Part Ia

Locally Compact Groups

1 | The Modular Function

By a locally compact group we mean a topological group that's locally compact and Hausdorff. Let $\mathcal{G}$ be a locally compact group. Let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on $\mathcal{G}$. Let $C_c^+(\mathcal{G})$ denote the compactly supported continnuous functions that are always non-negative. Note that $C_c(\mathcal{G})$ is the linear span of $C_c^+(\mathcal{G})$.

A left Haar measure on $\mathcal{G}$ is a non-zero Radon measure μ, such that for all $x \in \mathcal{G}$,

$$\mu(xE) = \mu(E).$$

where E is a Borel subset of $\mathcal{G}$ and $x \in \mathcal{G}$. Radon measures are measures that respect the topology of the underlying space. In case of locally compact spaces Riesz representation theorem tells us that Radon measures are in bijection with linear functionals on the space of compactly supported continuous functions, given by $f \mapsto \int f d\mu$, so the invariance translated in this setting means,

$$\int L_x f d\mu = \int f d\mu.$$

where $L_x(f(y)) = f(x^{-1}y)$. Haar measures are very useful, and they allow much of the analysis that was possible with Lebesgue measures on $\mathbb{R}^n$.

Theorem 1.1. [2] *Every locally compact group has a left Haar measure, unique upto scaling.*

If $\mathcal{G}$ is non-commutative, the left Haar measures allows us to quantify the 'amount of non-commutativity'. For each $x \in \mathcal{G}$, define, $\lambda_x(E) = \lambda(Ex)$. This is again a left Haar measure, and by uniqueness of Haar measures, there exists a number $\Delta(x) > 0$ such that,

$$\lambda_x = \Delta(x)\lambda.$$

This is independent of the choice of λ again by uniqueness. So, $x \mapsto \Delta(x)$ is a well defined function called the modular function. For any $x, y \in \mathcal{G}$, and $E \subset \mathcal{G}$, we have, $\lambda_{xy}(E) = \lambda(Exy) = \Delta(y)\Delta(x)\lambda(E)$, so,

$$xy \mapsto \Delta(x)\Delta(y),$$

or, Δ is a group homomorphism from the group $\mathcal{G}$ to the multiplicative group of positive numbers denoted by $\mathbb{R}_\times$. If χ_E is the characteristic function on E, then we have, $\chi_E(xy) = \chi_{Ey^{-1}}(x)$. So,

$$\int \chi_E(xy)d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1})\int \chi_E(x)d\lambda(x).$$

Since a general function $f \in L^1(\mathcal{G})$ can be approximated by simple functions, we have,

$$\int R_y f d\lambda = \Delta(y^{-1})\int f d\lambda.$$

or equivalently, this can be abbreviated as,

$$d\lambda(xy) = \Delta(y)d\lambda(x). \tag{right translation}$$

Since the map $y \mapsto R_y f$ is continuous, the map $y \mapsto \int R_y f d\lambda$ is a continuous map. So, Δ is continuous. So, the homomorphism Δ also respects the topological structure.

To each left Haar measure, we can associate a right Haar measure, $\rho(E) = \lambda(E^{-1})$. The modular modular function relates the two measures. If $f \in \mathcal{C}_c(\mathcal{G})$, then we have,

$$\int R_y f(x)\Delta(x^{-1})d\lambda(x) = \Delta(y)\int f(xy)\Delta(xy^{-1})d\lambda(x) = \int f(x)\Delta(x^{-1})d\lambda(x).$$

In the last step we used right translation. Thus the functional $f \mapsto \int f(x)\Delta(x^{-1})d\lambda(x)$ is right invariant, and hence its associated Radon measure is a right Haar measure. Since ρ is a right Haar measure it must vary from the Radon measure associated to this functional by a constant, say κ. On symmetric neighborhoods, the left and right Haar measures are equal. If the constant κ is not 1, then by continuity of Δ, we can choose a small enough symmetric neighborhood of $1 \in \mathcal{G}$ such that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|\kappa - 1|$ on the neighborhood.

$$|\kappa - 1|\lambda(U) = |\kappa\rho(U) - \lambda(U)| = \left|\int_U [\Delta(x^{-1}) - 1]d\lambda(x)\right| \leq \tfrac{1}{2}|\kappa - 1|\lambda(U).$$

which can only happen if $\kappa = 1$. This gives us,

$$d\rho(x) = \Delta(x^{-1})d\lambda(x)$$

This can be restated in a more convinient way as,

$$d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x). \tag{inversion}$$

If $\Delta \equiv 1$, in such a case the left Haar measures do not measure any non-commutativity, such a locally compact group $\mathcal{G}$ is called unimodular. Clearly, abelian groups are unimodular. For discrete groups, every element of the group has same measure, and hence whether we left translate or right translate, it would still have the same measure. So, discrete groups are also unimodular. Since Δ is a continuous homomorphism, it takes compact groups to compact subgroup of $\mathbb{R}_\times$, which can only be $\{1\}$. So, compact groups are unimodular, and if $K \subset \mathcal{G}$ is a compact subgroup, then $\Delta|_K \equiv 1$.

Let $[\mathcal{G}, \mathcal{G}]$ be the smallest closed subgroup of $\mathcal{G}$ containing all elements of the form, $[x, y] = xyx^{-1}y^{-1}$, called the commutator subgroup of $\mathcal{G}$. It's a normal subgroups since, $z[x, y]z^{-1} = zxyx^{-1}y^{-1}z^{-1} = [zxz^{-1}, zyz^{-1}]$. Since Δ is a group homomorphism, we have,

$$\Delta([x, y]) = [\Delta(x), \Delta(y)]$$

since $\mathbb{R}_\times$ is abelian, we have $[\Delta(x), \Delta(y)] = 1$. So, by isomorphism theorem of groups, the group homomorphism $\Delta : \mathcal{G} \to \mathbb{R}_\times$ must factor through $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$. This implies that if $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ is compact, then $\mathcal{G}$ is unimodular. This is intuitively obvious, the modular function was built to measure the non-commutativity and this just means that, $[\mathcal{G}, \mathcal{G}]$ doesn't contain any information about the non-commutativity contained in the group $\mathcal{G}$. Intuitively, the important thing to note about modular function is that the left Haar measures can only measure non-commutativity of 'large' groups in some sense.

