

By Alaoglu's theorem, **??**, $\sigma(\mathcal{A})$ is a compact Hausdorff space.

A left (or right, in our case it's irrelevant as we are dealing with commutative algebras) ideal of $\mathcal{A}$ is a subalgebra $\mathcal{I} \subset \mathcal{A}$ such that $AB \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and for all $B \in \mathcal{A}$. $\mathcal{I}$ is a proper ideal if $\mathcal{I} \neq \mathcal{A}$, and $\mathcal{I}$ is a maximal ideal if it's not contained in any proper ideal. If an ideal contains invertible an element, say A then $AA^{-1} = 1 \in \mathcal{I}$ which means that $B \in \mathcal{I}$ for all $B \in \mathcal{A}$, or $\mathcal{I} = \mathcal{A}$. If $A \in \mathcal{A}$ is not invertible then $\mathcal{I}_A = \{BA \mid B \in \mathcal{A}\}$ is an ideal containing A. Let $\overline{\mathcal{I}}$ be the closure of $\mathcal{I}$. Since the invertible elements of $\mathcal{A}$ form a group and is an open set in $\mathcal{A}$. $\overline{\mathcal{I}}$ cannot contain the identity of $\mathcal{A}$. $\overline{\mathcal{I}}$ is a proper ideal. Every ideal is contained in some maximal ideal, and since the closure of a proper ideal is also a proper ideal, the maximal ideals are closed. The collection of all maximal ideals of $\mathcal{A}$ will be denoted by $\mathcal{M}(\mathcal{A})$. Every non invertible element is contained in some maximal ideal.

Let $\varphi \in \sigma(\mathcal{A})$, for $A \in \ker(\varphi)$, and for all $B \in \mathcal{A}$,

$$\varphi(AB) = \varphi(A)\varphi(B) = 0,$$

so $AB \in \ker(\varphi)$. So it's an ideal. Since $\varphi(1) = 1 \notin \ker(\varphi)$ it's a proper ideal. Suppose $\ker(\varphi)$ is not a maximal ideal, and let $\ker(\varphi) \subsetneq \mathcal{I}$ with $\mathcal{I}$ a proper ideal.

Let $A \in \mathcal{I} \setminus \ker(\varphi)$, then we have, $A = (A - \varphi(A) \cdot 1) + \varphi(A) \cdot 1$. So, we can write $A = A' + \lambda \cdot 1$, for some $A' = A - \varphi(A) \cdot 1 \in \ker(\varphi)$ and $\lambda \in \mathbb{C}$. So, 1 is in the span of A and $\ker(\varphi)$. Equivalently, $\mathcal{I} = \mathcal{A}$ (!). $\ker(\varphi)$ is indeed a maximal ideal. Our goal is to relate the maximal ideals and multiplicative linear functionals.

Theorem 1.5.

$$\varphi \mapsto \ker(\varphi),$$

is a one-to-one correspondence between $\sigma(\mathcal{A})$ *and* $\mathcal{M}(\mathcal{A})$*.*

Proof

Suppose $\ker(\varphi) = \ker(\varkappa)$, every $A \in \mathcal{A}$ can be written as, $A = \varphi(A) \cdot 1 + B$ for some $B \in \ker(\varphi)$. So we have, $\varkappa(A) = \varphi(A)\varkappa(1) + \varkappa(B)$. Since $\ker(\varphi) = \ker(\varkappa)$ we have $\varkappa(B) = 0$ and hence for all $A \in \mathcal{A}$,

$$\varphi(A) = \varkappa(A),$$

or $\varphi = \varkappa$. Hence the mapping $\varphi \mapsto \ker(\varphi)$ is injective.

Suppose $\mathcal{I}$ is a maximal ideal. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the quotient map. $\mathcal{A}/\mathcal{I}$ inherits algebra structure from $\mathcal{A}$ and also inherits a norm $\|A + \mathcal{I}\| = \inf\{\|A + I\| \mid I \in \mathcal{I}\}$ making it a Banach algebra.

