

# PART III

## GELFAND THEORY

In these notes we go through the basic theory of  $C^*$ -algebras and spectral theory. The topics in this document will include spectral mapping theorem, Gelfand-Naimark theory.

### 1 | SPECTRAL MAPPING THEOREM

Our goal is now to abstract out the properties of operators on Hilbert spaces and study them. This will help us study more general quantum systems. Let  $\mathcal{A}$  be a Banach space over  $\mathbb{C}$  i.e.,  $\mathcal{A}$  is a vector space with a norm such that it's also complete under this norm. It's called a Banach algebra if it has a product structure such that,

$$\|AB\| \leq \|A\| \|B\|.$$

Since  $\|A_1B_1 - A_2B_2\| \leq \|A_1\| \|B_1 - B_2\| + \|B_2\| \|A_1 - A_2\|$  the multiplication map is continuous. An involutive Banach algebra or a  $*$ -algebra is a Banach algebra with a  $*$ -operation,

$$A \mapsto A^*$$

such that,  $(A^*)^* = A$ ,  $(A + B)^* = A^* + B^*$ ,  $(\lambda A)^* = \bar{\lambda}A^*$ ,  $(AB)^* = B^*A^*$ , and  $\|A\| = \|A^*\|$ . All these properties are imported from what we expect from the adjoint operation on operators on Hilbert spaces.

A  $*$ -algebra that satisfies,

$$\|A^*A\| = \|A^*\| \|A\| = \|A\|^2,$$

is called a  $C^*$ -algebra. It can be checked that the algebra of bounded operators on a Hilbert space  $\mathcal{H}$  forms a  $C^*$ -algebra with respect to the adjoint operation. By embedding into the operators on the algebra every  $C^*$ -algebra can be made to contain the unit element. Hence we will assume every  $C^*$ -algebra to be unital in this document.

An element  $A \in \mathcal{A}$  is said to be invertible if there exists a unique element  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = 1$ . The set of all invertible elements forms a group and is called the general linear group of  $\mathcal{A}$ . Denoted by  $\mathcal{G}(\mathcal{A})$ . Our aim is to show that  $\mathcal{G}(\mathcal{A})$  is open in  $\mathcal{A}$ .

Consider the unit ball around  $1 \in \mathcal{A}$ , i.e.,  $B_1(1) = \{A \mid \|A - 1\| \leq 1\}$ . Since  $\|A - 1\| \leq 1$ ,  $\sum_{n \geq 0} \|A - 1\|^n < \infty$ , so,  $A' = \sum_{n \geq 0} (A - 1)^n$  converges.

$$\begin{aligned} AA' &= A'A = (1 - (1 - A))A' = A' - (1 - A)A' = \sum_{n \geq 0} (1 - A)^n - (1 - A)A' \\ &= \sum_{n \geq 0} (1 - A)^n - \sum_{n \geq 1} (1 - A)^n = 1. \end{aligned}$$

So, every  $A \in B_1(1)$  is invertible. As a corollary, if  $\|A\| < |\lambda|$ , then  $(A - \lambda)$  is invertible with the inverse,  $(A - \lambda)^{-1} = -\sum_{n \geq 0} A^n / \lambda^{n+1}$ . Since left multiplication by an element  $L_B(A) = BA$  is continuous, for  $B \in \mathcal{G}(\mathcal{A})$ ,  $L_B$  is invertible with inverse  $L_{B^{-1}}$ .

Since the open unit ball around 1 is invertible 1 is in the interior of  $\mathcal{G}(\mathcal{A})$ . Using this we can obtain open balls around every element  $B \in \mathcal{G}(\mathcal{A})$  using translations.  $B \in \mathcal{G}(\mathcal{A})$ , then the continuous map  $L_B$  takes the open ball around 1 to an open ball around  $B$  i.e.,  $L_B(B_1(1))$  is an open ball around  $B$  entirely contained in  $\mathcal{G}(\mathcal{A})$ . Hence  $\mathcal{G}(\mathcal{A})$  is open.