2 | Convolution & Involution of Functions

Given any two radon measures μ, ν on $\mathcal{G}$, we can define a functional,

$$\mu * \nu : f \mapsto \iint f(xy) d\mu(x) d\mu(y).$$

Clearly this is a linear map, and satisfies, $|(\mu * \nu)(f)| \leq \|f\|_{\text{sup}} \|\mu\| \, \|\nu\|$, so it's a bounded linear functional, and hence define a measure. This new measure is called the convolution of μ and ν.

$$\int f d(\mu * \nu) = \iint f(xy) d\mu(x) d\nu(y).$$

The order of the variables is important, the order of integration is not. If σ is some other measure, fubini's rule guarantees that convolution is an associative operation,

$$\begin{aligned}
\int f d[\mu * (\nu * \sigma)] &= \iint f(xy) d\mu(x) d(\nu * \sigma)(y) \\
&= \iiint f(xyz) d\mu(x) d\nu(y) d\sigma(z) \\
&= \iint f(yz) d(\mu * \nu)(y) d\sigma(z) \\
&= \int f d[(\mu * \nu) * \sigma].
\end{aligned}$$

If $\mathcal{G}$ is abelian, $f(xy) = f(yx)$ and by definition, the convolution will be commutative, and conversely consider the delta measures, $\int f d(\delta_x * \delta_y) = \int \int f(uv) d\delta_x(u) d\delta_y(v) = f(xy) = \int f d\delta_{xy}$. So, if convolutions are commutative, we would have $\delta_{xy} = \delta_x * \delta_y = \delta_y * \delta_x = \delta_{yx}$. Or, equivalently $xy = yx$. So, $\mathcal{G}$ is abelian if and only if convolutions commute.

Convolutions define a product structure on the Radon measures on $\mathcal{G}$. The estimate, $\|\mu * \nu\| \leq \|\mu\| \, \|\nu\|$ turns the collection of all Radon measures $\mathcal{M}(\mathcal{G})$ on $\mathcal{G}$ into a Banach algebra, called the measure algebra on $\mathcal{G}$. The delta measure at identity, δ_1 determines the identity on this algebra.

Consider the operation on $\mathcal{M}(\mathcal{G})$ defined by,

$$\mu^*(E) = \overline{\mu(E^{-1})}.$$

If $\mu, \nu \in \mathcal{M}(\mathcal{G})$ then we have,

$$\begin{aligned}
\int f d(\mu * \nu)^* &= \int f(x^{-1}) d\bar{(}\mu * \nu) = \iint f((xy)^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) \\
&= \iint f(y^{-1}x^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) = \iint f(yx) d\mu^*(x) d\nu^*(y) = \int f d(\nu^* * \mu^*).
\end{aligned}$$

So, $(\mu * \nu)^* = \nu^* * \mu^*$. The operation $\mu \mapsto \mu^*$ is an involution operation.

Given a left Haar measure λ, each function $f \in L^1(\mathcal{G})$ can be identified with the measure $f(x)d\lambda(x) \in \mathcal{M}(\mathcal{G})$. The convolution operation on $\mathcal{M}(\mathcal{G})$ gives a convolution operation on $L^1(\mathcal{G})$ given by,

$$f * g(x) = \int f(y)g(y^{-1}x)d\lambda(y).$$

With some manipulation and using right translation, this is the same as,

$$f * g(x) = \int f(xy^{-1})g(y)\Delta(y^{-1})d\lambda(y). \tag{convolution}$$

The involution on $\mathcal{M}(\mathcal{G})$, restricted to $L^1(\mathcal{G})$ is defined by the relation,

$$f^*(x)d\lambda(x) = \overline{f(x^{-1})}d\lambda(x^{-1}).$$

Which after some manipulations, and using inversion, gives,

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \tag{involution}$$

$L^1(\mathcal{G})$ with the convolution product and involution is a Banach algebra, called L^1 group algebra of $\mathcal{G}$. These structures also can be defined for $L^p(\mathcal{G})$, for $p \in [1, \infty)$. $L^p(\mathcal{G})$ consists of functions such that $|f|^p$ is integrable. In such a case we set,

$$\|f\|_p = \left(\int_{x \in \mathcal{G}} |f|^p d\lambda(x) \right)^{1/p}$$

Theorem 2.1. (Minkowski's inequality)

2.1 | Approximate Identities

3 | Representations & Group Algebra

The action of groups of interest in quantum theory are as unitary operators, so, the objects of interest to us are unitary representations of locally compact groups. The physically important topology on operator algebras is the weak operator topology. The weak topology and strong topology coincide on unitary operators, so either topology works fine for us. [1]

A unitary representation $(\mathcal{H}_\pi, \pi)$ of a locally compact group $\mathcal{G}$ is a strongly continuous group homomorphism

$$\pi : \mathcal{G} \to U(\mathcal{H}_\pi).$$

where $\mathcal{H}_\pi$ is some Hilbert space. and $U(\mathcal{H}_\pi)$ is the group of unitary operators on $\mathcal{H}_\pi$. The dimension of the Hilbert space $\mathcal{H}_\pi$ is called the degree of the representation π. When there is no confusion, the representation will just be denoted by π. If π is an isomorphism between the $\mathcal{G}$ and $U(\mathcal{H}_\pi)$ it is called a faithful representation.

