# Part I

# Tangent & Cotangent Bundles

What we need are topological spaces on which we can do calculus. We make them locally look like Euclidean spaces and import calculus from the them. These notes are based on Ramanan's algebraic approach to differential geometry, as in the book [2]. I have tried to make it a bit more intuitive.

## 1 | Differentiable Manifolds

Our starting point is a topological space $\mathcal{M}$ that's Hausdorff and admits a countable base for the topology. This condition is to make sure there are no pathological spaces we should be worried about. The Hausdorff condition makes the points distinguishable by the topology itself. The countable basis allows us to do analysis. On this topological space we want a differentiable structure, i.e., the structure that allows us to do calculus.

The differentiable structure allows us to define differentiable functions. We expect differentiable functions to have some form of local nature, similar to continuous functions. The notion of a sheaf axiomatizes this 'local nature'. Given a topological space $X$, a sheaf is a way of describing a class of objects on $X$ that have a local nature. To motivate the definition, consider the set of continuous functions on the space $X$. Denote by $CU$ the set of real-valued continuous functions on $U$. Then every function, $f \in CU$ has the following local properties,

If $V \subset U$ then $f$ restricted to $V$ is a continuous map, $f|_V : V \to \mathbb{R}$. The map, $f \mapsto f|_V$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_V)|_W = f|_W.$$

This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of $X$ and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\mathrm{op}}$ is the dual category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\mathrm{op}}$ we get restriction morphism in $\mathbf{Sets}$, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_V$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps and $\mathbf{Sets}$ has a class of morphisms called restriction maps $|_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that, $|_{VW} \circ |_{UV} = |_{UW}$.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions, et $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map $f$ if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an $I$-indexed family of functions $(f_i)_{i \in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of $i$ and $j$ is important here and thats what distinguishes the two maps. The above property of existence of the function $f$ implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map $e$ from $CU$ to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$CU \overset{e}{\dashrightarrow} \prod_i CU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map $e$ is called the equalizer of $p$ and $q$.

**Definition 1.1.** A sheaf of sets $\mathcal{F}$ on a topological space $X$ is a functor,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i \in I} U_i$ of an open set $U$ of $X$ yields an equalizer diagram.

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

We start with what we expect from the 'differentiable' functions. The differentiable functions are continuous functions and hence satisfy the locality requirements and should form a sheaf. The sheaf of 'differentiable functions' is our starting point.

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}.$$

Since each differentiable function is expected to be a continuous function as well we have, $\mathcal{A}^{\mathcal{M}}(U) \subseteq C^{\mathcal{M}}(U)$, where $C^{\mathcal{M}}$ is the sheaf of continuous functions, $C^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}$, on $\mathcal{M}$ i.e., $\mathcal{A}^{\mathcal{M}}$ is a subsheaf of $C^{\mathcal{M}}$.

Let $\mathcal{F}_n$ be the sheaf of differentiable functions on the Euclidean space $\mathbb{R}^n$. The 'locally looks like Euclidean space' means that the sheaf $\mathcal{A}^{\mathcal{M}}$ locally looks like differentiable functions over a Euclidean space.

**Definition 1.2.** A differentiable manifold is a Hausdorff, second countable topological space $\mathcal{M}$ together with a sheaf

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

of subalgebras of $C^{\mathcal{M}}$ such that for any $x \in \mathcal{M}$ there is an open neighborhood $x \in U$ with a homeomorphism $U \cong_{\varphi} V \subseteq \mathbb{R}^n$, such that

$$(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{F}_n(V),$$

where $(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{A}^{\mathcal{M}}(\varphi^{-1}(V))$. This is easier to see in the diagram,

$$V \xrightarrow{\varphi^{-1}} U \xrightarrow{\mathcal{A}^{\mathcal{M}}} \mathcal{A}^{\mathcal{M}}(U).$$

The pair $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ is called a differentiable manifold. The homeomorphisms $\varphi$s are called coordinate charts and $\mathcal{A}^{\mathcal{M}}$ is called the differentiable structure. The homeomorphisms transfer the smooth on euclidean space to the manifold. The sections of $\mathcal{A}^{\mathcal{M}}$ over an open set $U \subset \mathcal{M}$ are called differentiable functions on $U$, and we can do calculus on them. Recall that a section of $\mathcal{A}^{\mathcal{M}}$ over $U$ is just an element of $\mathcal{A}^{\mathcal{M}}U$.

Clearly the Euclidean space is a differentiable manifold. For an open set $U \subset \mathcal{M}$ the pair $(U, \mathcal{A}^{\mathcal{M}}|_U)$ is a differentiable manifold. If $\cup_i U_i$ is an open cover of $\mathcal{M}$ then $(U_i, \mathcal{A}^{\mathcal{M}}|_{U_i})$ are open manifolds. Let $U_i \cong_{\varphi_i} V_i$ and $U_i$ and $U_j$ intersect, let $\varphi_i(U_i \cap U_j) = V_{ij}$ and $\varphi_j(U_i \cap U_j) = V_{ji}$ then,

$$V_{ij} \cong_{\varphi_j \circ \varphi_i^{-1}} V_{ji}.$$

So, a collection of differentiable manifolds $(U_i, \mathcal{A}_i)$ can be glued together to form a differentiable manifold if the homeomorphisms $\varphi_j \circ \varphi_i^{-1}$ map the restriction $\mathcal{A}_i|_{U_i \cap U_j}$ to $\mathcal{A}_j|_{U_i \cap U_j}$ i.e., differentiable maps are mapped to differentiable maps. This means that $\varphi_j \circ \varphi_i^{-1}$ is differentiable for every $i$, $j$.

$\mathcal{M}$ may be obtained by taking all the open sets $U_i$ and pasting $U_{ij} \subset U_i$ to $U_{ji} \subset U_j$ together by the transition functions.

$$\coprod_{i,j} U_i \cap U_j \underset{q}{\overset{p}{\rightrightarrows}} \coprod_i U_i \xrightarrow{c} \mathcal{M}.$$

The map $c$ sends all the points $x \in U_i$ to the same point $x \in \mathcal{M}$. $c$ is the coequalizer of $p$ and $q$ in the category **Top** of topological spaces. This is parallel to the definition of sheaf.