$\mathcal{A}/\mathcal{I}$ has no non-trivial ideals, because otherwise if $\mathcal{I}'$ is an ideal of $\mathcal{A}/\mathcal{I}$ then, consider $\pi^{-1}(\mathcal{I}')$. For all $J \in \pi^{-1}(\mathcal{I}')$ and $A \in \mathcal{A}$ since $\pi(J) \in \mathcal{I}'$ we have,

$$\pi(JA) = \pi(J)\pi(A) \in \mathcal{I}'.$$

So, $JA \in \pi^{-1}(\mathcal{I}')$ and hence $\pi^{-1}(\mathcal{I}')$ is an ideal. Since $\mathcal{I} \subsetneq \pi^{-1}(\mathcal{I}')$ it cannot be a maximal ideal. This is a contradiction as we assumed it to be a maximal ideal. Hence every non-zero element of $\mathcal{A}/\mathcal{I}$ is invertible because otherwise we can construct an ideal containing the element. By Gelfand-Mazur theorem we have,

$$\mathcal{A}/\mathcal{I} \cong \mathbb{C} \cdot 1$$

Let the above isomorphism be φ. The composition, $\varphi \circ \pi$ is in $\sigma(\mathcal{A})$ with $\ker(\varphi \circ \pi) = \mathcal{I}$. The map $\varphi \mapsto \ker(\varphi)$ is surjective. $\square$

This allows us to think of $\mathcal{M}(\mathcal{A})$ as a compact Hausdorff space. For every $A \in \mathcal{A}$ we have a map, $\widehat{A}(\varphi) = \varphi(A)$. With the weak* topology on $\sigma(\mathcal{A})$, $\widehat{A}$ is a continuous map on $\sigma(\mathcal{A})$. The map,

$$\Gamma : A \mapsto \widehat{A} \qquad \text{(Gelfand transform)}$$

is called Gelfand tranformation on $\mathcal{A}$. It's a map from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$. Here $C(X)$ means continuous maps on X to $\mathbb{C}$. If $A, B \in \mathcal{A}$ then we have,

$$\widehat{AB}(\varphi) = \varphi(AB) = \varphi(A)\varphi(B) = \widehat{A}(\varphi)\widehat{B}(\varphi).$$

So, the Gelfand transformation is an algebra homomorphism, and $\widehat{1}(\varphi) = \varphi(1) = 1$, so $\widehat{1}$ is a constant function. If A is invertible then for all $\varphi \in \sigma(\mathcal{A})$ we have, $\varphi(AA^{-1}) = 1$ or $\varphi(A)$ is non vanishing. Conversely suppose $\widehat{A}$ is never vanishing, and suppose A is not invertible, then there exists a maximal ideal $\mathcal{I}_A$ containing A. Let the associate multiplicative functional be φ_A such that $\ker \varphi_A = \mathcal{I}_A$. So we have,

$$\varphi_A(A) = \widehat{A}(\varphi_A) = 0$$

this is a contradiction as we started with the assumption that $\widehat{A}$ is non-vanishing. Hence A is invertible if and only if $\widehat{A}$ is non-vanishing. A *-algebra $\mathcal{A}$ is said to be symmetric if

$$\Gamma(A^*) = \widehat{A^*} = \overline{\widehat{A}}.$$

Our goal is to show that for commutative C^*-algebras the Gelfand transform is an isometric isomorphism.

Theorem 1.6.

$$\|\widehat{A}\|_{sup} \leq \|A\|.$$

Proof

Let $\lambda \in \sigma(A)$, i.e., $A - \lambda$ is not invertible. There exists φ_A such that $\varphi_A(A - \lambda) = 0$. So, we have,

$$\varphi_A(A) = \lambda.$$

So, λ is in the range of $\widehat{A}$. Conversely, suppose μ is in the range of $\widehat{A}$, then there exists $\varphi \in \sigma(\mathcal{A})$ such that $\widehat{A}(\varphi) = \mu$, or $\varphi(A - \lambda) = 0$, which means that $A - \lambda$ is not invertible. So, range of $\widehat{A}$ is same as spectrum of $\sigma(A)$.

Now, $\|\widehat{A}\|_{sup} = \sup_{\varphi \in \sigma(\mathcal{A})}\{|\widehat{A}(\varphi)|\}$. So, $\|\widehat{A}\|_{sup} = \rho(A) \leq \|A\|$. $\square$

Suppose $\mathcal{A}$ is symmetric, i.e., $\widehat{A^*} = \overline{\widehat{A}}$, then for all self-adjoint elements, $A = A^*$, $\widehat{A} = \overline{\widehat{A}}$. $\widehat{A}$ is a real valued function. Conversely, every element A can be written as a combination of self-adjoint operators, $A = A_1 + iA_2$, so we have, $A^* = A_1^* - iA_2^*$, and hence,

$$\widehat{A^*} = \widehat{A_1} - i\widehat{A_2} = \overline{\widehat{A}}.$$

So, $\mathcal{A}$ is symmetric if and only if $\widehat{A}$ is real valued function for self-adjoint A.

