# Part IV

# Covering spaces & Monodromy

In these notes we start looking at the étale space of sheaf of holomorphic functions. The étale space of holomorphic functions has special properties that allow us to use the techniques of covering spaces and elementary homotopy theory. We will assume some basic knowledge of homotopy and fundamental groups.

## 1 | Covering Spaces

Covering spaces have the same local topological properties as the base space but different global topological properties. Covering spaces can have less global constraints compared to base spaces (covering spaces with least constraints are called universal covers), and this allows the functions to them to have a bit more freedom to be weird compared to functions to the base space. This allows us to study certain properties of functions that wouldn't have been possible within the base space. This structure is again lost as we go down to the base space.

**Definition 1.1.** A local homeomorphism $\pi : C \to X$ is called a covering space if each $x \in X$ has a connected neighborhood $U_x$ such every,

$$\pi^{-1}(U_x) = \coprod_{i \in I} \widehat{U}_{x_i},$$

such that, $\pi(\widehat{U}_{x_i}) \cong U_x$.

$X$ is called the base space and $C$ the cover. The neighborhoods $U_x$ are called evenly covered neighborhoods. Let $x \in X$ and let $|\pi^{-1}(x)|$ be the number of elements in $\pi^{-1}(x)$. Let

$$\mathcal{P}_{|\pi^{-1}(x)|} = \{y \in X \mid |\pi^{-1}(y)| = |\pi^{-1}(x)|\},$$

be the set of all points in $X$ which have $|\pi^{-1}(x)|$ inverse images. This set is open because of local homeomorphism and since the complement is also open the set $\mathcal{P}_{|\pi^{-1}(x)|}$ is also closed. Since $x \in \mathcal{P}_{|\pi^{-1}(x)|}$, it's nonempty.

**Theorem 1.1.** $\deg(\pi) = |\pi^{-1}(x)|$ *is a constant.* □

Since covering spaces are locally homeomorphic we can lift a function $f : Y \to X$ to a function $\widehat{f} : Y \to C$ using the local homeomorphism. These lifts allow us to study certain properties of the function $f$ that were not possible within the base space due to constraints coming from its topology.

**Definition 1.2.** $\widehat{f} : Y \to C$ is called a lift of $f : Y \to X$ if $f = \widehat{f} \circ \pi$.

The local homeomorphism aspect of covering spaces allows us to lift paths and homotopies. The goal is to divide up the path into smaller paths such that each of these individual path belongs entirely to some evenly covered neighborhood and then lift each of them. Similarly for the homotopy, we can divide up the square of homotopy into smaller squares such that each smaller square belongs entirely to some evenly covered neighborhood and lift them.

**Theorem 1.2. (Path Lifting)** *Every path and homotopy of paths in $X$ can be lifted to paths and homotopies in $C$.* □

Sketch of Proof

Let $\pi : C \to X$ be a covering space and $C$ be path connected. Let $\eta : I \to X$ be a path in $X$. The space $X$ can be written as

$$X = \bigcup_{i \in J} U_i,$$

where $U_i$s are evenly covered neighborhoods. The interval $I$ can be finitely partitioned $I = \cup_{i \in K} I_i$ such that the image of each part $I_i$ under $\eta$ belongs entirely in one of $U_i$.[1] The initial point $\eta(0) \in X$ belongs to some neighborhood $U_i$ we can lift the path using the local homeomorphism to get a lift of the path and continue from the end of this part. Without using the Lebesgue lemma this can be proved via induction.

Similarly, we can lift homotopies to the covering space. So if two paths are homotopic in the base space then their lifts are also homotopic if they have the same initial point. □

If $\pi : C \to X$ is a covering space then $\Pi_1(C)$ is a subgroup of $\Pi_1(X)$ and the degree of the map is given by,

$$\deg(\pi) = |\Pi_1(X) : \Pi_1(C)|.$$

the proof is routine well-definedness checks and injective verification. If $\pi : C \to X$ is a local homeomorphism with the curve lifting property then we can lift every path, and hence to each element in the fundamental group we get an element. $\pi : C \to X$ is a covering map if and only if it has the curve lifting property.

