Part IV
Residue Theorem

1 | Winding number

The starting point is to study how many times a loop winds around a point. Finding this is however not possible within the base space. To do this we will lift the loop to the universal cover and then find a way to study the number of times it winds the point. This is similar to calculating the fundamental group of circle.

The starting point is to find a universal covering space. Consider the map,

$$\exp : z \mapsto e^z.$$

This is a local homeomorphism of $\mathbb{C}$ and $\mathbb{C}^*$ and around each point z the neighborhoods $B_\epsilon(z)$ with $\epsilon < 2\pi$ then,

$$B_\epsilon(z) \cong e^{B_\epsilon(z)}.$$

So the map $z \mapsto e^z$ and hence the map $z \mapsto x + e^z$ are covering maps as we have around each point a connected neighborhood thats homeomorphic to its image. Since the covering space is simply connected, it's the universal cover. Now we can start lifting paths, homotopies to the covering space. We will denote by $\mathbb{C}^*$ the complex plane without the point x.

Let $\eta : I \to \mathbb{C}$ be a loop or closed curve in $\mathbb{C}$ and let $x \in \mathbb{C}$ that's not in the image of η. So η is a loop in the base space and we can lift this loop to $\mathbb{C}$ using the above covering map $p : z \mapsto x + e^z$. Let $\widehat{\eta}_1$ and $\widehat{\eta}_2$ be two lifts then, $p \circ \widehat{\eta}_1 = p \circ \widehat{\eta}_2 = \eta$, so we have,

$$x + e^{\widehat{\eta}_1(0)} = x + e^{\widehat{\eta}_2(0)} = \eta(0)$$

So, $\widehat{\eta}_1(0) = \widehat{\eta}_2(0) + 2\pi ik$ for some $k \in \mathbb{Z}$. By uniqueness, of lifts, we have, $\widehat{\eta}_1 = \widehat{\eta}_2 + 2\pi ik$. So, the difference, $\widehat{\eta}(1) - \widehat{\eta}(0)$ is well defined, i.e., it's independent of the lift. Since η is a loop we have, $\eta(0) = \eta(1)$, and hence we have, $x + e^{\widehat{\eta}}(0) = x + e^{\widehat{\eta}(1)}$ or

$$\widehat{\eta}(0) = \widehat{\eta}(1) + 2\pi i n(\eta, x)$$

for some integer $n(\eta, x)$. The winding number of η with respect to x is defined to be,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)].$$

Note that this depends on the covering map, and hence on the point x. Every time the curve η winds a circle the argument changes by 2π, since the starting point and the end point of the curve are the same the radius is the same for both start and end. Hence $1/2\pi i[\widehat{\eta}(1) - \widehat{\eta}(0)]$ represents the number of times the loop winds around the point x.

Lemma 1.1. *If η is a loop in $\mathbb{C}$ then $x \mapsto n(\eta, x)$ is locally constant on $\mathbb{C}\backslash\{Im(\eta)\}$ i.e., it's constant on each connected component.*

Proof

The proof goes by showing the map $x \mapsto n(\eta, x)$ is continuous map to $\mathbb{C}$. Since $n(\eta, x) \in \mathbb{Z}$, it has to be constant on every connected component. To compute the winding number $n(\eta, x)$ we have to lift the curve η to $\widehat{\eta}$ with respect to the covering map $z \mapsto e^z$, and compute $\widehat{\eta}(1) - \widehat{\eta}(0)$.

For every loop η in $\mathbb{C}$, we have to define new curves,

$$\eta_a(t) = \eta(t) - a$$

is a loop in $\mathbb{C}^*$. Let $B_\epsilon(w) = \{|w - x| < \epsilon\}$ be a disc that's entirely inside $\mathbb{C}\backslash\{Im(\eta)\}$. This now gives us a map,

$$(t, x) \mapsto \eta(t) - x$$

This is a continuous map from $I \times B_\epsilon(w)$ to $\mathbb{C}^*$. $I \times B_\epsilon(w)$ is simply connected, i.e., the fundamental group is zero and hence by lifting criterion, every continuous map can be lifted, to a continuous map,

$$\lambda : I \times B_\epsilon(w) \to \mathbb{C}$$

such that

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z\mapsto e^z"] \\ I \times B_\epsilon(w) \arrow[ur, "\lambda"] \arrow[r, "\eta(t)-x"'] & \mathbb{C}^* \end{tikzcd}$$