If $\mathcal{A}$ is a C^*-algebra then we have $\|B^*B\| = \|B\|^2$ for all $B \in \mathcal{A}$. Let $A \in \mathcal{A}$ be self-adjoint, consider $B = A + it$, then we have,

$$\|B\|^2 = \|B^*B\| = \|A\|^2 + t^2$$

Since, $\varphi(B)^2 \leq \|B\|^2 = \|A\|^2 + t^2$, we get,

$$\begin{aligned}\varphi(A + it)^2 &= (Re(\varphi(A)) + iIm(\varphi(A)) + it)^2 \\ &= Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t + t^2 \leq \|A\|^2 + t^2.\end{aligned}$$

Which means $Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t \leq \|A\|^2$ i.e., right side is independent of t, so on the left side $Im(\varphi(A))$ must be zero. Hence $\varphi(A)$ is real valued for all $\varphi \in \sigma(\mathcal{A})$ or equivalently $\widehat{A}$ is real valued for all $A = A^*$. Hence C^*-algebras are symmetric.

Theorem 1.7. *If $\mathcal{A}$ is symmetric then $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$.*

Proof

The proof is an application of Stone-Weierstrass theorem, [?]. If $\mathcal{A}$ is symmetric then $\Gamma(\mathcal{A})$ is closed under conplex conjugation because,

$$\Gamma(A)^* = \Gamma(A^*).$$

So, $\Gamma(\mathcal{A})$ is a self-adjoint subalgebra. $\Gamma(1) = 1$, so $\Gamma(\mathcal{A})$ contains constant functions, and $\Gamma(\mathcal{A})$ separates the points on $\sigma(\mathcal{A})$, because if $\varphi, \varkappa \in \sigma(\mathcal{A})$ with $\varphi \neq \varkappa$ then there exists $A \in \mathcal{A}$ such that $\varphi(A) \neq \varkappa(A)$ i.e., $\Gamma(A)$ is such that $\Gamma(A)(\varphi) \neq \Gamma(A)(\varkappa)$.

So by Stone-Weierstrass theorem $\Gamma(\mathcal{A})$ is a dense subset of $C(\sigma(\mathcal{A}))$. $\square$

Suppose $A \in \mathcal{A}$, let $\sigma(A)$ be the spectrum of the operator, i.e., $\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not intertible}\}$. Suppose $\lambda \in \sigma(A)$ then $A - \lambda$ is not invertible, hence there exists some maximal ideal $\mathcal{I}_\lambda$ containing $A - \lambda$. Let $\varphi_\lambda \in \sigma(\mathcal{A})$ such that $\ker(\varphi_\lambda) = \mathcal{I}_\lambda$. Or equivalently, $\varphi_\lambda(A - \lambda) = 0$, or

$$\varphi_\lambda(A) = \lambda$$

So, to each $\lambda \in \sigma(A)$ we have a multiplicative functional φ_λ such that $\varphi_\lambda(A) = \lambda$.

If $\mathcal{A} = [A, 1]$, i.e., if $\mathcal{A}$ is generated by the identity and the operator A then $\varphi \in \sigma(\mathcal{A})$ is determined by its action on A. Since $\varphi(A^{-1}) = \varphi(A)^{-1}$ and $\varphi(A^*) = \overline{\varphi(A)}$ we have, $\widehat{A}(\varphi_1) = \widehat{A}(\varphi_2) \implies \varphi_1 = \varphi_2$. The map,

$$\widehat{A} : \sigma([A, 1]) \to \sigma(A)$$

is injective and surjective.

