Part II

Meromorphic Functions

We can use the tools developed about holomorphic functions to study certain functions that are holomorphic except at a few points. We can study such functions by studying the functions around the 'singularities'. The important tool to study such functions is the Laurent series expansion. With analytic functions we generalised polynomials to power series, with meromorphic functions, our goal is to similarly generalise rational functions.

1 | Singularities and Residues

We are interested in functions that are holomorphic in a neighborhood, except at some isolated points. These are similar to rational functions of the form $1/P(z)$. The zeros of the $P(z)$ are the problematic parts. To study such functions we study the behavior of the function in an annulus around the point where it's holomorphic.

Let Ω be the annulus $\rho_1 < |z| < \rho_2$, then for any function $f \in \mathcal{H}(\Omega)$, and a loop, $\gamma_r = re^{2\pi it}$, for $t \in [0,1]$, we have,

$$\int_{\gamma_r} f dz = \int_{[0,1]} f(re^{2\pi it})(2\pi i)re^{2\pi it} dt = 2\pi i \int_{[0,1]} g(re^{2\pi it}) dt$$

where $g(z) = zf(z)$. So,

$$\tfrac{d}{dr} \int_{\gamma_r} f dz = 2\pi i \int_{[0,1]} g'(re^{2\pi it}) \cdot e^{2\pi it} dt = r^{-1} \int_{[0,1]} \tfrac{d}{dt} g(re^{2\pi it}) dt = r^{-1}[g(r) - g(r)] = 0.$$

So the integral, $\int_{\gamma_r} f dz$ is independent of $\rho_1 < r < \rho_2$. For any $w \in \Omega$, define a holomorphic function $g \in \mathcal{H}(\Omega)$ by,

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \in \Omega, w \neq z \\ f'(w) & z = w \end{cases}$$

$$\int_{\gamma_r} \tfrac{f(z)-f(w)}{z-w} dz = \int_{\gamma_r} \tfrac{f(z)}{z-w} dz - \int_{\gamma_r} \tfrac{f(w)}{z-w} dz.$$

The second term equals $2\pi i f(w)$ if $|w| < r$ and is zero for $|w| > r$. For all $w \in \Omega$, we can find r_1, r_2 with $\rho_1 < r_1 < |w| < r_2 < \rho_2$, by the independence of $\int_{\gamma_r} g dz$ on r for all $\rho_1 < r < \rho_2$, we get,

$$f(w) = \frac{1}{2\pi i}\left[\int_{\gamma_{r_2}} \frac{f(z)}{z-w} dz - \int_{\gamma_{r_1}} \frac{f(z)}{z-w} dz\right]$$

we will exploit this formula to study these functions with singularities. Whenever its not problematic, we will assume $w = 0$.

Theorem 1.1. (Laurent Series) *Let $f \in \mathcal{H}(\Omega)$, where Ω is the annulus with $\rho_1 < |z| < \rho_2$. Then f can be uniquely written as,*

$$f(z) = \textstyle\sum_{n=-\infty}^{\infty} a_n z^n.$$

The series converges uniformly and absolutely for any compact set in Ω.

Proof

Similar to the proof of showing holomorphic functions are analytic, the proof involves expanding $1/(z-w)$. Let a_n be defined by,

$$a_n = \tfrac{1}{2\pi i} \int_{\gamma_r} \tfrac{f(z)}{z^{n+1}} dz$$

For $|w| < |z| = r_2$, we have, $1/(z-w) = \sum_{n=0}^{\infty} w^n/z^{n+1}$ and for $|w| > |z| = r_1$ we have, $1/(z-w) = -\sum_{m=0}^{\infty} z^m/w^{m+1} = -\sum_{n=-\infty}^{-1} w^n/z^{n-1}$, where $n = -m-1$. This gives us,

$$\tfrac{1}{2\pi i} \int_{\gamma_{r_2}} \tfrac{f(z)}{(z-w)} dz = \textstyle\sum_{n=0}^{\infty} a_n w^n \quad \text{and} \quad \tfrac{1}{2\pi i} \int_{\gamma_{r_1}} \tfrac{f(z)}{(z-w)} dz = -\textstyle\sum_{n=-\infty}^{-1} a_n w^n.$$