If $\widehat{\eta_x}$ is a lift of η_x with respect to the map, $z \mapsto e^z$, and in terms of λ it's $\widehat{\eta_x} = \lambda(\cdot, x)$. Hence the index is given by,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta_x}(1) - \widehat{\eta_x}(0)] = \frac{1}{2\pi i}[\lambda(1, x) - \lambda(0, x)].$$

So, the map $x \mapsto n(\eta, x)$ is a continuous map of $B_\epsilon(w)$ to $\mathbb{C}$. $\square$

Now we need a way to compute the winding number exploiting the structure of complex numbers. We will use the structure of holomorphic functions to compute index of loops. To begin this, we construct a loop in $\mathcal{EH}$ using the loop η. For each point, z in we will associate a germ in $\mathcal{EH}$ and the compose with the loop η.

Theorem 1.2.

$$n(\eta, x) = \frac{1}{2\pi i}\int_\eta \frac{dz}{z - x}.$$

Proof

Consider the map

$$f_x(z) = 1/z - x.$$

This is the derivative of the logarithm function which when composed with the covering map $z \mapsto x + e^z$ yields identity. So, we can use this to construct lifts. Consider the function,

$$\nu : z \mapsto \mathrm{germ}_z\, f_x$$

The composition of this map with η gives us a loop in $\mathcal{EH}$.

$$\Gamma = \nu \circ \eta : I \to \mathcal{EH}.$$

Our goal is to lift this map using the covering space $d : \mathcal{EH} \to \mathcal{EH}$ and use the primitive to construct a lift of the loop η, and thus relate the integral along η of the function $1/z - x$ to index.

Let $\widehat{\Gamma}$ be the lift of Γ with respect to the covering map $d : \mathcal{EH} \to \mathcal{EH}$. This associates to each $t \in I$, the germ $\widehat{\Gamma}\eta(t) \in \mathcal{H}_{\eta(t)}$. Let F with domain $\widehat{U}_{\eta(t)}$ be the representative of the germ $\widehat{\Gamma}(\eta(t))$. By definition of the derivative map d we have,

$$F'(z) = 1/(z - x).$$

So, we have, $F'(z)(z - x) - 1 = 0$. Multiplying by $e^{-F(z)}$ we have, $\frac{d}{dz}\left[(z - x)e^{-F}\right] = (1 - F'(z)(z - x))e^{-F(z)} = 0$. So, the term $(z - x)e^{-F(z)}$ is locally constant. We will use this function to construct a lift of the loop η. Consider the valuation map,

$$\begin{aligned} \Xi :\ & I \longrightarrow \mathcal{EH} \longrightarrow \mathbb{C} \\ & t \mapsto \widehat{\Gamma}(t)(\eta(t)) \end{aligned}$$

which evaluates the germ $\widehat{\Gamma}(t)$ at $\eta(t)$. Since $(z - x)e^{-F(z)}$ is locally constant on $U_{\eta(t)}$ with value α, so, the map, $t \mapsto (\eta(t) - x)e^{-\Xi(t)}$ is constant because it's local constant and I is compact. Let c be such that $e^c = \alpha$. Consider the map,

$$\widehat{\eta} : t \mapsto \Xi(t) + c$$

We claim that this is a lift of η. This is a simple check and we have to verify that $x + e^{\Xi(t)+c} = \eta(t)$. $x + e^{\Xi(t)+c} = x + e^{\Xi(t)}e^c = x + \alpha e^{\Xi(t)} = x + (\eta(t) - x)e^{-\Xi(t)}e^{\Xi(t)} = x + \eta(t) - x = \eta(t)$, i.e., $p \circ \widehat{\eta} = \eta$ or $\widehat{\eta}$ is a lift of η. So,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)] = \frac{1}{2\pi i}\int_\eta \frac{1}{(z - x)}dz$$

as $\int_\eta f(z)dz = F(1)(\eta(1)) - F(0)(\eta(0))$ for F primitive of f along η. □

We can now start listing some immediate conseuences of this new relation between the line integral $\int_\eta \frac{dz}{z-x}$ and the winding number of the curve η with respect to x.