A continuous map $f$ of a differentiable manifold $\mathcal{M}$ into a differentiable manifold $\mathcal{N}$,

$$f : \mathcal{M} \to \mathcal{N},$$

is said to be differentiable if it locally maps differentiable functions to differentiable functions, i.e., for all $x \in \mathcal{M}$ if $g$ is a differentiable function in a neighborhood $U$ of $f(x)$ then $g \circ f$ is differentiable function on $f^{-1}(U)$. If $g \in \mathcal{A}^{\mathcal{N}}(U)$ then $g \circ f \in \mathcal{A}^{\mathcal{M}}(f^{-1}(U))$. Hence to each differentiable maps there is a homomorphism of the sheaf $\mathcal{A}_{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$ given by the map,

$$g \mapsto g \circ f.$$

This is the map

$$C^{\mathcal{N}} \to f_*(C^{\mathcal{M}}),$$

of sheaves on $\mathcal{N}$ which sends the subsheaf $\mathcal{A}^{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$. Differentiable manifolds together with morphisms like is the category of smooth manifolds. $f_*$ is called the structure homomorphism associated to $f$. A differentiable map $f : \mathcal{M} \to \mathcal{N}$ of differentiable manifolds is called a diffeomorphism if there is a differentiable inverse.

# 2 | Tangent and Cotangent Bundles

What we want to do is give a linear approximation of a manifold at each point. In order to do this, we use curves passing through the point, and linearize them, and then study them.

Around each point $x \in \mathcal{M}$, consider all the smooth functions $f \in \mathcal{A}^{\mathcal{M}}(U)$, i.e., $f : U \to \mathbb{R}$ defined in some open neighborhood $U$ of $x \in \mathcal{M}$. For each smooth path $h : \mathbb{R} \to U$ which passes through $x$ with $h(0) = x$ we can define a smooth map,

$$f \circ h : \mathbb{R} \to \mathbb{R},$$

which has a first derivative at 0. This gives us a pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0}, \tag{pairing}$$

To remove redundant information, we define the equivalences $f \equiv f'$ at $x$ if $\langle f, h \rangle_x = \langle f', h \rangle_x$ for all $h$ and $h \equiv h'$ at $x$ if $\langle f, h \rangle_x = \langle f, h' \rangle_x$ for all $f$. Under addition and scalar multiplication of functions, this set of all equivalence classes of functions $f$ forms a real vector space, denoted $T^x$. Each function $f$ in the neighborhood of $x$ determines a vector $[f]$, usually written as $df$.

The sheaf of differentiable functions has more algebraic structure. It's a sheaf of algebras over $\mathbb{R}$ or the sheaf of module over the ring $\mathbb{R}$. The category of modules over some ring possess direct limits. Inclusion is a preorder on the collection of open sets given by,

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For the pre-sheaf $\mathcal{A}^{\mathcal{M}} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in $\mathbf{Sets}$ given by $\{\mathcal{A}^{\mathcal{M}} U\}_{U \in \mathcal{D}}$. We will focus on this particular directed system.

The stalk $\mathcal{A}^{\mathcal{M}}_x$ of a pre-sheaf $\mathcal{A}^{\mathcal{M}}$ at $x$ is the direct limit of the directed system $\{\mathcal{A}^{\mathcal{M}} U_i\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is a directed set of open neighborhoods of $x$.

$$\mathcal{A}^{\mathcal{M}}_x = \varinjlim_{x \in U} \mathcal{A}^{\mathcal{M}} U.$$

Stalks are functors,

$$\begin{aligned} \mathrm{Stalk}_x : \mathrm{PSh}(\mathcal{M}) &\to \mathbf{Sets} \\ \mathcal{A}^{\mathcal{M}} &\mapsto \mathcal{A}^{\mathcal{M}}_x. \end{aligned}$$

The elements of $\mathcal{A}^{\mathcal{M}}_x$ are called germs at $x$. If a germ is a direct limit of some element $f \in \mathcal{A}^{\mathcal{M}} U$ then we denote it by $\mathrm{germ}_x f$. Note that $\mathcal{A}^{\mathcal{M}}_x$ is an algebra, and in particular a ring. $\mathrm{germ}_x : \mathcal{A}^{\mathcal{M}} U \to \mathcal{A}^{\mathcal{M}}_x$, is a homomorphism of the respective category for each $U$.

An ideal $\mathcal{I}^{\mathcal{M}}_x \subset \mathcal{A}^{\mathcal{M}}_x$ is a subalgebra such that if $\mathrm{germ}_x f \in \mathcal{I}^{\mathcal{M}}_x$ then $\mathrm{germ}_x f \, \mathrm{germ}_x g \in \mathcal{I}^{\mathcal{M}}_x$ for all $\mathrm{germ}_x g \in \mathcal{A}^{\mathcal{M}}_x$. A proper ideal cannot contain the identity because that would mean the whole algebra is contained in the ideal. For each $\mathrm{germ}_x f \in \mathcal{A}^{\mathcal{M}}_x$, we have the evaluation map, $\mathrm{germ}_x f \to f(x)$, gives us an algebra homomorphism,

$$ß : \mathcal{A}^{\mathcal{M}}_x \to \mathbb{R}.$$

The kernel of this evaluation map $\mathcal{I}^{\mathcal{M}}_x = \ker(ß)$ is an ideal of $\mathcal{A}^{\mathcal{M}}_x$, consisting of all germs that vanish at $x$, i.e., $f(x) = 0$ and hence $f(x)g(x) = 0$ for all $g \in \mathcal{A}^{\mathcal{M}}$. Hence,

$$\mathcal{A}^{\mathcal{M}}_x / \mathcal{I}^{\mathcal{M}}_x \cong \mathbb{R}.$$

Evaluation can hence be thought of as taking quotient with the maximal ideal $\mathcal{I}_x^{\mathcal{M}}$. This is also the only maximal ideal, because all other functions have local inverse, because if a function $f$ is non-zero in a small neighborhood it has an inverse, defined by, $\mathrm{germ}_x(1/f)$, and hence this would mean the constant function belongs to the ideal which means it's not proper ideal. So, no other proper ideal can exist.

