Part I

Holomorphic Functions

Main aim of this part is to show holomorphic functions are analytic. Topics for these notes include, holomorphic functions, analytic functions, integration along paths, Cauchy's theorem, Cauchy integral formula, Liouville theorem, maximum modulus principle.

1 | Holomorphic Functions

Let U be an open subset of $\mathbb{C}$, let f be a complex valued function on U. The function f is said to be complex differentiable function at a point $z \in U$ if,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1D}$$

exists. In such a case the limit $f'(z)$ is called the derivative of f at z. If f is differentiable for all $z \in U$ it's called complex differentiable on U and the map $z \mapsto f'(z)$ is called the derivative of f. We need some simplifications and ways to verify complex differentiability using coordinates.

To do this we can start to look at f as a function from U as an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$. Then f can be written as, $f = u + iv$. The first requirement for complex differentiability of f at $z \in U$ is that it's is real differentiable. This means that there exists a real linear map, $df(z) : \mathbb{R}^2 \to \mathbb{R}^2$ such that,

$$\lim_{h \to 0} \|f(z+h) - f(z) - df(z)(h)\| / \|h\| \to 0, \tag{2D}$$

where the norm $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^2$. The unique real linear map $df(z)$ is called the differential of f at z. The matrix form of $df(z)$ is called the Jacobian and in terms of partial derivatives in the standard basis, it is given by,

$$J_f(z) = (\partial_j f_i(z))_{i,j}$$

In addition to real differentiability the function f for the function to be complex differentiable, the differential should should be complex linear inaddition to being real linear.

A real linear map T is complex linear if $T(i) = iT(1)$. This condition puts the required constraint. So the Jacobian matrix $J_f(z)$ will be of the form,

$$J_f(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{bmatrix} \partial_x u(z) & \partial_y u(z) \\ \partial_x v(z) & \partial_y v(z) \end{bmatrix}. \tag{3D}$$

This is called the Cauchy-Riemann equation.

A function f is said to be holomorphic on U if it's complex differentiable for all $z \in U$. An orientation preserving transformation i.e., the order of basis is preserved after transformation. Such linear transformations have positive determinant. An angle preserving function is a complex differentiable function whose differential preserves angles. An angle preserving, orientation preserving function is called conformal. Every non-constant holomorphic function is conformal. The set of all holomorphic functions on U is denoted by $\mathcal{H}U$.

Some simple properties of complex differentiable functions follow through simple application of the definition. If f, and g are two complex differentiable functions on U and $\lambda \in \mathbb{C}$ then, $f + g$, $f \cdot g$, and $\lambda \cdot f$ are also complex differentiable.

If f is a complex differentiable function on U and g is a complex differentiable function on V then the composition map $g \circ f$ is a complex differentiable function on U and,

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

which is the chain rule.

Given a sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$, a formal power series is a series $\sum_{n\geq 1} a_n z^n$. The set of formal power series is denoted by $\mathbb{C}[z]$. Let $\sum_{n\geq 1} a_n z^n$ be a formal power series. Let R be the suppremum of all $r \geq 0$ such that $|a_n|r^n$ is bounded. If $|z| > R$ then the series $\sum_{n\geq 1} a_n z^n$ cannot converge.

For every $\rho < r < R$, there exists some M_r such that, $|a_n|r^n \leq M_r$ or $|a_n| \leq M_r(1/r)^n$ by definition of R. Or,

$$|a_n|\rho^n \leq M_r(\rho/r)^n.$$

So, if C is a compact subset of $B(R)$ it's strictly contained in some ball of radius ρ less than R i.e., $C \subset B(\rho)$. So, for any $z \in C$ we have,

$$|a_n z^n| \leq |a_n|\rho^n \leq M_r(\rho/r)^n$$

So the series is bounded by $M_r \sum_{n\geq 1}(\rho/r)^n$. So, $\sum_{n\geq 1} a_n z^n$ converges uniformly in C. R is called the radius of convergence of $\sum_{n\geq 1} a_n z^n$.

Theorem 1.1. (Abel's theorem) $\sum_{n\geq 1} a_n z^n \in \mathbb{C}[z]$, $\exists R \geq 0$ *such that* $\sum_{n\geq 1} a_n z^n$ *uniformly converges for compact subsets* $C \subset B(R)$. □

The formal differentiation and integration of a power series $\sum_{n\geq 1} a_n z^n$ are defined to be, $\sum_{n\geq 1} n a_n z^{n-1}$ and $\sum_{n\geq 1} a_n/(n+1) z^{n+1}$ respectively. It can be showed that the radius of convergence of the formal differentiation and integration are the same. If a power series $\sum_{n\geq 1} a_n z^n$ has a positive radius of convergence then it's indefinitely complex differentiable in the ball of radius same as the radius of convergence. If a power series $\sum_{n\geq 1} a_n z^n$ converges to $f(z)$ in a ball $B_a(R)$, then it can be showed that $f(z)$ is differentiable with derivative $f'(z) = \sum_{n\geq 1} n a_n z^{n-1}$.

