Part II

Required Functional Analysis

Hilbert spaces and bounded operators between them lie at the foundations of the mathematics of quantum theory. Some important commonly used theorems are also discussed.

1 | Hilbert Spaces

Inner product spaces are vector spaces with additional structure. This extra structure allows us to measure angle between vectors and distance between vectors. The inner product simultaneously encodes the notion of angle as well as distance. he interpretation to keep in mind is that $\langle\varphi|\varkappa\rangle = \|\varphi\|\|\varkappa\|\cos\theta$, where θ represents the angle between the vectors φ and $\varkappa$. The inner product is a map, $(\varphi,\varkappa) \mapsto \langle\varphi|\varkappa\rangle$. We list below the properties we expect from such a function.

The first property we expect from inner product is positive definiteness i.e., $\langle\varphi|\varphi\rangle \geq 0$, this says that the norm square of any non-zero vector should be positive and should be zero only if the vector is zero vector.

$$\langle\varphi|\varphi\rangle \geq 0, \quad \langle\varphi|\varphi\rangle = 0 \iff \varphi = 0.$$

We require it to be conjugate symmetric so that the number $\langle\varphi|\varphi\rangle$ is real, i.e.,

$$\langle\varphi|\varkappa\rangle = \overline{\langle\varkappa|\varphi\rangle}.$$

Finally we require linearity in first variable and conjugate linearity in the second. By previous requirement the conjugate linearity in second variable is just equivalent to the linearity in first variable,

$$\langle\alpha\varphi_1 + \beta\varphi_2|\varkappa\rangle = \alpha\langle\varphi_1|\varkappa\rangle + \beta\langle\varphi_2|\varkappa\rangle.$$

An inner product space is a complex vector space equipped with an inner product as above. Using the inner product we can define a norm on the vector space by,

$$\|\varphi\| = \langle\varphi|\varphi\rangle^{\frac{1}{2}}$$

Consider the vector $\varphi + \lambda\varkappa$, by positive definiteness we have,

$$\langle\varphi + \lambda\varkappa|\varphi + \lambda\varkappa\rangle \geq 0.$$

On expanding we get,

$$|\lambda|^2\langle\varkappa|\varkappa\rangle - \overline{\lambda}\langle\varkappa|\varphi\rangle + \lambda\langle\varphi|\varkappa\rangle + \langle\varphi|\varphi\rangle \geq 0.$$

By letting $\lambda = \langle \varkappa | \varphi \rangle / \langle \varkappa | \varkappa \rangle$ we obtain $0 \leq [|\langle \varkappa | \varphi \rangle|^2 / |\langle \varkappa | \varkappa \rangle|^2] \langle \varkappa | \varkappa \rangle - 2[|\langle \varkappa | \varphi \rangle|^2 / \langle \varkappa | \varkappa \rangle] + \langle \varphi | \varphi \rangle$. This gives us,

$$|\langle \varkappa | \varphi \rangle|^2 \leq \langle \varkappa | \varkappa \rangle \langle \varphi | \varphi \rangle$$

with the equality holding if and only if the vectors φ and $\varkappa$ are linearly dependent.

Theorem 1.1. (Cauchy-Schwarz inequality)

$$|\langle \varkappa | \varphi \rangle| \leq \|\varkappa\| \, \|\varphi\|.$$

$\square$

Since $\|\varphi + \varkappa\|^2 = \|\varphi\|^2 + \|\varkappa\|^2 + 2\mathrm{Re}\langle \varphi | \varkappa \rangle \leq \|\varphi\|^2 + \|\varkappa\|^2 + 2\|\varphi\| \, \|\varkappa\|$, i.e.,

$$\|\varphi + \varkappa\| \leq \|\varphi\| + \|\varkappa\|.$$

So $\|\varphi\| = \langle \varphi | \varphi \rangle^{\frac{1}{2}}$ defines a norm. An inner product space that's complete in the norm coming from the inner product is called a Hilbert space. Given an inner product space V one can complete the inner product space by adding all Cauchy sequences and this Hilbert space is called the completion of V, denoted by $\overline{V}$.

The Cauchy-Schwarz inequality holds for any positive definite sesquilinear form on the vector space i.e., if $B : V \times V \to \mathbb{C}$ is a positive semidefinite i.e., $B(\varphi, \varphi) \geq 0$ and satisfies Hermitian symmetry and linearity in the first variable then we will have,

$$|B(\varphi, \varkappa)|^2 \leq B(\varphi, \varphi) \cdot B(\varkappa, \varkappa).$$

The proof is exactly same as above.