Going back to the pairing, the set of equivalence classes of paths $h$ are called tangent vectors at $x$ denoted by $T_x\mathcal{M}$. Each smooth path through $x$ has a tangent vector denoted by $\tau_h$. Using the pairing, the tangent vector $\tau_h$ determines a linear map, $D_{\tau_h} : T^x \to \mathbb{R}$ given by the action,

$$D_{\tau_h}([f]) = \langle f, h\rangle_x.$$

We would like to understand what $T^x$ is. The set $T_x\mathcal{M}$ of all tangent vectors at $x$ is isomorphic to the set of all linear maps $T^x \to \mathbb{R}$. That's to say, $T_x\mathcal{M}$ is the dual space of $T^x$, and hence is itself a vector space. We will hence denote $T^x$ by $T_x^\vee\mathcal{M}$ or $\mathrm{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R})$.

## 2.1 | Tangent Space and Cotangent Space

The derivative of a product, in the pairing, the map $D = D_{\tau_h}$ satisfies the following product rule,

$$D(fg) = f(x)D(g)(x) + g(x)D(f)(x). \tag{Leibniz}$$

for all $f, g \in \mathcal{A}^{\mathcal{M}}$. This is called the Leibniz property, and all the maps $D$ with the Leibniz property are called derivations. Conversely, every derivation there is a corresponding curve $h$ such that $D_{\tau_h} = D$.[1] The linear maps,

$$D : \mathcal{A}_x^{\mathcal{M}} \to \mathbb{R},$$

with the above Leibniz property at $x$ are called derivations, denoted by $T_x\mathcal{M}$.

Our goal is to describe the collection of equivalence classes of functions $f$. The equivalence relation is $f \equiv f'$ iff $\langle f, h\rangle_x = \langle f', h\rangle_x$, for all $h$ and these uniquely determine the map $D_{\tau_h}$. The equivalence classes are called cotangent vectors at $x$. By plugging in the constant 1, it can be checked that $D$ annihilates constant functions.

$$D(\lambda) = 0 \quad \forall \lambda \in \mathbb{R}.$$

Hence all functions that differ by constant belong to the same equivalence class. Hence, for every $\mathrm{germ}_x f \in \mathcal{A}_x^{\mathcal{M}}$, we can consider the functions $\mathrm{germ}_x(f - f(x))$. These functions vanish at $x$, the action of $D$ on the ideal $\mathcal{I}_x^{\mathcal{M}}$ is sufficient to describe the map $D$. So, we have a surjection of the ideal $\mathcal{I}_x^{\mathcal{M}}$ to the set of equivalence classes.

$$\mathcal{I}_x^{\mathcal{M}} \twoheadrightarrow T_x^\vee\mathcal{M}.$$

Now we have to remove all the redundant information from $\mathcal{I}_x^{\mathcal{M}}$. The kernel of the map is the ideal $(\mathcal{I}_x^{\mathcal{M}})^2 = \{\sum_{i,j} g_i f_j \ : \ f_i, g_j \in \mathcal{I}_x^{\mathcal{M}}\}$. Hence, we can quotient it out of $\mathcal{I}_x^{\mathcal{M}}$ and we have,

$$T_x^\vee\mathcal{M} \cong \mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2,$$

or that the equivalence class for the function $f$ only contains the first order information. From the definition of a differentiable manifold around each $x$, there is a neighborhood $U$, such that,

$$\mathcal{A}^{\mathcal{M}}(\varphi^{-1}(U)) = \mathcal{F}_n(V)$$

[1] The idea is to express it in local charts, and this should be of the form $\sum_i h_i \partial/\partial x_i$, and using this define a curve $h : t \mapsto \varphi^{-1}(t(c_i x_i))$. This works.

for the local chart, $\varphi$. Since $\mathcal{F}_n$ consist of smooth functions on $\mathbb{R}^n$, we can describe them in terms of their Taylor expansion. If we denote the local coordinates by $x_i$, we have,

$$f(y) = f(x) + \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(x)\right](\varphi_i(x) - x_i) + \sum_{i,j} \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi(x))\right](\varphi_i(x) - x_i)(\varphi_j(x) - x_j) + \cdots$$

Since the elements should be in $\mathcal{I}_x$, we have, $f(x) = 0$, and since we are quotienting out by $\mathcal{I}_x^2$, the higher order terms will go. The equivalence classes $[\varphi_i(x) - x_i]$ form a basis for the cotangent space. We denote them by $dx_i$.

The tangent space then is,

$$T_x\mathcal{M} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2, \mathbb{R}).$$

In local coordinates, the dual basis for the equivalence classes $[\varphi_i(x) - x_i]$ will be the equivalence classes $\partial/\partial x_i$. However we want to understand the structure of tangent spaces from a sheaf theoretic perspective.

For any derivation, $D \in T_x\mathcal{M}$, and $h \in \mathcal{A}^{\mathcal{M}}$,

$$h(x)D(fg) = h(x)f(x)D(g) + h(x)g(x)D(f).$$

If $D, D' \in T_x\mathcal{M}$, then their sum $D + D' \in T_x\mathcal{M}$. $hD \in T_x\mathcal{M}$. So, $T_x\mathcal{M}$ is an $\mathcal{A}_x^{\mathcal{M}}$-module. Using these derivations we can define the tangent sheaf.

Define $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ to be the set of all derivations.That is to say $D \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, if for all $f, g \in \mathcal{A}^{\mathcal{M}}U$,

$$D(fg) = fD(g) + gD(f).$$

Such operators $D : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}$ are called first order linear homogeneous differential operators. If $D, D' \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, then we can define a new operator $[D, D']$ defined by,

$$[D, D'](f) = D(D'(f)) - D'(D(f)). \tag{Lie bracket}$$

The geometric meaning of Lie bracket will become clear later. The tangent sheaf is the sheaf,

$$\begin{aligned} \mathcal{T} = \mathcal{D}(\mathcal{A}^{\mathcal{M}}) :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to \mathbf{Sets} \\ U &\mapsto \mathcal{D}(\mathcal{A}^{\mathcal{M}}U). \end{aligned}$$

is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules and sheaf of $\mathbb{R}$-Lie algebras.

$\{f_i\}_{i=1}^n \mapsto \sum_i f_i \frac{\partial}{\partial x_i}$ is an isomorphism of modules $(\mathcal{A}^{\mathcal{M}}(U))^{\oplus n}$ and $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ for a chart $(\varphi, U)$. Such sheaves of modules are called locally free modules. The tangent sheaf consists of the sections of the tangent bundle where the tangent bundle $T\mathcal{M}$ is the disjoint union,

$$T\mathcal{M} = \coprod_{x \in \mathcal{M}} T_x\mathcal{M}.$$

These sections correspond to vector fields. The stalks of this sheaf consist of germs of vector fields.

