Part III

Kan Extensions & Adjoint Functor Theorem

Adjoints and limits are some of the most important concepts in category theory. In this part we will discuss the relation between limits and adjoints. The goal is to prove the adjoint functor theorem and some an existence theorem for Kan extension.

1 | Adjoint-Limit Relations

Categories are compared by means of functors, and functors themselves are compared via natural transformations. Equivalence of categories allows us to basically think of the two categories as the same thing. This is however too restrictive. The relaxation of the notion of equivalence gives us the notion of adjoint.

1.1 | Adjoint Functors

The philosophy of adjoint functors is the following; when we want to study an object in mathematics, belonging to some weird category, we can take it, via a functor to some well understood category. But now this new category will not have the same meaning to the objects as the original category. So we would like a functor to get back to the original category. This functor is the adjoint functor.

An adjuntion from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functors,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B},$$

such that there is a natural isomorphisms of bifunctors $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y)$ and $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \tag{adjoint}$$

for all $X \in \mathcal{A}, Y \in \mathcal{B}$. Denote by $\mathcal{F} \dashv \mathcal{G}$. Since composition of natural isomorphisms is also a natural isomorphism if $\mathcal{F}$ has two adjoints $\mathcal{G}$ and $\widehat{\mathcal{G}}$, then we have,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, \widehat{\mathcal{G}}Y).$$

So, by Yoneda principle, adjoints if they exist are unique upto isomorphism. Consider the following two adjoint situations,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B} \underset{\mathcal{K}}{\overset{\mathcal{H}}{\rightleftarrows}} \mathcal{C},$$

By definition, we have for all $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{H} \circ \mathcal{F}X, Y)$, hence,

$$\mathcal{F} \circ \mathcal{H} \dashv \mathcal{G} \circ \mathcal{K}$$

When we are working with locally small categories, we can exploit the properties of the category of sets. We can look at adjoints from a functor category perspective, and representable functors.

Given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, for each $X \in \mathcal{B}$, we have the composite functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) &:= h^X \circ \mathcal{F} : \mathcal{A} \to \mathcal{B} \to \mathbf{Sets} \\ & \qquad A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

So, $\widehat{\mathcal{F}}$ is a functor to the functor category,

$$\widehat{\mathcal{F}} : \mathcal{B} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}},$$

which sends $X \mapsto \widehat{\mathcal{F}}(X)$. For each morphism $f : X \to Y$ in $\mathcal{B}$, the functor $\widehat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \to X$,

$$\widehat{\mathcal{F}}(f) : g \mapsto f \circ g$$

So, $\widehat{\mathcal{F}}(f \circ h) = \widehat{\mathcal{F}}(f) \circ \widehat{\mathcal{F}}(h)$, i.e., it's a covariant functor.

Lemma 1.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$ admits a left adjoint iff for all $X \in \mathcal{B}$,*

$$\widehat{\mathcal{F}}(X) : A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$$

is representable.

Proof

$\Leftarrow$ Suppose $\widehat{\mathcal{F}}(X)$ is representable for all $X \in \mathcal{B}$, then, $\exists\ \mathcal{G}X \in \mathcal{A}$ with, $\widehat{\mathcal{F}}(X) \cong h^{\mathcal{G}X}$, i.e.,

$$\widehat{\mathcal{F}}(X)(A) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X)$$

We have to make sure this is functorial, i.e., $X \mapsto \mathcal{G}X$ is a functor from $\mathcal{B}$ to $\mathcal{A}$. So, now we have to show that for each morphism $f : X \to Y$ there exists a morphism in the functor category, a natural transformation of functors, $\mathcal{G}f : \widehat{\mathcal{F}}(X) \to \widehat{\mathcal{F}}(Y)$, defined to be the maps that makes the following diagram commute.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X) \arrow[r] \arrow[d, "\mathcal{G}(f)"'] & \widehat{F}(X)(A) \arrow[d, "\widehat{\mathcal{F}}(f)"] \\ \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}Y) \arrow[r] & \widehat{F}(Y)(A) \end{tikzcd}$$

This also satisfies the composition needs by construction. By Yoneda lemma, this determines the functor $\mathcal{G}$ uniquely upto isomorphism.