Since, $f(w) = \frac{1}{2\pi i}\big[\int_{\gamma_{r_2}} \frac{f(z)}{z-w} dz - \int_{\gamma_{r_1}} \frac{f(z)}{z-w} dz\big]$, we have,

$$f(w) = \textstyle\sum_{n=-\infty}^{\infty} a_n z^n.$$

convergence follows from the convergence of $\sum_{n=-\infty}^{0} a_n z^n$ and $\sum_{n=0}^{\infty} a_n z^n$. For the uniqueness, let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, we can determine c_n, by the uniform convergence of $\sum_{n=-\infty}^{\infty} c_n z^n$, consider the integral

$$\int_{[0,1]} f(re^{2\pi it}) e^{2\pi mt} dt = \textstyle\sum_{n=-\infty}^{\infty} c_n \int_{[0,1]} r^n e^{2\pi i(n-m)t} dt = c_m r^m.$$

or we can write $c_n = r^{-n} \int_{[0,1]} f(re^{2\pi it}) e^{2\pi imt} dt = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz$ which is the same as a_n. $\square$

Theorem 1.2. (Riemann Extension Theorem) *Let Ω be a disc of radius ρ around 0, and let $f \in \mathcal{H}(\Omega^*)$, $\Omega^* = \Omega \backslash \{0\}$. If*

$$zf(z) \to 0$$

as $z \to 0$, then there exists $F \in \mathcal{H}(\Omega)$ such that $F|_{\Omega^*} = f$.

Proof

For $w \in \Omega^*$ we have,

$$f(w) = \textstyle\sum_{n=-\infty}^{\infty} a_n z^n$$

where $a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz$ for $\gamma_r \subset \Omega$. Let $M(r) = \sup_{|z|=r} |f(z)|$, by assumption we have, $rM(r) \to 0$ as $r \to 0$. So we have,

$$|a_n| = \big|\tfrac{1}{2\pi i} \int_{\gamma_r} \tfrac{f(z)}{z^{n+1}} dz\big| = \big| \int_{[0,1]} f(re^{2\pi it}) r^{-n} e^{-2\pi int} dt\big| \leq r^{-n} M(r).$$

for $n \leq -1$ the term $r^{-n-1} \cdot rM(r) \to 0$ as $r \to 0$. Since a_n is independent of r, a_n must be identically zero. Thus we have,

$$f(w) = \textstyle\sum_{n=0}^{\infty} a_n z^n.$$

By Weirstrass theorem, $\{\sum_{n=0}^{m} a_n z^n\}_{m \geq 0} \subset \mathcal{H}(\Omega)$ converges to $\sum_{n=0}^{\infty} a_n z^n$ and hence $F := \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\Omega) \in \mathcal{H}(\Omega)$ with $F|_{\Omega^*} = f$. $\square$

1.1 | Meromorphic Functions

A rational function is a function of the form, $\frac{Q(z)}{P(z)}$, where $Q(z)$ and $P(z)$ are polynomials. Analytic functions were a generalization of polynomial functions, and we considered all power series. Now our goal is to study functions of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where h and g are analytic functions, i.e., they can be locally written as power series.

A function f on Ω is meromorphic if it's holomorphic on Ω except at a finite number of points $\mathcal{E}$, such that around each point in $\mathcal{E}$, there exists a small disc $D \subset \Omega$ such that

$$f \cdot h|_D = g|_D.$$

with h and g being holomorphic functions on D.

Lemma 1.3. *Let Ω is a disc around 0, and let $f \in \mathcal{H}(\Omega^*)$, let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be its Laurent expansion at 0. f is meromorphic on D iff there exists an integer N such that $c_n = 0$ for $n < -N$.*

Proof

Suppose f is meromorphic on Ω, let B_ρ be a disc around 0 for which there exists two holomorphic functions $g, h \in \mathcal{H}(B_\rho)$ such that

$$f \cdot h|_{B_\rho} = g|_{B_\rho}.$$

Since $h \in \mathcal{H}(B_\rho)$ we can write it as, $h(z) = \sum_{n=0}^{\infty} h_n z^n$, let $N = \inf\{n \,|\, h_n \neq 0\}$. Now $h(z)$ can be written as, $h(z) = \sum_{n=0}^{\infty} h_n z^n = z^N \varphi(z)$, so by definition of N we have that $\varphi(0) = h_N$. So there exists some neighborhood U of 0 for which $\varphi(z) \neq 0$ and hence $g/\varphi \in \mathcal{H}(V)$.

If $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is the power series expansion of g then we have,

$$f(z) = \sum_{n=0}^{\infty} g_n z^{n-N}$$

By uniqueness of Laurent expansion we have that $a_n = g_{n+N}$.

