Part I

Modular Theory

In this chapter we will apply the tools of Tomita-Takesaki modular theory of von Neumann algebras in the quantum field theories setting. We will give an introduction to Tomita-Takesaki modular theory with an emphasis on heuristics and motivation. The idea is to study von Neumann algebras as they are 'seen' by states on them.

1 | Tomita-Takesaki Theory

Von Neumann algebras have algebraic and topological information. To be able to study a von Neumann algebra by its states we must ensure that the state carries both the algebraic and topological information. Faithful states 'see' every element of the von Neumann algebra. Normal states respect the topological information of the von Neumann algebra. The starting point in the analysis is faithful normal states.

1.1 | σ-Finite von Neumann Algebras

Since we expect the values of every single measurement instrument to be modeled using real numbers we expect the collection of effects corresponding to single measuring instrument to inherit these properties of real numbers. In this case the effects correspond to whether a measurement value lies in an interval. Since rational numbers are dense in real numbers, every interval has a rational number. Hence we expect any collection of mutually disjoint intervals to be of at most countable cardinality.

A von Neumann algebra is said to be σ-finite, if every collection of mutually orthogonal projections has atmost a countable cardinality. For the reason discussed above, we will assume all von Neumann algebras of interest to us to be σ-finite.

Suppose $\mathcal{A}$ is a σ-finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$. Suppose $\{\kappa_i\}_{i\in\mathcal{I}} \subseteq \mathcal{H}$ be a maximal family of non-zero vectors such that $P_i \perp P_j$ whenever i and j are different. Here P_i is the projection to the subspace spanned by $\mathcal{A}'\kappa_i$. Since $\mathcal{A}$ is a von Neumann algebra, by von Neumann's density theorem we have $\mathcal{A}'' = \mathcal{A}$, and hence it follows that the projection P_i lies in $\mathcal{A}$. By maximality of $\{\kappa_i\}_{i\in\mathcal{I}}$ we have

$$\sum_{i\in\mathcal{I}} P_i = 1.$$

Hence $\{\kappa_i\}_{i\in\mathcal{I}}$ is cyclic for $\mathcal{A}'$, and equivalently separating for $\mathcal{A}$.[1] This is a mutually orthogonal collection of projection operators, and since $\mathcal{A}$ is σ-finite it follows that $\{P_i\}$ is countable,

[1] A collection of vectors $\mathcal{K} \subset \mathcal{H}$ is separating for $\mathcal{A}$ if for every non-zero $A \in \mathcal{A}$ there exists some $\kappa \in \mathcal{K}$

and hence the collection $\{\kappa_i\}_{\mathcal{I}}$ is countable. We will hence assume $\mathcal{I} \equiv \mathbb{N}$.

Normalise the collection of vectors $\{\kappa_i\}$ such that $\sum_{\mathbb{N}} \|\kappa_i\|^2 = 1$. Using this collection we can construct a faithful normal state by

$$\omega(A) = \sum_{i \in \mathbb{N}} \langle \kappa_i | A \kappa_i \rangle.$$

Hence we have proved the following theorem,

Theorem 1.1. (Existence) *If $\mathcal{A}$ is σ-finite then $\mathcal{A}$ has faithful normal states.*

σ-finite von Neumann algebras are convenient to study, because they have faithful normal states and one can use the Hilbert space structure to study them. Since this assumption is physically justified as well we will assume all von Neumann algebras appearing in the future to be σ-finite.

Let $\mathcal{A}$ be a von Neumann algebra together with a faithful normal state ω. Since ω is faithful, the Gelfand ideal is trivial. The state ω defines an inner product on the von Neumann algebra $\mathcal{A}$ given by,

$$\langle [A] | [B] \rangle_\omega := \omega(A^\dagger B).$$

where $[A]$ denotes the equivalence class determined by A, in this case each element of the von Neumann algebra determines an equivalence class uniquely. The Gelfand vector $[1]$ will be denoted by Ω. If $\mathcal{H}_\omega$ is the completion with respect to the inner product, then the left multiplication operator uniquely determines a representation π_ω of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\omega$

$$\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega).$$

By normality of the state ω, $\pi_\omega(\mathcal{A})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\omega)$ and hence

$$\pi_\omega(\mathcal{A})'' = \pi_\omega(\mathcal{A}).$$

Since ω is faithful $\omega(A^\dagger A) = \langle [A] | [A] \rangle_\omega$ is non-zero whenever A is non-zero, and hence it follows that $[A]$ is non-zero for every $A \in \mathcal{A}$. The Gelfand vector Ω is separating for $\pi_\omega(\mathcal{A})$ hence is cyclic for the commutant $\pi_\omega(\mathcal{A})'$.

By construction $\pi_\omega(\mathcal{A})\Omega$ is dense in $\mathcal{H}_\omega$, or Ω is cyclic for $\pi_\omega(\mathcal{A})$, or separating for the commutant $\pi_\omega(\mathcal{A})'$. Ω is cyclic and separating for both $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})'$. Hence we have proved the following lemma,

Lemma 1.2. *The vector Ω is cyclic and separating for both $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})'$.*

The information about the von Neumann algebra $\mathcal{A}$, as 'seen' by the state ω is contained in the dense subset $\pi_\omega(\mathcal{A})\Omega$, and similarly, the information about the commutant $\mathcal{A}'$ is contained in $\pi_\omega(\mathcal{A})'\Omega$. Modular theory studies the relation between a von Neumann algebra and its commutant by relating $\pi_\omega(\mathcal{A})\Omega$ and $\pi_\omega(\mathcal{A})'\Omega$.

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such that $A\kappa$ is nonzero. $\mathcal{K} \subset \mathcal{H}$ is cyclic for $\mathcal{A}$ if $\mathcal{A}\mathcal{K}$ is dense in $\mathcal{H}$. It can be shown that $\mathcal{K}$ is separating for $\mathcal{A}$ if and only if it is cyclic for the commutant $\mathcal{A}'$.

1.2 | The Modular Group

We will assume that $\mathcal{A}$ is a von Neumann subalgebra on $\mathcal{H}_\omega$ with a cyclic and separating vector Ω, and denote $A\Omega$ by $[A]$. The approach of modular theory is to study the relation between $\mathcal{A}$ and $\mathcal{A}'$ by studying the relation between the dense subsets $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ using the inner product. The idea is to use the inner product $\langle\cdot|\cdot\rangle_\omega$ on $\mathcal{H}_\omega$ and exploit the fact that for every closed linear operator T on $\mathcal{H}_\omega$ we must have,

$$\left\langle \kappa | T\nu \right\rangle_\omega = \left\langle T^\dagger \kappa | \nu \right\rangle_\omega .$$

for every ν in the domain of T and κ in the domain of $T^\dagger$. Or similarly, for closed anti-linear operator S we must have

$$\left\langle \kappa | S\nu \right\rangle_\omega = \left\langle \nu | S^\dagger \kappa \right\rangle_\omega .$$

If one of the arguments is from $\mathcal{A}\Omega$ and the other is from $\mathcal{A}'\Omega$, and if there exists an operator of this type, we would have related the two dense subsets. The goal is to try to find such a map S on $\mathcal{H}_\omega$. Observe that for any $A \in \mathcal{A}$ and $B \in \mathcal{A}'$ we have

$$\left\langle [B^\dagger] | [A] \right\rangle_\omega = \omega(BA) = \omega(AB) = \left\langle [A^\dagger] | [B] \right\rangle_\omega .$$

So, we should expect the map T or S to be related to the involutions $[A] \mapsto [A^\dagger]$ for $A \in \mathcal{A}$ and $[B] \mapsto [B^\dagger]$ for $B \in \mathcal{A}'$. Hence the starting point is to study the two maps,

$$\underline{S}_\omega : [A] \mapsto [A^\dagger], \quad \underline{F}_\omega : [B] \mapsto [B^\dagger],$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{A}'$, with domains $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ respectively.