Let $\pi : E \to X$ be a covering space and let $f : Y \to X$ be a continuous map with $Y$ connected. Let $\widehat{f}_1, \widehat{f}_2$ be two lifts of $f$. Let

$$\mathcal{F} = \{y \mid \widehat{f}_1(y) = \widehat{f}_2(y)\}$$

This is an open set of $Y$ because around each $\widehat{f}_1(y)$ we can choose a uniformly covered neighborhood $U$ such that $U \cong \pi(U)$. Since $f$ is continuous $f^{-1}(U)$ is open with $x \in f^{-1}(U)$ where $\widehat{f}_1|_U = \widehat{f}_2|_U$.

[1] this requires the Lebesgue lemma,

Consider $Y \backslash \mathcal{F}$. Suppose $E$ is Hausdorff, let $y \in Y \backslash \mathcal{F}$ i.e., $\widehat{f}_1(y) \neq \widehat{f}_2(y)$ choose disjoint neighborhoods $\widehat{U}_1, \widehat{U}_2$ of $\widehat{f}_1(y)$ and $\widehat{f}_2(y)$ respectively such that

$$\pi|_{\widehat{U}_i} : U_i \cong U \subset X.$$

Then $V = f^{-1}(U) \subseteq Y \backslash \mathcal{F}$ or i.e., $Y \backslash \mathcal{F}$ is open. So, $\mathcal{F}$ is both open and closed. If there exists some $y \in Y$ for which $\widehat{f}_1(y) = \widehat{f}_2(y)$ then the lifts are the same.

$$\begin{tikzcd}
 & E \arrow[d, "\pi"] \\
Y \arrow[ur, bend left=40, "\widehat{f}_2"] \arrow[ur, bend left=10, "\widehat{f}_1"] \arrow[r, "f"] & X
\end{tikzcd}$$

**Theorem 1.3. (Uniqueness)** *Let $Y$ be connected, and $X, E, Y$ be Hausdorff. If $\widehat{f}_1, \widehat{f}_2$ are two lifts of $f : Y \to X$ such that $\widehat{f}_1(y) = \widehat{f}_2(y)$ for some $y \in Y$. Then $\widehat{f}_1 = \widehat{f}_2$.* $\square$

**Theorem 1.4. (Lifting Criterion)**

## 2 | Sheaf of Holomorphic Functions

The sheaf of holomorphic functions has some special properties. As we will show, the étale space of the sheaf of holomorphic functions is a Hausdorff space and hence the uniqueness theorems in the previous section are applicable. This allows us to apply theorems of algebraic topology to the study of holomorphic functions.

To each open set $U \subseteq X$ corresponds the associated collection of holomorphic functions $\mathcal{H}U$. Each $\mathcal{H}U$ has an algebra structure. Consider the sheaf of holomorphic functions,

$$\mathcal{H} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that for each open covering $U = \bigcup_{i \in I} U_i$ of an open set $U$ of $X$ comes with an equilizer diagram,

$$\begin{tikzcd}
\mathcal{H}U \arrow[r, dashed, "e"] & \prod_i \mathcal{H}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{H}(U_i \cap U_j).
\end{tikzcd}$$

where $p$ and $q$ are the maps,

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

The collection of open sets has a preorder given by, $V \geq U$ if $V \subset U$. For any directed collection $\mathcal{D}$ of open sets we have a directed system in **Sets** given by $\{\mathcal{H}U\}_{U \in \mathcal{D}}$. The stalk $\mathcal{H}_x$ of the sheaf $\mathcal{H}$ at $x$ is the direct limit of the directed system $\{\mathcal{H}U_i\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is a directed set of open neighborhoods of $x$.

$$\mathcal{H}_x = \varinjlim_{x \in U} \mathcal{H}U.$$

stalks are functors, $\mathrm{Stalk}_x : \mathrm{PSh}(X) \to \mathbf{Sets}$ with, $\mathcal{H} \mapsto \mathcal{H}_x$. The elements of $\mathcal{H}_x$ are called germs of holomorphic functions at $x$.

$$\begin{aligned} \mathrm{germ}_x : \mathcal{H}(U) &\to \mathcal{H}_x \\ f &\mapsto \mathrm{germ}_x(f). \end{aligned}$$

$\mathrm{germ}_x : \mathcal{H}U \to \mathcal{H}_x$, is a homomorphism of the respective category for each $U$. If $f, g \in \mathcal{H}U$ such that $\mathrm{germ}_x f = \mathrm{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset U$ such that $f|_{U_x} = g|_{U_x}$ i.e., they have the same power series expansion around $x$.