Let  $A \in \mathcal{A}$ , the spectrum of  $A$  in  $\mathcal{A}$  is defined as,

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda) \text{ is not invertible}\}.$$

$\sigma(A)$  is a closed subset of the disk  $\{\lambda \mid |\lambda| \leq \|A\|\}$ . For any  $\lambda \notin \sigma(A)$ , the resolvent of  $A$  is defined as,

$$R_A(\lambda) = (\lambda - A)^{-1}$$

where  $R_A : \mathbb{C} \setminus \sigma(A) \rightarrow \mathcal{A}$ . If  $\lambda, \mu \notin \sigma(A)$  then we have,

$$\begin{aligned} (\mu - \lambda)\mathbb{I} &= (\mu - A) - (\lambda - A) \\ &= (\lambda - A)(\lambda - A)^{-1}(\mu - A) - (\lambda - A)(\mu - A)(\mu - A)^{-1} \\ &= (\lambda - A)R_A(\lambda)(\mu - A) - (\lambda - A)R_A(\mu)(\mu - A)^{-1} \\ &= (\lambda - A)[R_A(\lambda) - R_A(\mu)](\mu - A) \end{aligned}$$

So we have,

$$R_A(\lambda)(\mu - \lambda)R_A(\mu) = R_A(\lambda)(\lambda - A)[R_A(\lambda) - R_A(\mu)](\mu - A)R_A(\mu)$$

$$\frac{R_A(\lambda) - R_A(\mu)}{\mu - \lambda} = R_A(\lambda)R_A(\mu)$$

So, as  $\lambda \rightarrow \mu$ ,  $R'_A(\lambda)$  exists and is equal to  $-R_A(\lambda)^2$ .  $R_A(\lambda)$  is continuous in  $\lambda$ .  $R_A(\lambda)$  is analytic  $\mathcal{A}$  valued function on  $\mathbb{C} \setminus \sigma(A)$ , i.e., complex derivative  $R'_A(\lambda)$  exists and is continuous.

Suppose  $\sigma(A)$  is empty, then  $R_A$  is an analytic function on all of  $\mathbb{C}$ . As  $\lambda \rightarrow \infty$  we have,

$$\|R_A(\lambda)\| = |\lambda|^{-1} \|(1 - \lambda^{-1}A^{-1})^{-1}\|$$

Since  $(1 - \lambda^{-1}A^{-1}) \rightarrow 1$  as  $\lambda \rightarrow \infty$  we have,  $\|R_A(\lambda)\| \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Since  $\lim_{\lambda \rightarrow \mu} [\varphi(R_A(\lambda)) - \varphi(R_A(\mu))]/(\lambda - \mu) = \lim_{\lambda \rightarrow \mu} (\varphi(R_A(\lambda) - R_A(\mu)))/(\lambda - \mu)$ . So,  $\varphi \circ R_A$  is a bounded analytic function. Since bounded entire functions are constant by Liouville's theorem  $R_A$  is a constant function, equal to zero which is a contradiction.  $\sigma(A)$  is also closed and bounded hence it's closed.

**LEMMA 1.1.** *If  $A \in \mathcal{A}$  then  $\sigma(A) \subset \mathbb{C}$  is nonempty and compact.*

Suppose there exists  $A \neq \lambda 1$ , then  $A - \lambda 1 \neq 0$ , if every element of  $\mathcal{A}$  is invertible we have,  $(A - \lambda)$  is invertible for all  $\lambda \in \mathbb{C}$  or  $\sigma(A)$  is empty which cannot happen by previous lemma.

**THEOREM 1.2. (GELFAND-MAZUR)** *If  $\mathcal{A}$  is a Banach algebra in which every non-zero element is invertible, then  $\mathcal{A} \cong \mathbb{C}$ .*

If  $p(z)$  is a polynomial, then the map  $p(z) \mapsto p(A)$  is a homomorphism from  $\mathbb{C}[z]$  to the algebra generated by 1 and  $A$  denoted by  $[1, A]$ .

