Sheaves in Differential Geometry

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Introduction

These notes provide an abstract introduction to sheaf theory and sheaf cohomology. The goal is to understand Grothendieck's six operations, abstractly.[1] The abstract machinery divides the difficulty into multiple smaller steps, so the difficulty of the theorems is not felt. I have tried to provide intuitive motivation for the categorical definitions. The purpose of these notes is to prepare the reader for Kashiwara-Schapira's books.

The notion of a pre-sheaf axiomatizes 'local nature' of certain mathematical objects. Given a topological space X, a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X. Denote by CU the set of real-valued continuous functions on an open set U. If $V \subset U$ then f restricted to V is a continuous map, $f|_{UV} : V \to \mathbb{R}$. The map, $f \mapsto f|_{UV}$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_{UV})|_{VW} = f|_{UW}.$$

So, this is functorial in nature, i.e., $|_{VW} \circ |_{UV} = |_{UW}$. This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\mathrm{op}}$ is the opposite category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\mathrm{op}}$ the functor assigns a morphism, the restriction morphism in **Sets**, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_{UV}$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}, \tag{pre-sheaf}$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps $U \subset V$ and the corresponding maps in **Sets** are called restriction maps, $|_{UV} : \mathcal{F}U \to \mathcal{F}V$.

Denote by $\mathrm{PSh}(X)$ the collection of all pre-sheaves over a topological space X. Each pre-sheaf is a functor from $\mathcal{O}(X)^{\mathrm{op}}$ to **Sets** can be considered an object and the natural transformation between the two pre-sheaves as morphism between these objects. So, the category of pre-sheaves is the functor category,

$$\mathrm{PSh}(X) = \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}}$$

So, to understand pre-sheaves, we could use tools for studying functor categories. The goal of the first half of this document will be to develop the tools needed to study functor categories, and hence pre-sheaves.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions. Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

[1] This is in the spirit of Fosco's comment on mathoverflow that 'If it's not possible to derive these statements from purely Kan-extensional arguments, then sheaves do belong to Algebraic Geometry'.

So, what we have is an I-indexed family of functions $(f_i)_{i\in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i\cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i\cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i\cap U_j} = f|_{U_i}|_{U_i\cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$.

$$\begin{tikzcd} CU \arrow[r, dashed, "e"] & \prod\limits_i CU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j} C(U_i\cap U_j). \end{tikzcd}$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

A sheaf of sets $\mathcal{F}$ on a topological space X is a functor,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i\in I} U_i$ of an open set U of X yields an equalizer diagram.

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \prod\limits_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j} \mathcal{F}(U_i\cap U_j). \end{tikzcd} \tag{collation property}$$

The category of sheaves, $\mathrm{Sh}(X)$, is a subcategory of the functor category,

$$\mathrm{Sh}(X) \rightarrowtail \mathrm{PSh}(X) = \widehat{\mathcal{O}(X)} = \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}},$$

which is the category consisting of all functors from $\mathcal{O}(X)^{\mathrm{op}}$ to $\mathbf{Sets}$, that satisfy the collation. The obstruction of going from local to global is carefully studied via cohomology. The goal of this document is to develop the tools needed to study cohomology of sheaves.

The starting point is the develop the necessary tools from category theory to study these objects. The main concepts of category theory is the representables-limits-adjoints triad. While Kashiwara-Schapira use a lot of abstract categorical tools, their approach still contains lots of difficult proofs. Our goal is to introduce a bit more abstract tools than Kashiwara-Schapira, but significantly reduce the difficulty of proofs. I feel the 'derived' concepts can be introduced much more neatly and intuitively with slightly more abstraction than the approach in Kashiwara-Schapira.

Category theory allows us to formalize patterns, when a collection of theorems indicate the existence of such a pattern. Category theory itself is useless to find such patterns. However, when the patterns are noted, and formalized already, abstract categorical way of stating and proving the theorems can be very helpful. The goal of these notes to understand and formalize already noticed patterns.

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Part I

The Six Functors

1 | Category of Pre-Sheaves

This chapter is an abstract categorical study of pre-sheaves. Category theory develops very general tools, which can be applied to many mathematical phenomena. The use of category theory in sheaf theory was started by Grothendieck based on the following philosophical position,

Yoneda-Grothendieck. An object is determined by its relation to other objects.

The Yoneda lemma makes this precise and exploitable. Yoneda lemma embeds a given category inside the category of functors from the given category to the category of sets, via the functor, 'maps to the given object' or 'maps from the given object'. This allows us to utilize the nice properties of the target category in our case the category of sets.

1.1 | Category of Functors

A set is a collection of 'elements'. A category $\mathcal{A}$ is more sophisticated, it possesses 'objects' similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{A}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects. Studying objects by their relations to other objects could be called the Yoneda-Grothendieck philosophy.

A functor $\mathcal{F}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of a mapping of objects of $\mathcal{A}$ to objects of $\mathcal{B}$, $X \mapsto \mathcal{F}X$ together with a map of the set of homomorphisms,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y).$$

the image of $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ denoted by $\mathcal{F}(f)$. That takes identity to identity and respects composition[1] i.e.,

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ can be thought of as a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. A functor $\mathcal{F}$ is faithful if the map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$ is injective for all X, Y. It's full if the map is surjective. If it's a bijection the functor is called fully faithful.

A natural transformation κ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, denoted by,

$$\kappa : \mathcal{F} \Rightarrow \mathcal{G},$$

[1]The composition $f \circ g$ assumes they are composable.

is a collection of mappings κ_X for every $X \in \mathcal{A}$, such that for all $f : X \to Y$, the diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & \mathcal{F}X \arrow[r, "\kappa_X"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}X \arrow[d, "\mathcal{G}(f)"] \\ Y & \mathcal{F}Y \arrow[r, "\kappa_Y"'] & \mathcal{G}Y \end{tikzcd} \qquad \text{(natural transformation)}$$

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors $\mathcal{F}$ and $\mathcal{G}$ is denoted by,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

We say two functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $\mathcal{F} \cong \mathcal{G}$. The collections of all functors from $\mathcal{A}$ to $\mathcal{B}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{B}^{\mathcal{A}}$. The nice thing about functor category $\mathcal{B}^{\mathcal{A}}$ is that it inherits many of the useful properties of the category $\mathcal{B}$.

Equivalence of two categories can be thought of as giving two complementary description of same mathematical object. We can compare two categories $\mathcal{A}$ and $\mathcal{B}$ via the functors between them. The starting point is the functor category $\mathcal{B}^{\mathcal{A}}$.

A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if there is a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that

$$\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}, \quad \text{and} \quad \mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}},$$

where the identity functor $\mathbb{1}_{\mathcal{A}}$ sends objects of $\mathcal{A}$ to the same objects, and morphisms to the same morphisms. $\mathcal{G}$ is called quasi-inverse functor. In such a case, $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent. Quotient categories can be defined when we have an equivalence relation on the collection of morphisms. The objects remain the same, and the hom-sets get quotiented.

1.2 | Representable Functors

Each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ tells us about all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{A}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{A}}(-, X)$. These are called hom functors.

$$\begin{aligned} h^X : \ & \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X). \end{aligned}$$

which maps each morphism $f : Y \to Z$ to a morphism of hom sets given by the composition,

$$Y \xrightarrow{f} Z \xrightarrow{g} X$$

We will denote this by,

$$\begin{aligned} h^X(f) : \ & h^X(Y) \to h^X(Z) \\ & g \mapsto g \circ f. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{A}}(Y, X)$s are all sets. Such categories are called locally small categories.

A contravariant functor $\mathcal{F} : \mathcal{A}^{\text{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be $\mathcal{G} \cong h_X$. Where $\cong$ stands for natural isomorphism.

1.2.1 | Yoneda Embedding

Yoneda embedding and representable functors allow us to use the nice properties (ability to take limits) of the category of sets to study more complex categories that are not so nice. We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\mathrm{Hom}_{\mathcal{A}}(-, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, -)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{A}$ and the functor $h^X \in \mathbf{Sets}^{\mathcal{A}^{\text{op}}}$.

Theorem 1.2.1. (Yoneda Lemma) *For a functor $\mathcal{F} : \mathcal{A}^{\text{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{A}$, there is a natural bijection,*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) \cong \mathcal{F}X \tag{Yoneda}$$

such that $\kappa \in \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) \leftrightarrow \kappa_X(\mathbb{1}_X) \in \mathcal{F}X$.

Proof

In the natural transformation diagram, replace $\mathcal{F}$ by h^X, and $\mathcal{G}$ by $\mathcal{F}$. $\kappa_X : h^X X \to \mathcal{F}X$. Now, $h^X X = \mathrm{Hom}_{\mathcal{A}}(X, X)$, which contains $\mathbb{1}_X$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) &\to \mathcal{F}X \\ \kappa &\mapsto \kappa_X(\mathbb{1}_X). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $Y \in \mathcal{A}$. For any $f : Y \to X$, we have,

$$\begin{tikzcd} X & h^X X \arrow[r, "\kappa_X"] \arrow[d, "h^X(f)"'] & \mathcal{F}X \arrow[d, "\mathcal{F}(f)"] & \mathbb{1}_X \arrow[r, mapsto, "\kappa_X"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ Y \arrow[u, "f"] & h^X Y \arrow[r, "\kappa_Y"'] & \mathcal{F}Y & f \arrow[r, mapsto, "\kappa_Y"'] & \kappa_Y(f) \end{tikzcd}$$

Hence $\kappa_Y(f) = \mathcal{F}(f)(\mu(\kappa))$, or the action of κ_Y is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_Y(f) = \varphi_Y(f)$ for all $Y \in \mathcal{A}$, so it's injective.

For surjectivity we have to show that for all sets $x \in \mathcal{F}X$, there exists a natural transformation φ such that $\varphi_X(\mathbb{1}_X) = x$. For $x \in \mathcal{F}X$, and $f : Y \to X$, construct the map,

$$\begin{aligned} \varphi : h^X &\to \mathcal{F} \\ f &\mapsto \mathcal{F}(f)(x). \end{aligned}$$

this satisfies the requirement that $\varphi_X(\mathbb{1}_X) = x$, because clearly, $\mathbb{1}_X \mapsto \mathcal{F}(\mathbb{1}_X)(x) = \mathbb{1}_x(x) = x$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} Y & h^X Y \arrow[r, "\varphi_Y"] \arrow[d, "h^X(g)"'] & \mathcal{F}Y \arrow[d, "\mathcal{F}(g)"] \\ Z \arrow[u, "g"] & h^X Z \arrow[r, "\varphi_Z"'] & \mathcal{F}Z \end{tikzcd}$$

commutes for all $Y, Z \in \mathcal{A}$, $g \in \mathrm{Hom}_{\mathcal{A}}(Z, Y)$. For $f : Y \to X$, by definition of φ,

$$\mathcal{F}(g) \circ (\varphi_Y(f)) = \mathcal{F}(g) \circ \mathcal{F}(f)(x)$$

which by functoriality of $\mathcal{F}$ is $= \mathcal{F}(f \circ g)(x)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_Z \circ (h_X(g)(f)) = \varphi_Z(h_X(f \circ g))$$

which again by definition of φ is $= \mathcal{F}(f \circ g)(x)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \to \mathcal{F}X$ is a bijection. □

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof covariant version is exactly the same, just have to reverse the arrows on the category $\mathcal{A}$. The Yoneda lemma gives us an embedding of the category $\mathcal{A}$ inside the functor category $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, given by,

$$X \mapsto h^X.$$

This embedding is called the Yoneda embedding $h^{(-)} : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, which sends an object $X \in \mathcal{A}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{A}}(-, X)$. These functors are fully faithful by Yoneda lemma, because by replacing the functor $\mathcal{F}$ by h^Y we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, h^Y) \cong h^Y(X) = \mathrm{Hom}_{\mathcal{A}}(X, Y). \tag{weak Yoneda}$$

Similarly for the covariant embedding, in which case this will be $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{A}}(Y, X)$.

Given a contravariant functor, $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$, the Yoneda tells us that we can think of the action of $\mathcal{F}$ on the element X as natural transformations to the hom functor h^X in the functor category. So, every functor $\mathcal{F}$ can extended and be thought of as a representable functor,

$$\begin{aligned} h^{\mathcal{F}} : (\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{G} &\mapsto \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(\mathcal{G}, \mathcal{F}) \end{aligned}$$

where elements $X \in \mathcal{A}$ are to be interpreted as the elements $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$. The following frequently used corollary of Yoneda lemma allows us to compare objects of locally small category by their hom-functors, and hence using the properties of the category of sets.

Lemma 1.2.2. (Yoneda Principle)

$$h^X \cong h^Y \Rightarrow X \cong Y. \tag{Yoneda principle}$$

□

Note that, Yoneda associates to each set in $\mathcal{F}X$ a natural transformation between h^X and $\mathcal{F}$. If the functor $\mathcal{F}$ is representable, i.e., there exists $Y \in \mathcal{A}$ such that there exists a natural isomorphism,

$$\mathcal{F} \xrightarrow[\alpha]{\cong} h^Y$$

Let $\mu(\alpha)$ be the corresponding element in $\mathcal{F}Y = \mathrm{Hom}_{\mathcal{A}}(Y, Y)$. The pair $(Y, \mu(\alpha))$ is called a universal object for $\mathcal{F}$. It's such that for any other object $Z \in \mathcal{A}$, and each $g \in \mathcal{F}X = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ there exists a unique morphism $f : X \to Y$ such that,

$$\mathcal{F}(f)(\mu(\alpha)) = g.$$

Digression: Enriched Categories

The hom-sets in practice usually are richer than merely being sets, they come equipped with additional structure. Enriched categories are categories, where the hom-sets have additional structure. Many of the constructions we can make in **Sets** can also be done in many enriched categories. Here will informally discuss the minimal necessary stuff from the theory of enriched categories so that the reader doesn't feel out of place.

An enriched category is a category in which the hom-sets come equipped with additional structure, that is, the hom-sets are objects in some *enriching* category, usually denoted by $\mathcal{V}$. $\mathcal{V}$ is called the base for enrichment. This already requires the category $\mathcal{V}$ to have some special properties.

Given any two composable morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$, and $g \in \mathrm{Hom}_{\mathcal{A}}(Y,Z)$ in the category $\mathcal{A}$, we can consider the composition of the morphisms,

$$g \circ f \in \mathrm{Hom}_{\mathcal{A}}(X,Z)$$

If $\mathcal{A}$ is a $\mathcal{V}$ enriched category, then $f, g, g \circ f \in \mathcal{V}$. So the notion of composition of morphisms in the category $\mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}}(Y,Z) \times \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{A}}(X,Z)$$

should correspond to a notion of 'composition of objects' or a product in the enriching category,

$$\odot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

There should be unit object corresponding to the identity morphisms, denoted by $\mathbb{1}_{\mathcal{V}}$ such that multiplication by the unit object leaves every object unchanged. This can be formalized by saying there exist natural transformations such that,

$$\mathbb{1}_{\mathcal{V}} \odot V \cong V \odot \mathbb{1}_{\mathcal{V}} \cong V \tag{identity}$$

for all $V \in \mathcal{V}$. Since composition of morphisms is associative, we want $\odot$ to also be associative. This can be formalized by saying there exist natural transformations such that,

$$U \odot (V \odot W) \cong (U \odot V) \odot W \tag{associativity}$$

These properties can also be formalized in terms of a commutative diagram but we will skip that. A category $\mathcal{V}$ with a 'product' $\odot$ with identity, and associativity is called a monoidal category. $\odot$ is called the monoidal product. If in addition the product is such that $U \odot V \cong V \odot U$ it's called a symmetric monoidal category.

A category $\mathcal{A}$ is said to be enriched by a monoidal category $\mathcal{V}$ if the hom-sets belong to $\mathcal{V}$ and the composition corresponds to the monoidal product $\odot$. We will denote the hom-sets of a $\mathcal{V}$-enriched category $\mathcal{A}$ by,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X,Y)$$

for all $X, Y \in \mathcal{A}$. When there is no confusion we will drop the superscript $\mathcal{V}$, or sometimes $\mathcal{A}^{\mathcal{V}}(X,Y)$

A closed monoidal category is monoidal category $\mathcal{V}$ where the functors $- \odot V : \mathcal{V} \to \mathcal{V}$ admits a right adjoint denoted by $\mathcal{H}om^{\mathcal{V}}(V,-)$. The family of right adjoints assemble in a unique way to give a bifunctor,

$$\mathcal{H}om^{\mathcal{V}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}.$$

such that,

$$\mathrm{Hom}_{\mathcal{V}}^{\mathcal{V}}(U \times V, W) \cong \mathrm{Hom}_{\mathcal{V}}^{\mathcal{V}}(U, \mathcal{H}om^{\mathcal{V}}(V, W))$$

for all $U, V, W \in \mathcal{V}$. $\mathcal{H}om^{\mathcal{V}}$ is called the internal hom. The internal homs act as the product $\odot$, and hence, $\mathcal{V}$ together with $\mathcal{H}om^{\mathcal{V}}$ is an enriched category over itself. A category is called cartesian closed if it's locally small, that is, it's enriched by the category of sets, and if $\odot$ is the cartesian product in **Sets**.

The small $\mathcal{V}$-categories themselves form a category. A $\mathcal{V}$-functor is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between $\mathcal{V}$-enriched categories such that the morphisms,

$$\mathrm{Hom}_{\mathcal{A}}^{\mathcal{V}}(X, Y) \xrightarrow{\mathcal{F}_{X,Y}} \mathrm{Hom}_{\mathcal{B}}^{\mathcal{V}}(\mathcal{F}X, \mathcal{F}Y)$$

commute with the $\odot$ operation and the identity. Similarly a $\mathcal{V}$-natural transformation between a pair of $\mathcal{V}$-functors $\mathcal{F}$ and $\mathcal{G}$ consist of natural transformation between $\mathcal{F}$ and $\mathcal{G}$ such that it commutes with composition by the natural transformations in identity and associativity. The collection of all $\mathcal{V}$-natural transformations between $\mathcal{F}$ and $\mathcal{G}$ will be denoted by,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{B}}}^{\mathcal{V}}(\mathcal{F}, \mathcal{G})$$

We will now state the enriched Yoneda lemma without proof.

Theorem 1.2.3. (Enriched Yoneda Lemma) *$\mathcal{A}$ be a small $\mathcal{V}$-category, then, for all $\mathcal{V}$-functors $\mathcal{F} : \mathcal{A} \to \mathcal{V}$,*

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{V}}}^{\mathcal{V}}(\mathrm{Hom}_{\mathcal{A}}^{\mathcal{V}}(X, -), \mathcal{F}) \cong \mathcal{F}X.$$

□

The ideas of the proofs are the same as those discussed the generic ones but with more verification (verifying the natural transformations preserve the enriching structure and so on.). Similarly we will have a notion of limits and colimits for the enriched categories.

1.2.2 | Representable Constructions

The Yoneda-Grothendieck philosophy now has a precise formulation; the properties of a category can be thought of as representability properties in functor category. A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be

$$\mathcal{G} \cong h_X$$

where $\cong$ stands for natural isomorphism. We will now see this in the following examples.

1.2.2.1 | (Co)Products

Let $\mathcal{A}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{A}$ indexed by a set I, then we can consider the contravariant functor,

$$\mathcal{G} : Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $\mathcal{G}(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, P)$. This is called the product, denoted by, $\prod_{i\in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
 & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"'] \arrow[dr, "f_j"] & \\
X_i & \prod_{i\in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I. So, the property that the category $\mathcal{A}$ has products, is translated into a statement that certain functor is representable.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $\mathcal{F}(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i\in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\
 & Y &
\end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

The categorical notions of product and coproduct correspond to the arithematic operations such as multiplication and addition. We can similarly talk about exponentiation. In the category of sets, **Sets**, for $X, Z \in \mathcal{A}$, Z^X is the function set consisting of all functions $h : X \to Z$. Here we have the bijection,

$$\mathrm{Hom}_{\mathbf{Sets}}(Y \times X, Z) \to \mathrm{Hom}_{\mathbf{Sets}}(Y, Z^X).$$

for a function, $f : Y \times X \to Z$, this map sends each $y \in Y$ to the function $f(y, -) \in Z^X$. Conversely given a function $f' : Y \to Z^X$, we can define a map $f(y, x) = f'(y)(x)$. So,

$$\mathrm{Hom}_{\mathbf{Sets}}(Y \times X, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(Y, Z^X)$$

or equivalently, $(-)^X$ is the right adjoint of $(-) \times X$. By setting $Y = 1$, we obtain,

$$Z^X \cong \mathrm{Hom}_{\mathbf{Sets}}(1, Z^X) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, Z).$$

Exponentiation can be representably defined in terms of products. Suppose $\mathcal{A}$ has products, it has exponentiation if for every Y, Z, the functor,

$$X \mapsto \mathrm{Hom}_{\mathcal{A}}(Y \times X, Z)$$

is representable.

1.2.2.2 | (Co)Kernel

For sets, the kernel of two maps s, t is defined as the set $\ker(s, t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{A}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"'] \arrow[drr, bend left, "j"] & & \\
\ker(f,g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object Z is called a zero object if for any object A, there exists a unique morphism $Z \to A$ and a unique morphism $A \to Z$. It's unique upto isomorphism and denoted by 0. Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \to \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd} X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\ \ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{A}}(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(Z, X)\big).$$

This can be visualized by the diagram,

$$\begin{tikzcd} Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f,g) \arrow[d, dashed, "e"] \\ & & X \end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd} Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\ & & X \end{tikzcd}$$

1.2.2.3 | Pullback or Fibered Product

Let $\mathcal{I}$ be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \to Z$ then for functors $\mathcal{F} : \mathcal{I} \to \mathcal{A}$, pullback $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$\mathcal{F}Y \xrightarrow{f} \mathcal{F}X \xleftarrow{g} \mathcal{F}Z$$

in a category $\mathcal{A}$ is the commutative square with vertex $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ such that any other commutative square factors through it, i.e.,

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_P"] \arrow[ddr, bend right, "g_P"'] & & \\ & \mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}Z \arrow[d, "g"] \\ & \mathcal{F}Y \arrow[r, "f"'] & \mathcal{F}X \end{tikzcd}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ as above,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{F}Z \arrow[d, dashed] \arrow[ddr, bend left, "f_P"] & \\
\mathcal{F}Y \arrow[r, dashed] \arrow[drr, bend right, "g_P"'] & \mathcal{F}Y \coprod_{\mathcal{F}X} \mathcal{F}Z \arrow[dr, dashed, "\exists!"] & \\
 & & C
\end{tikzcd}$$

1.3 | Adjoint Situations

Categories are compared by means of functors, and functors themselves are compared via natural transformations. Equivalence of categories allows us to basically think of the two categories as the same thing. This is however too restrictive. The relaxation of the notion of equivalence gives us the notion of adjoint.

The philosophy of adjoint functors is the following; when we want to study an object in mathematics, belonging to some weird category, we can take it, via a functor to some well understood category. But now this new category will not have the same meaning to the objects as the original category. So we would like a functor to get back to the original category. This functor is the adjoint functor.

An adjuntion from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functors,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B},$$

such that there is a natural isomorphisms of bifunctors $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y)$ and $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \tag{adjoint}$$

for all $X \in \mathcal{A}, Y \in \mathcal{B}$. Denote by $\mathcal{F} \dashv \mathcal{G}$. Since composition of natural isomorphisms is also a natural isomorphism if $\mathcal{F}$ has two adjoints $\mathcal{G}$ and $\widehat{\mathcal{G}}$, then we have,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, \widehat{\mathcal{G}}Y).$$

So, by Yoneda principle, adjoints if they exist are unique upto isomorphism. Consider the following two adjoint situations,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B} \underset{\mathcal{K}}{\overset{\mathcal{H}}{\rightleftarrows}} \mathcal{C},$$

By definition, we have for all $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{H} \circ \mathcal{F}X, Y)$, hence,

$$\mathcal{F} \circ \mathcal{H} \dashv \mathcal{G} \circ \mathcal{K}$$

When we are working with locally small categories, we can exploit the properties of the category of sets. We can look at adjoints from a functor category perspective, and representable functors.

Given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, for each $X \in \mathcal{B}$, we have the composite functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) := h^X \circ \mathcal{F} : \mathcal{A} \to \mathcal{B} &\to \mathbf{Sets} \\ A &\mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

So, $\widehat{\mathcal{F}}$ is a functor to the functor category,

$$\widehat{\mathcal{F}} : \mathcal{B} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}},$$

which sends $X \mapsto \widehat{\mathcal{F}}(X)$. For each morphism $f : X \to Y$ in $\mathcal{B}$, the functor $\widehat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \to X$,

$$\widehat{\mathcal{F}}(f) : g \mapsto f \circ g$$

So, $\widehat{\mathcal{F}}(f \circ h) = \widehat{\mathcal{F}}(f) \circ \widehat{\mathcal{F}}(h)$, i.e., it's a covariant functor.

Lemma 1.3.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$ admits a left adjoint iff for all $X \in \mathcal{B}$,*

$$\widehat{\mathcal{F}}(X) : A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$$

is representable.

Proof

$\Leftarrow$ Suppose $\widehat{\mathcal{F}}(X)$ is representable for all $X \in \mathcal{B}$, then, $\exists\ \mathcal{G}X \in \mathcal{A}$ with, $\widehat{\mathcal{F}}(X) \cong h^{\mathcal{G}X}$, i.e.,

$$\widehat{\mathcal{F}}(X)(A) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X)$$

We have to make sure this is functorial, i.e., $X \mapsto \mathcal{G}X$ is a functor from $\mathcal{B}$ to $\mathcal{A}$. So, now we have to show that for each morphism $f : X \to Y$ there exists a morphism in the functor category, a natural transformation of functors, $\mathcal{G}f : \widehat{\mathcal{F}}(X) \to \widehat{\mathcal{F}}(Y)$, defined to be the maps that makes the following diagram commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X) \arrow[r] \arrow[d, "\mathcal{G}(f)\circ"'] & \widehat{F}(X)(A) \arrow[d, "\widehat{\mathcal{F}}(f)"] \\
\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}Y) \arrow[r] & \widehat{F}(Y)(A)
\end{tikzcd}$$

This also satisfies the composition needs by construction. By Yoneda lemma, this determines the functor $\mathcal{G}$ uniquely upto isomorphism.