Theorem 1.8. (Gelfand-Naimark theorem) *If $\mathcal{A}$ is a unital commutative C^*-algebra then Γ is an isometric *-isomorphism of $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.*

Sketch of Proof

Suppose $\mathcal{A}$ is a commutative Banach algebra, we will show that $\|\widehat{A}\|_{\text{sup}} = \|A\|$ if and only if $\|A^{2^k}\| = \|A\|^{2^k}$ for $k \geq 1$. If $\|\widehat{A}\|_{\text{sup}} = \|A\|$ then,

$$\|A^{2^k}\| \leq \|A\|^{2^k} = \|\widehat{A}\|_{\text{sup}}^{2^k} = \|\widehat{A}^{2^k}\|_{\text{sup}} \leq \|A^{2^k}\|.$$

Here in the first step we used the product norm inequality, in the second step the assumption that $\|\widehat{A}\|_{\text{sup}} = \|A\|$, in the third step the definition of sup norm, and in the fourth step the fact that $\varphi(A) \leq \|A\|$ for all $\varphi \in \sigma(\mathcal{A})$. So,

$$\|\widehat{A}\|_{\text{sup}} = \|A\| \implies \|A^{2^k}\| = \|A\|^{2^k}.$$

Conversely, if $\|A^{2^k}\| = \|A\|^{2^k}$ for all $k \geq 1$, we have, $\|A^{2^k}\|^{1/2^k} = \|A\|$, but since $\lim_k \|A^{2^k}\|^{1/2^k} = \rho(A)$ and since $\|\widehat{A}\|_{sup} = \rho(A)$, we have,

$$\|A^{2^k}\| = \|A\|^{2^k} \implies \|\widehat{A}\|_{sup} = \|A\|.$$

Now for the case of commutative C^*-algebra $\mathcal{A}$, for any $B \in \mathcal{A}$, the element $A = B^*B$ is self-adjoint and hence,

$$\|A^{2^k}\| = \|(A^{2^k-1})^*(A^{2^k-1})\| = \|A^{2^k-1}\|^2.$$

So, we have $\|A^{2^k}\| = \|A\|^{2^k}$ and hence $\|\widehat{A}\|_{sup} = \|A\|$. Since $\mathcal{A}$ is a C^*-algebra we also have, $\|B^*B\| = \|B^2\|$, so we have,

$$\|B\|^2 = \|A\| = \|\widehat{A}\|_{sup} = \|\widehat{|B|^2}\|_{sup} = \|\widehat{B}\|^2_{sup}.$$

Γ is an isometry with closed, dense and injective range. □

2 | Spectral Theorem

Let $\mathcal{A}$ be a commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ containing I. By Gelfand-Naimark theorem, we have an isometric isomorphism between $C(\sigma(\mathcal{A}))$ and $\mathcal{A}$ given by the Gelfand transform. Denote the inverse Gelfand transform of $f \in C(\sigma(\mathcal{A}))$ by $T_f \in \mathcal{A}$, we have $\|T_f\| = \|f\|_{\text{sup}}$.

For every $\varphi, \varkappa \in \mathcal{H}$ we have the map,

$$f \mapsto \langle T_f \varphi | \varkappa \rangle$$

This is a bounded linear functional on $C(\sigma(\mathcal{A}))$ because,

$$|\langle T_f \varphi | \varkappa \rangle| \leq \|T_f\| \, \|\varphi\| \|\varkappa\| = \|f\|_{\text{sup}} \|\varphi\| \|\varkappa\|.$$

Riesz representation theorem says that bounded linear functionals on locally compact Hausdorff spaces correspond to unique Borel measures. Since $\sigma(\mathcal{A})$ is a locally compact Hausdorff space, to each bounded linear functional $f \mapsto |\langle T_f \varphi | \varkappa \rangle|$ there exists a unique complex Borel measure $\mu_{\varphi,\varkappa}$ on $\sigma(\mathcal{A})$ such that,

$$\langle T_f \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} f d\mu_{\varphi,\varkappa}$$

with $\|\mu_{\varphi,\varkappa}\| \leq \|\varphi\| \|\varkappa\|$. So the assignment, $(\varphi, \varkappa) \to \mu_{\varphi,\varkappa}$, is a map from $\mathcal{H} \times \mathcal{H}$ to $\mathcal{M}(\sigma(\mathcal{A}))$. Where $\mathcal{M}(\sigma(\mathcal{A}))$ is the set of all measures on $\sigma(\mathcal{A})$. Since the Gelfand transform takes adjoint to complex conjugate of the function, we have, $T_f^* = T_{\overline{f}}$ and for all $f \in C(\sigma(\mathcal{A}))$,

$$\int_{\sigma(\mathcal{A})} f d\mu_{\varphi,\varkappa} = \langle T_f \varphi | \varkappa \rangle = \langle \varphi | T_f^* \varkappa \rangle = \overline{\langle T_f^* \varkappa | \varphi \rangle} = \overline{\int_{\sigma(\mathcal{A})} \overline{f} d\mu_{\varkappa,\varphi}} = \int_{\sigma(\mathcal{A})} f d\overline{\mu_{\varphi,\varkappa}}.$$

Hence, we have, $\mu_{\varphi,\varkappa} = \overline{\mu_{\varkappa,\varphi}}$. For any positive function $f = \overline{g}g$ we have,

$$\int f d\mu_{\varphi,\varphi} = \langle T_f \varphi | \varphi \rangle = \langle T_g^* T_g \varphi | \varphi \rangle = \|T_g \varphi\|^2 \geq 0.$$

So $\mu_{\varphi,\varphi}$ is a positive measure for all φ.