Algebraic Properties

By construction and the assumption that ω is faithful and normal, both these sets are dense in $\mathcal{H}_\omega$. Hence $\underline{S}_\omega$ and $\underline{F}_\omega$ are densely defined anti-linear operators on $\mathcal{H}_\omega$. Let $\underline{S}_\omega^\dagger$ and $\underline{F}_\omega^\dagger$ be the adjoints of $\underline{S}_\omega$ and $\underline{F}_\omega$ respectively. For any $B \in \mathcal{A}'$ or $[B]$ is in the domain of $\underline{F}_\omega$ and for any $A \in \mathcal{A}$ we have $\langle \underline{F}_\omega[B] | [A] \rangle_\omega = \langle [B^\dagger] | [A] \rangle_\omega = \langle [A^\dagger] | [B] \rangle_\omega = \langle \underline{S}_\omega [A] | [B] \rangle_\omega$. By the anti-linearity of $\underline{S}_\omega$ we have $\left\langle \underline{S}_\omega [A] | [B] \right\rangle_\omega = \left\langle \underline{S}_\omega^\dagger [B] | [A] \right\rangle_\omega$. Hence we get

$$\left\langle \underline{F}_\omega [B] | [A] \right\rangle_\omega = \left\langle \underline{S}_\omega^\dagger [B] | [A] \right\rangle_\omega .$$

So, whenever $[B]$ is in the domain of $\underline{F}_\omega$ it is also in the domain of $\underline{S}_\omega^\dagger$. Hence we have

$$\underline{F}_\omega \subseteq \underline{S}_\omega^\dagger .$$

Similarly we have $\underline{S}_\omega \subseteq \underline{F}_\omega^\dagger$. In particular they are densely defined. An operator is closeable if and only if it has a densely defined adjoint. Hence $\underline{S}_\omega$ and $\underline{F}_\omega$ are closable with closures S_ω and F_ω respectively. Since adjoints of densely defined operators are closed we have,

$$\underline{F}_\omega \subseteq F_\omega \subseteq \underline{S}_\omega^\dagger ,$$

and similarly we have $\underline{F}_\omega \subseteq F_\omega \subseteq \underline{S}_\omega^\dagger$.

We can immediately describe some of the algebraic properties of S_ω which will be convenient for manipulations later. Let $S_\omega^\dagger$ be the adjoint of S_ω. Since $\underline{S}_\omega = \underline{S}_\omega^{-1}$, S_ω is injective and we have

$$S_\omega^2 = 1$$

as an unbounded operator. Similarly we have $(S_\omega^\dagger)^2 = 1$. Both S_ω and $S_\omega^\dagger$ are involutions on $\mathcal{H}_\omega$ with dense ranges. They possess unique polar decompositions. Let the polar decomposition of S_ω be given by

$$S_\omega = J_\omega \Delta_\omega^{\frac{1}{2}}$$

where $\Delta_\omega = S_\omega^\dagger S_\omega$ and J_ω is an anti-unitary operator on $\mathcal{H}_\omega$, that is $J_\omega J_\omega^\dagger = J_\omega^\dagger J_\omega = 1$.

Since S_ω is an involution it follows that Δ_ω is non-singular, positive and hence self-adjoint. Using the fact that $S_\omega^2 = 1$ and $(S_\omega^\dagger)^2 = 1$ we have $S_\omega^\dagger S_\omega S_\omega S_\omega^\dagger = 1$. Hence we have, $S_\omega S_\omega^\dagger = \Delta_\omega^{-1}$, and we get,

$$J_\omega \Delta_\omega^{\frac{1}{2}} J_\omega = \Delta_\omega^{-\frac{1}{2}}.$$

Now we also have, $\Delta_\omega^{-1/2} = J_\omega^2 J_\omega^\dagger \Delta_\omega^{1/2} J_\omega$. Since J_ω^2 is a unitary operator and $J_\omega^\dagger \Delta^{1/2} J_\omega$ is a positive self-adjoint operator. By the uniqueness of polar decomposition it follows that $J_\omega^2 = 1$ hence we have

$$J_\omega = J_\omega^\dagger.$$

Taking adjoint of the polar decomposition we get,

$$S_\omega^\dagger = \left(J_\omega \Delta_\omega^{\frac{1}{2}}\right)^\dagger = \Delta_\omega^{\frac{1}{2}} J_\omega = J_\omega \Delta_\omega^{-\frac{1}{2}}.$$

To study the properties of Δ_ω, it is convinient to consider its spectral decomposition. Let $E_\Delta : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}_\omega)$ be the spectral measure of Δ_ω, that is,

$$\left\langle \eta \,|\Delta_\omega \nu \right\rangle_\omega = \int_{\mathbb{R}} \lambda d \left\langle \eta \,|E_\lambda \nu \right\rangle_\omega.$$

for every κ in the domain of Δ_ω where $E_\epsilon = E_\Delta(\epsilon)$. The spectral measure for Δ_ω^{-1} corresponds to $J_\omega E_\Delta J_\omega$, and hence for any bounded Borel function f on $\mathbb{C}$, we have,

$$\begin{aligned}
\left\langle f(\Delta_\omega^{-1})\nu|\nu \right\rangle_\omega &= \int_{\mathbb{R}} f(\lambda) d \left\langle J_\omega E_\lambda J_\omega \nu \,|\nu \right\rangle_\omega \\
&= \int_{\mathbb{R}} f(\lambda_\omega) d \left\langle J_\omega \nu \,|E_\lambda J_\omega \nu \right\rangle_\omega \\
&= \int_{\mathbb{R}} f(\lambda_\omega) d \left\langle J_\omega \nu \,|E_\lambda J_\omega \nu \right\rangle_\omega \\
&= \left\langle f(\Delta_\omega) J_\omega \nu \,|J_\omega \nu \right\rangle_\omega = \left\langle J_\omega \nu \,|\tilde{f}(\Delta_\omega) J_\omega \nu \right\rangle_\omega.
\end{aligned}$$

where $\tilde{f}(x) = \overline{f(x)}$. By the anti-unitarity of J_ω we have,

$$\left\langle f(\Delta_\omega^{-1})\nu|\nu \right\rangle_\omega = \left\langle J_\omega \tilde{f}(\Delta_\omega) J_\omega \nu \,|\nu \right\rangle_\omega.$$

Hence it follows that, $f(\Delta_\omega^{-1}) = J_\omega \tilde{f}(\Delta_\omega) J_\omega$. By taking $f(x) = x^{-it}$ we obtain,

$$J_\omega \Delta_\omega^{it} = \Delta_\omega^{it} J_\omega.$$

J_ω is called the modular conjugation and Δ_ω is called the modular operator of ω.