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) = \varinjlim_{x \in U} \mathcal{D}(\mathcal{A}^{\mathcal{M}}U).$$

So, $\mathrm{germ}_x D : \mathcal{A}_x^{\mathcal{M}} \to \mathcal{A}_x^{\mathcal{M}}$. The evaluation of the germs at the point $x$ should give us vectors of the tangent space and they do. We evaluate

$$\mathcal{A}_x^{\mathcal{M}} \xrightarrow{\mathrm{germ}_x D} \mathcal{A}_x^{\mathcal{M}} \xrightarrow{\ß} \mathbb{R}.$$

So the composition of the derivation with this evaluation map corresponds to a derivation at $x$ which are tangent vectors at $x$.

The evaluation map gave us an isomorphism, $\mathcal{A}^{\mathcal{M}}_x/\mathcal{I}^{\mathcal{M}}_x \cong \mathbb{R}$. For locally free sheaves, for every $x \in \mathcal{M}$, there exists a neighborhood $U$ such that $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) = (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$. So, we have,

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}^{\mathcal{M}}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) \cong \mathbb{R}^n.$$

In this sense $\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}^{\mathcal{M}}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$ is the space of valuation of the sections at $x$. This is the same as the tangent space at $x$.

$$T_x\mathcal{M} = \mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}^{\mathcal{M}}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$$

We have a projection from the tangent bundle to the base space,

$$\pi : T\mathcal{M} \to \mathcal{M}$$

which sends $T_x\mathcal{M} \mapsto x$. Since $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$. So, the set $\pi^{-1}(U)$ can be given a differentiable structure of a product. These can be patched up to get a differentiable structure on $T\mathcal{M}$. This topology is Hausdorff and has a countable basis because locally it's a product of Hausdorff spaces with countable basis.[2] Smooth sections of this bundle are called vector fields.

Similarly we can consider the cotangent pre-sheaf,

$$\begin{aligned} \mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2 :\ & \mathcal{O}(\mathcal{M})^{\text{op}} \to \textbf{Sets} \\ & \qquad\qquad U \mapsto \mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2 \end{aligned}$$

where $\mathcal{I}^{\mathcal{M}}U$ is the maximal ideal of $\mathcal{A}^{\mathcal{M}}U$. This might not be a sheaf however. So we might have to sheafify such pre-sheafs. The stalks of this pre-sheaf are given by,

$$\text{Stalk}_x : \mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2 \mapsto \mathcal{I}^{\mathcal{M}}_x/(\mathcal{I}^{\mathcal{M}}_x)^2.$$

We can now 'bundle' these stalks together,

$$\mathcal{ET}^{\vee} = \coprod_x \mathcal{I}^{\mathcal{M}}_x/(\mathcal{I}^{\mathcal{M}}_x)^2,$$

and define the map, $\pi : \mathcal{ET}^{\vee} \to \mathcal{M}$ that sends each $\text{germ}_x f$ to the point $x$. Each $f \in \mathcal{I}U/(\mathcal{I}U)^2$ determines a function $\hat{f} : U \to \mathcal{ET}^{\vee}$ given by,

$$\hat{f} : x \mapsto \text{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{EF}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{EF}$. This topology makes both $\pi$ and $\hat{f}$ continuous by construction.

For the pre-sheaf $\mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2$, consider the collection of sections of the bundle $\mathcal{ET}^{\vee}$, denoted $\Gamma\mathcal{ET}^{\vee}$, i.e., is a continuous map $\hat{s} : \mathcal{M} \to \mathcal{ET}^{\vee}$ such that $\pi \circ \hat{s} = Id$. This collection of sections is a pre-sheaf over $\mathcal{M}$ because, it assigns to each open subset $U \subset \mathcal{M}$ the corresponding set of sections over $U$ and we have the obvious restriction map, i.e., restriction of

[2] Note that in the case of Etale space, the properties of individual elements of the sheaf are used to get a topology, in the case of tangent bundle we used the properties of the sheaf itself to get a topology. We first quotiented the stalks with the maximal ideal of the ring of functions and then bundled them, and didn't care about the properties of the individual elements for the topology.

the continuous map to the smaller domain. It's also a sheaf because $s_i$ are sections of $U_i$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section $s$ defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{ET}^\vee$ is a sheaf over $\mathcal{M}$.

$$\Gamma\mathcal{ET}^\vee : \mathcal{O}(\mathcal{M})^{\text{op}} \to \mathbf{Sets},$$

that assigns to each $U \in \mathcal{O}(\mathcal{M})$ the set $\Gamma\mathcal{ET}^\vee(U) = \coprod_{x \in U} \mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2$. For each open subset $U \subset \mathcal{M}$ there is a function,

$$\eta_U : \mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2 \to \Gamma\mathcal{ET}^\vee(U),$$
$$f \mapsto \hat{f}.$$

This natural transformation of functors maps pre-sheaves to sheaves. The sheafification $\Gamma\mathcal{ET}^\vee$ of the pre-sheaf $\mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2$ is called the cotangent sheaf, denoted by $\mathcal{T}^\vee$. This is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. It's also locally free, with the local isomorphism,

$$\{f_i\}_{i=1}^n \mapsto \sum_i f_i dx_i$$

of modules $(\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$ and $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ for a chart $(\varphi, U)$. Smooth sections of the cotangent bundle, or elements of the cotangent sheaf are called differential forms.

**Definition 2.1.** A sheaf of $\mathcal{A}^{\mathcal{M}}$-modules $\mathcal{D}$ is said to be locally free of rank $n$ if for every $x \in \mathcal{M}$ has a neighborhood $U$ such that,

$$\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$$

Locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules in general give rise to vector bundles. Conversely to each vector bundle we can associate the sheaf of differentiable sections of $\pi$ which is a locally free sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. There is a natural bijection between the sheaves of locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules and vector bundles. Every $\mathcal{A}^{\mathcal{M}}$-linear sheaf homomorphism gives a homomorphism of vector bundles. It's an equivalence of categories.