Once we have $\mu_{\varphi,\varkappa}$ we can define the integral for any Borel measurable function $f \in B(\sigma(\mathcal{A}))$ by $\int f d\mu_{\varphi,\varkappa}$. Now,

$$\left| \int_{\sigma(\mathcal{A})} f d\mu_{\varphi,\varkappa} \right| \leq \|f\|_{sup} \|\mu_{\varphi,\varkappa} \leq \|f\|_{sup} \|\varphi\| \, \|\varkappa\|.$$

and hence it defines a unique bounded operator $T_f \in \mathcal{B}(\mathcal{H})$,

$$\langle T_f \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} f d\mu_{\varphi,\varkappa}.$$

and it clearly agrees with the definition for $f \in C(\sigma(\mathcal{A}))$.

$$\langle T_{\overline{f}} \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} \overline{f} d\mu_{\varphi,\varkappa} = \overline{\int_{\sigma(\mathcal{A})} f d\mu_{\varkappa,\varphi}} = \overline{\langle T_f \varkappa | \varphi \rangle} = \langle \varphi | T_f \varkappa \rangle = \langle T_f^* \varphi | \varkappa \rangle.$$

So it maps $\overline{f}$ to T_f^*. Now, consider T_{fg}, to start, assume $f, g \in C(\sigma(\mathcal{A}))$, we have by definition of $\mu_{\varphi,\varkappa}$,

$$\int_{\sigma(\mathcal{A})} fg d\mu_{\varphi,\varkappa} = \langle T_f T_g \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} f d\mu_{T_g \varphi,\varkappa}.$$

So, we have $g d\mu_{\varphi,\varkappa} = d\mu_{T_g\varphi,\varkappa}$ for all $g \in C(\sigma(\mathcal{A}))$. Now using this, we have for all $f \in B(\sigma(\mathcal{A}))$,

$$\int_{\sigma(\mathcal{A})} fg d\mu_{\varphi,\varkappa} = \int_{\sigma(\mathcal{A})} f d\mu_{T_g\varphi,\varkappa} = \langle T_f T_g \varphi | \varkappa \rangle = \langle T_g \varphi | T_f^* \varkappa \rangle = \int_{\sigma(\mathcal{A})} g d\mu_{\varphi, T_f^* \varkappa}.$$

So, we have for all $f \in B(\sigma(\mathcal{A}))$, $f d\mu_{\varphi,\varkappa} = d\mu_{\varphi,T_f^*\varkappa}$. Now for $g \in B(\sigma(\mathcal{A}))$, we havem

$$\langle T_f T_g \varphi | \varkappa \rangle = \langle T_g \varphi | T_f^* \varkappa \rangle = \int_{\sigma(\mathcal{A})} g d\mu_{\varphi,T_f^*\varkappa} = \int_{\sigma(\mathcal{A})} fg d\mu_{\varphi,\varkappa} = \langle T_{fg} \varphi | \varkappa \rangle.$$

Which means that $T_{fg} = T_f T_g$, and hence it's an algebra homomorphism. The map $f \mapsto T_f$ is a $*$-homomorphism. Hence we have a representation of Borel functions on $\sigma(\mathcal{A})$ on the Hilbert space $\mathcal{H}$.

