Part II

Categories of Functors

Yoneda embedding and representable functors allow us to use the nice properties of the category of sets to study more complex categories that are not so nice.

1 | Preliminary Definitions

A set is a collection of 'elements'. A category $\mathcal{C}$ is more sophisticated, it possesses 'objects' similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{C}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects.

A functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of a mapping of objects of $\mathcal{C}$ to objects of $\mathcal{D}$, $X \mapsto FX$ together with a map of the set of homomorphisms,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY).$$

the image of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ denoted by $F(f)$. That takes identity to identity and respects composition i.e.,

$$F(f \circ g) = F(f) \circ F(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$F(f \circ g) = F(g) \circ F(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ can be thought of as a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. A functor F is faithful if the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is injective for all X, Y. It's full if the map is surjective. If it's a bijection the functor is called fully faithful.

A natural transformation κ between two functors F and G from category $\mathcal{C}$ to $\mathcal{D}$ is a collection of mappings κ_X for every $X \in \mathcal{C}$, such that for all $f : X \to Y$, the diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & FX \arrow[r, "\kappa_X"] \arrow[d, "F(f)"'] & GX \arrow[d, "G(f)"] \\ Y & FY \arrow[r, "\kappa_Y"'] & GY \end{tikzcd} \qquad \text{(natural transformation)}$$

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors F and G is denoted by,

$$\mathrm{Nat}(F, G).$$

We say two functors F and G are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $F \cong G$. The collections of all functors from $\mathcal{C}$ to $\mathcal{D}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{D}^{\mathcal{C}}$. The nice thing about functor category $\mathcal{D}^{\mathcal{C}}$ is that if $\mathcal{D}$ has some nice properties then $\mathcal{D}^{\mathcal{C}}$ inherits these useful properties.

1.1 | Equivalence of Categories

Two objects X and Y in a category $\mathcal{C}$ are isomorphic if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$ and $g \circ f = \mathbb{1}_X$. In such a case, both the objects carry the same information, so these objects are equivalent. We want a similar equivalence between categories. This allows us to study a new category using some already well understood category. Equivalence of two categories can be thought of as giving two complementary description of same matheamtical object. We can compare two categories $\mathcal{C}$ and $\mathcal{D}$ via the functors between them. The starting point is the functor category $\mathcal{D}^{\mathcal{C}}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$GF \cong \mathbb{1}_{\mathcal{C}}, \quad \text{and} \quad FG \cong \mathbb{1}_{\mathcal{D}},$$

where the identity functor $\mathbb{1}_{\mathcal{C}}$ sends objects of $\mathcal{C}$ to the same objects, and morphisms to the same morphisms. G is called quasi-inverse functor. In such a case, $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent.

Lemma 1.1. *$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories iff F is fully faithful and for every object $Y \in \mathcal{D}$ there exists an object X such that FX is isomorphic to Y.*

Proof

$\Rightarrow$ Suppose F is an equivalence of categories then there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathbb{1}_{\mathcal{D}}$ and $GF \cong \mathbb{1}_{\mathcal{C}}$. So, by this, there exists for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, isomorphisms,

$$\varphi_X : GFX \to X, \quad \varkappa_Y : FGY \to Y.$$

So, each object $Y \in \mathcal{D}$ is isomorphic to the object FX where $X = GY$. To show that this is fully faithful, we have to show it gives homset isomorphism. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ then we have the following diagram which commutes,

$$\begin{array}{ccc} GFX & \xrightarrow{\varphi_X} & X \\ {\scriptstyle GF(f)}\downarrow & & \downarrow{\scriptstyle f} \\ GFX' & \xrightarrow{\varphi_{X'}} & X' \end{array}$$

note here that φ_X is invertible. Hence we can construct f as,

$$f = \varphi_{X'} \circ GF(f) \circ \varphi_X^{-1}$$

So, each f can be constructed from $F(f)$. Given any map $g \in \mathrm{Hom}_{\mathcal{D}}(FX, FX')$, set,

$$f = \varphi_{X'} \circ G(g) \circ \varphi_X^{-1} \in \mathrm{Hom}_{\mathcal{C}}(X, X').$$

So we have $G(g) = GF(f)$ and this gives us a hom set isomorphism or that F is fully faithful.