A function $f : U \to \mathbb{C}$ is said to have a power series expansion at $c \in U$ if there exists a power series centered at c such that

$$f(z) = \sum_{n\geq 1} a_n (z - c)^n$$

for all z in a neighborhood of c and $\sum_{n\geq 1} a_n(z - c)^n$ is called the Taylor series expansion of f. A function f is called analytic on U if it has a power series expansion at every point in U. Every analytic function is hence indefinitely complex differentiable and holomorphic on U.

2 | Complex Integration

Let $\eta : [a,b] \to \mathbb{C}$ be a smooth curve. Let f be a continuous function defined atleast on the compact image $\eta([a,b])$. The path integral of f along η is defined by,

$$\int_\eta f(z)dz = \int_{[a,b]} f(\eta(t))\eta'(t)dt$$

since $(f \circ \eta) \cdot \eta'$ is continuous on $[a,b]$ the integral is well-defined. For a piecewise C^1-path the integral along $\eta = \eta_1 + \cdots + \eta_n$ is defined by, $\int_\eta f(z)dz = \sum_{i=1}^n \int_{\eta_i} f(z)dz$.

Suppose we have a reparametrization of the interval $[a,b]$, given by the C^1-map $\varphi : [a',b'] \to [a,b]$ then we have,

$$\int_{\eta\circ\varphi} f(z)dz = \int_{[a',b']} f(\eta(\varphi(t)))\eta'(\varphi(t))\varphi'(t)dt = \int_{[a,b]} f(\eta(s))\eta'(s)ds = \int_\eta f(z)dz.$$

where $s = \varphi(t)$, $ds = \varphi'(t)dt$. Hence the path integral is invariant under reparametrization. Length of a path $\eta : [a,b] \to \mathbb{C}$ is defined by,

$$L(\eta) = \int_{[a,b]} |\eta'(t)|dt.$$

It's easy to see the properties of the path integral,

If η is a path then, f, g in the domain containing η and $a, b \in \mathbb{C}$,

$$\int_\eta (af + bg)dz = a\int_\eta fdz + b\int_\eta gdz.$$

If the path η_1 starts at the end point of η_2 then,

$$\int_{\eta_1+\eta_2} fdz = \int_{\eta_1} fdz + \int_{\eta_2} fdz.$$

The reverse path can be written as $\widetilde{\eta}(s) = \eta(a+b-s)$, then by changing the variable, we have,

$$\int_{\widetilde{\eta}} fdz = \int_\eta f(\eta(a+b-s))\eta'(a+b-s)(-1)ds = -\int_\eta fdz.$$

If $\eta([a,b]) \subseteq U$ and $g \in \mathcal{H}U$ with continuous derivative g', we have,

$$\int_{g\circ\eta} fdz = \int_\eta f(g(z))g'(z)dz.$$

for all $f \in \mathcal{H}U$, $|\int_\eta fdz| \leq \int_{[a,b]} |f(\eta(t))||\eta'(t)|dt \leq \sup_{z\in(\eta([a,b]))} |f(z)| \int_{[a,b]} |\eta'(t)|dt = \|f\|_\eta L(\eta)$. and this can be extended to piecewise C^1 paths. So we have,

$$\left|\int_\eta fdz\right| \leq L(\eta)\|f\|_\eta.$$

where $\|f\|_\eta = \sup_{z\in\eta([a,b])} |f(z)|$. Using this we can easily show that if f_n converges to f uniformly in $\eta([a,b])$ then,

$$\lim_n \int_\eta f_n dz = \int_\eta fdz$$

Let $f \in \mathcal{H}U$ with continuous derivative, for a smooth curve η, we have,

$$\frac{df(\eta(t))}{dt} = f'(\eta(t))\eta'(t)$$

So, we have,

$$\int_\eta f' dz = \int_\eta f'(\eta(t))\eta'(t)dt = \int_{[a,b]} \frac{df(\eta(t))}{dt} dt = f(\eta(t))\Big|_a^b = f(\eta(b)) - f(\eta(a)).$$

If f has an anti-derivative F, i.e., $F' = f$ then,

$$\int_\eta f(z)dz = F(\eta(b)) - F(\eta(a)).$$

So, if f has an anti-derivative in U then the path integral is independent of the path, and only depends on the end points. If the integral is path independent then we can go back from one of the curves and for which the value of the integral will be negative of the original, and hence for every loop the value of the integral will be zero. Converse is also similar.

Theorem 2.1. (Cauchy's theorem) *$f \in \mathcal{H}U$, if η is a loop in U, then,*

$$\int_\eta f(z)dz = 0$$

□

Sketch of Proof

The proof is a simple application of Green's theorem together with the Cauchy-Riemann equation.

References