Suppose S commutes with all $T \in \mathcal{A}$, then S commutes with all T_f with $f \in C(\sigma(\mathcal{A}))$. So we have,

$$\int_{\sigma(\mathcal{A})} f d\mu_{\varphi,S^*\varkappa} = \langle T_f \varphi | S^* \varkappa \rangle = \langle S T_f \varphi | \varkappa \rangle = \langle T_f S \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} f d\mu_{S\varphi,\varkappa}$$

So, $\mu_{\varphi,S^*\varkappa} = \mu_{S\varphi,\varkappa}$. Hence for any $f \in B(\sigma(\mathcal{A}))$,

$$\langle T_f S \varphi | \varkappa \rangle = \int_{\sigma(\mathcal{A})} f d\mu_{S\varphi,\varkappa} = \int_{\sigma(\mathcal{A})} f d\mu_{\varphi,S^*\varkappa} = \langle T_f \varphi | S^* \varkappa \rangle = \langle S T_f \varphi | \varkappa \rangle.$$

Since this holds for all $\varphi, \varkappa \in \mathcal{H}$, S must commute with all T_f for $f \in B(\sigma(\mathcal{A}))$.

If $\{f_n\} \subset B(\sigma(\mathcal{A}))$ and $f_n \to f$ then $T_{f_n} \to T_f$ in weak operator topology, because $\int_{\sigma(\mathcal{A})} f_n d\mu_{\varphi,\varkappa} \to \int_{\sigma(\mathcal{A})} f d\mu_{\varphi,\varkappa}$ by dominated convergence theorem.

2.1 | Spectral Measure

Similar to how we used characteristic functions in integration theory, we will do the same with operators. Let $\epsilon \subset \sigma(\mathcal{A})$ be a Borel set, let χ_ϵ be the characteristic function of ϵ, i.e., $\chi_\epsilon(x) = 1$ if $x \in \epsilon$ and zero otherwise.

$$E(\epsilon) := T_{\chi_\epsilon}.$$

we can now list some immediate properties of spectral measures.

$E(\epsilon)$ is an orthogonal projection. This is because clearly the characteristic function satisfies $\chi_\epsilon = \chi_\epsilon^2 = \overline{\chi_\epsilon}$, the conjugation is because it's a real valued function. This gives us,

$$E(\epsilon) = T_{\chi_\epsilon} = T_{\chi_\epsilon} T_{\chi_\epsilon} = E(\epsilon)^2 = T_{\overline{\chi_\epsilon}} = E(\epsilon)^* \qquad \text{(projection)}$$

Hence $E(\epsilon)$ is a projection.

$E(\varnothing)$ corresponds to the constant zero function, since $f \mapsto T_f$ is an algebra homomorphism, it sends the zero map to zero map, and identity to identity and hence,

$$E(\varnothing) = T_{\chi_\varnothing} = 0, \quad E(\sigma(\mathcal{A})) = \mathbb{1} \qquad \text{(empty sets \& whole set)}$$

For intersection of two sets ϵ, ϵ', we have, $\chi_{\epsilon\cap\epsilon'} = \chi_\epsilon \chi_{\epsilon'}$, and hence we have,

$$E(\epsilon \cap \epsilon') = T_{\chi_{\epsilon\cap\epsilon'}} = T_{\chi_\epsilon \chi_{\epsilon'}} = E(\epsilon)E(\epsilon'). \qquad \text{(disjoint intersection)}$$

If ϵ_i are disjoint then we have for any finite unions,

$$\chi_{\coprod_i \epsilon_i} = \textstyle\sum_{i=1}^n \chi_{\epsilon_i}$$

So,

$$E(\coprod_i \epsilon_i) = \textstyle\sum_{i=1}^n E(\epsilon_i).$$

Now for the infinite case, let $\upsilon_n = \coprod_{i=1}^n \epsilon_i$, and $\upsilon = \coprod_{i\geq 0} \epsilon_i$, then $\chi_{\upsilon_n} \to \chi_\upsilon$ so, we have,

$$\textstyle\sum_{i=1}^n E(\epsilon_i) = E(\upsilon_n) \to E(\upsilon).$$

$\upsilon = \upsilon_n \coprod (\upsilon \backslash \upsilon_n)$, and hence $E(\upsilon) = E(\upsilon_n) + E(\upsilon\backslash\upsilon_n)$. For $\varphi \in \mathcal{H}$,

$$\|[E(\upsilon) - E(\upsilon_n)]\varphi\|^2 = \|E(\upsilon\backslash\upsilon_n)\varphi\|^2 = \langle E(\upsilon\backslash\upsilon_n)\varphi | E(\upsilon\backslash\upsilon_n)\varphi\rangle = \langle E(\upsilon\backslash\upsilon_n)\varphi|\varphi\rangle \to 0$$

Hence the series strongly converges.

$$E(\coprod_i \epsilon_i) = \sum_i E(\epsilon_i) \qquad \text{(convergence)}$$

Let ϵ and ϵ' be disjoint, then we have,

$$\langle E(\epsilon)\varphi | E(\epsilon')\varkappa\rangle = \langle E(\epsilon')E(\epsilon)\varphi|\varkappa\rangle = \langle E(\epsilon' \cap \epsilon)\varphi|\varkappa\rangle = \langle E(\varnothing)\varphi|\varkappa\rangle = 0.$$

So, $E(\epsilon)$ and $E(\epsilon')$ are mutually orthogonal.