**THEOREM 1.3. (SPECTRAL MAPPING THEOREM)**  $p(z) = \sum_{i=0}^N a_i z^i$ . Then,

$$\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) \mid \lambda \in \sigma(A)\}$$

PROOF

Fix  $\lambda \in \mathbb{C}$ , without loss of generality assume  $a_N \neq 0$ . Then we have by fundamental theorem of algebra,  $p(z) - \lambda = a_N \prod_{i=1}^N (z - \lambda_i)$  since  $p(z) \mapsto p(A)$  is an algebra homomorphism we have,

$$p(A) - \lambda = a_N \prod_{i=1}^N (A - \lambda_i)$$

So,  $\lambda \notin \sigma(p(A))$  if and only if  $\lambda_i \notin \sigma(A)$  and  $\lambda \notin p(\sigma(A))$ . □

The spectrum depends on the ambient algebra. If  $A - \lambda$  is invertible in  $\mathcal{A}$  with inverse  $(A - \lambda)^{-1}$  but  $(A - \lambda)^{-1}$  might not be in  $\mathcal{B} \subsetneq \mathcal{A}$ . So we have,

$$\sigma_{\mathcal{B}}(A) \supset \sigma_{\mathcal{A}}(A).$$

where  $\sigma_{\mathcal{B}}(A)$  is the spectrum with respect to  $\mathcal{B}$ . The spectral radius is defined as follows,

$$\rho(A) = \sup_{\lambda \in \sigma(A)} \{|\lambda|\}$$

Clearly  $\rho(A) \leq \|A\|$  because otherwise there exists some  $\lambda \in \mathbb{C}$  with  $|\lambda| > \|A\|$  or  $(A - \lambda)$  is invertible. The spectral radius is given by the following formula (hard proof which I will skip here),

**THEOREM 1.4. (SPECTRAL RADIUS FORMULA)**

$$\rho(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

□

## 2 | GELFAND-NAIMARK THEORY

The Gelfand-Naimark theorem gives a Hilbert nullstellensatz type relation between geometric objects and commutative  $C^*$ -algebras. All algebras in this section will be assumed unital and commutative.

Let  $\mathcal{A}$  be a commutative Banach algebra, a multiplicative functional  $\varphi$  is a linear functional that's also an algebra homomorphism,  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ ,

$$\varphi : AB \mapsto \varphi(A)\varphi(B).$$

The set of all multiplicative functionals will be called the spectrum of  $\mathcal{A}$  denoted by,  $\sigma(\mathcal{A})$ . The reason for this name will soon become clear. Multiplicative linear functionals are also called characters in some books.

Let  $\varphi \in \sigma(\mathcal{A})$ , for any  $A \in \mathcal{A}$ , we have,  $\varphi(A) = \varphi(1 \cdot A) = \varphi(1)\varphi(A)$ , or  $\varphi(1) = 1$ . If  $A$  is invertible then  $\varphi(A^{-1})\varphi(A) = \varphi(A^{-1}A) = 1$  or  $\varphi(A)$  is non-zero. Suppose  $|\varphi(A)| \not\leq \|A\|$ , then,  $A - |\varphi(A)|$  is invertible.

$$\varphi(A - |\varphi(A)|) = \varphi(A) - |\varphi(A)|$$

adjusting the phase of  $A$  this term can be made zero. This is however a contradiction as  $\varphi$  is non-zero for invertible elements of  $\mathcal{A}$ . So for every  $\varphi \in \sigma(\mathcal{A})$ , we have  $|\varphi(A)| \leq \|A\|$ . Equipped with the weak\* topology,  $\sigma(\mathcal{A})$  is a closed subset of the closed unit ball  $B$  of  $\mathcal{A}^*$ .

$$\sigma(\mathcal{A}) \subset B, \text{ is closed}$$

By Alaoglu's theorem,  $\sigma(\mathcal{A})$  is a compact Hausdorff space.

