Part I

Grothendieck Topologies

These notes discuss sheaves on general categories with a Grothendieck topology, i.e., a category which carry a notion of intersection and coverings. The sheafification of pre-sheaves on Grothendieck topology is also studied. For the discussion using sieves see [1].

1 | Grothendieck Topology; Sites

The only topological properties we needed in defining a sheaf on a topological space X were that of intersection and covering. We can isolate these properties, reformulate them categorically. These operations can be generalized further and hence allowing generalization of the notion of a sheaf. To motivate the definition, consider a topological space X. We use the topology to construct a category $\mathcal{O}(X)$ in which the open sets are the objects, and the morphisms $V \to U$ are inclusion maps $V \subset U$. So, the hom sets, if non-empty contain only one element, the inclusion map between the open sets. The space X is the final object of the category $\mathcal{O}(X)$.

Let $\{U_i\}$ be a covering of an open set $U \in \mathcal{O}(X)$. This first means that $U_i \subseteq U$, i.e., there exists an inclusion of U_i to U. So, the covering can be thought of as a collection of morphisms,

$$\mathcal{U} = \{U_i \xrightarrow{i} U\}_{i \in \mathcal{I}} \tag{covering}$$

For any open set U the trivial inclusion $U \subseteq U$ must be a covering of U,

$$\mathcal{U} = \{U \subseteq U\} \tag{isomorphism}$$

If we had a covering $\{U_i\}$ of U, and for each U_i a covering $\{U_{ij}\}$, these smaller open sets together must cover U. So, in terms of morphisms, the first covering is the collection of inclusions, $\mathcal{U}_i = \{ij : U_{ij} \to U_i\}_{j \in \mathcal{I}_i}$, and the second covering is the collection of inclusions $\mathcal{U} = \{i : U_i \to U\}_{i \in \mathcal{I}}$. We obtain that the composition,

$$\widehat{\mathcal{U}} = \{U_{ij} \xrightarrow{i \circ ij} U\}_{(i,j) \in \prod_i i \times \mathcal{I}_i} \tag{locality}$$

must be a covering.

Now to we need generalize the notion of an interesection. Given two open sets, U_i and U_j of $\mathcal{O}(X)$ we are interested in describing the intersection categorically. The open sets U_i and U_j come equipped with an inclusion maps i, j, into the space X,

$$U_i \xrightarrow{i} X \xleftarrow{j} U_j,$$

and for the intersection, we have two more morphisms from the intersection, $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$, So this gives us the following commutative square,

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "i"] \\ U_j \arrow[r, "j"'] & X \end{tikzcd} . \tag{intersection}$$

Intersection is the largest set, containing elements of both U_i and U_j, this amounts to saying, the above square must be the universal square, i.e., $U_i \cap U_j$ must be the pullback of the maps $U_i \to X \leftarrow U_j$.

Given a covering $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ and the inclusion $V \subseteq U$, then we can use the intersection and the cover $\{U_i\}_{i\in\mathcal{I}}$ to construct a covering of V, by taking intersections $V \cap U_i$. Using the maps $U_i \xrightarrow{i} U \leftarrow V$, we can construct the pullback,

$$\begin{tikzcd} V \cap U_i \arrow[r, "\hat{i}"] \arrow[d] & V \arrow[d] \\ U_i \arrow[r, "i"'] & U \end{tikzcd} .$$

So, the covering of V can be constructed with the morphisms $\hat{i}$, is given by,

$$\mathcal{V} = \{U_i \cap V \xrightarrow{\hat{i}} V\}_{i\in\mathcal{I}} \tag{base change}$$

These are easily generalized to any category $\mathcal{C}$. A Grothendieck topology on a category is a map,

$$U \mapsto \mathrm{Cov}(U)$$

such that the covering of an object U in the category $\mathcal{C}$, the collection of inclusion maps, in covering, is replaced with a collection of morphisms $\{U_i \to U\}_{i\in\mathcal{I}}$, the trivial inclusion in isomorphism is replaced by an isomorphism of objects. Intersection of open sets, in intersection is replaced by the pullback or fibered product of two maps. An inclusion morphism, $V \to U$ in base change is replaced with the collection of pullback morphisms $\{V \prod U_i \to V\}$.