Let $\mathrm{germ}_x f$, $\mathrm{germ}_x g \in \mathcal{H}_x$. Let $f$ and $g$ be the corresponding representatives with domains $U_f$ and $U_g$ respectively. We can define $\mathrm{germ}_x f + \mathrm{germ}_x g$ as the germ defined by the function $\mathrm{germ}_x(f + g)$ with representative $f + g$ on the domain $U_f \cap U_g$.

$$\mathrm{germ}_x f + \mathrm{germ}_x g = \mathrm{germ}_x(f + g).$$

Similarly, we can define multiplication by,

$$\mathrm{germ}_x f \cdot \mathrm{germ}_x g = \mathrm{germ}_x(f \cdot g).$$

$\mathcal{H}_x$ is a commutative ring with the above notion of addition, multiplication. The ring also contains the unit element given by the constant function. So $\mathcal{H}_x$ is an unital commutative ring. We can also define scaling operation as follows, for every $\lambda \in \mathbb{C}$, $\lambda \cdot \mathrm{germ}_x f = \mathrm{germ}_x(\lambda \cdot f)$. With these operations, the set $\mathcal{H}_x$ is a complex vector space.

Consider the set $\mathcal{I}_x$ of all non-units, i.e., non invertible elements in $\mathcal{H}_x$. Suppose $\mathrm{germ}_x f \in \mathcal{I}_x$ and suppose $f(x) \neq 0$ then there exists some neighborhood $U$ of $x$ such that $f|_U \neq 0$. Then $1/f$ on the neighborhood $U$ is a function such that

$$\mathrm{germ}_x f \cdot \mathrm{germ}_x(1/f) = 1,$$

or $\mathrm{germ}_x f$ is a unit, contradicting the assumption. Conversely, if $\mathrm{germ}_x f$ is a unit element then there exists some $g$ on some neighborhood $U_g$ such that $\mathrm{germ}_x f \, \mathrm{germ}_x g = 1$. This means that $f(z) \cdot g(z) = 1$ on some neighborhood of $x$, or $f(x) \neq 0$. So the set of non-units is given,

$$\mathcal{I}_x = \{\mathrm{germ}_x f \mid f(x) = 0\} \subset \mathcal{H}_x.$$

Since $(f \cdot g)(x) = f(x)g(x) = 0$ whenever $f(x) = 0$ we have, $\mathrm{germ}_x f \cdot \mathrm{germ}_x g \in \mathcal{I}_x$ for all $\mathrm{germ}_x g$, whenever $\mathrm{germ}_x f \in \mathcal{I}_x$. The set $\mathcal{I}_x$ is an ideal in $\mathcal{H}_x$. Let $\mathcal{I}$ be any proper ideal, it can't contain any units because otherwise it $1 \in \mathcal{I}$ which in turn means that $\mathcal{I} = \mathcal{H}_x$ or that $\mathcal{I}$ is not a proper ideal. Since $\mathcal{I}_x$ is the collection of all non-units it must be a maximal ideal.

The evaluation map,

$$\mathrm{germ}_x f \mapsto f(x),$$

is a homomorphism of $\mathcal{H}_x$ onto $\mathbb{C}$ with kernel $\mathcal{I}_x$. Hence by isomorphism theorem of group theory we have, $\mathcal{H}_x/\mathcal{I}_x = \mathbb{C}$.

**Theorem 2.1.**

$$\mathcal{H}_x/\mathcal{I}_x \cong \mathbb{C}.$$

□

Bundle the sets $\mathcal{H}_x$ into a disjoint union and define the map,

$$\mathcal{E}\mathcal{H} = \coprod_{x \in X} \mathcal{H}_x \xrightarrow{\ \pi\ } X,$$

that sends each $\mathrm{germ}_x f$ to the point $x$. Each $f \in \mathcal{H}U$ determines a function $\widehat{f} : U \to \mathcal{E}\mathcal{H}$ that maps

$$\widehat{f} : x \mapsto \mathrm{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{H}$ by taking as base of open sets all the image sets $\widehat{f}(U) \subset \mathcal{E}\mathcal{H}$. This topology makes both $\pi$ and $\widehat{f}$ continuous by construction. Each point $\mathrm{germ}_x f$ in $\mathcal{E}\mathcal{H}$ has an open neighborhood $\widehat{f}(U)$. $\pi$ restricted $\widehat{f}(U)$ is a homeomorphism of $\widehat{f}(U)$ and $U$. So $\pi : \mathcal{E}\mathcal{H} \to X$ is a local homeomorphism.

