# PART I

# LOCALLY COMPACT GROUPS

## 1 | THE MODULAR FUNCTION

By a locally compact group we mean a topological group that is locally compact and Hausdorff. Let $\mathcal{G}$ be a locally compact group. Let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on $\mathcal{G}$. Let $C_c^+(\mathcal{G})$ denote the compactly supported continuous functions that are always non-negative. Note that $C_c(\mathcal{G})$ is the linear span of $C_c^+(\mathcal{G})$.

A left Haar measure on $\mathcal{G}$ is a non-zero Radon measure $\lambda$, such that for all $x \in \mathcal{G}$,

$$\lambda(xE) = \lambda(E).$$

where $E$ is a Borel subset of $\mathcal{G}$ and $x \in \mathcal{G}$. Radon measures are measures that respect the topology of the underlying space. In case of locally compact spaces the Riesz representation theorem tells us that Radon measures are in bijection with linear functionals on the space of compactly supported continuous functions, given by $f \mapsto \int f d\lambda$, so the invariance translated to this setting means,

$$\int L_x f d\lambda = \int f d\lambda.$$

where $(L_x f)(y) = f(x^{-1}y)$. Haar measures are very useful, and they allow much of the analysis that was possible with Lebesgue measures on $\mathbb{R}^n$.

**THEOREM 1.1.** [2] *Every locally compact group has a left Haar measure unique up to scaling.*

If $\mathcal{G}$ is non-commutative, the left Haar measures allows us to quantify the 'amount of non-commutativity'. For each $x \in \mathcal{G}$, define, $\lambda_x(E) = \lambda(Ex)$. This is again a left Haar measure, and by uniqueness of Haar measures, there exists a number $\Delta(x) > 0$ such that,

$$\lambda_x = \Delta(x)\lambda.$$

The number $\Delta(x)$ is independent of the choice of $\lambda$ by the uniqueness Haar measure. So, $x \mapsto \Delta(x)$ is a well defined function called the modular function. For any $x, y \in \mathcal{G}$, and $E \subset \mathcal{G}$, we have, $\lambda_{xy}(E) = \lambda(Exy) = \Delta(y)\Delta(x)\lambda(E)$, so,

$$xy \mapsto \Delta(x)\Delta(y),$$

or, $\Delta$ is a group homomorphism from the group $\mathcal{G}$ to the multiplicative group of positive numbers denoted by $\mathbb{R}_\times$. If $\chi_E$ is the characteristic function on $E$, then we have, $\chi_E(xy) = \chi_{Ey^{-1}}(x)$. So,

$$\int_\mathcal{G} \chi_E(xy)d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1})\int_\mathcal{G} \chi_E(x)d\lambda(x).$$

Since a general function $f \in L^1(\mathcal{G})$ can be approximated by simple functions, we have,

$$\int R_y f d\lambda = \Delta(y^{-1})\int f d\lambda.$$

or equivalently, this can be abbreviated as,

$$d\lambda(xy) = \Delta(y)d\lambda(x). \qquad\qquad \text{(right translation)}$$

Since the map $y \mapsto R_y f$ is uniformly continuous, the map $y \mapsto \int R_y f d\lambda$ is a continuous map. So, $\Delta$ is continuous. So, the homomorphism $\Delta$ also respects the topological structure.

To each left Haar measure, we can associate a right Haar measure, $\rho(E) = \lambda(E^{-1})$. The modular modular function relates the two measures. If $f \in \mathcal{C}_c(\mathcal{G})$, then we have,

$$\int_\mathcal{G} R_y f(x)\Delta(x^{-1})d\lambda(x) = \Delta(y)\int_\mathcal{G} f(xy)\Delta(xy^{-1})d\lambda(x) = \int_\mathcal{G} f(x)\Delta(x^{-1})d\lambda(x).$$

In the last step we used (right translation). Thus the functional $f \mapsto \int f(x)\Delta(x^{-1})d\lambda(x)$ is right invariant, and hence its associated Radon measure is a right Haar measure. Since $\rho$ is a right Haar measure it must vary from the Radon measure associated to this functional by a constant, say $\kappa$. On symmetric neighborhoods, the left and right Haar measures are equal. If the constant $\kappa$ is not 1, then by continuity of $\Delta$, we can choose a small enough symmetric neighborhood of $1 \in \mathcal{G}$ such that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|\kappa - 1|$ on that neighborhood.

$$|\kappa - 1|\lambda(U) = |\kappa\rho(U) - \lambda(U)| = \left|\int_U [\Delta(x^{-1}) - 1]d\lambda(x)\right| \leq \frac{1}{2}|\kappa - 1|\lambda(U).$$

which can only happen if $\kappa = 1$. This gives us,

$$d\rho(x) = \Delta(x^{-1})d\lambda(x)$$

This can be restated in a more convinient way as,

$$d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x). \qquad\qquad \text{(inversion)}$$

If $\Delta \equiv 1$, in such a case the left Haar measures do not measure any non-commutativity, such a locally compact group $\mathcal{G}$ is called unimodular. Clearly, abelian groups are unimodular. For discrete groups, every element of the group has same measure, and hence whether we left translate or right translate, it would still have the same measure. So, discrete groups are also unimodular. Since $\Delta$ is a continuous homomorphism, it takes compact groups to compact subgroups of $\mathbb{R}_\times$, which can only be $\{1\}$. So, compact groups are unimodular, and if $K \subset \mathcal{G}$ is a compact subgroup, then $\Delta|_K \equiv 1$.

Let $[\mathcal{G}, \mathcal{G}]$ be the smallest closed subgroup of $\mathcal{G}$ containing all elements of the form, $[x, y] = xyx^{-1}y^{-1}$, called the commutator subgroup of $\mathcal{G}$. It is a normal subgroups since, $z[x, y]z^{-1} = zxyx^{-1}y^{-1}z^{-1} = [zxz^{-1}, zyz^{-1}]$. Since $\Delta$ is a group homomorphism, we have,

$$\Delta([x, y]) = [\Delta(x), \Delta(y)]$$

since $\mathbb{R}_\times$ is abelian, we have $[\Delta(x), \Delta(y)] = 1$. So, by the isomorphism theorem of groups, the group homomorphism $\Delta : \mathcal{G} \to \mathbb{R}_\times$ must factor through $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$. This implies that if $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ is compact, then $\mathcal{G}$ is unimodular. This is intuitively obvious, the modular function was built to measure the non-commutativity and this just means that, $[\mathcal{G}, \mathcal{G}]$ does not contain any information about the non-commutativity contained in the group $\mathcal{G}$. Intuitively, the important thing to note about the modular function is that the left Haar measures can only measure non-commutativity of 'large' groups in some sense.

## 2 | Convolution & Involution of Functions

Given any two Radon measures $\mu, \nu$ on $\mathcal{G}$, we can define a functional,

$$\mu * \nu : f \mapsto \int_\mathcal{G} \int_\mathcal{G} f(xy) d\mu(x) d\mu(y).$$

for $f \in C_c(\mathcal{G})$. Clearly this is a linear map, and satisfies, $|(\mu * \nu)(f)| \leq \|f\|_{\sup} \|\mu\| \, \|\nu\|$, so it is a bounded linear functional, and hence defines a measure. This new measure is called the convolution of $\mu$ and $\nu$, we may write,

$$\int f d(\mu * \nu) = \int_\mathcal{G} \int_\mathcal{G} f(xy) d\mu(x) d\nu(y).$$

The order of the variables is important, the order of integration is not. If $\sigma$ is some other measure, Fubini's rule guarantees that convolution is an associative operation,

$$\begin{aligned}
\int f d[\mu * (\nu * \sigma)] &= \int_\mathcal{G} \int_\mathcal{G} f(xy) d\mu(x) d(\nu * \sigma)(y) \\
&= \int_\mathcal{G} \int_\mathcal{G} \int_\mathcal{G} f(xyz) d\mu(x) d\nu(y) d\sigma(z) \\
&= \int_\mathcal{G} \int_\mathcal{G} f(yz) d(\mu * \nu)(y) d\sigma(z) \\
&= \int f d[(\mu * \nu) * \sigma].
\end{aligned}$$

If $\mathcal{G}$ is abelian, $f(xy) = f(yx)$ and by definition, the convolution will be commutative. Conversely, consider the delta measures, $\int f d(\delta_x * \delta_y) = \int \int f(uv) d\delta_x(u) d\delta_y(v) = f(xy) = \int f d\delta_{xy}$. So, if convolutions are commutative, we would have $\delta_{xy} = \delta_x * \delta_y = \delta_y * \delta_x = \delta_{yx}$. Or, equivalently $xy = yx$. So, $\mathcal{G}$ is abelian if and only if convolutions commute.