A category with a Grothendieck topology is called a site. Let $\mathcal{O}(X)$ be a category equipped with a Grothendieck topology $\mathcal{T}_X$. A pre-sheaf of sets on a site $\mathcal{O}(X)$ is a contravariant functor,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}\,.$$

For the sake of simplicity we will denote the maps $\mathcal{F}(i)$ by $|_{U_i}$ like in the case of pre-sheaves on topological spaces. The pre-sheaf $\mathcal{F}$ is called separated if given a covering $\{U_i \to U\}$, and two sections $f, g \in \mathcal{F}U$, whose pullbacks to each $\mathcal{F}U_i$ coincide, it follows that $f = g$. That is for all covers $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$,

$$\forall i,\ \ f|_{U_i} = g|_{U_i} \Rightarrow f = g \tag{separated}$$

Some examples of pre-sheaves that are not separated can be found here.

A pre-sheaf is a sheaf if the sections can be patched up. That's to say, given a covering $\{U_i \to U\}_{ß\in\mathcal{I}} \in \mathrm{Cov}(U)$ and a set of elements $f_i \in \mathcal{F}U_i$ which coincide on the 'intersection' i.e., if f_i and f_j coincide on the fibered product,

$$\begin{tikzcd} U_i \prod_{i,j} U_j \arrow[r] \arrow[d] & U_i \arrow[d, "i"] \\ U_j \arrow[r, "j"'] & U \end{tikzcd} .$$

That is to say if $f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j}$, then there must exist unique section $f \in \mathcal{F}U$ such that $f|_{U_i} = f_i$.

$$f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j} \Rightarrow \exists f \in \mathcal{F}U, f|_{U_i} = f_i.$$

This can be traslated into saying that there exists the following equalizer.

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \textstyle\prod_{i,j} U_j). \tag{collation}$$

where,

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

This is an equalizer similar to the case of sheaves on topological spaces. This is also called gluing or patching. The meaning of separated is that there is at most one way to patch up the pre-sheaf. Here the big $\prod$s are product of sets, and small $\prod_{i.j}$s are fibered products in the category $\mathcal{O}(X)$. In the classical case of a topological space we have $U_i \prod_{i,i} U_i = U_i \cap U_i = U_i$, so the two possible pullbacks from $U_i \prod_{i,i} U_i$ coincide; but if the map $U_i \to U$ is not injective, then the two projections $U_i \prod_{i,i} U_i \to U_i$ will be different.

2 | Sheafification Functor

Sheafification introduces extra structure on the pre-sheaf using the structure of the underlying topological space or site. In particular, the covers of any open set forms a category which is cofiltered. Sheafification utilizes this structure to construct a sheaf. This section will be informal, for a more formal discussion, see [1].

The category of sheaves $\mathrm{Sh}(X)$ over a topological space X is a full subcategory of the category of pre-sheaves $\mathrm{PSh}(X)$ over the topological space X,

$$\mathrm{Sh}(X) \rightarrowtail \mathrm{PSh}(X)$$

A sheaf is a contravariant functor, $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, together with the collation property. We have the forgetful functor from sheaves to presheaves that forgets about the patching properties of the sheaf. We can forget about the collation property, and this forgetful functor associates with the sheaf the underlying pre-sheaf. Denote this forgetful functor by,

$$\iota_X : \mathrm{Sh}(X) \to \mathrm{PSh}(X).$$

The left-adjoint to this functor is the sheafification functor,

$$^{\mathrm{Sh}} : \mathrm{PSh}(X) \coloneqq \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}} \to \mathrm{Sh}(X).$$

We want to generalize sheafification we did for the category of open sets on a topological space to sites. We should be careful, and all the steps in the process should be carefully motivated. Although we will still work withing the category of open sets corresponding to a topological space, we will do it in the more general language. We will then state this construction as a theorem without making things too formal.

The problem with pre-sheaves is that they might not be separated and even if they are separated, might not patch up, so the first step is to divide up the covers into smaller and smaller refinements, separate it out and manually patch it up. This is the same as bundling up stalks like we did for the case of topological spaces.