Let $(\mathcal{H}_\pi, \pi)$ be a representation of $\mathcal{G}$, a subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is said to be invariant under π if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in \mathcal{G}$. If $\mathcal{M}$ is closed and $P_{\mathcal{M}}$ is the orthogonal projection onto the closed subspace $\mathcal{M}$ the invariance implies that,

$$P_{\mathcal{M}}\pi(x) = \pi(x)P_{\mathcal{M}},$$

[1] Suppose the net of unitary operators $\{T_\alpha\}$ converge to T strongly, then for any $|\varphi\rangle \in \mathcal{H}$, $\|(T_\alpha - T)\varphi\|^2 = \|T_\alpha\varphi\|^2 - 2\mathrm{Re}\langle T_\alpha\varphi|T\varphi\rangle + \|T\varphi\|^2 = 2\|\varphi\|^2 - 2\mathrm{Re}\langle T_\alpha\varphi|T\varphi\rangle$ So, the $\{T_\alpha\}$ converges to T in strong topology only if the last term converges to $2\|T\varphi\|^2$.

for all $x \in \mathcal{G}$. We can define a new representation $(\mathcal{M}, \pi^{\mathcal{M}})$ by defining,

$$\pi^{\mathcal{M}}(x) = P_{\mathcal{M}}\pi(x)P_{\mathcal{M}},$$

It is called a subrepresentation of $(\mathcal{H}_\pi, \pi)$. This procedure of going to subrepresentation gives a decomposition of π. If $\mathcal{M}$ is invariant under π then so is $\mathcal{M}^\perp$. This gives us a second subrepresentation. Now the original Hilbert space $\mathcal{H}$ can be written as a direct sum, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and each operator $\pi(x)$ then decomposes as a direct sum $\pi(x) = \pi^{\mathcal{M}}(x) \oplus \pi^{\mathcal{M}^\perp}(x)$. So the representation can be written as $(\mathcal{H}_\pi, \pi) = (\mathcal{M}, \pi^{\mathcal{M}}) \oplus (\mathcal{M}^\perp, \pi^{\mathcal{M}^\perp})$.

Given a family of representations $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha\in I}$ of $\mathcal{G}$ the direct sum of representations $\{\pi_\alpha\}$ is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha\in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ such that $\lim_F[\sum_{\alpha\in F} \|\varphi_\alpha\|^2] < \infty$ where F is a finite subset of I. The purpose of this definition is so that norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha\in I} \pi_\alpha,$$

is called direct sum of representations $\{(\mathcal{H}_\alpha, \pi_\alpha)\}_{\alpha\in I}$, denoted by, $\sum^{\oplus}_{\alpha\in I}\{(\mathcal{H}_\alpha, \pi_\alpha)\}$. A representation is trivial if $\pi(x) = 1$ for every $x \in \mathcal{G}$. These are uninteresting representations. A representation can however have a trivial part. If the set of all elements of $\mathcal{G}$ whose image under π is 1, is exactly the identity element, then the representation is called non-degenerate.

A vector φ in a Hilbert space $\mathcal{H}_\pi$ is called cyclic for $\mathcal{A}$ if $\{\pi(x)|\varphi\rangle\}_{x\in\mathcal{G}}$ is dense in $\mathcal{H}_\pi$. A cyclic representation of $\mathcal{G}$ is a triple $(\mathcal{H}, \pi, |\varphi\rangle)$ where $(\mathcal{H}_\pi, \pi)$ is a representation of $\mathcal{G}$ and φ is a cyclic for $\pi(\mathcal{G})$.

Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of $\mathcal{A}$. Take a maximal family of nonzero vectors $|\{\Omega_\alpha\rangle\}_{\alpha\in I}$ in $\mathcal{H}$ such that,

$$\langle\pi(A)\Omega_\alpha|\pi(B)\Omega_\beta\rangle = 0,$$

for all $A, B \in \mathcal{A}$ and $\alpha \neq \beta$. Define, $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}}_{A\in\mathcal{A}}$. This is an invariant subspace of $\mathcal{H}$. Define $\pi_\alpha = P_{\mathcal{H}_\alpha}\pi P_{\mathcal{H}_\alpha}$ where $P_{\mathcal{H}_\alpha}$ is projection onto $\mathcal{H}_\alpha$. Then by construction each $\mathcal{H}_\alpha$ are mutually orthogonal and hence the representation $(\mathcal{H}, \pi)$ is of the form,

$$\mathcal{H} = \bigoplus_{\alpha\in I}\{(\mathcal{H}_\alpha, \pi_\alpha)\}$$

By Zorn's lemma, every non-degenerate representation can be written as a direct sum of a family of cyclic subrepresentations. If no invariant subspaces the representation $(\mathcal{H}_\pi, \pi)$ of $\mathcal{G}$ is called irreducible.

Given two representations π_i and π_j of $\mathcal{G}$, an intertwining operator for π_i and π_j is a bounded linear operator,

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

such that for every $x \in \mathcal{G}$, $T\pi_i(x) = \pi_j(x)T$. The collection of all intertwiners between π_i and π_j is denoted by $\mathrm{Hom}(\pi_i, \pi_j)$. If $\mathrm{Hom}(\pi_i, \pi_j)$ contains a unitary operator, the two representations are called unitarily equivalent.

The collection $\mathcal{C}(\pi)$ of bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(x)$ for all $x \in \mathcal{G}$ is called the commutant of π is an algebra. Since product operation is weakly continuous, it is weakly closed. Since $T^*\pi(x) = [\pi(x^{-1})T]^* = [T\pi(x^{-1})]^* = \pi(x)T^*$, it's also closed under taking adjoints. Thus $\mathcal{C}(\pi)$ is a von Neumann algebra.