$\Rightarrow$ The other direction is obvious and follows directly from the definition of adjoint, i.e., if there exists a left adjoint $\mathcal{G} \dashv \mathcal{F}$ each functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ is representable with representative $\mathcal{G}X$. $\square$

1.3.1 | Adjoints as Reflections

The notion of reflection of a functor provides a bit more intuitive meaning of what adjoints are doing. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor. We want to associate to each object $B \in \mathcal{B}$ and object $R_B \in \mathcal{A}$ such that $\mathcal{F}R_B$ is the best estimate of B in $\mathcal{A}$. In categorical terms, the estimation is done with morphisms. So, the 'best estimate' is a morphism,

$$\kappa_B : B \to \mathcal{F}R_B$$

such that for any other $A \in \mathcal{A}$, with an estimation $\varkappa : B \to \mathcal{F}A$ factors uniquely through R_B. R_B together with the morphism κ_B is called the reflection of B along $\mathcal{F}$. Visualised by the diagram,

$$\begin{tikzcd}
R_B \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_B \arrow[d, "\mathcal{F}(f)"'] & B \arrow[l, "\kappa_B"'] \arrow[dl, "\varkappa"] \\
A & \mathcal{F}A &
\end{tikzcd} \tag{reflection}$$

that's to say there exists a unique morphism $f : R_B \to A$ such that,

$$\mathcal{F}(f) \circ \kappa_B = \varkappa.$$

Intuitively κ_B is a better estimate than $\varkappa$. We can't have two best estimates κ_B and κ'_B because in that case we have two maps $f : R_B \to R'_B$ and $f' : R'_B \to R_B$, such that,

$$\mathcal{F}(f) \circ \kappa_B = \kappa'_B, \quad \mathcal{F}(f') \circ \kappa'_B = \kappa_B$$

So we get,

$$\mathcal{F}(f \circ f') \circ \kappa'_B = \kappa'_B,$$

By uniqueness this means $f \circ f' = \mathbb{1}_{R'_B}$, so any two reflections are isomorphic.

Lemma 1.3.2. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, suppose reflection exists for each $B \in \mathcal{B}$, then there exists a unique functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{G}B = R_B$ and a natural transformation κ such that,*

$$\kappa_B : B \to \mathcal{F} \circ \mathcal{G}B.$$

Proof

The existence of reflection to each object in $\mathcal{B}$ gives us an associated object for each object in $\mathcal{B}$, we want to understand what happens to the morphisms.

Let $f : X \to Y$ be a morphism in $\mathcal{B}$. Then we have,

$$\begin{tikzcd} R_X \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_X \arrow[d, "\mathcal{F}(f)"'] & X \arrow[l, "\kappa_X"'] \arrow[d, "g"] \\ R_Y & \mathcal{F}R_Y & Y \arrow[l, "\kappa_Y"] \end{tikzcd}$$

In this diagram, $\kappa_Y \circ g : X \to \mathcal{F}R_Y$ is an estimate, and hence there must exist a morphism $f : R_X \to R_Y$ such that $\mathcal{F}(f) \circ \kappa_X = \kappa_Y \circ g$. So, we define,

$$\mathcal{G}(g) \coloneqq f.$$

By construction this makes κ a natural transformation which is determined by the components κ_X. By exploiting uniqueness we can show that this is functorial, that is $\mathcal{G}(g \circ h) = \mathcal{G}(g) \circ \mathcal{G}(h)$. $\square$

If the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a reflection, by definition, to each morphism $\varkappa : X \to \mathcal{F}A$ there exists a unique morphism $f : \mathcal{G}X \to A$. Conversely, any map f uniquely determines $\varkappa$ by,

$$\mathcal{F}(f) \circ \kappa_X = \varkappa.$$

Which means we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, A) \cong \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}A).$$

Since this holds for every X and A it is the adjoint condition. So, adjoint and reflection are the same functor. Note that in the construction of the 'reflection functor' we assumed every object in $\mathcal{B}$ has a reflection. The adjoint functor theorems try to simplify this condition, under additional constraints.

1.4 | (Co)Limits

The notion of limits and colimits is very important as they allow us to construct new objects and functors. They are also very closely related to adjoint functors. To heuristacally motivate, limits is the categorical notion of 'closest' object to or from a system. Here, we intend to find an object that's nearest to a category. The notion of nearness comes from the morphisms, so the idea is to put the starting category in some other category, where the morphisms provide some sort of 'categorical distance', and then use this notion of distance of the target category to describe the 'limit'.

Let $\mathcal{I}$ and $\mathcal{A}$ be two categories. An inductive system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{F} : \mathcal{I} \to \mathcal{A}.$$

Intuitively, the limit of a system is an object in $\mathcal{A}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $X \in \mathcal{A}$ the constant functor $\Delta X : \mathcal{I} \to \mathcal{A}$ that sends everything in $\mathcal{I}$ to X, and each morphism in $\mathcal{I}$ to the identity on X. A relation between an object X and the system $\mathcal{F}$ is a natural transformation between ΔX and $\mathcal{F}$. Such a natural transformation is called a cone. The collection of all such cones is the set of all natural transformations,

$$\begin{aligned} C_{\mathcal{F}} : \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}). \end{aligned}$$

It's a contravariant functor from $\mathcal{A}$ to **Sets**. If the functor $C_{\mathcal{F}}$ is representable, there exists an object $L_{\mathcal{F}} \in \mathcal{A}$ such that,

$$C_{\mathcal{F}} \cong h^{L_{\mathcal{F}}}.$$

So, in such case $C_{\mathcal{F}}(X) \cong \mathrm{Hom}_{\mathcal{A}}(X, L_{\mathcal{F}})$. The representative $L_{\mathcal{F}}$ if it exists is called the colimit of the system $\mathcal{F}$, and is denoted by $\varprojlim \mathcal{F} := L_{\mathcal{F}}$ i.e.,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \tag{colimit}$$

and hence every cone must factor through $L_{\mathcal{F}}$. Intuitively the limit is the 'closest' object to the system. The notion of closeness must come from morphisms, so if there exists any other object with morphisms to the system, then it must be 'farther' than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

This means that for all objects $X \in \mathcal{A}$ and all family of morphisms $f_I : X \to \mathcal{F}I$, in $\mathcal{A}$ such that for all $f \in \mathrm{Hom}_{\mathcal{I}}(I, J)$, with $f_J = f_I \circ \mathcal{F}(f)$ factors uniquely through $\varprojlim \mathcal{F}$.

$$\begin{tikzcd}
 & X \arrow[d, dashed] \arrow[ddl, "f_I"'] \arrow[ddr, "f_J"] & \\
 & \varprojlim \mathcal{F} \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_J"'] & \\
\mathcal{F}I \arrow[rr, "\mathcal{F}(f)"'] & & \mathcal{F}J
\end{tikzcd}$$

A projective system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}.$$

Similar to the inductive system, for projective system $\mathcal{G} : \mathcal{I}^{\text{op}} \to \mathcal{A}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^{\mathcal{G}} : \mathcal{A} &\to \textbf{Sets} \\ X &\mapsto \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}^{\text{op}}}}(\mathcal{G}, \Delta X). \end{aligned}$$

If it's representable with representative $L^{\mathcal{G}}$,

$$C^{\mathcal{G}} \cong h_{L^{\mathcal{G}}}.$$

Denote the representative by $\varinjlim \mathcal{G} := L^{\mathcal{G}}$ is called the limit of the projective system. If the limit exists, we have,

$$\operatorname{Hom}_{\mathcal{A}^{\mathcal{I}^{\text{op}}}}(\mathcal{G}, \Delta X) \cong \operatorname{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, X) \tag{limit}$$

Projective limits can be written in terms of universal property as,

$$\begin{tikzcd}
\mathcal{G}I \arrow[rr, "\mathcal{G}(f)"] \arrow[dr, "\varphi_I"] \arrow[ddr, "f_I"'] & & \mathcal{G}J \arrow[dl, "\varphi_J"'] \arrow[ddl, "f_J"] \\
& \varinjlim \mathcal{G} \arrow[d, dashed] & \\
& X &
\end{tikzcd}$$

Note that if $\mathcal{I}$ admits initial object 0, then the limit $\varprojlim \mathcal{F}$ corresponds to the object $\mathcal{F}(0)$. Similarly for colimit, with terminal object.

1.4.1 | (Co)limit Calculus

A category $\mathcal{A}$ is cocomplete with respect to $\mathcal{I}$ if for all inductive systems indexed by $\mathcal{I}$, the colimit exists, if $\mathcal{I}$ is not explicitly said, then it means that $\mathcal{A}$ is cocomplete with respect to all small categories. $\mathcal{A}$ is complete with respect to $\mathcal{I}$ if it has all limits for all projective systems indexed by $\mathcal{I}$. $\mathcal{A}$ is called bicomplete if it's both complete and cocomplete.

Theorem 1.4.1. *$\mathcal{A}$ is cocomplete $\Rightarrow$ $\mathcal{A}^{\mathcal{K}}$ is cocomplete.*

Proof

Given an inductive system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$, the goal is to construct a new functor $\varprojlim \mathcal{F}$ in $\mathcal{A}^{\mathcal{I}}$ such that,

$$\operatorname{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

where $\Delta^{\mathcal{K}}$ is the constant functor in $\mathcal{A}^{\mathcal{K}}$ which sends $A \in \mathcal{A}$ to the constant functor $\Delta^{\mathcal{K}}A$. This is then by definition the colimit of the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$.

In order to do this we have to describe where $\varprojlim \mathcal{F}$ sends elements of $\mathcal{I}$. The construction involves argument wise assignment of objects in $\mathcal{A}$ for each object in $\mathcal{I}$, and then showing the functoriality, that is verify it respects composition of morphisms.

Construction. By definition of colimit, for every $I \in \mathcal{I}$, we have,

$$\operatorname{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}I, \mathcal{F}I) \cong \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}I, \varprojlim(\mathcal{F}I)) \tag{isomorphism}$$

here $\mathcal{F}I : \mathcal{K} \to \mathcal{A}$ is a fixed functor, an inductive system and $\varprojlim(\mathcal{F}I)$ its inductive limit. So we define the associated object by,

$$\varprojlim \mathcal{F}(I) := \varprojlim(\mathcal{F}I).$$

Functoriality.

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}(\cdot), \mathcal{F}(\cdot)) : \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathbf{Sets}$$

is a bifunctor, so for each natural transformation $\kappa : \Delta^{\mathcal{K}}\mathcal{H} \Rightarrow \mathcal{F}$,

$$\begin{tikzcd}
\mathcal{I} \arrow[rr, bend left, "\mathcal{F}"] \arrow[dr, "\mathcal{H}"'] & {} \arrow[d, phantom, "\Uparrow\kappa"] & \mathcal{A}^{\mathcal{K}} \\
& \mathcal{A} \arrow[ur, "\Delta^{\mathcal{K}}"'] &
\end{tikzcd}$$

and morphism $f : I \to J$, we have maps,

$$\begin{tikzcd}
I \arrow[d, "f"'] & \Delta^{\mathcal{K}}\mathcal{H}I \arrow[r, "\kappa_I"] \arrow[d, "\Delta^{\mathcal{K}}\mathcal{H}(f)"'] & \mathcal{F}I \arrow[d, "\mathcal{F}(f)"] \\
J & \Delta^{\mathcal{K}}\mathcal{H}J \arrow[r, "\kappa_J"'] & \mathcal{F}J
\end{tikzcd}$$

The commutative square gives us, $\kappa_J \circ \Delta^{\mathcal{K}}\mathcal{H}(f) = \mathcal{F}(f) \circ \kappa_I$, and the isomorphism of sets gives us a morphisms, $\widehat{\kappa}_I$ and $\widehat{\kappa}_J$ such that,

$$\widehat{\kappa}_I \circ \mathcal{H}(f) = \varprojlim(\mathcal{F}(f)) \circ \widehat{\kappa}_J.$$

So, $\widehat{\kappa} : \mathcal{H} \to \varprojlim \mathcal{F}(\cdot)$ is a natural transformation. So we get,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

Since this natural transformation is defined using the natural transformation κ it will satisfy the required compositions. So we have, $\varprojlim \mathcal{F}(f \circ g) = (\varprojlim \mathcal{F}(f)) \circ (\varprojlim \mathcal{F}(g))$. $\square$

This gives us an adjoint situation, where colimit is left-adjoint to the constant functor and the constant functor is left-adjoint to the limit,

$$\varinjlim \dashv \Delta \dashv \varprojlim \qquad \text{(limit-diagonal adjointness)}$$

Theorem 1.4.2.

$$\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}),$$

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, A) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, A).$$

Proof

The idea is to study an appropriate functor category, get hom-set isomorphisms and then apply Yoneda principle. So, we have to show for each set $X \in \mathbf{Sets}$, we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$$

Using the limit-diagonal adjointness, this reduces to showing $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$ and $\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$ are isomorphic. Δs are in the appropriate categories.

Let $\kappa : \Delta X \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$ be a natural transformation, then κ is determined by its components

$$\kappa_I : (\Delta X)I \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I).$$

Each $(\Delta X)I$ is a set, and hence the maps κ_I is itself determined by its action on the elements of the set X, So, for each $x \in X$, $\kappa_I(x)$ is a morphism in $\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I)$.

If we think of A as the constant functor, we can define using $\varphi_x(\cdot) := \kappa_{(\cdot)}(x)$ defines an element $\varphi \in \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$. This is a bijection and hence,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$$

Applying the limit-diagonal adjointness again to this and the Yoneda principle, we get that $\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$. The other isomorphism is also similar. □

So, hom-functor takes limits to colimits in the first argument and takes limits to limits in the second argument. This can now be used to prove many things easily.

Intuitively, limits take us to the 'last object' in a diagram and colimits take us to the 'first object' in a diagram. Here 'most' is quantified in terms of morphisms, and first and last are quantified by the direction of the morphisms. So, if the indexing category is a product category, then we can think of limits as trying to find the bottom right corner of the rectangle diagram, and similarly the colimit is the top left corner. So, whether we go to right most first and then go to the bottom or whether go to the bottom first and then go to the right shouldn't change which object we reach after doing this. We now formalize this.

Theorem 1.4.3. (Fubini for limits)

$$\varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathcal{F} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F} \cong \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}$$

Proof

This can be proved using the relation between the constant functors. Let $\Delta^{\mathcal{I} \times \mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I} \times \mathcal{J}}$, $\Delta^{\mathcal{I}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ and $\Delta^{\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ be the constant functors in the appropriate functor category. Then we have, $\Delta^{\mathcal{I} \times \mathcal{J}} = \Delta^{\mathcal{I}} \Delta^{\mathcal{J}}$. This gives us,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}) &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I} \times \mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}} \Delta^{\mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F}). \end{aligned}$$

By Yoneda principle we have the required isomorphism. Since $\mathcal{I} \times \mathcal{J} \cong \mathcal{J} \times \mathcal{I}$, the other isomorphism also follows. □

Lemma 1.4.4. *Right adjoints preserve limits.*

Proof

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint say $\mathcal{G} : \mathcal{B} \to \mathcal{A}$, and suppose $\mathcal{H} : \mathcal{I} \to \mathcal{A}$ is an inductive system with a limit $\varprojlim \mathcal{H}$, we must prove that $\mathcal{F}(\varprojlim \mathcal{H})$ is the limit of the inductive system $\mathcal{F} \circ \mathcal{H}$.

$\mathcal{F}\mathcal{H} : \mathcal{I} \to \mathcal{B}$ is an inductive system in $\mathcal{B}$ indexed by $\mathcal{I}$. For all $X \in \mathcal{B}$,

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{B}}(X, \mathcal{F} \varprojlim \mathcal{H}) &\cong \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}X, \varprojlim \mathcal{H}) \\
&\cong \varprojlim \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}X, \mathcal{H}) \\
&\cong \varprojlim \operatorname{Hom}_{\mathcal{B}}(X, \mathcal{F}\mathcal{H}) \\
&\cong \operatorname{Hom}_{\mathcal{B}}(X, \varprojlim \mathcal{F}\mathcal{H})
\end{aligned}$$

By Yoneda principle, we have, $\mathcal{F}(\varprojlim \mathcal{H}) \cong \varprojlim \mathcal{F} \circ \mathcal{H}$. □

Right adjoints preserve limits, can be remembered by the acronym, RAPL. Under additional conditions on the category $\mathcal{A}$, the converse holds, these theorems are called adjoint functor theorems.

1.4.2 | (Co)end Calculus

Similar to how natural transformations relate functors between two categories, dinatural transformations relate bifunctors. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors. Bifunctors are functors when one of the arguments are fixed. We will denote $\mathcal{F}^X$ for the contravariant functor $\mathcal{F}(\cdot, X)$, and $\mathcal{F}_X$ for the covariant functor $\mathcal{F}(X, \cdot)$.

Each morphism $f : X \to Y$ gives rise to the following morphisms in $\mathcal{B}$,

$$\begin{tikzcd}
X \arrow[dd, "f"'] & & \mathcal{F}(X,X) \arrow[dl, "\mathcal{F}^X(f)"'] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\
& \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \\
Y & & \mathcal{F}(Y,Y) & \mathcal{G}(Y,Y) \arrow[ur, "\mathcal{G}^Y(f)"'] &
\end{tikzcd}$$

A dinatural transformation $\kappa : \mathcal{F} \to \mathcal{G}$ consists of a family of morphisms,

$$\kappa_X : \mathcal{F}(X,X) \to \mathcal{G}(X,X)$$

such that for any $f : X \to Y$ the following commutes,

$$\begin{tikzcd}
X \arrow[dd, "f"'] & & \mathcal{F}(X,X) \arrow[dl, "\mathcal{F}^X(f)"'] \arrow[r, "\kappa_X"] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\
& \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \\
J & & \mathcal{F}(Y,Y) \arrow[r, "\kappa_Y"] & \mathcal{G}(Y,Y) \arrow[ur, "\mathcal{G}^Y(f)"'] &
\end{tikzcd}$$

Similar to the case of limits and cones, we can describe 'doubly indexed limits'. Suppose we are given a system $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$, intuitively, the end of the system is an object in $\mathcal{B}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $E \in \mathcal{B}$ the constant bifunctor $\Delta E : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ that sends everything in $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ to E, and each morphism in $\mathcal{B}$ to the identity on E. A relation between an object X and the system $\mathcal{F}$ is a dinatural transformation δ between ΔE and $\mathcal{F}$. Represented by the diagram,

$$
\begin{tikzcd}
X \arrow[d, "f"'] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F}(X,X) \arrow[dr, "\mathcal{F}^X(f)"] & \\
E \arrow[ur, "\delta^E_X"] \arrow[dr, "\delta^E_Y"'] & & \mathcal{F}(X,Y) \\
 & \mathcal{F}(Y,Y) \arrow[ur, "\mathcal{F}_Y(f)"'] &
\end{tikzcd}
$$

Such a dinatural transformation is called a wedge. The collection of all such wedges is the set of all such dinatural transformations,

$$
\begin{aligned}
W_{\mathcal{F}} : \mathcal{B} &\to \mathbf{Sets} \\
E &\mapsto \mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}).
\end{aligned}
$$

The end of the system $\mathcal{F}$ is intuitively the object which is closest to the system. A relation between an object $E \in \mathcal{B}$ and the system $\mathcal{F}$ consists of a dinatural transformation from the constant bifunctor ΔE to $\mathcal{F}$. So, the 'closest' would be such that any other dinatural transformation should factor through the 'closest' one.

The end of a system $\mathcal{F}$, is an object $\int_{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\omega : \Delta \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$ such that every other wedge factors through it.

$$
\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(E, \int_{\mathcal{A}} \mathcal{F}) \tag{end}
$$

Also denoted by $\int_{A \in \mathcal{A}} \mathcal{F}A$. Expressed in a commutative diagram by,

$$
\begin{tikzcd}
E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\kappa_X"] \arrow[ddr, bend right, "\kappa_Y"'] & & \\
 & \int_{\mathcal{A}} \mathcal{F} \arrow[r, dashed, "\omega_X"] \arrow[d, dashed, "\omega_Y"'] & \mathcal{F}(X,X) \arrow[d, "\mathcal{F}_X(f)"] \\
 & \mathcal{F}(Y,Y) \arrow[r, "\mathcal{F}^Y(f)"'] & \mathcal{F}(X,Y)
\end{tikzcd}
$$

The integral notation corresponds to the intuition that the equalizer given in $\int_{\mathcal{A}}$ can be thought of as an averaging operation on the functor where we run through the objects of $\mathcal{A}$.

The coend of a system $\mathcal{F}$, is an object $\int^{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\sigma : \mathcal{F} \to \Delta \int^{\mathcal{A}} \mathcal{F}$, such that every other cowedge factors from it.

$$
\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\mathcal{F}, \Delta E) \cong \mathrm{Hom}_{\mathcal{B}}(\int^{\mathcal{A}} \mathcal{F}, E) \tag{coend}
$$

Expressed in a commutative diagram by,

$$
\begin{tikzcd}
\mathcal{F}(Y,X) \arrow[r, "\mathcal{F}_Y(f)"] \arrow[d, "\mathcal{F}^X(f)"'] & \mathcal{F}(Y,Y) \arrow[d, dashed] \arrow[ddr, bend left, "\kappa_Y"] & \\
\mathcal{F}(X,X) \arrow[r, dashed] \arrow[drr, bend right, "\kappa_X"'] & \int^{\mathcal{A}} \mathcal{F} \arrow[dr, dashed, "\exists!"] & \\
 & & E
\end{tikzcd}
$$

Functoriality. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors and let $\eta : \mathcal{F} \to \mathcal{G}$ be a dimatural transformation. Any connection from an object E to the system $\mathcal{F}$ gives rise to a connection from the object to the system $\mathcal{G}$, given by the composition.

In particular we have the composition for the connection $\omega : \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$. So, this connection must factor through $\int_{\mathcal{A}} \mathcal{G}$, and hence we have a map, whose components are given by the composition,

$$\begin{tikzcd}
\int_{\mathcal{A}} \mathcal{F} \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\eta_X \circ \omega_X"] \arrow[ddr, bend right, "\eta_Y \circ \omega_Y"'] & & \\
& \int_{\mathcal{A}} \mathcal{G} \arrow[r, dashed, "\varphi_X"] \arrow[d, dashed, "\varphi_Y"'] & \mathcal{G}(X,X) \arrow[d, "\mathcal{G}_X(f)"] \\
& \mathcal{G}(Y,Y) \arrow[r, "\mathcal{G}^Y(f)"'] & \mathcal{G}(X,Y)
\end{tikzcd}$$

This is functorial since the $\int_{\mathcal{A}} \eta : \int_{\mathcal{A}} \mathcal{F} \to \int_{\mathcal{A}} \mathcal{G}$ are defined component wise and so the composition will be component wise. So $\int$ is functorial, that is, $\int_{\mathcal{A}} (\eta \circ \kappa) = \int_{\mathcal{A}} \eta \circ \int_{\mathcal{A}} \kappa$.

Fubini Rule. Given a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$, the end in the first two coordinates, gives rise to a functor,

$$\int_{\mathcal{A}} \mathcal{F} : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}.$$

The end of this functor is then,

$$\int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F} \in \mathcal{C}$$

With some tedious checking it turns out that,

$$\int_{\mathcal{A} \times \mathcal{B}} \mathcal{F} \cong \int_{\mathcal{A}} \int_{\mathcal{B}} \mathcal{F} \cong \int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F}. \tag{Fubini for ends}$$

1.4.2.1 | (Co)ends as (Co)limits

The motivation for the definition of ends/coends was very similar to that of limits/colimits. So we should expect the two concepts to be closely related. To intuitively motivate the relation between ends and limits, we have to intuitively understand how ends describe average of a system and limits describe closeness to the system. Limits only take into account the relation an object has 'with' the system, that is, a natural transformation from constant functor to the system. In case of ends, the relation 'between' the objects of the system is important, this is encoded in the dinatural transformation. So, the 'average' should also be expected to be some sort of limit taken over morphisms between objects of the indexing category. This is what makes it average over the relations between objects of the system.

We can think of ends and coends as limits and colimits. The first step is then to associate to each bifunctor system a functor, and turn the ends/coends of bifunctor systems into limits/colimits of systems. This involves estalishing an equivalence between the category of bifunctors and an appropriate category of functors. Consider a bifunctor

$$\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$$

attaches to pairs of objects in $\mathcal{A}$ objects in $\mathcal{B}$ such that the relation between the pair of objects is preserved. So, we could think of the functor as assigning to each morphism between pairs fo objects in $\mathcal{A}$ an object of $\mathcal{B}$. So, we can start with a new category where objects are morphisms of the category $\mathcal{A}$.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$, the bifunctor assigns to the pair X, Y the object $\mathcal{F}(X, Y)$, so instead we could assign to the morphism f, the object $\mathcal{F}(\mathrm{src}(f), \mathrm{tgt}(f))$. Where src is the source object of f, and $\mathrm{tgt}(f)$ is the target. To make sure the bifunctoriality transfers to functoriality of this new association we have to define the morphisms suitably. For two every morphism $f, g \in \mathrm{Hom}_{\mathcal{A}}$ define a morphism between them to be a pair of morphisms,

$$\begin{tikzcd} X \arrow[d, "f"'] & \widehat{X} \arrow[l] \arrow[d, "g"] \\ Y \arrow[r] & \widehat{Y}. \end{tikzcd} \tag{bimorph}$$

Note that the reverse direction of the connection from $\widehat{X}$ to X is necessary to make the association a functor. Because the bifunctor is contravariant in the first argument, with the second argument fixed. So with this, we have a new functor,

$$\widehat{\mathcal{F}} : \widehat{\mathcal{A}} \to \mathcal{B}$$

where $\widehat{\mathcal{A}}$ is the category consisting of morphisms of $\mathcal{A}$ as objects and the above rule for morphisms, bimorph. So the functor,

$$\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}} \to \mathcal{B}^{\widehat{\mathcal{A}}}, \quad \mathcal{F} \mapsto \widehat{\mathcal{F}},$$

is an equivalence of categories, and must respect initial/terminal objects. Since limits/colimits and ends/coends are initial/terminal objects in the respective categories we have,

$$\int_{\mathcal{A}} \mathcal{F} \cong \varprojlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}, \quad \int^{\mathcal{A}} \mathcal{F} \cong \varinjlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}. \tag{(co)ends as (co)limits}$$

For a detailed proof, see [?]. This immediately leads to the following observation that if $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ preserves all limits then it preserves all the ends that exist. If $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$ is a system, and $\mathcal{G}$ is a continuous functor then

$$\mathcal{G}\Big(\int_{\mathcal{A}} \mathcal{F}\Big) \cong \int_{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

If $\mathcal{G}$ is a contravariant functor and cocontinuous, then,

$$\mathcal{G}\Big(\int_{\mathcal{A}} \mathcal{F}\Big) \cong \int^{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

An immediate corollary is that Hom functors preserve ends in the second argument and maps ends to coends in the first argument.

