Part IV

Runge's Theorem, Homology

Runge's theorem says that if K doesn't have hole inside it, then every function analytic in a neighborhood of K can be approxmated on K by holomorphic functions on Ω. In case there is a hole, there could be some meromorphic function with the pole inside this hole, and it'll not be possible to approximate such functions with holomorphic functions on Ω.

The standard theory for detecting holes in spaces is cohomology and homology in algebraic topology, and hence we should expect Runge's theorem to have some deep relation to homology and cohomology.

1 | Runge's Theorem

Using the Taylor expansion, every analytic function on a disc can be approximated uniformly by polynomials in z on any smaller disc. Every entire function can be approximated by polynomials uniformly on every compact set. Runge's theorem generalizes this.

Let $\Omega \subseteq \mathbb{C}$ and let $K \subset \Omega$ be a compact subset. For any continuous function $f \in C(K)$, we define the norm of the function f on K by,

$$|f|_K = \sup_{z \in K} |f(z)|.$$

With this norm $C(K)$ is a Banach space. Using this we can define a topology on $\mathcal{H}(\Omega)$ by taking as neighborhoods the sets,

$$B_{\epsilon,K}(f) = \{g \in \mathcal{H}(\Omega) \mid |f - g|_K < \epsilon\}$$

With this topology, a sequence $\{f_n\}$ converges in $\mathcal{H}(\Omega)$ if and only if $\{f_n\}$ converges uniformly on any compact set in Ω. The topology is called compact open topology. Let $\mathcal{O}(K)$ denote the space of all continuous functions on K, that are restrictions of holomorphic functions on some bigger domain containing K, i.e., for all $f \in \mathcal{O}(K)$ there exists $U \supset K$, and $g \in \mathcal{H}(U)$ such that $g|_K = f$. Hence $\mathcal{O}(K) \subset C(K)$.

We now have the restriction map,

$$\rho : \mathcal{H}(\Omega) \to \mathcal{O}(K), \qquad \rho(f) = f|_K.$$

Runge's theorem says that the image of $\mathcal{H}(\Omega)$ under ρ is dense in $\mathcal{O}(K)$ iff K has no holes. Before going further, we note a few topological definitions, an open set $U \subset \Omega$ is said to be relatively compact if the closure $\overline{U}$ is compact. The boundary of an open set U with respect to Ω, $\partial_\Omega U$, is defined to be the set of all points z such that every neighborhood of z intersects with U and $\Omega \backslash U$.

Theorem 1.1. (Runge Approximation theorem) *Let $K \subset \Omega$ be a compact subset. Then the following conditions on Ω and K are equivalent.*

1. *$\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$.*
2. *No connected component of $\Omega \backslash K$ is relatively compact in Ω.*
3. *For every $z \in \Omega \backslash K$, there is a function $f \in \mathcal{H}(\Omega)$ such that,*

$$|f(z)| > |f|_K.$$

Sketch of Proof

The proof is a bit difficult, even though the ideas involved are simple and intuitive. We will only sketch the proof, and tell what is happening intuitively. This should make it less scary. For a detailed proof along these lines follow [1].

Note first that *2* is saying that K has no holes. Observe that *2* and *3* are the same statements. The maximum modulus principle allows us to say that K has no holes in terms of holomorphic functions. So, we just have to show *1* and *2* are equivalent.

1 $\Rightarrow$ *2.*, Suppose *1* holds, i.e., every function in $\mathcal{O}(K)$ can be approximated by elements of $\rho(\mathcal{H}(\Omega))$. Now suppose *2* doesn't hold, i.e., the compact subset has a connected hole O, then we can choose any point inside O, w, and consider the function,

$$f(z) = 1/(z - w).$$

This belongs to $\mathcal{O}(K)$. But this cannot be approximated by functions in $\rho(\mathcal{H}(\Omega))$.

Suppose it can be approximated by holomorphic functions on Ω, then there exists, $\{f_n\} \subset \mathcal{H}(\Omega)$ such that $f_n|_K \to f|_K$ uniformly. Since each f_n are holomorphic, so is $f_n - f_m$, and by maximum modulus principle,

$$|f_n(z) - f_m(z)| \leq \sup_{z \in \overline{O}} |f_n(z) - f_m(z)| = \sup_{z \in \partial O} |f_n(z) - f_m(z)| \leq |f_n - f_m|_K$$

Since $|f_n - f_m|_K \to 0$, we have $f_n(z) - f_m(z) \to 0$ for all $z \in O$. Hence, $\{f_n|_O\} \to g \in \mathcal{H}(O)$. But we know that $f(z)(z - w) = 1$ and hence we must have $f_n(z)(z - w) \to 1$ uniformly on the boundary of the hole O, and hence on O. This means $g(z)(z - w) = 1$, which cannot be a holomorphic function. Hence there must not exist any connected holes O of K.

