Part I

Objects of Quantum Theory

The path to the construction of a new physical theory is usually a complicated one. In its initial stages, lots of ideas are tried and tested. The surviving ideas are revised, modified, and more clearly defined. These become the postulates of the theory. The purpose of this chapter is however not to study the historical construction of quantum theory but to give instrumentalist reconstruction of the formal structure of quantum theory. The purpose of reconstruction is to translate primary scientific ideas into a logically impeccable language such that the new presentation is superior concerning clarity and precision. Reconstruction makes the core underlying structure of physical theory clearer and could expose the problems. Many believe that quantum physics is so different from the usual way of thinking that its meaning cannot be communicated directly. People belonging to this group have a 'shut up and calculate' attitude towards the subject and they intend to acquire an intuitive feeling for the subject through calculation and solving problems. We attempt, through reconstruction, try to show them that some of the foundations of quantum mechanics can be clarified and the problems can be clearly laid out. We will divide quantum theory into two parts, the semantics part and the evolution part. The semantic part will consist of the physical objects of the theory which in our case are observables and states and their interpretation. The evolution part is about the dynamic processes of the physical theory. This chapter will act as fixing the notation, philosophical setting and bring the reader on the same page.

The presentation here will be closest to Günther Ludwig school with some stuff borrowed from quantum logic literature. The Ludwig school has the advantage of being compatible with most interpretations if suitably reformulated. See [1]-[10], for operational quantum theory. Usually when one tries to formulate quantum theory one starts with a pre-theory such as classical mechanics then 'quantizes' the theory. This makes the theory very messy and the underlying physical ideas hidden and unclear.

1 | Semantics Part I: Physical Objects

A physical theory is in some sense to be interpreted from outside in terms of pre-theories not belonging to the theory in question itself. To minimize going to these pre-theories we adopt a purely instrumentalist view of physics. The construction and behavior of instruments will not be of interest to us. Any changes occurring in the instruments during 'measurements' will be accepted as objective events. According to this point of view, the fundamental notions of quantum mechanics have to be defined operationally in terms of macroscopic instruments and prescriptions for their application. Quantum mechanics is then interpreted entirely in terms of such instruments and events. These instruments and events are our links to 'objective reality'.

1.1 | Effects and Ensembles

From this instrumentalist or operational point of view, the notion of 'state' can be defined in terms of the preparation procedure. A preparation procedure is characterized by the kind of system it prepares. The other important thing is the existence of a measuring instrument that is capable of undergoing changes upon their interaction. The observable change in the instrument is called an effect in the Ludwig school.

To simplify the procedure consider instruments that record 'hits'. These instruments perform simple 'yes-no' measurements. Any measurement can be interpreted as a combination of yes-no measurements. These yes-no instruments can be used to build any general instrument. Suppose we have such an instrument, label its registration procedure by R. If the experiment is conducted a lot of times, we get a relative frequency of occurrence of 'yes'. To every preparation procedure ρ and registration procedure R_i there exists a probability $\mu(\rho, R_i)$ of occurrence of 'yes' associated with the pair.

$$(\rho, R_i) \longrightarrow \mu(\rho|R_i).$$

The numbers $\mu(\rho|R_i)$ are called operational statistics. Two completely different preparation procedures may give the same probabilities for all experiments R. Such preparation procedures must be considered equivalent. Such preparation procedures are called operationally equivalent preparations. A precursor to the notion of a state of the system is an equivalence class of preparations procedures yielding the same result. They are called ensembles.

The basic mathematical structure of ensembles and effects can be understood using purely mathematical reasons, without introducing any new physical law. Denote the class of ensembles by S and the class of effects by E. The maps of interest to us are the following,

$$S \times E \xrightarrow{\mu} [0, 1].$$

There may be two experiments that give the same probabilities for every ensemble. Such apparatuses must be considered equivalent. They are called operationally equivalent effects. An effect is the equivalence class of apparatuses yielding the same result. In general, a registration procedure R for an experiment will have outcomes $\{R_i\}$. For an outcome, R_i of the registration procedure R, denotes the corresponding equivalence class of measurement procedures by E_{R_i}. Each outcome R_i of the registration procedure corresponds to a functional E_{R_i} called the effect of R_i that acts on the ensemble of the system to yield the corresponding probability.

