Part I

Operational Theories

The path to the construction of a new physical theory is usually a complicated one. In its initial stages, lots of ideas are tried and tested. The surviving ideas are revised, modified, and more clearly formulated. These become the postulates of the theory. The purpose of this chapter is however not to study the historical construction of quantum theory but to give reconstruction of the formal structure of quantum theory so that the new presentation is superior concerning clarity and precision. Reconstruction makes the core underlying structure of physical theory clearer and could expose the problems. The presentation here is closest to Günther Ludwig school. The Ludwig school has the advantage of being compatible with most interpretations of quantum theory if suitably reformulated.

A physical theory is in some sense to be interpreted from outside in terms of pre-theories not belonging to the theory in question itself. To minimize going to these pre-theories we adopt a purely instrumentalist view of physics. The construction and behavior of instruments will not be of interest to us. Any changes occurring in the instruments during 'measurements' will be accepted as objective events. According to this point of view, the fundamental notions of quantum mechanics have to be defined operationally in terms of macroscopic instruments and prescriptions for their application. Quantum mechanics can then be interpreted entirely in terms of such instruments and events. These instruments and events are our links to 'objective reality'.

1 | Structure of Operational Theories

From an instrumentalist or operational point of view, the notion of 'state' can be defined in terms of the preparation procedure. A preparation procedure is characterized by the kind of system it prepares. The other important thing is the existence of a measuring instrument that is capable of undergoing changes upon their interaction. The observable change in the instrument is called an effect in the Ludwig school.

1.1 | Effects and Ensembles

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To simplify the procedure consider instruments that record 'hits'. These instruments perform simple 'yes-no' measurements. Any measurement can be interpreted as a combination of yes-no measurements. These yes-no instruments can be used to build any general instrument. Suppose we have such an instrument, label its registration procedure by R. If the experiment is conducted a lot of times, we get a relative frequency of occurrence of 'yes'. To every preparation procedure ρ and registration procedure R_i there exists a probability $\mu(\rho, R_i)$ of occurrence of 'yes' associated with the pair.

$$(\rho, R_i) \longrightarrow \mu(\rho|R_i).$$

The numbers $\mu(\rho|R_i)$ are called operational statistics. Two completely different preparation procedures may give the same probabilities for all experiments R. Such preparation procedures must be considered equivalent. Such preparation procedures are called operationally equivalent preparations. A precursor to the notion of a state of the system is an equivalence class of preparations procedures yielding the same result. They are called ensembles.

The basic mathematical structure of ensembles and effects can be understood using purely mathematical reasons, without introducing any new physical law. Denote the class of ensembles by S and the class of effects by E. The maps of interest to us are the following,

$$S \times E \xrightarrow{\mu} [0,1].$$

There may be two experiments that give the same probabilities for every ensemble. Such apparatuses must be considered equivalent. They are called operationally equivalent effects. An effect is the equivalence class of apparatuses yielding the same result. In general, a registration procedure R for an experiment will have outcomes $\{R_i\}$. For an outcome, R_i of the registration procedure R, denotes the corresponding equivalence class of measurement procedures by E_{R_i}. Each outcome R_i of the registration procedure corresponds to a functional E_{R_i} called the effect of R_i that acts on the ensemble of the system to yield the corresponding probability.

$$E_{R_i} : \rho \mapsto E_{R_i}(\rho) = \mu(\rho|R_i).$$

Maps of interest to us will be those that assign to each of its outcomes R_i its associated effect E_{R_i}. Since each ensemble fixes a probability distribution we have,

$$\mu_\rho : R_i \mapsto \mu_\rho(R_i) = \mu(\rho|R_i).$$

The above-given map μ_ρ is determined by the instrument and the registration procedure. Accounting to the fact that preparation procedures can be combined to produce a mixed ensemble, the set of ensembles is taken to be a convex set. Since a mixture of ensembles corresponds to a convex combination of probabilities each functional E_{R_i} preserves the convex structure. Since two preparations giving the same result on every effect represent the same ensemble and two measurement procedures that can't distinguish ensemble represent the same effect, ensembles and effects are mutually separating. A generalized probabilistic theory is an association of a convex state space and effect vectors to a given system, such that the states and effects are uniquely determined by the probabilities they produce. This is known as the principle of tomography. The aim is to obtain a GPT from an operational theory. We are interested in embedding the ensembles inside the vector space of linear functionals on the effects and embed effects inside the vector space of linear functionals on the ensembles. More generally, one takes an operational theory and 'quotients' with operational equivalences to obtain a GPT.