2 $\Rightarrow$ *1.*, Firstly we note that,

$$\rho(\mathcal{H}(\Omega)) \subset \mathcal{O}(K) \subset C(K).$$

If a subspace W is dense in V then the action of a continuous functional is determined by it's action on the subspace W. Hahn-Banach theorem allows us to extend bounded linear functionals defined on subspaces to the whole space, the extension is unique if the subspace is dense. So, $\rho(\mathcal{H}(K))$ is dense in $\mathcal{O}(K)$ if and only if for every continuous linear form λ on $C(K)$, whenever $\lambda|_{\rho(\mathcal{H}(\Omega))} = 0$ implies $\lambda|_{\mathcal{O}(K)} = 0$. Continuous linear functional λ on functions on locally compact spaces correspond to measures μ. So, to show $\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$, we have to show that if for a measure $\mu(f) = 0$ for all $f \in \rho(\mathcal{H}(\Omega))$ then $\mu(g) = 0$ for all $g \in \mathcal{O}(K)$.

What we are trying to do is, if a function f has poles outside K, then we can push it to infinity, so that the function can be approximated by elements in $\rho(\mathcal{H}(\Omega))$. It's enough to show that $f(z,w) = (z-w)^{-1} \in \mathcal{O}(K)$, for $w \in \mathbb{C}\backslash K$, can be approximated by elements in $\rho(\mathcal{H}(\Omega))$, i.e., $f(z,w) \in \overline{\rho(\mathcal{H}(\Omega))}$. Because then, by taking products, we can approximate functions with higher order poles outside K.

Let λ be a continuous linear functional on $C(K)$ that vanishes on $\rho(\mathcal{H}(\Omega))$. Consider the function $f(z) = (z-w)^{-1}$ for $w \in \mathbb{C}\backslash K$. If $|w| > \sup_{z\in K}|z|$, then we have,

$$f(z,w) = -\sum_{i=0}^{\infty} \frac{z^n}{w^{n+1}}, \quad \forall\, z \in K.$$

Since λ vanishes on $\rho(\mathcal{H}(\Omega))$, λ vanishes for each term of the series, by continuity of λ we can take the summation inside. By continuity of λ,

$$\lambda\big(\lim_{h\to 0} \tfrac{f(z,w+h)-f(z,w)}{h}\big) = \lim_{h\to 0} \tfrac{\lambda(f(z,w+h))-\lambda(f(z,w))}{h}.$$

So, $\lambda(f(-,w))$ is holomorphic on K with, $\frac{\partial(\lambda(f(-,w)))}{\partial w} = \lambda\big(\frac{\partial(f(-,w))}{\partial w}\big)$.

Suppose $f(z,w) \notin \overline{\rho(\mathcal{H}(\Omega))}$, consider the bounded linear functional defined on the span of $f(z,w)$ and $\overline{\rho(\mathcal{H}(\Omega))}$,

$$\varphi\big(g(-) + tf(-,w)\big) \coloneqq t.$$

this bounded linear functional is such that $\varphi|_{\rho(\mathcal{H}(\Omega))} \equiv 0$, and not zero for $f(z,w)$. By Hahn-Banach theorem this can be extended to $C(K)$. But this is a contradiction because from our assumption any continuous functional λ with $\lambda|_{\rho(\mathcal{H}(\Omega))} \equiv 0$ must also vanish for $\mathcal{O}(K)$. But this functional isn't vanishing for $f(z,w) \in \mathcal{O}(K)$. Hence the assumption

$$f(z,w) \notin \overline{\rho(\mathcal{H}(\Omega))},$$

must be false. □

Let $K \subset \Omega$ be a compact set. We define,

$$K_{\mathcal{H}(\Omega)} \coloneqq \{z \in \Omega \mid |f(z)| \le |f|_K \text{ for all } f \in \mathcal{H}(\Omega)\}.$$

This set is called the holomorphic convex hull of K. This includes all points in K.

Intuitively this would be K if we fill up all the interior holes, because on this filled up compact space, by maximum modulus principle, the maxima can only be attained on the boundary. Intuitively, we will have for each point on the outer boundary a holomorphic functions that attain maxima at the point. The Runge's approximation is applicable to this filled up compact set.

1.1 | Homology form of Cauchy's Theorem

As we stated before, Runge's theorem relates to holes inside a compact set K, and naturally we should expect this to be related to homology and cohomology. We give here a version of Cauchy's theorem using Runge's theorem.

Let Ω be a connected open set in $\mathbb{C}$, two loops η_1, η_2 are homologous if they are boundary of the same surface. A loop $\eta : I \to \Omega$ in Ω, is said to be homologous to zero, denoted $\eta \sim_\Omega 0$, if there exists a surface whose boundary is η. We can now use complex analysis to understand when this is possible. Firstly we cannot have a surface whose boundary is the loop if there is an obstruction in Ω, i.e., a hole 'inside' the loop η. We can make this precise in terms of winding number.