Convolutions define a product structure on the Radon measures on $\mathcal{G}$. The estimate, $\|\mu * \nu\| \leq \|\mu\| \, \|\nu\|$ turns the collection of all Radon measures $\mathcal{M}(\mathcal{G})$ on $\mathcal{G}$ into a Banach algebra, called the measure algebra on $\mathcal{G}$. The delta measure at identity, $\delta_1$ determines the identity on this algebra.

Consider the operation on $\mathcal{M}(\mathcal{G})$ defined by,

$$\mu^*(E) = \overline{\mu(E^{-1})}.$$

If $\mu, \nu \in \mathcal{M}(\mathcal{G})$ then we have,

$$\begin{aligned}
\int f d(\mu * \nu)^* &= \int_\mathcal{G} f(x^{-1}) \overline{d(\mu * \nu)} = \int_\mathcal{G} \int_\mathcal{G} f((xy)^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) \\
&= \int_\mathcal{G} \int_\mathcal{G} f(y^{-1}x^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) = \int_\mathcal{G} \int_\mathcal{G} f(yx) d\mu^*(x) d\nu^*(y) = \int f d(\nu^* * \mu^*).
\end{aligned}$$

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So, $(\mu * \nu)^* = \nu^* * \mu^*$. The operation $\mu \mapsto \mu^*$ is an involution operation.

Given a left Haar measure $\lambda$, each function $f \in L^1(\mathcal{G})$ can be identified with the measure $f(x)d\lambda(x) \in \mathcal{M}(\mathcal{G})$. The convolution operation on $\mathcal{M}(\mathcal{G})$ gives a convolution operation on $L^1(\mathcal{G})$ given by,

$$f * g(x) = \int_{\mathcal{G}} f(y)g(y^{-1}x)d\lambda(y).$$

With some manipulation and using (right translation), this is the same as,

$$f * g(x) = \int_{\mathcal{G}} f(xy^{-1})g(y)\Delta(y^{-1})d\lambda(y). \qquad \text{(convolution)}$$

The involution on $\mathcal{M}(\mathcal{G})$, restricted to $L^1(\mathcal{G})$ is defined by the relation,

$$f^*(x)d\lambda(x) = \overline{f(x^{-1})}d\lambda(x^{-1}).$$

Which after some manipulations, and using (inversion), gives,

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \qquad \text{(involution)}$$

$L^1(\mathcal{G})$ with the convolution product and involution is a Banach algebra, called the $L^1$ group algebra of $\mathcal{G}$. These structures also can be defined for $L^p(\mathcal{G})$, for $p \in [1, \infty)$. $L^p(\mathcal{G})$ consists of functions such that $|f|^p$ is integrable. In such a case we set,

$$\|f\|_p = \left( \int_{\mathcal{G}} |f|^p d\lambda(x) \right)^{1/p}$$

## 3 | REPRESENTATIONS & GROUP ALGEBRA

The action of groups of interest in quantum theory are as unitary operators, so, the objects of interest to us are unitary representations of locally compact groups. The physically important topology on operator algebras is the weak operator topology. The weak topology and strong topology coincide on unitary operators, so either topology works fine for us. [1]

A unitary representation $(\mathcal{H}_\pi, \pi)$ of a locally compact group $\mathcal{G}$ is a strongly continuous group homomorphism

$$\pi : \mathcal{G} \to U(\mathcal{H}_\pi),$$

where $\mathcal{H}_\pi$ is some Hilbert space, and $U(\mathcal{H}_\pi)$ is the group of unitary operators on $\mathcal{H}_\pi$. The dimension of the Hilbert space $\mathcal{H}_\pi$ is called the degree of the representation $\pi$. When there is no confusion, the representation will just be denoted by $\pi$. If $\pi$ is an injection of $\mathcal{G}$ into $U(\mathcal{H}_\pi)$ it is called a faithful representation.

Let $(\mathcal{H}_\pi, \pi)$ be a representation of $\mathcal{G}$, a subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is said to be invariant under $\pi$ if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in \mathcal{G}$. If $\mathcal{M}$ is closed and $P_{\mathcal{M}}$ is the orthogonal projection onto the closed subspace $\mathcal{M}$ the invariance implies that,

$$P_{\mathcal{M}}\pi(x) = \pi(x)P_{\mathcal{M}},$$

for all $x \in \mathcal{G}$. We can define a new representation $(\mathcal{M}, \pi^{\mathcal{M}})$ by defining,

$$\pi^{\mathcal{M}}(x) = P_{\mathcal{M}}\pi(x)P_{\mathcal{M}},$$

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[1] Suppose the net of unitary operators $\{T_\alpha\}$ converge to $T$ strongly, then for any $|\varphi\rangle \in \mathcal{H}$, $\|(T_\alpha - T)\varphi\|^2 = \|T_\alpha\varphi\|^2 - 2\text{Re}\langle T_\alpha\varphi|T\varphi\rangle + \|T\varphi\|^2 = 2\|\varphi\|^2 - 2\text{Re}\langle T_\alpha\varphi|T\varphi\rangle$ So, the $\{T_\alpha\}$ converges to $T$ in the strong topology only if the last term converges to $2\|T\varphi\|^2$.

It is called a subrepresentation of $(\mathcal{H}_\pi, \pi)$. This procedure of going to subrepresentation gives a decomposition of $\pi$. If $\mathcal{M}$ is invariant under $\pi$ then so is $\mathcal{M}^\perp$. This gives us a second subrepresentation. Now the original Hilbert space $\mathcal{H}$ can be written as a direct sum, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and each operator $\pi(x)$ then decomposes as a direct sum $\pi(x) = \pi^{\mathcal{M}}(x) \oplus \pi^{\mathcal{M}^\perp}(x)$. So the representation can be written as $(\mathcal{H}_\pi, \pi) = (\mathcal{M}, \pi^{\mathcal{M}}) \oplus (\mathcal{M}^\perp, \pi^{\mathcal{M}^\perp})$.

Given a family of representations $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ of $\mathcal{G}$ the direct sum of the representations $\{\pi_\alpha\}$ is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form $\varphi = \{\varphi_\alpha\}_{\alpha \in I}$ such that $\lim_F [\sum_{\alpha \in F} \|\varphi_\alpha\|^2] < \infty$ where $F$ is a finite subset of $I$. The purpose of this definition is so that the norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha \in I} \pi_\alpha,$$

is called direct sum of representations $\{(\mathcal{H}_\alpha, \pi_\alpha)\}_{\alpha \in I}$, denoted by, $\sum_{\alpha \in I}^{\oplus} \{(\mathcal{H}_\alpha, \pi_\alpha)\}$. A representation is trivial if $\pi(x) = 1$ for every $x \in \mathcal{G}$. These are uninteresting representations. A representation can however have a trivial part.

A vector $\varphi$ in a Hilbert space $\mathcal{H}_\pi$ is called cyclic for $\mathcal{A}$ if $\{\pi(x)|\varphi\rangle\}_{x \in \mathcal{G}}$ is dense in $\mathcal{H}_\pi$. A cyclic representation of $\mathcal{G}$ is a triple $(\mathcal{H}, \pi, |\varphi\rangle)$ where $(\mathcal{H}_\pi, \pi)$ is a representation of $\mathcal{G}$ and $\varphi$ is a cyclic vector for $\pi(\mathcal{G})$.

Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of $\mathcal{A}$. Take a maximal family of nonzero vectors $|\{\Omega_\alpha\}\rangle_{\alpha \in I}$ in $\mathcal{H}$ such that,

$$\langle \pi(A)\Omega_\alpha | \pi(B)\Omega_\beta \rangle = 0,$$

for all $A, B \in \mathcal{A}$ and $\alpha \neq \beta$. Define, $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}}_{A \in \mathcal{A}}$. This is an invariant subspace of $\mathcal{H}$. Define $\pi_\alpha = P_{\mathcal{H}_\alpha} \pi P_{\mathcal{H}_\alpha}$ where $P_{\mathcal{H}_\alpha}$ is projection onto $\mathcal{H}_\alpha$. Then by construction the $\mathcal{H}_\alpha$ are mutually orthogonal and hence the representation $(\mathcal{H}, \pi)$ is of the form,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \{(\mathcal{H}_\alpha, \pi_\alpha)\}$$

By Zorn is lemma, every non-degenerate representation can be written as a direct sum of a family of cyclic subrepresentations. If it has no nontrivial closed invariant subspaces the representation $(\mathcal{H}_\pi, \pi)$ of $\mathcal{G}$ is called irreducible.

Given two representations $\pi_i$ and $\pi_j$ of $\mathcal{G}$, an intertwining operator for $\pi_i$ and $\pi_j$ is a bounded linear operator,

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

such that for every $x \in \mathcal{G}$, $T\pi_i(x) = \pi_j(x)T$. The collection of all intertwiners between $\pi_i$ and $\pi_j$ is denoted by $\mathcal{C}(\pi_i, \pi_j)$. If $\mathcal{C}(\pi_i, \pi_j)$ contains a unitary operator, the two representations are called unitarily equivalent.

The collection $\mathcal{C}(\pi)$ of bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(x)$ for all $x \in \mathcal{G}$ is called the commutant of $\pi$ is an algebra. Since product operation is weakly continuous, $\mathcal{C}(\pi)$ is weakly closed. Since $T^*\pi(x) = [\pi(x^{-1})T]^* = [T\pi(x^{-1})]^* = \pi(x)T^*$, it is also closed under taking adjoints. Thus $\mathcal{C}(\pi)$ is a von Neumann algebra.

If $\pi$ is reducible, $\mathcal{C}(\pi)$ contains non-trivial projections. Conversely, if $\mathcal{C}(\pi)$ contains bounded operators that are not multiples of identity, say $T$, then we can split it into the

canonical sum of two self-adjoint operators. Since these are linear combinations of the operators $T$ and $T^*$ which belong to $\mathcal{C}(\pi)$, they also belong to $\mathcal{C}(\pi)$. If an operator commutes with the self-adjoint operator, it also commutes with its spectral projections. So, $\pi(x)$ must also commute with the spectral projections of some self-adjoint operator. So, the subspace corresponding to the projection operator is an invariant subspace.

Let $\pi_i$ and $\pi_j$ be two unitary representations of $\mathcal{G}$. Then for every intertwining operator

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

for $\pi_i$ and $\pi_j$, the adjoint of the operator $T^* : \mathcal{H}_{\pi_j} \to \mathcal{H}_{\pi_i}$ is an intertwining operator for $\pi_j$ and $\pi_i$ because,

$$T^*\pi_j(x) = [\pi_j(x^{-1})T]^* = [T\pi_i(x^{-1})]^* = \pi_i(x)T^*.$$

So, $T^*T \in \mathcal{C}(\pi_i)$ and $TT^* \in \mathcal{C}(\pi_j)$. If $\pi_i$ and $\pi_j$ are two irreducible unitary representations of $\mathcal{G}$, then both of these should be a multiple of the identity, that is, $T^*T = \kappa I$ or equivalently, $\kappa^{-1/2}T$ must be a unitary operator or $T$ is zero. So, two irreducible representations $\pi_i$ and $\pi_j$ are equivalent if $\mathcal{C}(\pi_i, \pi_j)$ is one dimensional. These facts about irreducibility of representations are summarized in Schur's lemma, which we now state.

**THEOREM 3.1. (SCHUR'S LEMMA)** *If $\pi_i$ and $\pi_j$ are irreducible representations of $\mathcal{G}$ then either they are equivalent representations, or they have no intertwiners.*

If $\mathcal{G}$ is abelian, then for every representation $\pi$ of $\mathcal{G}$, the operators $\pi(x)$ commutes with all operators in $\pi(\mathcal{G})$, so $\pi(x) \in \mathcal{C}(\pi)$ for every $x \in \mathcal{G}$. If $\pi$ is irreducible, then we should have $\pi(x) = \kappa_x I$, so every irreducible representation of a locally compact abelian group is one dimensional. Although this looks very surprising, it is because spaces of operators on a vector space contain a lot more information than groups. This is an important corollary of Schur's lemma.

**COROLLARY 3.2.** *If $\mathcal{G}$ is abelian and $\pi$ is irreducible, then $\mathcal{H}_\pi \cong \mathbb{C}$.*

The unitary representations of the group $\mathcal{G}$ are closely related to the $*$-representations of the algebra of integrable functions $L^1(\mathcal{G})$. Once we have fixed a left Haar measure $\lambda$, let $f \in L^1(\mathcal{G})$, then for any unitary representation $\pi$ of the group $\mathcal{G}$, we can define an operator $\pi(f)$ on the Hilbert space $\mathcal{H}_\pi$ by,

$$\langle \pi(f)\varphi | \varkappa \rangle = \int_{\mathcal{G}} f(x) \langle \pi(x)\varphi | \varkappa \rangle d\lambda(x).$$

Or, more compactly,

$$\pi(f) = \int_{\mathcal{G}} f(x)\pi(x) d\lambda(x),$$

in the sense of the strong operator topology. This satisfies,

$$|\langle \pi(f)\varphi | \varkappa \rangle| \leq \|f\|_1 \|\varphi\| \|\varkappa\|.$$

So, $\pi(f)$ is a bounded linear operator on $\mathcal{H}_\pi$ with the norm $\|\pi(f)\| \leq \|f\|_1$. For any two $f, g \in L^1(\mathcal{G})$,

$$\pi(f * g) = \int_\mathcal{G} \int_\mathcal{G} f(y)g(y^{-1}x)\pi(x)d\lambda(y)d\lambda(x) = \int_\mathcal{G} \int_\mathcal{G} f(y)g(x)\pi(yx)d\lambda(x)d\lambda(y)$$
$$= \int_\mathcal{G} \int_\mathcal{G} f(y)g(x)\pi(y)\pi(x)d\lambda(x)d\lambda(y) = \pi(f)\pi(g).$$

So, we obtain an algebra homomorphism. Moreover,

$$\pi(f^*) = \int_\mathcal{G} \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)d\lambda(x) = \int_\mathcal{G} \overline{f(x)}\pi(x^{-1})d\lambda(x) = \int_\mathcal{G} [f(x)\pi(x)]^* d\lambda(x) = \pi(f)^*,$$

hence, $\pi$ is a $*$-homomorphism. The left translation by $x$ of the function $f$ is given by,

$$\pi(L_x f) = \int_\mathcal{G} f(x^{-1}y)\pi(y)d\lambda(y) = \int_\mathcal{G} f(y)\pi(xy)d\lambda(y)$$
$$= \int_\mathcal{G} f(y)\pi(x)\pi(y)d\lambda(y) = \pi(x)\pi(f).$$

That is,
$$\pi(L_x f) = \pi(x)\pi(f)$$

Similarly, for the right translation,

$$\Delta(x^{-1})\pi(R_{x^{-1}}f) = \Delta(x^{-1}) \int_\mathcal{G} f(yx^{-1})\pi(y)d\lambda(y) = \int_\mathcal{G} f(y)\pi(xy)d\lambda(y)$$
$$= \int_\mathcal{G} f(y)\pi(y)\pi(x)d\lambda(y) = \pi(f)\pi(x).$$

That is,
$$\Delta(x^{-1})\pi(R_{x^{-1}}f) = \pi(f)\pi(x).$$

This will be reversed if we had chosen a right Haar measure to define integration. So, the map $f \mapsto \pi(f)$ is a $*$-homomorphism from $L^1(\mathcal{G})$ to $\mathcal{B}(\mathcal{H}_\pi)$. Since $\pi$ is strongly continuous, we can find a neighborhood $V$ of 1, such that for all $x \in V$,

$$\|\pi(x)\varphi - \varphi\| < \|\varphi\|.$$

Now consider the function, $f = |V|^{-1}\chi_V$, we get,

$$\|\pi(f)\varphi - \varphi\| = |V|^{-1}\left\|\int_V [\pi(x)\varphi - \varphi]d\lambda(x)\right\| < \|\varphi\|$$

So, $\|\pi(x)\varphi\| \neq 0$, or $\pi(x)\varphi \neq 0$. Hence,

$$\pi : f \mapsto \pi(f)$$

is a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on $\mathcal{H}_\pi$. So, a representation of the locally compact group $\mathcal{G}$ gives rise to a representation of integrable functions on the same Hilbert space. Conversely, a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on a Hilbert space $\mathcal{H}_\pi$ gives rise to a unique unitary representation of $\mathcal{G}$. The idea is to use the functions $f$ approaching the delta function at $x$, to get an operator for $\pi(x)$. See §3.2, [2].