Topological Properties

We will now prove that it is possible to talk about the von Neumann algebra $\mathcal{A}$ by talking about the dense subset $\mathcal{A}\Omega$, and similarly for its commutant. Note that much of the dirty work lies in going back from $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ to $\mathcal{A}$ and $\mathcal{A}'$ respectively. The elements of $\mathcal{A}$ and

$\mathcal{A}'$ act as bounded operators on the Hilbert space $\mathcal{H}_\omega$. The goal is to understand them by the properties of the operators F_ω and S_ω. We also prove that $S_\omega = F_\omega^\dagger$, as we had anticipated in the beginning of this section.

Whenever $A \in \mathcal{A}$ and $B \in \mathcal{A}'$ we have,

$$A[B] = [AB] = [BA] = B[A].$$

So, we can equivalently think of the vector $[AB]$ as the action of an element $A \in \mathcal{A}$ on the vector $[B]$ in the domain of F_ω or as the action of an element in $B \in \mathcal{A}'$ on the vector $[A]$ in the domain of S_ω.

One can obtain the vector $[BA]$ from the von Neumann algebra $\mathcal{A}$ by the continuous map $A \mapsto B[A]$ from the von Neumann algebra $\mathcal{A}$ to the Hilbert space $\mathcal{H}_\omega$, or by the action of the bounded operator A on the vector $[B]$. This can be summarised with the following diagram:

$$\begin{array}{ccc} [B] & \longrightarrow & A[B] \\ & & \| \\ [A] & \longrightarrow & B[A] \end{array}$$

If a vector ν can be written as an equivalence class $[B]$ for some $B \in \mathcal{A}'$ then the vector ν must be in the domain of F_ω, and hence we must also have

$$\begin{aligned} B^\dagger[A] &= [B^\dagger A] = [AB^\dagger] \\ &= A[B^\dagger] = A(F_\omega[B]). \end{aligned}$$

We now show that the converse is also true, that is, every vector in the domain of F_ω satisfying these two continuity requirements will be of the form $[B]$. Consider the collection of all vectors ν in the domain of F_ω, such that the map

$$[A] \mapsto A\nu$$

is continuous and there exists a bounded operator[2] denoted by B_ν with

$$A\nu = B_\nu[A]$$

for every $A \in \mathcal{A}$. Let $\mathcal{A}'_\omega$ denote the collection of all operators B_ν as above.

We now show that $\mathcal{A}'_\omega = \mathcal{A}'$. We have to show that for every $B_\nu \in \mathcal{A}'_\omega$ and $A \in \mathcal{A}$, $B_\nu A = AB_\nu$. and we have to show that $B_\nu^\dagger \in \mathcal{A}'_\omega$. Let the dense subset be $\mathcal{A}\Omega$. For any $A, C \in \mathcal{A}$ we have $AB_\nu[C] = A(C\nu) = (AC)\nu = B_\nu[AC] = B_\nu A[C]$. Hence it follows that $B_\nu A = AB_\nu$ on the dense subset $\mathcal{A}\Omega$. Since B_ν is bounded it is also continuous, and hence it follows that $B_\nu A = AB_\nu$ on all of $\mathcal{H}_\omega$ for all $A \in \mathcal{A}$. Since $B_{\nu_1}B_{\nu_2}[A] = B_{\nu_1}AB_{\nu_2} = AB_{\nu_1}B_{\nu_2}$, we also have $B_{\nu_1}B_{\nu_2} \in \mathcal{A}'$ whenever $B_{\nu_1}, B_{\nu_2} \in \mathcal{A}'$.

To show that $B_\nu^\dagger \in \mathcal{A}'_\omega$ we will show that $B_\nu^\dagger = B_{F_\omega \nu} \in \mathcal{A}'_\omega$. That is, $F_\omega \nu$ is in the domain of F_ω and there exists a bounded operator $B_{F_\omega\nu}$ such that,

$$A\big(F_\omega \nu\big) = B_{F_\omega \nu}[A]$$

for every $A \in \mathcal{A}$. We will prove this again for the dense subset $\mathcal{A}\Omega$. Let $A, C \in \mathcal{A}$ we have $\langle [C] | B_\nu^\dagger [A] \rangle_\omega = \langle B_\nu[C] | [A] | \rangle_\omega = \langle C\nu | [A] \rangle_\omega = \langle \nu | [C^\dagger A] \rangle_\omega$. Since ν is already in the domain of

[2] A linear operator on a normed space is continuous if only if it is bounded

F_ω and using the fact that $\underline{F}_\omega \subset \underline{S}_\omega^\dagger$ we have $\langle\nu|[C^\dagger A]\rangle_\omega = \langle\nu|\underline{S}_\omega[A^\dagger C]\rangle_\omega = \langle[A^\dagger C]|F_\omega\nu\rangle_\omega = \langle[C]|A(F_\omega\nu)\rangle_\omega$. Hence we showed that

$$A(F_\omega\nu) = B_\nu^\dagger[A].$$

Hence we have, $\mathcal{A}'_\omega = \mathcal{A}'$. It also follows that, $[B_\nu] = B_\nu[1] = \nu$. We can similarly define $\mathcal{A}_\omega$ to be the collection of all η in the domain of S_ω such that the map

$$[B] \mapsto B\eta$$

is continuous for every $B \in \mathcal{A}'$ and there exist a bounded operator A_η such that, $B\eta = A_\eta[B]$. Similarly, it follows that $\mathcal{A}_\omega = \mathcal{A}$. We have proved the following lemma,

Lemma 1.3.

$$\mathcal{A}'_\omega = \mathcal{A}', \ \textit{and}\ \mathcal{A}_\omega = \mathcal{A}.$$

We now generalise this further. We now show that if the map $\underline{B}_\nu : [A] \mapsto A\nu$, with ν in the domain of F_ω, is closable, then its closure, denoted by B_ν is affiliated with the commutant $\mathcal{A}'$. We follow a similar path as for the previous lemma and start with what we expect out of the closure, i.e., $B_\nu^\dagger = B_{F_\omega\nu}$ and that

$$B_\nu^\dagger[A] = A\big(F_\omega\nu\big).$$

Let us define

$$B_{F_\omega\nu}[A] \coloneqq A(F_\omega\nu)$$

This is a densely defined operator with domain $\mathcal{A}\Omega$. We will show that $B_{F_\omega\nu}$ is the adjoint of the map $\underline{B}_\nu$, which proves that $\underline{B}_\nu$ is closable.

We now take a similar route as the previous lemma. For every $A, C \in \mathcal{A}$ we have $\langle\underline{B}_\nu[C]|[A]\rangle_\omega = \langle C\nu|[A]\rangle_\omega = \langle\nu|[C^\dagger A]\rangle_\omega$. Since ν is in the domain of F_ω and $S_\omega^\dagger \subseteq \underline{F}_\omega \subseteq F_\omega$, we have, $\langle\nu|[C^\dagger A]\rangle_\omega = \langle\nu|\underline{S}_\omega[A^\dagger C]\rangle_\omega = \langle[A^\dagger C]|F_\omega\nu\rangle_\omega = \langle[C]|A(F_\omega\nu)\rangle_\omega$. We have,

$$\big\langle\underline{B}_\nu[C]|[A]\big\rangle_\omega = \big\langle[C]|B_{F_\omega\nu}[A]\big\rangle_\omega$$

Hence we have $B_{F_\omega\nu} \subseteq \underline{B}_\nu^\dagger$. Or, $\underline{B}_\nu$ is closable, we denote the closure by B_ν. We have, $B_\nu[A] = A\nu$ and $B_\nu^\dagger[A] = A\big(F_\omega\nu\big)$.