$\Rightarrow$ The other direction is obvious and follows directly from the definition of adjoint, i.e., if there exists a left adjoint $\mathcal{G} \dashv \mathcal{F}$ each functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ is representable with representative $\mathcal{G}X$. $\square$

1.1.1 | Exponentiation and Cartesian Closedness

1.2 | Limits & Colimits

The notion of limits and colimits is very important as they allow us to construct new objects and functors. They are also very closely related to adjoint functors.

Let $\mathcal{I}$ and $\mathcal{A}$ be two categories. An inductive system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{F} : \mathcal{I} \to \mathcal{A}.$$

Intuitively, the limit of a system is an object in $\mathcal{A}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $X \in \mathcal{A}$ the constant functor $\Delta X : \mathcal{I} \to \mathcal{A}$ that sends everything in $\mathcal{I}$ to X, and each morphism in $\mathcal{I}$ to the identity on X. A relation between an object X and the system $\mathcal{F}$ is a natural transformation between ΔX and $\mathcal{F}$. Such a natural transformation is called a cone. The collection of all such cones is the set of all natural transformations,

$$\begin{aligned} C_{\mathcal{F}} : \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}). \end{aligned}$$

It's a contravariant functor from $\mathcal{A}$ to **Sets**. If the functor $C_{\mathcal{F}}$ is representable, there exists an object $L_{\mathcal{F}} \in \mathcal{A}$ such that,

$$C_{\mathcal{F}} \cong h^{L_{\mathcal{F}}}.$$

So, in such case $C_{\mathcal{F}}(X) \cong \mathrm{Hom}_{\mathcal{A}}(X, L_{\mathcal{F}})$. The representative $L_{\mathcal{F}}$ if it exists is called the colimit of the system $\mathcal{F}$, and is denoted by $\varinjlim \mathcal{F} := L_{\mathcal{F}}$ i.e.,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}}(X, \varinjlim \mathcal{F}) \tag{colimit}$$

and hence every cone must factor through $L_{\mathcal{F}}$. Intuitively the limit is the 'closest' object to the system. The notion of closeness must come from morphisms, so if there exists any other object with morphisms to the system, then it must be 'farther' than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

This means that for all objects $X \in \mathcal{A}$ and all family of morphisms $f_I : X \to \mathcal{F}I$, in $\mathcal{A}$ such that for all $f \in \mathrm{Hom}_{\mathcal{I}}(I, J)$, with $f_J = f_I \circ \mathcal{F}(f)$ factors uniquely through $\varinjlim \mathcal{F}$.

$$\begin{tikzcd} & X \arrow[d, dashed] \arrow[ddl, "f_I"'] \arrow[ddr, "f_J"] & \\ & \varinjlim \mathcal{F} \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_J"'] & \\ \mathcal{F}I \arrow[rr, "\mathcal{F}(f)"'] & & \mathcal{F}J \end{tikzcd}$$

A projective system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}.$$

Similar to the inductive system, for projective system $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^{\mathcal{G}} : \mathcal{A} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X). \end{aligned}$$

If it's representable with representative $L^{\mathcal{G}}$,

$$C^{\mathcal{G}} \cong h_{L^{\mathcal{G}}}.$$

Denote the representative by $\varinjlim \mathcal{G} := L^{\mathcal{G}}$ is called the limit of the projective system. If the limit exists, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X) \cong \mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, X) \tag{limit}$$

Projective limits can be written in terms of universal property as,

$$\begin{tikzcd} \mathcal{G}I \arrow[rr, "\mathcal{G}(f)"] \arrow[dr, "\varphi_I"] \arrow[ddr, "f_I"'] & & \mathcal{G}J \arrow[dl, "\varphi_J"'] \arrow[ddl, "f_J"] \\ & \varinjlim \mathcal{G} \arrow[d, dashed] & \\ & X & \end{tikzcd}$$

Note that if $\mathcal{I}$ admits initial object 0, then the limit $\varprojlim \mathcal{F}$ corresponds to the object $\mathcal{F}(0)$. Similarly for colimit, with terminal object.

A category $\mathcal{A}$ is cocomplete with respect to $\mathcal{I}$ if for all inductive systems indexed by $\mathcal{I}$, the colimit exists, if $\mathcal{I}$ is not explicitly said, then it means that $\mathcal{A}$ is cocomplete with respect to all small categories. $\mathcal{A}$ is complete with respect to $\mathcal{I}$ if it has all limits for all projective systems indexed by $\mathcal{I}$.