Two vectors φ and $\varkappa$ are said to be orthogonal if $\langle \varphi | \varkappa \rangle = 0$. If $\{\varphi_i\}_{i \in \{1, \ldots, n\}}$ is a set of mutually orthogonal vectors then its easy to cheack that $\| \sum_{i=1}^n \varphi_i \|^2 = \sum_{i=1}^n \|\varphi_i\|^2$. A collection $\{\varphi_i\}_{i \in I}$ in an inner product space is said to be orthonormal if $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$ for all $i, j \in I$. This is a collection of unit vectors that are mutually orthogonal. For a finite orthonormal set $\{\varphi_i\}_{i \in \{1, \ldots, n\}}$ and a vector φ is a linear combination of these φ_is then we have, $\varphi = \sum_{i=1}^n \alpha_i \varphi_i$, then by orthonormality of φ_is we get,

$$\varphi = \textstyle\sum_{i=1}^n \alpha_i \varphi_i \implies \alpha_i = \langle \varphi | \varphi_i \rangle.$$

Now for any random vector φ, not necessarily a linear combination of the vectors $\{\varphi_i\}_{i \in \{1, \ldots n\}}$, consider the vector $\varphi - \sum_{i=1}^n \langle \varphi | \varphi_i \rangle \varphi_i$. This vector is orthogonal to every vector $\varphi_j \in \{\varphi_i\}_{i \in \{1, \ldots n\}}$. Let $\varphi' = \sum_{i=1}^n \langle \varphi | \varphi_i \rangle \varphi_i$ then taking the norm on $\varphi - \varphi'$ we have,

$$\|\varphi\|^2 = \|\varphi'\|^2 + \|\varphi - \varphi'\|^2$$

This is due to the fact that φ' and $\varphi - \varphi'$ are orthogonal and for orthogonal vectors we have $\| \sum_{i=1}^n \varphi_i \|^2 = \sum_{i=1}^n \|\varphi_i\|^2$. So we have,

$$\|\varphi\|^2 = \|\varphi'\|^2 + \|\varphi - \varphi'\|^2 \geq \|\varphi'\|^2 = \textstyle\sum_{i=1}^n |\langle \varphi | \varphi_i \rangle|^2.$$

Theorem 1.2. (Bessel's inequality)

$$\sum_{i=1}^n |\langle \varphi | \varphi_i \rangle|^2 \leq \|\varphi\|^2.$$

$\square$

Denote by

$$\vee\{\varphi_i\}_{i\in I},$$

the vector space spanned by $\{\varphi_i\}_{i\in I}$. Given a subset W of the Hilbert space $[W] = \overline{\vee W}$ is the smallest closed subspace containing W. We want to extend the notion of sums and linear combinations to infinite sets.

To do this we need some notion of wellbehavedness. Let I be an indexing set, and $\mathcal{F}(I)$ denote the finite subsets of I. For $F \in \mathcal{F}(I)$, let, $\alpha_F = \sum_{i\in F}\alpha_i$. $\mathcal{F}(I)$ is a directed set with the order defined by inclusion. The family $\{\alpha_i\}_{i\in I}$ is said to be unconditionally summable if $\varinjlim_{F\in\mathcal{F}(I)} \alpha_F$ exists and if $\varinjlim_{F\in\mathcal{F}(I)} \alpha_F = \alpha$ denoted by,

$$\alpha = \textstyle\sum_{i\in I} \alpha_i.$$

An orthonormal basis of a Hilbert space is a maximal orthonormal set. If $[\{\varphi_i\}_{i\in I}]$ is the span of the orthonormal basis. Then any φ in the Hilbert space can be written as,

$$\varphi = \sum_{i\in I}\langle\varphi|\varphi_i\rangle\varphi_i.$$

It requires some work to show this and can be found in [2]. The set of orthonormal sets is a partially ordered set and by Zorn's lemma there exists a maximal element. Every Hilbert space admits an orthonormal basis. The Hilbert space of interest to us will be one with countable orthonormal basis. We will denote Hilbert space with a countable infinite orthonormal basis by $\mathcal{H}$.