**THEOREM 3.3.** *The von Neumann algebras generated by $\pi(\mathcal{G})$ and $\pi(L^1(\mathcal{G}))$ are identical.*

SKETCH OF PROOF

The idea is to approximate functions in $L^1(\mathcal{G})$ by compactly supported functions, and then approximate the representatives of these compactly supported functions using the representatives of the group elements.

If $g \in \mathcal{C}_c(\mathcal{G})$, we can divide up the support of $g$ into a finite partition, say $E = \{E_i\}$. Now we can approximate the function $g$, using the value $g$ at some point $x_i \in E_i$, by,

$$\Sigma_E = \sum_i g(x_i)\pi(x_i)|E_i|$$

Given any $\epsilon > 0$, by compactness of the support of $g$, and continuity of the map $x \mapsto g(x)\pi(x)\varphi$, for each $\varphi$, there exists a partition $E = \{E_i\}$ of the support of $g$ such that, $\|g(x)\pi(x)\varphi - g(y)\pi(y)\varphi\| < \epsilon$, whenever $x, y \in E_j$ for some $j$. Thus,

$$\|\Sigma_E\varphi - \pi(g)\varphi\| < \epsilon|\cup_i E_i|.$$

Thus every neighborhood of $\pi(g)$ with respect to the strong topology, contains sums $\Sigma_E$.

Now every function $f \in L^1(\mathcal{G})$ is the $L^1$ limit of functions in $\mathcal{C}_c(\mathcal{G})$. So, by continuity of $\pi$, $\pi(f)$ is a norm limit of operators in $\pi(\mathcal{C}_c(\mathcal{G}))$. So, the von Neumann algebra generated by $\pi(L^1(\mathcal{G}))$ is contained in the von Neumann algebra generated by $\pi(\mathcal{G})$. Conversely, each $\pi(x)$ is the strong limit of $\pi(L_x\phi_U)$, as $U \to \{1\}$. Where $\phi_U$ is an 'approximate identity'. See [2] §2.5 for approximate identities, and §3.2 for a complete proof. $\square$

## 3.1 | FUNCTIONS OF POSITIVE TYPE

In the $C^*$ algebra case representations and states are closely related to each other, we need to find analogues of states for group algebras. These are provided by functions of positive type. We can do similar constructions with group algebras that we did with $C^*$-algebras.

Let $L^\infty(\mathcal{G})$ denote the collection of all classes of bounded measurable functions on $\mathcal{G}$. With the sup-norm, $\|f\|_\infty := \sup_{x \in \mathcal{G}}|f(x)|$, $L^\infty(\mathcal{G})$ is a Banach space.

Every function $\omega \in L^\infty(\mathcal{G})$ defines a linear functional on $L^1(\mathcal{G})$ by

$$f \mapsto \widehat{\omega}(f) := \int_{\mathcal{G}} f(x)\omega(x)d\lambda(x)$$

This is bounded because,

$$|\widehat{\omega}(f)| = \left|\int_{\mathcal{G}} f(x)\omega(x)d\lambda(x)\right| \leq \|\omega\|_\infty \int_{\mathcal{G}}\left|f(x)\right|d\lambda(x) = \|\omega\|_\infty\|f\|_1.$$

So, $\widehat{\omega}$ is a bounded linear functional on the group algebra $L^1(\mathcal{G})$. A function $\omega \in L^\infty(\mathcal{G})$ is said to be a function of positive type if for any function $f \in L^1(\mathcal{G})$,

$$\widehat{\omega}(f^* * f) = \int (f^* * f)\omega \geq 0.$$

Expanding the integral, by (convolution) and (involution), we have,

$$\int (f^* * f)\omega = \int_{\mathcal{G}}\int_{\mathcal{G}} \Delta(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\omega(x)d\lambda(y)d\lambda(x)$$

$$= \int_{\mathcal{G}}\int_{\mathcal{G}} \overline{f(y)}f(yx)\omega(x)d\lambda(x)d\lambda(y) = \int_{\mathcal{G}}\int_{\mathcal{G}} f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

We can restate that a function $\omega \in L^\infty(\mathcal{G})$ is of positive type if,

$$\omega(f^* * f) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y) \geq 0 \qquad \text{(positive type)}$$

Let $\omega$ be a function of positive type, consider the function complex conjugate, $\overline{\omega}$,

$$\int (f^* * f)\overline{\omega} = \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{f(y)}f(yx)\overline{\omega(x)}d\lambda(y)d\lambda(x) = \overline{\int [(\overline{f})^* * \overline{f}]\omega} \geq 0$$

So, if $\omega$ is a function of positive type, then so is $\overline{\omega}$. The set of all continuous functions of positive type on $\mathcal{G}$ is denoted by $\mathcal{S}(\mathcal{G})$.

Given a representation $\pi$ of $\mathcal{G}$, every vector $\varphi \in \mathcal{H}_\pi$ defines a function on the group,

$$\Phi(x) = \langle \pi(x)\varphi|\varphi \rangle.$$

By the continuity requirements of the representation, this is a continuous function on $\mathcal{G}$. $\Phi(y^{-1}x) = \langle \pi(y^{-1}x)\varphi|\varphi \rangle = \langle \pi(y^{-1})\pi(x)\varphi|\varphi \rangle = \langle \pi(x)\varphi|\pi(y)\varphi \rangle$. So for every $f \in L^1(\mathcal{G})$,

$$\widehat{\Phi}(f^* * f) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)\overline{f(y)}\Phi(y^{-1}x)d\lambda(x)d\lambda(y)$$

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} \langle f(x)\pi(x)\varphi|f(y)\pi(y)\varphi \rangle d\lambda(x)d\lambda(y) = \|\pi(f)\varphi\| \geq 0,$$

i.e., $\Phi \in \mathcal{S}(\mathcal{G})$. Consider the space $L^2(\mathcal{G})$ of all square integrable functions on $\mathcal{G}$, consisting of functions $g$ such that $\int |g|^2 d\lambda < \infty$. Equipped with the inner product,

$$\langle h, g \rangle = \int_{\mathcal{G}} \overline{h(x)}g(x)d\lambda(x),$$

$L^2(\mathcal{G})$ is a Hilbert space. On this space, we have a representation of the group $\mathcal{G}$, given by,

$$[\pi_L(x)g](y) = L_x g(y) = g(x^{-1}y).$$

The left invariance of the left Haar measure ensures that this is a unitary representation. It is called the left regular representation of $\mathcal{G}$ on $L^2(\mathcal{G})$. The right regular representation is similarly defined. For any $g \in L^2(\mathcal{G})$, let $\tilde{g}(x) = \overline{g(x^{-1})}$, then we have,

$$\langle \pi_L(x)g|g \rangle = \int_{\mathcal{G}} g(x^{-1}y)\overline{g(y)}d\lambda(y) = \overline{g * \tilde{g}(x)}.$$

Since every function of the form $\langle \pi(x)\varphi|\varphi \rangle$ arising from a representation $\pi$ is a function of positive type, the above defined function is one too. Since conjugates of functions of positive type are also functions of positive type, we have,

$$g * \tilde{g} \in \mathcal{S}(\mathcal{G}).$$

In particular, $\mathcal{S}(\mathcal{G})$ is non-empty. So, for each $g \in L^2(\mathcal{G})$, $g * \tilde{g}$ defines a function of positive type. In particular, functions of positive type do indeed exist.

### 3.1.1 | CONSTRUCTION OF REPRESENTATIONS

Every unitary representations gives rise to functions of positive type. The converse is also true, that is every function of positive type arises from a unitary representation. The key ingredient to proving this is to use functions of positive type to construct Hilbert spaces, much like GNS construction in the case of $C^*$-algebras.