1.2.1 | Limit-Colimit Calculus

Theorem 1.2. *$\mathcal{A}$ be cocomplete with respect to $\mathcal{I} \Rightarrow \mathcal{A}^{\mathcal{K}}$ is cocomplete with respect to $\mathcal{I}$.*

Proof

Given an inductive system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$, the goal is to construct a new functor $\varprojlim \mathcal{F}$ in $\mathcal{A}^{\mathcal{I}}$ such that,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

where $\Delta^{\mathcal{K}}$ is the constant functor in $\mathcal{A}^{\mathcal{K}}$ which sends $A \in \mathcal{A}$ to the constant functor $\Delta^{\mathcal{K}}A$. This is then by definition the colimit of the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$.

In order to do this we have to describe where $\varprojlim \mathcal{F}$ sends elements of $\mathcal{I}$. The construction involves argument wise assignment of objects in $\mathcal{A}$ for each object in $\mathcal{I}$, and then showing the functoriality, that is verify it respects composition of morphisms.

Construction. By definition of colimit, for every $I \in \mathcal{I}$, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}I, \mathcal{F}I) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}I, \varprojlim(\mathcal{F}I)) \tag{isomorphism}$$

here $\mathcal{F}I : \mathcal{K} \to \mathcal{A}$ is a fixed functor, an inductive system and $\varprojlim(\mathcal{F}I)$ its inductive limit. So we define the associated object by,

$$\varprojlim \mathcal{F}(I) := \varprojlim(\mathcal{F}I).$$

Functoriality.

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}(\cdot), \mathcal{F}(\cdot)) : \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathbf{Sets}$$

is a bifunctor, so for each natural transformation $\kappa : \Delta^{\mathcal{K}}\mathcal{H} \Rightarrow \mathcal{F}$,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, bend left, "\mathcal{F}"] \arrow[dr, "\mathcal{H}"'] & & \mathcal{A}^{\mathcal{K}} \\ & \mathcal{A} \arrow[u, phantom, "\Uparrow\kappa"] \arrow[ur, "\Delta^{\mathcal{K}}"'] & \end{tikzcd}$$

and morphism $f : I \to J$, we have maps,

$$\begin{tikzcd} I \arrow[d, "f"'] & \Delta^{\mathcal{K}}\mathcal{H}I \arrow[r, "\kappa_I"] \arrow[d, "\Delta^{\mathcal{K}}\mathcal{H}(f)"'] & \mathcal{F}I \arrow[d, "\mathcal{F}(f)"] \\ J & \Delta^{\mathcal{K}}\mathcal{H}J \arrow[r, "\kappa_J"'] & \mathcal{F}J \end{tikzcd}$$

The commutative square gives us, $\kappa_J \circ \Delta^{\mathcal{K}}\mathcal{H}(f) = \mathcal{F}(f) \circ \kappa_I$, and the isomorphism of sets gives us a morphisms, $\widehat{\kappa}_I$ and $\widehat{\kappa}_J$ such that,

$$\widehat{\kappa}_I \circ \mathcal{H}(f) = \varprojlim(\mathcal{F}(f)) \circ \widehat{\kappa}_J.$$

So, $\widehat{\kappa} : \mathcal{H} \to \varprojlim \mathcal{F}(\cdot)$ is a natural transformation. So we get,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

Since this natural transformation is defined using the natural transformation κ it will satisfy the required compositions. So we have, $\varprojlim \mathcal{F}(f \circ g) = (\varprojlim \mathcal{F}(f)) \circ (\varprojlim \mathcal{F}(g))$. □

This gives us an adjoint situation, where colimit is left-adjoint to the constant functor and the constant functor is left-adjoint to the limit,

$$\varprojlim \dashv \Delta \dashv \varinjlim \qquad \text{(limit-diagonal adjointness)}$$

Theorem 1.3.

$$\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}).$$

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, A) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, A),$$

Proof

The idea is to study an appropriate functor category, get hom-set isomorphisms and then apply Yoneda principle. So, we have to show for each set $X \in \mathbf{Sets}$, we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$$

Using the limit-diagonal adjointness, this reduces to showing $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$ and $\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$ are isomorphic. Δs are in the appropriate categories.