$$E_{R_i} : \rho \mapsto E_{R_i}(\rho) = \mu(\rho|R_i).$$

Maps of interest to us will be those that assign to each of its outcomes R_i its associated effect E_{R_i}. Since each ensemble fixes a probability distribution we have,

$$\mu_\rho : R_i \mapsto \mu_\rho(R_i) = \mu(\rho|R_i).$$

The above-given map μ_ρ is determined by the instrument and the registration procedure. Accounting to the fact that preparation procedures can be combined to produce a mixed ensemble, the set of ensembles is taken to be a convex set. Since a mixture of ensembles corresponds to a convex combination of probabilities each functional E_{R_i} preserves the convex structure. Since two preparations giving the same result on every effect represent the same ensemble and two measurement procedures that can't distinguish ensemble represent the same effect, ensembles and effects are mutually separating. A generalized probabilistic theory is an association of a convex state space and effect vectors to a given system, such that the

states and effects are uniquely determined by the probabilities they produce. This is known as the principle of tomography. The aim is to obtain a GPT from an operational theory. We are interested in embedding the ensembles inside the vector space of linear functionals on the effects and embed effects inside the vector space of linear functionals on the ensembles. More generally, one takes an operational theory and 'quotients' with operational equivalences to obtain a GPT.

Denote by $\mathcal{S}$ the set of maps, $f : E \longrightarrow \mathbb{R}$ such that $f(X) = \sum_i \alpha_i \mu(\rho_i|X)$ and denote by $\mathcal{E}$ the set of maps, $g : S \longrightarrow \mathbb{R}$ such that $g(\rho) = \sum_i \beta_i \mu(\rho|R_i)$ where ρ_i and R_i are ensembles and effects respectively and $\alpha_i, \beta_i \in \mathbb{R}$. Clearly $\mathcal{S}$ and $\mathcal{E}$ are real vector spaces. We can embed ensembles inside $\mathcal{S}$ with the map,

$$\rho \longmapsto \mu_\rho,$$

and similarly embed effects inside $\mathcal{E}$ with the map,

$$R_i \longmapsto E_{R_i}.$$

The bilinear map $\langle \cdot|\cdot \rangle : \mathcal{S} \times \mathcal{E} \to \mathbb{R}$ which coincides with μ is then uniquely determined. $\langle \mathcal{S} \,|\, \mathcal{E} \rangle$ becomes a dual pair. The completions of $\mathcal{S}$ and $\mathcal{E}$ will provide us the necessary mathematical structure for ensembles and effects. We will denote $\langle \cdot|\cdot \rangle$ by μ.

A registration procedure E_R is an effect valued function that assigns to each possible outcome R_i its effect E_{R_i},

$$E_R : \; R_i \longmapsto E_{R_i}.$$

It's important to find a mathematical structure that describes the registration procedure E_R beyond this basic vector space structure. The purpose of this section is to study the mathematical representatives of effects and ensembles in quantum formalism.

To get the mathematical representatives of physical observables one has to study the logical relations of a set of propositions that are considered meaningful and empirically verifiable according to the theory that describes the physical system. The logic of a physical system will mean the algebraic structure that represents the equivalence classes of the elementary sentences. To simplify the procedure one initially reduces the elementary sentences of the system to simple 'yes-no' questions called propositions. For the development of any mathematical theory, the first step is the idealization of the registrations. Here we are satisfied with the usage of real numbers to label the outcomes.

The concept of observable which is one of the main physical objects of quantum theory can be obtained from a certain idealization of the registration procedure. Consider a registration procedure E_A whose outcomes $\{A_i\}$ are measured using the same equipment. The events of such a registration procedure should form a Boolean ring. The aim is to arrive at the notion of observable from these special kinds of registration procedures. What we seek are maps from Boolean rings to the effects. A mapping A of a Boolean ring Σ into an ordered interval $[0, \epsilon]$ of a vector space, such that, $A(\mathbb{I}) = \epsilon$ where $\mathbb{I}$ is unit of Σ and

$$A(\sigma_1 \vee \sigma_2) = A(\sigma_1) + A(\sigma_2) \quad \text{for all} \;\; \sigma_1 \wedge \sigma_2 = 0,$$

is called an additive measure on Σ. A set $F \subset \mathcal{E}$ is called a set of coexistent effects if there exists a Boolean ring Σ_A with an additive measure $A : \Sigma_A \to \mathcal{E}$ such that $F \subset A\Sigma_A$.

