

# PART VI

## REPRESENTATIONS

We are interested in representing an abstract  $C^*$ -algebra as operators on Hilbert space. The important part is the GNS construction, that allows us to construct a representation of an abstract  $C^*$ -algebra using a state acting on it. The tools in this part will be very important in quantum field theory.

### 1 | REPRESENTATIONS OF $C^*$ -ALGEBRAS

A  $*$ -homomorphism between two  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is a mapping

$$\pi : A \rightarrow \pi(A),$$

such that  $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$ ,  $\pi(AB) = \pi(A)\pi(B)$  and  $\pi(A^*) = (\pi(A))^*$ . It's an algebra homomorphism which also preserves the  $*$ -operation.

Given a  $*$ -homomorphism  $\pi : \mathcal{A} \rightarrow \mathcal{B}$ , we have,  $1 = \pi(AA^{-1}) = \pi(A)\pi(A^{-1})$  so,  $\pi$  maps invertible elements to invertible elements, and  $\pi(A^{-1}) = \pi(A)^{-1}$ . Hence we have,

$$\sigma(\pi(A)) \subset \sigma(A).$$

This immediately tells us that, for self-adjoint operators,  $\|\pi(A)\| = \rho(\pi(A)) \leq \rho(A) = \|A\|$ . Since  $A^*A$  is self-adjoint, we have,

$$\|\pi(A)\|^2 = \|\pi(A^*)\pi(A)\| = \|\pi(A^*A)\| \leq \|A^*A\| = \|A\|^2.$$

A representation  $(\mathcal{H}, \pi)$  of a unital  $C^*$ -algebra  $\mathcal{A}$  is a  $*$ -homomorphism,

$$\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$$

which is unital, i.e.,  $\pi(1) = 1$  and  $\mathcal{H}$  is some Hilbert space. Two representation  $(\mathcal{H}_1, \pi_1)$  and  $(\mathcal{H}_2, \pi_2)$  of an algebra  $\mathcal{A}$  are said to be equivalent if there exists a unitary operation  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that

$$\pi_1(A) = U\pi_2(A)U^*,$$

for all  $A \in \mathcal{A}$ .

If the mapping  $\pi$  is an isomorphism between  $\mathcal{A}$  and  $\pi(\mathcal{A})$  it's called a faithful representation. Suppose we have a faithful representation of  $\mathcal{A}$  then by injectivity we have,  $\ker \pi = \{0\}$ . There exists  $\pi^{-1}$  from range of  $\pi$  into  $\mathcal{A}$ .

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\| \leq \|A\|.$$

So whenever  $\pi$  is a faithful representation, then for every  $A \in \mathcal{A}$ ,

$$\|A\| = \|\pi(A)\|.$$

If  $(\mathcal{H}, \pi)$  is a representation of  $\mathcal{A}$ , a subspace  $\mathcal{H}_1$  of  $\mathcal{H}$  is said to be invariant under  $\pi$  if  $\pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1$  for all  $A \in \mathcal{A}$ . If  $\mathcal{H}_1$  is closed and  $P_{\mathcal{H}_1}$  is the orthogonal projection with range  $\mathcal{H}_1$  then the invariance implies,

$$P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1} = \pi(A)P_{\mathcal{H}_1}.$$

for all  $A \in \mathcal{A}$ . Hence,

$$\begin{aligned}\pi(A)P_{\mathcal{H}_1} &= (P_{\mathcal{H}_1}\pi(A^*)P_{\mathcal{H}_1})^* \\ &= (\pi(A^*)P_{\mathcal{H}_1})^* \\ &= P_{\mathcal{H}_1}\pi(A).\end{aligned}$$

for all  $A \in \mathcal{A}$ . Hence  $\mathcal{H}_1$  is invariant under  $\pi$  if and only if,  $\pi(A)P_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(A)$  for all  $A \in \mathcal{A}$ . If we define  $\pi_1$  by,

$$\pi_1(A) = P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1},$$

then  $(\mathcal{H}_1, \pi_1)$  is a representation of  $\mathcal{A}$ . It's called a subrepresentation of  $(\mathcal{H}, \pi)$ . This procedure of going to subrepresentation gives a decomposition of  $\pi$ . If  $\mathcal{H}_1$  is invariant under  $\pi$  then so is  $\mathcal{H}_1^\perp$ . Setting  $\mathcal{H}_2 = \mathcal{H}_1^\perp$  one can define a second subrepresentation. Now the original Hilbert space  $\mathcal{H}$  can be written as a direct sum,  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  and each operator  $\pi(A)$  then decomposes as a direct sum  $\pi(A) = \pi_1(A) \oplus \pi_2(A)$ . So the representation can be written as  $(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2)$ .

Given a family of representations  $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$  of  $\mathcal{A}$  the direct sum of representations  $\mathcal{H}_\alpha$  is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form  $\varphi = \{\varphi_\alpha\}_{\alpha \in I}$  such that  $\lim_F [\sum_{\alpha \in F} \|\varphi_\alpha\|^2] < \infty$  where  $F$  is a finite subset of  $I$ . The purpose of this definition is so that norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha \in I} \pi_\alpha.$$

The operators  $\pi(A)$  on  $\mathcal{H}$  are bounded because  $\|\pi_\alpha(A)\| \leq \|A\|$  for each  $\alpha \in I$ .