Lemma 1.3. *Let η_1, η_2 be two homotopic loops in $\mathbb{C}^*$, then, $n(\eta_1, 0) = n(\eta_2, x)$*

Proof

This is a simple application of the Cauchy's theorem for the map $z \mapsto (z)^{-1}$ which is holomorphic on $\mathbb{C}^*$. □

Lemma 1.4. *Let U be the unique connected component of $\mathbb{C}\backslash Im(\eta)$ that's unbounded, then, $n(\eta, x) = 0$ for all $x \in U$.*

Proof

Since for any loop η in a simply connected region U and $f \in \mathcal{H}U$, $\int_\eta f dz = 0$, we have,

$$\int_\eta \frac{dz}{z-x} = 0$$

for $x \in U$. Let $B_R(w)$ be the disc around w such that $Im(\eta)$ lies inside $B_R(w)$. Then for any $x \notin B_R(w)$, x lies 'outside' η. So, η can be collapsed in $B_R(w)$. So, $\int_\eta \frac{dz}{z-x} = 0$. Hence, $n(\eta, x) = 0$. Since it's constant on connected components, it must be zero for all of U. □

Lemma 1.5. *Let $\Omega \subset \mathbb{C}$ and $x, y \in \mathbb{C}\backslash\Omega$, then, there exists $f \in \mathcal{H}(\Omega)$ such that,*

$$e^{f(z)} = \frac{z-x}{z-y}, \quad z \in \Omega.$$

Sketch of Proof

Let $g(z) = \frac{z-x}{z-y}$, we will show that g'/g has a primitive and this will determine an $f \in \mathcal{H}(\Omega)$ with the required property.

$$g'/g = \frac{(x-y)/(z-y)^2}{(z-x)/(z-y)} = \frac{x-y}{(z-x)(z-y)} = \frac{1}{(z-x)} - \frac{1}{(z-y)}.$$

So, we have,

$$\int_\eta \left(\frac{1}{(z-x)} - \frac{1}{(z-y)}\right)dz = 2\pi i[n(\eta, x) - n(\eta, y)]$$

Since the winding number is constant on connected components, we have that $\int_\eta g'/g dz = 0$ for all η. Hence by Morera's theorem there exists a primitive h such that, $h' = g'/g$. Now we have,

$$e^{-h(z)}[-h'(z)g(z) - g'(z)] = 0$$

or,

$$g = \alpha e^h$$

Or, $g = e^{h+c}$ for some appropriate c. $f(z) = h(z) + c$ is the required function. □

Note that if the loop goes around a point x only once, the winding number will be 1. To prove this, we will have to create a small loop around the point and create a new loop, and reduce working with the harder outer loop to working with this small loop. We will however not prove this, as it's a bit irritating to track all the paths.

2 | Residue Theorem

Suppose $f \in \mathcal{H}(\Omega\backslash E)$ where E is a discrete set. Let the Laurent series expansion of f at x be $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$. The residue of f at $x \in E$ is defined to be,

$$\text{res}_f(x) = a_{-1}.$$

The principal part of f at x is defined to be,

$$g(z) = \sum_{-\infty}^{-1} a_n(z-a)^n.$$

The principal part is holomorphic on $\Omega\backslash\{x\}$. The residue has the nice property that,

$$f(z) - \frac{\operatorname{res}_f(x)}{(z-x)}$$

has a primitive on some small annulus around x, given by, $\sum_{n\neq -1} a_n(z-a)^{n+1}/(n+1)$. Conversely, whenever there is a holomorphic function g such that, $g'(z) = f(z) - \frac{\alpha}{(z-x)}$, we have that,

$$0 = \int_\eta g'(z)dz = \int_\eta f(z)dz - \underbrace{\int_\eta \frac{\alpha}{(z-x)}dz}_{2\pi i\alpha}.$$