For a locally free sheaf $\mathcal{D}$, at each point $x \in \mathcal{M}$, we have a neighborhood $U$ such that $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$. This composed with the evaluation gives us,

$$\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x = \mathbb{R}^n.$$

We can bundle the stalks $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x$ together,

$$\mathcal{VD} = \coprod_{x \in \mathcal{M}} \mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x,$$

together with the natural projection $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x \mapsto x$. Since $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$ and using this identification, the topology and a differentiable structure can be provided to the bundle $\mathcal{VD}$. $\mathcal{VD}$ is the vector bundle associated with the locally free sheaf $\mathcal{D}$. This will become important in the future.

## 2.2 | Tensor Algebra, Exterior Algebra

We understand linear maps quite well, what we want to do is study multilinear maps using linear algebra. The idea of tensor products is to study multilinear maps as linear maps. Suppose we have a collection of $\mathcal{A}$-modules $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and a multilinear map,

$$\beta : \prod_i \mathcal{V}_i \to \mathcal{W},$$

where $\mathcal{W}$ is an $\mathcal{A}$-module. What we want to do is study all such multilinear maps from the collection $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ to $\mathcal{W}$ as linear maps from the 'tensor product' $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ to $\mathcal{W}$ as a linear map. The algebraic tensor product of $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ is an $\mathcal{A}$-module $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ together with a multilinear map,

$$t : \prod_{i\in\mathcal{I}}\mathcal{V}_i \to \bigotimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i,$$

such that every other multilinear map from $\prod_{i\in\mathcal{I}}\mathcal{V}_i$ to $\mathcal{W}$ uniquely factors through $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$. This is the universal property of tensor products. This can be expressed in a commutative diagram by,

$$\begin{tikzcd}
\prod_{i\in\mathcal{I}}\mathcal{V}_i \arrow[r, "t"] \arrow[dr, "\beta"'] & \bigotimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i \arrow[d, "\exists!\ \tilde{\beta}"] \\
& \mathcal{W}
\end{tikzcd}$$

For the construction of the tensor product check wikipedia.

If $\mathcal{E}$ and $\mathcal{F}$ are two locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules corresponding to vector bundles, then we can form the presheaf of $\mathcal{A}^{\mathcal{M}}U$-module, $\mathcal{E}(U)\otimes_{\mathcal{A}^{\mathcal{M}}U}\mathcal{F}(U)$. Whose stalk at each point $x$ is given by $\mathcal{E}_x\otimes_{\mathcal{A}^{\mathcal{M}}_x}\mathcal{F}_x$.

$$\begin{aligned}
\mathcal{E}\otimes_{\mathcal{A}^{\mathcal{M}}}\mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to \mathbf{Sets} \\
U &\mapsto \mathcal{E}(U)\otimes_{\mathcal{A}^{\mathcal{M}}U}\mathcal{F}(U)
\end{aligned}$$

Suppose $\mathcal{E}$ and $\mathcal{F}$ be locally free, then around each $x\in\mathcal{M}$ there exist neighborhoods $U$ and $V$ such that $\mathcal{E}(U)\cong(\mathcal{A}^{\mathcal{M}}U)^{\oplus k}$ and $\mathcal{F}(V)\cong(\mathcal{A}^{\mathcal{M}}V)^{\oplus l}$. In particular, on the intersection $U\cap V$,

$$\mathcal{E}(U\cap V)\cong(\mathcal{A}^{\mathcal{M}}(U\cap V))^{\oplus k},\quad \mathcal{F}(U\cap V)\cong(\mathcal{A}^{\mathcal{M}}(U\cap V))^{\oplus l}$$

For the sake of simplicity we will assume $U=V$. We have,

$$\mathcal{E}U\otimes_{\mathcal{A}^{\mathcal{M}}}\mathcal{F}U\cong(\mathcal{A}^{\mathcal{M}}U)^{\oplus kl}.$$

The tensor products of interest to us will be the tensor products of tangent sheaf $\mathcal{T}$ of $\mathcal{A}^{\mathcal{M}}$-modules and cotangent sheaf $\mathcal{T}^\vee$ of $\mathcal{A}^{\mathcal{M}}$-modules. We will denote $\mathcal{T}^{(k,l)}$ the sheaf consisting of tensor product of $k$ tangent and $l$ cotangent sheaves.

$$\mathcal{T}^{(k,l)}=\mathcal{T}^{\otimes_{\mathcal{A}^{\mathcal{M}}}k}\otimes_{\mathcal{A}^{\mathcal{M}}}(\mathcal{T}^\vee)^{\otimes_{\mathcal{A}^{\mathcal{M}}}l}.$$

The sections of such tensor products are called tangent fields, $(k,l)$-type tensor field in particular. After evaluation at each stalk this will correspond to $k$ times tensor product of tangent space, and $l$ time tensor product of cotangent space.

The tensor algebra of $\mathcal{T}$ is defined as the direct sum,

$$T^\bullet_{\mathcal{A}^{\mathcal{M}}}\mathcal{T}=\bigoplus_{i\geq 0}\mathcal{T}^i,$$

together with the multiplication defined by tensor product and extending linearly.

We usually encounter multilinear maps with additional properties. These will be the usual types of multilinear functionals we encounter while doing calculus. These are bilinear forms that are alternating, i.e., when the entries repeat the form should be zero. An example of such a multilinear map is the oriented area.

A $k$-linear map $\alpha : \mathcal{V} \times \cdots \mathcal{V} \to \mathcal{W}$ is called alternating, if the value is zero whenever two entries are the same. An exterior power of degree $l$ is the universal vector space $\bigwedge^{l}_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ together with an alternating multilinear map $i : \mathcal{V} \times \cdots \times \mathcal{V} \to \bigwedge^{l}_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ such that for all alternating multilinear maps $\alpha$, there exists a unique linear map $\tilde{\alpha}$ such that the following diagram commutes,

$$\begin{tikzcd} \prod^{l} \mathcal{V} \arrow[r, "i"] \arrow[dr, "\alpha"'] & \bigwedge^{l}_{\mathcal{A}^{\mathcal{M}}} \mathcal{V} \arrow[d, "\exists!\ \tilde{\alpha}"] \\ & \mathcal{W} \end{tikzcd}$$

An exterior algebra of $\mathcal{A}^{\mathcal{M}}$-algebra, is the direct sum of all exterior powers, denoted by $\bigwedge^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{M}$. This is also the quotient of the tensor algebra by the two-sided ideal $\mathcal{K}$ generated by all elements of the form $\tau \otimes \tau$ for all $\tau \in \mathcal{T}$. This quotient is called the exterior algebra,

$$\bigwedge^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{M} = T^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T}/\mathcal{K}.$$