Let κ be in the domain of B_ν. Since B_ν is the closure of $\underline{B}_\nu$ and since the domain of $\underline{B}_\nu$ is $\mathcal{A}\Omega$, it follows that there exists a sequence $\{[A_i]\} \subset \mathcal{A}\Omega$ with $[A_i] \to \kappa$ such that $\underline{B}_\nu[A_i] \to B_\nu\kappa$. Let $A \in \mathcal{A}$, we have $\underline{B}_\nu A[A_i] = \underline{B}_\nu[AA_i] = (AA_i)\nu = A(A_i\nu) = A(\underline{B}_\nu[A_i])$, and by continuity of the operator A, $A[A_i] \to A\kappa$ and hence it follows that

$$\underline{B}_\nu A[A_i] \to A(B_\nu\kappa)$$

Again since B_ν is the closure of $\underline{B}_\nu$ it follows that $(B_\nu A)\kappa = (AB_\nu)\kappa$. So $A\kappa$ is in the domain of B_ν whenever κ is in the domain of B_ν, and we have $B_\nu A\kappa = AB_\nu\kappa$ for all $A \in \mathcal{A}$. Hence B_ν is affiliated with $\mathcal{A}'$.

Lemma 1.4. *For every ν in the domain of F_ω, there exists B_ν, affiliated with $\mathcal{A}'$ such that*

$$\nu = B_\nu\Omega.$$

As stated in the beginning of this section, the goal is to find a closed (anti-linear) operator that allows us to relate the dense subsets $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$. We expected that this relation comes from the maps $\underline{S}_\omega$ and $\underline{F}_\omega$ on $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ respectively. We now use the lemmas discussed so far to show that this is indeed the case.

Corollary 1.5. (Takesaki)

$$S_\omega^\dagger = F_\omega = J_\omega \Delta_\omega^{-\frac{1}{2}}.$$

Proof

We already have $F_\omega \subseteq \underline{S}_\omega^\dagger = S_\omega^\dagger$. To show that F_ω is indeed equal to $S_\omega^\dagger$ we must show that whenever ν is in the domain of $S_\omega^\dagger$, it is also in the domain of F_ω. By definition of $S_\omega^\dagger$, we have,

[[Stuck Here]]

Lemma 1.6. *Intersection of domain of S_ω and F_ω is dense in $\mathcal{H}_\omega$.*

Proof

1.2.1 | The Modular Relation

We now relate the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$ in terms of the modular relations. We will then solve this relation in terms of the modular operator and the modular conjugation. The goal is to use the Hilbert space $\mathcal{H}_\omega$ to relate the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$, and then remove any terms involving the Hilbert space directly. We will follow [1], since both $\mathcal{A}$ and $\mathcal{A}'$ are treated on an equal footing.

For any $A, B, C \in \mathcal{A}$, we have $BCA^\dagger = (AC^\dagger B^\dagger)^\dagger$. Since we can view the algebra $\mathcal{A}$ in terms of the dense subset $\mathcal{A}\Omega$, viewing C as the vector $[C]$ in $\mathcal{A}\Omega$, we have,

$$BS_\omega AS_\omega[C] = S_\omega AS_\omega B[C].$$

So, when viewed from the subset $\mathcal{A}\Omega$, the operator $S_\omega AS_\omega$ acts as if it is in the commutant of $\mathcal{A}$. We use this heuristic relation as motivation to construct an element of $\mathcal{A}'$ for each element in the von Neumann algebra $\mathcal{A}$.

Sakai-Radon-Nikodym Theorem

We prove a useful technical lemma, due to Sakai, which will provide us a link between the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$ on the Hilbert space $\mathcal{H}_\omega$. Note that if ω is a faithful normal state on $\mathcal{A}$, it contains non-trivial information about every operator in $\mathcal{A}$.

Now, suppose $\varphi \ll \omega$. This by definition means that for every operator $C \in \mathcal{A}$,

$$(\omega - \varphi)\big(C^\dagger C\big) \geq 0.$$

Heuristically this means that the state ω contains more information about $\mathcal{A}$ than the information contained in φ about $\mathcal{A}$. Or that φ has 'less' information about the von Neumann algebra $\mathcal{A}$ than the information contained in ω. Quantifying 'how much?' usually give rise to a Radon-Nikodym type relation between φ and ω. We will describe below a Radon-Nikodym type relation due to Sakai.

Let φ be a state on $\mathcal{A}$. Then, $\widehat{\varphi}(C, B) = \varphi(C^\dagger B)$ defines a sesquilinear form. Since φ a positive linear functional we have, $\widehat{\varphi}\big(\mu C - B, \mu C - B\big) \geq 0$. Upon expanding and taking

$$\mu = \widehat{\varphi}(C, B)/\widehat{\varphi}(C, C)$$

we obtain the Cauchy-Schwarz inequality,

$$|\varphi(C^\dagger B)|^2 \leq \varphi(C^\dagger C)\varphi(B^\dagger B).$$

If $\varphi \ll \omega$, we have, $\varphi(C^\dagger C) \leq \omega(C^\dagger C)$. Hence we must have

$$\big|\varphi(C^\dagger B)\big|^2 \leq \omega(C^\dagger C)\omega(B^\dagger B).$$

We now use the fact that for a von Neumann algebra $\mathcal{A}$, $\mathcal{A}_*$ is the predual of $\mathcal{A}$, and hence every continous linear functional on $\mathcal{A}_*$ can be thought of as an element of $\mathcal{A}$. This allows us to 'quantify' the extra information in ω using an operator A in $\mathcal{A}$.

The idea is to exploit this fact about von Neumann algebras, along with the Hahn-Banach separation theorem to show that φ can be constructed using ω and some operator A. A in somesense 'quantifies' the extra information contained in ω.

Theorem 1.7. (Sakai-Radon-Nikodym) *Let φ be such that*

$$\left|\varphi(C^\dagger B)\right|^2 \leq \omega(C^\dagger C)\omega(B^\dagger B)$$

then for any $\lambda \in \mathbb{R}_+$ fixed, $\exists A \in \mathcal{A}$, with $\|A\| \leq 1$ such that

$$\varphi(X) = \omega\big(\lambda AX + \lambda^{-1}XA\big).$$

Proof

The first step is to prepare to use Hahn-Banach separation theorem. Fix λ. Construct a parametrized collection of linear functionals on $\mathcal{A}$ given by,

$$\omega_{\lambda,A} : X \mapsto \omega(\lambda AX + \lambda^{-1}XA).$$

for every $A \in \mathcal{A}$. The role of λ here is to absorb some of the wildness and to ensure A is well behaved. Since $\omega_{\lambda,A}$ is constructed out of ω, it clearly has 'less information' about the von Neumann algebra $\mathcal{A}$ than ω itself. If A was the identity, then $\omega_{\lambda,I}$ is only a scaling of ω, or would have the same information once normalised.

By normality of the state ω it follows that whenever A is near to C in $\mathcal{A}$, $\omega_{\lambda,A}(X)$ must be near to $\omega_{\lambda,C}(X)$ as complex numbers. By linearity of ω we also have, $\omega_{\lambda,(A+C)}(X) = \omega_{\lambda,A}(X) + \omega_{\lambda,C}(X)$. So the mapping

$$A \mapsto \omega_{\lambda,A}$$

defines a continuous linear map from $\mathcal{A}$ to $\mathcal{A}_*$. Hence the set $\omega_{\lambda,1} \equiv \{\omega_{\lambda,A} \mid \|A\| \leq 1\}$ is a convex subset of $\mathcal{A}_*$, since it is the image of a closed set it is weakly compact in $\mathcal{A}_*$. We now prove that φ belongs to this set, which would prove the lemma.

