Part IV

Derived Categories & Functors

The goal of cohomology is to measure the obstruction to exactness. So, if a morphism of complexes keeps the obstruction to exactness the same, then we have not lost the information about the obstruction by going in between such complexes.

A morphism of complexes,

$$\begin{array}{ccc} A^\bullet & & \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots \\ \downarrow f & & \quad\quad \downarrow f^{i-1} \quad\quad \downarrow f^i \quad\quad \downarrow f^{i+1} \\ B^\bullet & & \cdots \xrightarrow[\partial^B_{i-2}]{} B^{i-1} \xrightarrow[\partial^B_{i-1}]{} B^i \xrightarrow[\partial^B_i]{} B^{i+1} \xrightarrow[\partial^B_{i+1}]{} \cdots \end{array}$$

is called a quasi-isomorphism ($\cong_{\mathcal{Q}}$) if the induced morphism at cohomology, $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all i. We want to treat quasi-isomorphic objects as isomorphic objects. A derived category is a category in which such quasi-isomorphic complexes are treated as the same object, i.e., quasi-isomorphisms are isomorphisms.

Derived functors are

The ideal use of these notes is to skim through the material to get an idea of what's happening before actually reading the material carefully from textbooks.

1 | Derived Categories

Given an additive category $\mathcal{A}$, a differential object in $\mathcal{A}$ is a sequence,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

where each $A^i \in \mathcal{A}$ and homomorphisms $\partial_i : A^i \to A^{i+1}$ are morphisms in the category $\mathcal{A}$. The morphisms between two such differential objects $A^\bullet$ and $B^\bullet$, $A^\bullet \xrightarrow{u} B^\bullet$ consists of morphisms $u^i : A^i \to B^i$ such that the following commutative diagram commutes,

$$\begin{array}{ccc} A^\bullet & & \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots \\ \downarrow u & & \quad\quad \downarrow u^{i-1} \quad\quad \downarrow u^i \quad\quad \downarrow u^{i+1} \\ B^\bullet & & \cdots \xrightarrow[\partial^B_{i-2}]{} B^{i-1} \xrightarrow[\partial^B_{i-1}]{} B^i \xrightarrow[\partial^B_i]{} B^{i+1} \xrightarrow[\partial^B_{i+1}]{} \cdots \end{array}$$

The set of morphisms $A^\bullet \to B^\bullet$ is denoted by $\mathrm{Hom}(A^\bullet, B^\bullet)$. The category with differential objects as objects and $\mathrm{Hom}(A^\bullet, B^\bullet)$ is the functor category $\mathcal{A}^{\mathbb{Z}}$.

1.1 | Complexes & Cohomology Functors

A sequence of morphisms in $\mathcal{A}$,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

is called complex if for all i,

$$\partial^A_i \circ \partial^A_{i-1} = 0,$$

The category of complexes over an abelian category $\mathcal{A}$ is an abelian subcategory of $\mathcal{A}^{\mathbb{Z}}$. Denoted by $\mathcal{C}(\mathcal{A})$. Since $\mathcal{A}$ is an abelian category we can split $A^{i-1} \xrightarrow{\partial^{i-1}} A^i$ as

$$\ker \partial^{i-1} \to A^{i-1} \to \mathrm{Im}\ \partial^{i-1} \to A^i \to \mathrm{coker}\ \partial^{i-1}.$$

Then using the definition of ker and coker, we get the following diagram,

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] & & \ker \partial^A_i \arrow[d] & & \ker \partial^A_{i+1} \arrow[d, dashed] & \\
\cdots \arrow[ur, dashed, tail] \arrow[r] & A^{i-1} \arrow[r] \arrow[d, dashed] & \mathrm{Im}\ \partial^A_{i-1} \arrow[r] \arrow[ur, dashed, tail, "f_i"] & A^i \arrow[r] \arrow[d] & \mathrm{Im}\ \partial^A_i \arrow[r] \arrow[ur, dashed, tail] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \mathrm{coker}\ \partial^A_{i-1} \arrow[ur, dashed] & & \mathrm{coker}\ \partial^A_{i-1} \arrow[ur, dashed, "c_i"'] & & \mathrm{coker}\ \partial^A_i. \arrow[ur, dashed] &
\end{tikzcd} \tag{split}$$