1.1 | Closed Subspaces and Projections

Let W be a closed subspace of a Hilbert space $\mathcal{H}$, hence a Hilbert space on its own. Every vector φ can be written uniquely as, $\varphi = \varphi_W + \varphi_{W^\perp}$. Let $W \subset \mathcal{H}$ be a subset, the orthogonal complement of W, denoted by $W^\perp$ is the set of all vectors $\varphi \in \mathcal{H}$ such that $\langle\varphi|\varkappa\rangle = 0$ for all $\varkappa \in W$. Using some Cauchy-Schwarz manipulation it can be showed that orthogonal complements are closed subspaces. Let $V \subset W \subset \mathcal{H}$, some basic set theory tells us,

$$V \subset W \implies W^\perp \subset V^\perp.$$

Let W be a closed subspace of a Hilbert space $\mathcal{H}$, then $W^\perp$ is a closed subspace. Some set theory manipulation tell us,

$$(W^\perp)^\perp = W.$$

Any vector $\varphi \in \mathcal{H}$ can be written as $\varphi = \varphi_W + \varphi_{W^\perp}$ where $\varphi_W \in W$ and $\varphi_{W^\perp} \in W^\perp$ and this decomposition is unique. Let $\varphi_W = P_W\varphi$ by uniqueness the map P_W is a linear map on $\mathcal{H}$. Since φ_W and $\varphi_{W^\perp}$ are orthogonal we have,

$$\|\varphi\|^2 = \|\varphi_W\|^2 + \|\varphi_{W^\perp}\|^2 = \|P_W\varphi\|^2 + \|\varphi - P_W\varphi\|^2,$$

so we have, $\|P_W\varphi\| \leq \|\varphi\|$ for all $\varphi \in \mathcal{H}$. So P_W is a bounded operator on $\mathcal{H}$. Furthermore we have,

$$\|P_W\varphi\|^2 = \|\varphi_W\|^2 = \langle\varphi_W|\varphi_W + \varphi_{W^\perp}\rangle = \langle P_W\varphi|\varphi\rangle = \langle\varphi|P_W\varphi\rangle.$$

The bounded linear operator P_W is called the orthogonal projection onto the closed subspace W. Instead of studying the properties of closed subspaces we can study the properties of these operators. By definition we have,

$$P_W^2 = P_W.$$

Since every vector φ can be written as $\varphi = \varphi_W + \varphi_{W^\perp} = P_W\varphi + (1 - P_W)\varphi$ by uniqueness. So we have,

$$P_{W^\perp} = 1 - P_W.$$

If V and W are closed subspaces and $V \subset W$. It can be checked that,

$$P_W P_V = P_V$$

and similarly,

$$P_V P_W = P_V$$

If V and W are orthogonal then we have $V \subset W^\perp$ so, $(1 - P_W)P_V = P_V$ or equivalently

$$P_W P_V = 0, \quad P_V P_W = 0.$$

Viewing in terms of the decomposition it can be seen that in this case the orthogonal projection onto the subspace $V + W$ is the operator $P_V + P_W$.

1.2 | Adjoint Operator

The starting point of a discussion on adjoints is the study of bounded linear functionals on $\mathcal{H}$, i.e., the elements of the Banach dual space $\mathcal{H}^*$ of $\mathcal{H}$. All the definitions in this section rely on the Riez lemma. The aim is to think of bounded linear operators as inner products with some vector. Consider the equation,

$$\phi_\varphi(\varkappa) = \langle \varkappa | \varphi \rangle.$$

Clearly this is a linear map because inner product is linear in the first argument. By Cauchy-Schwarz inequality we have,

$$\phi_\varphi(\varkappa) = \langle \varkappa | \varphi \rangle \leq \|\varphi\| \, \|\varkappa\|.$$

Since $\|\phi_\varphi\|_{\mathcal{H}^*} = \sup_{\|\varkappa\| \leq 1}\{\phi_\varphi(\varkappa)\}$ we have that $\|\phi_\varphi\|_{\mathcal{H}^*} \leq \|\varphi\|$. So it's a bounded operator. The inequality $\|\phi_\varphi\|_{\mathcal{H}^*} \leq \|\varphi\|$ is actually an equality because $\phi_\varphi(\varphi/\|\varphi\|) = \langle \varphi|\varphi\rangle/\|\varphi\| = \|\varphi\|$. Conversely, let ϕ be a non-zero bounded linear functional and let $\mathcal{L}$ be the kernel of ϕ. Since it's a bounded linear functional, it's continuous, and hence the kernel is closed subspace of $\mathcal{H}$. So we have,

$$\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp.$$

Note that this is where the completeness of Hilbert space is used, though we are not going to explain this further. Now ϕ maps $\mathcal{L}^\perp$ to $\mathbb{C}$. If $\phi(\varphi) = \phi(\varkappa)$ then we have $\phi(\varphi - \varkappa) = 0$ or $\varphi - \varkappa \in \mathcal{L}$. Since this can't be the case it must be zero. So ϕ maps $\mathcal{L}^\perp$ one-to-one into $\mathbb{C}$ i.e., $\mathcal{L}^\perp$ is one dimensional.