Lemma 2.1. (Residue Theorem) *$\Omega \subset \mathbb{C}$, E be a discrete set in Ω. Let η be a loop in $\Omega\backslash E$ that's is homotopic to a point as a curve in Ω. Then for any $f \in \mathcal{H}(\Omega\backslash E)$, $\{x \in E : n(\eta, x) \neq 0\}$ is finite and,*

$$\frac{1}{2\pi i}\int_\eta f(z)dz = \sum_{x\in E} \operatorname{res}_f(x) n(\eta, x). \qquad \text{(Residue)}$$

Proof

Let $H : I \times I \to \Omega$ be the homotopy of η to the constant map. Let $K = H(I \times I)$, be the compact image of the homotopy. Since K is compact, $K \cap E$ is a finite set. If $x \notin K$, then η is homotopic to a point in $\mathbb{C}\backslash\{x\}$ and hence $n(\eta, x) = 0$. So $n(\eta, x) \neq 0$ for only finitely many points $x \in E$.

Let g_i be the principle part of f at $x_i \in K \cap E$. Then the function $f - \sum_i g_i$ is holomorphic on an open set U that contains the compact set K. So,

$$\int_\eta (f - \textstyle\sum_i g_i)dz = 0$$

or, $\int_\eta f dz = \int_\eta \sum_i g_i dz$. Now we will compute $\int_\eta \sum_i g_i dz$. Each $g_i(z) = \sum_{-\infty}^{-1} a_n(z-x_i)^n$. In this every term except $a_{-1}(z-x_i)^{-1}$ has a primitive, and hence the integral becomes,

$$\int_\eta g_i dz = 2\pi i a_{-1}\int_\eta \frac{dz}{z - x_i} = 2\pi i \operatorname{res}_f(x_i) n(\eta, x_i).$$

Hence we have,

$$\frac{1}{2\pi i}\int_\eta f(z)dz = \textstyle\sum_{x\in E} \operatorname{res}_f(x) n(\eta, x).$$

□

Let f be a meromorphic function on Ω. In a small neighborhood U, we can write $f(z)$ as,

$$f(z) = (z-x)^k g(z)$$

where $g \in \mathcal{H}(U)$. Where k is the order of the pole at x. Suppose $f(x) \neq 0$, the f'/f is a meromorphic function and, $\frac{f'(z)}{f(z)} = \frac{k}{z-x} + \frac{g'(z)}{g(z)}$. g'/g is holomorphic at x, and hence we have that

$$\operatorname{res}_{f'/f} x = \operatorname{ord}_x(f).$$

Lemma 2.2. (**Global Cauchy's Formula**) *$f \in \mathcal{H}(\Omega)$, $x \in \Omega \backslash \{x\}$, then*

$$n(\eta, x) f(x) = \frac{1}{2\pi i} \int_{\eta} \frac{f(z)}{z - x} dz.$$

Sketch of Proof

Since $f \in \mathcal{H}(\Omega)$, $g(z) = f(z)/(z - x) \in \mathcal{H}(\Omega \backslash \{x\})$ with residue, $\operatorname{res}_g x = f(x)$. Hence we have by Residue theorem,

$$n(\eta, x) f(x) = \frac{1}{2\pi i} \int_{\eta} \frac{f(z)}{z - x} dz.$$

$\square$.

Lemma 2.3. (**Argument Principle**) *f be meromorphic on Ω, Z_f be zeros of f. P_f be the poles of f. Then assuming the poles and zeros are not in the image of a loop η,*

$$n(f \circ \eta, 0) = \frac{1}{2\pi i} \int_{f \circ \eta} \frac{1}{z} dz = \frac{1}{2\pi i} \int_{\eta} \frac{f'}{f} dz = \sum_{x \in Z_f \cup P_f} \operatorname{ord}_x(f) n(\eta, x).$$

Sketch of Proof

This directly follows from the Residue theorem and the fact that, $\operatorname{res}_{f'/f} x = \operatorname{ord}_x f$. $\square$

A holomorphic function $f : \Omega \to \Omega'$ is called an analytic isomorphism of Ω onto Ω' if there exists a holomorphic function $g : \Omega' \to \Omega$ such that $g \circ f$ and $f \circ g$ are identity maps on Ω and Ω' respectively. If $\Omega = \Omega'$ it's called an analytic automorphism.

References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000