Let $\kappa : \Delta X \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$ be a natural transformation, then κ is determined by its components

$$\kappa_I : (\Delta X)I \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I).$$

Each $(\Delta X)I$ is a set, and hence the maps κ_I is itself determined by its action on the elements of the set X, So, for each $x \in X$, $\kappa_I(x)$ is a morphism in $\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I)$.

If we think of A as the constant functor, we can define using $\varphi_x(\cdot) := \kappa_{(\cdot)}(x)$ defines an element $\varphi \in \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$. This is a bijection and hence,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$$

Applying the limit-diagonal adjointness again to this and the Yoneda principle, we get that $\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$. The other isomorphism is also similar. □

So, hom-functor takes limits to colimits in the first argument and takes limits to limits in the second argument. This can now be used to prove many things easily.

Intuitively, limits take us to the 'last object' in a diagram and colimits take us to the 'first object' in a diagram. Here 'most' is quantified in terms of morphisms, and first and last are quantified by the direction of the morphisms. So, if the indexing category is a product category, then we can think of limits as trying to find the bottom right corner of the rectangle diagram, and similarly the colimit is the top left corner. So, whether we go to right most first and then go to the bottom or whether go to the bottom first and then go to the right shouldn't change which object we reach after doing this. We now formalize this.

Theorem 1.4.

$$\varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathcal{F} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F} \cong \varinjlim_{\mathcal{I}\times\mathcal{J}} \mathcal{F}$$

Proof

This can be proved using the relation between the constant functors. Let $\Delta^{\mathcal{I}\times\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}\times\mathcal{J}}$, $\Delta^{\mathcal{I}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ and $\Delta^{\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ be the constant functors in the appropriate functor category. Then we have, $\Delta^{\mathcal{I}\times\mathcal{J}} = \Delta^{\mathcal{I}}\Delta^{\mathcal{J}}$. This gives us,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{I}\times\mathcal{J}} \mathcal{F}) &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}\times\mathcal{J}} A, \mathcal{F}) \\
&\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}}\Delta^{\mathcal{J}} A, \mathcal{F}) \\
&\cong \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F}).
\end{aligned}$$

By Yoneda principle we have the required isomorphism. Since $\mathcal{I} \times \mathcal{J} \cong \mathcal{J} \times \mathcal{I}$, the other isomorphism also follows. □

Lemma 1.5. *If $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint, $\mathcal{F}$ preserves all limits in $\mathcal{A}$.*

Proof

If a left-adjoint $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ exists for $\mathcal{F}$, and suppose $\mathcal{H} : \mathcal{I} \to \mathcal{A}$ is an inductive system with a limit $\varprojlim \mathcal{H}$, we must prove that $\mathcal{F}(\varprojlim \mathcal{H})$ is the limit of the inductive system $\mathcal{F} \circ \mathcal{H}$.

$\mathcal{F} \circ \mathcal{H} : \mathcal{I} \to \mathcal{B}$ is an inductive system in $\mathcal{B}$ indexed by $\mathcal{I}$. For all $X \in \mathcal{B}$,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F} \varprojlim \mathcal{H}) &\cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \varprojlim \mathcal{H}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \mathcal{H}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F} \circ \mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathcal{B}}(X, \varprojlim \mathcal{F} \circ \mathcal{H})
\end{aligned}$$

By Yoneda principle, we have, $\mathcal{F}(\varprojlim \mathcal{H}) \cong \varprojlim \mathcal{F} \circ \mathcal{H}$. $\square$

Right adjoints preserve limits, can be remembered by the acronym, RAPL. Under additional conditions on the category $\mathcal{A}$, the converse holds, these theorems are called adjoint functor theorems.

2 | Kan Extension of Functors

Given two functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$, the goal of Kan extension is to find a new functor $\mathcal{K}$ such that the manipulation done by composite functor $\mathcal{K} \circ \mathcal{F}$ is close to the manipulation done by the functor $\mathcal{G}$.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd}
 & \mathcal{B} & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathcal{K}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C}
\end{tikzcd}$$

The left Kan extension of $\mathcal{G}$ along $\mathcal{F}$ is a functor is a functor $\operatorname{Lan}_{\mathcal{F}} \mathcal{G} : \mathcal{B} \to \mathcal{C}$ such that there exists a natural transformation $\alpha : \mathcal{G} \to (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F}$, such that any other extension $\mathcal{K}$ with natural transformation $\mathcal{G} \to \mathcal{K} \circ \mathcal{F}$ factors through $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$, i.e., $\mathcal{G} \Rightarrow (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F}$.