Denote by $\mathcal{S}$ the set of maps, $f : E \longrightarrow \mathbb{R}$ such that $f(X) = \sum_i \alpha_i \mu(\rho_i|X)$ and denote by $\mathcal{E}$ the set of maps, $g : S \longrightarrow \mathbb{R}$ such that $g(\rho) = \sum_i \beta_i \mu(\rho|R_i)$ where ρ_i and R_i are ensembles and effects respectively and $\alpha_i, \beta_i \in \mathbb{R}$. Clearly $\mathcal{S}$ and $\mathcal{E}$ are real vector spaces. We can embed ensembles inside $\mathcal{S}$ with the map,

$$\rho \longmapsto \mu_\rho,$$

and similarly embed effects inside $\mathcal{E}$ with the map,

$$R_i \longmapsto E_{R_i}.$$

The bilinear map $(\cdot|\cdot) : \mathcal{S} \times \mathcal{E} \to \mathbb{R}$ which coincides with μ is then uniquely determined. $(\mathcal{S}|\mathcal{E})$ becomes a dual pair. The completions of $\mathcal{S}$ and $\mathcal{E}$ will provide us the necessary mathematical structure for ensembles and effects. We will denote $(\cdot|\cdot)$ by μ.

A registration procedure E_R is an effect valued function that assigns to each possible outcome R_i its effect E_{R_i},

$$E_R : \; R_i \longmapsto E_{R_i}.$$

It's important to find a mathematical structure that describes the registration procedure E_R beyond this basic vector space structure. The purpose of this section is to study the mathematical representatives of effects and ensembles in quantum formalism.

1.2 | Observables and States

To get the mathematical representatives of physical observables one has to study the logical relations of a set of propositions that are considered meaningful and empirically verifiable according to the theory that describes the physical system. The logic of a physical system will mean the algebraic structure that represents the equivalence classes of the elementary sentences. To simplify the procedure one initially reduces the elementary sentences of the system to simple 'yes-no' questions called propositions. For the development of any mathematical theory, the first step is the idealization of the registrations. Here we are satisfied with the usage of real numbers to label the outcomes.

The concept of observable which is one of the main physical objects of quantum theory can be obtained from a certain idealization of the registration procedure. Consider a registration procedure E_A whose outcomes $\{A_i\}$ are measured using the same equipment. The events of such a registration procedure should form a Boolean ring. The aim is to arrive at the notion of observable from these special kinds of registration procedures. What we seek are maps from Boolean rings to the effects. A mapping A of a Boolean ring Σ into an ordered interval $[0, \epsilon]$ of a vector space, such that, $A(\mathbb{I}) = \epsilon$ where $\mathbb{I}$ is unit of Σ and

$$A(\sigma_1 \vee \sigma_2) = A(\sigma_1) + A(\sigma_2) \quad \text{for all} \;\; \sigma_1 \wedge \sigma_2 = 0,$$

is called an additive measure on Σ. A set $F \subset \mathcal{E}$ is called a set of coexistent effects if there exists a Boolean ring Σ_A with an additive measure $A : \Sigma_A \to \mathcal{E}$ such that $F \subset A\Sigma_A$.

An observable is a special kind of registration procedure where the outcomes form a complete Boolean ring. An observable is a pair (Σ_A, A), where Σ_A is a Boolean ring and A is an additive measure,

$$A : \; \Sigma_A \to \mathcal{E}.$$

We will denote the observable by the map A. The complete Boolean lattice structure of Σ_A is the idealization of the registration procedure. Observables are effect-valued functions where outcomes have a Boolean lattice structure. Döring and Isham [8], have generalized this where they replace the Boolean lattice with much more general mathematical objects called Heyting algebras, a reader interested in foundations might find this interesting.

Suppose we have two observables A and B and there exists a homomorphism h of the Boolean ring Σ_A into the Boolean ring Σ_B then intuitively the observable B measures more than A since the measurements of the observable A is contained in the observable B. Two observables are equivalent if the homomorphism h is an isomorphism. Two observables A and B are said to coexist if there exists an observable AB and two homomorphisms h and i such that $h : \Sigma_A \to \Sigma_{AB}$ and $i : \Sigma_B \to \Sigma_{AB}$. Denote by Ξ the effects that coexist with every other effect. Two observables A and B are mutually complementary if every coexistent effect is in Ξ. If two effects $E_{A_i} \in A$ and $E_{B_j} \in B$ are coexistent then at least one of them is in Ξ. The existence of such observables is a feature of quantum mechanics that wasn't the case in classical mechanics.

Similar to effects we can study idealizations of ensembles. Since effects and ensembles are closely related objects we would see similar conditions on the notion of state coming from observables. The state should provide for each observable a probability distribution. In the formulations of quantum mechanics, the question of whether it is possible to make joint measurements of pairs of observables is important. It's this question that leads to all the subtleties of quantum mechanics. The question is regarding the possibility of joint preparation. This helps us separate the question about the possibility of making joint preparations from the problem of registration. We are interested in decomposing the ensemble and studying the relation between different decompositions. Maps of interest to us are of the form,