The complex $A^\bullet$ is called exact if the morphisms are such that,

$$\mathrm{Im}\ \partial^A_{i-1} \cong \ker \partial^A_i.$$

Given a complex $A^\bullet$, the cohomology is the 'objects' by which sequence fails to be exact. If the complex $A^\bullet$ is not exact at the node A_i, then the $\ker \partial^A_i$ is 'larger' than $\mathrm{Im}\ \partial^A_{i-1}$. The extra stuff in $\ker \partial^A_i$ that's not in $\mathrm{Im}\ \partial^A_{i-1}$ is the cokernel of the morphism f_i. So, we have by universal property of kernels, there exists a morphism,

$$\mathrm{Im}\ \partial^A_{i-1} \xrightarrow{f_i} \ker \partial^A_i \longrightarrow \mathrm{coker}\ f_i \coloneqq \overline{H}^i(A^\bullet).$$

The extra stuff in $\ker \partial^A_i$ that's not in the image of ∂^A_{i-1} is also the kernel of cokernel $\mathrm{coker}\ \partial^A_{i-1}$. By universal property of cokernels, there exists a morphism,

$$\underline{H}^i(A^\bullet) = \ker c_i \longrightarrow \mathrm{coker}\ \partial^A_{i-1} \xrightarrow{c_i} \mathrm{Im}\ \partial^A_i.$$

This is the object we want and it's this part which is making the complex not exact.

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] & & \ker \partial^A_i \arrow[d] \arrow[r, dashed] & \overline{H}^i(A^\bullet) & \ker \partial^A_{i+1} \arrow[d, dashed] & \\
\cdots \arrow[ur, dashed, tail] \arrow[r] & A^{i-1} \arrow[r] \arrow[d, dashed] & \mathrm{Im}\ \partial^A_{i-1} \arrow[r] \arrow[ur, dashed, tail, "f_i"] & A^i \arrow[r] \arrow[d] & \mathrm{Im}\ \partial^A_i \arrow[r] \arrow[ur, dashed, tail] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \mathrm{coker}\ \partial^A_{i-1} \arrow[ur, dashed] & \underline{H}^i(A^\bullet) \arrow[r, dashed] & \mathrm{coker}\ \partial^A_{i-1} \arrow[ur, dashed, "c_i"'] & & \mathrm{coker}\ \partial^A_i. \arrow[ur, dashed] &
\end{tikzcd}$$

In this diagram, the sequences,

$$\ker \partial^A_i \to A^i \to \mathrm{coker}\ \partial^A_{i-1}$$

need not be exact. It's this non-exactness we are trying to capture. Let the composite morphism $\ker \partial_i^A \to A^i \to \operatorname{coker} \partial_{i-1}^A$ be h^i, the cohomology can be defined as the image of the morphism h^i,

$$\ker \partial_i^A \to H^i(A^\bullet) \coloneqq \mathrm{Im}\ h^i \to \operatorname{coker} \partial_{i-1}^A$$

So, $H^i(A^\bullet)$ is the replacement of A^i in split that makes the split into an exact sequence. The three definitions are equivalent, see [?] for the proof where they give equivalence of five equivalent definitions. So, $\mathrm{Im}\ h_i = \ker c_i = \operatorname{coker} f_i$.

So, $H^i(A^\bullet)$ is well defined and we get an exact sequence of morphisms,

$$\begin{tikzcd}
 & \ker \partial_i^A \arrow[d, dashed] & \\
\mathrm{Im}\ \partial_{i-1}^A \arrow[ur, dashed, tail, bend left, "f_i"] & H^i(A^\bullet) \arrow[d, dashed] & \mathrm{Im}\ \partial_i^A \\
 & \operatorname{coker} \partial_{i-1}^A \arrow[ur, dashed, bend right, "c_i"'] &
\end{tikzcd}$$