**Theorem 2.2.** *$\pi : \mathcal{E}\mathcal{H} \to X$ is a local homeomorphism.* □

The disjoint union $\coprod_{x \in X} \mathcal{H}_x$ together with the topology just described is the étale space of holomorphic functions on $X$. For $X = \mathbb{C}$ the local homeomorphism above makes the étale space of holomorphic functions a two-dimensional manifold.

Let $\mathrm{germ}_x f \in \mathcal{H}_x$, $\mathrm{germ}_y g \in \mathcal{H}_y$ such that $\mathrm{germ}_x f \neq \mathrm{germ}_y g$.

If $x \neq y$, find disjoint neighborhoods $U_f$ and $U_g$ for the representatives of $\mathrm{germ}_x f$ and $\mathrm{germ}_y g$. Then we have $\mathrm{germ}_x f \in \widehat{f}(U_f)$ and $\mathrm{germ}_y g \in \widehat{g}(U_g)$ are disjoint neighborhoods. So whenever $\mathrm{germ}_x f \neq \mathrm{germ}_y g$ we can find disjoint open neighborhoods around them.

If $x = y$, then $\mathrm{germ}_x f \neq \mathrm{germ}_y g$ only if they have different power series expansion around $x$ i.e., there exists a neighborhood $U$ such that $f|_U \neq g|_U$. So

$$\widehat{f}(U) \cap \widehat{g}(U) = \varnothing,$$

because otherwise if there existed some $\mathrm{germ}_z h \in \widehat{f}(U) \cap \widehat{g}(U)$ it means $\mathrm{germ}_z f = \mathrm{germ}_z g$ which is a contradiction by choice of $U$.

So, whenever $\mathrm{germ}_x f \neq \mathrm{germ}_y g$, we can find disjoint open sets $U, V$ such that $\mathrm{germ}_x f \in U$ and $\mathrm{germ}_y g \in V$. In other words, $\mathcal{E}\mathcal{H}$ is Hausdorff.

**Theorem 2.3. (Hausdorff)** *$\mathcal{E}\mathcal{H}$ is a Hausdorff topological space.* □

The complex derivative induces a map on the étale space of holomorphic functions. Let $\mathrm{germ}_x f \in \mathcal{H}_x$ with the representative function $f$ with a domain $U$. Define the derivative $d$ as the map,

$$\begin{aligned} d :\ & \mathcal{E}\mathcal{H} \longrightarrow \mathcal{E}\mathcal{H} \\ & \mathrm{germ}_x f \mapsto \mathrm{germ}_x(f') := d\,\mathrm{germ}_x f. \end{aligned}$$

where $f'$ is the complex derivative of the function $f$.

Let $\mathrm{germ}_x f \in \mathcal{H}_x$ and let $f$ be a representative of $\mathrm{germ}_x f$ with domain $U$. Let $F$ be a primitive of $f$, i.e., $F' = f$ for some $B \subset U$. Clearly,

$$d(\widehat{F + c})(U) = \widehat{f}(U),$$

or $\widehat{(F+c)}(U) \subset d^{-1}(\widehat{f}(U))$ for all $c \in \mathbb{C}$.

If $d\widehat{g}(z) = \widehat{f}(z)$, then $\operatorname{germ}_z g' = \operatorname{germ}_x f$ or $g' = f$ in a neighborhood of $z$, so, $d/dz(g - F) = 0$ in the neighborhood or $g = F + c$ in the neighborhood. So we have for some neighborhood $U$,

$$d^{-1}\widehat{f}(U) = \coprod_{c\in\mathbb{C}} \widehat{(F+c)}(U).$$

Each $\widehat{(F+c)}(U)$ maps injectively onto $\widehat{f}(U)$, since $d$ is a map from $\mathcal{EH}$ to $\mathcal{EH}$ it's hence an homeomorphism. So, $d : \mathcal{EH} \to \mathcal{EH}$ is a covering map.