This quotient puts all the elements in the tensor algebra that have same two entries into the equivalence class of zero. The image of the tensor product $\mathcal{T}^k$ in $\bigwedge \mathcal{M}$ is denoted by $\bigwedge^k \mathcal{M}$. The image of $\tau_1 \otimes \cdots \otimes \tau_n$ is denoted by $\tau_1 \wedge \cdots \wedge \tau_n$. This is a graded algebra. By expanding $(\tau_1 + \tau_2) \otimes (\tau_1 + \tau_2)$ we see that,

$$\tau_1 \otimes \tau_2 \sim -\tau_2 \otimes \tau_1,$$

in $\bigwedge^2 \mathcal{M}$. Similarly, in $\bigwedge^l \mathcal{M}$,

$$\tau_1 \otimes \cdots \otimes \tau_l \sim (-1)^{\operatorname{sgn}(\sigma)} \tau_{\sigma(1)} \otimes \cdots \tau_{\sigma(l)},$$

where $\sigma$ is a permutation of $\{1, \ldots, l\}$. The exterior algebra is a skew-commutative algebra.

A differential $k$-form or $k$-form is an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree $k$ on the space of vector fields. Or equivalently, sections of $k$th exterior power of cotangent sheaf. The exterior product of two differential forms $\omega$ and $\kappa$ is defined to be the differential form,

$$\omega \wedge \kappa(X_1, \ldots, X_{k+l}) = \sum_{\sigma} \epsilon_{\sigma} \omega(X_{\sigma(1), \ldots, X_{\sigma(k)}}) \kappa(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}).$$

here $\sigma$ is a permutation of the set $\{1, \ldots k + l\}$ and $\epsilon_\sigma$ is the sign of the permutation. I will motivate the need and intution for the definition in the next part.

## 2.3 | Differential of a map

If the tangent space at $x$ is interpreted as the linear approximation of the manifold $\mathcal{M}$ at $x$, then the differential of a map is interpreted as the linear approximation of the map $\varkappa$.

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map of manifold $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ into $(\mathcal{N}, \mathcal{A}^{\mathcal{N}})$. Let $x \in \mathcal{M}$, the tangent space is defined to be the collection of the equivalence classes of such curves passing through $x$. Using the map $\varkappa$, we can push forward the curve $h$ to $\mathcal{N}$, given by the composition, $\varkappa \circ h : \mathbb{R} \to \varkappa(U)$.

$$\mathbb{R} \xrightarrow{h} \mathcal{M} \xrightarrow{\varkappa} \mathcal{N}$$

By a differential of the function $\varkappa$, we mean a map $D\varkappa(x)$ that takes the equivalence class $\tau_h$ to the equivalence class $\tau_{\varkappa\circ h}$. The new pairing that arises from the map is,

$$\langle g, \varkappa \circ h\rangle_{\varkappa(x)} = \frac{d(g \circ \varkappa \circ h(t))}{dt}\bigg|_{t=0},$$

for all $g \in \mathcal{A}^{\mathcal{N}}(\varkappa(U))$.

$$\begin{aligned} D\varkappa(x) :\ T_x\mathcal{M} &\to T_{\varkappa(x)}\mathcal{N} \\ \tau_h &\mapsto \tau_{\varkappa\circ h} \end{aligned}$$

It gives a vector bundle homomorphism of $T\mathcal{M}$ into $\varkappa^*(T\mathcal{N})$, usually denoted by $D\varkappa$. In terms of local coordinates this will be the Jacobian of the map.

Let $[f] \in \mathcal{I}_x^{\mathcal{N}}/(\mathcal{I}_x^{\mathcal{N}})^2$, with the representative $f \in \mathcal{A}^{\mathcal{N}}U$, Then we have the pull back, given by the composition,

$$\mathcal{M} \xrightarrow{\ \varkappa\ } \mathcal{N} \xrightarrow{\ f\ } \mathbb{R}$$

Now, $f \circ \varkappa \in \mathcal{A}^{\mathcal{M}}(\varkappa^{-1}U)$. The new pairing that arises from the map is,

$$\langle f \circ \varkappa, h\rangle_x = \frac{d(f \circ \varkappa \circ h(t))}{dt}\bigg|_{t=0},$$

for all curves $h : \mathbb{R} \to \mathcal{M}$ with $h(0) = x$. This gives us the pullback map,

$$\begin{aligned} T^\vee_{\varkappa(x)}\mathcal{N} &\to T^\vee_x\mathcal{M} \\ [f] &\mapsto [f \circ \varkappa]. \end{aligned}$$

It gives a vector bundle homomorphism of $\varkappa_*(T^\vee\mathcal{N})$ into $T^\vee\mathcal{M}$, and usually denoted by $D\varkappa^\dagger$. In terms of local coordinates this will be the adjoint of the Jacobian.

A map $\varkappa : \mathcal{M} \to \mathcal{N}$ corresponds to a corresponding linear map on the tensor product bundle, described by its action on the tangent vectors and the cotangent vectors as above. So, in terms of local coordinates it will be a tensor product of the Jacobians and adjoints of the Jacobians. We will denote this map by $\varkappa^*$.

A smooth function $\varkappa : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if $D\varkappa(x)$ is invertible for all $x \in \mathcal{M}$. This would mean that there exists an smooth inverse $\varkappa^{-1}$. The set of all diffeomorphisms of a manifold, i.e., diffeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ is a group. We will call such maps diffeomorphism 'of' $\mathcal{M}$. A one parameter group of diffeomorphisms of $\mathcal{M}$ is a collection of diffeomorphisms,

$$\varkappa : t \mapsto \varkappa_t$$

where each $\varkappa_t$ is a diffeomorphism of $\mathcal{M}$ such that, $\varkappa_0 = \mathbb{I}_{\mathcal{M}}$,

$$\varkappa_t \circ \varkappa_s = \varkappa_{t+s} \quad \forall t, s \in \mathbb{R},$$

and $\varkappa_t$ is a smooth as a map from $\mathcal{M}\times\mathbb{R}$ to $\mathcal{M}$. Where $\mathcal{M}\times\mathbb{R}$ has the differentiable structure of a product manifold. The one parameter group $t \mapsto \varkappa_t$ determines at each $x \in \mathcal{M}$ smooth curves,

$$t \mapsto \varkappa_t(x).$$

At each point $x$, this gives a tangent vector, the equivalence class of curves $[\varkappa_t(x)]$. Hence we obtain at each point $x \in \mathcal{M}$ a vector in the tangent space at $x$.