An observable is a special kind of registration procedure where the outcomes form a complete Boolean ring. An observable is a pair (Σ_A, A), where Σ_A is a Boolean ring and A is an additive measure,

$$A : \; \Sigma_A \to \mathcal{E}.$$

We will denote the observable by the map A. The complete Boolean lattice structure of Σ_A is the idealization of the registration procedure. Observables are effect-valued functions where outcomes have a Boolean lattice structure.

Suppose we have two observables A and B and there exists a homomorphism h of the Boolean ring Σ_A into the Boolean ring Σ_B then intuitively the observable B measures more than A since the measurements of the observable A is contained in the observable B. Two observables are equivalent if the homomorphism h is an isomorphism. Two observables A and B are said to coexist if there exists an observable AB and two homomorphisms h and i such that $h : \Sigma_A \to \Sigma_{AB}$ and $i : \Sigma_B \to \Sigma_{AB}$. Denote by Ξ the effects that coexist with every other effect. Two observables A and B are mutually complementary if every coexistent effect is in Ξ. If two effects $E_{A_i} \in A$ and $E_{B_j} \in B$ are coexistent then at least one of them is in Ξ. The existence of such observables is a feature of quantum mechanics that wasn't the case in classical mechanics.

Similar to effects we can study idealizations of ensembles. Since effects and ensembles are closely related objects we would see similar conditions on the notion of state coming from observables. The state should provide for each observable a probability distribution. In the formulations of quantum mechanics, the question of whether it is possible to make joint measurements of pairs of observables is important. It's this question that leads to all the subtleties of quantum mechanics. The question is regarding the possibility of joint preparation. This helps us separate the question about the possibility of making joint preparations from the problem of registration. We are interested in decomposing the ensemble and studying the relation between different decompositions. Maps of interest to us are of the form,

$$w : \Sigma_A \to \mathcal{S},$$

such that $w(1) = \mu_\rho$. The structure of the observable A would be contained in the Boolean lattice Σ_A. The condition $w(1) = \mu_\rho$ contains the structure of the ensemble μ_ρ. These maps are called preparators in Ludwig's approach. A preparator of the ensemble μ_ρ is a map $w_i : \Sigma_i \to \mathcal{S}$ such that $w_i(1) = \mu_\rho$. Preparators represent the information the state contains about an observable. A preparator w_i of the ensemble μ_ρ is more comprehensive than the preparator ω_j if there exists a homomorphism $h : \Sigma_j \to \Sigma_i$. w_i and w_j coexist if there is a preparator w which is more comprehensive than both. Using a preparator $w : \Sigma \to \mathcal{S}$ of μ_ρ, new preparators can be obtained as follows: Let $[0, \epsilon] \subset \Sigma$ then,

$$w_\epsilon : [0, \epsilon] \to \mathcal{S},$$

where $[1/\mu(w(\epsilon), 1)]w := w_\epsilon$ is a preparator of the ensemble $[1/\mu(w(\epsilon), 1)]w(\epsilon) := \mu_{\rho_\epsilon}$. We will call this the preparator of $[0, \epsilon]$. Suppose we have two preparators, w_i and w_j, we call them mutually exclusive if there doesn't exist sections $[0, \epsilon_i] \subset \Sigma_i$ and $[0, \epsilon_j] \subset \Sigma_j$ such that $\mu_{\rho_{\epsilon_i}} = \mu_{\rho_{\epsilon_j}}$ and the canonical preparators of $[0, \epsilon_i]$ and $[0, \epsilon_j]$ coexist. Two preparators w_i and w_j of μ_ρ are complementary if whenever there is a homomorphism $h : \Sigma_i|_h \to \Sigma_j$ the corresponding new restricted preparators are mutually exclusive. These properties allow us to extract the mathematical properties of ensembles.

Suppose $A : \Sigma_A \to \mathcal{E}$ is an observable then a state μ_ρ gives us a map,

$$\mu_\rho^A : \Sigma_A \to [0, 1],$$

such that $\mu_\rho^A(0) = 0$, $\mu_\rho^A(E^\perp) = 1 - \mu_\rho^A(E)$ and whenever E_i are mutually orthogonal,

$$\mu_\rho^A(\vee_i E_i) = \sum_i \mu_\rho^A(E_i).$$

For all practical purposes, we will assume the measurement scale is separable. This assumption gives us all the nice properties needed to do mathematics.