Corollary 1.4.5. (Hom-(Co)End relations)

$$\mathrm{Hom}_{\mathcal{B}}\Big(\int_{\mathcal{A}} \mathcal{F}, A\Big) \cong \int^{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, A)$$

$$\mathrm{Hom}_{\mathcal{B}}\Big(A, \int_{\mathcal{A}} \mathcal{F}\Big) \cong \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(A, \mathcal{F}).$$

□

Theorem 1.4.6.

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F},\mathcal{G}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A).$$

Proof

We have to show that $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F},\mathcal{G})$ satisfies the universal property of ends. That's, any collection of morphisms from an object $E \in \mathbf{Sets}$ to the system $\mathcal{H} \equiv \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-),\mathcal{G}(-))$ factors through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F},\mathcal{G})$.

Suppose we have a wedge $\tau : E \to \mathcal{H}$, then for each $A \in \mathcal{A}$ we have a morphisms $\tau_A : E \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A)$. Since E is a set this is a set map, which maps each $x \in E$ to a morphism between the objects $\mathcal{F}A$ and $\mathcal{G}A$. That's to say $\tau_A(x) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A)$.

For a morphism $f : A \to B$, the wedge condition says that,

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\tau_A"] \arrow[ddr, bend right, "\tau_B"'] & & \\
& \displaystyle\int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A) \arrow[r, dashed, "\omega_A"] \arrow[d, dashed, "\omega_B"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A) \arrow[d, "\circ\mathcal{F}(f)"] \\
& \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}B,\mathcal{G}B) \arrow[r, "\mathcal{G}(f)\circ"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}B)
\end{tikzcd}$$

So, we have,

$$\mathcal{G}(f) \circ \tau_A(x) = \tau_B(x) \circ \mathcal{F}(f).$$

This means that $\tau_{(-)}(x)$ is a natural transformation,

$$\begin{tikzcd}
A \arrow[d, "f"] & \mathcal{F}A \arrow[r, "\tau_A(x)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}A \arrow[d, "\mathcal{G}(f)"] \\
B & \mathcal{F}B \arrow[r, "\tau_B(x)"'] & \mathcal{G}B
\end{tikzcd}$$

So, it must factor through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F},\mathcal{G})$. This is precisely the universal property of ends. So,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F},\mathcal{G}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A).$$

$\square$

The collection of all natural transformations from $\mathcal{F}$ to $\mathcal{G}$ can be thought of as taking an average of elements of $\mathcal{A}$ as 'measured' by the functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A,\mathcal{G}A)$.

1.4.2.2 | Functor Tensor Product

1.4.3 | Weighted (Co)limits

1.4.4 | Density theorem for Pre-Sheaves

We now have the necessary tools to prove the famous result about the category of pre-sheaves, $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Every pre-sheaf can be canonically presented as a colimit of representable functors. For the sake of familiarity we will assume that $\mathcal{C} = \mathcal{O}(X)$.

THEOREM 1.4.7. (CO-YONEDA/DENSITY FORMULA) $\mathcal{F} \in \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$,

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} \mathcal{F}U \times \mathrm{Hom}_{\mathcal{C}}(V,U) \cong \int_{U\in\mathcal{C}} (\mathcal{F}U)^{\mathrm{Hom}_{\mathcal{C}}(U,V)}.$$

PROOF

By Yoneda applied to the functor

$$\mathcal{H} : V \mapsto \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W),$$

we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^V, \mathcal{H}) \cong \mathcal{H}V.$$

The following chain of isomorphisms lets us sneak in Yoneda principle,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W) &\cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}\left(h^V, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}(-), W)\right) \\ &\cong \int_{U\in\mathcal{C}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}U, W)\right) \\ &\cong \int_{U\in\mathcal{C}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U \times \mathcal{F}U, W\right) \\ &\cong \mathrm{Hom}_{\mathbf{Sets}}\left(\int^{U\in\mathcal{C}} h^V U \times \mathcal{F}U, W\right) \end{aligned}$$

Here, in the second step we used 1.4.6, and in the third step we used hom-tensor adjointness in **Sets**. The last step follows from hom-sets taking coends to ends in the first argument. So, we have by Yoneda principle,

$$\mathcal{F}V \cong \int^{U\in\mathcal{C}} h^V U \times \mathcal{F}U$$

This holds when the pre-sheaf is to any category that's cartesian closed, and the product are replaced with the correct tensor product in the target category. In case of modules this corresponds to the internal hom-tensor product adjointness. □

When $\mathcal{C}$ is the category of open sets of a space X, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(U,V)$ is nonempty, only when $V \subseteq U$, in which case its a singleton set, with the inclusion map as the only map. If we consider continuous functions as the pre-sheaf, and $V \subset U$, any 'continuous function' belonging to U will give rise to a 'continuous function' in V. Intuitively, we want to include all such functions and it will be a limit in this sense.

For the enriched case, the product will be replaced by tensor product $\odot$ in the enriching category $\mathcal{V}$. Limits will be replaced by weighted limits and using these we get the enriched density formula,

THEOREM 1.4.8. (ENRICHED DENSITY FORMULA) *Let $\mathcal{F} \in \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$, then,*

$$\mathcal{F}V \cong \int^{U\in\mathcal{C}} h^V U \odot \mathcal{F}U$$

□

For proofs, and details, see [?],[?] and the references therein.

1.5 | Kan Extension of Functors

Given two functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$, the goal of Kan extension is to find a new functor $\mathcal{K}$ such that the manipulation done by composite functor $\mathcal{K} \circ \mathcal{F}$ is closest to the manipulation done by the functor $\mathcal{G}$. The obstruction to $\mathcal{K} \circ \mathcal{F}$ being the same as $\mathcal{G}$ comes from $\mathcal{F}$ losing information that $\mathcal{G}$ preserves.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd} & \mathcal{B} & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C} \end{tikzcd}$$

The left Kan extension of $\mathcal{G}$ along $\mathcal{F}$ is a functor is a functor $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G} : \mathcal{B} \to \mathcal{C}$ such that there exists a natural transformation $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F}$, such that any other extension $\mathcal{K}$ with natural transformation $\mathcal{G} \to \mathcal{K} \circ \mathcal{F}$ factors through $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$, i.e., $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F}$.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathcal{K}"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C} \end{tikzcd}$$

So, in terms of a commutative diagram a Kan extension is the functor that makes the diagram as close to commutative as possible.

The left Kan extension $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ can be visualised by the diagram,

$$\mathcal{G} \Longrightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{K} \circ \mathcal{F} \Longrightarrow \cdots$$

This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{K} \circ \mathcal{F}$ there exists a natural transformation from $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ to $\mathcal{K}$, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}, \mathcal{K})$$

Intuitively the left Kan extension is the left most functor to the functor in the above visualisation. Here 'most' is quantified in terms of natural transformations. The left Kan extension is the left most (as explained above) extension of the functor $\mathcal{G}$ with respect to $\mathcal{F}$. We should expect the construction of the 'left most' functor to be related to taking colimits in an appropriate category. If $\mathcal{C}$ is cocomplete, the functor category to $\mathcal{C}$ will also be cocomplete, and $\mathcal{A}$ has some 'nice properties' then we could think of these as a system in the functor category and intuitively, the left 'most' should exist.

The right Kan extension is defined similarly, and only the direction of the natural transformation is changed from $\mathcal{K} \circ \mathcal{F}$ to $\mathcal{G}$.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathcal{H}"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} & \mathcal{C} \end{tikzcd}$$

The right Kan extension is denoted by $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$. It's the right most functor in the following visualization,

$$\cdots \Longrightarrow \mathcal{H} \circ \mathcal{F} \Longrightarrow (\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{G}$$

Similar to left Kan extensions, we should expect the construction of the 'right most' functor to be related to taking limits in an appropriate category. This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{H} \circ \mathcal{F}$ there exists a natural transformation from $\mathcal{H}$ to $\mathrm{Ran}_{\mathcal{F}} \mathcal{G}$, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{H}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathcal{H}, \mathrm{Ran}_{\mathcal{F}} \mathcal{G}).$$

Kan extension are functors in the appropriate functor category.

Lemma 1.5.1. *$\forall \mathcal{G}$, $\mathrm{Lan}_{\mathcal{F}} \mathcal{G}$, $\mathrm{Ran}_{\mathcal{F}} \mathcal{G}$ exist, then,*

$$\mathrm{Lan}_{\mathcal{F}} \dashv \circ\mathcal{F} \dashv \mathrm{Ran}_{\mathcal{F}} .$$

The proof of this adjointness with precomposition has already been described, and directly follows from definition of Kan extensions, for this to make sense we only need the fact that $\mathrm{Lan}_{\mathcal{F}}$ and $\mathrm{Ran}_{\mathcal{F}}$ are functors, i.e., $\mathrm{Lan}_{\mathcal{F}} \mathcal{G}$ and $\mathrm{Ran}_{\mathcal{F}} \mathcal{G}$ exist $\forall \mathcal{G}$, and satisfy some composition rules, which it will due to the definition.

If $\mathcal{C}$ is a locally small category, a Kan extension is said to be pointwise Kan extension if it is preserved by representable functors $\mathrm{Hom}_{\mathcal{C}}(C, -)$ for all $C \in \mathcal{C}$. It's called absolute if any functor $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ preserves the Kan extension.

Lemma 1.5.2. *If $\mathcal{L}$ is a left adjoints, then,*

$$\mathrm{Lan}_{\mathcal{F}}(\mathcal{L} \circ \mathcal{G}) \cong \mathcal{L} \circ \mathrm{Lan}_{\mathcal{F}} \mathcal{G}.$$

Proof

Suppose $\mathcal{G}$ has a left Kan extension along $\mathcal{F}$,

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}} \mathcal{G}"] \arrow[drr, bend left, "\mathcal{K}"] & & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & {} \arrow[u, phantom, "\Uparrow"] & \mathcal{C} \arrow[r, shift left, "\mathcal{L}"] & \mathcal{D} \arrow[l, shift left, "\mathcal{R}"]
\end{tikzcd}$$

Let $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ be a functor that's left adjoint, i.e., there exists a functor $\mathcal{R} : \mathcal{D} \to \mathcal{C}$ to which $\mathcal{L}$ is a left adjoint, then we have,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^{\mathcal{B}}}(\mathcal{L}\,\mathrm{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K}) &\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathrm{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{R}\mathcal{K}) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, (\mathcal{R}\mathcal{K})\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{A}}}(\mathcal{L}\mathcal{G}, \mathcal{K}\mathcal{F})
\end{aligned}$$

So, by definition of Kan extension, we have that $\mathcal{L}\,\mathrm{Lan}_{\mathcal{F}} \mathcal{G} \cong \mathrm{Lan}_{\mathcal{F}}(\mathcal{L}\mathcal{G})$. □

What's happening is that left adjoints preserve colimits, and this guarantees the existence of the Kan extenstion for the composition. So, when the required colimits exist, the notion of closest makes sense. The closest functor to $\mathcal{L}$ is $\mathcal{L}$, so once we know the Kan extension exists, it must be the one described above.

1.5.1 | Kan Extensions as Coends

Certain additional constraints on the starting categories guarantees the existence of Kan extensions. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd}
 & \mathcal{B} \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C}
\end{tikzcd}$$

Theorem 1.5.3. (Existence/Coend Formula)

$$\mathrm{Lan}_{\mathcal{F}}\, \mathcal{G}B \cong \int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, B) \odot \mathcal{G}A$$

whenever $\mathcal{A}$ is small, $\mathcal{B}$ is locally small, $\mathcal{C}$ is cocomplete.

Proof

Note that the local smallness of $\mathcal{B}$, or being enriched by the category of sets is needed for the existence of tensor product $\odot$, which is needed for the existence of extensions, this condition maybe replaced with an appropriate enriched category with a tensor product. Once they exist, the cocompleteness is needed for the existence of the limit, and to take the limits, we need the starting category to be small. So the conditions we require are to be expected.

The proof is again find a chain of isomorphisms so we can sneak in Yoneda principle. Firstly, by the end formula for natural transformations, we have,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}\mathcal{F}A\right)$$

Applying Yoneda to the functor $\mathcal{H} : X \mapsto \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}X\big)$, we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}\mathcal{F}A\big) = \mathcal{H}(\mathcal{F}A)$$

Applying the end formula for natural transformations again, we get,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}(h^{\mathcal{F}A}B, \mathcal{H}B)$$

This gives us the following double 'integral',

$$\int_{A\in\mathcal{A}} \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}B\right)\right)$$

By using the hom-tensor adjointness we get,

$$\begin{aligned}
\int_{A\in\mathcal{A}} \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\left(\mathcal{G}A, \mathcal{K}B\right)\right) &\cong \int_{A\in\mathcal{A}} \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\left(h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\right) \\
&\cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\left(\int_{A\in\mathcal{A}} h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\right) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}\left(\int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, -) \odot \mathcal{G}A, \mathcal{K}\right)
\end{aligned}$$

Here, we used the Fubini rule for ends, and hom-(co)end relations in the first step, and end formula for natural transformations in the next step. Now by Yoneda principle proves the theorem. $\square$

Assume the conditions of coend formula are satisfied. For each functor $\mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, the left Kan extension along $\mathcal{F}$ is the functor, $\mathrm{Lan}_{\mathcal{F}}(-) : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{B}}$, $\mathcal{G} \mapsto \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$. Intuitively we expect 'nearest' composed with 'nearest' to be the 'nearest'. So we will have,

$$\mathrm{Lan}_{\mathcal{F}\circ\mathcal{E}}\,\mathcal{G} = \mathrm{Lan}_{\mathcal{F}}(\mathrm{Lan}_{\mathcal{E}}\,\mathcal{G}).$$

The proof is a simple application of the coend formula and some end-coend calculus.

If a Kan extension at each point $B \in \mathcal{B}$ can be written as a limit such as the one above, then it's always pointwise Kan extension, as Hom-functor preserves limits. Conversely, if $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ is pointwise, then for each $C \in \mathcal{C}$ we have, $\mathrm{Lan}_{\mathcal{F}}(h_C \circ \mathcal{G}) \cong h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$.

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}"] \arrow[drr, bend left, "h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}"] & & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[r, phantom, "\Uparrow"] & & \mathcal{C} \arrow[r, "h_C"'] & \mathbf{Sets}
\end{tikzcd}$$

Applying Yoneda to the functor $\mathcal{H} \coloneqq h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ we get,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{B}}}(h^B, \mathcal{H}) \cong \mathcal{H}B = \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}B)$$

Since **Sets** is cocomplete, we can express the Kan extension as a coend formula. Now, by Yoneda principle this means that the Kan extension $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ can itself be written as a coend.

1.5.2 | All Concepts are Kan Extensions

Any concept that talks about some notion of nearest/closest/best should be expected to be related to Kan extensions. Kan extensions is a very general construction which unites limits, adjoints, and other constructions. Many of the categorical constructions involve limits or adjunctions. Limits correspond to nearest objects to/from a system. Adjoints when viewed as reflections also say that the associated objects are the nearest to the original object when transformed back. So both of these concepts should be expected to be related to Kan extensions.

1.5.2.1 | (Co)Limits as Kan Extensions

Limits are the nearest object to a system. We can now think of objects in a category $\mathcal{A}$ as functors from a terminal category to the category $\mathcal{A}$. Here the terminal category is a category with one object $*$ and one morphism, the identity morphism on the object $\mathbb{1}_*$. This category will be denoted by $\mathbb{1}$. Now each object A in $\mathcal{A}$ can be thought of as a functor $\widehat{A} : \mathbb{1} \to \mathcal{A}$. Since we want the 'nearest' object to a system, we want a nearest approximation of the system by a functor that corresponds to such an object. This is a Kan extension situation.

The terminal category $\mathbb{1}$ is the unique object in the category of categories, such that there exist only one functor from any category $\mathcal{I}$ to $\mathbb{1}$, which sends everything to the only object in the category, and every morphism to the only morphism in $\mathbb{1}$. Denote this functor by $\mathcal{T}$. Then we have the following Kan extension problem,

$$\begin{tikzcd}
& \mathbb{1} \arrow[dr, dashed, "\mathcal{H}"] & \\
\mathcal{I} \arrow[ur, "\mathcal{T}"] \arrow[rr, "\mathcal{F}"'] & \Downarrow & \mathcal{A}
\end{tikzcd}$$

For each functor $\mathcal{H} : \mathbb{1} \to \mathcal{A}$ corresponds to an object H in the category $\mathcal{A}$, and $\mathcal{H} \circ \mathcal{T}$ represents the constant system which maps everything in the indexing category $\mathcal{I}$ to a constant object. The natural transformations from $\mathcal{H} \circ \mathcal{T}$ to $\mathcal{F}$ represent the cones, from the object H to the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$.

The limit of a system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ is the right Kan extension of $\mathcal{F}$ along $\mathcal{T}$,

$$\varprojlim \mathcal{F} = \mathrm{Ran}_{\mathcal{T}} \mathcal{F}(*).$$

Similarly the colimit of a system is the object that's nearest from the system. So, the colimit will be the left Kan extension of the system along $\mathcal{T}$.

$$\varinjlim \mathcal{F} \cong \mathrm{Lan}_{\mathcal{T}} \mathcal{F}(*).$$

1.5.2.2 | Adjoint Functors as Kan Extensions

Adjoint functors when viewed as reflections §1.3.1, bring with them some notion of 'nearness'. So adjoints should be expected to be related to Kan extensions. Consider an adjoint situation $\mathcal{F} \dashv \mathcal{G}$,

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathcal{F}"] & \mathcal{B} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd},$$

If we think of objects in $\mathcal{A}$ as having some information, and after performing $\mathcal{F}$ we lose information in the category $\mathcal{A}$, then the functor $\mathcal{G}$ is such that $\mathcal{G}\mathcal{F}$ is the 'best' recovery of the original information about $\mathcal{A}$. So, $\mathcal{G}\mathcal{F}$ is the 'nearest' to the identity functor $\mathbb{1}_{\mathcal{A}}$, 'along' $\mathcal{F}$. This is a natural transformation,

$$\mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}\,.$$

The nearest means that any other 'recovery' will have to factor through this that's to say, $\cdots \Rightarrow \mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}$. Similarly, the left adjoint is the 'nearest' from the identity functor on $\mathcal{B}$, so we have,

$$\mathbb{1}_{\mathcal{A}} \Rightarrow \mathcal{F}\mathcal{G}.$$

So, we get the following Kan extension way of thinking about adjoints.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}} \cong \mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \arrow[dr, dashed, "\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F}"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{B} \arrow[ur, "\mathcal{G}"] \arrow[rr, "\mathbb{1}_{\mathcal{B}}"'] & {} & \mathcal{B} \end{tikzcd}$$

So, the right adjoint $\mathcal{G}$ is the left Kan extension of $\mathbb{1}_{\mathcal{A}}$ along $\mathcal{F}$. Similarly, the left adjoint $\mathcal{F}$ is the right Kan extension of $\mathbb{1}_{\mathcal{B}}$ along $\mathcal{G}$. An adjoint situation can be expressed as the following pair of Kan extensions,

$$\begin{tikzcd} & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \arrow[ur, "\mathcal{F}"'] & \end{tikzcd} = \begin{tikzcd} & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] & & \mathcal{B} \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \arrow[ul, Rightarrow] \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \\ & \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \end{tikzcd} = \begin{tikzcd} \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & & \mathcal{B} \arrow[dr, "\mathcal{G}"] & \\ & \mathcal{A} \arrow[ur, Rightarrow] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \end{tikzcd}$$

such that any other extension $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ factors through the Kan extension,

$$\cdots \Longrightarrow \mathcal{H} \circ \mathcal{G} \Longrightarrow (\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}) \circ \mathcal{G} \Longrightarrow \mathbb{1}_{\mathcal{B}}$$

visualised by the diagram,

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \\
& \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \\
& \mathcal{A} \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}$$

Similarly, any extension $\mathcal{K}$ of $\mathcal{F}$ along $\mathbb{1}_{\mathcal{A}}$ must factor through the adjoint $\mathcal{G}$,

$$\mathbb{1}_{\mathcal{A}} \Longrightarrow (\mathrm{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}) \circ \mathcal{F} \Longrightarrow \mathcal{K} \circ \mathcal{F} \Longrightarrow \cdots$$

visualised by the diagram,

$$\begin{tikzcd}
& \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] & \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \arrow[dr, "\mathcal{K}"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow"] & \, & \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow"] & \, & \mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{B} \arrow[dr, "\mathcal{K}"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] \arrow[r, phantom, "\Uparrow"] & \, & \mathcal{A}
\end{tikzcd}$$

Lemma 1.5.4.

$$\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F} \dashv \mathcal{G} \cong \mathrm{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}} \,.$$

□

So, $\mathcal{G}$ has a left adjoint if and only if $\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}$ exists. Similarly, $\mathcal{F}$ has a right adjoint if and only if $\mathrm{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}$ exists. The adjoint functor theorems usually are some conditions for the existence of these Kan extensions. When the categories are small, we can express the Kan extension as a coend. So in such cases, adjoints do exist.

Formula Sheet

2 | Grothendieck Topology

The only topological properties we needed in describing a sheaf on a topological space X, in the introduction were that of intersection and covering. We can isolate these properties, reformulate them categorically. This gives us the notion of a Grothendieck topology, a category with a Grothendieck topology is a sites.

2.1 | Grothendieck Topology; Sites

To motivate the definition, consider a topological space X. We use the topology to construct a category $\mathcal{O}(X)$ in which the open sets are the objects, and the morphisms $V \to U$ are inclusion maps $V \subset U$. So, the hom sets, if non-empty contain only one element, the inclusion map between the open sets. The space X is the final object of the category $\mathcal{O}(X)$.

Let $\{U_i\}$ be a covering of an open set $U \in \mathcal{O}(X)$. This first means that $U_i \subseteq U$, i.e., there exists an inclusion of U_i to U. So, the covering can be thought of as a collection of morphisms,

$$\mathcal{U} = \{U_i \xrightarrow{i} U\}_{i \in \mathcal{I}} \tag{covering}$$

For any open set U the trivial inclusion $U \subseteq U$ must be a covering of U,

$$\mathcal{U} = \{U \subseteq U\} \tag{isomorphism}$$

If we had a covering $\{U_i\}$ of U, and for each U_i a covering $\{U_{ij}\}$, these smaller open sets together must cover U. So, in terms of morphisms, the first covering is the collection of inclusions, $\mathcal{U}_i = \{ij : U_{ij} \to U_i\}_{j \in \mathcal{I}_i}$, and the second covering is the collection of inclusions $\mathcal{U} = \{i : U_i \to U\}_{i \in \mathcal{I}}$. We obtain that the composition,

$$\widehat{\mathcal{U}} = \{U_{ij} \xrightarrow{i \circ ij} U\}_{(i,j) \in \prod_i i \times \mathcal{I}_i} \tag{locality}$$

must be a covering.

Now to we need generalize the notion of an interesection. Given two open sets, U_i and U_j of $\mathcal{O}(X)$ we are interested in describing the intersection categorically. The open sets U_i and U_j come equipped with an inclusion maps i, j, into the space X,

$$U_i \xrightarrow{i} X \xleftarrow{j} U_j,$$

and for the intersection, we have two more morphisms from the intersection, $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$, So this gives us the following commutative square,

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "i"] \\ U_j \arrow[r, "j"'] & X \end{tikzcd} . \tag{intersection}$$

Intersection is the largest set, containing elements of both U_i and U_j, this amounts to saying, the above square must be the universal square, i.e., $U_i \cap U_j$ must be the pullback of the maps $U_i \to X \leftarrow U_j$.

Given a covering $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ and the inclusion $V \subseteq U$, then we can use the intersection and the cover $\{U_i\}_{i\in\mathcal{I}}$ to construct a covering of V, by taking intersections $V \cap U_i$. Using the maps $U_i \xrightarrow{i} U \leftarrow V$, we can construct the pullback,

$$\begin{tikzcd}
V \cap U_i \arrow[r, "\hat{i}"] \arrow[d] & V \arrow[d] \\
U_i \arrow[r, "i"'] & U
\end{tikzcd} .$$

So, the covering of V can be constructed with the morphisms $\hat{i}$, is given by,

$$\mathcal{V} = \{U_i \cap V \xrightarrow{\hat{i}} V\}_{i\in\mathcal{I}} \tag{base change}$$

These are easily generalized to any category $\mathcal{C}$. A Grothendieck topology on a category is a map,

$$U \mapsto \mathrm{Cov}(U)$$

such that the covering of an object U in the category $\mathcal{C}$, the collection of inclusion maps, in covering, is replaced with a collection of morphisms $\{U_i \to U\}_{i\in\mathcal{I}}$, the trivial inclusion in isomorphism is replaced by an isomorphism of objects. Intersection of open sets, in intersection is replaced by the pullback or fibered product of two maps. An inclusion morphism, $V \to U$ in base change is replaced with the collection of pullback morphisms $\{V \prod U_i \to V\}$.