The new exact sequence we have is,

$$\begin{tikzcd}[column sep=small]
 & \ker \partial_i^A \arrow[d] & & \ker \partial_i^A \arrow[d] & & \ker \partial_{i+1}^A \arrow[d] & \\
\cdots \arrow[ur, bend left] & H^{i-1}(A^\bullet) \arrow[d] & \mathrm{Im}\ \partial_{i-1}^A \arrow[ur, bend left] & H^i(A^\bullet) \arrow[d] & \mathrm{Im}\ \partial_i^A \arrow[ur, bend left] & H^{i+1}(A^\bullet) \arrow[d] & \cdots \\
 & \operatorname{coker} \partial_{i-1}^A \arrow[ur, bend right] & & \operatorname{coker} \partial_{i-1}^A \arrow[ur, bend right] & & \operatorname{coker} \partial_i^A \arrow[ur, bend right] &
\end{tikzcd}$$

By replacing the A^is with $H^i(A^\bullet)$ we get an exact sequence. This gives us a collection of functors from $\mathcal{C}(\mathcal{A})$ to $\mathcal{A}$, called the cohomology functors,

$$H^i : C(\mathcal{A}) \to \mathcal{A},\ H^i(A^\bullet) = \ker \partial_i^A / \mathrm{Im}\ \partial_{i-1}^A.$$

A morphism of complexes,

$$\begin{tikzcd}
A^\bullet \arrow[d, dashed, "f"] & \cdots \arrow[r, "\partial_{i-2}^A"] & A^{i-1} \arrow[r, "\partial_{i-1}^A"] \arrow[d, dashed, "f^{i-1}"] & A^i \arrow[r, "\partial_i^A"] \arrow[d, dashed, "f^i"] & A^{i+1} \arrow[r, "\partial_{i+1}^A"] \arrow[d, dashed, "f^{i+1}"] & \cdots \\
B^\bullet & \cdots \arrow[r, "\partial_{i-2}^B"'] & B^{i-1} \arrow[r, "\partial_{i-1}^B"'] & B^i \arrow[r, "\partial_i^B"'] & B^{i+1} \arrow[r, "\partial_{i+1}^B"'] & \cdots
\end{tikzcd}$$

is called a quasi-isomorphism (qis) if the induced morphism at cohomology,

$$H^i(f) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$$

is an isomorphism for all i. Induced morphism makes sense because cohomology is a functor.

Quasi-isomorphisms manipulate the complexes so that the obstruction to exactness is preserved. So the collection of quasi-isomorphic complexes have same obstructions and could be treated to be the same. We want to treat quasi-isomorphic objects as isomorphic objects. For the sake of simplicity, we will stop writing the bullets for the remainder of this note.

1.2 | Localization of Categories

Localization is the process of adding all the formal inverses to a collection of morphisms. Note that the objects remain the same, but there will be more relation between these objects. Let $\mathcal{Q}$ denote a collection of morphisms in $\mathcal{C}$, the aim of localization is to construct a new category $\mathcal{C}[\mathcal{Q}^{-1}]$ and a functor,

$$\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$$

which sends morphisms belonging to $\mathcal{Q}$ to isomorphisms in $\mathcal{C}[\mathcal{Q}^{-1}]$. This construction being 'universal' i.e., for any other category $\mathcal{D}$ with a functor $\widehat{\mathcal{L}_{\mathcal{Q}}} : \mathcal{C} \to \mathcal{D}$, which sends morphisms in $\mathcal{Q}$ to isomorphisms gets factored through $\mathcal{L}_{\mathcal{Q}}$, that's to say there exists a functor

$$\mathcal{H} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$$

such that $\widehat{\mathcal{L}_{\mathcal{Q}}} = \mathcal{H} \circ \mathcal{L}_{\mathcal{Q}}$, so we have the following commutative diagram of functors,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widehat{\mathcal{L}_{\mathcal{Q}}}"] \arrow[d, "\mathcal{L}_{\mathcal{Q}}"'] & \mathcal{D} \\
\mathcal{C}[\mathcal{Q}^{-1}] \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}$$

The extra things $\mathcal{D}$ has that's not already in $\mathcal{C}[\mathcal{Q}^{-1}]$ should not related to the localization process, i.e adding inverses to the morphisms in $\mathcal{Q}$. This can be formalized by saying precomposition of functors with $\mathcal{L}_{\mathcal{Q}}$ is an isomorphism of the respective natural transformations. If we have two functors $\mathcal{F}, \mathcal{G} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$, then,