A left (or right, in our case it's irrelevant as we are dealing with commutative algebras) ideal of  $\mathcal{A}$  is a subalgebra  $\mathcal{I} \subset \mathcal{A}$  such that  $AB \in \mathcal{I}$  whenever  $A \in \mathcal{I}$  and for all  $B \in \mathcal{A}$ .  $\mathcal{I}$  is a proper ideal if  $\mathcal{I} \neq \mathcal{A}$ , and  $\mathcal{I}$  is a maximal ideal if it's not contained in any proper ideal. If an ideal contains invertible an element, say  $A$  then  $AA^{-1} = 1 \in \mathcal{I}$  which means that  $B \in \mathcal{I}$  for all  $B \in \mathcal{A}$ , or  $\mathcal{I} = \mathcal{A}$ . If  $A \in \mathcal{A}$  is not invertible then  $\mathcal{I}_A = \{BA \mid B \in \mathcal{A}\}$  is an ideal containing  $A$ . Let  $\overline{\mathcal{I}}$  be the closure of  $\mathcal{I}$ . Since the invertible elements of  $\mathcal{A}$  form a group and is an open set in  $\mathcal{A}$ .  $\overline{\mathcal{I}}$  cannot contain the identity of  $\mathcal{A}$ .  $\overline{\mathcal{I}}$  is a proper ideal. Every ideal is contained in some maximal ideal, and since the closure of a proper ideal is also a proper ideal, the maximal ideals are closed. The collection of all maximal ideals of  $\mathcal{A}$  will be denoted by  $\mathcal{M}(\mathcal{A})$ . Every non invertible element is contained in some maximal ideal.

Let  $\varphi \in \sigma(\mathcal{A})$ , for  $A \in \ker(\varphi)$ , and for all  $B \in \mathcal{A}$ ,

$$\varphi(AB) = \varphi(A)\varphi(B) = 0,$$

so  $AB \in \ker(\varphi)$ . So it's an ideal. Since  $\varphi(1) = 1 \notin \ker(\varphi)$  it's a proper ideal. Suppose  $\ker(\varphi)$  is not a maximal ideal, and let  $\ker(\varphi) \subsetneq \mathcal{I}$  with  $\mathcal{I}$  a proper ideal.

Let  $A \in \mathcal{I} \setminus \ker(\varphi)$ , then we have,  $A = (A - \varphi(A) \cdot 1) + \varphi(A) \cdot 1$ . So, we can write  $A = A' + \lambda \cdot 1$ , for some  $A' = A - \varphi(A) \cdot 1 \in \ker(\varphi)$  and  $\lambda \in \mathbb{C}$ . So,  $1$  is in the span of  $A$  and  $\ker(\varphi)$ . Equivalently,  $\mathcal{I} = \mathcal{A}$  (!).  $\ker(\varphi)$  is indeed a maximal ideal. Our goal is to relate the maximal ideals and multiplicative linear functionals.

### THEOREM 2.1.

$$\varphi \mapsto \ker(\varphi),$$

is a one-to-one correspondence between  $\sigma(\mathcal{A})$  and  $\mathcal{M}(\mathcal{A})$ .

### PROOF

Suppose  $\ker(\varphi) = \ker(\varkappa)$ , every  $A \in \mathcal{A}$  can be written as,  $A = \varphi(A) \cdot 1 + B$  for some  $B \in \ker(\varphi)$ . So we have,  $\varkappa(A) = \varphi(A)\varkappa(1) + \varkappa(B)$ . Since  $\ker(\varphi) = \ker(\varkappa)$  we have  $\varkappa(B) = 0$  and hence for all  $A \in \mathcal{A}$ ,

$$\varphi(A) = \varkappa(A),$$

or  $\varphi = \varkappa$ . Hence the mapping  $\varphi \mapsto \ker(\varphi)$  is injective.

Suppose  $\mathcal{I}$  is a maximal ideal. Let  $\pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$  be the quotient map.  $\mathcal{A}/\mathcal{I}$  inherits algebra structure from  $\mathcal{A}$  and also inherits a norm  $\|A + \mathcal{I}\| = \inf\{\|A + I\| \mid I \in \mathcal{I}\}$  making it a Banach algebra.