A category with a Grothendieck topology is called a site. Let $\mathcal{O}(X)$ be a category equipped with a Grothendieck topology $\mathcal{T}_X$. A pre-sheaf of sets on a site $\mathcal{O}(X)$ is a contravariant functor,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}\,.$$

For the sake of simplicity we will denote the maps $\mathcal{F}(i)$ by $|_{U_i}$ like in the case of pre-sheaves on topological spaces. The pre-sheaf $\mathcal{F}$ is called separated if given a covering $\{U_i \to U\}$, and two sections $f, g \in \mathcal{F}U$, whose pullbacks to each $\mathcal{F}U_i$ coincide, it follows that $f = g$. That is for all covers $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$,

$$\forall i,\ \ f|_{U_i} = g|_{U_i} \Rightarrow f = g \tag{separated}$$

Some examples of pre-sheaves that are not separated can be found here.

A pre-sheaf is a sheaf if the sections can be patched up. That's to say, given a covering $\{U_i \to U\}_{i\in\mathcal{I}} \in \mathrm{Cov}(U)$ and a set of elements $f_i \in \mathcal{F}U_i$ which coincide on the 'intersection' i.e., if f_i and f_j coincide on the fibered product,

$$\begin{tikzcd}
U_i \prod_{i,j} U_j \arrow[r] \arrow[d] & U_i \arrow[d, "i"] \\
U_j \arrow[r, "j"'] & U
\end{tikzcd} .$$

That is to say if $f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j}$, then there must exist unique section $f \in \mathcal{F}U$ such that $f|_{U_i} = f_i$.

$$f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j} \Rightarrow \exists f \in \mathcal{F}U, f|_{U_i} = f_i.$$

This can be traslated into saying that there exists the following equalizer.

$$\mathcal{F}U \dashrightarrow^{e} \prod_i \mathcal{F}U_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \prod_{i,j} U_j). \tag{collation}$$

where,

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

This is an equalizer similar to the case of sheaves on topological spaces. This is also called gluing or patching. The meaning of separated is that there is at most one way to patch up the pre-sheaf. Here the big $\prod$s are product of sets, and small $\prod_{i.j}$s are fibered products in the category $\mathcal{O}(X)$. In the classical case of a topological space we have $U_i \prod_{i,i} U_i = U_i \cap U_i = U_i$, so the two possible pullbacks from $U_i \prod_{i,i} U_i$ coincide; but if the map $U_i \to U$ is not injective, then the two projections $U_i \prod_{i,i} U_i \to U_i$ will be different.

2.2 | Sieves

It's useful to get a functorial way of expressing inclusion of open sets. Subset inclusions are special relation between open sets of a topological space. A functorial description of such relations will be useful for the case of sites. A general category usually contains a lot more relations between objects. These morphisms don't necessarily correspond to subset relation. So, we would like to throw out these extra morphisms.

The notion of sieve formalizes what is meant by allowed relations. Given two objects in a category $U, V \in \mathcal{C}$, the allowed morphisms from V to U should be a subset of all morphisms between V and U. If $S^U(V)$ represents the allowed morphisms from V to U, we have,

$$S^U(V) \subseteq \mathrm{Hom}_{\mathcal{C}}(V, U).$$

Let this subset inclusion be denoted by $\kappa_V : S^U(V) \to \mathrm{Hom}_{\mathcal{C}}(V, U)$. Let $V \subseteq U$ and W be some other open set which is related to V. Then topologically it's irrelevant how W is related to V. The composite relation would still be a subset relation. Let $\coprod S^U$ be the union of all $S^U(V)$, corresponding to all the allowed relations to U.

$$g \in \coprod S^U \Rightarrow g \circ f \in \coprod S^U$$

whenever the composition makes sense. This means that S^U is functorial.

$$\begin{tikzcd} W \arrow[d, "f"] & S^U V \arrow[r, "\kappa_W"] \arrow[d, "S^U(f)"'] & h^U(W) \arrow[d, "\circ f"] \\ V & S^U W \arrow[r, "\kappa_V"'] & h^U(V). \end{tikzcd}$$

A sieve S^U on U is a subfunctor of the hom-functor h^U, denoted by $S^U \subseteq h^U$. This means that there is a natural transformation from S^U to h^U whose components are monic.

If W is some other object in $\mathcal{C}$ with a morphism to V, if f represents a subset relation between V and U that is $f \in S^U(V)$ and $g \in h^V(W)$ is any morphism, then we should have, the composition would still yield a subset. Topologically it's irrelevant how W is related to V. So,

2.3 | Sheafification Functor

Sheafification introduces extra structure on the pre-sheaf using the structure of the underlying topological space or site. In particular, the covers of any open set forms a category which is cofiltered. Sheafification utilizes this structure to construct a sheaf. This section will be informal, for a more formal discussion, see [**?**].

The category of sheaves $\mathrm{Sh}(X)$ over a site X is a full subcategory of the category of pre-sheaves $\mathrm{PSh}(X)$ over the site X,

$$\mathrm{Sh}(X) \rightarrowtail \mathrm{PSh}(X)$$

A sheaf is a contravariant functor, $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, together with the collation property. We have the forgetful functor from sheaves to presheaves that forgets about the patching properties of the sheaf. We can forget about the collation property, and this forgetful functor associates with the sheaf the underlying pre-sheaf. Denote this forgetful functor by,

$$\iota_X : \mathrm{Sh}(X) \to \mathrm{PSh}(X).$$

The left-adjoint to this functor is the sheafification functor,

$$^{\mathrm{Sh}} : \mathrm{PSh}(X) \coloneqq \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}} \to \mathrm{Sh}(X).$$

We want to generalize sheafification we did for the category of open sets on a topological space to sites. We should be careful, and all the steps in the process should be carefully motivated. Although we will still work withing the category of open sets corresponding to a topological space, we will do it in the more general language. We will then state this construction as a theorem without making things too formal.

2.3.1 | Sheafification on Topological Spaces

2.3.1.1 | Etale Spaces

2.3.2 | Sheafification on Sites

2.3.2.1 | Filtrant (Co)Limits

Filtrant categories or filtered categories generalize the notion of a directed set, and hence the category of open sets of a topological space. This allows us to talk about restriction or corestriction on these general categories. Intuitively, a filtrant category should be a category such that for every two objects, there exists a larger object containing both. Largeness is to be expressed in terms of morphisms.

Categorically, this can be formalized by saying for every two objects V and W, there exists an object U together with morphisms from the two objects.

$$V \to U \leftarrow W. \tag{container object}$$

In case of directed sets, the maps described above are inclusion maps. In the categorical counterpart, there can be multiple maps $V \rightrightarrows W$ describing different ways in which W is 'larger' than V. Intuitively, we want there to be an object that's larger in a common way. This means that for any two morphisms $V \rightrightarrows W$ there exists an morphism $h : W \to U$, such that $h \circ f = h \circ g$. Visualised by the diagram,

$$\begin{tikzcd} V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, dashed] & W \arrow[d, dashed, "h"] \\ & U \end{tikzcd} \tag{directedness}$$

This does not have to be coequaliser. A filtrant category $\mathcal{C}$ is a non-empty category that has container object and directedness properties. It's cofiltrant if $\mathcal{C}^{\mathrm{op}}$ is filtrant.

The category of open sets of a topological space is an example of a filtrant category. Given any two open sets, there exists a larger open set containing them, since between any two objects, there can only be one map, the directedness condition is trivially satisfied. The topological space X itself will be the largest open set.

Lemma 2.3.1. *$\mathcal{C}$ is a small filtrant category iff for any finite diagram, there exists a cocone.*

Proof

Suppose every finite diagram has a cocone, then we have to show that it will be non-empty and satisfy container object and directedness. For non-emptyness, considering the empty diagram combined with the existence of a cocone gives us an object, showing that the category is non-empty. For container object, we can consider a discrete diagram, which consists of two objects V, W, with no morphisms between them, in such a case, cocone corresponds to an object U such that, there are morphisms, $V \to U$ and $W \to U$. So, we get that,

$$V \to U \leftarrow W.$$

Similarly, for directedness consider the diagram, $V \rightrightarrows W$, it should have a cocone with vertex U, is precisely the directedness condition.

Conversely, if the small category $\mathcal{C}$ is filtering, we have to show for every finite diagram there exists a cocone. The proof is by induction on the number of arrows, the container object condition gives for each two objects, a larger object, and directedness guarantees a common larger object for any finite collection of objects.

If there are no arrows at all, then the diagram consists of a discrete collection of objects $\{V_i\}_{i \in N}$, and repeated application of container object gives us a cocone. For the induction step, we have a cocone for a diagram, we have to show that if the diagram is extended by an arrow, there exists a cocone for the new diagram. The existence of such a larger object is guaranteed by container object, directedness guarantees that there exists some other larger object, such that it satisfies the commutative diagram of cocone.

[Make this nicer]

This means that if $\mathcal{C}$ is a small filtrant category, for any finite collection of objects there exist an object that's 'larger' all the objects in the collection. If a category has a final object it will always be filtrant.

2.3.2.2 | The $^+$ Functor

The problem with pre-sheaves is that they might not be separated and even if they are separated, might not patch up, so the first step is to divide up the covers into smaller and smaller refinements, separate it out and manually patch it up. This is the same as bundling up stalks like we did for the case of topological spaces.

Separation

Let $U \in \mathcal{O}(X)$, and

$$\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$$

be a covering of U. By definition of a pre-sheaf we have the association, $U_i \mapsto \mathcal{F}U_i$. The set,

$$\mathrm{Eq}(\mathcal{U}, \mathcal{F}) = \Big\{(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i \;\mid\; f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j}\Big\},$$

consists of all tuples of 'functions' that can be patched up whenever there is an intersection, i.e., they must agree on the 'intersection' $U_i \prod_{i,j} U_j$. So we are throwing out all the functions that cannot be patched up. But now it might happen that things are patchable on refinements but not on bigger open sets. So we must refine.

To motivate refinement, consider the example of a topological space, X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of an open set $U \subset X$. A refinement of $\mathcal{V} = \{V_j\}_{j \in J}$ of the open cover $\mathcal{U}$ is a set of open subsets $V_j \subset X$ which is itself an open cover, and is such that each for each $j \in J$, there exists an $i \in I$ such that $V_j \subset U_i$. Hence there exists a map $f : J \to I$ such that,

$$\begin{tikzcd} V_j \arrow[rr, "f"] \arrow[dr] & & U_{f(j)} \arrow[dl] \\ & U & \end{tikzcd} .$$

Now, by going to refinements, the intersections are smaller, the elements of the pre-sheaves have 'less space' to disagree on. So, the refinements give us a map,

$$\mathrm{Eq}(\mathcal{U}, \mathcal{F}) \to \mathrm{Eq}(\mathcal{V}, \mathcal{F}).$$

By taking limit over such refinements, we include the 'essense' of all functions in the pre-sheaf $\mathcal{F}$ similar to how in case of topological space, the stalks of the pre-sheaf was the same as the stalks of the sheafification. So if $\mathrm{Cov}(U)$ is the category of all coverings of U, with refinements as morphisms, then we have a functor,

$$\begin{aligned} \mathrm{Eq}_{\mathcal{F}}U : \mathrm{Cov}(U)^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{U} &\mapsto \mathrm{Eq}(\mathcal{U}, \mathcal{F}). \end{aligned}$$

Since the category of pre-sheaves **Sets** has limits, we can take the limit of this functor. Now using this, we can associate to each pre-sheaf $\mathcal{F}$, a new pre-sheaf with,

$$\mathcal{F}^+U := \varinjlim_{\mathrm{Cov}(U)} \mathrm{Eq}_{\mathcal{F}}U$$

The construction so far can be stated as follows,

Lemma 2.3.2. *If $\mathcal{F}$ is a pre-sheaf, then $\mathcal{F}^+$ is a separated pre-sheaf.* $\square$

Note here that if $\mathcal{F}$ is a sheaf then the equalisers already exist, i.e., we don't have to remove elements or take restrictions of elements that don't patch up. So in such a case $\mathcal{F}^+ = \mathcal{F}$.

Patch-up

Once we have a separated pre-sheaf, it means that there is only one way it can be patched up. Or the patch up is unique. So now we have to patch up $\mathcal{F}^+$.

use the functor $\mathcal{F}$ to construct a functor on the category of covers of U for any $U \subseteq X$. Firstly we already have for each $U_i \in \mathrm{Cov}(U)$ the association $\mathcal{F}U_i$ coming from the functor $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$.

So, for every inclusion $U_i \subseteq U_j$ we have to construct a new restriction map,

Theorem 2.3.3. *ι_X admits a left-adjoint $^{\mathrm{Sh}}$, i.e.,*

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(\mathcal{F}, \iota_X \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}^{\mathrm{Sh}}, \mathcal{G}).$$

[1]

Note that we didn't use any properties of the category $\mathcal{A}$ here, we only used the properties of the initial category to add structure.

[1]When we are working with general sites, the cover $\mathrm{Cov}(U)$ is cofiltered i.e., has a notion similar to directed set, and hence the adjoint functor, sheafification is possible. We will not discuss the filtered category or sheafification for general sites here.

3 | Abelian Sheaves

Many of the properties of the category of pre-sheaves depend on the target category, to say this more abstractly, the properties of the functor category $\mathcal{A}^{\mathcal{B}}$ are very closely related to the properties of the category $\mathcal{A}$. The goal of this chapter is to discuss abelian categories, which have some of the nice properties the category **Sets** has. Many of the interesting pre-sheaves take values in such abelian categories.

The abelian categories have the necessary objects to talk about sequences of morphisms, and exactness of sequences. The goal of this chapter is to study exactness of sequences of morphisms. The obstruction to exactness of a sequence is called cohomology, and the primary tools for the study of such obstruction is the theory of derived categories and derived functors.

3.1 | Abelian Categories

3.2 | Diagram Chasing in Abelian Categories

Diagram chases usually lead to simple connection between distant objects in a complex. In this chapter we discuss some of the diagram chase lemmas. The main goal here is to prove the Salamander lemma due to [6], which provides a more visual understanding of diagram chases and the standard diagram chase lemmas become corollaries.

3.2.1 | Salamander Lemma

3.2.1.1 | Snake Lemma

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \bullet \arrow[r, dashed] \arrow[d, tail] & \bullet \arrow[r, dashed] \arrow[d, tail] & \bullet \arrow[d, tail] \arrow[ddll, dashed, rounded corners] & \\
 & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bullet \arrow[r] \arrow[d, two heads] & \bullet \arrow[r] \arrow[d, two heads] & \bullet \arrow[d, two heads] & \\
 & \bullet \arrow[r, dashed] \arrow[d] & \bullet \arrow[r, dashed] \arrow[d] & \bullet \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 3.2.1. (Snake Lemma) *$\mathcal{A}$ abelian, if in **??**, the solid exact rows and columns are exact then there exists a sequence of dashed arrows which is exact.*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker f_{i-1} \arrow[r, dashed] \arrow[d, tail] & \ker f_i \arrow[r, dashed] \arrow[d, tail] & \ker f_{i+1} \arrow[d, tail] \arrow[ddll, dashed, rounded corners, "\delta"] & \\
 & A^{i-1} \arrow[r, "\partial_A^{i-1}"] \arrow[d, "f_{i-1}"] & A^i \arrow[r, "\partial_A^i"] \arrow[d, "f_i"] & A^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & 0 \\
0 \arrow[r] & B^{i-1} \arrow[r, "\partial_B^{i-1}"] \arrow[d, two heads] & B^i \arrow[r, "\partial_B^{i-1}"] \arrow[d, two heads] & B^{i+1} \arrow[d, two heads] & \\
 & \operatorname{coker} f_{i-1} \arrow[r] \arrow[d] & \operatorname{coker} f_i \arrow[r] \arrow[d] & \operatorname{coker} f_{i+1} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The starting point is a double complex in an abelian category $\mathcal{A}$,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where each $\bullet$ is an object in $\mathcal{A}$. Focusing on what happens around an object A in this double complex, consider the following morphisms to and from the object,

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

3.2.2 | Corollaries of Salamander Lemma

Lemma 3.2.2. (Five Lemma) *Consider two exact sequences $A^\bullet$ and $B^\bullet$,*

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r] \arrow[d, "u_i"] & A^{i+1} \arrow[r] \arrow[d, "u_{i+1}"] & A^{i+2} \arrow[r] \arrow[d, "u_{i+2}"] & A^{i+3} \arrow[r] \arrow[d, "u_{i+3}"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & B^{i+2} \arrow[r] & B^{i+3} \arrow[r] & \cdots
\end{tikzcd}$$

3.2.3 | Category of Complexes

Given an additive category $\mathcal{A}$, a differential object in $\mathcal{A}$ is a sequence,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

where each $A^i \in \mathcal{A}$ and homomorphisms $\partial_i : A^i \to A^{i+1}$ are morphisms in the category $\mathcal{A}$. The morphisms between two such differential objects $A^\bullet$ and $B^\bullet$, $A^\bullet \xrightarrow{u} B^\bullet$ consists of morphisms $u^i : A^i \to B^i$ such that the following commutative diagram commutes,

$$\begin{tikzcd}
A^\bullet \arrow[d, "u"'] & \cdots \arrow[r, "\partial^A_{i-2}"] & A^{i-1} \arrow[r, "\partial^A_{i-1}"] \arrow[d, "u^{i-1}"'] & A^i \arrow[r, "\partial^A_i"] \arrow[d, "u^i"'] & A^{i+1} \arrow[r, "\partial^A_{i+1}"] \arrow[d, "u^{i+1}"'] & \cdots \\
B^\bullet & \cdots \arrow[r, "\partial^B_{i-2}"'] & B^{i-1} \arrow[r, "\partial^B_{i-1}"'] & B^i \arrow[r, "\partial^B_i"'] & B^{i+1} \arrow[r, "\partial^B_{i+1}"'] & \cdots
\end{tikzcd}$$

The set of morphisms $A^\bullet \to B^\bullet$ is denoted by $\mathrm{Hom}(A^\bullet, B^\bullet)$. The category with differential objects as objects and $\mathrm{Hom}(A^\bullet, B^\bullet)$ is the functor category $\mathcal{A}^{\mathbb{Z}}$.

3.2.3.1 | Cohomology Functors

A sequence of morphisms in $\mathcal{A}$,

$$A^\bullet \equiv \cdots \xrightarrow{\partial^A_{i-2}} A^{i-1} \xrightarrow{\partial^A_{i-1}} A^i \xrightarrow{\partial^A_i} A^{i+1} \xrightarrow{\partial^A_{i+1}} \cdots$$

is called complex if for all i,

$$\partial^A_i \circ \partial^A_{i-1} = 0,$$

The category of complexes over an abelian category $\mathcal{A}$ is an abelian subcategory of $\mathcal{A}^{\mathbb{Z}}$. Denoted by $\mathcal{C}(\mathcal{A})$. Since $\mathcal{A}$ is an abelian category every morphism $A^{i-1} \to A^i$ splits as,

$$\ker \partial^{i-1} \longrightarrow A^{i-1} \longrightarrow \Im \partial^{i-1} \longrightarrow A^i \longrightarrow \mathrm{coker}\, \partial^{i-1}$$

The sequence of morphisms can then be written as,

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] & & \ker \partial^A_i \arrow[d] & & \ker \partial^A_{i+1} \arrow[d] & \\
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & \Im \partial^A_{i-1} \arrow[r] & A^i \arrow[r] \arrow[d] & \Im \partial^A_i \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \mathrm{coker}\, \partial^A_{i-1} & & \mathrm{coker}\, \partial^A_{i-1} & & \mathrm{coker}\, \partial^A_i. &
\end{tikzcd}$$

Then by the universal property of ker and coker, we get the following connections,

$$\begin{tikzcd}
 & \ker \partial^A_{i-1} \arrow[d] \arrow[dl, dashed] & & \ker \partial^A_i \arrow[d] \arrow[dl, dashed, "f_i"'] & & \ker \partial^A_{i+1} \arrow[d] \arrow[dl, dashed] & \\
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & \Im \partial^A_{i-1} \arrow[r] & A^i \arrow[r] \arrow[d] & \Im \partial^A_i \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \mathrm{coker}\, \partial^A_{i-1} \arrow[ur, dashed] & & \mathrm{coker}\, \partial^A_{i-1} \arrow[ur, dashed, "c_i"'] & & \mathrm{coker}\, \partial^A_i. \arrow[ur, dashed] &
\end{tikzcd} \qquad \text{(split)}$$

The complex $A^\bullet$ is called exact if the morphisms are such that,

$$\mathrm{Im}\ \partial^A_{i-1} \cong \ker \partial^A_i.$$

Given a complex $A^\bullet$, the cohomology is the 'objects' by which sequence fails to be exact. If the complex $A^\bullet$ is not exact at the node A_i, then the $\ker \partial^A_i$ is 'larger' than $\mathrm{Im}\ \partial^A_{i-1}$. The extra stuff in $\ker \partial_i^A$ that's not in $\Im \partial_{i-1}^A$ is the cokernel of the morphism f_i. So, we have by universal property of kernels, there exists a morphism,

$$\text{Im } \partial_{i-1}^A \xrightarrow{f_i} \ker \partial_i^A \longrightarrow \text{coker } f_i \coloneqq \overline{H}^i(A^\bullet).$$

The extra stuff in $\ker \partial_i^A$ that's not in the image of ∂_{i-1}^A is also the kernel of cokernel $\text{coker } \partial_{i-1}^A$. By universal property of cokernels, there exists a morphism,

$$\underline{H}^i(A^\bullet) = \ker c_i \longrightarrow \text{coker } \partial_{i-1}^A \xrightarrow{c_i} \Im \partial_i^A.$$

This is the object we want and it's this part which is making the complex not exact.

$$\begin{tikzcd}
 & \ker \partial_{i-1}^A \arrow[d] & & \ker \partial_i^A \arrow[d] \arrow[r, dashed] & \overline{H}^i(A^\bullet) & \ker \partial_{i+1}^A \arrow[d, dashed] & \\
\cdots \arrow[ur, dashed, tail] \arrow[r] & A^{i-1} \arrow[r] \arrow[d, dashed] & \Im \partial_{i-1}^A \arrow[r] \arrow[ur, dashed, tail, "f_i"] & A^i \arrow[r] \arrow[d] & \Im \partial_i^A \arrow[r] \arrow[ur, dashed, tail] & A_{i+1} \arrow[r] \arrow[d] & \cdots \\
 & \text{coker } \partial_{i-1}^A \arrow[ur, dashed] & \underline{H}^i(A^\bullet) \arrow[r, dashed] & \text{coker } \partial_{i-1}^A \arrow[ur, dashed, "c_i"'] & & \text{coker } \partial_i^A. \arrow[ur, dashed] &
\end{tikzcd}$$

In this diagram, the sequences,

$$\ker \partial_i^A \to A^i \to \text{coker } \partial_{i-1}^A$$

need not be exact. It's this non-exactness we are trying to capture. Let the composite morphism $\ker \partial_i^A \to A^i \to \text{coker } \partial_{i-1}^A$ be h^i, the cohomology can be defined as the image of the morphism h^i,

$$\ker \partial_i^A \to H^i(A^\bullet) \coloneqq \Im h^i \to \text{coker } \partial_{i-1}^A$$

So, $H^i(A^\bullet)$ is the replacement of A^i in split that turns it into an exact sequence. The three definitions are equivalent, see [3] for the proof where they give two more equivalent definitions. So, $\Im h_i = \ker c_i = \text{coker } f_i$.

So, $H^i(A^\bullet)$ is well defined and we get an exact sequence of morphisms,

$$\begin{tikzcd}
 & \ker \partial_i^A \arrow[d, dashed] & \\
\text{Im } \partial_{i-1}^A \arrow[ur, dashed, tail, "f_i"'] & H^i(A^\bullet) \arrow[d, dashed] & \Im \partial_i^A \\
 & \text{coker } \partial_{i-1}^A \arrow[ur, dashed, "c_i"] &
\end{tikzcd}$$

The new exact sequence we have is,

$$\begin{tikzcd}
 & \ker \partial_i^A \arrow[d] & & \ker \partial_i^A \arrow[d] & & \ker \partial_{i+1}^A \arrow[d] & \\
\cdots \arrow[ur, bend left] & H^{i-1}(A^\bullet) \arrow[d] & \text{Im } \partial_{i-1}^A \arrow[ur, bend left] & H^i(A^\bullet) \arrow[d] & \Im \partial_i^A \arrow[ur, bend left] & H^{i+1}(A^\bullet) \arrow[d] & \cdots \\
 & \text{coker } \partial_{i-1}^A \arrow[ur, bend right] & & \text{coker } \partial_{i-1}^A \arrow[ur, bend right] & & \text{coker } \partial_i^A \arrow[ur, bend right] &
\end{tikzcd}$$

By replacing the A^is with $H^i(A^\bullet)$ we get an exact sequence. This gives us a collection of functors from $\mathcal{C}(\mathcal{A})$ to $\mathcal{A}$, called the cohomology functors,

$$H^i : C(\mathcal{A}) \to \mathcal{A},\ H^i(A^\bullet) = \ker \partial_i^A / \mathrm{Im}\ \partial_{i-1}^A.$$

A morphism of complexes,

$$\begin{tikzcd}
A^\bullet \arrow[d, dashed, "f"] & \cdots \arrow[r, "\partial_{i-2}^A"] & A^{i-1} \arrow[r, "\partial_{i-1}^A"] \arrow[d, dashed, "f^{i-1}"] & A^i \arrow[r, "\partial_i^A"] \arrow[d, dashed, "f^i"] & A^{i+1} \arrow[r, "\partial_{i+1}^A"] \arrow[d, dashed, "f^{i+1}"] & \cdots \\
B^\bullet & \cdots \arrow[r, "\partial_{i-2}^B"'] & B^{i-1} \arrow[r, "\partial_{i-1}^B"'] & B^i \arrow[r, "\partial_i^B"'] & B^{i+1} \arrow[r, "\partial_{i+1}^B"'] & \cdots
\end{tikzcd}$$

is called a quasi-isomorphism ($\cong_{\mathcal{Q}}$) if the induced morphism at cohomology,

$$H^i(f) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$$

is an isomorphism for all i. We want to treat quasi-isomorphic objects as isomorphic objects. Quasi-isomorphisms manipulate the complexes so that the obstruction to exactness is preserved. So the collection of quasi-isomorphic complexes have same obstructions and could be treated to be the same. For the sake of simplicity, I will stop writing the bullets for the remainder of this note.