A representation is trivial if  $\pi(A) = 0$  for every  $A \in \mathcal{A}$ . These are uninteresting representations. A representation can however have a trivial part.

$$\mathcal{D} = \{\varphi \in \mathcal{H} \mid \pi(A)\varphi = 0 \forall A \in \mathcal{A}\}.$$

It follows that  $\pi_{\mathcal{D}} = P_{\mathcal{D}}\pi P_{\mathcal{D}} = 0$  where  $P_{\mathcal{D}}$  is the projection onto the subspace  $\mathcal{D}$ . A representation  $(\mathcal{H}, \pi)$  is non degenerate if  $\mathcal{D} = \{0\}$ .

A vector  $|\Omega\rangle$  in a Hilbert space  $\mathcal{H}$  is called cyclic for  $\mathcal{A}$  if  $\{A|\Omega\rangle\}_{A \in \mathcal{A}}$  is dense in  $\mathcal{H}$ . A cyclic representation of  $\mathcal{A}$  is a triple  $(\mathcal{H}, \pi, |\Omega\rangle)$  where  $(\mathcal{H}, \pi)$  is a representation of  $\mathcal{A}$  and  $|\Omega\rangle$  is a cyclic for  $\pi(\mathcal{A})$ .

Let  $(\mathcal{H}, \pi)$  be a nondegenerate representation of  $\mathcal{A}$ . Take a maximal family of nonzero vectors  $\{|\Omega_\alpha\rangle\}_{\alpha \in I}$  in  $\mathcal{H}$  such that,

$$\langle \pi(A)\Omega_\alpha | \pi(B)\Omega_\beta \rangle = 0,$$

for all  $A, B \in \mathcal{A}$  and  $\alpha \neq \beta$ . Define,  $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}}_{A \in \mathcal{A}}$ . This is an invariant subspace of  $\mathcal{H}$ . Define  $\pi_\alpha = P_{\mathcal{H}_\alpha} \pi P_{\mathcal{H}_\alpha}$  where  $P_{\mathcal{H}_\alpha}$  is projection onto  $\mathcal{H}_\alpha$ . Then by construction each  $\mathcal{H}_\alpha$  are mutually orthogonal and hence the representation  $(\mathcal{H}, \pi)$  is of the form,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha, \quad \pi = \bigoplus_{\alpha \in I} \pi_\alpha.$$

So every nondegenerate representation can be written as a direct sum of a family of cyclic subrepresentations. If no invariant subspaces the representation  $(\mathcal{H}, \pi)$  of  $\mathcal{A}$  is called irreducible. If  $\mathcal{A}$  be a self-adjoint algebra of operators on Hilbert space  $\mathcal{H}$ ,

### 1.1 | GELFAND-NAIMARK-SEGAL CONSTRUCTION

Denote the dual space of  $\mathcal{A}$  i.e., the set of all continuous linear functionals over  $\mathcal{A}$  by  $\mathcal{A}^\#$ . The norm of a functional  $f$  over  $\mathcal{A}$  is defined by,  $\|f\| = \sup_{\|A\|=1} \{|f(A)|\}$ . A linear functional  $\omega$  over the algebra  $\mathcal{A}$  is called positive if,

$$\omega(A^*A) \geq 0$$

for all  $A \in \mathcal{A}$ . A positive linear functional over  $\mathcal{A}$  with  $\|\omega\| = 1$  is called a state. The state is called faithful if  $\omega(A^*A) = 0$  implies  $A = 0$ .

If  $\omega$  is a positive linear functional over  $\mathcal{A}$  then we can define a sesquilinear form,  $\varrho(B, A) = \omega(B^*A)$ . i.e.,  $\varrho(\mu A, \lambda B) = \bar{\mu}\lambda\varrho(A, B)$ , and  $\varrho(A, B) = \overline{\varrho(B, A)}$ . Since  $\omega$  a positive linear functional we have,

$$\varrho(\lambda A - B, \lambda A - B) \geq 0.$$

On expanding it, we obtain,

$$|\lambda|^2 \varrho(A, A) - \bar{\lambda} \varrho(A, B) - \lambda \varrho(B, A) + \varrho(B, B) \geq 0$$

By letting  $\lambda = \varrho(A, B)/\varrho(A, A)$  we obtain,  $0 \leq [|\varrho(A, B)|^2/\varrho(A, A)^2]\varrho(A, A) - 2[|\varrho(A, B)|^2/\varrho(A, A)] + \varrho(B, B)$ . This gives us,

$$|\varrho(A, B)|^2 \leq B_\omega(A, A)B_\omega(B, B).$$

If  $\omega$  is a positive linear functional then it satisfies the Cauchy-Schwarz inequality,

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B).$$

If  $\omega_1$  and  $\omega_2$  are two positive linear functionals we write  $\omega_1 \geq \omega_2$  if  $\omega_1 - \omega_2$  is positive. This gives an order on positive linear functionals. If  $\omega_1$  and  $\omega_2$  are two states over  $\mathcal{A}$  and  $0 < \lambda < 1$  then  $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$  is also a state such that  $\omega \geq \lambda\omega_1$  and  $\omega \geq (1 - \lambda)\omega_2$ . The set of all states is a convex subset of  $\mathcal{A}^\#$  and we will denote it by  $S(\mathcal{A})$ . The extreme points of this convex set are called pure states. They are such that  $\omega > \lambda\omega_1$  iff  $\omega_1 = \omega$ .