The left Kan extension $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$ can be visualised by the diagram,

$$\mathcal{G} \implies (\operatorname{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \implies \mathcal{K} \circ \mathcal{F} \implies \cdots$$

This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{K} \circ \mathcal{F}$ there exists a natural transformation from $\operatorname{Lan}_{\mathcal{F}} \mathcal{G}$ to $\mathcal{K}$, i.e.,

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K} \circ \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{C}^{\mathcal{B}}}(\operatorname{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K})$$

Intuitively the left Kan extension is the left most functor to the functor in the above visualisation. Here 'most' is quantified in terms of natural transformations. The left Kan extension is the left most (as explained above) extension of the functor $\mathcal{G}$ with respect to $\mathcal{F}$. We should expect the construction of the 'left most' functor to be related to taking colimits in an appropriate category. If $\mathcal{C}$ is cocomplete, the functor category to $\mathcal{C}$ will also be cocomplete, and $\mathcal{A}$ has some 'nice properties' then we could think of these as a system in the functor category and intuitively, the left 'most' should exist.

The right Kan extension is defined similarly, and only the direction of the natural transformation is changed from $\mathcal{K} \circ \mathcal{F}$ to $\mathcal{G}$.

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathcal{H}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C}
\end{tikzcd}$$

The right Kan extension is denoted by $\operatorname{Ran}_{\mathcal{F}} \mathcal{G}$. It's the right most functor in the following visualization,

$$\cdots \implies \mathcal{H} \circ \mathcal{F} \implies (\operatorname{Ran}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \implies \mathcal{G}$$

Similar to left Kan extensions, we should expect the construction of the 'right most' functor to be related to taking limits in an appropriate category. This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{H} \circ \mathcal{F}$ there exists a natural transformation from $\mathcal{H}$ to $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{H} \circ \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathcal{H}, \mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}).$$

Kan extension are functors in the appropriate functor category.

For each functor $\mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, the left Kan extension is the functor,

$$\mathrm{Lan}_{\mathcal{F}}(\cdot) : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{B}},$$

which sends $\mathcal{G}$ to $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$. Clearly, $\mathcal{G}$ is the closest to $\mathcal{G}$, similarly we expect 'nearest' composed with 'nearest' to be the 'nearest'.[1] So we will have,

$$\mathrm{Lan}_{\mathcal{F}}(\mathcal{G} \circ \mathcal{H}) = \mathcal{G} \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{H}, \quad \mathrm{Lan}_{\mathcal{F}\circ\mathcal{E}}\,\mathcal{G} = \mathrm{Lan}_{\mathcal{F}}(\mathrm{Lan}_{\mathcal{E}}\,\mathcal{G}).$$

Similarly for the right Kan extension.

Lemma 2.1.

$$\mathrm{Lan}_{\mathcal{F}} \dashv \circ\mathcal{F} \dashv \mathrm{Ran}_{\mathcal{F}}\,.$$

The proof of this adjointness with precomposition has already been described, and directly follows from definition of Kan extensions. Certain additional constraints on the starting categories guarantees the existence of Kan extensions.

2.1 | End-Coend Calculus

2.1.1 | Kan Extensions as Coends

Theorem 2.2. (Kan) *If $\mathcal{A}, \mathcal{B}$ small, and $\mathcal{C}$ cocomplete, $\Rightarrow \forall \mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ exists.*

The idea is to consider all the left extensions of the functor $\mathcal{G}$ along $\mathcal{F}$. Use the assumptions about $\mathcal{A}$ and $\mathcal{B}$ being small to think of the category of left extensions as a inductive system, and then use the cocompleteness of the functor category $\mathcal{C}^{\mathcal{A}}$ to get the existence.

To formalize this idea, we need the notion of comma category.

for the proof.

Proof

The goal is to think of the collection of all left extensions as an inductive system, and show that the indexing category is small. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd} & \mathcal{B} & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathcal{K}_i"'] \arrow[dr, dashed, bend left=60, "\mathcal{K}_j"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[r, phantom, "\Uparrow"] & & \mathcal{C} \end{tikzcd}$$

Consider the free category whose objects consist of left extensions of $\mathcal{G}$ along $\mathcal{F}$, and morphisms between these objects consist of the natural transformations between these extensions. The composition defined by concatenation. Denote this category by, $(\mathcal{F}|\mathcal{G})$.