3.3 | Indization of Categories

An ind-object in a category $\mathcal{A}$ is a functor $\in \mathbf{Sets}^{\mathcal{A}}$ which is a limit of a system

3.3.1 | Ind Objects

3.4 | Localization of Categories

Localization is the process of adding all the formal inverses to a collection of morphisms. Note that the objects remain the same, but there will be more relation between these objects. Let $\mathcal{Q}$ denote a collection of morphisms in $\mathcal{C}$, the aim of localization is to construct a new category $\mathcal{C}[\mathcal{Q}^{-1}]$ and a functor,

$$\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$$

which sends morphisms belonging to $\mathcal{Q}$ to isomorphisms in $\mathcal{C}[\mathcal{Q}^{-1}]$. This construction being 'universal' i.e., for any other category $\mathcal{D}$ with a functor $\widehat{\mathcal{L}_{\mathcal{Q}}} : \mathcal{C} \to \mathcal{D}$, which sends morphisms in $\mathcal{Q}$ to isomorphisms gets factored through $\mathcal{L}_{\mathcal{Q}}$, that's to say there exists a functor

$$\mathcal{H} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$$

such that $\widehat{\mathcal{L}_{\mathcal{Q}}} = \mathcal{H} \circ \mathcal{L}_{\mathcal{Q}}$, so we have the following commutative diagram of functors,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widehat{\mathcal{L}_{\mathcal{Q}}}"] \arrow[d, "\mathcal{L}_{\mathcal{Q}}"'] & \mathcal{D} \\
\mathcal{C}[\mathcal{Q}^{-1}] \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}$$

The extra things $\mathcal{D}$ has that's not already in $\mathcal{C}[\mathcal{Q}^{-1}]$ should not related to the localization process, i.e adding inverses to the morphisms in $\mathcal{Q}$. This can be formalized by saying precomposition of functors with $\mathcal{L}_{\mathcal{Q}}$ is an isomorphism of the respective natural transformations. If we have two functors $\mathcal{F}, \mathcal{G} : \mathcal{C}[\mathcal{Q}^{-1}] \to \mathcal{D}$, then,

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}[\mathcal{Q}^{-1}]}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{D}}}(\mathcal{F} \circ \mathcal{L}_{\mathcal{Q}}, \mathcal{G} \circ \mathcal{L}_{\mathcal{Q}}).$$

Or equivalently, the functor,

$$\circ \mathcal{L}_{\mathcal{Q}} : \mathcal{C}[\mathcal{Q}^{-1}]^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}.$$

is fully faithful. $\mathcal{C}[\mathcal{Q}^{-1}]$ together with the functor $\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}[\mathcal{Q}^{-1}]$ is called the localization of $\mathcal{C}$ with $\mathcal{Q}$.

Localization, if it exists is unique upto equivalence of categories. The problem now is to understand what constraint on $\mathcal{Q}$ that guarantees the existence of localization of $\mathcal{C}$ with respect to $\mathcal{Q}$. Every category $\mathcal{D}$ that has the needed inverses to $\mathcal{Q}$ will be such that, there exists a functor from the localization $\mathcal{C}[\mathcal{Q}^{-1}]$ to it,

$$\mathcal{C}[\mathcal{Q}^{-1}] \xrightarrow{\mathcal{H}} \mathcal{D}$$

So the localization is the 'left most' category that satisfies certain properties, contains formal inverses. We should expect some colimit type thing happening here in the category of categories.

Sketch of Construction

The goal is to add the 'inverses' and turn it into a category. Let $\mathcal{Q}^{-1}$ be the set in $\mathcal{C}^{\mathrm{op}}$ corresponding to the collection of morphisms $\mathcal{Q}$. Note here that we are assuming the hom-sets are small sets. The new category should include these extra morphisms.

Consider first the graph with objects of $\mathcal{C}$ as vertices, and the arrows of the graph consists of morphisms in $\mathcal{C}$ together with the collection $\mathcal{Q}^{-1}$. So, the new collection of morphisms is given by, $\mathrm{Hom}_{\mathcal{C}} \coprod \mathcal{Q}^{-1}$. With concatenation as composition this is a category, denoted by $\mathcal{FC}[\mathcal{Q}^{-1}]$. The identity morphisms given by the empty path from and to the same vertex. To turn this into a category we need we have to define equivalences that make the compositions $f \circ f^{-1}$ into identities for all $f \in \mathcal{Q}$.

So the localization of $\mathcal{C}$ with $\mathcal{Q}$ is,

$$\mathcal{C}[\mathcal{Q}^{-1}] \coloneqq \mathcal{FC}(\mathcal{Q}^{-1})/\sim$$

where the quotient consists of same objects and morphisms are quotiented by the above equivalence. This quotient is the colimit we needed. $\square$

For $\mathcal{Q}$ with special properties more direct formulas for the hom-sets of $\mathcal{C}[\mathcal{Q}^{-1}]$ can be obtained.

3.5 | Derived Categories

The goal of cohomology is to measure the obstruction to exactness. So, if a morphism of complexes keeps the obstruction to exactness the same, then we have not lost the information about the obstruction by going in between such complexes.

A morphism of complexes,

$$\begin{tikzcd} A^\bullet \arrow[d, dashed, "f"] & \cdots \arrow[r, "\partial^A_{i-2}"] & A^{i-1} \arrow[r, "\partial^A_{i-1}"] \arrow[d, dashed, "f^{i-1}"] & A^i \arrow[r, "\partial^A_i"] \arrow[d, dashed, "f^i"] & A^{i+1} \arrow[r, "\partial^A_{i+1}"] \arrow[d, dashed, "f^{i+1}"] & \cdots \\ B^\bullet & \cdots \arrow[r, "\partial^B_{i-2}"'] & B^{i-1} \arrow[r, "\partial^B_{i-1}"'] & B^i \arrow[r, "\partial^B_i"'] & B^{i+1} \arrow[r, "\partial^B_{i+1}"'] & \cdots \end{tikzcd}$$

is a quasi-isomorphism ($\cong_\mathcal{Q}$) if the induced morphism at cohomology, $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all i. We want to treat quasi-isomorphic objects as isomorphic objects. A derived category is a category in which such quasi-isomorphic complexes are treated as the same object, i.e., quasi-isomorphisms are isomorphisms.

3.5.1 | Localization with Quasi-Isomorphisms

The localization of the category of complexes with the collection of quasi-isomorphisms is called the derived category.

$$\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$$

where $\mathcal{Q}$ is the collection of quasi-isomorphisms. Our goal is to obtain a more direct formula for localization of the category of complexes with quasi-isomorphisms. We want to get an explicit description of localization for the case of quasi-isomorphisms. This can be done with the special properties the collection of quasi-isomorphisms comes equipped with. Composition of morphisms that preserve cohomology also preserve cohomology, so composition of quasi-isomorphisms is also a quasi-isomorphism. Identity morphisms on the complexes preserve cohomology, so identity morphism should be a quasi-isomorphism.

Let $f : A \to B$ be a morphism, and suppose $q : B \to C$ is a quasi-isomorphism, then it's invertible in the localized category. So, $\mathcal{L}(q)$ is an isomorphism, so we could just invert and get a map $A \to C$.

$$\begin{tikzcd} & \mathcal{L}(C) \arrow[d, "\cong"] \\ \mathcal{L}(A) \arrow[r] \arrow[ur, dashed] & \mathcal{L}(B). \end{tikzcd}$$

So, in the original category, there should exist a quasi-isomorphism, $A \xrightarrow{\cong_\mathcal{Q}} D$ such that the following diagram commutes,

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed, "\cong_\mathcal{Q}"] & C \arrow[d, "\cong_\mathcal{Q}"] \\ A \arrow[r] & B. \end{tikzcd} \tag{extension}$$

Similarly for the arrows reversed, for the same reason. Suppose we have two morphisms, $f, g : B \to C$ and a quasi-isomorphism $\tau : C \to D$ such that $\tau \circ f = \tau \circ g$ then it means that f and g manipulate the information about cohomology similarly. So, what τ did was rearrage the relevant information to make the two morphisms equal. It should also be possible to rearrange this information before manipulation by f and g. So, there should exist a quasi-isomorphism $\sigma : A \to B$ such that $f \circ \sigma = g \circ \sigma$.

$$A \overset{\cong_{\mathcal{Q}}}{\dashrightarrow} B \rightrightarrows C \overset{\cong_{\mathcal{Q}}}{\dashrightarrow} D. \tag{symmetry}$$

A collection of morphisms $\mathcal{Q} \subset \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}$ is called a right multiplicative system if it contains every isomorphism in $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}$, closed under composition, and satisfies extension and symmetry.

The explicit construction of the localization of a right multiplicative system is as follows, $\mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$ consists of the same objects as $\mathcal{C}(\mathcal{A})$. For morphisms, we should think of objects that are quasi-isomorphic as the same object. So, the morphisms in the derived category will be represented by pairs of morphisms in the original category of complexes, visualised by the diagram,

$$\begin{tikzcd} \mathcal{L}(C) \arrow[r, "f"] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd}$$

The important part is the transformation described by the map f. Consider two morphisms between $\mathcal{L}(A)$ and $\mathcal{L}(B)$,

$$\begin{tikzcd} \mathcal{L}(C) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}(D) \arrow[r] \arrow[d, "\cong"] & \mathcal{L}(B) \\ \mathcal{L}(A) & \end{tikzcd}$$

Qualitatively, these two morphisms represent the same morphism if they transform same quasi-isomorphic information. So the non-quasi-isomorphic maps are the important parts. Two morphisms are equivalent if their non-quasi-isomorphic representative factor through the same morphism. Visualised by the diagram,

$$\begin{tikzcd} & C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[d, dashed] \arrow[dr] & \\ A & E \arrow[l, dashed, "\cong_{\mathcal{Q}}"'] \arrow[r, dashed] & B \\ & D \arrow[ul, "\cong_{\mathcal{Q}}"] \arrow[u, dashed] \arrow[ur] & \end{tikzcd} \tag{derived equivalence}$$

By the symmetry of the diagram this is clearly an equivalence relation.

For the composition of morphisms in the derived category, for two morphisms, $\mathcal{L}(A) \to \mathcal{L}(B)$ and $\mathcal{L}(B) \to \mathcal{L}(C)$ the composition of the morphisms corresponds to the following representative in the original category of complexes,

$$\begin{tikzcd} & & E \arrow[dl, dashed, "\cong_{\mathcal{Q}}"'] \arrow[dr, dashed] & & \\ & C \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & & D \arrow[dl, "\cong_{\mathcal{Q}}"'] \arrow[dr] & \\ A & & B & & C \end{tikzcd} \tag{derived composition}$$

Note that the above square is possible due to extension property. The category with same objects as the category of complexes, with morphisms described by the equivalence classes of morphisms in the category of complexes as described in derived equivalence, together with composition as described in derived composition is the derived category $\mathcal{D}(\mathcal{A})$.

For proof that this is indeed the localization of $\mathcal{C}(\mathcal{A})$ with $\mathcal{Q}$ see, [1]. What needs to be checked is that any other functor which takes quasi-isomorphisms to isomorphisms must factor through this category.

3.5.2 | Structure of Derived Category

To the category of complexes we added a bunch of new morphisms to obtain the derived category. So we cannot expect these newly added morphisms to have kernels and cokernels. The derived category of an abelian category will not in general be an abelian category. It also doesn't have to be a locally small category as we have added new morphisms to the hom-sets.

Each object $A \in \mathcal{A}$ defines a complex,

$$\underset{\bullet}{A} \equiv \cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

where the A is at the 0th position. So, we have an embedding of the category $\mathcal{A}$ in $\mathcal{C}(\mathcal{A})$ given by,

$$\mathcal{A} \to \mathcal{C}(\mathcal{A}), \quad A \mapsto \underset{\bullet}{A}.$$

By definition of zero objects, there can only be one map 0 to and from any object. This implies that the functor $\mathcal{A} \to \mathcal{C}(\mathcal{A})$ is fully faithful. That's to say,

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(\underset{\bullet}{A}, \underset{\bullet}{B})$$

So, $\mathcal{A}$ can be considered as a full subcategory of $\mathcal{C}(\mathcal{A})$, denote the subcategory by $\iota(\mathcal{A})$.

The main point of doing all the localization stuff was to throw out unnecessary information and focus the study on the information preserved by cohomology functors H^i. The cohomology functors transform quasi-isomorphisms to isomorphisms, by definition of quasi-isomorphism. Going from $\mathcal{C}(\mathcal{A})$ to $\mathcal{D}(\mathcal{A})$, we have added new morphisms, making the objects more 'connected', so more objects will be isomorphic.

A complex $A^\bullet$ is said to be H^0-complex if all the cohomological information is contained in the zeroth position, that is to say, $H^i(A^\bullet) = 0$ for all is except at 0. Denote the subcategory of all H^0-complexes by $H^0(\mathcal{A})$.

Lemma 3.5.1.

$$\mathcal{A} \underset{H^0}{\overset{\mathcal{L}\circ\iota}{\rightleftarrows}} H^0(\mathcal{A}) \rightarrowtail \mathcal{D}(\mathcal{A}).$$

is an equivalence of categories.

Proof

The goal is to identify each H^0-complexe $A^\bullet$ with $H^0(A^\bullet)$. This gives us a functor $H^0 : \mathcal{D}(\mathcal{A}) \to \mathcal{A}$.

3.5.2.1 | Triangulated Categories

3.6 | Derived Functors

Functors from one category to other transform the information about the objects and morphisms between them. The information we are interested in here is regarding the obstruction to exactness, in complexes. For this to make sense, we have to restrict ourselves to functors that take objects from a category where we can talk about complexes, and exactness, to categories where we can talk about complexes and exactness. So, the functors will be assumed to be additive functors between abelian categories.

3.6.1 | via Kan Extensions

Given a functor what we want is a new functor that takes quasi-isomorphisms to isomorphisms. This can be formulated cleanly, via Kan extensions. So the idea is to find a functor that is along the localization functor, so by making it pass through the derived category the quasi-isomorphisms are transformed to isomorphisms and then any functor from the derived category will have to map isomorphisms to isomorphisms, by functoriality. Let

$$\mathcal{L} : \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}),$$

be the localization of $\mathcal{C}(\mathcal{A})$ with $\mathcal{Q}$. Let $\mathcal{F} : \mathcal{C}(\mathcal{A}) \to \mathcal{D}$ be an arbitrary functor. A right derived functor of $\mathcal{F}$ is a functor $R\mathcal{F}$ together with a natural transformation,

$$\begin{tikzcd}
 & \mathcal{D}(\mathcal{A}) \arrow[dr, dashed, "R\mathcal{F}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{C}(\mathcal{A}) \arrow[ur, "\mathcal{L}"] \arrow[rr, "\mathcal{F}"'] & {} & \mathcal{D}
\end{tikzcd}$$

such that every other functor $\mathcal{K} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}$ factors through $R\mathcal{F}$, i.e., there exists a natural transformation from $R\mathcal{F} \Rightarrow \mathcal{K}$.

$$\begin{tikzcd}
 & \mathcal{D}(\mathcal{A}) \arrow[dr, dashed, "R\mathcal{F}"'] \arrow[dr, bend left=50, dashed, "\mathcal{K}"] \arrow[dr, phantom, bend left=20, "\Nearrow"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{C}(\mathcal{A}) \arrow[ur, "\mathcal{L}"] \arrow[rr, "\mathcal{F}"'] & {} & \mathcal{D}
\end{tikzcd}$$

This means that $R\mathcal{F}$ is exactly the left Kan extension of $\mathcal{F}$ along the localization functor $\mathcal{L}$. $R\mathcal{F}$ is called absolute right derived functor if for every functor $\mathcal{H} : \mathcal{D} \to \mathcal{E}$ the right derived functor of the composition $\mathcal{H} \circ \mathcal{F}$ is given by $\mathcal{H} \circ R\mathcal{F}$.

Note that existence of derived functors is not guaranteed here. The starting category that is $\mathcal{C}(\mathcal{A})$ need not be small, and $\mathcal{D}(\mathcal{A})$ need not be locally small. So even though the target category is cocomplete, other requirements of coend formula are not met.

Theorem 3.6.1.

Corollary 3.6.2. (Quillen Adjunction theorem)

3.6.2 | Resolutions

3.6.3 | Spectral Sequences

3.6.4 | Derived Functors as Kan Extensions

4 | The Six Functors

In these notes we study certain adjoint pairs that don't need the language of derived categories. These can be called internal functors or external depending on whether we stay within that category of sheaves over a topological space or we leave the category. These are four of Grothendieck's 'six operations'.

4.1 | Direct and Inverse Image Sheaves

Continuous maps $X \to Y$ gives rise to an adjoint pair of functors $\mathrm{Sh}(X) \rightleftharpoons \mathrm{Sh}(Y)$ called direct image, and inverse image, these are external operations, i.e., we are moving to a different topological space. A topological space X determines a category $\mathrm{PSh}(X)$ of sheaves on X. A continuous map of spaces $f : X \to Y$ will induce functors in both directions, forward and backward, on the associated category of pre-sheaves $\mathrm{PSh}(X)$ and $\mathrm{PSh}(Y)$.

Let $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ be a sheaf on X. Using this sheaf we can construct a sheaf on Y as follows, the sheaf assigns to each open set $U \subset X$ a set $\mathcal{F}U$, we can associate the same set $\mathcal{F}U$ to the open set whose image under f was U. The continuous function f gives us a functor of categories of open sets,

$$f^{-1} : \mathcal{O}(Y)^{\mathrm{op}} \to \mathcal{O}(X)^{\mathrm{op}}.$$

This gives rise to a new sheaf, the induced sheaf $f_*\mathcal{F}$ on Y, defined as the composition of functors,

$$\mathcal{O}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}.$$

Defined for each open set V of Y, i.e., $V \in \mathcal{O}(Y)$ by,

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

This is defined because $f^{-1}V$ is an open set by definition of continuous functions. $f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$ under f. This gives us a functor,

$$f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y).$$

This respects the composition of functions, i.e.,

$$(fg)_* = f_*g_*$$

So, if we set $\mathrm{PSh}(f) = f_*$, then PSh is a functor from the category of topological spaces to the category of sheaves. The direct image functor f_* has a left exact, left adjoint f^*. The construction of this left adjoint is considerably more complex.

4.1.1 | Inverse Image Sheaves via Étale Space

To construct inverse image, we make use of pullbacks. Let $E \to Y$ be a bundle over Y, we can then pullback E along a function $f : X \to Y$ giving us the bundle, $f^*E \to X$,

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

f^* is a functor $f^* : \mathbf{Bund}\ Y \to \mathbf{Bund}\ X$.

Suppose E is an étale bundle over Y, i.e., around each point $e \in E$ has a neighborhood U_e that is homeomorphic to its image $\pi(U_e)$. By definition of pullback, the space f^*E consists of points which we can label using points of X and E by $\langle x, e\rangle$ such that $fx = \pi e$, i.e., the pullback is the universal equalizer of the two maps.

By definition of étale spaces, there is a neighborhood U_e of e that's mapped homeomorphically to its image. Using this image we can construct an open neighborhood of x, by taking $f^{-1}(\pi(U_e))$. This is possible because by definition of pullback we have $fx = \pi e$. So, $\langle f^{-1}(\pi(U_e)), U_e\rangle$ is an open neighborhood of $\langle x, r\rangle$ that is mapped homeomorphically onto $f^{-1}(\pi(U_e))$ of X. Hence f^*E is étale.

If we start with a sheaf $\mathcal{F}$ on Y, a point in the pullback $f^*\mathcal{F}$ is of the form, $\langle x, \mathrm{germ}_{f(x)}\, s\rangle$ where $s \in \mathcal{F}V$ is an element of the sheaf $\mathcal{F}$.

Lemma 4.1.1. *Pullback of an étale space under a continuous map is étale.* □

This gives us a map of sheaves,

$$\mathrm{Sh}(Y) \xrightarrow{\mathcal{E}} \mathbf{Etale}\, Y \xrightarrow{f^*} \mathbf{Etale}\, X \xrightarrow{\Gamma} \mathrm{Sh}(X).$$

Here the first map is the bundling of the stalks of a sheaf to an étale space, and the last map is taking the sheaf of sections of the bundle. The composition gives us a functor of sheaves, which we denote again by f^*,

$$\mathrm{Sh}(Y) \xrightarrow{f^*} \mathrm{Sh}(X).$$

Theorem 4.1.2.

$$\begin{tikzcd} \mathrm{Sh}(X) \arrow[r, shift left, "f_*"] & \mathrm{Sh}(Y) \arrow[l, shift left, "f^*"] \end{tikzcd}, \qquad f^* \dashv f_*.$$

Sketch of Proof

So, what we need to prove is that, $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*F, G) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(F, f_*G)$, which for the sake of brevity we will denote by,

$$\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Sh}(Y)(F, f_*G).$$

Since we have an equivalence between the category of étale spaces and category of sheaves over X, we have that $\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$, where $\mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$ is the collection of morphisms between étale spaces $\mathcal{E}f^*F$ and $\mathcal{E}G$.

Let $K(\mathcal{E}f^*F, \mathcal{E}G)$ be the set of functions $k : \mathcal{E}f^*F \to \mathcal{E}G$ over X.

4.1.2 | via Kan Extension

4.2 | Category of Abelian Sheaves

Sheaves encountered often in geometry are abelian sheaves. Let X be a topological space, and consider continuous functions $f : X \to \mathbb{R}$ on the topological space X. It forms a sheaf,

$$C : \mathcal{O}(X)^{\text{op}} \to \textbf{Sets}.$$

Continuous functions can be added, substracted, multiplied and scaled to obtain new continuous functions. For each open set $U \in \mathcal{O}(X)$, the collection of all continuous functions on it CU forms an $\mathbb{R}$-algebra. Each CU are $\mathbb{R}$-module objects in the category **Sets**.

Since we can encounter lot of sheaves which take values in an abelian category, it's justified to give them special attention.[1] Let $\mathcal{A}$ be an abelian category, an abelian pre-sheaf on a topological space X is a functor $\mathcal{F}$,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathcal{A}, \tag{pre-sheaf}$$

The category of all pre-sheaves on a topological space X with values in $\mathcal{A}$ is denoted by $\text{PSh}(X, \mathcal{A})$. The functor category,

$$\text{PSh}(X, \mathcal{A}) := \mathcal{A}^{\mathcal{O}(X)^{\text{op}}}.$$

admits small limits since the abelian category $\mathcal{A}$ admits small limits , small colimits, and the small filtered limits are exact and limits in functor category are computed pointwise. Once we have small limits we can start doing all sorts of things like take pullbacks, pushforward, products, coproducts, etc.

A pre-sheaf is a sheaf if the local associations $\mathcal{F}U_i$ are restrictions of global association. So, given an open covering $U = \bigcup_{i \in I} U_i$, if $f_i \in \mathcal{F}U_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it should mean that there exists a section $f \in \mathcal{F}U$ such that $f_i = f|_{U_i}$. The maps $f_i \in \mathcal{F}U_i$ and $f_j \in \mathcal{F}U_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from $\mathcal{F}U$ to $\prod_i \mathcal{F}U_i$ such that $pe = qe$. $\mathcal{F}U \to \prod_i \mathcal{F}U_i$

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \prod_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{F}(U_i \cap U_j). \end{tikzcd} \tag{collation}$$

This is the collation property. For general sites, the intersection $\cap$ will be replaced by the fibered product $\prod_{i,j}$ and the open covers are replaced by covers on sites, injective maps replaced by monic maps and so on. We are interested in exploiting the target category now, i.e., $\mathcal{A}$.

[1] Abelian categories are categories where we can do homological algebra, i.e., kernels, cokernels, images, coimages, direct sums, products, etc exist. The important result we need about them is that if $\mathcal{A}$ is an abelian category then so is the functor category $\mathcal{A}^{\mathcal{C}}$ for any category $\mathcal{C}$. For a discussion on abelian categories see [1].

4.2.1 | Internal $\mathcal{H}om$

The operations of interest to us now are within a category of pre-sheaves $\mathrm{PSh}(X, \mathcal{A})$ or sheaves $\mathrm{Sh}(X, \mathcal{A})$ on a given topological space X or site. Since we are not going to use or do anything special with the underlying topological space or site, we will for the sake of simplicity assume it to be a topological space X.

Consider two pre-sheaves, $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(X, \mathcal{A})$, for any $U \subset X$, consider the new pre-sheaves, the restructions, $\mathcal{F}|_U, \mathcal{G}|_U \in \mathrm{PSh}(U, \mathcal{A})$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{tikzcd} U \arrow[d, "|_V"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_V)"'] & \mathcal{G}U \arrow[d, "G(|_V)"] \\ V & \mathcal{F}V \arrow[r, "\kappa_V"'] & \mathcal{G}V. \end{tikzcd}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom, denoted by,

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(X, \mathcal{A}).$$

Sometimes also written as $\mathcal{G}^{\mathcal{F}}$. We will now show that the collation property holds for the $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, and hence $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, i.e., for any cover $\{U_i\}$ of U, the following is an exact sequence,

$$0 \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{G})U \overset{e}{\dashrightarrow} \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i \textstyle\prod_{i,j} U_j).$$

To show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{F}, \mathcal{G})U$ and at $\prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 4.2.1. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then so is $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i \in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})U$, and $U_i \subset U$, we have,

$$\begin{tikzcd} U \arrow[d, "|_{U_i}"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_{U_i})"'] & \mathcal{G}U \arrow[d, "G(|_{U_i})"] \\ U_i & \mathcal{F}U_i \arrow[r, "\kappa_{U_i}"'] & \mathcal{G}U_i. \end{tikzcd}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{F}(U_i)$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \prod V} = 0.$$

Now, $\{W \prod U_i\}$ is a cover of W, and $\mathcal{G}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i\in I} \in \prod_i \mathcal{H}om(\mathcal{F},\mathcal{G})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \prod_{i,j} U_j} = \kappa_j|_{U_i \prod_{i,j} U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{G}$ is a sheaf to patch these natural transformations.