For a closed curve η and $x \in \mathbb{C}$, the index $n(\eta, x)$ is zero if the point x is outside the curve, i.e., the curve doesn't wind around any point outside. So, the set of all points for which the winding number is nonzero must be inside Ω. So, we define that $\eta \sim_\Omega 0$ if the set,

$$S = \{x \in \mathbb{C} \backslash \mathrm{Im}(\eta) \mid n(\eta, x) \neq 0\} \subset \Omega$$

Lemma 1.2. *If η is homotopic to a point in Ω then $\eta \sim_\Omega 0$.*

Proof

We have to show if η is homotopic to a point in Ω then the set $S = \{x \mid n(\eta, x) \neq 0\}$ is contained in Ω. Suppose not, i.e., suppose there exists some $x \in \mathbb{C}$ not in Ω such that $n(\eta, x) \neq 0$. We have to arrive at a contradiction if η is homotopic to a point.

Let $x \in \mathbb{C}$ that's not in Ω. Since η is homotopic to a point in $\Omega \backslash \{x\}$. For homotopic loops η_1, η_2, we must have,

$$n(\eta_1, x) = n(\eta_2, x)$$

By applying Cauchy's theorem for the function $z \mapsto (z - x)^{-1}$ and monodromy theorem. Hence we have that $n(\eta, x) = 0$, or that it cannot be nonzero. $\square$

The converse is not true, there can be loops homologous to zero but not homotopic to a point. For example, look up Pochhammer contour. This makes the homology form of Cauchy's theorem below more general than the homotopy form.

Theorem 1.3. (Cauchy's theorem; Homology Form) *Let Ω be connected open set. $\eta : I \to \Omega$ be a loop. If $\eta \sim_\Omega 0$ then $\forall f \in \mathcal{H}(\Omega)$,*

$$\int_\eta f(z)dz = 0.$$

Proof

Firstly we will sketch why any loop is homotopic to a piecewise differentiable curve. Since the loop η (compact set) is inside Ω, we can choose an $\epsilon > 0$ such that $B_\epsilon(\eta(t)) \subset \Omega$ for all $t \in I$, i.e., there is an ϵ width patch around η entirely contained in Ω. Now we can partition the interval $I = [0, t_1] \cup \cdots \cup [t_n, 1]$ such that each $[t_i, t_{i+1}]$ is inside one of the ϵ balls, and then connect the end points by a straight line which is a differentiable curve. So, we obtained a piecewise differentiable curve. Let this curve be Γ. Now we have a homotopy between the curves,

$$F(t, s) = (1 - s)\eta(t) + s\Gamma(t).$$

By the homotopy form of Cauchy's theorem we have reduced calculating the integral for η to calculating the integral for a piecewise differentiable curve. So, the index,

$$n(\eta, x) = n(\Gamma, x) = \int_\Gamma (z - x)^{-1}dz.$$

So, we have $\Gamma \sim_\Omega 0$. We now have to show that $\int_\Gamma f(z)dz = 0$ for $f \in \mathcal{H}(\Omega)$.

For any $f \in \mathcal{H}(\Omega)$, we can find, by Runge's theorem, a sequence of rational functions with poles outside Ω that converge uniformly to f on compact sets (in our case $\text{Im}(\eta)$). Then,

$$\lim_{n\to\infty} \int_\Gamma R_n dz = \lim_{n\to\infty} \int_{[0,1]} R_n(\eta(t))\eta'(t)dt = \int_\Gamma f(z)dz.$$

By Residue theorem,

$$\tfrac{1}{2\pi i} \int_\Gamma R_n dz = \sum_{x \in E_n} n(\Gamma, x) \operatorname*{res}_{R_n}(x)$$

where E_n is the set of poles of R_n. Since $x \notin \Omega$, $n(\Gamma, x) = 0$, and hence the integral is zero. Thus $\int_\Gamma f dz = \lim_{n\to\infty} \int_\Gamma R_n dz$. $\square$

This type of uniform convergence is very useful trick. Suppose $\Omega \subset \mathbb{C}$ such that $\mathbb{C}\backslash\Omega$ has no compact connected components, i.e., has no holes, then Runge's theorem is applicable. For any function $f \in \mathcal{H}(\Omega)$, there is a sequence $\{f_n\} \subset \mathcal{H}(\mathbb{C})$ such that $f_n \to f$ uniformly on any compact subset of Ω. For any loop η in Ω,

$$\int_\eta f dz = \lim_{n\to\infty} \int_\eta f_n dz = 0$$

By Morera's theorem, f must have a primitive.

References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000