It is important to note that preparation and registration procedures producing the same ensembles and effects are not always equal, in fact, the notion of equality won't even make sense. The transition from preparation and registration procedures to ensembles and effects is a transition from the real world to the abstract mathematical world. It should be noted that it doesn't make sense to 'prepare' closed systems, one has to assume such systems start off in some state a priori.

1.2 | Observables and States

By the end of the nineteenth century, it was clear that elementary processes obeyed some 'discontinuous' laws. There existed no mathematical formalism of quantum theory that would provide a unified structure. Heisenberg's solution to this problem was to use linear operators as a starting point. What von Neumann took away from Heisenberg's idea was that the mathematical objects needed for the description of observables is found in Hilbert spaces and operator algebras acting on the Hilbert spaces.

1.2.1 | Hilbert Spaces and Operators

The space of functions on both discrete and continuous spaces have the same Hilbert space structure, see [8] for a discussion.[1] The coexistence of discrete and continuous observables is possible. The necessary structure for the abstract mathematical framework of quantum theory is found in Hilbert spaces and operator algebras.

Let $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be a complex Hilbert space. $\mathcal{P}(\mathcal{H})$ denote the set of all closed subspaces. Denote $\mathcal{H}_i \leq \mathcal{H}_j$ if and only if $\mathcal{H}_i \subseteq \mathcal{H}_j$. The relation $\leq$ is a partial ordering in $\mathcal{P}(\mathcal{H})$. Join $\vee$ of a family $\{\mathcal{H}_i\}_{i\in I}$ is the linear span of the family denoted $\vee_i \mathcal{H}_i$. Meet $\wedge$ of a family $\{\mathcal{H}_i\}_{i\in I}$ is the intersection of the family, denoted $\wedge_i \mathcal{H}_i$. The orthocomplement of $\mathcal{H}_i$ in $\mathcal{P}(\mathcal{H})$ denoted by $\mathcal{H}_i^\perp$ is the closed subspace of vectors $\varphi \in \mathcal{H}$ such that $\langle\varphi|\mathcal{H}_i\rangle = 0$. Since there is a bijection between closed subspaces of a Hilbert space and projection operators acting on the Hilbert space, the set of all projection operators on the Hilbert space inherits a lattice structure from the lattice of closed subspaces. Abusing notation, we will denote the projection operators on $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. The orthocomplement of the projection E is the projection onto the orthogonal complement of the subspace corresponding to the projection operator E and is denoted by $E^\perp$. The lattice structure of $\mathcal{P}(\mathcal{H})$ coming from the above relations gives us the necessary structure to get the mathematical representatives of physical observables. The non-Boolean lattice $\mathcal{P}(\mathcal{H})$ of projections should act as the space of effects.

$$\mathcal{E} \equiv \mathcal{P}(\mathcal{H})$$

For a family of projection operators to represent an observable, we should make sure that the family forms a Boolean algebra. A quantum mechanical observable is an additive measure of the form,

$$E_A : \Sigma_A \to \mathcal{P}(\mathcal{H}),$$

a projection valued function. Usually in physical experiments, the statements that can be made are of the type 'the value of the observable lies in some set ϵ_i of real numbers'. To accommodate the fact that the measurement scale is composed of real numbers, we identify

[1] I have not discussed the motivation here as I felt it will be beneficial for the reader to go through von Neumann's textbook instead reading a copy of his discussion of the subject here

Σ_A with the Borel sets of $\mathbb{R}$. It should be noted that the observables need not be real, the physics community has historically decided to use real numbers to label the outcomes of experiments. Any other labeling should work equally well. Döring and Isham have done an interesting generalization of this scheme [**?**]. Their idea seems to be to replace the Boolean structure in Σ_A with a more general propositional language system and question if values of the system should be more general than 'real'. Though we find this to be a beautiful generalization for the future of quantum theory we don't think this is the part needing fixing for solving the foundational problems in quantum theory. We believe we can get a lot of work done with real measurement scales themselves.