If π is reducible, $\mathcal{C}(\pi)$ contains non-trivial projections. Conversely, if $\mathcal{C}(\pi)$ contains bounded operators that are not multiples of identity, say T, then we can split it into the canonnical sum of two self-adjoint operators. Since these are linear combinations of the operators T and T^* which belong to $\mathcal{C}(\pi)$, they also belong to $\mathcal{C}(\pi)$. If an operator commutes with the self-adjoint operator, it also commutes with its spectral projections. So, $\pi(x)$ must also commute with the spectral projections of some self-adjoint operator. So, the subspace corresponding to the projection operator is an invariant subspace.

Let π_i and π_j be two unitary representations of $\mathcal{G}$. Then for every intertwining operator

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

for π_i and π_j, the adjoint of the operator $T^* : \mathcal{H}_{\pi_j} \to \mathcal{H}_{\pi_i}$ is an intertwining operator for π_j and π_i because,

$$T^*\pi_j(x) = [\pi_j(x^{-1})T]^* = [T\pi_i(x^{-1})]^* = \pi_i(x)T^*.$$

So, $T^*T \in \mathcal{C}(\pi_i)$ and $TT^* \in \mathcal{C}(\pi_j)$. If π_i and π_j are two irreducible unitary representations of $\mathcal{G}$, then both of these should be a multiple of identity, that is, $T^*T = \kappa I$ or equivalently, $\kappa^{-1/2}T$ must be a unitary operator or T is zero. So, two irreducible representations π_i and π_j are equivalent if $\mathcal{C}(\pi_i, \pi_j)$ is one dimensional, or they are unitarily inequivalent. These two facts about irreducibility of representations is called Schur's lemma, which we will now state below.

Theorem 3.1. (Schur's Lemma) *If π_i and π_j are irreducible representations of $\mathcal{G}$ then either they are equivalent representations, or they have no intertwiners.* □

If $\mathcal{G}$ is abelian, then for every representation π of $\mathcal{G}$, the operators $\pi(x)$ commutes with all operators in $\pi(\mathcal{G})$, so $\pi(x) \in \mathcal{C}(\pi)$ for every $x \in \mathcal{G}$. If π is irreducible, then we should have $\pi(x) = \kappa_x I$, so every irreducible representation of an locally compact abelian group is one dimensional. Although this looks very surprising, it's because vector spaces contain a lot more information than groups. This is an important corollary of Schur's lemma.

Corollary 3.2. *If $\mathcal{G}$ is abelian, and π is irreducible, then $\mathcal{H}_\pi = \mathbb{C}$.*

The unitary representations of the group $\mathcal{G}$ are closely related to the $*$-representations of the algebra of integrable functions $L^1(\mathcal{G})$. Once we have fixed a left Haar measure λ, let $f \in L^1(\mathcal{G})$, then for any unitary representation π of the group $\mathcal{G}$, we can define a operator $\pi(f)$ on the Hilbert space $\mathcal{H}_\pi$ by,

$$\langle \pi(f)\varphi | \varkappa \rangle = \int f(x) \langle \pi(x)\varphi | \varkappa \rangle d\lambda(x).$$

Or, more compactly,

$$\pi(f) = \int f(x)\pi(x) d\lambda(x).$$

This satisfies,

$$|\langle \pi(f)\varphi | \varkappa \rangle| \leq \|f\|_1 \|\varphi\| \|\varkappa\|.$$

So, $\pi(f)$ is a bounded linear operator on $\mathcal{H}_\pi$ with the norm $\|\pi(f)\| \leq \|f\|_1$. For any two $f, g \in L^1(\mathcal{G})$,

$$\begin{aligned}\pi(f * g) &= \iint f(y)g(y^{-1}x)\pi(x)d\lambda(y)d\lambda(x) = \iint f(y)g(x)\pi(yx)d\lambda(x)d\lambda(y) \\ &= \iint f(y)g(x)\pi(y)\pi(x)d\lambda(x)d\lambda(y) = \pi(f)\pi(g).\end{aligned}$$

So, it's an algebra homomorphism.

$$\pi(f^*) = \int \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)d\lambda(x) = \int \overline{f(x)}\pi(x^{-1})d\lambda(x) = \int [f(x)\pi(x)]^* d\lambda(x) = \pi(f)^*.$$

So, π is a $*$-homomorphism. The left translation of x of the function f is given by,

$$\begin{aligned}\pi(L_x f) &= \int f(x^{-1}y)\pi(y)d\lambda(y) = \int f(y)\pi(xy)d\lambda(y) \\ &= \int f(y)\pi(x\pi(y))d\lambda(y) = \pi(x)\pi(f).\end{aligned}$$

That is,

$$\pi(L_x f) = \pi(x)\pi(f)$$

Similarly, for the right translation,

$$\begin{aligned}\Delta(x^{-1})\pi(R_{x^{-1}}f) &= \Delta(x^{-1})\int f(yx^{-1})\pi(y)d\lambda(y) = \int f(y)\pi(xy)d\lambda(y) \\ &= \int f(y)\pi(y)\pi(x)d\lambda(y) = \pi(f)\pi(x).\end{aligned}$$

That is,

$$\Delta(x^{-1})\pi(R_{x^{-1}}f) = \pi(f)\pi(x).$$

This will be reversed if we had chosen a right Haar measure to define integration. So, the map $f \mapsto \pi(f)$ is a $*$-homomorphism from $L^1(\mathcal{G})$ to $\mathcal{B}(\mathcal{H}_\pi)$. Since π is strongly continuous, we can find a neighborhood V of 1, such that for all $x \in V$,

$$\|\pi(f)\varphi - \varphi\| < \|\varphi\|.$$

Now consider the function, $f = |V|^{-1}\chi_V$, we get,

$$\|\pi(f)\varphi - \varphi\| = |V|^{-1}\left\| \int_V [\pi(x)\varphi - \varphi]d\lambda(x)\right\| < \|\varphi\|$$

So, $\|\pi(x)\varphi - \varphi\| \neq 0$, or $\pi(x)\varphi \neq 0$. Hence,

$$\pi : f \mapsto \pi(f)$$

is a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on $\mathcal{H}_\pi$. So, a representation of the locally compact group $\mathcal{G}$ gives rise to a representation of integrable functions on the same Hilbert space. Conversely, given a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on a Hilbert space $\mathcal{H}_\pi$ gives rise to a unique unitary representation of $\mathcal{G}$. The idea is to use the functions f approaching to delta function at x, to get an operator for $\pi(x)$. See §3.2, [2].