## 2.4 | Vector Fields; Lie Derivative

Each one parameter group of diffeomorphisms determines a vector field. The converse also holds locally. Given a vector field $X$ on a differentiable manifold $\mathcal{M}$ there exists a one-parameter group of diffeomorphisms $\varkappa$ such that $X_\varkappa = X$. This is due to the existence and uniqueness of solutions to ODEs.

For any $f \in \mathcal{A}^{\mathcal{M}}$ we have the smooth composition, $f \circ \varkappa_t : \mathcal{M} \to \mathbb{R}$. For fixed $x \in \mathcal{M}$, and varying $t$, this also corresponds to the smooth map,

$$\begin{aligned} f(\varkappa_{(\cdot)}(x)) :\ & \mathbb{R} \to \mathbb{R} \\ & t \mapsto f(\varkappa_t(x)). \end{aligned}$$

So, we can differentiate this function.

$$(X_\varkappa f)(x) = \lim_{t \to 0} \frac{f(\varkappa_t(x)) - f(x)}{t} = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0} = \langle f, h \rangle_x,$$

where $h = \varkappa_t(x)$. This can also be thought as the function,

$$\begin{aligned} X_\varkappa(\cdot) : \mathcal{A}^{\mathcal{M}} & \to \mathcal{A}^{\mathcal{M}}. \\ f & \mapsto (X_\varkappa f). \end{aligned}$$

It is called the differentiation of the function $f$ with respect to $\varkappa$ at $x$. $\varkappa_t$ moves $x$ to $\varkappa_t(x)$, so what the above limit is doing is measuring the infinitesimal change to the function $f$ as 'along' $\varkappa_t$. It's also called the Lie derivative of the function $f$ along the vector field $X$, denoted by,

$$L_{X_\varkappa}(f) = X_\varkappa(f).$$

For $f, g \in \mathcal{A}^{\mathcal{M}}$, by directly plugging into the definition, we find that,

$$X_\varkappa(f+g)(x) = X_\varkappa(f)(x) + X_\varkappa(g)(x),$$

and for the product,

$$X_\varkappa(fg)(x) = f(x)(X_\varkappa g)(x) + g(x)(X_\varkappa f)(x).$$

So, $X_\varkappa$ is linear map, and

$$X_\varkappa(fg) = f(X_\varkappa g) + g(X_\varkappa f).$$

Hence, $X_\varkappa$ is a homogeneous first order operator. This abstract definition, while less intuitive can be extended to tensors product case later.

The flow of a vector field $X$ is the one parameter group of diffeomorphisms $\varkappa$ such that $X_\varkappa = X$. A flow is said to be a global flow if it's defined for all $\mathbb{R}$ and at every point $x \in \mathcal{M}$. If a vector field gives rise to a global flow, it's called complete. For compact manifolds they do exist as we can always find local solutions and patch them up for finite cover. Let $X$ be a vector field and $\varkappa_t$ be its corresponding flow, then the orbit of a point $x \in \mathcal{M}$, $t \mapsto \varkappa_t(x)$ is called the integral curve for $X$.

The integral curve is the constant map if and only if the vector field is zero. Such points are called singularities. $\varkappa_t$ fixes a point $x$ if and only if $x$ is a singularity of $X$. When $x$ is not a singularity, $X(x) \neq 0$, and hence by continuity, the integral curve has injective differential nearby. Hence the integral curve is an immersed one dimensional manifold.

## 2.5 | Tensor Fields; Lie Derivative

Let $X$ be a vector field with the corresponding flow $\varkappa$. These are diffeomorphisms, $\varkappa_t : \mathcal{M} \to \mathcal{M}$. Let $\mathcal{T}^{(k,l)}$ be a tensor sheaf, consisting of tensor product of $k$ tangent and $l$ cotangent sheaves. A tensor fields are sections of the tensor sheaf. The diffeomorphism gives an isomorphism of the sheaf. If $\mathcal{T}_x^{(k,l)}$ is the stalk of $\mathcal{T}^{(k,l)}$ at $x$, then we have the induced isomorphism,

$$\varkappa_t^* : \mathcal{T}_x^{(k,l)}/\mathcal{I}_x^{\mathcal{M}} \to \mathcal{T}_{\varkappa_t(x)}^{(k,l)}/\mathcal{I}_{\varkappa_t(x)}^{\mathcal{M}}.$$

So, using the flow of the vector field we can push forward the tensor field $R$. Since the association $t \mapsto \varkappa_t^*$ is also smooth the map[3], we have,

$$\varkappa_{(\cdot)}^*(R(x)) : \; t \mapsto \varkappa_t^*(R(\varkappa_t(x))).$$

from $\mathbb{R}$ to the vector bundle $\mathcal{V}\mathcal{T}^{(k,l)}$ associated with the locally free sheaf $\mathcal{T}^{(k,l)}$. In particular we can talk about taking the limit,

$$L_X(R)(x) = \lim_{t\to 0} \frac{\varkappa_t^*(R(\varkappa_t(x))) - R(x)}{t}, \qquad \text{(Lie derivative)}$$

called the Lie derivative of the tensor field $R$ with respect to the vector field $X$ at $x \in \mathcal{M}$.

To study how the Lie derivative of the tensor product, exterior product and symmetric product of tensor fields, we can study Lie derivative of multilinear maps. Let $\beta$ be an $\mathcal{A}^{\mathcal{M}}$-bilinear sheaf homomorphism,

$$\beta : \mathcal{V} \times \mathcal{V}' \to \mathcal{W},$$

where $\mathcal{V}, \mathcal{V}'$ and $\mathcal{W}$ are tensor sheaves. The action of $\varkappa_t$ is given by,

$$\varkappa_t^*(\beta(R, R')) = \beta(\varkappa_t^* R, \varkappa_t^* R').$$

Plugging this in the Lie derivative, we get,

$$\begin{aligned}
L_X(\beta(R,R'))(x) &= \lim_{t\to 0} \tfrac{\varkappa_t^*\beta(R,R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} = \lim_{t\to 0} \tfrac{\beta(\varkappa_t^* R, \varkappa_t^* R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} \\
&= \beta\big(\lim_{t\to 0} \tfrac{\varkappa_t^* R(\varkappa_t(x)) - R(x)}{t}, \lim_{t\to 0} \varkappa_t^* R'(\varkappa_t(x))\big) + \beta\big(R, \lim_{t\to 0} \tfrac{\varkappa_t^* R'(\varkappa_t(x)) - R'(x)}{t}\big) \\
&= \beta(L_X R, R')(x) + \beta(R, L_X R')(x).
\end{aligned}$$

Here, we used the added and subtracted a term, and then took the limit inside. To take the limit inside, we would need the bilinear form to be continuous.