The converse is much simpler, given $f(z) = \sum_{n=-N}^{\infty} a_n z^n$, we can write it as,

$$\underbrace{(z)^N}_{h(z)} f(z) = \underbrace{\sum_{n=0}^{\infty} a_{n-N} z^n}_{g(z)}.$$

$\square$

Theorem 1.4. *Let $f \in \mathcal{H}(\Omega \backslash \mathcal{E})$. f is meromorphic on Ω iff for every $z_0 \in \mathcal{E}$, there exists a neighborhood U of z_0 with $U \cap \mathcal{E} = \{z_0\}$ such that*

$f|_{U\backslash\{z_0\}}$ *is bounded, or,* $|f(z)| \to \infty$ *as* $z \to z_0$.

Proof

If $f|_{U\setminus\{z_0\}}$ is bounded then by Riemann extension theorem, there exists a holomorphic function $g \in \mathcal{H}(U)$ such that $f|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}$, and hence we have,

$$1 \cdot f|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}.$$

If $|f(z)| \to \infty$ as $z \to z_0$ then by continuity there exists some disc B_ρ around 0 for which $|f(z)| \geq 1$, or $1/|f(z)|$ is bounded. So, applying Riemann extension to $1/f(z)$, we have, $1 \cdot \frac{1}{f(z)} = g(z)$, or

$$f \cdot g|_{B_\rho} = 1|_{B_\rho}.$$

Hence f is meromorphic.

For the other side, let f be a meromorphic function, let U be a neighborhood of $z_0 \in \mathcal{E}$ such that $U \cap \mathcal{E} = \{z_0\}$. Since f is meromorphic, we have,

$$hf|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}.$$

h, g holomorphic on U. So we can write them as $h(z) = \sum_{n=0}^{\infty} h_n(z-z_0)^n = (z-z_0)^k\varphi(z)$ and $g(z) = \sum_{n=0}^{\infty} g_n(z-z_0)^n = (z-z_0)^l\varkappa(z)$, with $\varphi(z), \varkappa(z) \neq 0$. So, we have,

$$f(z) = (z-z_0)^{k-l}\varphi(z)/\varkappa(z)$$

if $k \geq l$, then f is bounded otherwise $|f(z)| \to \infty$ as $z \to z_0$. $\square$

Let f be a meromorphic function on an open set Ω, a point z_0 is said to be a pole of f if $|f(z)| \to \infty$ as $z \to z_0$. If it's not a pole then it can be extended to a holomorphic function by Riemann extension theorem.

For a meromorphic function f on Ω, let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series expansion at z_0. The order of the meromorphic function f at z_0 is defined as

$$\mathrm{ord}_{z_0}(f) := \inf\{n \ : \ c_n \neq 0\}.$$

If $f \equiv 0$ at z_0 then we set $\mathrm{ord}_{z_0} := \infty$. For the meromorphic function $1/(z-z_0)^n$, the order is n. This is an easy way to remember the order. It's also the smallest $(z-z_0)^n$ we have to multiply to remove the singularity. Clearly, z_0 is a pole of f if and only if the $\mathrm{ord}_{z_0}(f) < 0$. If the $\mathrm{ord}_{z_0}(f) = -1$ it's called a simple pole, example is $1/z$.

If f is a holomorphic function, then it has a zero at z_0 if and only if $\mathrm{ord}_{z_0}(f) > 0$. An example is $(z-z_0)^n$. Here $\mathrm{ord}_{z_0} > n$. This is called the order of zero at z_0. f is holomorphic at z_0 and $f(z_0) \neq 0$ if and only if $\mathrm{ord}_{z_0}(f) = 0$. If the $\mathrm{ord}_{z_0}(f) = 1$ it's called a simple zero, an example is z.

Some basic properties of order can be quickly checked. Let f, g be meromorphic on Ω,

$$\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$$

For $\lambda \in \mathbb{C}$,

$$\mathrm{ord}_{z_0}(\lambda f) = \mathrm{ord}_{z_0}(f)$$

For the sums,

$$\mathrm{ord}_{z_0}(f+g) \geq \min(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)).$$

A pole is called essential if there are infinitely many $a_n \neq 0$ for $n < 0$, i.e., f is not a meromorphic function.

Theorem 1.5. (Casorati-Weierstrass) *Let $f \in \mathcal{H}(\Omega^*)$, suppose 0 is an essential singularity then $f(\Omega^*)$ is dense in $\mathbb{C}$.*

References