Theorem 3.3. *The von Neumann algebras generated by $\pi(\mathcal{G})$ and $\pi(L^1(\mathcal{G}))$ are identical.*

Idea of Proof

The idea is to approximate functions in $L^1(\mathcal{G})$ by compactly supported functions, and then approximate the representatives of these compactly supported functions using the representatives of the group elements.

If $g \in \mathcal{C}_c(\mathcal{G})$, we can divide up the support of g into a finite partition, say $E = \{E_i\}$. Now we can approximate the function g, using the value g at some point $x_i \in E_i$, by,

$$\Sigma_E = \sum_i g(x_i)\pi(x_i)|E_i|$$

Given any $\epsilon > 0$, by compactness of the support of g, and continuity of the map $x \mapsto g(x)\pi(x)\varphi$, for each φ, there exists a partition $E = \{E_i\}$ of the support of g such that, $\|g(x)\pi(x)\varphi - g(y)\pi(y)\varphi\| < \epsilon$, whenever $x, y \in E_j$ for some j. Thus,

$$\|\Sigma_E\varphi - \pi(g)\varphi\| < \epsilon|\cup_i E_i|.$$

Thus every neighborhood of $\pi(g)$ with respect to the strong topology, contains sums Σ_E.

Now every function $f \in L^1(\mathcal{G})$ is the L^1 limit of functions in $\mathcal{C}_c(\mathcal{G})$. So, by continuity of π, $\pi(f)$ is norm limit of operators in $\pi(\mathcal{C}_c(\mathcal{G}))$. So, the von Neumann algebra generated by $\pi(L^1(\mathcal{G}))$ is contained in the von Neumann algebra generated by $\pi(\mathcal{G})$. Conversely, each $\pi(x)$ is the strong limit of $\pi(L_x\phi_U)$, as $U \to \{1\}$. Where ϕ_U is an 'approximate identity'. See [2] §2.5 for approximate identities, and §3.2 for a complete proof. □

3.1 | Functions of Positive Type

In the C^* algebra case representations and states are closely related to each other, we need to find analogues of states for group algebras. A function of positive type is analogous to states on operator algebras. We can do similar constructions with group algebras that we did with C^*-algebras.

Let $L^\infty(\mathcal{G})$ denote the collection of all bounded measurable functions on $\mathcal{G}$. With the sup-norm, $\|f\|_\infty := \sup_{x\in\mathcal{G}} |f(x)|$, $L^\infty(\mathcal{G})$ is a Banach space.

Every function $\omega \in L^\infty$ defines a linear functional on $L^1(\mathcal{G})$ defined by,

$$f \mapsto \widehat{\omega}(f) := \int_{x\in\mathcal{G}} f(x)\omega(x)d\lambda(x)$$

This is bounded because,

$$|\widehat{\omega}(f)| = \left|\int_{x\in\mathcal{G}} f(x)\omega(x)d\lambda(x)\right| \leq \|\omega\|_\infty \left|\int_{x\in\mathcal{G}} f(x)d\lambda(x)\right| = \|\omega\|_\infty\|f\|_1.$$

So, $\widehat{\omega}$ is a bounded linear functional on the group algebra $L^1(\mathcal{G})$. A function $\omega \in L^\infty(\mathcal{G})$ is said to be a function of positive type if for any function $f \in L^1(\mathcal{G})$,

$$\widehat{\omega}(f^* * f) = \int (f^* * f)\omega \geq 0.$$

Expanding the integral, by convolution and involution, we have,

$$\begin{aligned}\int (f^* * f)\omega &= \iint \Delta(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\omega(x)d\lambda(y)d\lambda(x)\\ &= \iint \overline{f(y)}f(yx)\omega(x)d\lambda(x)d\lambda(y) = \iint f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).\end{aligned}$$

We can restate that a function $\omega \in L^\infty(\mathcal{G})$ is of positive type if,

$$\omega(f^* * f) = \iint f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y) \geq 0 \qquad \text{(positive type)}$$

Let ω be a function of positive type, consider the function complex conjugate, $\overline{\omega}$,

$$\int (f^* * f)\overline{\omega} = \iint \overline{f(y)}f(yx)\overline{\omega(x)}d\lambda(y)d\lambda(x) = \overline{\int [(\overline{f})^* * \overline{f}]\omega} \geq 0$$

So, if ω is a function of positive type, then so is $\overline{\omega}$. The set of all continuous functions of positive type on $\mathcal{G}$ is denoted by $\mathcal{P}(\mathcal{G})$.

Given a representation π of $\mathcal{G}$, every vector $\varphi \in \mathcal{H}_\pi$ defines a function,

$$\Phi(x) = \langle \pi(x)\varphi|\varphi\rangle.$$

By the continuity requirements of the representation, this is a continuous function on $\mathcal{G}$. $\Phi(y^{-1}x) = \langle \pi(y^{-1}x)\varphi|\varphi\rangle = \langle \pi(y^{-1})\pi(x)\varphi|\varphi\rangle = \langle \pi(x)\varphi|\pi(y)\varphi\rangle$. So for every $f \in L^1(\mathcal{G})$,

$$\begin{aligned}\widehat{\Phi}(f^* * f) &= \iint f(x)\overline{f(y)}\Phi(y^{-1}x)d\lambda(x)d\lambda(y) \\ &= \iint \langle f(x)\pi(x)\varphi|f(y)\pi(y)\varphi\rangle d\lambda(x)d\lambda(y) = \|\pi(f)\varphi\| \geq 0\end{aligned}$$