$\Leftarrow$ Assuming to each $Y \in \mathcal{D}$ there corresponds $X_Y \in \mathcal{C}$ such that there exists an isomorphism $\varkappa_Y : FX_Y \to Y$. We have to construct a quasi-inverse functor using these isomorphisms. Set $GY = X_Y$, and for each morphism $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Y')$, set,

$$G(g) = \varkappa_{Y'}^{-1} \circ g \circ \varkappa_Y$$

then we have $G(g) \in \mathrm{Hom}_{\mathcal{D}}(FGY, FGY')$ which is same as $\mathrm{Hom}_{\mathcal{C}}(GY, GY')$ because we had assumed F is fully faithful, i.e., the hom sets are isomorphic. It's easyto check that G is a functor and is a quasi-inverse to F. $\square$

Let $\mathcal{C}$ be a category, and let for any pair of objects X, Y of $\mathcal{C}$ an equivalence relation $\sim_{X,Y}$ in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ be given then, we can define a new category, called the quotient category $\mathcal{D} = \mathcal{C}/\sim$, and the quotient functor $Q : \mathcal{C} \to \mathcal{D}$ such that,

$$f \sim f' \implies Qf = Qf'$$

and every functor $F : \mathcal{C} \to \mathcal{D}'$, with $Ff = Ff'$ whenever $f \sim f'$, factors through $\mathcal{D}$, i.e., there exists a unique functor $G : \mathcal{D} \to \mathcal{D}'$ such that $F = G \circ Q$. Note that in this category $\mathcal{D}$, the objects are the same, but the hom sets are reduced, by the equivalence relation.

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\sim_{X,Y} .$$

2 | Representable Functors

Each $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ tells us about all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{C}}(-, X)$. These are called hom functors.

$$\begin{aligned} h_X :\ & \mathcal{C} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y). \end{aligned}$$

which maps each morphism $f : Y \to Z$ to a morphism of hom sets given by the composition,

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

We will denote this by,

$$\begin{aligned} h_X(f) :\ & h_X(Y) \to h_X(Z) \\ & g \mapsto f \circ g. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$s are all sets. Such categories are called locally small categories.

A covariant functor $F : \mathcal{C} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{C}$,

$$F \cong h_X$$

in such a case, F is said to be represented by the object X. For contravariant functors, $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$, this will be $G \cong h^X$. Where $\cong$ stands for natural isomorphism.

2.1 | Yoneda Embedding

We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\mathrm{Hom}_{\mathcal{C}}(-, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, -)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{C}$ and the functor $h_X \in \mathbf{Sets}^{\mathcal{C}}$.

Theorem 2.1. (Yoneda Lemma) *For a functor $F : \mathcal{C} \to \mathbf{Sets}$ and any $A \in \mathcal{C}$, there is a natural bijection,*

$$\mathrm{Nat}(h_A, F) \cong F(A)$$

such that $\kappa \in \mathrm{Nat}(h_A, F) \leftrightarrow \kappa_A(\mathbb{1}_A) \in F(A)$.

Proof

In the natural transformation diagram, replace F by h_A, and G by F. For $A = X$, $\kappa_A : h_A A \to FA$. Now, $h_A A = \mathrm{Hom}_{\mathcal{C}}(A, A)$, which contains $\mathbb{1}_A$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Nat}(h_A, F) &\to FA \\ \kappa &\mapsto \kappa_A(\mathbb{1}_A). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $B \in \mathcal{C}$. For any $f : A \to B$, we have,

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \qquad \begin{tikzcd} h_A A \arrow[r, "\kappa_A"] \arrow[d, "h_A(f)"'] & FA \arrow[d, "F(f)"] \\ h_A B \arrow[r, "\kappa_B"'] & FB \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}_A \arrow[r, mapsto, "\kappa_A"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ f \arrow[r, mapsto, "\kappa_B"'] & \kappa_B(f) \end{tikzcd}$$

Hence $\kappa_B(f) = F(f)(\mu(\kappa))$, or the action of κ_B is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_B(f) = \varphi_B(f)$ for all $B \in \mathcal{C}$, so it's injective.

