Part IV

Residue Theorem

In this part we will study the residue theorem and related material from a topological viewpoint.

1 | Winding number and Index

The starting point is to study how many times a loop winds around a point. Finding this is however not possible within the base space. To do this we will lift the loop to the universal cover and then find a way to study the number of times it winds the point. This is similar to calculating the fundamental group of circle.

The starting point is to find a universal covering space. Consider the map,

$$\exp : z \mapsto e^z.$$

This is a local homeomorphism of $\mathbb{C}$ and $\mathbb{C}^*$ and around each point x the neighborhoods $B_\epsilon(x)$ with $\epsilon < 2\pi$ then,

$$B_\epsilon(x) \cong e^{B_\epsilon(x)}.$$

So the map $z \mapsto e^z$ and hence the map $z \mapsto x + e^z$ are covering maps as we have around each point a connected neighborhood thats homeomorphic to its image. Since the covering space is simply connected, it's the universal cover. Now we can start lifting paths, homotopies to the covering space. We will denote by $\mathbb{C}^*$ the complex plane without the point x.

Let $\eta : I \to \mathbb{C}$ be a loop or closed curve in $\mathbb{C}$ and let $x \in \mathbb{C}$ that's not in the image of η. So η is a loop in the base space and we can lift this loop to $\mathbb{C}$ using the above covering map $p : z \mapsto x + e^z$. Let $\widehat{\eta}_1$ and $\widehat{\eta}_2$ be two lifts then, $p \circ \widehat{\eta}_1 = p \circ \widehat{\eta}_2 = \eta$, so we have,

$$x + e^{\widehat{\eta}_1}(0) = x + e^{\widehat{\eta}_2(0)} = \eta(0)$$

So, $\widehat{\eta}_1(0) = \widehat{\eta}_2(0) + 2\pi i k$ for some $k \in \mathbb{Z}$. By uniqueness, of lifts, we have, $\widehat{\eta}_1 = \widehat{\eta}_2 + 2\pi i n$. So, the difference, $\widehat{\eta}(1) - \widehat{\eta}(0)$ is well defined, i.e., it's independent of the lift. Since η is a loop we have, $\eta(0) = \eta(1)$, and hence we have, $x + e^{\widehat{\eta}}(0) = x + e^{\widehat{\eta}(1)}$ or

$$\widehat{\eta}(0) = \widehat{\eta}(1) + 2\pi i n(\eta, x)$$

for some integer $n(\eta, x)$. The winding number of η with respect to x is defined to be,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)].$$

Note that this depends on the covering map, and hence on the point x. Every time the curve η winds a circle the argument changes by 2π, since the starting point and the end point of the curve are the same the radius is the same for both start and end. Hence $1/2\pi i[\widehat{\eta}(1) - \widehat{\eta}(0)]$ represents the number of times the loop winds around the point x. Now we need a way to compute it.

We will use the structure of holomorphic functions to compute index. To begin this, we construct a loop in $\mathcal{EH}$ using the loop η. For each point, z in we will associate a germ in $\mathcal{EH}$ and the compose with the loop η.

Theorem 1.1.

$$n(\eta, x) = \frac{1}{2\pi i} \int_\eta \frac{dz}{z - x}.$$

Sketch of Proof

Consider the map

$$f_x(z) = 1/z - x.$$

This is the derivative of the logarithm function which when composed with the covering map $z \mapsto x + e^z$ yields identity. So, we can use this to construct lifts. Consider the function,

$$\nu : z \mapsto \operatorname{germ}_z f_x$$

The composition of this map with η gives us a loop in $\mathcal{EH}$.

$$\Gamma = \nu \circ \eta : I \to \mathcal{EH}.$$

Our goal is to lift this map using the covering space $d : \mathcal{EH} \to \mathcal{EH}$ and use the primitive to construct a lift of the loop η, and thus relate the integral along η of the function $1/z - x$ to index.

Let $\widehat{\Gamma}$ be the lift of Γ with respect to the covering map $d : \mathcal{EH} \to \mathcal{EH}$. This associates to each $t \in I$, the germ $\widehat{\Gamma}\eta(t) \in \mathcal{H}_{\eta(t)}$. Let F with domain $\widehat{U}_{\eta(t)}$ be the representative of the germ $\widehat{\Gamma}(\eta(t))$. By definition of the derivative map d we have,

$$F'(z) = 1/(z - x).$$

So, we have, $F'(z)(z-x) - 1 = 0$. Multiplying by $e^{-F(z)}$ we have, $\frac{d}{dz}\left[(z-x)e^{-F}\right] = (1 - F'(z)(z-x))e^{-F(z)} = 0$. So, the term $(z-x)e^{-F(z)}$ is locally constant. We will use this function to construct a lift of the loop η. Consider the valuation map,

$$\begin{aligned} \Xi :\ I &\longrightarrow \mathcal{EH} \longrightarrow \mathbb{C} \\ t &\mapsto \widehat{\Gamma}(t)(\eta(t)) \end{aligned}$$

which evaluates the germ $\widehat{\Gamma}(t)$ at $\eta(t)$. Since $(z-x)e^{-F(z)}$ is locally constant on $U_{\eta(t)}$ with value α, so, the map, $t \mapsto (\eta(t) - x)e^{-\Xi(t)}$ is constant because it's local constant and I is compact. Let c be such that $e^c = \alpha$. Consider the map,

$$\widehat{\eta} : t \mapsto \Xi(t) + c$$

We claim that this is a lift of η. This is a simple check and we have to verify that $x + e^{\Xi(t)+c} = \eta(t)$. $x + e^{\Xi(t)+c} = x + e^{\Xi(t)}e^c = x + \alpha e^{\Xi(t)} = x + (\eta(t) - x)e^{-\Xi(t)}e^{\Xi(t)} = x + \eta(t) - x = \eta(t)$, i.e., $p \circ \widehat{\eta} = \eta$ or $\widehat{\eta}$ is a lift of η. So,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)] = \frac{1}{2\pi i} \int_\eta \frac{1}{(z-x)} dz$$

as $\int_\eta f(z)dz = F(1)(\eta(1)) - F(0)(\eta(0))$ for F primitive of f along η. $\square$

References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000