So, $\Phi \in \mathcal{P}(\mathcal{G})$. Consider the space of all square integrable functions on $\mathcal{G}$, consisting of functions g such that $\int |g|^2 d\lambda < \infty$. This together with the inner product,

$$\langle h, g\rangle = \int_{x\in\mathcal{G}} \overline{h(x)}g(x)d\lambda(x).$$

$L^2(\mathcal{G})$ is a Hilbert space. On this space, we have a representation of the group $\mathcal{G}$, given by,

$$[\pi_L(x)g](y) = L_x g(y) = g(x^{-1}y).$$

The left invariance of the left Haar measure ensures that this is a unitary representation. It is called the left regular representation of $\mathcal{G}$ on $L^2(\mathcal{G})$. The right regular representation is similarly defined. Let $g \in L^2(\mathcal{G})$, let $\tilde{g}(x) = \overline{g(x^{-1})}$, then we have,

$$\langle \pi_L(x)g|g\rangle = \int g(x^{-1}y)\overline{g(y)}d\lambda(y) = \overline{g * \tilde{g}(x)}.$$

For any $f \in L^1(\mathcal{G})$, we have,

$$(\widehat{g * \tilde{g}})(f^* * f) = \iint f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

[Clarify This]

So, for each $g \in L^2(\mathcal{G})$, $g * \tilde{g}$ defines a function of positive type. So, functions of positive type do indeed exist.

3.1.1 | Construction of Representations

Every unitary representations give rise to functions of positive type. The converse is also true, that is every function of positive type arises from a unitary representation. The key ingredient to proving this is to use functions of positive type to construct Hilbert spaces, much like GNS construction in the case of C^*-algebras.

The goal is to define an inner product using the function of positive type on $L^1(\mathcal{G})$ and consider the completion, similar to how it's done in GNS construction. Given a function ω of positive type, define a hermitian form on $L^1(\mathcal{G})$ by,

$$\langle f, g\rangle_\omega = \int (g^* * f)\omega = \iint f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

By Fubini's theorem this satisfies,

$$|\langle f, g\rangle_\omega| \leq \|\omega\|_\infty \|f\|_1 \|g\|_1. \tag{bounded}$$

This is positive semi-definite since ω is of positive type. So, $\langle , \rangle_\omega$ defines a positive definite Hermitian form on $L^1(\mathcal{G})$. So, this satisfies the Schwarz inequality,

$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle_\omega \langle g, g\rangle_\omega.$$

So, we can consider the ideal $\mathcal{J}_\omega = \{f \mid \langle f, f\rangle_\omega = 0\}$. By the above inequality, $f \in \mathcal{J}_\omega$ if and only if $\langle f, g\rangle_\omega = 0$ for all $g \in L^1(\mathcal{G})$. So, $\langle , \rangle_\omega$ defines an inner product on the quotient space $L^1(\mathcal{G})/\mathcal{J}_\omega$. If the image of $f \in L^1(\mathcal{G})$ in $\mathcal{H}_\omega$ by $[f]$, the inner product is given by,

$$\langle [f], [g]\rangle = \langle f, g\rangle_\omega$$

Denote the completion of this inner product space by $\mathcal{H}_\omega$. We have a natural action of the group $\mathcal{G}$ on the Hilbert space $\mathcal{H}_\omega$ given by the left translation.

$$\pi_\omega(x)[f] = [L_x f].$$

This action is a unitary operator because,

$$\begin{aligned}\langle L_x f, L_x g\rangle_\omega &= \iint f(x^{-1}y)\overline{g(x^{-1}z)}\omega(z^{-1}y)d\lambda(y)d\lambda(z) \\ &= \iint f(y)\overline{g(z)}\omega((xz)^{-1}(xy))d\lambda(y)d\lambda(z) = \langle f, g\rangle_\omega.\end{aligned}$$

In the last step, we used the left invariance of the Haar measure λ. This also tells us that the action keeps $\mathcal{J}_\omega$ invariant, and hence L_x gives rise to a unitary operator on $\mathcal{H}_\omega$. We also have an action of $L^1(\mathcal{G})$ on $\mathcal{H}_\omega$ given by,

$$\pi_\omega(g)[f] = [g * f].$$

Theorem 3.4.

$$\exists\ \Omega \in \mathcal{H}_\omega \ni \pi_\omega(f)|\Omega\rangle = [f], \omega(x) = \langle \pi_\omega(x)\Omega, \Omega\rangle.$$

almost everywhere.

Proof

We want to think of ω as a limit of functions $L^1(\mathcal{G})$ functions. The idea is that if $\psi \in L^1(\mathcal{G})$ and $\omega \in L^\infty(\mathcal{G})$ then $\psi * \omega \in L^1(\mathcal{G})$[2] and consider the product of ω with an approximate identity to obtain the required sequence. $\widehat{f}_i(f) \to \widehat{\omega}(f)$ because $\psi_{U_i} * \omega \to \omega$ in appropriate topologies. Now all we are left with is some routine checks, which can be found in §3.3 [2]. $\square$

The above theorem says that the information about the action of ω is determined by its infinitesimal behavior around the identity and $\Omega = [\{\psi_{U_i}\}]$. So, every function of positive type agrees with continuous functions locally almost everywhere.