Suppose there exist no A such that $\varphi = \omega_{\lambda,A}$. Then by Hahn-Banach separation theorem there exists a continuous real linear functional κ on $\mathcal{A}_*$ which separates the compact convex subset ω_λ in $\mathcal{A}_*$ from the point φ in $\mathcal{A}_*$. That is, the interval, $\kappa(\omega_{\lambda,A})$ for $\|A\| \leq 1$ and $\kappa(\varphi)$ are disjoint. Assume without loss of generality that $\kappa(\varphi) > \kappa(\omega_{\lambda,A})$ for every $\|A\| \leq 1$.

Since $\mathcal{A}$ is a von Neumann algebra, $\mathcal{A}_*$ is its predual, and hence κ corresponds to an element H in $\mathcal{A}$. Let $H \in \mathcal{A}$ correspond to the functional κ then we have,

$$\mathrm{Re}(\varphi(H)) > \left|\omega_{\lambda,1}(H)\right| \equiv \sup\Big\{\mathrm{Re}(\omega_{\lambda,A}(H)) \mid \omega_{\lambda,A} \in \omega_{\lambda,1}\Big\}.$$

Consider the polar decomposition, $H = U|H| = |H^\dagger|U$ where U is a unitary operator and both U and $|H|$ are in $\mathcal{A}$. We can now use U and $|H|$ to arrive at a contradiction. We have,

$$\omega_{\lambda,\frac{U}{2}^\dagger}(H) = \tfrac{1}{2}\lambda\omega\big(U^\dagger H\big) + \tfrac{1}{2}\lambda^{-1}\omega\big(HU^\dagger\big) = \tfrac{1}{2}\lambda\omega\big(|H|\big) + \tfrac{1}{2}\lambda^{-1}\omega\big(|H^\dagger|\big).$$

By unitarity of U we have $\|U\|^2 = \|U^\dagger U\| = 1$, and hence $\omega_{\lambda,\frac{U}{2}^\dagger} \in \omega_{\lambda,1}$. Hence we have

$$\mathrm{Re}\Big(\omega_{\lambda,\frac{U}{2}^\dagger}(H)\Big) \leq \left|\omega_\lambda(H)\right| < \mathrm{Re}(\varphi(H))$$

Combining this with the AM-GM inequality we have,

$$\begin{aligned}\mathrm{Re}(\varphi(H)) > |\omega_{\lambda,1}(H)| &\geq \tfrac{1}{2}\big(\lambda\omega\big(|H|\big) + \lambda^{-1}\omega\big(|H^\dagger|\big)\big)\\ &\geq \omega\big(|H|\big)^{\frac{1}{2}}\omega\big(|H^\dagger|\big)^{\frac{1}{2}}.\end{aligned}$$

Since we assumed that $|\varphi(C^\dagger B)| \leq \omega(C^\dagger C)^{1/2}\omega(B^\dagger B)^{1/2}$, for every $C, B \in \mathcal{A}$, we have,

$$\begin{aligned}\big|\mathrm{Re}(\varphi(H))\big| \leq |\varphi(H)| &= \big|\varphi(U|H|^{\frac{1}{2}}|H|^{\frac{1}{2}})\big| \\ &\leq \big|\omega(|H|)\big|^{\frac{1}{2}}\big|\omega(U|H|U)\big|^{\frac{1}{2}} = \big|\omega(|H|)\big|^{\frac{1}{2}}\big|\omega(|H^\dagger|)\big|^{\frac{1}{2}}.\end{aligned}$$

Which implies that $\omega(|H|)^{1/2}\omega(|H^\dagger|)^{1/2} > \omega(|H|)^{1/2}\omega(|H^\dagger|)^{1/2}$ which is absurd and we arrive at a contradiction. Hence there must exist some A such that $\varphi = \omega_{\lambda,A}$. □

Relation between $\mathcal{A}$ and $\mathcal{A}'$

We now use the Sakai-Radon-Nikodym theorem to relate the von Neumann algebra $\mathcal{A}$ with its commutant $\mathcal{A}'$. By construction, the Hilbert space $\mathcal{H}_\omega$ describes all the information about the von Neumann algebra $\mathcal{A}$ contained in the state ω. We can now ask how much of this information remains after the action of $\mathcal{A}'$. Instead of cyclic and separating vector Ω, we instead consider the action of $\mathcal{A}$ on elements of $\mathcal{A}'\Omega$, which heuristically is the information as 'seen' by the state ω assuming ω has already seen $\mathcal{A}'$. So, we are interested in information of the form, $\omega(XB)$.

To study such information consider the linear functionals of the form

$$\omega_B(X) = \big\langle [B] | [X] \big\rangle_\omega,$$

for $B \in \mathcal{A}'$ and note that $\omega(X) = \omega_I(X)$.

For any $C, D \in \mathcal{A}$, we have,

$$\omega_B(C^\dagger C) = \omega(B^\dagger C^\dagger C) \leq$$

$$\begin{aligned}\omega_B(C^\dagger D) = \omega(B^\dagger C^\dagger D) &\leq \big[\omega(B^\dagger B)\big]^{\frac{1}{2}}\big[\omega\big((C^\dagger D)D^\dagger C\big)\big]^{\frac{1}{2}} \\ &= \big[\langle [B]|[B]\rangle\big]^{\frac{1}{2}}\big[\omega(C^\dagger C)\omega(DD^\dagger)\big]^{\frac{1}{2}}. \qquad \text{(Why?)}\end{aligned}$$

If in addition B is such that $\langle [B]|[B]\rangle_\omega^{1/2} = \|B\| \leq 1$ then we will have

$$\omega_B(C^\dagger D) \leq \omega(C^\dagger C)^{\frac{1}{2}}\omega(D^\dagger D)^{\frac{1}{2}}.$$

In such a case, the condition of Sakai-Radon-Nikodym theorem is satisfied, and hence for every $\lambda \in \mathbb{R}_+$, there exists some A in $\mathcal{A}$ such that, $\omega_B(X) = \omega_{\lambda,A}(X) = \omega(\lambda XA + \lambda^{-1}AX)$. Expressing in terms of the inner product on $\mathcal{H}_\omega$, and rearranging using the action of $\mathcal{A}$ on $\mathcal{H}_\omega$ we have,

$$\begin{aligned}\big\langle [B]|[X]\big\rangle_\omega &= \lambda\big\langle [AX]|\Omega\big\rangle_\omega + \lambda^{-1}\big\langle [XA]|\Omega\big\rangle_\omega \\ &= \lambda\big\langle [X]|[A^\dagger]\big\rangle_\omega + \lambda^{-1}\big\langle [A]|[X^\dagger]\big\rangle_\omega.\end{aligned}$$

Hence we have,

$$\big\langle [B]|[X]\big\rangle_\omega = \lambda\big\langle [A^\dagger]|[X]\big\rangle_\omega + \lambda^{-1}\big\langle [A]|[X^\dagger]\big\rangle_\omega,$$

upon rearranging we get, $\big\langle [X^\dagger]|[A]\big\rangle_\omega = \lambda\big\langle [B] - \lambda[A^\dagger]|[X]\big\rangle_\omega$. Since $A, X \in \mathcal{A}$ we have $[A], [X] \in \mathcal{A}\Omega$ and in particular they are in the domain of S_ω. Since the right hand side is well defined, we conclude that $[A]$ is in the domain of $S_\omega^\dagger = F_\omega$.