So, we have to chose a vector φ' and scale it appropriately. Let $\varphi' \in \mathcal{L}^\perp$ be a unit vector, let $\varphi = \phi(\varphi')\varphi'$. Then we have,

$$\phi(\varkappa) = \langle \phi(\varphi')\varphi' | \varkappa \rangle.$$

The vector $\varphi = \phi(\varphi')\varphi'$ is a vector such that

$$\phi(\varkappa) = \langle \varphi | \varkappa \rangle.$$

The choice of φ is unique because if φ_1 is another vector that satisfies the above relation then we have, $0 = \|\phi_\varphi - \phi_{\varphi_1}\| = \|\varphi - \varphi_1\|$, which implies $\varphi - \varphi_1 = 0$ because of positive definiteness of norm.

Theorem 1.3. (Riesz Lemma) *If $\phi \in \mathcal{H}^*$, there exists a unique $\varphi \in \mathcal{H}$ with $\phi = \phi_\varphi$.* $\square$

The Banach space $\mathcal{H}^*$ can be turned into a Hilbert space with inner product defined by, $\langle \phi_\varphi | \phi_\varkappa \rangle = \langle \varphi | \varkappa \rangle$. However the map $\varphi \mapsto \phi_\varphi$ is not linear, it's conjugate linear i.e., $\phi_{\alpha\varphi} = \overline{\alpha}\phi_\varphi$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces and let $B : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ be a sesquilinear form. We say it's bounded if there exists $K \in \mathbb{R}_+$ such that,

$$|B(\varphi, \varkappa)| \leq K\|\varphi\| \, \|\varkappa\|.$$

For every $\varkappa \in \mathcal{H}$, the bounded sesquilinear form $B : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ defines a bounded linear functional on $\mathcal{K}$ given by, $\varphi \mapsto \overline{B(\varkappa, \varphi)}$, with norm at most $K\|\varkappa\|$. By Riesz lemma there exists a unique vector in $\mathcal{K}$, say $T\varkappa$ such that for all φ, $\overline{B(\varkappa, \varphi)} = \langle \varphi | T\varkappa \rangle$. So we have,

$$B(\varkappa, \varphi) = \langle T\varkappa | \varphi \rangle,$$

and $\|T\varphi\| \leq K\|\varphi\|$. Since B is sesquilinear, the mapping $T : \mathcal{H} \to \mathcal{K}$ is a linear map and since $\|T\varphi\| \leq K\|\varphi\|$ it's a bounded linear operator. Conversely, every bounded operator T defines a sesquilinear form given by,

$$B(\varphi, \varkappa) = \langle T\varphi | \varkappa \rangle.$$

Now consider the new sesquilinear form defined by, $B'(\varphi, \varkappa) = \langle \varphi | T\varkappa \rangle$. Let T^* be the corresponding bounded linear operator such that $\overline{B'(\varphi, \varkappa)} = \langle \varkappa | T^*\varphi \rangle$ or $\langle \varphi | T\varkappa \rangle = \langle T^*\varphi | \varkappa \rangle$.

Theorem 1.4. *For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\exists !$ $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\langle \varphi | T\varkappa \rangle = \langle T^*\varphi | \varkappa \rangle$.* $\square$

T^* is called the adjoint of the operator T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $T = T^*$. It follows from definition that,

$$(\alpha T_1 + T_2)^* = \overline{\alpha} T_1^* + T_2^*, \;\; (T^*)^* = T, \;\; \mathbb{I}_\mathcal{H}^* = \mathbb{I}_\mathcal{H}, \;\; (ST)^* = T^*S^*$$