For surjectivity we have to show that for all sets $u \in FA$, there exists a natural transformation φ such that $\varphi_A(\mathbb{1}_A) = u$. For $u \in FA$, and $f : A \to B$, construct the map,

$$\begin{aligned} \varphi : h_A &\to F \\ f &\mapsto F(f)(u). \end{aligned}$$

this satisfies the requirement that $\varphi_A(\mathbb{1}_A) = u$, because clearly, $\mathbb{1}_A \mapsto F(\mathbb{1}_A)(u) = \mathbb{1}_u(u) = u$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} B \arrow[d, "g"] \\ C \end{tikzcd} \qquad \begin{tikzcd} h_A B \arrow[r, "\varphi_B"] \arrow[d, "h_A(g)"'] & FB \arrow[d, "F(g)"] \\ h_A C \arrow[r, "\varphi_C"'] & FC \end{tikzcd}$$

commutes for all $B, C \in \mathcal{C}$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. For $f : A \to B$, by definition of φ,

$$F(g) \circ (\varphi_B(f)) = F(g) \circ F(f)(u)$$

which by functoriality of F is $= F(g \circ f)(u)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_C \circ (h_A(g)(f)) = \varphi_C(h_A(g \circ f))$$

which again by definition of φ is $= F(g \circ f)(u)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Nat}(h_A, F) \to FA$ is a bijection. $\square$

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof is same for the contravariant case. Immediate corollary is that if a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable then it's unique upto isomorphism. The Yoneda embedding functor,

$$h_{-} : \; \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}}$$

which sends an object $X \in \mathcal{C}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(-, X)$. Similarly we can define the contravariant Yoneda embedding,

$$h^{-} : \; \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

which sends an object $Y \in \mathcal{C}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(Y, -)$. These functors are fully faithful. Many of the definitions and properties of algebraic objects can be expressed in categorical language. Representable functor definitions are simpler to study and they inherit many interesting properties from the category of sets when we are dealing with locally small categories.

3 | Limits & Colimits

Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. An inductive system in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor,

$$F : \mathcal{I} \to \mathcal{C}.$$

The limit of a system is an object in $\mathcal{C}$ that is 'closest' to the system.

This can be formalised in the functor category as follows; Attach to each object $X \in \mathcal{C}$ the constant functor $c_X : \mathcal{I} \to \mathcal{C}$ that sends everything in $\mathcal{I}$ to X, and each morphism in $\mathcal{I}$ to the identity. A relation between an object X and the system F is a natural transformation between F and c_X. Such a natural transformation is called a cone. The collection of all such cones is the set of all naural transformations,

$$\begin{aligned} C_F : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, c_X). \end{aligned}$$

It's a contravariant functor from $\mathcal{C}$ to $\mathbf{Sets}$. If the functor C_F is representable, there exists an object L such that,

$$C_F \cong h^L$$

So, in such case $C_F(X) \cong \mathrm{Hom}_{\mathcal{C}}(X, L)$. In such a case we have, $\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, c_X) \cong \mathrm{Hom}_{\mathcal{C}}(X, L)$ and hence every cone must factor through L.[1] The representative L is called the limit of the inductive system, and is denoted by $\varinjlim F$.

A projective system in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor from $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$. Similar to the inductive system, for projective system $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^G : \mathcal{C} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}^{\mathrm{op}}}}(c_X, G). \end{aligned}$$

[1]Intuitively the limit is the 'closest' object to the system. The notion of closeness comes from morphisms, so if there exists any other object with morphisms to the system, then it must be 'farther' than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

If it's representable with representative M, $C^G(X) \cong \mathrm{Hom}_{\mathcal{C}}(M, X)$. The limit is denoted by $\varprojlim G$.

For the case of locally small categories, we can define limits using limits in the category of sets, making it easier to work with. In case of the category of sets, **Sets**, we can define the limit in terms of the initial/terminal object,

$$\varprojlim F := \mathrm{Cone}(1, F) = \mathrm{Nat}(1, F)$$

here the initial/terminal object is the set with one element. This is set, as we assumed the indexing category is small, i.e., the hom sets are small sets. Since we work with locally small categories, we could use this as definition for limit in the category of sets, then use this to define inductive and projective limits representably using hom sets of categories.