The quantum observables are analogous to classical random variables, namely, that of a projection valued measure,

$$E_A : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}).$$

This generalizes the classical case, for which mathematical representatives were the measure space $(\Omega, \Sigma(\Omega), \mu)$, where the σ-algebra, $\Sigma(\Omega)$ is a class of subsets of the set Ω which correspond to events and μ is a probability measure. A classical random variable is defined as a map $X : \Omega \to \mathbb{R}$. The map doing the work in assigning necessary probabilities is its inverse, considered as a set map,

$$X^{-1} : \mathcal{B}(\mathbb{R}) \to \Sigma(\Omega).$$

A spectral measure is a projection operator-valued function E defined on the sets of $\mathbb{R}$ such that, $E(\mathbb{R}) = I$ and $E(\sqcup_i \epsilon_i) = \sum_i E(\epsilon_i)$, where ϵ_is are disjoint Borel sets of $\mathbb{R}$. The spectral theorem says that every self-adjoint operator A corresponds to a spectral measure E_A such that,

$$A = \int \lambda \, dE_A(\lambda),$$

and conversely, every spectral measure corresponds to a self-adjoint operator. In the finite-dimensional case this reduces to $A = \sum_i \lambda_i E_i$ where E_is are projections onto eigenspaces of λ_is. Observables in quantum theories are represented by self-adjoint operators on some complex Hilbert space and the orthogonal projections of the self-adjoint operator correspond to the events. The values of the observable are the spectrum of the operator. The characteristic feature of quantum theory is that the space of effects is a non-commutative entity.

The mathematical representatives of the physical states for the quantum case are the maps, $\omega : \mathcal{P}(\mathcal{H}) \to [0, 1]$, such that $\omega(0) = 0$, $\omega(E^{\perp}) = 1 - \omega(E)$ and $\omega(\vee_i E_i) = \sum_i \omega(E_i)$ for mutually orthogonal E_i. For an observable with the associated self-adjoint operator A, the map

$$\mu^A = \omega \circ E_A : \Sigma_A \to [0, 1],$$

determines a classical probability measure. The existence and classification of such non-commutative probability measures on Hilbert spaces is given by the Gleason's theorem.

Theorem 1.1. (Gleason) *If the complex separable Hilbert spaces $\mathcal{H}$ of dimension greater than 2, then every ω is of the form*

$$\omega(E) = Tr(\rho E).$$

where ρ is a positive semidefinite self-adjoint operator of unit trace or density matrix. Conversely, every density matrix determines a state as defined in the above formula.

Sketch of Proof

Gleason's proof of the theorem is quite complicated. He starts by defining what he calls frame functions of weight W on separable Hilbert space $\mathcal{H}$. A frame function f is a real valued functions on unit sphere of $\mathcal{H}$ such that for any orthonormal basis, $\{|\varkappa_i\rangle\}_{i\in\mathbb{N}}$, $\sum_i f(|\varkappa_i\rangle) = W$. A frame function is regular if there exists a self-adjoint operator ρ such that for every unit vector $|\varkappa\rangle \in \mathcal{H}$,

$$f(|\varkappa\rangle) = \langle\varkappa|\rho\varkappa\rangle$$

Gleason proves that every frame function on two dimensional Hilbert spaces is regular. For Hilbert spaces of dimension greater than three the result holds for every two dimensional subspaces. Then he proves the continuity of frame functions. Every non-negative frame function on a Hilbert space of dimension greater than three is regular. Much of the hard work lies in this part. I will cheat and skip this hard part. A brave reader can go read Gleason's original paper [11] or H Granström's master's thesis [12] on Gleason's theorem.

Suppose $\omega : \mathcal{P}(\mathcal{H}) \to [0, 1]$ be a function as described above. Let E_φ be the projection onto the subspace spanned by the unit vector φ. $f(\varphi) = \omega(E_\varphi)$ defines a non-negative frame function. By regularity there exists a self-adjoint operator ρ such that,

$$f(\varphi) = \langle\varphi|\rho\varphi\rangle.$$

Since this holds for all unit vectors, ρ is positive semi-definite. Denote by $E_\mathcal{H}$ the projection onto the whole Hlibert space i.e., the identity operator. Given an orthonormal basis $\{|\varphi_i\rangle\}_{i\in\mathbb{N}}$ of $\mathcal{H}$ we have,

$$\omega(E_\mathcal{H}) = \sum_i \omega(E_{|\varphi_i\rangle}) = \sum_i \langle\varphi_i|\rho\varphi_i\rangle = Tr(\rho).$$