The goal is to define an inner product using the function of positive type on $L^1(\mathcal{G})$ and consider the completion, similar to how it is done in the GNS construction. Given a function $\omega$ of positive type, define a hermitian form on $L^1(\mathcal{G})$ by,

$$\langle f, g \rangle_\omega = \int (g^* * f)\omega = \int_\mathcal{G} \int_\mathcal{G} f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

By Fubini's is theorem this satisfies,

$$|\langle f, g \rangle_\omega| \leq \|\omega\|_\infty \|f\|_1 \|g\|_1. \tag{bounded}$$

The Hermitian form is positive semi-definite since $\omega$ satifies (positive type). So, $\langle, \rangle_\omega$ defines a positive definite or semi-definite Hermitian form on $L^1(\mathcal{G})$. It satisfies the Cauchy-Schwarz inequality,

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle_\omega \langle g, g \rangle_\omega.$$

Consider the ideal $\mathcal{J}_\omega = \{f \mid \langle f, f \rangle_\omega = 0\}$. By the above inequality, $f \in \mathcal{J}_\omega$ if and only if $\langle f, g \rangle_\omega = 0$ for all $g \in L^1(\mathcal{G})$. So, $\langle, \rangle_\omega$ defines an inner product on the quotient space $L^1(\mathcal{G})/\mathcal{J}_\omega$. The image of $f \in L^1(\mathcal{G})$ in $L^1(\mathcal{G})/\mathcal{J}_\omega$ is denoted by $[f]$. The inner product on $L^1(\mathcal{G})/\mathcal{J}_\omega$ is given by,

$$\langle [f], [g] \rangle = \langle f, g \rangle_\omega$$

Denote the completion of this inner product space by $\mathcal{H}_\omega$. We have a natural action of the group $\mathcal{G}$ on the Hilbert space $\mathcal{H}_\omega$ given by the left translation.

$$\pi_\omega(x)[f] = [L_x f].$$

This action is a unitary operator, because

$$\langle L_x f, L_x g \rangle_\omega = \int_\mathcal{G} \int_\mathcal{G} f(x^{-1}y)\overline{g(x^{-1}z)}\omega(z^{-1}y)d\lambda(y)d\lambda(z)$$

$$= \int_\mathcal{G} \int_\mathcal{G} f(y)\overline{g(z)}\omega((xz)^{-1}(xy))d\lambda(y)d\lambda(z) = \langle f, g \rangle_\omega.$$

In the last step, we used the left invariance of the Haar measure $\lambda$. This also tells us that the action keeps $\mathcal{J}_\omega$ invariant, and hence $L_x$ gives rise to a unitary operator on $\mathcal{H}_\omega$. We also have an action of $L^1(\mathcal{G})$ on $\mathcal{H}_\omega$ given by,

$$\pi_\omega(g)[f] = [g * f].$$

**THEOREM 3.4.** *There exists $\Omega \in \mathcal{H}_\omega$ such that for all $f \in L^1(\mathcal{G})$, $\pi_\omega(f)|\Omega\rangle = [f]$, and $\omega(x) = \langle \pi_\omega(x)\Omega, \Omega \rangle$ almost everywhere.*

### IDEA OF PROOF

We want to think of $\omega$ as a limit of $L^1(\mathcal{G})$ functions. The idea is that if $\psi \in L^1(\mathcal{G})$ and $\omega \in L^\infty(\mathcal{G})$ then $\psi * \omega \in L^1(\mathcal{G})$[2] and consider the product of $\omega$ with an approximate identity to obtain the required sequence. $\widehat{f_i}(f) \to \widehat{\omega}(f)$ because $\psi_{U_i} * \omega \to \omega$ in appropriate topologies. Now all we are left with is some routine checks, which can be found in §3.3 [2]. □

The above theorem says that the information about the action of $\omega$ is determined by its infinitesimal behavior around the identity and $\Omega = [\{\psi_{U_i}\}]$. So, every function of positive type agrees with continuous functions locally almost everywhere. So, there is a correspondence between cyclic representations and continuous functions of positive type.

Given a function of positive type $\omega \in \mathcal{S}(\mathcal{G})$, there exists some representation $\pi_\omega$ such that

$$|\omega(x)| = |\langle \pi_\omega(x)\Omega, \Omega \rangle| \leq \|\Omega\|^2 = \omega(1)$$

Similarly,

$$\omega(x^{-1}) = \langle \pi_\omega(x^{-1})\Omega, \Omega \rangle = \langle \Omega, \pi_\omega(x)\Omega \rangle = \overline{\omega(x)}$$

Suppose $\pi_i$ and $\pi_j$ are two cyclic representations of $\mathcal{G}$ with cyclic vectors $\Omega_i$ and $\Omega_j$ and such that $\langle \pi_i(x)\Omega_i, \Omega_i \rangle = \omega(x) = \langle \pi_j(x)\Omega_j, \Omega_j \rangle$, then it follows that $T\pi_i(x)\Omega_i = \pi_j(x)\Omega_j$ extends to a linear isometry on the spans, and by continuity to a unitary map between the Hilbert spaces $\mathcal{H}_{\pi_i}$ and $\mathcal{H}_{\pi_j}$. So, $\pi_i(x)T = T\pi_j(x)$ for all $x \in \mathcal{G}$. So, any two cyclic representations are unitarily equivalent. So, every cyclic representation $\pi$ such that $\omega(x) = \langle \pi(x)\Omega, \Omega \rangle$ is unitarily equivalent to the one we constructed above.

### 3.1.2 | STRUCTURE OF $\mathcal{S}(\mathcal{G})$

Let $\mathcal{S}(\mathcal{G})$ be the set of all continuous functions of positive type, then by linearity of integration, we have for every finite set $\{\omega_i\}_{i \in \mathcal{I}} \subset \mathcal{S}(\mathcal{G})$ and $\lambda_i \in [0,1]$ with $\sum_i \lambda_i = 1$,

$$\int (f^* * f)\left(\sum_i \lambda_i \omega_i\right) = \sum_{i \in \mathcal{I}} \lambda_i \int (f^* * f)\omega_i \geq 0.$$

for all $f \in L^1(\mathcal{G})$. So, the set $\mathcal{S}(\mathcal{G})$ is a convex set, and since scaling of functions of positive type by positive numbers also gives a function of positive type it is a convex cone. Some special subsets of $\mathcal{S}(\mathcal{G})$ can be singled out,

$$\mathcal{S}_1 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty = 1\}, \quad \mathcal{S}_0 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty \leq 1\}.$$

Let $\mathcal{E}(\mathcal{S}_i)$ be the extreme points of $\mathcal{S}_i$.

**THEOREM 3.5.** *Let $\omega \in \mathcal{S}(\mathcal{G})$, then $\pi_\omega$ is irreducible iff $\omega \in \mathcal{E}(\mathcal{S}_1)$.*

### PROOF

Suppose $\pi_\omega$ is a reducible representation of $\mathcal{G}$, say, $\mathcal{H}_\omega = \mathcal{M} \oplus \mathcal{M}^\perp$ where $\mathcal{M}$ is nontrivial and invariant under $\pi_\omega$. Let $\Omega$ be a cyclic vector for the representation $\pi_\omega$, then $\Omega$ cannot belong to either $\mathcal{M}$ or $\mathcal{M}^\perp$, because otherwise $\pi_\omega(\mathcal{G})\Omega$ can only belong to either $\mathcal{M}$ or $\mathcal{M}^\perp$. But, if $\Omega$ is a cyclic vector, then $\pi_\omega(\mathcal{G})\Omega$ must be dense in $\mathcal{H}_\omega$.

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[2]this is where almost everywhere is used.