Separation

Let $U \in \mathcal{O}(X)$, and

$$\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$$

be a covering of U. By definition of a pre-sheaf we have the association, $U_i \mapsto \mathcal{F}U_i$. The set,

$$\mathrm{Eq}(\mathcal{U}, \mathcal{F}) = \{(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i \mid f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j}\},$$

consists of all tuples of 'functions' that can be patched up whenever there is an intersection, i.e., they must agree on the 'intersection' $U_i \prod_{i,j} U_j$. So we are throwing out all the functions that cannot be patched up. But now it might happen that things are patchable on refinements but not on bigger open sets. So we must refine.

To motivate refinement, consider the example of a topological space, X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of an open set $U \subset X$. A refinement of $\mathcal{V} = \{V_j\}_{j \in J}$ of the open cover $\mathcal{U}$ is a set of open subsets $V_j \subset X$ which is itself an open cover, and is such that each for each $j \in J$, there exists an $i \in I$ such that $V_j \subset U_i$. Hence there exists a map $f : J \to I$ such that,

$$\begin{tikzcd} V_j \arrow[rr, "f"] \arrow[dr] & & U_{f(j)} \arrow[dl] \\ & U & \end{tikzcd}.$$

Now, by going to refinements, the intersections are smaller, the elements of the pre-sheaves have 'less space' to disagree on. So, the refinements give us a map,

$$\mathrm{Eq}(\mathcal{U}, \mathcal{F}) \to \mathrm{Eq}(\mathcal{V}, \mathcal{F}).$$

By taking limit over such refinements, we include the 'essense' of all functions in the pre-sheaf $\mathcal{F}$ similar to how in case of topological space, the stalks of the pre-sheaf was the same as the stalks of the sheafification. So if $\mathrm{Cov}(U)$ is the category of all coverings of U, with refinements as morphisms, then we have a functor,

$$\begin{aligned} \mathrm{Eq}_{\mathcal{F}}U : \mathrm{Cov}(U)^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{U} &\mapsto \mathrm{Eq}(\mathcal{U}, \mathcal{F}). \end{aligned}$$

Since the category of pre-sheaves **Sets** has limits, we can take the limit of this functor. Now using this, we can associate to each pre-sheaf $\mathcal{F}$, a new pre-sheaf with,

$$\mathcal{F}^+U := \varinjlim_{\mathrm{Cov}(U)} \mathrm{Eq}_{\mathcal{F}}U$$

The construction so far can be stated as follows,

Lemma 2.1. *If $\mathcal{F}$ is a pre-sheaf, then $\mathcal{F}^+$ is a separated pre-sheaf.* □

Note here that if $\mathcal{F}$ is a sheaf then the equalisers already exist, i.e., we don't have to remove elements or take restrictions of elements that don't patch up. So in such a case $\mathcal{F}^+ = \mathcal{F}$.

Patch-up

Once we have a separated pre-sheaf, it means that there is only one way it can be patched up. Or the patch up is unique. So now we have to patch up $\mathcal{F}^+$.

use the functor $\mathcal{F}$ to construct a functor on the category of covers of U for any $U \subseteq X$. Firstly we already have for each $U_i \in \mathrm{Cov}(U)$ the association $\mathcal{F}U_i$ coming from the functor $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$.

So, for every inclusion $U_i \subseteq U_j$ we have to construct a new restriction map,

Theorem 2.2. *ι_X admits a left-adjoint $^{\mathrm{Sh}}$, i.e.,*

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(\mathcal{F}, \iota_X \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}).$$

[1]

Note that we didn't use any properties of the category $\mathcal{A}$ here, we only used the properties of the initial category to add structure.

[1] When we are working with general sites, the cover $\mathrm{Cov}(U)$ is cofiltered i.e., has a notion similar to directed set, and hence the adjoint functor, sheafification is possible. We will not discuss the filtered category or sheafification for general sites here.

References

[1] S Mac Lane, L Moerdijk, Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer, 1992

[2] B Fantechi, L Göttsche, L Illusie, S Kleiman, N Nitsure, A Vistoli, Fundamental Algebraic Geometry 2005