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}[\mathcal{Q}^{-1}]}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{D}}}(\mathcal{F} \circ \mathcal{L}_{\mathcal{Q}}, \mathcal{G} \circ \mathcal{L}_{\mathcal{Q}}).$$

Or equivalently, the functor,

$$\circ \mathcal{L}_{\mathcal{Q}} : \mathcal{C}[\mathcal{Q}^{-1}]^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}.$$

is fully faithful. $\mathcal{C}[\mathcal{Q}^{-1}]$ together with the functor $\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$ is called the localization of $\mathcal{C}$ with $\mathcal{Q}$.

Localization, if it exists is unique upto equivalence of categories. The problem now is to understand what constraint on $\mathcal{Q}$ that guarantees the existence of localization of $\mathcal{C}$ with respect to $\mathcal{Q}$. Every category $\mathcal{D}$ that has the needed inverses to $\mathcal{Q}$ will be such that, there exists a functor from the localization $\mathcal{C}[\mathcal{Q}^{-1}]$ to it,

$$\mathcal{C}[\mathcal{Q}^{-1}] \xrightarrow{\mathcal{H}} \mathcal{D}$$

So the localization is the 'left most' category that satisfies certain properties, contains formal inverses. We should expect some colimit type thing happening here in the category of categories.

Sketch of Construction

The goal is to add the 'inverses' and turn it into a category. Let $\mathcal{Q}^{-1}$ be the set in $\mathcal{C}^{\mathrm{op}}$ corresponding to the collection of morphisms $\mathcal{Q}$. Note here that we are assuming the hom-sets are small sets. The new category should include these extra morphisms.

Consider first the graph with objects of $\mathcal{C}$ as vertices, and the arrows of the graph consists of morphisms in $\mathcal{C}$ together with the collection $\mathcal{Q}^{-1}$. So, the new collection of morphisms is given by, $\mathrm{Hom}_{\mathcal{C}} \coprod \mathcal{Q}^{-1}$. With concatenation as composition this is a category, denoted by

$\mathcal{FC}[\mathcal{Q}^{-1}]$. The identity morphisms given by the empty path from and to the same vertex. To turn this into a category we need we have to define equivalences that make the compositions $f \circ f^{-1}$ into identities for all $f \in \mathcal{Q}$.

So the localization of $\mathcal{C}$ with $\mathcal{Q}$ is,

$$\mathcal{C}[\mathcal{Q}^{-1}] \coloneqq \mathcal{FC}(\mathcal{Q}^{-1})/\sim$$

where the quotient consists of same objects and morphisms are quotiented by the above equivalence. This quotient is the colimit we needed. $\square$

For $\mathcal{Q}$ with special properties more direct formulas for the hom-sets of $\mathcal{C}[\mathcal{Q}^{-1}]$ can be obtained. The localization of the category of complexes with the collection of quasi-isomorphisms is called the derived category.

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$$

where $\mathcal{Q}$ is the collection of quasi-isomorphisms. Our goal is to obtain a more direct formula for localization of the category of complexes with quasi-isomorphisms.

1.2.1 | Localization with Quasi-Isomorphisms

We want to get an explicit description of localization for the case of quasi-isomorphisms. This can be done with the special properties the collection of quasi-isomorphisms comes equipped with. Composition of morphisms that preserve cohomology also preserve cohomology, so composition of quasi-isomorphisms is also a quasi-isomorphism. Identity morphisms on the complexes preserve cohomology.

Let $f : A \to B$ be a morphism, and suppose $q : B \to C$ is a quasi-isomorphism, then it's invertible in the localized category. So, $\mathcal{L}(q)$ is an isomorphism, so we could just invert and get a map $A \to C$.

$$\begin{tikzcd} & \mathcal{L}(C) \arrow[d, "\cong"] \\ \mathcal{L}(A) \arrow[r] \arrow[ur, dashed] & \mathcal{L}(B). \end{tikzcd}$$

So, in the original category, there should exist a quasi-isomorphism, $A \dashrightarrow^{\cong_{\mathcal{Q}}} D$ such that the following diagram commutes,