Hence we can express the vector $[B]$ in the set $\mathcal{A}'\Omega$ in terms of a vector $[A]$ in the set $\mathcal{A}\Omega$ by the equation,

$$[B] = (\lambda S_\omega + \lambda^{-1}F_\omega)[A].$$

This is a relation between vectors in the dense subsets $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$. We however want to study the relation between $\mathcal{A}$ and $\mathcal{A}'$ as operators on $\mathcal{H}_\omega$. So, we have to now go from the dense subsets $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ to the algebras $\mathcal{A}$ and $\mathcal{A}'$. In order to do this, we should again think about how $\mathcal{A}$ and $\mathcal{A}'$ act on the domains of S_ω and F_ω.

Let ν be in the domain of both S_ω and F_ω. By the expressions for S_ω and F_ω in terms of the modular operator and modular conjugation, this equivalently means that ν is in the domain of both $\Delta_\omega^{1/2}$ and $\Delta_\omega^{-1/2}$. Set

$$\eta = (S_\omega + F_\omega)\nu$$

Since $\nu \in \mathcal{D}(F_\omega)$, by 1.4, there exists a sequence $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{A}'$ with $[B_i] \to \eta$. By taking $\lambda = 1$ in the relation $[B] = (\lambda S_\omega + \lambda^{-1} F_\omega)[A]$, there exists to each $B_i \in \mathcal{A}'$ an associated $A_i \in \mathcal{A}$ such that $(S_\omega + F_\omega)[A_i] = [B_i]$. Hence we have,

$$\begin{array}{rcl} (S_\omega + F_\omega)[A_i] & = & [B_i] \\ \downarrow & & \downarrow \\ (S_\omega + F_\omega)\nu & = & \eta \end{array}$$

$(S_\omega + F_\omega)$ has a bounded inverse.

[[Why??]]

By letting $\nu_i = (S_\omega + F_\omega)^{-1}[B_i]$ we have, $[A_i] = (S_\omega + F_\omega)^{-1}[B_i] \to \nu$. Expressing S_ω and F_ω in terms of the polar decompositions, we have $(S_\omega + F_\omega)^{-1} = (J_\omega \Delta_\omega^{1/2} + J_\omega \Delta_\omega^{-1/2})^{-1} = (\Delta_\omega^{1/2} + \Delta_\omega^{-1/2})^{-1} J_\omega^{-1}$. By mulitiplying each side by $\Delta_\omega^{1/2}$, we have $\Delta_\omega^{1/2}[A_i] = \Delta_\omega^{1/2}(\Delta_\omega^{1/2} + \Delta_\omega^{-1/2})^{-1} J_\omega [B_i] = (\Delta_\omega^{-1} + 1)^{-1} J_\omega [B_i]$. The spectral decomposition of $(\Delta_\omega^{-1} + 1)^{-1}$ is given by,

$$(\Delta_\omega^{-1} + 1)^{-1} = \int_{\mathbb{R}_+} \left[1 + \lambda^{-1}\right]^{-1} dE_\lambda \leq \int_{\mathbb{R}_+} dE_\lambda.$$

where E is the spectral measure for Δ_ω. Hence it defines a bounded operator, and is hence continuous. So, $\Delta_\omega^{1/2}[A_i] \to \Delta_\omega^{1/2}(S_\omega + F_\omega)^{-1}(S_\omega + F_\omega)\nu = \Delta_\omega^{1/2}\nu$. Similarly we also have, $\Delta_\omega^{-1/2}[A_i] \to \Delta_\omega^{-1/2}\nu$. We can now replace elements in $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ with elements in the domain of S_ω and F_ω.

Consider the relation,

$$\big\langle [B] | [X] \big\rangle_\omega = \lambda \big\langle [A^\dagger] | [X] \big\rangle_\omega + \lambda^{-1} \big\langle [A] | [X^\dagger] \big\rangle_\omega,$$

We may replace $[A]$ by a vector ν in $\mathcal{D}(F_\omega)$

[[Finish up]]

Hence we obtain,

$$\lambda \big\langle \nu | S_\omega A S_\omega \eta \big\rangle_\omega + \lambda^{-1} \big\langle \nu | F_\omega A F_\omega \eta \big\rangle_\omega = \big\langle \nu | B \eta \big\rangle_\omega,$$

$\forall \eta, \nu \in \mathcal{D}(S_\omega) \cap \mathcal{D}(F_\omega)$.

We now restate this relation in a more convinient form, which we will call the modular relation between $\mathcal{A}$ and $\mathcal{A}'$. This provides cleaner formula and easier to manipulate.

Theorem 1.8. (The Modular Relation) *Let $\lambda \in \mathbb{R}_+$ then for all $B \in \mathcal{A}'$ there exists $A \in \mathcal{A}$ with*

$$\lambda\left\langle \Delta_\omega^{\frac{1}{2}}\nu | A\Delta_\omega^{-\frac{1}{2}}\eta\right\rangle_\omega + \lambda^{-1}\left\langle \Delta_\omega^{-\frac{1}{2}}\nu | A\Delta_\omega^{\frac{1}{2}}\eta\right\rangle_\omega = \left\langle \nu | J_\omega B J_\omega \eta\right\rangle_\omega,$$

$\forall \eta, \nu \in \mathcal{D}\left(\Delta_\omega^{-\frac{1}{2}}\right) \cap \mathcal{D}\left(\Delta_\omega^{\frac{1}{2}}\right)$.

Proof

Using the identity $J_\omega \Delta^{1/2} J_\omega = \Delta_\omega^{-1/2}$, it follows that $J_\omega \nu$ is in the domain of $\Delta_\omega^{1/2}$ or equivalently in the domain of S_ω, whenever ν is in the domain of $\Delta_\omega^{-1/2}$ or equivalently in the domain of $F_\omega = J_\omega \Delta_\omega^{-1/2}$. Similarly if η is in the domain of S_ω it follows that $J_\omega \eta$ is in the domain of F_ω. So, if ν is in the domain of both S_ω and F_ω, then $J_\omega \nu$ is also in the domain of both S_ω and F_ω.

As we had related before, let $A \in \mathcal{A}$ and $B \in \mathcal{A}'$ be such that

$$\lambda\left\langle \nu | S_\omega A S_\omega \eta\right\rangle_\omega + \lambda^{-1}\left\langle \nu | F_\omega A F_\omega \eta\right\rangle_\omega = \left\langle \nu | B\eta\right\rangle_\omega,$$

where ν and η are in the domain of both S_ω and F_ω. We now take $J_\omega \nu$ and $J_\omega \eta$ instead of ν and η. Subsituting $S_\omega = J_\omega \Delta_\omega^{1/2}$ and $F_\omega = J_\omega \Delta_\omega^{-1/2}$, and using the fact,

$$J_\omega^2 = 1,$$

it follows that,

$$\begin{aligned}
\left\langle J_\omega \nu | S_\omega A S_\omega J_\omega \eta\right\rangle_\omega &= \left\langle J_\omega \nu | J_\omega \Delta_\omega^{\frac{1}{2}} A J_\omega \Delta_\omega^{\frac{1}{2}} J_\omega \eta\right\rangle_\omega \\
&= \left\langle \Delta_\omega^{\frac{1}{2}} J_\omega^2 \nu | A\Delta_\omega^{-\frac{1}{2}} J_\omega^2 \eta\right\rangle_\omega \\
&= \left\langle \Delta_\omega^{\frac{1}{2}}\nu | A\Delta_\omega^{-\frac{1}{2}}\eta\right\rangle_\omega.
\end{aligned}$$