Since $\mathcal{G}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{F}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

here the first map comes from the natural transformation, $\mathcal{F}V \ni f \mapsto \kappa_i(f|_{V \prod U_i})$. Since $\mathcal{G}$ is a sheaf, this must uniquely factor through $\mathcal{G}V$, by definition of equaliser. Hence, we have,

$$\mathcal{F}V \xrightarrow{\exists !} \mathcal{G}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{F},\mathcal{G})(U)$ defines the patched up element that equalizes the diagram. $\square$

Note here that $\mathcal{F}$ need not be a sheaf, the above proposition also holds if $\mathcal{F}$ is a pre-sheaf, and $\mathcal{G}$ is a sheaf.

4.2.1.1 | Hom-Tensor Adjointness

Note here that the natural transformations κ respect the abelian structure. The exponentiation category $\mathcal{G}^{\mathcal{F}}$ has more structure than the standard exponentiation in **Sets**. Let $\mathcal{R}$ be a sheaf of commutative rings.

$$\begin{aligned} \mathcal{R} \;:\; \mathcal{O}(X)^{\text{op}} &\to \text{ CRings} \\ U &\mapsto \mathcal{R}U. \end{aligned}$$

Then we can consider the pre-sheaves which take values in the category ${}_{\mathcal{R}}\text{Mod}$ of $\mathcal{R}$-modules, i.e., U gets mapped to $\mathcal{R}U$-modules. We will denote such pre-sheaves by $\text{PSh}(X,\mathcal{R})$. It naturally inherits a tensor product from the module structure. So, for two pre-sheaves $\mathcal{F}$ and $\mathcal{G}$, we can construct the tensor product pre-sheaf,

$$\begin{aligned} \mathcal{F}\widehat{\otimes}_{\mathcal{R}}\mathcal{G} \;:\; \mathcal{O}(X)^{\text{op}} &\to {}_{\mathcal{R}}\text{Mod} \\ U &\mapsto \mathcal{F}U \otimes_{\mathcal{R}U} \mathcal{G}U \end{aligned}$$

We will denote the sheafification of this pre-sheaf by,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} \in \text{Sh}(X,\mathcal{R}).$$

This is a bifunctor,

$$\cdot \otimes_{\mathcal{R}} \cdot : \text{ Sh}(X,\mathcal{R}) \times \text{Sh}(X,\mathcal{R}) \to \text{Sh}(X,\mathcal{R}).$$

4.2.2 | Tensor Product as a Coend

4.3 | Verdier Duality

5 | Stacks & Descent Theory

Stacks are sheaves with values in a category. Pre-sheaves associate to each open set of a topological space or a site, an object of a category such as the category of sets. Stacks associate to a category a collection of categories. The functor of interest to us is of the form,

$$\mathcal{H} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Cat}.$$

The goal is to develop tools to study moduli problems. A typical moduli problem in geometry is to study some geometric objects associated to some geometric object, say bundles on a manifold. So, such problems correspond to functors,

$$\mathcal{F} : \mathbf{Man} \to \mathbf{Sets}$$

which assigns to each manifold $\mathcal{M}$ the 'set of geometric objects on the manifold $\mathcal{M}$'. This is called a moduli functor. A 'moduli space' corresponding to the moduli functor $\mathcal{F}$ is a manifold $\mathcal{M}_{\mathcal{F}}$ which contains information about all such geometric objects on all manifolds.

Suppose such a moduli space $\mathcal{M}_{\mathcal{F}}$ exists for $\mathcal{F}$, and suppose we have some set T of 'geometric objects' on a manifold $\mathcal{M}$, then, for each element t of this set T, we should have maps of the form,

$$\pi_t : t \to \mathcal{M}$$

Like projection. Now if a moduli space exists information about all such maps should be contained in the moduli space S. So, we should have a map from this collection of maps $\pi_t \to S$

So, this should mean (not sure why it must only be this way) we have something like,

t –>M | V S

Fix a category $\mathcal{C}$, the objects of $\mathcal{C}$ are to be thought of as open sets. We are interested in studying categories over $\mathcal{C}$, that is, categories equipped with 'projection' functors to $\mathcal{C}$.

5.1 | Fibered Categories

5.1.1 | Categories Fibered in Groupoids

5.1.2 | Categories Fibered in Sets

5.1.3 | Equivalence of Fibered Categories

5.2 | Stacks

5.3 | Descent Theory

Part II

Differential Geometry & Microlocalization

6 | Tangency & Lie Derivative

The goal here is to define topological spaces on which we can do calculus. We make them locally look like Euclidean spaces and import calculus from the them. I will follow Ramanan's approach to differential geometry via sheaves, as he develops in the book [2].

6.1 | Sheaf of Differentiable Functions

Our starting point is a topological space $\mathcal{M}$ that's Hausdorff and admits a countable base for the topology. This condition is to make sure there are no pathological spaces we should be worried about. The Hausdorff condition makes the points distinguishable by the topology itself. The countable basis allows us to do analysis. On this topological space we want a differentiable structure, i.e., the structure that allows us to do calculus.

The differentiable structure allows us to define differentiable functions. We expect differentiable functions to have some form of local nature, similar to continuous functions. The notion of a sheaf axiomatizes this 'local nature'.

6.1.1 | Sheaves

Given a topological space $\mathcal{M}$, a sheaf is a way of describing a class of objects on $\mathcal{M}$ that have a local nature. To motivate the definition, consider the set of continuous functions on the space $\mathcal{M}$. Denote by CU the set of real-valued continuous functions on U. Then every function, $f \in CU$ has the following local properties,

If $V \subset U$ then f restricted to V is a continuous map, $f|_V : V \to \mathbb{R}$. The map, $f \mapsto f|_V$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_V)|_W = f|_W.$$

This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(\mathcal{M})$ are open sets of $\mathcal{M}$ and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(\mathcal{M})^{\mathrm{op}}$ is the dual category of $\mathcal{O}(\mathcal{M})$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(\mathcal{M})^{\mathrm{op}}$ we get restriction morphism in **Sets**, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_V$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

where morphisms in $\mathcal{O}(\mathcal{M})$ are inclusion maps and **Sets** has a class of morphisms called restriction maps $|_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that, $|_{VW} \circ |_{UV} = |_{UW}$.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions. Let $U = \bigcup_{i\in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i\cap U_j} = f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i\in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i\cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i\cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i\cap U_j} = f|_{U_i}|_{U_i\cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$CU \overset{e}{\dashrightarrow} \prod_i CU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

A sheaf of sets $\mathcal{F}$ on a topological space $\mathcal{M}$ is a functor,

$$\mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i\in I} U_i$ of an open set U of $\mathcal{M}$ yields an equalizer diagram.

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

We start with what we expect from the 'differentiable' functions. The differentiable functions are continuous functions and hence satisfy the locality requirements and should form a sheaf. The sheaf of 'differentiable functions' is our starting point.

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}.$$

Since each differentiable function is expected to be a continuous function as well we have, $\mathcal{A}^{\mathcal{M}}(U) \subseteq C^{\mathcal{M}}(U)$, where $C^{\mathcal{M}}$ is the sheaf of continuous functions, $C^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}$, on $\mathcal{M}$ i.e., $\mathcal{A}^{\mathcal{M}}$ is a subsheaf of $C^{\mathcal{M}}$.

6.1.1.1 | Differentiable Manifolds

Let $\mathcal{F}_n$ be the sheaf of differentiable functions on the Euclidean space $\mathbb{R}^n$. The 'locally looks like Euclidean space' means that the sheaf $\mathcal{A}^{\mathcal{M}}$ locally looks like differentiable functions over a Euclidean space.

A differentiable manifold is a Hausdorff, second countable topological space $\mathcal{M}$ together with a sheaf,

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

of subalgebras of $C^{\mathcal{M}}$ such that for any $x \in \mathcal{M}$ there is an open neighborhood $x \in U$ with a homeomorphism $U \cong_{\varphi} V \subseteq \mathbb{R}^n$, such that

$$(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{F}_n(V),$$

where $(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{A}^{\mathcal{M}}(\varphi^{-1}(V))$. This is easier to see in the diagram,

$$V \xrightarrow{\varphi^{-1}} U \xrightarrow{\mathcal{A}^{\mathcal{M}}} \mathcal{A}^{\mathcal{M}}U.$$

The pair $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ is called a differentiable manifold. The homeomorphisms φs are called coordinate charts and $\mathcal{A}^{\mathcal{M}}$ is called the differentiable structure. We will assume that each $\mathcal{A}^{\mathcal{M}}U$ to be an $\mathbb{R}$-algebra. The homeomorphisms transfer the smoothness on euclidean space to the manifold. The sections of $\mathcal{A}^{\mathcal{M}}U$ are called differentiable functions on U, and we can do calculus on them.

Clearly the Euclidean space is a differentiable manifold. For an open set $U \subset \mathcal{M}$ the pair $(U, \mathcal{A}^{\mathcal{M}}|_U)$ is a differentiable manifold. If $\cup_i U_i$ is an open cover of $\mathcal{M}$ then $(U_i, \mathcal{A}^{\mathcal{M}}|_{U_i})$ are open manifolds. Let $U_i \cong_{\varphi_i} V_i$ and U_i and U_j intersect, let $\varphi_i(U_i \cap U_j) = V_{ij}$ and $\varphi_j(U_i \cap U_j) = V_{ji}$ then,

$$V_{ij} \cong_{\varphi_j \circ \varphi_i^{-1}} V_{ji}.$$

So, a collection of differentiable manifolds $(U_i, \mathcal{A}_i)$ can be glued together to form a differentiable manifold if the homeomorphisms $\varphi_j \circ \varphi_i^{-1}$ map the restriction $\mathcal{A}_i|_{U_i \cap U_j}$ to $\mathcal{A}_j|_{U_i \cap U_j}$ i.e., differentiable maps are mapped to differentiable maps. This means that $\varphi_j \circ \varphi_i^{-1}$ is differentiable for every i, j.

$\mathcal{M}$ may be obtained by taking all the open sets U_i and pasting $U_{ij} \subset U_i$ to $U_{ji} \subset U_j$ together by the transition functions.

$$\coprod_{i,j} U_i \cap U_j \overset{p}{\underset{q}{\rightrightarrows}} \coprod_i U_i \xrightarrow{c} \mathcal{M}.$$

The map c sends all the points $x \in U_i$ to the same point $x \in \mathcal{M}$. c is the coequalizer of p and q in the category **Top** of topological spaces. This is parallel to the definition of sheaf.

A continuous map f of a differentiable manifold $\mathcal{M}$ into a differentiable manifold $\mathcal{N}$,

$$f : \mathcal{M} \to \mathcal{N},$$

is said to be differentiable if it locally maps differentiable functions to differentiable functions, i.e., for all $x \in \mathcal{M}$ if g is a differentiable function in a neighborhood U of $f(x)$ then $g \circ f$ is differentiable function on $f^{-1}(U)$. If $g \in \mathcal{A}^{\mathcal{N}}(U)$ then $g \circ f \in \mathcal{A}^{\mathcal{M}}(f^{-1}(U))$. Hence to each differentiable maps there is a homomorphism of the sheaf $\mathcal{A}_{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$ given by the map,

$$g \mapsto g \circ f.$$

This is the map

$$C^{\mathcal{N}} \to f_*(C^{\mathcal{M}}),$$

of sheaves on $\mathcal{N}$ which sends the subsheaf $\mathcal{A}^{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$. Differentiable manifolds together with morphisms like this is called the category of smooth manifolds. f_* is called the structure homomorphism associated to f. A differentiable map $f : \mathcal{M} \to \mathcal{N}$ of differentiable manifolds is called a diffeomorphism if there is a differentiable inverse.

6.1.2 | Stalks, Étale Spaces & Sheafification

What we want to study is the behavior of a function in a neighborhood of a point. The starting point is the notion of direct limit. A directed system within a category $\mathcal{C}$ is a set of objects $\{C_i\}_{i\in I}$, where I has a preorder $\leq$, together with morphisms, $f_{ij} : C_i \to C_j$ such that $f_{ii} = \mathbb{1}_{C_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$.

A direct limit of a directed system in a category $\mathcal{C}$ is an object C together with morphisms $\varphi_i : C_i \to C$ with the universal property described by the following diagram,

$$\begin{tikzcd}
C_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr] & & C_j \arrow[dl, "\varphi_j"'] \arrow[ddl] \\
& C \arrow[d, dashed] & \\
& L &
\end{tikzcd}$$

All the categories of interest to us (abelian categories), such as the category of modules over some ring possess direct limits also called colimit or inductive limit. We will not prove this fact in this part. The direct limit as above will be denoted,

$$C = \varinjlim_{i\in I} C_i$$

Inclusion is a preorder on the collection of open sets given by

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For a pre-sheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in **Sets** given by $\{\mathcal{F}U\}_{U\in\mathcal{D}}$. We will focus on this particular directed system.

Definition 6.1.1. The stalk $\mathcal{F}_x$ of a pre-sheaf $\mathcal{F}$ at x is the direct limit of the directed system $\{\mathcal{F}U_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}U.$$

Stalks are functors,

$$\begin{aligned} \mathrm{Stalk}_x : \mathrm{PSh}(X) &\to \mathbf{Sets} \\ \mathcal{F} &\mapsto \mathcal{F}_x. \end{aligned}$$

The elements of $\mathcal{F}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{F}U$ then we denote it by $\mathrm{germ}_x f$. $\mathrm{germ}_x : \mathcal{F}U \to \mathcal{F}_x$, is a homomorphism of the respective category for each U.

If $f, g \in \mathcal{F}U$ such that $\mathrm{germ}_x f = \mathrm{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset U$ such that $f|_{U_x} = g|_{U_x}$. The neighborhoods U_x is an open cover of U and if $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ is a sheaf then,

$$\mathcal{F}U \to \prod_{x\in U} \mathcal{F}U_x,$$

is an injective map and hence we have $f = g$ on U. 'Bundle' the various sets $\mathcal{F}_x$ into a disjoint union,

$$\mathcal{E}\mathcal{F} = \coprod_x \mathcal{F}_x,$$

and define the map, $\pi : \mathcal{E}\mathcal{F} \to X$ that sends each $\mathrm{germ}_x f$ to the point x. Each $f \in \mathcal{F}U$ determines a function $\hat{f} : U \to \mathcal{E}\mathcal{F}$ given by,

$$\hat{f} : x \mapsto \mathrm{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{F}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{E}\mathcal{F}$. This topology makes both π and $\hat{f}$ continuous by construction. Each point $\mathrm{germ}_x f$ in $\mathcal{E}\mathcal{F}$ has an open neighborhood $\hat{f}(U)$. π restricted to $\hat{f} : U \to \hat{f}(U)$, is a homeomorphism. The space $\mathcal{E}\mathcal{F}$ together with the topology just defined is called the étale space of $\mathcal{F}$.

So we get a functor

$$\mathcal{E} : \mathrm{PSh}(X) \to \mathbf{Top},$$

which assigns to each pre-sheaf $\mathcal{F}$ of X a topological space $\mathcal{E}\mathcal{F}$. $\pi : \mathcal{E}\mathcal{F} \to X$ is a bundle. For a given pre-sheaf $\mathcal{F}$, consider the collection of sections of the bundle $\mathcal{E}\mathcal{F}$, denoted $\Gamma\mathcal{E}\mathcal{F}$. A section is a continuous map $\hat{s} : X \to \mathcal{E}\mathcal{F}$ such that $\pi \circ \hat{s} = Id$.

Note that a bundle over an object X in a category $\mathcal{C}$ is simply an object E of $\mathcal{C}$ equipped with a morphism p in $\mathcal{C}$ from E to X.

$$p : E \to X.$$

In our case, the category $\mathcal{C}$ is the category of topological spaces **Top**.

The collection of sections is a pre-sheaf over X because, it assigns to each open subset $U \subset X$ the corresponding set of sections over U and we have the obvious restriction map, i.e., restriction of the continuous map to the smaller domain. It's also a sheaf because s_i are sections of U_i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section s defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{E}\mathcal{F}$ is a sheaf over X.

$$\Gamma\mathcal{E}\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

that assigns to each $U \in \mathcal{O}(X)$ the set $\Gamma\mathcal{E}\mathcal{F}(U) = \coprod_{x \in U} \mathcal{F}_x$. For each open subset $U \subset X$ there is a function,

$$\eta_U : \mathcal{F}U \to \Gamma\mathcal{E}\mathcal{F}(U),$$
$$f \mapsto \hat{f}.$$

The natural transformation of functors, $\eta : \mathcal{F} \mapsto \Gamma\mathcal{E}\mathcal{F}$, maps pre-sheaves to sheaves. It's called sheafification of $\mathcal{E}$. We will denote the sheafification $\Gamma\mathcal{E}\mathcal{F}$ of $\mathcal{F}$, by $\mathcal{F}^{\mathrm{Sh}}$.

Theorem 6.1.1. *If the pre-sheaf $\mathcal{F}$ is a sheaf, then η is an isomorphism. $\mathcal{F} \cong \mathcal{F}^{\mathrm{Sh}}$.*

Sketch of proof

The injectivity part is simple, we have to show $\hat{f} = \hat{g}$ implies $f = g$. This is true because if $\hat{f} = \hat{g}$ then $\mathrm{germ}_x f = \mathrm{germ}_x g$ for every $x \in U$. So for each $x \in U$ we have a neighborhood

U_x for which $f|_{U_x} = g|_{U_x}$. Since $\mathcal{F}$ is a sheaf the collation property implies the uniqueness and we have $f = g$.

For surjectivity, we have to construct a function $f \in \mathcal{F}U$ for every continuous section h of $\mathcal{EF}$. Since h is a section, we have for each $x \in U$ a germ $\mathrm{germ}_x f_x \in \mathcal{EF}$ such that,

$$h(x) = \mathrm{germ}_x f_x,$$

where $f_x \in \mathcal{F}U_x$. Now since h is continuous and $\hat{f}_x(U_x)$ is an open set, so there must exist open set $V_x \subset U_x$ such that $h(V_x) \subset \hat{f}_x(U_x)$ i.e., $h = \hat{f}_x$ on V_x. Now we have to verify these functions agree on intersections. This is true because they give rise to the same germs. Then by collation property there exists a function f such that $f|_{V_x} = f_x$. $\square$

Note that the above proof also establishes an isomorphism between $\mathcal{F}_x$ and $\mathcal{F}_x^{\mathrm{Sh}}$ for all pre-sheaves. The stalkwise isomorphism holds for pre-sheaves. Sheaves are exactly the pre-sheaves that tie its stalks into a bundle. This stalkwise isomorphisms also guarantees that the sheafification is a universal solution i.e., $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, then it factors through $\mathcal{F}^{\mathrm{Sh}}$, this follows from our construction, where our starting point was stalks.

The identification of sheaves with the sheaves of sections of a bundle suggests that a sheaf $\mathcal{F}$ on X can be replaced by the corresponding bundle $\pi : \mathcal{EF} \to X$, and that this bundle is always a local homeomorphism. In this section, we show that the opposite is also true. Every 'étale bundle' can be interpreted as a sheaf.

Definition 6.1.2. A bundle $\pi : E \to X$ is said to be étale if π is a local homeomorphism i.e., to each $e \in \mathcal{E}$ there exists an open set $e \in V$ such that $\pi(V) \subset X$ is open and $\pi|_V$ is a homeomorphism.

Étale spaces of a pre-sheaf over X is clearly an étale bundle. The projection $\pi : X \times \mathbb{R} \to X$ is not a étale map because open sets in $X \times \mathbb{R}$ are of type $U \times V$ and this can never be homeomorphic to an open neighborhood of X. Similarly, vector bundles are not étale. Note that the definition of étale space is different from that of covering space, a covering space is a map $p : C \to X$ such that each point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ can be written as the disjoint union of homeomorphic open sets of C. Étale spaces generalize covering spaces. Every covering space is an étale space. Both étale spaces and covering spaces of topological manifolds are of same dimension as the base space.

A morphism between two bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ is a map φ_{12} such that the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi_{12}"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

The collection of all bundles over X with the above notion of morphism is a category. Denote by **Bund** X the category of all bundles over X. Denote by **Etale** X the collection of all étale bundles over X. **Etale** X is a full subcategory of **Bund** X.

In the previous subsection, we associated to each sheaf $\mathcal{F}$ a bundle $\mathcal{EF}$

$$\mathcal{E} : \mathcal{F} \mapsto \mathcal{EF},$$

and the sheaf of sections of this bundle $\Gamma\mathcal{E}\mathcal{F} = \mathcal{F}^{\text{Sh}}$ was identified with the sheaf itself. Now we are interested in is associating to each étale bundle $\mathcal{E}$ over X a sheaf. If $p : Y \to X$ is a bundle then $\Gamma : Y \to \Gamma Y$, maps the bundle Y to the sheaf of sections of Y. Associate to this sheaf the corresponding étale space $\mathcal{E}\Gamma Y$.

Theorem 6.1.2. *For any space X we have an equivalence of categories,*

$$\text{Sh}(X) \rightleftarrows \textbf{Etale}\, X \rightarrowtail \textbf{Bund}\, X$$

Sketch of Proof

Our aim is now to define a natural transformation of bundles, $\epsilon : \mathcal{E}\Gamma Y \mapsto Y$, and show that if the bundle $p : Y \to X$ is étale then ϵ is an isomorphism.

The étale space $\mathcal{E}\Gamma Y$ consists of elements of the form $\hat{s}(x)$ for some point $x \in X$ and some section $s : U \to Y$. Define ϵ as follows,

$$\epsilon(\hat{s}(x)) = s(x).$$

Note that this definition is independent of the choice of s because if t is some other representative of the same germ $\hat{s}(x)$ at x then $s = t$ in some neighborhood, so it would mean $s(x) = t(x)$. When the bundle is étale we need to show there exists an inverse to ϵ. Suppose $p : Y \to X$ is étale, to each point $y \in Y$ with $p(y) = x$ there is a neighborhood U of x and a section $s : U \to Y$ such that $s(x) = y$. Define the inverse θ to ϵ as,

$$\theta : y \mapsto \hat{s}(x).$$

This is well defined and is the inverse of ϵ. $\square$

6.2 | Tangent and Cotangent Bundles

What we want to do is give a linear approximation of a manifold at each point. In order to do this, we use curves passing through the point, and linearize them, and then study them.

Around each point $x \in \mathcal{M}$, consider all the smooth functions $f \in \mathcal{A}^{\mathcal{M}}U$, i.e., $f : U \to \mathbb{R}$ defined in some open neighborhood U of $x \in \mathcal{M}$. For each smooth path $h : \mathbb{R} \to U$ which passes through x with $h(0) = x$ we can define a smooth map,

$$f \circ h : \mathbb{R} \to \mathbb{R},$$

which has a first derivative at 0. This gives us a pairing,

$$\langle f, h\rangle_x = \left.\frac{d(f \circ h(t))}{dt}\right|_{t=0}, \qquad \text{(tangency pairing)}$$

To remove redundant information, we define the equivalences $f \equiv f'$ at x if $\langle f, h\rangle_x = \langle f', h\rangle_x$ for all h and $h \equiv h'$ at x if $\langle f, h\rangle_x = \langle f, h'\rangle_x$ for all f. Under addition and scalar multiplication of functions, this set of all equivalence classes of functions f forms a real vector space, denoted T^x. Each function f in the neighborhood of x determines a vector $[f]$.

The sheaf of differentiable functions has more algebraic structure. It's a sheaf of algebras over $\mathbb{R}$ or the sheaf of module over the ring $\mathbb{R}$. The category of modules over some ring possess direct limits. Inclusion is a preorder on the collection of open sets given by,

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For the pre-sheaf $\mathcal{A}^{\mathcal{M}} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Sets}$, we get a directed system in **Sets** given by $\{\mathcal{A}^{\mathcal{M}}U\}_{U\in\mathcal{D}}$. We will focus on this particular directed system.

The stalk $\mathcal{A}^{\mathcal{M}}_x$ of a pre-sheaf $\mathcal{A}^{\mathcal{M}}$ at x is the direct limit of the directed system $\{\mathcal{A}^{\mathcal{M}}U_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{A}^{\mathcal{M}}_x = \varinjlim_{x\in U} \mathcal{A}^{\mathcal{M}}U.$$

Stalks are functors,

$$\begin{aligned}\text{Stalk}_x : \text{PSh}(\mathcal{M}) &\to \mathbf{Sets}\\ \mathcal{A}^{\mathcal{M}} &\mapsto \mathcal{A}^{\mathcal{M}}_x.\end{aligned}$$

The elements of $\mathcal{A}^{\mathcal{M}}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{A}^{\mathcal{M}}U$ then we denote it by $\text{germ}_x f$. Note that $\mathcal{A}^{\mathcal{M}}_x$ is an algebra, and in particular a ring. $\text{germ}_x : \mathcal{A}^{\mathcal{M}}U \to \mathcal{A}^{\mathcal{M}}_x$, is a homomorphism of the respective category for each U.

