Part III

Abelian Sheaves

The goal of this document is to study abelian sheaves, i.e., sheaves with values in an abelian category, in particular construct operations such as hom and tensor, and direct and inverse images. These are four of Grothendieck's 'six operations'.

1 | Category of Abelian Sheaves

An abelian pre-sheaf on a topological space X is a functor $\mathcal{F}$,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathcal{A},$$

where $\mathcal{A}$ is an abelian category. Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in \mathcal{F}U_i$ such that $f_i x = f_j x$ for every $x \in U_i \prod U_j$ then it means that there exists a continuous function $f \in \mathcal{F}U$ such that $f_i = f|_{U_i}$. The maps $f_i \in \mathcal{F}U_i$ and $f_j \in \mathcal{F}U_j$ represent the restriction of same map f if,

$$f|_{U_i \prod U_j} = f_i|_{U_i \prod U_j} = f_j|_{U_i \prod U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \prod U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \prod U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \prod U_j} = f|_{U_i}|_{U_i \prod U_j}$ which means that there is a map e from $\mathcal{F}U$ to $\prod_i \mathcal{F}U_i$ such that $pe = qe$. $\mathcal{F}U \to \prod_i \mathcal{F}U_i$

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \prod_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{F}(U_i \prod U_j). \end{tikzcd}$$

This is the collation property. For general categories $\prod$ will be replaced by the fibered product and the covers are replaced by covers on sites.

1.1 | Direct & Inverse Image Sheaves

1.2 | Hom-Tensor Adjointness

1.2.1 | Monoidal Categories

A monoidal category is a category equipped with some notion of 'tensor product' of its objects

References