Now similar to how we define measures, we consider a measure space (Ω, Σ) consisting of a set Ω, together with a σ-algebra Σ. A $\mathcal{H}$-projection valued measure on (Ω, Σ) or spectral measure is a map,

$$E : \Sigma \to \mathcal{B}(\mathcal{H}).$$

that satisfy the above conditions, projection, empty sets & whole set, disjoint intersection, and convergence. For each $\varphi, \varkappa \in \mathcal{H}$, one can construct ordinary complex measures,

$$E_{\varphi,\varkappa}(\epsilon) = \langle E(\epsilon)\varphi|\varkappa\rangle.$$

this turns out to be a measure, because the above requirements force it. This is a 'measure valued inner product', $(\varphi, \varkappa) \mapsto E_{\varphi,\varkappa}$. $\|E_{\varphi,\varphi}\| = E_{\varphi,\varphi}(\Omega) = \|\varphi\|^2$. For any function $f \in B((\Omega, \Sigma))$, for any $\varphi, \varkappa$ with $\|\varphi\|^2 = \|\varkappa\|^2 = 1$, we have by polarization,

$$\left| \int f dE_{\varphi,\varkappa} \right| \leq \tfrac{1}{4}\|f\|_{\text{sup}} \left[\|\varphi + \varkappa\|^2 + \|\varphi - \varkappa\|^2 + \|\varphi + i\varkappa\|^2 + \|\varphi - i\varkappa\|^2 \right] \leq 4\|f\|_{sup}.$$

So it is bounded, and hence defines a bounded operator T, such that,

$$\langle T\varphi|\varkappa\rangle = \int_\Omega f dE_{\varphi,\varkappa}.$$

We will hence denote T by,

$$T = \int_\Omega f dE.$$

The map $f \mapsto \int f dE$ is linear and $|\int f dE| \leq 4\|f\|_{sup}$. Every Borel measurable function is a uniform limit of simple functions, i.e., functions of the form $f = \sum_{i=0}^n c_i \chi_{\epsilon_i}$, so it's enough to study simple functions. In such case,

$$\int_\Omega f dE_{\varphi,\varkappa} = \sum_i c_i E_{\varphi,\varkappa}(\epsilon_i) = \left\langle \sum_i c_i E(\epsilon_i)\varphi|\varkappa\right\rangle.$$

For any two simple functions, $f = \sum_{i=1}^n c_i \chi_{\epsilon_i}$ and $g = \sum_{j=1}^m d_j \chi_{\epsilon_j}$,

$$fg = \sum_{i,j} c_i d_j \chi_{\epsilon_i \cap \epsilon_j}.$$

This gives us,

$$\int_\Omega fg dE = \sum_{i,j} c_i d_j E(\epsilon_i \cap \epsilon_j) = \sum_{i,j} c_i d_j E(\epsilon_i)E(\epsilon_j) = \int f dE \int g dE.$$

So, $fg \mapsto \int f dE \int g dE$ i.e., it's an algebra homomorphism. It follows because $E(\epsilon) = E^*(\epsilon)$ that $\int \overline{f} dE = (\int f dE)^*$. Hence it's a $*$-homomorphism from $B(\Omega)$ to $\mathcal{B}(\mathcal{H})$.

Theorem 2.1. (Spectral Theorem) *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be commutative C^*-algebra, and let $\Omega = \sigma(\mathcal{A})$, then there exists a unique spectral measure E on Ω such that*

$$T = \int \widehat{T} dE.$$

where $\widehat{T}$ is the Gelfand transform of T. If S commutes with all $T \in \mathcal{A}$ then S commutes with all $E(\epsilon)$, for Borel set $\epsilon \subset \Omega = \sigma(\mathcal{A})$.

Proof

We only have to prove the uniqueness of the spectral measure, which holds by uniqueness of Riesz representation. The other assertion regarding S which commutes with $\mathcal{A}$ is also already proved. □

2.2 | Unbounded Operators

References

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