**Theorem 2.4. (Covering space)** *$d : \mathcal{EH} \to \mathcal{EH}$ is a covering space.* □

$\mathcal{EH}$ is a Hausdorff space and the map $d : \mathcal{EH} \to \mathcal{EH}$ is a covering map, so theorem 1.4 about uniqueness of lists is applicable here. Consider a curve $\eta : I \to U$, for every holomorphic function $f \in \mathcal{H}U$ the curve induces a map, $\Gamma : I \to \mathcal{EH}$ given by,

$$\begin{aligned} \Gamma :\ & [0,1] \to \mathcal{EH} \\ & t \mapsto \operatorname{germ}_{\eta(t)} f \in \mathcal{H}_{\eta(t)} \end{aligned}$$

So a primitive of $f$ along $\eta$ is the lifting of the function $\Gamma$ with respect to $d : \mathcal{EH} \to \mathcal{EH}$,

$$\begin{tikzcd} & \mathcal{EH} \arrow[d, "d"] \\ I \arrow[ur, "\widehat{\Gamma}"] \arrow[r, "\Gamma"'] & \mathcal{EH} \end{tikzcd}$$

For fixed initial value or any common value, they uniqueness of 1.4 applies. The primitive of $f$ along a curve $\eta$ as defined above, $F = \widehat{\Gamma} : I \to \mathcal{EH}$. To each germ $\operatorname{germ}_{\eta(t)} f$ we have an association $\widehat{\Gamma}(t) \in \mathcal{EH}$ such that $d(\widehat{\Gamma}(t)) = \operatorname{germ}_{\eta(t)} f$.

Let $t \in I$ and let $U_t$ be a neighborhood around $\eta(t)$ and $h$ be such that,

$$h(\eta(t)) = \int_{[0,t]} f(\eta(s))\eta'(s)ds.$$

Let $F(t) = \operatorname{germ}_{\eta(t)} h$, by definition of $d$ we have, $dF(t) = \operatorname{germ}_{\eta(t)} h' = \operatorname{germ}_{\eta(t)} f$. It can then be verified that $F$ is a continuous, and hence a lift of $f$ along $\eta$. This will work for any piecewise smooth continuous curve.

Since $F$ is a primitive, any other primitive would be of the form $F + c$ and hence,

$$F(1)(\eta(1)) - F(0)(\eta(0)) = \int_{[0,1]} f(\eta(s))\eta'(s)ds. \tag{I}$$

If $f \in \mathcal{H}U$, and $\eta : I \to U$ is a continuous curve, define,

$$\int_\eta f dz = F(1)(\eta(1)) - F(0)(\eta(0))$$

where $F : I \to \mathcal{EH}$ is a primitive of $f$ along $\eta$.

## 3 | Monodromy Theorem

The covering space and Hausdorff properties of the étale space of holomorphic functions lets us apply theorems of covering spaces and homotopy theory. In particular, we can lift curves, homotopies.

Let $f \in \mathcal{H}U$ and let $\eta_1$ and $\eta_2$ be two curves in $U$. Suppose they are homotopic with homotopy $H$, then for each $s \in I$ we have a curve $H_s$. Let $\Gamma_s$ be the map,

$$\begin{aligned} \Gamma_s : \ [0,1] &\to \mathcal{E}\mathcal{H} \\ t &\mapsto \operatorname{germ}_{H_s(t)} f, \end{aligned}$$

then $\Gamma_s$ is a homotopy between $\Gamma_0$ and $\Gamma_1$ with fixed end points. Let $\widetilde{\Gamma_s}$ be the homotopy lift. We have, $\widetilde{\Gamma_0}(0) = \widetilde{\Gamma_1}(0)$ and similarly $\widetilde{\Gamma_0}(1) = \widetilde{\Gamma_1}(1)$.

**Theorem 3.1. (Homotopy form of Cauchy's theorem)** *Let $f \in \mathcal{H}U$, suppose $\eta_1 \simeq \eta_2$ be two curves in $U$, then,*

$$\int_{\eta_1} f(z)dz = \int_{\eta_2} f(z)dz.$$

*If $U$ is simply connected and $\eta$ is a loop then,*

$$\int_{\eta} f(z)dz = 0.$$

□

The lifting properties can be applied to the local homeomorphism, $\pi : \mathcal{E}\mathcal{H} \to X$, if $\operatorname{germ}_a f \in \mathcal{H}_a$ and $\operatorname{germ}_a f$ can be continued analytically along $\eta_s = H_s$ then analytic continuation along $\eta_1$ and $\eta_2$ yield same germ at $b$.

## References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer Science+Business Media, LLC, 2000