Given a closed two-sided ideal  $\mathcal{J} \subseteq \mathcal{A}$ , the quotient algebra is defined by,

$$\mathcal{A}_{\mathcal{J}} = \mathcal{A}/\mathcal{J} = \{[A] = A + J \mid J \in \mathcal{J}\}$$

with the norm,  $\|[A]\| = \inf_{J \in \mathcal{J}} \{\|A + J\|\}$  the algebra  $\mathcal{A}_{\mathcal{J}}$  is a  $C^*$ -algebra.

The Gelfand-Naimark-Segal theorem constructs for a given  $C^*$ -algebra  $\mathcal{A}$  and a state  $\omega$  a representation of the algebra of observables  $\mathcal{A}$  on the set of bounded operators  $\mathcal{B}(\mathcal{H})$  for some  $\mathcal{H}$ .

**THEOREM 1.1. (GELFAND-NAIMARK-SEGAL)** *Let  $\omega$  be a state on a unital  $C^*$ -algebra  $\mathcal{A}$  then there exists a cyclic representation  $(\mathcal{H}_\omega, \pi_\omega, |\Omega\rangle)$  of unit norm such that*

$$\omega(A) = \langle \Omega | \pi_\omega(A) \Omega \rangle$$

*for all  $A \in \mathcal{A}$ . The representation is unique in the sense that if  $(\mathcal{H}, \pi, |\Omega_\varphi\rangle)$  is a cyclic representation such that,  $\varphi(A) = \langle \Omega_\varphi | \pi(A) \Omega_\varphi \rangle$  then there exists a unique unitary operator  $U : \mathcal{H} \rightarrow \mathcal{H}_\omega$ , such that,*

$$\pi_\omega(A) = U \pi(A) U^*,$$

*and  $U|\Omega_\varphi\rangle = |\Omega_\omega\rangle$ .*

### PROOF

Given a state  $\omega$  on  $\mathcal{A}$  one considers the set in  $\mathcal{A}$  defined by,

$$\mathcal{J}_\omega = \{A \mid \omega(A^*A) = 0\}.$$

By Cauchy-Schwarz inequality whenever  $A \in \mathcal{J}_\omega$  for any  $B \in \mathcal{A}$  we have,

$$|\omega((BA)^*BA)|^2 = |\omega(C^*A)|^2 \leq \omega(C^*C)\omega(A^*A) = 0,$$

where  $C = B^*BA$ . So,  $BA \in \mathcal{J}_\omega$ . So  $\mathcal{J}_\omega$  is an ideal. Factorizing  $\mathcal{A}$  by  $\mathcal{J}_\omega$  an inner product is introduced on the quotient space  $\mathcal{A}_{\mathcal{J}_\omega}$  defined by,

$$\langle [A] | [B] \rangle := \omega(A^*B).$$

where  $[A]$  and  $[B]$  denote the equivalence classes determined by  $A$  and  $B$  respectively. The new vector space is completed by adding all the Cauchy sequences and we denote the Hilbert space by  $\mathcal{H}_\omega$ . On this Hilbert space we have the representation of the algebra  $\mathcal{A}$ ,

$$\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega),$$

defined by,

$$\pi_\omega(A)[B] \equiv [AB].$$

Let  $[I] = |\Omega_\omega\rangle$ . The expectation of any observable can then be written as,

$$\omega(A) = \langle \Omega_\omega | A \Omega_\omega \rangle.$$

A state on the algebra can be represented as a vector in some Hilbert space. A vector  $\varphi \in \mathcal{H}$  is said to be cyclic for  $\mathcal{A}$  if the closure of  $\mathcal{A}\varphi$  is same as  $\mathcal{H}$ . A vector  $\varphi$  is separating for  $\mathcal{A}$  if  $A\varphi = 0$  implies  $A = 0$  for all  $A \in \mathcal{A}$ . The vector  $|\Omega_\omega\rangle$  is cyclic for  $\mathcal{A}$ .

For every state  $\omega$  on an algebra  $\mathcal{A}$  there exists a cyclic representation  $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ .

$$\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle, \quad \forall A \in \mathcal{A}.$$

If there is another cyclic representation  $(\mathcal{H}, \pi, |\Omega\rangle)$  then define a map,  $U\pi(A)|\Omega\rangle = \pi_\omega(A)|\Omega_\omega\rangle$ . This is an isometry with an inverse, hence it extends to a unitary map.  $\square$

The representation is faithful if the state is faithful. That's probably where the name faithful state comes from.

## REFERENCES

- [1] V S SUNDER, Functional Analysis: Spectral Theory, Birkhauser Advanced Texts, 1991