We clearly have an inductive system indexed by $(\mathcal{F}|\mathcal{G})$,

$$\mathcal{L} : (\mathcal{F}|\mathcal{G}) \to \mathcal{C}^{\mathcal{B}}$$

Note that natural transformations are determined by their components.

[1]Note that these follow directly from the definition of Kan extensions, but will remain mysterious. So I wanted to state it intuitively, and leave the proof.

In the following we will discuss these nice properties of $\mathcal{A}$ that guarantee the existence of Kan extensions.

Lemma 2.3. *$\mathcal{A}$ is small and $\mathcal{C}$ is finitely complete, locally small category, $\Rightarrow \mathcal{C}^{\mathcal{A}}$ is locally small.*

Idea of Proof

We have to show for any two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{C}$ the collection of all natural transformations $\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$ forms a set. Note that two natural transformations are the same if they have the same components. Since $\mathcal{C}$ is locally small, for each $A \in \mathcal{A}$, each natural transformation $\kappa : \mathcal{F} \to \mathcal{G}$ is determined by its components κ_A, since $\mathcal{A}$ is small and $\mathcal{C}$ is locally small, there are only a set of subobjects.

2.2 | All Concepts are Kan Extensions

Kan extensions is a very general construction which unites limits, adjoints, and other constructions.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor, where $\mathcal{A}$ is a small category, then for any other functor $\mathcal{H} : \mathcal{B} \to \mathcal{C}$, we can form composition functors,

$$\mathcal{H} \circ \mathcal{F} : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

This gives us a functor between functor categories $\circ\mathcal{F} : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$.

Theorem 2.4. *If $\mathcal{A}$ is small and $\mathcal{C}$ is cocomplete, then* $\mathrm{Lan}_{\mathcal{F}}\mathcal{G}$ *exists.*

Proof

To prove this we will do the following, first use the smallness of the category $\mathcal{A}$ to construct a functor $\mathcal{K} : \mathcal{B} \to \mathcal{C}$ that's an extension of $\mathcal{G}$ along $\mathcal{F}$, this gives us the existence of extensions, then use the cocompleteness to take the colimit.

Note that, to construct such an extension we mainly need some nice properties of the category $\mathcal{A}$. Using the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ we can define for each $X \in \mathcal{B}$, a contravariant functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) \coloneqq h^X \circ \mathcal{F} :\ & \mathcal{A} \to \mathbf{Sets} \\ & A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

This gives us a new category, the category of elements of $\mathcal{F}$, i.e., the category whose objects are images of objects of $\mathcal{A}$ under the functor $\mathcal{F}$. So, this category consists of objects which are pairs (A, a) where $A \in \mathcal{A}$ and $a \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ and the morphisms $f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ in $\mathcal{A}$, such that $\widehat{\mathcal{F}}(X)(f)(a) = b$.

$$\begin{tikzcd} A \arrow[r, "\mathcal{F}"] \arrow[d, "f"'] & \mathcal{F}A \arrow[d, "\mathcal{F}(f)"] & a \arrow[d, mapsto, "\mathcal{F}(f)"] \\ B \arrow[r, "\mathcal{F}"] & \mathcal{F}B & b \end{tikzcd}$$

This category contains all the information contained in the functor $\mathcal{F}$. Abusing notation, we will denote the category of elements of the functor by $\widehat{\mathcal{F}}(X)$. From this we have the forgetful functor $(A, a) \to A$,

$$\varphi : \widehat{\mathcal{F}}(X) \to \mathcal{A}$$

So, $\widehat{\mathcal{F}}$ is a functor to the functor category,

$$\widehat{\mathcal{F}} : \mathcal{B} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}},$$

which sends $X \mapsto \widehat{\mathcal{F}}(X)$. For each morphism $f : X \to Y$ in $\mathcal{B}$, the functor $\widehat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \to X$, Let $\mathcal{A}$ and $\mathcal{B}$ be two small categories, i.e., the collection of objects is a set. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor between them. The composition gives us a functor between functor categories,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{H}\circ\mathcal{F}"'] & \mathcal{B} \arrow[d, "\mathcal{H}"] \\ & \mathcal{C}. \end{tikzcd}$$

Denote the composition map by, $\circ\mathcal{F} : \mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$, such that $\mathcal{H} \mapsto \mathcal{H} \circ \mathcal{F}$. The existence of the left Kan extension of $\mathcal{G}$ along $\mathcal{F}$