An ideal $\mathcal{I}^{\mathcal{M}}_x \subset \mathcal{A}^{\mathcal{M}}_x$ is a subalgebra such that if $\text{germ}_x f \in \mathcal{I}^{\mathcal{M}}_x$ then $\text{germ}_x f\, \text{germ}_x g \in \mathcal{I}^{\mathcal{M}}_x$ for all $\text{germ}_x g \in \mathcal{A}^{\mathcal{M}}_x$. A proper ideal cannot contain the identity because that would mean the whole algebra is contained in the ideal. For each $\text{germ}_x f \in \mathcal{A}^{\mathcal{M}}_x$, the evaluation map, $\text{germ}_x f \to f(x)$, gives us an algebra homomorphism,

$$ß : \mathcal{A}^{\mathcal{M}}_x \to \mathbb{R}.$$

The kernel of this evaluation map $\mathcal{I}^{\mathcal{M}}_x = \ker(ß)$ is an ideal of $\mathcal{A}^{\mathcal{M}}_x$, consisting of all germs that vanish at x, i.e., $f(x) = 0$ and hence $f(x)g(x) = 0$ for all $g \in \mathcal{A}^{\mathcal{M}}$. Hence,

$$\mathcal{A}^{\mathcal{M}}_x / \mathcal{I}^{\mathcal{M}}_x \cong \mathbb{R}.$$

Evaluation can hence be thought of as taking quotient with the maximal ideal $\mathcal{I}^{\mathcal{M}}_x$. This is also the only maximal ideal, because all other functions have local inverse, because if a function f is non-zero in a small neighborhood it has an inverse, defined by, $\text{germ}_x(1/f)$, and hence this would mean the constant function belongs to the ideal which means it's not proper ideal. So, no other proper ideal can exist.

Going back to the tangency pairing, the set of equivalence classes of paths h are called tangent vectors at x denoted by $T_x\mathcal{M}$. Each smooth path through x has a tangent vector denoted by τ_h. Using the pairing, we get a pairing of the equivalence classes.

$$\langle [f], \tau_h \rangle = \langle f, h \rangle_x.$$

The tangent vector τ_h determines a linear map, $D_{\tau_h} : T^x \to \mathbb{R}$ given by the action,

$$D_{\tau_h}([f]) = \langle [f], \tau_h \rangle.$$

We would like to understand what T^x is. The set $T_x\mathcal{M}$ of all tangent vectors at x spans the set of all linear maps $T^x \to \mathbb{R}$. That's to say, $T_x\mathcal{M}$ is the dual space of T^x, and hence is itself a vector space. We will hence denote T^x by $T^{\vee}_x\mathcal{M}$ or $\text{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R})$.

6.2.1 | Tangent Sheaf

The derivative of a product, in the tangency pairing, the map $D = D_{\tau_h}$ satisfies the following product rule,

$$D(fg) = f(x)D(g)(x) + g(x)D(f)(x). \tag{Leibniz}$$

for all $f, g \in \mathcal{A}^{\mathcal{M}}$. This is called the Leibniz property, and all the maps D with the Leibniz property are called derivations. Conversely, every derivation there is a corresponding curve h such that $D_{\tau_h} = D$.[1] The linear maps,

$$D : \mathcal{A}_x^{\mathcal{M}} \to \mathbb{R},$$

with the above Leibniz property at x are called derivations, denoted by $T_x\mathcal{M}$.

The equivalence relation, $f \equiv f'$ iff $\langle f, h\rangle_x = \langle f', h\rangle_x$, for all h, uniquely determine the map D_{τ_h}. The equivalence classes are called cotangent vectors at x. By plugging in the constant 1, it can be checked that D annihilates constant functions.

$$D(\lambda) = 0 \quad \forall \lambda \in \mathbb{R}.$$

Hence all functions that differ by constant belong to the same equivalence class. Hence, for every $\mathrm{germ}_x\, f \in \mathcal{A}_x^{\mathcal{M}}$, we can consider the functions $\mathrm{germ}_x(f - f(x))$. These functions vanish at x, the action of D on the ideal $\mathcal{I}_x^{\mathcal{M}}$ is sufficient to describe the map D. So, we have a surjection of the ideal $\mathcal{I}_x^{\mathcal{M}}$ to the set of equivalence classes.

$$\mathcal{I}_x^{\mathcal{M}} \twoheadrightarrow T_x^{\vee}\mathcal{M}.$$

Now we have to remove all the redundant information from $\mathcal{I}_x^{\mathcal{M}}$. The kernel of the map is the ideal $(\mathcal{I}_x^{\mathcal{M}})^2 = \{\sum_{i,j} g_i f_j \ : \ f_i, g_j \in \mathcal{I}_x^{\mathcal{M}}\}$. Hence, we can quotient it out of $\mathcal{I}_x^{\mathcal{M}}$ and we have,

$$T_x^{\vee}\mathcal{M} \cong \mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2,$$

or that the equivalence class for the function f only contains the first order information of f. From the definition of a differentiable manifold around each x, there is a neighborhood U, such that,

$$\mathcal{A}^{\mathcal{M}}(\varphi^{-1}(U)) = \mathcal{F}_n(V)$$

for the local chart, φ. Since $\mathcal{F}_n$ consist of smooth functions on $\mathbb{R}^n$, we can describe them in terms of their Taylor expansion. If we denote the local coordinates by x_i, we have,

$$f(y) = f(\varphi(x)) + \sum_{i=1}^{n} \left[\tfrac{\partial f}{\partial x_i}(\varphi(x))\right](\varphi_i(x) - x_i) + \sum_{i,j} \tfrac{1}{2!}\left[\tfrac{\partial^2 f}{\partial x_i \partial x_j}(\varphi(x))\right](\varphi_i(x) - x_i)(\varphi_j(x) - x_j) + \cdots$$

Since the elements should be in $\mathcal{I}_x$, we have, $f(x) = 0$, and since we are quotienting out by $\mathcal{I}_x^2$, the higher order terms will go. The equivalence classes $[\varphi_i(x) - x_i]$ form a basis for the cotangent space. We denote them by dx_i.

The tangent space then is,

$$T_x\mathcal{M} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2, \mathbb{R}).$$

In local coordinates, the dual basis for the equivalence classes $[\varphi_i(x) - x_i]$ will be the equivalence classes $\partial/\partial x_i$. However we want to understand the structure of tangent spaces from a sheaf theoretic perspective.

For any derivation, $D \in T_x\mathcal{M}$, and $h \in \mathcal{A}^{\mathcal{M}}$,

$$h(x)D(fg) = h(x)f(x)D(g) + h(x)g(x)D(f).$$

If $D, D' \in T_x\mathcal{M}$, then their sum $D + D' \in T_x\mathcal{M}$. $hD \in T_x\mathcal{M}$. So, $T_x\mathcal{M}$ is an $\mathcal{A}_x^{\mathcal{M}}$-module. Using these derivations we can define the tangent sheaf.

[1] The idea is to express it in local charts, and this should be of the form $\sum_i h_i \partial/\partial x_i$, and using this define a curve $h : t \mapsto \varphi^{-1}(t(h_i x_i))$. This works.

Define $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ to be the set of all derivations. That is to say $D \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, if for all $f, g \in \mathcal{A}^{\mathcal{M}}U$,

$$D(fg) = fD(g) + gD(f).$$

Such operators $D : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}$ are called first order linear homogeneous differential operators. If $D, D' \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, then we can define a new operator $[D, D']$ defined by,

$$[D, D'](f) = D(D'(f)) - D'(D(f)). \tag{Lie bracket}$$

The geometric meaning of Lie bracket will become clear later. The tangent sheaf is the sheaf,

$$\begin{aligned} \mathcal{T} = \mathcal{D}(\mathcal{A}^{\mathcal{M}}) :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathcal{A}^{\mathcal{M}}}\mathrm{Mod} \\ U &\mapsto \mathcal{D}(\mathcal{A}^{\mathcal{M}}U). \end{aligned}$$

It is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

$\{f_i\}_{i=1}^{n} \mapsto \sum_i f_i \frac{\partial}{\partial x_i}$ is an isomorphism of modules $(\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$ and $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ for a chart (φ, U). Such sheaves of modules are called locally free modules. The tangent sheaf consists of the sections of the tangent bundle where the tangent bundle $T\mathcal{M}$ is the disjoint union,

$$T\mathcal{M} = \coprod_{x \in \mathcal{M}} T_x\mathcal{M}.$$

These sections correspond to vector fields. The stalks of this sheaf consist of germs of vector fields.

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) = \varinjlim_{x \in U} \mathcal{D}(\mathcal{A}^{\mathcal{M}}U).$$

So, $\operatorname{germ}_x D : \mathcal{A}_x^{\mathcal{M}} \to \mathcal{A}_x^{\mathcal{M}}$. The evaluation of the germs at the point x should give us vectors of the tangent space and they do. We evaluate

$$\mathcal{A}_x^{\mathcal{M}} \xrightarrow{\operatorname{germ}_x D} \mathcal{A}_x^{\mathcal{M}} \xrightarrow{\ \text{ß}\ } \mathbb{R}.$$

So, the composition of the derivation with this evaluation map corresponds to a derivation at x which are tangent vectors at x. The evaluation map gave us an isomorphism, $\mathcal{A}_x^{\mathcal{M}}/\mathcal{I}_x^{\mathcal{M}} \cong \mathbb{R}$. For locally free sheaves, for every $x \in \mathcal{M}$, there exists a neighborhood U such that $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) = (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$. So, we have,

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) \cong \mathbb{R}^n.$$

In this sense $\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$ is the space of valuation of the sections at x. This is the same as the tangent space at x.

$$T_x\mathcal{M} = \mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$$

We have a projection from the tangent bundle to the base space,

$$\pi : T\mathcal{M} \to \mathcal{M}$$

which sends $T_x\mathcal{M} \mapsto x$. Since $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$. So, the set $\pi^{-1}(U)$ can be given a differentiable structure of a product. These can be patched up to get a differentiable structure on $T\mathcal{M}$. This topology is Hausdorff and has a countable basis because locally it's a product of Hausdorff spaces with countable basis.[2] Smooth sections of this bundle are called vector fields.

[2]Note that in the case of Etale space, the properties of individual elements of the sheaf are used to get a topology, in the case of tangent bundle we used the properties of the sheaf itself to get a topology. We first quotiented the stalks with the maximal ideal of the ring of functions and then bundled them, and didn't care about the properties of the individual elements for the topology.

6.2.2 | Cotangent Sheaf

Similarly we can consider the cotangent pre-sheaf,

$$\mathcal{C} = \mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2 : \mathcal{O}(\mathcal{M})^{\text{op}} \to {}_{\mathcal{A}^{\mathcal{M}}}\text{Mod}$$
$$U \mapsto \mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$$

where $\mathcal{I}^{\mathcal{M}}U$ is the maximal ideal of $\mathcal{A}^{\mathcal{M}}U$. This might not be a sheaf however. So we might have to sheafify such pre-sheafs. The stalks of this pre-sheaf are given by,

$$\text{Stalk}_x : \mathcal{C} \mapsto \mathcal{C}_x = \mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2.$$

We can now 'bundle' these stalks together,

$$\mathcal{EC} = \coprod_x \mathcal{C}_x,$$

and define the map, $\pi : \mathcal{EC} \to \mathcal{M}$ that sends each $\text{germ}_x f$ to the point x. Each $f \in \mathcal{C}U$ determines a function $\hat{f} : U \to \mathcal{EC}$ given by,

$$\hat{f} : x \mapsto \text{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{EF}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{EF}$. This topology makes both π and $\hat{f}$ continuous by construction.

For the pre-sheaf $\mathcal{C}$, consider the collection of sections of the bundle $\mathcal{EC}$, denoted $\Gamma\mathcal{EC}$, i.e., is a continuous map $\hat{s} : \mathcal{M} \to \mathcal{EC}$ such that $\pi \circ \hat{s} = Id$. This collection of sections is a pre-sheaf over $\mathcal{M}$ because, it assigns to each open subset $U \subset \mathcal{M}$ the corresponding set of sections over U and we have the obvious restriction map, i.e., restriction of the continuous map to the smaller domain. It's also a sheaf because s_i are sections of U_i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section s defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{EC}$ is a sheaf over $\mathcal{M}$.

$$\Gamma\mathcal{EC} : \mathcal{O}(\mathcal{M})^{\text{op}} \to {}_{\mathcal{A}^{\mathcal{M}}}\text{Mod},$$

that assigns to each $U \in \mathcal{O}(\mathcal{M})$ the set $\mathcal{EC}U = \coprod_{x \in U} \mathcal{C}_x$. For each open subset $U \subset \mathcal{M}$ there is a function,

$$\eta_U : \mathcal{C}U \to \Gamma\mathcal{EC}U,$$
$$f \mapsto \hat{f}.$$

This natural transformation of functors maps pre-sheaves to sheaves called sheafification of the pre-sheaf $\mathcal{C}$.

$${}^{\text{Sh}} : \text{PSh} \to \text{Sh}\,.$$

We will denote the sheafification of the pre-sheaf $\mathcal{C}$, $\Gamma\mathcal{EC}$ by $\mathcal{C}^{\text{Sh}}$. For the cotangent pre-sheaf we will call the sheafification as the cotangent sheaf, denoted by $\mathcal{T}^{\vee}$. This is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. It's also locally free, with the local isomorphism,

$$\{f_i\}_{i=1}^n \mapsto \sum_i f_i dx_i$$

of modules $(\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$ and $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ for a chart (φ, U). Smooth sections of the cotangent bundle, or elements of the cotangent sheaf are called differential forms.

6.2.3 | Locally Free Sheaves and Vector Bundles

A sheaf of $\mathcal{A}^{\mathcal{M}}$-modules $\mathcal{D}$ is said to be locally free of rank n if for every $x \in \mathcal{M}$ has a neighborhood U such that,

$$\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}.$$

Locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules in general give rise to vector bundles. Conversely to each vector bundle we can associate the sheaf of differentiable sections of π which is a locally free sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. There is a natural bijection between the sheaves of locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules and vector bundles. Every $\mathcal{A}^{\mathcal{M}}$-linear sheaf homomorphism gives a homomorphism of vector bundles. It's an equivalence of categories.

For a locally free sheaf $\mathcal{D}$, at each point $x \in \mathcal{M}$, we have a neighborhood U such that $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$. This composed with the evaluation gives us,

$$\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x = \mathbb{R}^n.$$

We can bundle the stalks $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x$ together,

$$\mathcal{V}\mathcal{D} = \coprod_{x\in\mathcal{M}} \mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x,$$

together with the natural projection $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x \mapsto x$. Since $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$ and using this identification, the topology and a differentiable structure can be provided to the bundle $\mathcal{V}\mathcal{D}$. $\mathcal{V}\mathcal{D}$ is the vector bundle associated with the locally free sheaf $\mathcal{D}$. This will become important in the future.

6.2.3.1 | Tensor Algebra, Exterior Algebra

We understand linear maps quite well, what we want to do is study multilinear maps using linear algebra. The idea of tensor products is to study multilinear maps as linear maps. Suppose we have a collection of $\mathcal{A}$-modules $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and a multilinear map,

$$\beta : \textstyle\prod_i \mathcal{V}_i \to \mathcal{W},$$

where $\mathcal{W}$ is an $\mathcal{A}$-module. What we want to do is study all such multilinear maps from the collection $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ to $\mathcal{W}$ as linear maps from the 'tensor product' $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ to $\mathcal{W}$ as a linear map. The algebraic tensor product of $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ is an $\mathcal{A}$-module $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ together with a multilinear map,

$$t : \textstyle\prod_{i\in\mathcal{I}}\mathcal{V}_i \to \bigotimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i,$$

such that every other multilinear map from $\prod_{i\in\mathcal{I}} \mathcal{V}_i$ to $\mathcal{W}$ uniquely factors through $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$. This is the universal property of tensor products. This can be expressed in a commutative diagram by,

$$\begin{tikzcd}
\prod_{i\in\mathcal{I}} \mathcal{V}_i \arrow[r, "t"] \arrow[dr, "\beta"'] & \bigotimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i \arrow[d, "\exists!\ \tilde{\beta}"] \\
 & \mathcal{W}
\end{tikzcd}$$

For the construction of the tensor product check wikipedia.

If $\mathcal{E}$ and $\mathcal{F}$ are two locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules corresponding to vector bundles, then we can form the presheaf of $\mathcal{A}^{\mathcal{M}}U$-module, $\mathcal{E}(U) \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}(U)$. Whose stalk at each point x is given by $\mathcal{E}_x \otimes_{\mathcal{A}^{\mathcal{M}}_x} \mathcal{F}_x$.

$$\begin{aligned} \mathcal{E} \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathcal{A}^{\mathcal{M}}}\mathrm{Mod} \\ U &\mapsto \mathcal{E}(U) \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}(U) \end{aligned}$$

Suppose $\mathcal{E}$ and $\mathcal{F}$ be locally free, then around each $x \in \mathcal{M}$ there exist neighborhoods U and V such that $\mathcal{E}(U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus k}$ and $\mathcal{F}(V) \cong (\mathcal{A}^{\mathcal{M}}V)^{\oplus l}$. In particular, on the intersection $U \cap V$,

$$\mathcal{E}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus k}, \quad \mathcal{F}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus l}$$

For the sake of simplicity we will assume $U = V$. We have,

$$\mathcal{E}U \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus kl}.$$

The tensor products of interest to us will be the tensor products of tangent sheaf $\mathcal{T}$ of $\mathcal{A}^{\mathcal{M}}$-modules and cotangent sheaf $\mathcal{T}^{\vee}$ of $\mathcal{A}^{\mathcal{M}}$-modules. We will denote $\mathcal{T}^{(k,l)}$ the sheaf consisting of tensor product of k tangent and l cotangent sheaves.

$$\mathcal{T}^{(k,l)} = \mathcal{T}^{\otimes_{\mathcal{A}^{\mathcal{M}}} k} \otimes_{\mathcal{A}^{\mathcal{M}}} (\mathcal{T}^{\vee})^{\otimes_{\mathcal{A}^{\mathcal{M}}} l}.$$

The sections of such tensor products are called tangent fields, (k,l)-type tensor field in particular. After evaluation at each stalk this will correspond to k times tensor product of tangent space, and l time tensor product of cotangent space.

The tensor algebra of $\mathcal{T}$ is defined as the direct sum,

$$T^{\bullet}_{\mathcal{A}^{\mathcal{M}}}\mathcal{T} = \bigoplus_{i \geq 0} \mathcal{T}^i,$$

together with the multiplication defined by tensor product and extending linearly.

We usually encounter multilinear maps with additional properties. These will be the usual types of multilinear functionals we encounter while doing calculus. These are bilinear forms that are alternating, i.e., when the entries repeat the form should be zero. An example of such a multilinear map is the oriented area.

A k-linear map $\alpha : \mathcal{V} \times \cdots \mathcal{V} \to \mathcal{W}$ is called alternating, if the value is zero whenever two entries are the same. An exterior power of degree l is the universal vector space $\bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ together with an alternating multilinear map $i : \mathcal{V} \times \cdots \times \mathcal{V} \to \bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ such that for all alternating multilinear maps α, there exists a unique linear map $\tilde{\alpha}$ such that the following diagram commutes,

$$\begin{tikzcd} \prod^l \mathcal{V} \arrow[r, "i"] \arrow[dr, "\alpha"'] & \bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V} \arrow[d, "\exists!\ \tilde{\alpha}"] \\ & \mathcal{W} \end{tikzcd}$$

An exterior algebra of $\mathcal{A}^{\mathcal{M}}$-algebra $\mathcal{T}$, is the direct sum of all exterior powers, denoted by $\bigwedge^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T}$. This is also the quotient of the tensor algebra by the two-sided ideal $\mathcal{K}$ generated by all elements of the form $\tau \otimes \tau$ for all $\tau \in \mathcal{T}$. This quotient is called the exterior algebra,

$$\bigwedge^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T} = T^{\bullet}_{\mathcal{A}^{\mathcal{M}}}\mathcal{T}/\mathcal{K}.$$

This quotient puts all the elements in the tensor algebra that have same two entries into the equivalence class of zero. By doing this we are removing all the elements that can't be distinguished by alternating multilinear maps. We will stop writing the subscript $\mathcal{A}^{\mathcal{M}}$ when there is no confusion. The image of the tensor product $\mathcal{T}^k$ in $\bigwedge \mathcal{T}$ is denoted by $\bigwedge^k \mathcal{T}$. The image of $\tau_1 \otimes \cdots \otimes \tau_n$ is denoted by $\tau_1 \wedge \cdots \wedge \tau_n$. This is a graded algebra. By expanding $(\tau_1 + \tau_2) \otimes (\tau_1 + \tau_2)$ we see that,

$$\tau_1 \wedge \tau_2 = -\tau_2 \wedge \tau_1,$$

in $\bigwedge^2 \mathcal{T}$. Similarly, in $\bigwedge^l \mathcal{T}$,

$$\tau_1 \wedge \cdots \wedge \tau_l = (-1)^{\mathrm{sgn}(\sigma)} \tau_{\sigma(1)} \wedge \cdots \wedge \tau_{\sigma(l)},$$

where σ is a permutation of $\{1, \ldots, l\}$. The exterior algebra is a skew-commutative algebra.

A differential k-form or k-form is an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields. Or equivalently, sections of the sheafification of the kth exterior power of cotangent pre-sheaf. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$\omega \wedge \kappa(X_1, \ldots, X_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(X_{\sigma(1), \ldots, X_{\sigma(k)}}) \kappa(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}),$$

where σ is a permutation of the set $\{1, \ldots k + l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ is the sign of the permutation. This is a graded algebra.

6.2.4 | Differential of a map

If the tangent space at x is interpreted as the linear approximation of the manifold $\mathcal{M}$ at x, then the differential of a map is interpreted as the linear approximation of the map $\varkappa$.

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map of manifold $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ into $(\mathcal{N}, \mathcal{A}^{\mathcal{N}})$. Let $x \in \mathcal{M}$, the tangent space is defined to be the collection of the equivalence classes of such curves passing through x. Using the map $\varkappa$, we can push forward the curve h to $\mathcal{N}$, given by the composition, $\varkappa \circ h : \mathbb{R} \to \varkappa(U)$.

$$\mathbb{R} \xrightarrow{h} \mathcal{M} \xrightarrow{\varkappa} \mathcal{N}$$

By a differential of the function $\varkappa$, we mean a map $D\varkappa(x)$ that takes the equivalence class τ_h to the equivalence class $\tau_{\varkappa \circ h}$. The new pairing that arises from the map is,

$$\langle g, \varkappa \circ h \rangle_{\varkappa(x)} = \left. \frac{d(g \circ \varkappa \circ h(t))}{dt} \right|_{t=0},$$

for all $g \in \mathcal{A}^{\mathcal{N}}(\varkappa(U))$.

$$\begin{aligned} D\varkappa(x) : \ T_x\mathcal{M} &\to T_{\varkappa(x)}\mathcal{N} \\ \tau_h &\mapsto \tau_{\varkappa \circ h} \end{aligned}$$

It gives a vector bundle homomorphism of $T\mathcal{M}$ into $\varkappa^*(T\mathcal{N})$, usually denoted by $D\varkappa$. In terms of local coordinates this will be the Jacobian of the map. For compositions, we have,

$$(\varkappa \circ \varphi)^* = \varkappa^* \circ \varphi^*.$$

Let $[f] \in \mathcal{I}_x^{\mathcal{N}} / (\mathcal{I}_x^{\mathcal{N}})^2$, with the representative $f \in \mathcal{A}^{\mathcal{N}} U$, Then we have the pull back, given by the composition,

$$\mathcal{M} \xrightarrow{\varkappa} \mathcal{N} \xrightarrow{f} \mathbb{R}$$

Now, $f \circ \varkappa \in \mathcal{A}^{\mathcal{M}}(\varkappa^{-1}U)$. The new pairing that arises from the map is,

$$\langle f \circ \varkappa, h \rangle_x = \left. \frac{d(f \circ \varkappa \circ h(t))}{dt} \right|_{t=0},$$

for all curves $h : \mathbb{R} \to \mathcal{M}$ with $h(0) = x$. This gives us the pullback map,

$$\begin{aligned} T^{\vee}_{\varkappa(x)}\mathcal{N} &\to T^{\vee}_{x}\mathcal{M} \\ [f] &\mapsto [f \circ \varkappa]. \end{aligned}$$

It gives a vector bundle homomorphism of $\varkappa_*(T^{\vee}\mathcal{N})$ into $T^{\vee}\mathcal{M}$, and usually denoted by $D\varkappa^{\dagger}$. In terms of local coordinates this will be the adjoint of the Jacobian. For compositions, we have,

$$(\varkappa \circ \varphi)_* = \varphi_* \circ \varkappa_*.$$

A map $\varkappa : \mathcal{M} \to \mathcal{N}$ corresponds to a corresponding linear map on the tensor product bundle, described by its action on the tangent vectors and the cotangent vectors as above. So, in terms of local coordinates it will be a tensor product of the Jacobians and adjoints of the Jacobians. We will denote this map by $\varkappa^*$.

6.3 | Lie Derivative

A smooth function $\varkappa : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if $D\varkappa(x)$ is invertible for all $x \in \mathcal{M}$. This would mean that there exists an smooth inverse $\varkappa^{-1}$. The set of all diffeomorphisms of a manifold, i.e., diffeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ is a group. We will call such maps diffeomorphism 'of' $\mathcal{M}$. A one parameter group of diffeomorphisms of $\mathcal{M}$ is a collection of diffeomorphisms,

$$\varkappa : t \mapsto \varkappa_t$$

where each $\varkappa_t$ is a diffeomorphism of $\mathcal{M}$ such that, $\varkappa_0 = \mathbb{1}_{\mathcal{M}}$,

$$\varkappa_t \circ \varkappa_s = \varkappa_{t+s} \quad \forall t, s \in \mathbb{R},$$

and $\varkappa_t$ is a smooth as a map from $\mathcal{M} \times \mathbb{R}$ to $\mathcal{M}$. Where $\mathcal{M} \times \mathbb{R}$ has the differentiable structure of a product manifold. The one parameter group $t \mapsto \varkappa_t$ determines at each $x \in \mathcal{M}$ smooth curves,

$$t \mapsto \varkappa_t(x).$$

At each point x, this gives a tangent vector, the equivalence class of curves $[\varkappa_t(x)]$. Hence we obtain at each point $x \in \mathcal{M}$ a vector in the tangent space at x.