So, $\Omega = \alpha\varphi + \beta\varkappa$ where $\varphi$ and $\varkappa$ are unit vectors in $\mathcal{M}$ and $\mathcal{M}^\perp$ respectively. Let $\omega_\varphi(x) = \langle \pi_\omega(x)\varphi|\varphi\rangle$ and $\omega_\varkappa(x) = \langle \pi_\omega(x)\varkappa|\varkappa\rangle$. Then $\omega_\varphi, \omega_\varkappa \in \mathcal{S}(\mathcal{G})$. We have

$$\omega_\varphi(x) - \omega_\varkappa(x) = \langle \pi_\omega(x)\varphi|\varphi\rangle - \langle \pi_\omega(x)\varkappa|\varkappa\rangle$$
$$= \langle \pi_\omega(x)\Omega|\alpha^{-1}\varphi - \beta^{-1}\varkappa\rangle,$$

as can be seen by standard calculations using $\Omega = \alpha\varphi + \beta\varkappa$ and $\langle\varphi,\varkappa\rangle = 0$. Since $\Omega$ is cyclic, and $\alpha^{-1}\varphi - \beta^{-1}\varkappa$ is non-zero, there must exist some $x \in \mathcal{G}$ such that $\langle \pi_\omega(x)\Omega|\alpha^{-1}\varphi - \beta^{-1}\varkappa\rangle$ is non-zero. So, in particular for some $x \in \mathcal{G}$,

$$\omega_\varphi(x) - \omega_\varkappa(x) \neq 0,$$

in particular $\omega_\varphi \neq \omega_\varkappa$ as continuous functions on $\mathcal{G}$. Since $\omega_\varphi(1) = \omega_\varkappa(1) = 1$ they are linearly independent, because if not, we would have $\omega_\varphi = \delta\omega_\varkappa$, in which case we should have $\omega_\varphi(1) = \delta\omega_\varkappa(1) = \delta$, hence $\omega_\varphi = \omega_\varkappa$. We arrive at

$$\omega(x) = \langle \pi_\omega(x)\Omega|\Omega\rangle = \alpha^2\langle \pi_\omega\varphi|\varphi\rangle + \beta^2\langle \pi_\omega\varkappa|\varkappa\rangle = \alpha^2\omega_\varphi(x) + \beta^2\omega_\varkappa(x).$$

therefore $\omega$ is not an extreme point in $\mathcal{S}(\mathcal{G})$.

Suppose $\pi_\omega$ is irreducible, and suppose $\omega$ is not an extreme point of $\mathcal{S}(\mathcal{G})$. Then there exist $\omega_\varphi$ and $\omega_\varkappa$ in $\mathcal{S}(\mathcal{G})$ such that $\omega = \omega_\varphi + \omega_\varkappa$. For any $f, g \in L^1(\mathcal{G})$, we have,

$$\langle f, f\rangle_{\omega_\varphi} = \langle f, f\rangle_\omega - \langle f, f\rangle_{\omega_\varkappa} \leq \langle f, f\rangle_\omega.$$

So,
$$|\langle f, g\rangle_{\omega_\varphi}| \leq \langle f, f\rangle_{\omega_\varphi}^{1/2}\langle g, g\rangle_{\omega_\varphi}^{1/2} \leq \langle f, f\rangle_\omega^{1/2}\langle g, g\rangle_\omega^{1/2}.$$

This gives us a bounded Hermitian form on $\mathcal{H}_\omega$, $(f, g) \to \langle f, g\rangle_{\omega_\varphi}$. So by the Riesz lemma, there exists a bounded self-adjoint operator $T$ on $\mathcal{H}_\omega$ such that,

$$\langle f, g\rangle_{\omega_\varphi} = \langle T[f], [g]\rangle_\omega.$$

By the left action of $\mathcal{G}$ on $L^1(\mathcal{G})$, we have,

$$\langle T\pi_\omega(x)[f], [g]\rangle_\omega = \langle T[L_x f], [g]\rangle_\omega = \langle L_x f, g\rangle_{\omega_\varphi} = \langle f, L_{x^{-1}}g\rangle_{\omega_\varphi}$$
$$= \langle T[f], \pi(x^{-1})[g]\rangle_{\omega_\varphi} = \langle \pi_\omega(x)T[f], [g]\rangle_\omega.$$

So, $T \in \mathcal{C}(\pi_\omega)$. By Schur's lemma, this means that $T = c\mathbb{I}$ or that $\langle f, g\rangle_{\omega_\varphi} = c\langle f, g\rangle_\omega$. Since

$$\langle f, g\rangle_\omega = \int (g^* * f)\omega = \int_\mathcal{G}\int_\mathcal{G} f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y),$$

we must have $\omega_\varphi = c\omega$, or that $\omega$ is an extreme point of $\mathcal{S}(\mathcal{G})$. $\qquad\square$

If $\omega_i$ is a net of functions of positive types that converges weakly to $\omega$, then the weak* limit $\omega$ is also a function of positive type.[3] So, $\mathcal{S}_0$ is the weak* closed unit ball in $L^\infty(\mathcal{G})$. Now, by Alaoglu is theorem, $\mathcal{S}_0$ is weak* compact and by the Krein-Milman theorem it is the weak* closure of its extreme points.

The constant zero-function, which maps every element of the group to zero is a function of positive type by definition. Since $\|\omega\|_\infty = \omega(1)$, we have that $\|0\|_\infty = \omega(1) = 0$. This

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[3] recall that weak* limit implies that $\widehat{\omega}_i(f) \to \widehat{\omega}(f)$ for all $f$.

implies that the constant zero-function cannot be a mixture of other functions of positive type. Suppose $\omega$ be an extreme point in $\mathcal{S}_1$, and suppose it is a mixture of two other functions of positive type, that is, $\omega = \mu_i \omega_i + \mu_j \omega_j$ where $\mu_i + \mu_j = 1$. Then $\omega(1) = \mu_i \omega_i(1) + \mu_j \omega_j(1)$, this can only happen if $\omega_i(1) = \omega_j(1) = 1$. Which means that if $\omega \in \mathcal{S}_1$, then it can only be a mixture of other $\omega_i$'s in $\mathcal{S}_1$. Let $\omega \in \mathcal{S}(\mathcal{G})$, then we can construct the state $\omega/\omega(1) \in \mathcal{S}_1$. So, we can always write $\omega$ as a mixture,

$$\omega = (1 - \omega(1))0 + (\omega(1))\omega/\omega(1),$$

therefore we ahve shown

$$\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}.$$

**THEOREM 3.6.** *The convex hull of $\mathcal{E}(\mathcal{S}_1)$ is weak\* dense in $\mathcal{S}_1$.*

## PROOF

We have to show that every function of positive type, $\omega$ of norm 1, is a weak\* limit of mixtures of functions in $\mathcal{E}(\mathcal{S}_1)$. Note that by Alaoglu's theorem $\mathcal{S}_0$ is a compact set in the weak\* topology, in particular it is weak\* closed by the Krein-Milman theorem. There exists a net of functions of positive type $\{\omega_i\}$ in the convex hull of $\mathcal{E}(\mathcal{S}_0)$ such that $\omega_i \to \omega$ weakly. We must ensure that if $\omega \in \mathcal{S}_1$ then these $\omega_i$'s can themselves be described as mixtures $\omega_i = \sum_j \mu_j \psi_j$ such that each $\psi_j \in \mathcal{S}_1$.

Since $\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}$, we can write each $\omega_i$ as $\mu_1 \psi_1 + \cdots + \mu_{n_i} \psi_{n_i} + \mu_{n_i+1} 0$ where each $\psi_j \in \mathcal{E}(\mathcal{S}_1)$. We can now remove the 0 component of the mixture. We can consider the new net,

$$\omega_i' = \frac{1}{\omega_i(1)} \sum_{j=1}^{n_i} \mu_j \psi_j$$

Each of these $\omega_i' \in \mathcal{S}_1$, by construction, and hence $\omega_i'$ is in the convex hull of $\mathcal{E}(\mathcal{S}_1)$ and $\omega = \lim_i \omega_i'$. $\qquad\square$

If $g \in \mathcal{C}_c(\mathcal{G})$ then $g \in L^2(\mathcal{G})$ and hence $g * \tilde{g} \in \mathcal{S}(\mathcal{G})$. Now we can apply polarization identity, to say that for any $f, g \in \mathcal{C}_c(\mathcal{G})$,

$$f * \tilde{g} \in \mathcal{S}(\mathcal{G})$$

Using approximate identities it can be showed that $\{f * g \mid f, g \in \mathcal{C}_c(\mathcal{G})\}$ is dense in $\mathcal{C}_c(\mathcal{G})$. So, the linear span of $\mathcal{C}_c(\mathcal{G}) \cap \mathcal{S}(\mathcal{G})$ contains all functions of the form $f * g$ for $f, g \in \mathcal{C}_c(\mathcal{G})$. $[\mathcal{C}_c(\mathcal{G}) \cap \mathcal{S}(\mathcal{G})]$ is dense in $\mathcal{C}_c(\mathcal{G})$ in the uniform norm. So, every compactly supported function can be approximated by a functions of positive type that are compactly supported.