Similarly we have,

$$\left\langle J_\omega \nu | F_\omega A F_\omega J_\omega \eta\right\rangle_\omega = \left\langle \Delta_\omega^{-\frac{1}{2}}\nu | A\Delta_\omega^{\frac{1}{2}}\eta\right\rangle_\omega.$$

Hence we have,

$$\lambda\left\langle \Delta_\omega^{\frac{1}{2}}\nu | A\Delta_\omega^{-\frac{1}{2}}\eta\right\rangle_\omega + \lambda^{-1}\left\langle \Delta_\omega^{-\frac{1}{2}}\nu | A\Delta_\omega^{\frac{1}{2}}\eta\right\rangle_\omega = \left\langle \nu | J_\omega B J_\omega \eta\right\rangle_\omega.$$

whenever ν and η are in the domain of $\Delta_\omega^{1/2}$ and $\Delta_\omega^{-1/2}$. □

1.2.2 | Tomita's Theorem

The modular relation gives rise to a relation between $\mathcal{A}$ and $\mathcal{A}'$ for each $\lambda \in \mathbb{R}_+$. We now remove the dependence on λ. The idea is to think of the relation as the Fourier transform of a complex, rapidly decreasing function. The parameter λ can then be viewed in terms of characters. We need to develop certain tools to be able to do this.

Let f be an function bounded on the strip, $S \equiv \{-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$. Then we can define a meromorphic function by,

$$g(z) = \pi f(z)(\sin(\pi z))^{-1}.$$

This is a meromorphic function with a pole at $z = 0$. Since the power series expansion of $\sin(\pi z) = \pi z - \pi^3 z^3/3! + \cdots = z(\pi - \pi^3 z^2/3! + \cdots)$, it follows that it is a simple pole. Hence, the residue is given by,

$$\mathrm{res}_g(0) = \lim_{z\to 0} \pi f(z)\big(\pi - \tfrac{\pi^3 z^2}{3!} + \cdots\big)^{-1} = f(0).$$

By residue theorem, for any loop $\gamma \subseteq \mathcal{H}(S\backslash E)$, we have

$$\sum_{x\in E} \mathrm{res}_g(x) n(\gamma, x) = (2\pi i)^{-1} \int_\gamma g(z)dz,$$

where $n(\gamma, x)$ is the winding number for the loop γ around the point x, which corresponds to the number of times the loop 'winds' around the point. As z tends to infinity, since the function f is assumed to be bounded, the meromorphic function g rapidly tends to zero for $|\chi| < \pi$. Since there is only one pole for g, and at 0, for any loop γ not passing through 0 we have,

$$f(0)n(\gamma, 0) = (2\pi i)^{-1}\Big[\int g(\gamma(t))\gamma'(t)dt\Big].$$

Now, choose the loop to be the one along the infinite rectangle, then we can split the curve into 4 parts two of which are along the lines, $z = \pm\frac{1}{2}$, given by, $\gamma_\pm(t) = \pm it \pm \frac{1}{2}$, the other two curves are at infinity, and since g rapidly dies at infinity, they will not have any contribution. Hence we have,

$$\begin{aligned} f(0) &= (2\pi i)^{-1} \int_{\mathbb{R}} \Big[ig\big(it + \tfrac{1}{2}\big) - ig\big(it - \tfrac{1}{2}\big)\Big]dt \\ &= (2\pi i)^{-1} \int_{\mathbb{R}} i\pi\Bigg[\Big[\frac{f(it + \frac{1}{2})}{\sin\big(i\pi t + \frac{\pi}{2}\big)}\Big] - \Big[\frac{f(it - \frac{1}{2})}{\sin\big(i\pi t - \frac{\pi}{2}\big)}\Big]\Bigg]dt \end{aligned}$$

Using the fact that $\sin(i\pi t + \pi/2) = \cos(i\pi t) = \cosh(\pi t)$, and $\sin(i\pi t - \pi/2) = -\cos(i\pi t) = -\cosh(\pi t)$, and by substituting $\cosh(\pi t) = (e^{\pi t} + e^{-\pi t})/2$, we obtain,

$$f(0) = \int_{\mathbb{R}} \Big[\frac{f\big(it + \frac{1}{2}\big) + f\big(it - \frac{1}{2}\big)}{e^{\pi t} + e^{-\pi t}}\Big]dt.$$

We now use this formula to obtain an operator equation using the modular relation;

Lemma 1.9. (Operator Equation) *If $A \in \mathcal{A}, B \in \mathcal{A}'$ are such that*

$$\lambda\big\langle\Delta_\omega^{\frac{1}{2}}\nu|A\Delta_\omega^{-\frac{1}{2}}\eta\big\rangle_\omega + \lambda^{-1}\big\langle\Delta_\omega^{-\frac{1}{2}}\nu|A\Delta_\omega^{\frac{1}{2}}\eta\big\rangle_\omega = \big\langle\nu|J_\omega BJ_\omega\eta\big\rangle_\omega$$

for any $\eta, \nu \in \mathcal{D}(\Delta_\omega^{-\frac{1}{2}}) \cap \mathcal{D}(\Delta_\omega^{\frac{1}{2}})$, then,

$$A = \int_{\mathbb{R}} \lambda^{2it}\Big[\frac{\Delta_\omega^{it}J_\omega BJ_\omega\Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big]dt.$$

Proof

For every $\nu, \eta \in \mathcal{D}(\Delta_\omega^{-1/2}) \cap \mathcal{D}(\Delta_\omega^{1/2})$, consider the complex valued function

$$f_{\nu,\eta}(z) = \lambda^{2z} \Big\langle \Delta_\omega^{\overline{z}} \nu \mid A \Delta_\omega^{-z} \eta \Big\rangle_\omega.$$

By strong continuity of Δ_ω^z, normality of the action of $\mathcal{A}$ on $\mathcal{H}_\omega$, and by skew-linearity with respect to the first argument in the inner product, $f_{\nu,\eta}$ is an analytic function, and we have,

$$\begin{aligned} f_{\nu,\eta}\big(it + \tfrac{1}{2}\big) &= \lambda^{2(it+\frac{1}{2})} \Big\langle \Delta_\omega^{\overline{it+\frac{1}{2}}} \nu \;\Big|\; A \Delta_\omega^{-it-\frac{1}{2}} \eta \Big\rangle_\omega \\ &= \lambda^{2(it+\frac{1}{2})} \Big\langle \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \nu \;\Big|\; A \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \eta \Big\rangle_\omega \\ &= \lambda^{2(it+\frac{1}{2})} \Big\langle \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \nu \;\Big|\; A \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \eta \Big\rangle_\omega. \end{aligned}$$

Similarly we have,

$$\begin{aligned} f_{\nu,\eta}\big(it - \tfrac{1}{2}\big) &= \lambda^{2(it-\frac{1}{2})} \Big\langle \Delta_\omega^{\overline{it-\frac{1}{2}}} \nu \;\Big|\; A \Delta_\omega^{-it+\frac{1}{2}} \eta \Big\rangle_\omega \\ &= \lambda^{2(it-\frac{1}{2})} \Big\langle \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \nu \;\Big|\; A \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \eta \Big\rangle_\omega \\ &= \lambda^{2(it-\frac{1}{2})} \Big\langle \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \nu \;\Big|\; A \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \eta \Big\rangle_\omega. \end{aligned}$$