$\mathcal{A}/\mathcal{I}$  has no non-trivial ideals, because otherwise if  $\mathcal{I}'$  is an ideal of  $\mathcal{A}/\mathcal{I}$  then, consider  $\pi^{-1}(\mathcal{I}')$ . For all  $J \in \pi^{-1}(\mathcal{I}')$  and  $A \in \mathcal{A}$  since  $\pi(J) \in \mathcal{I}'$  we have,

$$\pi(JA) = \pi(J)\pi(A) \in \mathcal{I}'.$$

So,  $JA \in \pi^{-1}(\mathcal{I}')$  and hence  $\pi^{-1}(\mathcal{I}')$  is an ideal. Since  $\mathcal{I} \subsetneq \pi^{-1}(\mathcal{I}')$  it cannot be a maximal ideal. This is a contradiction as we assumed it to be a maximal ideal. Hence every non-zero element of  $\mathcal{A}/\mathcal{I}$  is invertible because otherwise we can construct an ideal containing the element. By Gelfand-Mazur theorem we have,

$$\mathcal{A}/\mathcal{I} \cong \mathbb{C} \cdot 1$$

Let the above isomorphism be  $\varphi$ . The composition,  $\varphi \circ \pi$  is in  $\sigma(\mathcal{A})$  with  $\ker(\varphi \circ \pi) = \mathcal{I}$ . The map  $\varphi \mapsto \ker(\varphi)$  is surjective.  $\square$

This allows us to think of  $\mathcal{M}(\mathcal{A})$  as a compact Hausdorff space. For every  $A \in \mathcal{A}$  we have a map,  $\widehat{A}(\varphi) = \varphi(A)$ . With the weak\* topology on  $\sigma(\mathcal{A})$ ,  $\widehat{A}$  is a continuous map on  $\sigma(\mathcal{A})$ . The map,

$$\Gamma : A \mapsto \widehat{A}$$

is called Gelfand transformation on  $\mathcal{A}$ . It's a map from  $\mathcal{A}$  to  $C(\sigma(\mathcal{A}))$ . Here  $C(X)$  means continuous maps on  $X$  to  $\mathbb{C}$ . If  $A, B \in \mathcal{A}$  then we have,

$$\widehat{AB}(\varphi) = \varphi(AB) = \varphi(A)\varphi(B) = \widehat{A}(\varphi)\widehat{B}(\varphi).$$

So, the Gelfand transformation is an algebra homomorphism, and  $\widehat{1}(\varphi) = \varphi(1) = 1$ , so  $\widehat{1}$  is a constant function. If  $A$  is invertible then for all  $\varphi \in \sigma(\mathcal{A})$  we have,  $\varphi(AA^{-1}) = 1$  or  $\varphi(A)$  is non vanishing. Conversely suppose  $\widehat{A}$  is never vanishing, and suppose  $A$  is not invertible, then there exists a maximal ideal  $\mathcal{I}_A$  containing  $A$ . Let the associate multiplicative functional be  $\varphi_A$  such that  $\ker \varphi_A = \mathcal{I}_A$ . So we have,

$$\varphi_A(A) = \widehat{A}(\varphi_A) = 0$$

this is a contradiction as we started with the assumption that  $\widehat{A}$  is non-vanishing. Hence  $A$  is invertible if and only if  $\widehat{A}$  is non-vanishing. A \*-algebra  $\mathcal{A}$  is said to be symmetric if

$$\Gamma(A^*) = \widehat{A^*} = \overline{\widehat{A}}.$$

Our goal is to show that for commutative  $C^*$ -algebras the Gelfand transform is an isometric isomorphism.

**THEOREM 2.2.**

$$\|\widehat{A}\|_{sup} \leq \|A\|.$$

PROOF

Let  $\lambda \in \sigma(A)$ , i.e.,  $A - \lambda$  is not invertible. There exists  $\varphi_A$  such that  $\varphi_A(A - \lambda) = 0$ . So, we have,

$$\varphi_A(A) = \lambda.$$

So,  $\lambda$  is in the range of  $\widehat{A}$ . Conversely, suppose  $\mu$  is in the range of  $\widehat{A}$ , then there exists  $\varphi \in \sigma(\mathcal{A})$  such that  $\widehat{A}(\varphi) = \mu$ , or  $\varphi(A - \lambda) = 0$ , which means that  $A - \lambda$  is not invertible. So, range of  $\widehat{A}$  is same as spectrum of  $\sigma(A)$ .