**THEOREM 3.7. (GELFAND-RAIKOV)** *Irreducible representations of $\mathcal{G}$ separate the points on $\mathcal{G}$.*

## IDEA OF PROOF

We have to show that for any $x \neq y$ there exists an irreducible representation $\pi$ such that $\pi(x) \neq \pi(y)$. The idea is to use the fact that $\mathcal{G}$ is locally compact and Hausdorff to find a compactly supported function. One can choose a function such that the support does not include one of the points. This function can be written as a linear combination of functions

of positive type. This linear combination can again be written as a convex combination of extreme points. Since these extreme points correspond to irreducible representations we would have the required irreducible representation that separates points on $\mathcal{G}$. The full proof requires a careful discussion regarding topology of compact convergence and it's equivalence with the weak$^*$ topology on $\mathcal{E}(\mathcal{S}_1)$. This can be found in [2].

Existence of Haar measures on locally compact groups is a very important tool for analysis of locally compact groups. We need to construct a functional on the space of compactly supported continuous functions that is translation invariant. All that we have to work with is the set of continuous functions on the locally commpact group. Compactly supported functions are well behaved, so, we can try to exploit their nice properties.

Let $f, g$ be two compactly supported positive functions. Note that the support of each of these functions can be covered by finitely many open sets. We can relate the two functions using this. We can dominate the function $f$ by a finite linear combination of scaled translations of $g$. Note that this is possible because $f$ attains a maximum, and $g$ is a non-zero positive function. So, there exists some $x \in \mathcal{G}$ such that $g(x) > 0$, and we can find a number $\lambda$ such that $\lambda g(x) > \|f\|_{\sup}$. By continuity of translation we can find a neighborhood around $x$ where this holds. We can now transfer via translation $L_y$ all the problematic parts to this neighborhood. Since the support of $f$ is compact, we can cover it with finitely many translations of this neighborhood, hence we can make sense of a sum. So, there exists a finite collection of real numbers $\lambda_i$ and translations $L_{x_i}$ such that,

$$f \leq \sum_i \lambda_i(L_{x_i}(g)).$$

Consider the set of all finite collections of numbers $\{\lambda_i\}$ such that $f \leq \sum_{i=1}^n \lambda_i(L_{x_i}(g))$. This is nonempty due to the above reason. Denote this set by $\mathcal{D}(f : g)$. The intuitive meaning here is that $\sum_i \lambda_i(L_{x_i}g)$ is an approximation of $f$ in terms of translations of the function $g$. Using this we can define the 'best' approximation based on how little scaling we have to do with the translations of $g$. This defines some sort of index of the function $f$ with respect to $g$.

Since each element of this set is a finite collection of real numbers, we can add them up. So, we have a 'sum' function on $\mathcal{D}(f : g)$,

$$\sum : \mathcal{D}(f : g) \to \mathbb{R}$$
$$\{\lambda_i\} \mapsto \sum_i \lambda_i.$$

Using this function, we can define a new object,

$$(f : g) \coloneqq \inf_{\{\lambda_i\} \in \mathcal{D}(f:g)} \left( \sum_i \lambda_i \right).$$

We will call this the 'approximation index' of the function $f$ with respect to $g$. This approximation index allows us to reduce the problem of constructing a special linear functional on the space of compactly supported continuous functions to doing some stuff with some 'reference function'. This will be made precise below. Note that this is where the 'locally compact' part has been important. The Riesz representation tells us that the construction of Radon measures is equivalent to constructing some special linear functionals on compactly supported continuous functions. Then the compactness of the supports of the functions ensure that the sets $\mathcal{D}(f : g)$ are non-empty for each compactly supported functions $f$ and $g$.

Note that the translation $L_y$ is a linear map on the space of continuous functions, so we have,

$$L_{y^{-1}}\left( \sum_i \lambda_i(L_{x_i}g) \right) = \sum_i \lambda_i(L_{y^{-1}x_i}g).$$

Since the approximation index is an infimum, the approximation index of any translation $L_y$ of the function $f$ with respect to $g$ will be the same. So, we have,

$$(f : g) = (L_y f : g). \tag{invariance}$$

Scaling the functions $f$ by $\lambda$ just scales the sets $\mathcal{D}(f : g)$, so we have,

$$(\lambda f : g) = \lambda(f : g). \tag{scaling}$$

If $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(h : g)$, then the disjoint union, $\{\lambda_i\} \coprod \{\mu_j\} \in \mathcal{D}(f + h : g)$. So, we have,

$$(f + h : g) \leq (f : g) + (h : g). \tag{subadditivity}$$

Note that if this was additive, we would be done, we would have constructed a linear functional that is invariant under left translation. It is however possible choose $g$ such that the inequality gets closer to an equality. The key lies in how well we can approximate the given function $f$, in terms of linear combinations of translations of $g$. So, the problem is now choosing an appropriate $g$. We will need some further properties of the functionals $(f : g)$ to do this.

If $f \leq h$ then any upper approximation of $h$ by translations of $g$ is also an upper approximation of $f$ by translations of $g$. Hence we have,

$$f \leq h \Rightarrow (f : g) \leq (h : g). \tag{monotonicity}$$

Let $f$ attain a maximum at $x$, that is $f(x) = \|f\|_{\sup}$ and $g$ attain a maximum at $y$, that is, $g(y) = \|g\|_{\sup}$. Then the translation $L_{yx^{-1}}(f)$ attains a maximum at $y$, by $(f : g) = (L_{yx^{-1}}f : g)$, we obtain,

$$(f : g) \geq \|f\|_{\sup}/\|g\|_{\sup}. \tag{boundedness}$$

Suppose $f \leq \sum_i \lambda_i(L_{x_i}g)$ and $g \leq \sum_j \mu_j(L_{y_j}h)$, then by linearity of translations, we have, $f \leq \sum_{i,j} \lambda_i\mu_j(L_{x_iy_j}h)$. That is to say, $\{\lambda_i\mu_j\} \in \mathcal{D}(f : h)$ whenever $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(g : h)$. So, we have,

$$(f : h) \leq (f : g)(g : h). \tag{product}$$

Now we choose a 'reference function' $F$, and define a normalized index,

$$I_g(f) := (f : g)/(F : g).$$

This gives us a functional $I_g : C_c^+(\mathcal{G}) \to \mathbb{R}$, that is left invariant, subadditive and monotone. By, $(F : g) \leq (F : f)(f : g)$ and $(f : g) \leq (f : F)(F : g)$ we have,

$$f_F \equiv (F : f)^{-1} \leq I_g(f) \leq (f : F) \equiv f^F \tag{rescaling}$$

Now the problem is to choose an appropriate $g$ such that (subadditivity) becomes an equality. In order to choose such a $g$, we have to understand what makes the approximations $(f : g)$ not 'close' to the function $f$. The function is determined by its value at each point. So, given a number, we can scale it appropriately at each point to get $f$. What we are trying to do is to approximate the function $f$, by scaling a bunch of functions on some open sets. So, smaller open sets means better approximation. So, to make the approximation closer to the actual value of $f$, we could use $g$ that has smaller support.