Given a function of positive type $\omega \in \mathcal{P}(\mathcal{G})$, there exists some representation π_ω such that

$$|\omega(x)| = |\langle \pi_\omega(x)\Omega, \Omega\rangle| \leq \|\Omega\|^2 = \omega(1)$$

Similarly,

$$\omega(x^{-1}) = \langle \pi_\omega(x^{-1})\Omega, \Omega\rangle = \langle \Omega, \pi_\omega(x)\Omega\rangle = \overline{\omega(x)}$$

Suppose π_i and π_j are two cyclic representations of $\mathcal{G}$ with cyclic vectors Ω_i and Ω_j then it follows that $T\pi_i(x)\Omega_i = \pi_j(x)\Omega_j$ extends to a linear isometry on the spans, and by continuity to a unitary map between the Hilbert spaces $\mathcal{H}_{\pi_i}$ and $\mathcal{H}_{\pi_j}$. So, $\pi_i(x)T = T\pi_j(x)$ for all $x \in \mathcal{G}$. So, any two cyclic representations are unitarily equivalent. So, every cyclic representation is unitarily equivalent to the one we constructed above.

3.1.2 | Gelfand-Raikov Theorem

[2] this is where almost everywhere is used.

Theorem 3.5. (Gelfand-Raikov)

Existence of Haar Measures

Existence of Haar measure on locally compact groups is a very important tool for analysis of locally compact groups. We need to construct a functional on the space of compactly supported continuous functions that's translation invariant. All that we have to work with is the set of continuous functions on the locally commpact group. Compactly supported functions are well behaved, so, we can try to exploit the nice properties.

Let f, g be two compactly supported positive functions. Note that the support of each of these functions can be covered by finitely many open sets. We can relate the two functions using this. We can dominate the function f by a finite linear combination of scaled translations of g. Note that this is possible because f attains a maxima, and g is non-zero positive function. So, there exists some $x \in \mathcal{G}$ such that $g(x) > 0$, and we can find a number λ such that $\lambda g(x) > \|f\|_{\sup}$. By continuity of translation we can find a neighborhood around x where this holds. We can now transfer via translation L_y all the problematic parts to this neighborhood. Since the support of f is compact, we can cover it with finitely many translations of this neighborhood, hence we can make sense of a sum. So, there exist a finite collection of real numbers λ_i and translations L_{x_i} such that

$$f \leq \textstyle\sum_i \lambda_i (L_{x_i}(g)).$$

Consider the set of all finite collection of numbers $\{\lambda_i\}$ such that $f \leq \sum_{i=1}^n \lambda_i (L_{x_i}(g))$. This is nonempty due to the above reaon. Denote this set by $\mathcal{D}(f : g)$. The intuitive meaning here is that $\sum_i \lambda_i (L_{x_i} g)$ is an approximation of f in terms of translations of the function g. Using this we can define 'best' approximation based on how little scaling we have to do with the translations of g. This defines some sort of index of the function f with respect to g.

Since each element of this set is a finite collection of real numbers, we can add them up. So, we have a 'sum' function on $\mathcal{D}(f : g)$,

$$\begin{aligned} \textstyle\sum : \mathcal{D}(f : g) &\to \mathbb{R} \\ \{\lambda_i\} &\mapsto \textstyle\sum_i \lambda_i. \end{aligned}$$

Using this function, we can define a new object,

$$(f : g) \coloneqq \inf_{\{\lambda_i\} \in \mathcal{D}(f:g)} \big(\textstyle\sum_i \lambda_i \big).$$

We will call this the 'approximation index' of the function f with respect to g. This approximation index allows us to reduce the problem of constructing a special linear functional on the space of compactly supported continuous functions to doing some stuff with some 'reference function'. This will be made precise below. Note that this is where the 'locally compact' part has been important. The Riesz representation tells us that construction of Radon measures is equivalent to constructing some special linear functionals on compactly supported continuous functions. Then the compactness of the supports of functions makes sure that the sets $\mathcal{D}(f : g)$ are non-empty for each compactly supported functions f and g.

Note that the translation L_y is a linear map on the space of continuous functions, so we have,

$$L_{y^{-1}} \big(\textstyle\sum_i \lambda_i (L_{x_i} g) \big) = \textstyle\sum_i \lambda_i (L_{y^{-1} x_i} g).$$

Since the approximation index is an infimum, the approximation index of any translation L_y of the function f with respect to g will be the same. So, we have,

$$(f : g) = (L_y f : g). \tag{invariance}$$

Scaling functions f by λ just scales the sets $\mathcal{D}(f : g)$, so we have,

$$(\lambda f : g) = \lambda(f : g). \tag{scaling}$$

If $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(h : g)$, then $\{\lambda_i\} \coprod \{\mu_j\} \in \mathcal{D}(f + h : g)$. So, we have,

$$(f + h : g) \leq (f : g) + (h : g). \tag{subadditivity}$$

Note that if this was additive, we would be done, we would have constructed a linear functional that is invariant under left translation. It is however possible choose g such that the inequality gets closer to an equality. The key lies in how well we can approximate the given function f, in terms of linear combinations of translations of g. So, the problem is now choosing appropriate g. We will need some other properties of the functionals $(f : g)$ to do this.

If $f \leq h$ then any upper approximation of h by translations of g is also an upper approximation of f by translations of g. Hence we have,

$$f \leq h \Rightarrow (f : g) \leq (h : g). \tag{monotonicity}$$

Let f attain maxima at x, that is $f(x) = \|f\|_{\text{sup}}$ and g attain maxima at y, that is, $g(y) = \|g\|_{\text{sup}}$. Then the translation $L_{yx^{-1}}(f)$ attains maxima at y, by $(f : g) = (L_{yx^{-1}} f : g)$, we obtain,

$$(f : g) \geq \|f\|_{\text{sup}} / \|g\|_{\text{sup}}. \tag{boundedness}$$

Suppose $f \leq \sum_i \lambda_i (L_{x_i} g)$ and $g \leq \sum_j \mu_j (L_{x_j} h)$, then by linearity of translations, we have, $f \leq \sum_{i,j} \lambda_i \mu_j (L_{x_i x_j} h)$. That is to say, $\{\lambda_i \mu_j\} \in \mathcal{D}(f : h)$ whenever $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(g : h)$. So, we have,