$\|T\| = \sup_{\|\varphi\| \leq 1}\{\|T\varphi\|\}$, and $|\langle T\varphi | \varkappa \rangle| \leq \|T\varphi\| \, \|\varkappa\|$ so for $\|\varphi\| \leq 1$ and $\|\varkappa\| \leq 1$ we have, $|\langle T\varphi | \varkappa \rangle| \leq \|T\|$. Let $\varkappa = T\varphi / \|T\varphi\|$, we have, $|\langle T\varphi | T\varphi / \|T\varphi\| \rangle| = \|T\varphi\|$. So the bound is actually an equality. So we have, $\|T\| = \sup_{\|\varphi\| \leq 1, \|\varkappa\| \leq 1}\{|\langle T\varphi | \varkappa \rangle|\}$. This tells us,

$$\|T\| = \|T^*\|.$$

Furthermore, $\|T\varphi\|^2 = \langle T\varphi | T\varphi \rangle = \langle T^*T\varphi | \varphi \rangle \leq \|T^*T\varphi\|\|\varphi\| \leq \|T^*T\|\|\varphi\|$, but $\|T^*T\| \leq \|T^*\|\|T\|$ so we get,

$$\|T\|^2 = \|T^*T\|.$$

Suppose $T \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator, then we have, $\langle T\varphi | \varphi \rangle = \langle \varphi | T^*\varphi \rangle = \overline{\langle T^*\varphi | \varphi \rangle}$ or $\langle \varphi | T^*\varphi \rangle \in \mathbb{R}$. By choosing $T_1 = T + T^*/2$ and $T_2 = T - T^*/2i$ we can write every operator can be decomposed in terms of self-adjoint operators,

$$T = T_1 + iT_2.$$

Let P_W be the projection onto a closed subspace $W \subset \mathcal{H}$. Since $\langle P_W\varphi | \varphi \rangle = \langle \varphi | P_W\varphi \rangle$, we have $P_W = P_W^* = P_W^2$. Conversely suppose $P = P^2 = P^*$, let W be the range of P. If $\varphi \in W$ then there exists some $\varkappa \in \mathcal{H}$ such that $\varphi = P\varkappa$ and

$$P\varphi = P^2\varkappa = P\varkappa = \varphi.$$

If $\varkappa \in W^{\perp}$ then we have,

$$\langle P\varkappa|\nu\rangle = \langle \varkappa|P\nu\rangle = 0 \ \forall \nu \in \mathcal{H}$$

or $P\varkappa = 0$. Thus P is a projection onto closed subspace W.

Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ where $\mathcal{H}, \mathcal{K}$ are Hilbert spaces. For any $\varphi \in \mathcal{H}$, we have,

$$\|T\varphi\|^2 = \langle T^*T\varphi|\varphi\rangle = \langle (T^*T)^{\frac{1}{2}}\varphi|(T^*T)^{\frac{1}{2}}\varphi\rangle = \|(T^*T)^{\frac{1}{2}}\varphi\|^2.$$

$\ker T = \ker(T^*T)^{\frac{1}{2}}$. Suppose $\varphi \in (\operatorname{ran}(T^*))^{\perp}$, i.e., $\langle \varphi|T^*\varkappa\rangle = 0$ for all $\varkappa \in \mathcal{K}$.

$$\langle \varphi|T^*\varkappa\rangle = \langle T\varphi|\varkappa\rangle = 0$$

for all $\varkappa \in \mathcal{K}$. Hence $\ker T = (\operatorname{ran}(T^*))^{\perp}$, and taking perp on this we get, $(\ker T)^{\perp} = \overline{\operatorname{ran} T^*}$.

1.3 | Polar Decomposition

Let $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, it's called a partial isometry if it's an isometry on whatever it acts on i.e., $U|_{(\ker U)^{\perp}}$ is an isometry. Let $\mathcal{L} = (\ker U)^{\perp}$ any $\varphi \in \mathcal{H}$ can be written as a unique sum,

$$\varphi = \varphi_{\mathcal{L}} + \varphi_{\mathcal{L}^{\perp}}.$$

where $\varphi_{\mathcal{L}} \in \mathcal{L}$ and $\varphi_{\mathcal{L}^{\perp}} \in \mathcal{L}^{\perp}$. Consider the map U^*U,

$$\begin{aligned}\langle U^*U\varphi|\varkappa\rangle &= \langle U\varphi|U\varkappa\rangle = \langle U\varphi_{\mathcal{L}}|U\varkappa_{\mathcal{L}}\rangle \\ &= \langle \varphi_{\mathcal{L}}|\varkappa_{\mathcal{L}}\rangle = \langle \varphi_{\mathcal{L}}|\varkappa\rangle.\end{aligned}$$

This holds for all $\varkappa \in \mathcal{H}$, so, U^*U is the projection onto $\mathcal{L}$. Conversely, let $\mathcal{L}$ be the range of a projection of the form $P = U^*U$, then for every $\varphi \in \mathcal{H}$,

$$\|P\varphi\|^2 = \langle P\varphi|\varphi\rangle = \langle U^*U\varphi|\varphi\rangle = \langle U\varphi|U\varphi\rangle = \|U\varphi\|^2.$$

So, $\ker U = \ker P = \mathcal{L}^{\perp}$ and hence U is isometric on $\mathcal{L}$ as $P|_{\mathcal{L}}$ is identity.