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed, "\cong_{\mathcal{Q}}"'] & C \arrow[d, "\cong_{\mathcal{Q}}"] \\ A \arrow[r] & B. \end{tikzcd} \tag{extension}$$

Similarly for the arrows reversed, for the same reason. Suppose we have two morphisms, $f, g : B \to C$ and a quasi-isomorphism $\tau : C \to D$ such that $\tau \circ f = \tau \circ g$ then it means that f and g manipulate the information about cohomology similarly. So, what τ did was rearrage the relevant information to make the two morphisms equal. It should also be possible to rearrange this information before manipulation by f and g. So, there should exist a quasi-isomorphism $\sigma : A \to B$ such that $f \circ \sigma = g \circ \sigma$.

$$\begin{tikzcd} A \arrow[r, dashed, "\cong_{\mathcal{Q}}"] & B \arrow[r, shift left] \arrow[r, shift right] & C \arrow[r, dashed, "\cong_{\mathcal{Q}}"] & D. \end{tikzcd} \tag{symmetry}$$

A collection of morphisms $\mathcal{Q} \subset \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}$ is called a right multiplicative system if it contains every isomorphism in $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}$, closed under composition, and satisfies extension and symmetry.

The explicit construction of the localization of a right multiplicative system is as follows, $\mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$ consists of the same objects as $\mathcal{C}(\mathcal{A})$. For morphisms, we should think of objects that are quasi-isomorphic as the same object. So, the morphisms in the derived category will be represented by pairs of morphisms in the original category of complexes, visualised by the diagram,

$$\begin{tikzcd}
\mathcal{L}(C) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\
\mathcal{L}(A) &
\end{tikzcd}$$

The important part is the transformation described by the map f. Consider two morphisms between $\mathcal{L}(A)$ and $\mathcal{L}(B)$,

$$\begin{tikzcd}
\mathcal{L}(C) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) & \mathcal{L}(D) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\
\mathcal{L}(A) & & \mathcal{L}(A) &
\end{tikzcd}$$

Qualitatively, these two morphisms represent the same morphism if they transform same quasi-isomorphic information. So the non-quasi-isomorphic maps are the important parts. Two morphisms are equivalent if their non-quasi-isomorphic representative factor through the same morphism. Visualised by the diagram,

$$\begin{tikzcd}
& C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[d, dashed] \arrow[dr] & \\
A & E \arrow[l, dashed, "\cong_{\mathcal{Q}}"] \arrow[r, dashed] & B \\
& D \arrow[ul, "\cong_{\mathcal{Q}}"] \arrow[u, dashed] \arrow[ur] &
\end{tikzcd} \qquad \text{(derived equivalence)}$$

By the symmetry of the diagram this is clearly an equivalence relation.

For the composition of morphisms in the derived category, for two morphisms, $\mathcal{L}(A) \to \mathcal{L}(B)$ and $\mathcal{L}(B) \to \mathcal{L}(C)$ the composition of the morphisms corresponds to the following representative in the original category of complexes,

$$\begin{tikzcd}
& & E \arrow[dl, dashed, "\cong_{\mathcal{Q}}"'] \arrow[dr, dashed] & & \\
& C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & & D \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & \\
A & & B & & C
\end{tikzcd} \qquad \text{(derived composition)}$$

Note that the above square is possible due to extension property. The category with same objects as the category of complexes, with morphisms described by the equivalence classes of morphisms in the category of complexes as described in derived equivalence, together with composition as described in derived composition is the derived category $\mathcal{D}(\mathcal{A})$.

For proof that this is indeed the localization of $\mathcal{C}(\mathcal{A})$ with $\mathcal{Q}$ see, [1]. What needs to be checked is that any other functor which takes quasi-isomorphisms to isomorphisms must factor through this category.

1.3 | Structure of Derived Category

To the category of complexes we added a bunch of new morphisms to obtain the derived category. So we cannot expect these newly added morphisms to have kernels and cokernels. The abelianness of the category of complexes will go in the derived category.

1.3.1 | Triangulated Categories

2 | Derived Functors

2.1 | Resolutions

2.2 | Spectral Sequences

2.3 | Derived Functors as Kan Extensions

References

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