$f_{\nu,\eta}$ is also bounded in the strip $S = \{-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$, so $g(z) = \pi f_{\nu,\mu}(z)(\sin(\pi z))^{-1}$ decreases rapidly as z tends to infinity. Hence by the calculation prior this lemma, we have,

$$f_{\nu,\eta}(0) = \int_{\mathbb{R}} \left[\frac{f_{\nu,\eta}(it + \frac{1}{2}) + f_{\nu,\eta}(it - \frac{1}{2})}{e^{\pi t} + e^{-\pi t}} \right] dt.$$

Since $f_{\nu,\eta}(0) = \big\langle \nu | A\eta \big\rangle_\omega$, we have,

$$\Big\langle \nu | A\eta \Big\rangle_\omega = \int_{\mathbb{R}} \lambda^{2(it+\frac{1}{2})} \left[\frac{\langle \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \nu | A \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \eta \rangle_\omega}{e^{\pi t} + e^{-\pi t}} \right] dt + \int_{\mathbb{R}} \lambda^{2(it-\frac{1}{2})} \left[\frac{\langle \Delta_\omega^{-\frac{1}{2}} \Delta_\omega^{-it} \nu | A \Delta_\omega^{\frac{1}{2}} \Delta_\omega^{-it} \eta \rangle_\omega}{e^{\pi t} + e^{-\pi t}} \right] dt.$$

Note that $\Delta_\omega^{-it}\nu, \Delta_\omega^{-it}\eta \in \mathcal{D}(\Delta_\omega^{-1/2}) \cap \mathcal{D}(\Delta_\omega^{1/2})$. Hence using the modular relation;

$$\lambda \big\langle \Delta_\omega^{\frac{1}{2}} \nu | A \Delta_\omega^{-\frac{1}{2}} \eta \big\rangle_\omega + \lambda^{-1} \big\langle \Delta_\omega^{-\frac{1}{2}} \nu | A \Delta_\omega^{\frac{1}{2}} \eta \big\rangle_\omega = \big\langle \nu | J_\omega B J_\omega \eta \big\rangle_\omega,$$

we get,

$$\begin{aligned} \big\langle \nu | A\eta \big\rangle_\omega &= \int_{\mathbb{R}} \lambda^{it} \Big\langle \Delta_\omega^{-it} \nu \;\Big|\; \Big[\frac{J_\omega B J_\omega}{e^{\pi t} + e^{-\pi t}} \Big] \Delta_\omega^{-it} \eta \Big\rangle_\omega dt \\ &= \Big\langle \nu \;\Big|\; \Big[\int_{\mathbb{R}} \lambda^{it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}} \Big] dt \Big] \eta \Big\rangle_\omega. \end{aligned}$$

Since the choice of $\nu, \eta \in \mathcal{D}(\Delta_\omega^{-1/2}) \cap \mathcal{D}(\Delta_\omega^{1/2})$ was arbitrary, it follows that,

$$A = \int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}} \Big] dt.$$

$\square$

With this lemma, we got rid of ν and η and showed that A and B can be related directly as an operator equation. We now have all the ingredients to prove Tomita's theorem, which establishes a relation between von Neumann algebras and their commutants in terms of the modular operator and the modular conjugation.

For every $B \in \mathcal{A}'$,

$$\int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt \in \mathcal{A}.$$

Consider the function,

$$f_{\nu,\eta}(t) = \Big\langle \eta \,\Big|\Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] \nu \Big\rangle_\omega$$

The damping factor $1/(e^{\pi t} + e^{-\pi t})$ ensures that this function decreases faster than any polynomial in t, and allows us to do Fourier analysis. We now use the positivity of λ. Since $\lambda \in \mathbb{R}_+$ there exists χ such that $\lambda = e^{\chi/2}$. The Fourier transform of $f_{\nu,\eta}(t)$ is

$$\begin{aligned}
\mathcal{F} f_{\nu,\eta}(\chi) &= \int_{\mathbb{R}} \lambda^{2it} \Big\langle \eta \,\Big|\Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] \nu \Big\rangle_\omega dt \\
&= \Big\langle \eta \,\Big|\Big[\int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt\Big] \nu \Big\rangle_\omega .
\end{aligned}$$

We now prove Tomita's theorem via injectivity of Fourier transforms.

Theorem 1.10. (Tomita) *If J_ω and Δ_ω are as before, then*

$$J_\omega \mathcal{A} J_\omega = \mathcal{A}', \quad \Delta_\omega^{it} \mathcal{A} \Delta_\omega^{-it} = \mathcal{A} \ \forall t \in \mathbb{R}.$$

Proof

Similar to $f_{\nu,\eta}$, consider the function,

$$g_{\nu,\eta}(t) = \Big\langle \eta \,\Big| U \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] U^\dagger \nu \Big\rangle_\omega,$$

where U is a unitary operator in $\mathcal{A}'$. The Fourier transform of $g_{\nu,\eta}$ is given by,

$$\begin{aligned}
\mathcal{F} g_{\nu,\eta}(\chi) &= \int_{\mathbb{R}} \lambda^{2it} \Big\langle \eta \,\Big| U \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] U^\dagger \nu \Big\rangle_\omega dt \\
&= \Big\langle U^\dagger \eta \,\Big|\Big[\int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt\Big] U^\dagger \nu \Big\rangle_\omega .
\end{aligned}$$

Since $\langle U\nu | U\eta \rangle_\omega = \langle \nu | \eta \rangle_\omega$ it follows that,

$$\mathcal{F} g_{\nu,\eta}(\chi) = \Big\langle \eta \,\Big|\Big[\int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt\Big] \nu \Big\rangle_\omega = \mathcal{F} f_{\nu,\eta}(\chi).$$

By injectivity of Fourier transforms it follows that $f_{\nu,\eta} = g_{\nu,\eta}$. Since the choice of ν and η was also arbitrary it follows that $U(\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it}) = (\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it})U$. Since the choice of the unitary $U \in \mathcal{A}'$ was arbitrary we conclude that

$$\Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it} \in \mathcal{A} \ \ \forall t \in \mathbb{R}.$$

By taking $t = 0$ we have, $J_\omega \mathcal{A}' J_\omega \subseteq \mathcal{A}$ and by symmetry we also have, $J_\omega \mathcal{A} J_\omega \subseteq \mathcal{A}'$. Hence we have,

$$J_\omega \mathcal{A} J_\omega = \mathcal{A}'.$$

Hence every $A \in \mathcal{A}$ is of the form, $A = J_\omega B J_\omega$ for some $B \in \mathcal{A}'$. We have, $\Delta_\omega^{it} A \Delta_\omega^{-it} = \Delta_\omega^{it} J_\omega B J_\omega \Delta_\omega^{-it} \in \mathcal{A}$. Hence it follows that,

$$\Delta_\omega^{it} \mathcal{A} \Delta_\omega^{-it} \in \mathcal{A}.$$

This completes the proof of the theorem. $\sigma_t^\omega(A) := \Delta_\omega^{it} A \Delta_\omega^{-it}$ is called the modular automorphism of $\mathcal{A}$ associated with ω. □

1.3 | The KMS Condition

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