Theorem 1.5. (Polar Decomposition) *$T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has a unique decomposition,*

$$T = U|T|$$

where U is a partial isometry, and $|T|$ is positive.

Sketch of Proof

Every positive operator can be written as a square of another positivel operator $T = B^2$ where B is positive. Since T^*T is positive, we can make sense of $|T| = \sqrt{T^*T}$. For every $\varphi, \varkappa \in \mathcal{H}$, we have,

$$\langle T\varphi|T\varkappa\rangle = \langle T^*T\varphi|\varkappa\rangle = \langle |T|^2\varphi|\varkappa\rangle = \langle |T|\varphi||T|\varkappa\rangle,$$

so there exists a unique unitary operator $U_{|T|}$ on the range of $|T|$ such that, $U_{|T|}(|T|\varkappa) = T\varkappa$. If the range of $|T|$ is $\mathcal{L}$ and $P_{\mathcal{L}}$ be the projection onto $\mathcal{L}$, then $U = U_{|T|}P$ defines a partial isometry and furthermore,

$$T = U|T|$$

□

2 | Locally Convex Spaces

2.1 | Alaoglu's Theorem

Theorem 2.1. (Alaoglu's Theorem) *$\mathcal{A}$ is a normed space,*

$$B_1(\mathcal{A}^*) = \{\varphi \in \mathcal{A}^* \mid \|\varphi\| = 1\}$$

is a compact Hausdorff space with respect to weak *-topology.

Sketch of Proof

The idea is to embed $B_1(\mathcal{A}^*)$ in $Z = \overline{D}^{B_1(\mathcal{A})}$, where,

$$\overline{D}^{B_1(\mathcal{A})} = \prod_{i \in B_1(\mathcal{A})} \overline{D} = \{f : B_1(\mathcal{A}) \to \overline{D}\}.$$

which is compact because $\overline{D}$ is compact and by Tychonoff's theorem product of compact sets is compact.

If $\varphi \in B_1(\mathcal{A}^*)$, define a map, $F : B_1(\mathcal{A}^*) \to Z$, given by,

$$\varphi \mapsto \{\varphi(A)\}_{A \in B_1(\mathcal{A})} \in \overline{D}^{B_1(\mathcal{A})}$$

With $B_1(\mathcal{A}^*)$ equipped with the weak*-topology, and Z equipped with the product topology, the above map is continuous, i.e., if $\{\varphi_i\}_i \to \varphi$ if and only if $\{\varphi_i(A)\} \to \varphi(A)$ for all $A \in B_1(\mathcal{A})$. Since F is 1-1, it maps homeomorphically onto its image in Z.

Since Z is compact Hausdorff space, it's enough to show that range(F) is closed in Z.

$$K_{A,B,\alpha,\beta} = \{f \in Z \mid f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)\}$$

is a closed set. If f is in the range of F then there exists some φ such that $\{\varphi(A)\}_{A \in B_1(\mathcal{A})} = f$. $f \in K_{A,B,\alpha,\beta}$ when defined.

$$\text{range}(F) = \bigcap_{A,B,\alpha,\beta} K_{A,B,\alpha,\beta}$$

and it's closed as it's the intersection of closed sets. □

2.2 | Stone-Weierstrass Theorem

Theorem 2.2. (Stone-Weierstrass Theorem) *Let X be a compact Hausdorff space, $\mathcal{A} \subseteq C(X)$, self-adjoint subalgebra. If $\mathcal{A}$ contains all constant functions, i.e., it's a unital subalgebra, and $\mathcal{A}$ separates points in X, i.e., $x \neq y$ implies there exists some $A \in \mathcal{A}$ such that $A(x) \neq A(y)$ then $\mathcal{A}$ is dense in $C(X)$.*

Sketch of Proof

References

[1] P R Halmos, Finite Dimensional Vector Spaces, Princeton University Press, 1942

[2] V S Sunder, Functional Analysis: Spectral Theory, Birkhauser Advanced Texts, 1991