$$(f : h) \leq (f : g)(g : h). \tag{product}$$

Now we choose a 'reference function' F, and define a normalized index,

$$I_g(f) \coloneqq (f : g)/(F : g).$$

This gives us a functional $I_g : C_c^+(\mathcal{G}) \to \mathbb{R}$, that is left invariant, subadditive and monotone. By, $(F : g) \leq (F : f)(f : g)$ and $(f : g) \leq (f : F)(F : g)$ we have,

$$f_F \equiv (F : f)^{-1} \leq I_g(f) \leq (f : F) \equiv f^F \tag{rescale}$$

Now the problem is to choose an appropriate g such that subadditivity becomes an equality. In order to choose such a g, we have to understand what makes the approximations $(f : g)$ not 'close' to the function f. The function is determined by its value at each point. So, given a number, we can scale it appropriately at each point to get f. What we are trying to do is approximate the function f, by scaling a bunch of functions on some open sets. So, smaller open sets means better approximation. So, to make the approximation closer to the actual value of f, we could use g that has smaller support.

Let f_i, f_j be two compactly supported continuous functions. Let g be a positive compactly supported continuous function such that $g \equiv 1$ on the support of the functions $(f_i + f_j)$. Let $\delta > 0$ be a number (will be specified a bit later). Then we can consider the function,

$$h = f_i + f_j + \delta \cdot g$$

Using this we can construct new functions,

$$h_i = f_i / h, \quad h_j = f_j / h.$$

This implies that $h_i \in \mathcal{C}_c^+(\mathcal{G})$, and $h_i \equiv 0$ whenever $f_i \equiv 0$. By continuity of h_i, for every $\delta > 0$ we can find a small enough neighborhood V of $1 \in \mathcal{G}$ such that

$$|h_i(x) - h_i(y)| \leq \delta$$

For any function φ with support in V, consider an approximate of h by finitely many translations of φ, $h \leq \sum_k \lambda_k L_{x_k}\varphi$, then we have,

$$f_i(x) = h(x)h_i(x) \leq \sum_j \lambda_k L_{x_k}\varphi(x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta)$$

whenever $x, x_k \in V$. Since $h_i + h_j \leq 1$, we get,

$$(f_i : \varphi) + (f_j : \varphi) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta) + \sum_k \lambda_k \varphi(x_k^{-1}x)(h_j(x_k) + \delta) \leq (1 + 2\delta)\sum_k \lambda_k.$$

Hence, by taking infimum of all such $\sum_k \lambda_k$, we get,

$$\begin{aligned} I_\varphi(f_i) + I_\varphi(f_j) &\leq (1 + 2\delta)I_\varphi(h) \leq (1 + 2\delta)(I_\varphi(f_i) + I_\varphi(f_j) + \delta I_\varphi(g)) \\ &= (I_\varphi(f_i + f_j)) + \delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g). \end{aligned}$$

The last step is by subadditivity and scaling. Then by rescale we can choose δ small enough such that the last part is less than ϵ, $\delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g) < \epsilon$. Hence we can choose a function φ with a small support such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

The choice of δ determines the support of the function used for approximating. This can be chosen such that $I_\varphi(f_i) + I_\varphi(f_j)$ is as close to $I_\varphi(f_i + f_j)$ as we want. However this choice of δ depends on the functions f_i and f_j.

Each function $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ gives us a map $f \mapsto I_\varphi(f) \in \mathbb{R}$. The goal is to find a functional that's linear. For each φ the value of $I_\varphi(f)$ lies in the interval $X_f \equiv [f_F, f^F]$. So, we can embed all such functionals inside the compact Hausdorff space,

$$X = \prod_{f \in \mathcal{C}_c^+(\mathcal{G})} X_f.$$

This is compact Hausdorff space because it is a product of compact Hausdorff spaces. It consists of all real valued functions from the space of compactly supported functions whose value at f lies in the interval X_f.

Consider for each neighborhood V of $1 \in \mathcal{G}$, the compact subsets $K(V)$ of X, consisting of closures of the collection of all functionals of the form I_φ with support of φ inside V.

$$X = \bigcup_{1 \in V} K(V)$$

Clearly, $V \subseteq W$ implies $K(V) \subseteq K(W)$. So, these sets satisfy the 'finite intersection property' $\cap_{i=1}^n K(V_i) \supset K(\cap_{i=1}^n V_i)$. This is a strict subset because there always exist functions with domain that's larger than $\cap_i V_i$. Since X is compact, the intersection $\cap_i K(V_i)$ is nonempty.[3] So there exists some point I in X that lies in all of $K(V)$s.

[3] This is an alternate characterization of compactness. The equivalence of the two definitions is basically rewriting the compactness definition of open sets and unions in terms of closed sets and intersections using De Morgan's laws.

Every neighborhood of I contains I_φs. So, for any collection of functions $\{f_i\}_{i=1}^n \subset \mathcal{C}_c^+(\mathcal{G})$ and $\epsilon > 0$, there exists some $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ such that $|I(f_i) - I_\varphi(f_i)| < \epsilon$.

[Why?]

This implies that I is left translation invariant $L_y I = I$, commutes with scaling. Since by adjusting the support for all $\epsilon > 0$, we can find a function φ such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

This implies that, $I(f_i) + I(f_j) \leq I(f_i + f_j) + \epsilon$ for all ϵ, and since I belongs to every $K(V)$, ϵ can be arbitrarily small, hence,

$$I(f_i) + I(f_j) = I(f_i + f_j).$$

This has a unique extension to $\mathcal{C}_c(\mathcal{G})$, because any function can be written as $f = g - h$ for $g, h \in \mathcal{C}_c^+(\mathcal{G})$, and if $f = g' - h'$, then we have $g + h' = g' + h$ and hence, $I(g) + I(h') = I(g') + I(h)$, or, $I(f) = I(g) - I(h)$ is well defined. This linear functional corresponds to a left Haar measure.

Uniqueness

References

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