Consider two functors, $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$. For any object $X \in \mathcal{C}$ we can construct the composite functor, $\widehat{F}_X := \mathrm{Hom}_{\mathcal{C}}(F(-), X)$, and $\widehat{G}_X := \mathrm{Hom}_{\mathcal{C}}(X, G(-))$ this is a inductive system in the category of sets,

$$\widehat{F}_X : \mathcal{I}^{\mathrm{op}} \to \textbf{Sets},$$

It's easy to see that it's a functor from $\mathcal{I}^{\mathrm{op}}$ to **Sets** using the following diagram,

$$\begin{tikzcd} i \arrow[d, "f"] \\ j \end{tikzcd} \qquad \begin{tikzcd} F_i \arrow[d, "F(f)"'] \arrow[dr] & \\ F_j \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} G_i \arrow[r] & X \\ G_j \arrow[u, "G(f)"] \arrow[ur] & \end{tikzcd}$$

and the projective limit exists. The limit of this inductive system F, denoted by $\varinjlim F$, can be defined as the representative of the functor,

$$X \mapsto \varprojlim \widehat{F}_X.$$

So, we have directly by definition,

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim F, X) \cong \varprojlim(\mathrm{Hom}_{\mathcal{C}}(F, X)).$$

and similarly for the projective systems,

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim G) \cong \varprojlim(\mathrm{Hom}_{\mathcal{C}}(X, G)).$$

This can be translated as follows, for all objects $C \in \mathcal{C}$ and all family of morphisms $f_X : C \to FX$, in $\mathcal{C}$ such that for all $s \in \mathrm{Hom}_{\mathcal{I}}(X, Y)$, with $f_Y = f_X \circ F(s)$ factors uniquely through $\varprojlim F$.

$$\begin{tikzcd} & C \arrow[dl, "f_X"'] \arrow[d, dashed] \arrow[dr, "f_Y"] & \\ & \varprojlim F \arrow[dl, "\varphi_X"] \arrow[dr, "\varphi_Y"'] & \\ FX \arrow[rr, "F(s)"'] & & FY \end{tikzcd}$$

This might be the reason for naming it cones. Similarly, projective limits can be written in terms of universal property as,

$$\begin{tikzcd}
FX \arrow[rr, "F(s)"] \arrow[dr, "\varphi_X"] \arrow[ddr, "f_X"'] & & FY \arrow[dl, "\varphi_Y"'] \arrow[ddl, "f_Y"] \\
& \varinjlim G \arrow[d, dashed] & \\
& L &
\end{tikzcd}$$

Note that if $\mathcal{I}$ admits terminal object t, then the limit $\varprojlim F$ corresponds to the object $F(t)$. Note that indexing sets usually have terminal objects.

A category $\mathcal{C}$ is called complete if it has all small limits, it's called cocomplete if it has all small colimits. In such a case, limit is the functor,

$$\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$$

Limits are essential tool to construct new objects and new functors. When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit correpsonds to products and coproducts.

4 | Adjoint Functors; Kan Extension

Given a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$ is called left-adjoint to F if,

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. F is called left-adjoint to G, and G is right-adjoint to F. Denoted by $F \dashv G$. Adjoints are unique upto isomorphism and is the representative of the functor,

$$X \mapsto \mathrm{Hom}_{\mathcal{D}}(FX, Y).$$

The isomorphism gives us,

$$\mathrm{Hom}_{\mathcal{C}}(GX, GY) \cong \mathrm{Hom}_{\mathcal{D}}((F \circ G)X, Y)$$

and similarly,

$$\mathrm{Hom}_{\mathcal{D}}(FX, FY) \cong \mathrm{Hom}_{\mathcal{D}}(X, (G \circ F)Y)$$

4.1 | Kan Extension of Functors

Consider three categories $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{C}$, if $H : \mathcal{J} \to \mathcal{I}$ is a functor, then we can construct a functor H_*,

$$H_* : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathcal{J}}$$

given by the composition $\circ H$, $\mathcal{J} \xrightarrow{H} \mathcal{I} \xrightarrow{F} \mathcal{C}$ If it admits an left adjoint $H^* : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{I}}$, then we should have an isomorphism,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(H^* F, G) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, H_* G).$$

4.1.1 | Yoneda Extension

References

[1] S Mac Lane, L Moerdijk, Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer, 1992

[2] M Kashiwara, P Schapira, Categories and Sheaves, Springer, 2000