3 | Adjoint Functor Theorems

Here we will prove the general adjoint functor theorem due to Freyd. The condition needed by the GAFT is called solution set condition. We begin by explaining what adjoints are doing intuitively by defining them via reflections.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor. We want to associate with each $X \in \mathcal{B}$ an object $R_X \in \mathcal{A}$ such that $\mathcal{F}R_X$ is the best estimate of $X \in \mathcal{B}$. In categorical terms, the estimation is done with morphisms. So the 'best estimate' is a morphism,

$$\kappa_X : X \to \mathcal{F}R_X$$

such that for any other $A \in \mathcal{A}$, with an estimation $\varkappa : X \to \mathcal{F}A$ factors uniquely through $\mathcal{F}R_X$. R_X together with the morphism κ_X is called the reflection of X along $\mathcal{F}$. Visualized by the diagram,

$$\begin{tikzcd} R_X \arrow[d, dashed, "\exists!\, f"'] & \mathcal{F}R_X \arrow[d, "\mathcal{F}(f)"'] & X \arrow[l, "\kappa_X"'] \arrow[dl, "\varphi"] \\ A & \mathcal{F}A & \end{tikzcd} \tag{reflection}$$

that's to say, there exists a unique $f : R_X \to A$ such that,

$$\mathcal{F}(f) \circ \kappa_X = \varphi$$

Intuitively, κ_X is a better estimate than φ. We can't have two best estimates κ_X and κ'_X because in such a case, we have, two maps, $f : R_X \to R'_X$ and $f' : R'_X \to R_X$, such that

$$\mathcal{F}(f) \circ \kappa_X = \kappa'_X, \quad \mathcal{F}(f') \circ \kappa'_X = \kappa_X$$

So we get that,

$$\mathcal{F}(f \circ f') \circ \kappa'_X = \kappa'_X$$

By uniqueness, $f \circ f' = \mathbb{1}_{R'_X}$. Hence any two reflections must be isomorphic.

Lemma 3.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, and suppose reflection exists for each $X \in \mathcal{B}$, there exists a unique reflection functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that, $\mathcal{G}X = R_X$ and a natural transformation κ such that,*

$$\kappa_X : X \to \mathcal{F} \circ \mathcal{G}X.$$

Proof

Let $g : X \to Y$ be a morphism, then we have,

$$\begin{tikzcd}
R_X \arrow[d, dashed, "\exists!\, f"'] & \mathcal{F}R_X \arrow[d, "\mathcal{F}(f)"'] & X \arrow[l, "\kappa_X"'] \arrow[d, "g"] \\
R_Y & \mathcal{F}R_Y & Y \arrow[l, "\kappa_Y"]
\end{tikzcd}$$

In this diagram, $\kappa_Y \circ g : X \to \mathcal{F}R_Y$ is an estimate, and hence there must exist and $f : R_X \to R_Y$ such that $\mathcal{F}(f) \circ \kappa_X = \kappa_Y \circ g$. So, we define,

$$\mathcal{G}(g) \coloneqq f$$

By construction this makes κ a natural transformation. By exploiting uniqueness we can show that this is functorial, i.e., $\mathcal{G}(g \circ h) = \mathcal{G}(g) \circ \mathcal{G}(h)$. □

If the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a reflection, by definition, to each morphism $\varphi : X \to \mathcal{F}A$ there exists a unique map $f : \mathcal{G}X \to A$. Conversely, any map f uniquely determines φ by,

$$\mathcal{F}(f) \circ \kappa_X = \varphi$$

Which means, we have an isomorphism,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, A) \cong \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}A)$$

Since this holds for each X and A, it is the adjoint condition. So, adjoint and reflection are the same functors.

3.0.1 | Solution Set Condition

Adjoint condition is equivalent to the existence of reflection for each object. The solution set condition is a much weaker requirement than the existence of a reflection, and it turns out to be enough to guarantee the existence of an adjoint.

3.0.2 | Freyd Adjoint Functor Theorem

Theorem 3.2. *Let $\mathcal{A}$ be a complete category, a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint iff it preserves small limits, and $\mathcal{F}$ satisfies the* **??** *for every object $X \in \mathcal{B}$.*

3.0.3 | Special Adjoint Functor Theorem

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