Let $f_i, f_j$ be two compactly supported continuous functions. Let $g$ be a positive compactly supported continuous function such that $g \equiv 1$ on the support of the functions $(f_i + f_j)$. Let $\delta > 0$ be a number (will be specified a bit later). Then we can consider the function,

$$h = f_i + f_j + \delta \cdot g$$

Using this we can construct new functions,

$$h_i = f_i/h, \quad h_j = f_j/h.$$

This implies that $h_i \in \mathcal{C}_c^+(\mathcal{G})$, and $h_i \equiv 0$ whenever $f_i \equiv 0$. By continuity of $h_i$, for every $\delta > 0$ we can find a small enough neighborhood $V$ of $1 \in \mathcal{G}$ such that

$$|h_i(x) - h_i(y)| \leq \delta$$

whenever $x, y \in V$. For any function $\varphi$ with support in $V$, consider an approximate of $h$ by finitely many translations of $\varphi$, $h \leq \sum_k \lambda_k L_{x_k} \varphi$, then we have,

$$f_i(x) = h(x)h_i(x) \leq \sum_j \lambda_k L_{x_k}\varphi(x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta)$$

whenever $x, x_k \in V$. Since $h_i + h_j \leq 1$, we get,

$$(f_i : \varphi) + (f_j : \varphi) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta) + \sum_k \lambda_k \varphi(x_k^{-1}x)(h_j(x_k) + \delta) \leq (1 + 2\delta)\sum_k \lambda_k.$$

Hence, by taking the infimum of all such $\sum_k \lambda_k$, we get,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq (1 + 2\delta)I_\varphi(h) \leq (1 + 2\delta)(I_\varphi(f_i) + I_\varphi(f_j) + \delta I_\varphi(g))$$
$$= (I_\varphi(f_i + f_j)) + \delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g).$$

The last step is by (subadditivity) and (scaling). Then by (rescaling) we can choose $\delta$ small enough such that the last part is less than $\epsilon$, $\delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g) < \epsilon$. Hence we can choose a function $\varphi$ with a small support such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

The choice of $\delta$ determines the support of the function used for approximation. This can be chosen such that $I_\varphi(f_i) + I_\varphi(f_j)$ is as close to $I_\varphi(f_i + f_j)$ as we want. However this choice of $\delta$ depends on the functions $f_i$ and $f_j$.

Each function $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ gives us a map $f \mapsto I_\varphi(f) \in \mathbb{R}$. The goal is to find a functional that is linear. For each $\varphi$ the value of $I_\varphi(f)$ lies in the interval $X_f \equiv [f_F, f^F]$. So, we can embed all such functionals inside the compact Hausdorff space,

$$X = \prod_{f \in \mathcal{C}_c^+(\mathcal{G})} X_f.$$

This is compact Hausdorff space because it is a product of compact Hausdorff spaces. It consists of all real valued functions from the space of compactly supported functions whose value at $f$ lies in the interval $X_f$.

Consider for each neighborhood $V$ of $1 \in \mathcal{G}$, the compact subsets $K(V)$ of $X$, consisting of closures of the collection of all functionals of the form $I_\varphi$ with support of $\varphi$ inside $V$.

$$X = \bigcup_{1 \in V} K(V)$$

Clearly, $V \subseteq W$ implies $K(V) \subseteq K(W)$. So, these sets satisfy the 'finite intersection property' $\bigcap_{i=1}^n K(V_i) \supset K(\bigcap_{i=1}^n V_i)$. This is a strict subset because there always exist functions with

domain that is larger than $\bigcap_i V_i$. Since $X$ is compact, the intersection $\bigcap_i K(V_i)$ is nonempty.[4] So there exists some point $I$ in $X$ that lies in all of the $K(V)$'s.

Every neighborhood of $I$ contains $I_\varphi$'s. So, for any collection of functions $\{f_i\}_{i=1}^n \subset \mathcal{C}_c^+(\mathcal{G})$ and $\epsilon > 0$, there exists some $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ such that $|I(f_i) - I_\varphi(f_i)| < \epsilon$, by definition of neighborhoods in product topology.

This implies that $I$ is left translation invariant $L_y I = I$, and commutes with scaling. Since by adjusting the support for all $\epsilon > 0$, we can find a function $\varphi$ such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

This implies that, $I(f_i) + I(f_j) \leq I(f_i + f_j) + \epsilon$ for all $\epsilon$, and since $I$ belongs to every $K(V)$, $\epsilon$ can be arbitrarily small, hence,

$$I(f_i) + I(f_j) = I(f_i + f_j).$$

This has a unique extension to $\mathcal{C}_c(\mathcal{G})$, because any function can be written as $f = g - h$ with $g, h \in \mathcal{C}_c^+(\mathcal{G})$, and if $f = g' - h'$, then we have $g + h' = g' + h$ and hence, $I(g) + I(h') = I(g') + I(h)$, or, $I(f) = I(g) - I(h)$ is well defined. This linear functional corresponds to a left Haar measure.

## UNIQUENESS

Suppose $\lambda$ and $\mu$ are two left Haar measures on $\mathcal{G}$. To prove uniqueness we have to show that the ratio $\int f d\lambda / \int f d\mu$ is the same for all compactly supported functions $f$. So, it is sufficient to show for any $\epsilon > 0$,

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| < \epsilon$$

Let $f$ be a compactly supported function, let $K_f = \text{supp}(f)$. Since $f$ is a continuous function, for every $\epsilon > 0$ there exists a neighborhood $U_x$ of $x$ such that,

$$|f(z) - f(x)| < \tfrac{1}{2}\epsilon.$$

for any $z \in U_x$. We can now translate this neighborhood to the identity $L_{x^{-1}} U_x$, and hence we have for every $\epsilon$, a neighborhood of $1$ such that $|f(yx) - f(x)| < \tfrac{1}{2}\epsilon$ for all $y$ in the neighborhood. Choose a symmetric neighborhood $V_x$ of $1$ such that $V_x V_x \subset L_{x^{-1}} U_x$. The sets $x V_x$ cover the compact set $K_f = \text{supp}(f)$, and hence there must exist a finite subcover, $K \subseteq \bigcup_{i=1}^n x_i V_{x_i}$.

Consider the neighborhood of $1$, $V = \bigcap_{i=1}^n V_{x_i}$. Let $y \in V$. If $x \in K_f$ then there exists some $i$ such that $x_i^{-1} x \in V_{x_i}$, so that $xy = x_i(x_i^{-1} x) \in U_{x_i}$. Then,

$$|f(xy) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon.$$

If both $x$ and $yx$ are not in $K$ then $f(x) = f(yx) = 0$. Hence if $f$ is a compact supported function, then it's left uniformly continuous, that is $\|L_y f - f\|_{\sup} \to 0$ uniformly.

Let $f, g \in \mathcal{C}_c^+(\mathcal{G})$, let $V_0$ be a symmetric, compact neighborhood of $1$, and let $V_f = V_0 K_f \cup K_f V_0$, and $V_g = V_0 K_g \cup K_g V_0$. $V_f$ and $V_g$ are compact sets. For any $y \in V_0$, we have

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[4]This is an alternate characterization of compactness. The equivalence of the two definitions is basically rewriting the compactness definition of open sets and unions in terms of closed sets and intersections using De Morgan is laws.

continuous functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto g(xy) - g(yx)$ with supports $V_f$ and $V_g$ respectively. Note that their value at 1 is zero.

Hence for any $\epsilon > 0$, there exists a neighborhood $V \subset V_0$ of 1 such that,

$$|f(xy) - f(yx)| < \epsilon, \quad |g(xy) - g(yx)| < \epsilon$$

Pick a nice function $h$ such that $h(x) = h(x^{-1})$, with support in $V$. Then,

$$\int h d\mu \int f d\lambda = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(x) d\lambda(x) d\mu(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(yx) d\lambda(x) d\mu(y).$$

Using $h(x) = h(x^{-1})$ we get,

$$\int h d\lambda \int f d\mu = \int_{\mathcal{G}} \int_{\mathcal{G}} h(x) f(y) d\lambda(x) d\mu(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}x) f(y) d\lambda(x) d\mu(y)$$

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} h(x^{-1}y) f(y) d\mu(y) d\lambda(x) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(yx) d\mu(y) d\lambda(x)$$

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(xy) d\lambda(x) d\mu(y).$$

Therefore,

$$\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| = \left| \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) \left[ f(xy) - f(yx) \right] d\lambda(x) d\mu(y). \right| \leq \epsilon \lambda(V_f) \int h d\mu.$$

Similarly,

$$\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| = \leq \epsilon \lambda(V_f) \int h d\mu.$$

Dividing these inequalities by $\int h d\mu \int f d\mu$ and $\int h d\mu \int g d\mu$ and adding them, we get,

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| < \epsilon \left[ \frac{\lambda(V_f)}{\int f d\mu} + \frac{\lambda(V_g)}{\int g d\mu} \right].$$

Since $\epsilon$ is arbitrary, the difference must be zero, and hence $\lambda$ and $\mu$ are unique upto a constant multiple. $\qquad \square$

## References

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