For any subspace $\mathcal{K} \subset \mathcal{H}$ denote by $E_\mathcal{K}$ the corresponding projection operator. Take an orthonormal basis $\{\varkappa_i\}_{i\in I}$ for $\mathcal{K}$ and extend it to $\mathcal{H}$. Then we can write,

$$\omega(E_\mathcal{K}) = \sum_{i\in I} \omega(\varkappa_i) = \sum_{i\in I} \langle E_\mathcal{K}\varkappa_i|\rho\varkappa_i\rangle = Tr(\rho E_\mathcal{K}).$$

So we have for all projection operators $E \in \mathcal{P}(\mathcal{H})$, we have,

$$\omega(E) = Tr(\rho E).$$

□

The proof of Gleason's theorem is unimportant. The proof requires patience to read through and high amount of problem solving skill, intelligence and insight to come up with. But we don't need those to understand what it is saying. We probably will never need the methods used in the proof of Gleason's theorem for understanding quantum mechanics.

More general quantum experiments correspond to positive operator-valued measures. The effects E are given by positive operators, $O \leq E \leq I$ as probabilities are positive quantities. Since these should sum to 1 for an experiment, it will be a resolution of identity $\sum_i E_{A_i} = I$, where E_{A_i}s are effects. The resolution of identity $E_A : A_i \to E_{A_i}$ is called positive operator-valued measure (POVM). General quantum mechanical experiments are represented by pairs (ρ, E_A). For Gleason's theorem in this setting see [15].

We call ω a Gleason measure. Every state corresponds to a positive semidefinite self-adjoint operator of unit trace. We denote the set of all states on the Hilbert space $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$. The mathematical representatives of ensembles are states.

$$\mathcal{S} \equiv \mathcal{S}(\mathcal{H})$$

The extreme points of this convex set are called pure states, pure states are of the form, $\rho = \rho^2$ and corresponds to some vector $|\varphi\rangle$ in the Hilbert space $\mathcal{H}$ and ρ is the projection onto the subspace generated by $|\varphi\rangle$. Such a state is denoted by $|\varphi\rangle\langle\varphi|$.

For an observable with an associated self-adjoint operator A the probability that the observable takes a value lying in the interval ϵ is given by,

$$\mu_\rho^A(\epsilon) = Tr(\rho E_A(\epsilon)).$$

The expectation value of the observable will be,

$$\langle A\rangle = \int \lambda \, d\mu_\rho^A(\lambda) = Tr(\rho A).$$

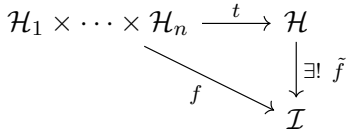
All the Hilbert spaces will be assumed to be separable, complex.

1.2.2 | Tensor Product

Given a finite number of Hilbert spaces $\mathcal{H}_i$, for n quantum systems, the problem is to describe the Hilbert space appropriate to the 'product' system. Let $\mathcal{I}$ denote a possible solution to this problem: that is the states of $\mathcal{I}$ are supposed to be states for the product system. Then, at the very least, some of the preparation procedures for the product system should be obtainable by arranging in some manner the preparation procedures on the individual systems. We should be able to construct a certain function,

$$f : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{I}.$$

The interpretation of f is that, it introduces a component from each individual system into the product system. Accounting to the superpositions, the product system should inherit the structure from the components. The map f must be linear for each component. The universal solution $\mathcal{H}$ to this problem is the algebraic tensor product. It's the vector space $\mathcal{H}$ together with an n-linear map t such that, for any n-linear map $f : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{I}$, there exists a unique linear map $\tilde{f} : \mathcal{H} \to \mathcal{I}$,



This vector space inherits a canonical inner product from the component Hilbert spaces. The completion of $\mathcal{H} = \otimes_{i\in I}\mathcal{H}_i$ under the canonical inner product will serve as the Hilbert spaces for product systems.

It's important to note that when we are given a closed system, the notion of preparation of state doesn't make sense. So in such cases, we are stuck with states given by nature.

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