Now,  $\|\widehat{A}\|_{sup} = \sup_{\varphi \in \sigma(\mathcal{A})} \{|\widehat{A}(\varphi)|\}$ . So,  $\|\widehat{A}\|_{sup} = \rho(A) \leq \|A\|$ .  $\square$

Suppose  $\mathcal{A}$  is symmetric, i.e.,  $\widehat{A^*} = \overline{\widehat{A}}$ , then for all self-adjoint elements,  $A = A^*$ ,  $\widehat{A} = \overline{\widehat{A}}$ .  $\widehat{A}$  is a real valued function. Conversely, every element  $A$  can be written as a combination of self-adjoint operators,  $A = A_1 + iA_2$ , so we have,  $A^* = A_1^* - iA_2^*$ , and hence,

$$\widehat{A^*} = \widehat{A_1} - i\widehat{A_2} = \overline{\widehat{A}}.$$

So,  $\mathcal{A}$  is symmetric if and only if  $\widehat{A}$  is real valued function for self-adjoint  $A$ .

If  $\mathcal{A}$  is a  $C^*$ -algebra then we have  $\|B^*B\| = \|B\|^2$  for all  $B \in \mathcal{A}$ . Let  $A \in \mathcal{A}$  be self-adjoint, consider  $B = A + it$ , then we have,

$$\|B\|^2 = \|B^*B\| = \|A\|^2 + t^2$$

Since,  $\varphi(B)^2 \leq \|B\|^2 = \|A\|^2 + t^2$ , we get,

$$\begin{aligned} \varphi(A + it)^2 &= (Re(\varphi(A)) + iIm(\varphi(A)) + it)^2 \\ &= Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t + t^2 \leq \|A\|^2 + t^2. \end{aligned}$$

Which means  $Re(\varphi(A))^2 + Im(\varphi(A))^2 + 2Im(\varphi(A))t \leq \|A\|^2$  i.e., right side is independent of  $t$ , so on the left side  $Im(\varphi(A))$  must be zero. Hence  $\varphi(A)$  is real valued for all  $\varphi \in \sigma(\mathcal{A})$  or equivalently  $\widehat{A}$  is real valued for all  $A = A^*$ . Hence  $C^*$ -algebras are symmetric.

**THEOREM 2.3.** *If  $\mathcal{A}$  is symmetric then  $\Gamma(\mathcal{A})$  is dense in  $C(\sigma(\mathcal{A}))$ .*

PROOF

The proof is an application of Stone-Weierstrass theorem, [2]. If  $\mathcal{A}$  is symmetric then  $\Gamma(\mathcal{A})$  is closed under complex conjugation because,

$$\Gamma(A)^* = \Gamma(A^*).$$

So,  $\Gamma(\mathcal{A})$  is a self-adjoint subalgebra.  $\Gamma(1) = 1$ , so  $\Gamma(\mathcal{A})$  contains constant functions, and  $\Gamma(\mathcal{A})$  separates the points on  $\sigma(\mathcal{A})$ , because if  $\varphi, \varkappa \in \sigma(\mathcal{A})$  with  $\varphi \neq \varkappa$  then there exists  $A \in \mathcal{A}$  such that  $\varphi(A) \neq \varkappa(A)$  i.e.,  $\Gamma(\mathcal{A})$  is such that  $\Gamma(\mathcal{A})(\varphi) \neq \Gamma(\mathcal{A})(\varkappa)$ .

So by Stone-Weierstrass theorem  $\Gamma(\mathcal{A})$  is a dense subset of  $C(\sigma(\mathcal{A}))$ .  $\square$

Suppose  $A \in \mathcal{A}$ , let  $\sigma(A)$  be the spectrum of the operator, i.e.,  $\sigma(A) = \{\lambda \mid (A - \lambda) \text{ is not invertible}\}$ . Suppose  $\lambda \in \sigma(A)$  then  $A - \lambda$  is not invertible, hence there exists some maximal ideal  $\mathcal{I}_\lambda$  containing  $A - \lambda$ . Let  $\varphi_\lambda \in \sigma(\mathcal{A})$  such that  $\ker(\varphi_\lambda) = \mathcal{I}_\lambda$ . Or equivalently,  $\varphi_\lambda(A - \lambda) = 0$ , or

$$\varphi_\lambda(A) = \lambda$$

So, to each  $\lambda \in \sigma(A)$  we have a multiplicative functional  $\varphi_\lambda$  such that  $\varphi_\lambda(A) = \lambda$ .