The Lie derivative of $\beta(R, R')$ with respect to $X$ is,

$$L_X(\beta(R,R'))(x) = \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \qquad \text{(product rule)}$$

Here we will have to add and subtract $\beta(\varkappa_t^* R(\varkappa_t(x)), R'(x))$ and use $\mathcal{A}^{\mathcal{M}}$-bilinearity of $\beta$ to get the Lie derivative inside, this is similar to how product rule is proved. The result is also valid for just $\mathbb{R}$-bilinearity. So, by taking the bilinear map $\beta$ to be the tensor product, $(R, R') \mapsto R \otimes R'$, we have,

$$L_X(R \otimes R') = L_X R \otimes R' + R \otimes L_X R'.$$

Similarly, for the differential forms, which are sections of exterior powers of cotangent sheaf,

$$L_X(\omega \wedge \omega') = L_X\omega \wedge \omega' + \omega \wedge L_X\omega'.$$

[3] we have to carefully look at a bunch of maps, and this will turn out to be smooth

Let $Y$ be a vector field, it's a derivation,

$$Y : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}.$$

The map $(Y, f) \mapsto Y(f)$ is a $\mathbb{R}$-bilinear map. So, the product rule is applicable. Note that $Y(f) \in \mathcal{A}^{\mathcal{M}}$, and the Lie derivative of functions is given by,

$$L_X(g) = X(g)$$

Hence, by plugging in $g = Y(f)$, we have,

$$\underbrace{L_X(Y(f))}_{X(Y(f))} = (L_X(Y))(f) + Y(\underbrace{L_X(f)}_{X(f)})$$

So, we have, $(L_X(Y))(f) = X(Y(f)) - Y(X(f)) = [X, Y](f)$. This is the Lie bracket, and it should be interpreted as the Lie derivative of $Y$ with respect to the vector field $X$. Now, we can consider the bilinear map, $(Y, Z) \mapsto [Y, Z]$. By applying product rule, we get,

$$L_X([Y, Z]) = [[X, Y], Z]] + [Y, [X, Z]].$$

But by previous calculation of Lie derivative of vector fields, we have, $L_X([Y, Z]) = [X, [Y, Z]]$. This yields us the so called Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \qquad \text{(Jacobi identity)}$$

This can also be written as,

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

For $f, g \in \mathcal{A}^{\mathcal{M}}U$, by expanding it can be verified that,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f).$$

So, $[X, Y] \in \mathcal{T}U$. It can also be checked that $[,]$ is $\mathbb{R}$-bilinear. So, $\mathcal{T}U$ is an $\mathbb{R}$-algebra for all $U$. It's however not a $\mathcal{A}^{\mathcal{M}}U$-algebra because it's not $\mathcal{A}^{\mathcal{M}}U$-bilinear. Because for $g \in \mathcal{A}^{\mathcal{M}}U$,

$$\begin{aligned} [X, gY](f) &= X(gY(f)) - gY(X(f)) \\ &= X(g)Y(f) + gX(Y(f)) - gY(X(f)) \\ &= (X(g)Y + g[X, Y])(f). \end{aligned}$$

So, it's not $\mathcal{A}^{\mathcal{M}}$-bilinear, and hence can't be an $\mathcal{A}^{\mathcal{M}}U$-algebra. The tangent sheaf is an $\mathbb{R}$-algebra. If $X$ and $Y$ are two commuting vector fields, i.e., $[X, Y] = 0$ then the flows of the corresponding vector fields commute in the group of diffeomorphisms of $\mathcal{M}$.

A Lie algebras over a commutative ring $\mathcal{A}$ is an $\mathcal{A}$-module together with a bilinear operation, $(X, Y) \mapsto [X, Y]$, such that, $[X, X] = 0$ and satisfies the Jacobi identity. A homomorphism of Lie algebras is an algebra homomorphism $f : \mathcal{V} \to \mathcal{W}$ such that,

$$f([X, Y]) = [f(X), f(Y)].$$

The sheaf $\mathcal{T}U$ is a sheaf of Lie algebras over $\mathbb{R}$. The restriction map is a Lie algebra homomorphism.

Consider the $\mathbb{R}$-bilinear map, $(\omega, Y) \mapsto \omega(Y) \in \mathcal{A}^{\mathcal{M}}$, applying the product rule we have,

$$L_X(\omega(Y)) = \beta(L_X\omega, Y) + \beta(\omega, L_X Y)(x) = (L_X(\omega))(Y) + \omega(L_X(Y)).$$

So, the Lie derivative of a 1-form $\omega$ is given by,

$$(L_X(\omega))(Y) = X(\omega(Y)) - \omega([X, Y]).$$

Similarly, the Lie derivative of a $k$-form is given by,

$$(L_X(\alpha))(X_1, \ldots, X_k) = X(\alpha(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).$$

The Lie derivative allows us to differentiate a tensor field with respect to a vector field. What we want is a notion of differentiation of a tensor field with respect to a tangent vector at a point. This we cannot do with Lie derivative, the behavior of the vector field in a neighborhood was important as we took the limit. Or more algebraicly speaking,

$$(L_{fX}\omega)(Y) = (fX)(\omega(Y)) - \omega([fX, Y]) = f(L_X\omega)(Y) + (Y(f))\omega(X)$$

or, $L$ is not $\mathcal{A}^{\mathcal{M}}$-linear. Changing the vector field $X$ at $x$ with a function $f \in \mathcal{A}^{\mathcal{M}}$, also depends on the behavior of the function $f$ in the neighborhood and not just its value at $x$. In order to differentiate tensor fields with respect to a tangent vector we need the notion of connection, which we will discuss later.

## References


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