$$w :\ \Sigma_A \to \mathcal{S},$$

such that $w(1) = \mu_\rho$. The structure of the observable A would be contained in the Boolean lattice Σ_A. The condition $w(1) = \mu_\rho$ contains the structure of the ensemble μ_ρ. These maps are called preparators in Ludwig's approach. A preparator of the ensemble μ_ρ is a map $w_i : \Sigma_i \to \mathcal{S}$ such that $w_i(1) = \mu_\rho$. Preparators represent the information the state contains about an observable. A preparator w_i of the ensemble μ_ρ is more comprehensive than the preparator ω_j if there exists a homomorphism $h : \Sigma_j \to \Sigma_i$. w_i and w_j coexist if there is a preparator w which is more comprehensive than both. Using a preparator $w : \Sigma \to \mathcal{S}$ of μ_ρ, new preparators can be obtained as follows: Let $[0, \epsilon] \subset \Sigma$ then,

$$w_\epsilon : [0, \epsilon] \to \mathcal{S},$$

where $[1/\mu(w(\epsilon), 1)]w := w_\epsilon$ is a preparator of the ensemble $[1/\mu(w(\epsilon), 1)]w(\epsilon) := \mu_{\rho_\epsilon}$. We will call this the preparator of $[0, \epsilon]$. Suppose we have two preparators, w_i and w_j, we call them mutually exclusive if there doesn't exist sections $[0, \epsilon_i] \subset \Sigma_i$ and $[0, \epsilon_j] \subset \Sigma_j$ such that $\mu_{\rho_{\epsilon_i}} = \mu_{\rho_{\epsilon_j}}$ and the canonical preparators of $[0, \epsilon_i]$ and $[0, \epsilon_j]$ coexist. Two preparators w_i and w_j of μ_ρ are complementary if whenever there is a homomorphism $h : \Sigma_i|_h \to \Sigma_j$ the corresponding new restricted preparators are mutually exclusive. These properties allow us to extract the mathematical properties of ensembles.

Suppose $A : \Sigma_A \to \mathcal{E}$ is an observable then a state μ_ρ gives us a map,

$$\mu_\rho^A :\ \Sigma_A \to [0, 1],$$

such that $\mu_\rho^A(0) = 0$, $\mu_\rho^A(E^\perp) = 1 - \mu_\rho^A(E)$ and whenever E_i are mutually orthogonal,

$$\mu_\rho^A(\vee_i E_i) = \sum_i \mu_\rho^A(E_i).$$

For all practical purposes, we will assume the measurement scale is separable, usually real number scales. This assumption gives us all the nice mathematical properties needed.

It is important to note that preparation and registration procedures producing the same ensembles and effects are not always equal, in fact, the notion of equality won't even make sense. The transition from preparation and registration procedures to ensembles and effects is a transition from the real world to the abstract mathematical world. It should also be noted that it doesn't make sense to 'prepare' closed systems, one has to assume such systems start off in some state a priori.

By the end of the nineteenth century, it was clear that elementary processes obeyed some 'discontinuous' laws. There existed no mathematical formalism of quantum theory that would provide a unified structure. Heisenberg's solution to this problem was to use linear operators as a starting point. The space of functions on both discrete and continuous spaces have the same Hilbert space structure. The coexistence of discrete and continuous observables is possible. The necessary structure for the abstract mathematical framework of quantum theory is found in Hilbert spaces and operator algebras. We will later revisit the motivation for why Hilbert space and operator algebras provide us with the required generalization of observables. Let $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be a complex Hilbert space. $\mathcal{P}(\mathcal{H})$ denote the set of all closed subspaces. Denote $\mathcal{H}_i \leq \mathcal{H}_j$ if $\mathcal{H}_i \subseteq \mathcal{H}_j$. The relation $\leq$ is a partial ordering in $\mathcal{P}(\mathcal{H})$. Join $\vee$ of a family $\{\mathcal{H}_i\}_{i\in I}$ is the linear span of the family denoted $\vee_i\mathcal{H}_i$. Meet $\wedge$ of a family $\{\mathcal{H}_i\}_{i\in I}$ is the intersection of the family, denoted $\wedge_i\mathcal{H}_i$. The orthocomplement of $\mathcal{H}_i$ in $\mathcal{P}(\mathcal{H})$ denoted by $\mathcal{H}_i^{\perp}$ is the closed subspace of vectors $\varphi \in \mathcal{H}$ such that $\langle\varphi|\mathcal{H}_i\rangle = 0$. Since there is a bijection between closed subspaces of a Hilbert space and projection operators acting on the Hilbert space, the set of all projection operators on the Hilbert space inherits a lattice structure from the lattice of closed subspaces. We will denote the projection operators on $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. The orthocomplement of the projection E is the projection onto the orthogonal complement of the subspace corresponding to the projection operator E and is denoted by $E^{\perp}$. The lattice structure of $\mathcal{P}(\mathcal{H})$ coming from the above relations gives us the necessary structure to get the mathematical representatives of physical observables. The non-Boolean lattice $\mathcal{P}(\mathcal{H})$ of projections should act as the space of effects. The first step is to study the structure of lattice of projection operators on Hilbert spaces.

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