If  $\mathcal{A} = [A, 1]$ , i.e., if  $\mathcal{A}$  is generated by the identity and the operator  $A$  then  $\varphi \in \sigma(\mathcal{A})$  is determined by its action on  $A$ . Since  $\varphi(A^{-1}) = \varphi(A)^{-1}$  and  $\varphi(A^*) = \overline{\varphi(A)}$  we have,  $\widehat{A}(\varphi_1) = \widehat{A}(\varphi_2) \implies \varphi_1 = \varphi_2$ . The map,

$$\widehat{A}: \sigma([A, 1]) \rightarrow \sigma(A)$$

is injective and surjective.

**THEOREM 2.4. (GELFAND-NAIMARK THEOREM)** *If  $\mathcal{A}$  is a unital commutative  $C^*$ -algebra then  $\Gamma$  is an isometric  $*$ -isomorphism of  $\mathcal{A}$  to  $C(\sigma(\mathcal{A}))$ .*

#### SKETCH OF PROOF

Suppose  $\mathcal{A}$  is a commutative Banach algebra, we will show that  $\|\widehat{A}\|_{\text{sup}} = \|A\|$  if and only if  $\|A^{2^k}\| = \|A\|^{2^k}$  for  $k \geq 1$ . If  $\|\widehat{A}\|_{\text{sup}} = \|A\|$  then,

$$\|A^{2^k}\| \leq \|A\|^{2^k} = \|\widehat{A}\|_{\text{sup}}^{2^k} = \|\widehat{A^{2^k}}\|_{\text{sup}} \leq \|A^{2^k}\|.$$

Here in the first step we used the product norm inequality, in the second step the assumption that  $\|\widehat{A}\|_{\text{sup}} = \|A\|$ , in the third step the definition of sup norm, and in the fourth step the fact that  $\varphi(A) \leq \|A\|$  for all  $\varphi \in \sigma(\mathcal{A})$ . So,

$$\|\widehat{A}\|_{\text{sup}} = \|A\| \implies \|A^{2^k}\| = \|A\|^{2^k}.$$

Conversely, if  $\|A^{2^k}\| = \|A\|^{2^k}$  for all  $k \geq 1$ , we have,  $\|A^{2^k}\|^{1/2^k} = \|A\|$ , but since  $\lim_k \|A^{2^k}\|^{1/2^k} = \rho(A)$  and since  $\|\widehat{A}\|_{\text{sup}} = \rho(A)$ , we have,

$$\|A^{2^k}\| = \|A\|^{2^k} \implies \|\widehat{A}\|_{\text{sup}} = \|A\|.$$

Now for the case of commutative  $C^*$ -algebra  $\mathcal{A}$ , for any  $B \in \mathcal{A}$ , the element  $A = B^*B$  is self-adjoint and hence,

$$\|A^{2^k}\| = \|(A^{2^k-1})^*(A^{2^k-1})\| = \|A^{2^k-1}\|^2.$$

So, we have  $\|A^{2^k}\| = \|A\|^{2^k}$  and hence  $\|\widehat{A}\|_{\text{sup}} = \|A\|$ . Since  $\mathcal{A}$  is a  $C^*$ -algebra we also have,  $\|B^*B\| = \|B\|^2$ , so we have,

$$\|B\|^2 = \|A\| = \|\widehat{A}\|_{\text{sup}} = \|\widehat{|B|^2}\|_{\text{sup}} = \|\widehat{B}\|_{\text{sup}}^2.$$

$\Gamma$  is an isometry with closed, dense and injective range. □

#### REFERENCES

- [1] V S SUNDER, Functional Analysis: Spectral Theory, Birkhauser Advanced Texts, 1991
- [2] Stone-Weierstrass theorem, Wikipedia