6.3.1 | Lie Derivative of Functions

Each one parameter group of diffeomorphisms determines a vector field. The converse also holds locally. Given a vector field X on a differentiable manifold $\mathcal{M}$ there exists a one-parameter group of diffeomorphisms $\varkappa$ such that $X_{\varkappa} = X$. This is due to the existence and uniqueness of solutions to ODEs.

For any $f \in \mathcal{A}^{\mathcal{M}}$ we have the smooth composition, $f \circ \varkappa_t : \mathcal{M} \to \mathbb{R}$. For fixed $x \in \mathcal{M}$, and varying t, this also corresponds to the smooth map,

$$\begin{aligned} f(\varkappa_{(\cdot)}(x)) :\ \mathbb{R} &\to \mathbb{R} \\ t &\mapsto f(\varkappa_t(x)). \end{aligned}$$

So, we can differentiate this function.

$$(X_\varkappa f)(x) = \lim_{t\to 0} \frac{f(\varkappa_t(x)) - f(x)}{t} = \left.\frac{d(f\circ h(t))}{dt}\right|_{t=0} = \langle f, h\rangle_x,$$

where $h = \varkappa_t(x)$. This can also be thought as the function,

$$\begin{aligned} X_\varkappa(\cdot) : \mathcal{A}^{\mathcal{M}} &\to \mathcal{A}^{\mathcal{M}}. \\ f &\mapsto (X_\varkappa f). \end{aligned}$$

It is called the differentiation of the function f with respect to $\varkappa$ at x. $\varkappa_t$ moves x to $\varkappa_t(x)$, so what the above limit is doing is measuring the infinitesimal change to the function f 'along' $\varkappa_t$. It's also called the Lie derivative of the function f along the vector field X, denoted by,

$$L_{X_\varkappa}(f) = X_\varkappa(f).$$

For $f, g \in \mathcal{A}^{\mathcal{M}}$, by directly plugging into the definition, we find that,

$$X_\varkappa(f+g)(x) = X_\varkappa(f)(x) + X_\varkappa(g)(x),$$

and for the product,

$$X_\varkappa(fg)(x) = f(x)(X_\varkappa g)(x) + g(x)(X_\varkappa f)(x).$$

So, $X_\varkappa$ is linear map, and

$$X_\varkappa(fg) = f(X_\varkappa g) + g(X_\varkappa f).$$

Hence, $X_\varkappa$ is a homogeneous first order operator. This abstract definition, while less intuitive can be extended to tensors product case later.

The flow of a vector field X is the one parameter group of diffeomorphisms $\varkappa$ such that $X_\varkappa = X$. A flow is said to be a global flow if it's defined for all $\mathbb{R}$ and at every point $x \in \mathcal{M}$. If a vector field gives rise to a global flow, it's called complete. For compact manifolds they do exist as we can always find local solutions and patch them up for finite cover. Let X be a vector field and $\varkappa_t$ be its corresponding flow, then the orbit of a point $x \in \mathcal{M}$, $t \mapsto \varkappa_t(x)$ is called the integral curve for X.

The integral curve is the constant map if and only if the vector field is zero. Such points are called singularities. $\varkappa_t$ fixes a point x if and only if x is a singularity of X. When x is not a singularity, $X(x) \neq 0$, and hence by continuity, the integral curve has injective differential nearby. Hence the integral curve is an immersed one dimensional manifold.

6.3.2 | Lie Derivative of Tensor Fields

Let X be a vector field with the corresponding flow $\varkappa$. These are diffeomorphisms, $\varkappa_t : \mathcal{M} \to \mathcal{M}$. Let $\mathcal{T}^{(k,l)}$ be a tensor sheaf, consisting of tensor product of k tangent and l cotangent sheaves. A tensor fields are sections of the tensor sheaf. The diffeomorphism gives an isomorphism of the sheaf. If $\mathcal{T}_x^{(k,l)}$ is the stalk of $\mathcal{T}^{(k,l)}$ at x, then we have the induced isomorphism,

$$\varkappa_t^* : \mathcal{T}_x^{(k,l)}/\mathcal{I}_x^{\mathcal{M}} \to \mathcal{T}_{\varkappa_t(x)}^{(k,l)}/\mathcal{I}_{\varkappa_t(x)}^{\mathcal{M}}.$$

So, using the flow of the vector field we can push forward the tensor field R. Since the association $t \mapsto \varkappa_t^*$ is also smooth the map[3], we have,

$$\varkappa_{(\cdot)}^*(R(x)) :\ t \mapsto \varkappa_t^*(R(\varkappa_t(x))).$$

[3]we have to carefully look at a bunch of maps, and this will turn out to be smooth

from $\mathbb{R}$ to the vector bundle $\mathcal{VT}^{(k,l)}$ associated with the locally free sheaf $\mathcal{T}^{(k,l)}$. In particular we can talk about taking the limit,

$$L_X(R)(x) = \lim_{t\to 0} \frac{\varkappa_t^*(R(\varkappa_t(x))) - R(x)}{t}, \tag{Lie derivative}$$

called the Lie derivative of the tensor field R with respect to the vector field X at $x \in \mathcal{M}$.

To study how the Lie derivative of the tensor product, exterior product and symmetric product of tensor fields, we can study Lie derivative of multilinear maps. Let β be an $\mathcal{A}^{\mathcal{M}}$-bilinear sheaf homomorphism,

$$\beta : \mathcal{V} \times \mathcal{V}' \to \mathcal{W},$$

where $\mathcal{V}, \mathcal{V}'$ and $\mathcal{W}$ are tensor sheaves. The action of $\varkappa_t$ is given by,

$$\varkappa_t^*(\beta(R, R')) = \beta(\varkappa_t^* R, \varkappa_t^* R').$$

Plugging this in the Lie derivative, we get,

$$\begin{aligned} L_X(\beta(R,R'))(x) &= \lim_{t\to 0} \tfrac{\varkappa_t^*\beta(R,R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} = \lim_{t\to 0} \tfrac{\beta(\varkappa_t^* R, \varkappa_t^* R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} \\ &= \beta\big(\lim_{t\to 0} \tfrac{\varkappa_t^* R(\varkappa_t(x)) - R(x)}{t}, \lim_{t\to 0} \varkappa_t^* R'(\varkappa_t(x))\big) + \beta\big(R, \lim_{t\to 0} \tfrac{\varkappa_t^* R'(\varkappa_t(x)) - R'(x)}{t}\big) \\ &= \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \end{aligned}$$

Here, we added and subtracted a term, and then took the limit inside. To take the limit inside, we would need the bilinear form to be continuous.

The Lie derivative of $\beta(R, R')$ with respect to X is,

$$L_X(\beta(R,R'))(x) = \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \tag{product rule}$$

Here we will have to add and subtract $\beta(\varkappa_t^* R(\varkappa_t(x)), R'(x))$ and use $\mathcal{A}^{\mathcal{M}}$-bilinearity of β to get the Lie derivative inside, this is similar to how product rule is proved. The result is also valid for just $\mathbb{R}$-bilinearity. So, by taking the bilinear map β to be the tensor product, $(R, R') \mapsto R \otimes R'$, we have,

$$L_X(R \otimes R') = L_X R \otimes R' + R \otimes L_X R'.$$

Similarly, for the differential forms, which are sections of exterior powers of cotangent sheaf,

$$L_X(\omega \wedge \omega') = L_X\omega \wedge \omega' + \omega \wedge L_X\omega'.$$

Let Y be a vector field, it's a derivation,

$$Y : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}.$$

The map $(Y, f) \mapsto Y(f)$ is a $\mathbb{R}$-bilinear map. So, the product rule is applicable. Note that $Y(f) \in \mathcal{A}^{\mathcal{M}}$, and the Lie derivative of functions is given by,

$$L_X(g) = X(g)$$

Hence, by plugging in $g = Y(f)$, we have,

$$\underbrace{L_X(Y(f))}_{X(Y(f))} = (L_X(Y))(f) + Y(\underbrace{L_X(f)}_{X(f)})$$

So, we have, $(L_X(Y))(f) = X(Y(f)) - Y(X(f)) = [X,Y](f)$. This is the Lie bracket, and it should be interpreted as the Lie derivative of Y with respect to the vector field X. Now, we can consider the bilinear map, $(Y, Z) \mapsto [Y, Z]$. By applying product rule, we get,

$$L_X([Y,Z]) = [[X,Y],Z]] + [Y,[X,Z]].$$

But by previous calculation of Lie derivative of vector fields, we have, $L_X([Y,Z]) = [X,[Y,Z]]$. This yields us the so called Jacobi identity,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{Jacobi identity}$$

This can also be written as,

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

For $f, g \in \mathcal{A}^{\mathcal{M}}U$, by expanding it can be verified that,

$$[X,Y](fg) = f[X,Y](g) + g[X,Y](f).$$

So, $[X,Y] \in \mathcal{T}U$. It can also be checked that $[,]$ is $\mathbb{R}$-bilinear. So, $\mathcal{T}U$ is an $\mathbb{R}$-algebra for all U. It's however not a $\mathcal{A}^{\mathcal{M}}U$-algebra because it's not $\mathcal{A}^{\mathcal{M}}U$-bilinear. Because for $g \in \mathcal{A}^{\mathcal{M}}U$,

$$\begin{aligned}[X, gY](f) &= X(gY(f)) - gY(X(f)) \\ &= X(g)Y(f) + gX(Y(f)) - gY(X(f)) \\ &= (X(g)Y + g[X,Y])(f).\end{aligned}$$

So, it's not $\mathcal{A}^{\mathcal{M}}$-bilinear, and hence can't be an $\mathcal{A}^{\mathcal{M}}U$-algebra. The tangent sheaf is an $\mathbb{R}$-algebra. If X and Y are two commuting vector fields, i.e., $[X,Y] = 0$ then the flows of the corresponding vector fields commute in the group of diffeomorphisms of $\mathcal{M}$.

A Lie algebras over a commutative ring $\mathcal{A}$ is an $\mathcal{A}$-module together with a bilinear operation, $(X,Y) \mapsto [X,Y]$, such that, $[X,X] = 0$ and satisfies the Jacobi identity. A homomorphism of Lie algebras is an algebra homomorphism $f : \mathcal{V} \to \mathcal{W}$ such that,

$$f([X,Y]) = [f(X), f(Y)].$$

The sheaf $\mathcal{T}U$ is a sheaf of Lie algebras over $\mathbb{R}$. The restriction map is a Lie algebra homomorphism.

Consider the $\mathbb{R}$-bilinear map, $(\omega, Y) \mapsto \omega(Y) \in \mathcal{A}^{\mathcal{M}}$, applying the product rule we have,

$$L_X(\omega(Y)) = \beta(L_X\omega, Y) + \beta(\omega, L_X Y)(x) = (L_X(\omega))(Y) + \omega(L_X(Y)).$$

So, the Lie derivative of a 1-form ω is given by,

$$(L_X(\omega))(Y) = X(\omega(Y)) - \omega([X,Y]).$$

Similarly, the Lie derivative of a k-form is given by,

$$(L_X(\alpha))(X_1, \dots, X_k) = X(\alpha(X_1, \dots, X_k)) - \sum_{i=1}^{k} \alpha(X_1, \dots, [X, X_i], \dots, X_k).$$

It follows that,

$$L_{[X,Y]}\omega = L_X L_Y \omega - L_Y L_X \omega.$$

and,

$$L_X(h\omega) = L_X(h) \cdot \omega + h \cdot L_X(\omega).$$

The Lie derivative allows us to differentiate a tensor field with respect to a vector field. What we want is a notion of differentiation of a tensor field with respect to a tangent vector at a point. This we cannot do with Lie derivative, the behavior of the vector field in a neighborhood was important as we took the limit. Or more algebraicly speaking,

$$(L_{fX}\omega)(Y) = (fX)(\omega(Y)) - \omega([fX, Y]) = f(L_X\omega)(Y) + (Y(f))\omega(X)$$

or, L is not $\mathcal{A}^{\mathcal{M}}$-linear. Changing the vector field X at x with a function $f \in \mathcal{A}^{\mathcal{M}}$, also depends on the behavior of the function f in the neighborhood and not just its value at x. In order to differentiate tensor fields with respect to a tangent vector we need the notion of connection, which we will discuss later.

6.4 | Algebra of Differential Operators

7 | $\mathcal{D}$-Modules, Jets & Connections

8 | Integration & Exterior Derivative

To develop calculus, we need the ability to integrate on manifolds. As it turns out integration on manifolds is very closely related to differential forms.

8.1 | Integration on Manifolds

So, to start with we need a measure on the manifold. Since we expect the measure to respect the topology of the manifold, it should be a Borel measure. By Riesz duality theorem, measure on a locally compact Hausdorff space can be identified with a linear functional,

$$\mu : \mathcal{C}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{C}_c^{\mathcal{M}}$ with $f \geq 0$. Where $\mathcal{C}_c^{\mathcal{M}}$ are all compactly supported continuous functions. The positivity condition makes the functional continuous.

Differentiable functions with compact support form a self-adjoint subalgebra of this algebra of compactly supported continuous functions, and separate points of the space $\mathcal{M}$. Hence by Stone-Weierstrass theorem, the algebra $\mathcal{A}_c^{\mathcal{M}}$ of compactly supported differentiable functions is dense in $\mathcal{C}_c^{\mathcal{M}}$. Since by Hahn-Banach theorem, linear functionals on subalgebras uniquely extend to the whole algebra, we can study functionals,

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{A}_c^{\mathcal{M}}$ with $f \geq 0$.

So, by a Borel measure on a differentiable manifold $\mathcal{M}$ we mean a linear form μ on the vector space $\mathcal{A}_c^{\mathcal{M}}$ of differentiable functions with a compact support on $\mathcal{M}$ which satisfies certain continuity requirement i.e., for a sequence of compactly supported differentiable functions, $\{f_i\}$, with support contained in the compact set K, if $\sup\{|f_i|\} \xrightarrow[i\to\infty]{} 0$ then $\mu(f_i) \to 0$.

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

The scalar $\mu(f)$ is denoted by $\int f d\mu$. On the space of compactly supported differentiable functions, we can define the sup norm making it into a Banach space. We can then start doing functional analysis. The measures under our consideration will be continuous linear functionals on $\mathcal{A}_c^{\mathcal{M}}$.

8.1.1 | Differentiable Measures

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map. For any function $g \in \mathcal{A}_c^{\mathcal{N}}$, the composition, $g \circ \varkappa \in \mathcal{A}_c^{\mathcal{M}}$. The composition has compact support because the manifold is Hausdorff, and hence the inverse image of compact set is compact. The image measure can then defined by,

$$(\varkappa^*(\mu))(g) = \mu(g \circ \varkappa).$$

The continuity and linearity follow from continuity of differentiable functions.

Now with this composition, we can start defining the Lie derivative. Let $\varkappa_t$ be the flow of a vector field X, The Lie derivative of a measure μ with respect to the vector field X is the functional,

$$f \mapsto \lim_{t\to 0} \frac{((\varkappa_t^{-1})^*(\mu))(f)-\mu(f)}{t} = \mu\Big(\underbrace{\lim_{t\to 0} \frac{f\circ\varkappa_t^{-1}-f}{t}}_{-L_X(f)=-X(f)} \Big).$$

We needed the continuity of the measure to take the limit inside. So we have,

$$L_X(\mu)(f) = -\mu(X(f)). \qquad \text{(Lie derivative)}$$

This will be our notion of differentiation of measures. Multiple differentiations will be defined as multiple iterations of the Lie derivative of the measure. We say a Borel measure μ is indefinitely differentiable if the k times differentiations is a Borel measure for all k.

If μ is a measure and $h \in \mathcal{A}^{\mathcal{M}}$, then the map,

$$f \mapsto \mu(hf)$$

is a linear form on $\mathcal{A}_c^{\mathcal{M}}$ and satisfies the continuity requirement i.e., if $\sup\{|hf_n|\}$ tends to zero then so does $\mu(hf_n)$. We will denote this measure by $h \cdot \mu$. Together with this notion of multiplication, $\mathcal{B}^{\mathcal{M}}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

Using $(L_X\mu)(f) = -\mu(Xf)$, the Lie derivative of μ with respect to hX is,

$$(L_{hX}\mu)(f) = -\mu(hX(f))$$

by using $X(hf) = fX(h) + hX(f)$, this is $= -\mu(X(hf) - fX(h)) = -\mu(X(hf)) + X(h)\mu(f)$.

$$(L_{hX}\mu)(f) = \underbrace{(L_X\mu)(hf)}_{h\cdot L_X(\mu)(f)} + X(h)\mu(f)$$

Since $h \cdot \mu(f) = \mu(hf)$, we have, $-\mu(hX(f)) = h - \cdot\mu(X(f)) = h \cdot L_X(\mu)$. So,

$$L_X(h\cdot\mu) = L_{hX}\mu = h \cdot L_X(\mu) + L_X(h)\cdot\mu. \qquad \text{(Leibniz rule)}$$

Similarly by expanding we get that,

$$L_{[X,Y]}(\mu) = L_X L_Y(\mu) - L_Y L_X(\mu). \qquad \text{(Lie bracket)}$$

So, the behavior of differentiable measures under Lie derivative is similar to that of differential forms. We are interested in measures that are locally translation invariant. Let V be a vector space, and μ be a measure. It's said to be translation invariant if $\varkappa_t^*\mu = \mu$ for $\varkappa_t = tv$ or equivalently, $L_{\partial_v}\mu = 0$ for all v.

8.1.2 | The Sheaf of Differentiable Measures

We now start looking at the collection of all indefinitely differentiable measures. Let $U \subseteq V$, then we have the natural inclusion of compactly supported functions functions $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$, by setting the functions to be equal to zero outside U.

Let $\mathcal{B}^{\mathcal{M}}U$ be the set of all differentiable measures on U, we have used $\mathcal{B}$ here for Borel. The inclusion $U \subset V$ gives rise to a restriction map of differentiable measures $\mu \mapsto \mu|_V$. The action of $\mu|_U$ on $\mathcal{A}_c^{\mathcal{M}}U$, is given by the action of μ on $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$. So,

$$\mathcal{B}^{\mathcal{M}} : U \mapsto \mathcal{B}^{\mathcal{M}}U$$

is a presheaf. Now, to patch these measures up, we need the notion of partition of unity. This is an important tool. What we intend to do is restrict the domain of functions to some regions so we can forget about the behaviour of the function outside some region.

Let $\{U_i\}_{i\in I}$ be a locally finite open cover of a differentiable manifold $\mathcal{M}$. The locally finite open covers exist because manifolds are locally compact. A partition of unity with respect to the cover $\{U_i\}_{i\in I}$ is a family of smooth functions $\{\varphi_i\}_{i\in I}$ with values in $[0,1]$ such that

$$\textstyle\sum_{i\in I}\varphi_i = 1$$

with support of φ_i contained in U_i. Once we have such a partition of unity, we can study the function $(\sum_{i\in I}\varphi_i)f$ instead of the function f. To show the existence, let $\{V_i\}_{i\in I}$ be an open cover with $\overline{V_i}\subset U_i$, we can construct functions ψ_i that have support in U_i.[1] Since the cover is locally finite the sum makes sense.

$$\varphi_i = \psi_i/(\textstyle\sum_{i\in I}\psi_i)$$

then acts as a partition of unity. This allows us to study the functions using the charts.

Let $\{U_i\}$ be a locally finite familly of open sets and μ_i be Borel measure on them. Suppose $\mu_i|_{U_i\cap U_j} = \mu_i|_{U_i\cap U_j}$ for all i, j, then we can define a measure μ on $U = \cup\{U_i\}$ by multiplying any function $f\in\mathcal{A}_c^{\mathcal{M}}U$ with a partition of unity associated with $\{U_i\}$, and then define,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f).$$

Since $\{U_i\}$ is locally finite, the sum is welldefined. Now, suppose $f\in\mathcal{A}_c^{\mathcal{M}}U_i$, then $\mu(f) = \sum_i\mu_i(\varphi_i f)$, since the support of f is contained in U_i, we have for every $U_i\cap U_j$, $\mu_i|_{U_i\cap U_j} = \mu_j|_{U_i\cap U_j}$ and hence we have,

$$\mu(f) = \textstyle\sum_i\mu_i(\varphi_i f) = \sum_i\mu_j(\varphi_i f)$$

By linearity of measures this is

$$= \mu_j((\textstyle\sum_i\varphi_i)f) = \mu_j(f).$$

Hence the collation property holds, i.e., there exists an equilizer map e such that,

$$\mathcal{B}^{\mathcal{M}}U \overset{e}{\dashrightarrow} \prod_i \mathcal{B}^{\mathcal{M}}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j}\mathcal{B}^{\mathcal{M}}(U_i\cap U_j).$$

Since the partition of unity is a differentiable map, the linear map μ is also continuous, and hence is a Borel measure. The differentiable measures on a differentiable manifold $\mathcal{M}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

8.2 | Differential Forms

Differential forms are closely related to 'oriented volume'. The multilinear maps of interest to us are alternating multilinear maps, alternating multilinear maps also carry with them information about orientation. Orientation makes the order of vectors important. To study

[1] Let $S_1\subset S_2$ be concentric spheres in $\mathbb{R}^n$ centered at 0 then we need to show that there exist differentiable function which is zero outside S_2 and non zero everywhere inside S_1. If we take S_2 to be the unit ball, the function, $\Phi(x) = \exp\left(\frac{1}{\sum_i x_i^2 - 1}\right)$ for x in the unit ball and zero outside works.

such multilinear maps we can restrict ourselves to the study of exterior powers instead of studying the much larger tensor product. So the starting point is the cotangent pre-sheaf.

$$\mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends each open set U to $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ which we will denote by $\mathcal{C}U$. We can consider the exterior algebra of this cotangent pre-sheaf.

$$\bigwedge^k \mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends U to $\wedge^k \mathcal{C}U$. The elements of the stalks of this pre-sheaf will be called k-forms. We can now bundle these stalks and consider the sheaf of sections of this bundle.

$$(\bigwedge^k \mathcal{C})^{\mathrm{Sh}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

This is a sheaf of vector spaces over $\mathbb{R}$ usually denoted by Ω^k. We can similarly consider the exterior algebra, which consists of the direct sum of all the exterior powers.

$$\Omega^\bullet = \bigoplus_{i=0}^{\infty} \Omega^k$$

Note that $\Omega^k = 0$ for $k > n$ where n is the dimension of the manifold and $\Omega^0 = \mathcal{A}^{\mathcal{M}}$. A differential k-form is a section of sheafification of kth exterior power of cotangent pre-sheaf. Equivalently it's an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields.

8.2.1 | Exterior Product

Although we haven't yet described the relation between differential forms and volumes, it's useful and use it to motivate other definitions involving differential forms. We want to be able to multiply two lengths and find out area. This is the idea of exterior product. Given two differential forms which intuitively measure some sort of length, we want to define an 'oriented area'. Let ω and κ be two differential forms. These give us a map,

$$\tau \mapsto (\omega(\tau), \kappa(\tau))$$

for each tangent vector τ. Now, we can define $(\omega \wedge \kappa)(\tau_1, \tau_2)$ to be the area of the parallelogram with sides $(\omega(\tau_1), \kappa(\tau_1))$ and $(\omega_1(\tau_2), \omega_2(\tau_2))$. Now, the area of the parallelogram is given by,

$$(\omega \wedge \ \kappa)(\tau_1, \tau_2) = \begin{vmatrix} \omega(\tau_1) & \kappa(\tau_1) \\ \omega(\tau_2) & \kappa(\tau_2) \end{vmatrix}.$$

This is called the exterior product of the differential forms ω and κ. We can generalize this to more general volumes. Let ω be a differential k-form and κ be differential l-form. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$(\omega \wedge \kappa)(\tau_1, \dots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)}) \kappa(\tau_{\sigma(k+1)}, \dots, \tau_{\sigma(k+l)}),$$

where σ is a partition[2] of the set $\{1, \dots k+l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ where $\mathrm{sgn}(\sigma)$ is the sign of the partition. Note that $\mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$. This is a bilinear map. The exterior

[2] a partition is a permutation such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

algebra with the above product is a $\mathbb{Z}$-graded algebra. We can now list the basic properties of the exterior product.

Consider,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \kappa(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)}) \omega(\tau_{\sigma(k+1)}, \dots, \tau_{\sigma(k+l)}),$$

We can act this by a permutation, γ, such that, $(\gamma(1), \gamma(2), \dots, \gamma(k+l)) = (k+1, \dots, k+l, 1, \dots, k)$. The sign of this permutation is, $\mathrm{sgn}(\gamma) = (-1)^{kl}$. So we have,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = (-1)^{kl} (\omega \wedge \kappa)(\tau_1, \dots, \tau_{k+l}).$$

Some basic combinatorics argument shows us that,

$$(\omega \wedge \kappa) \wedge \xi = \omega \wedge (\kappa \wedge \xi)$$

At each point x, every cotangent vector can be written in terms of local coordinates φ as, $[f] = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(x)\right] dx_i$. where dx_i is the equivalence class corresponding to the function $\varphi_i(x) - x_i$. Since we expect differential forms to be smooth sections of the cotangent sheaf, every differential form can be written as,

$$\omega = \textstyle\sum_{i=1}^{n} a_i dx_i.$$

where $a_i \in \mathcal{A}^{\mathcal{M}}$. Similarly, differential k-forms can be written as,

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1, \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

where each $a_{i_1, \dots i_k} \in \mathcal{A}^{\mathcal{M}}$.

Using the tangency pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0},$$

we can pair a differential form and a vector field pointwise which measures the length of X using the differential form ω at each point. This is called a contraction or interior product of ω with X. Denoted by

$$\iota_X \omega := \langle \omega, X \rangle$$

This can also be extended to differential k-forms.

$$\iota_X \omega(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}).$$

We can now start listing down the algebraic properties of the contraction. For exterior product of differential forms is given by,

$$\iota_X(\omega \wedge \kappa) = (\iota_X \omega) \wedge \kappa + (-1)^p \omega \wedge (\iota_X \kappa).$$

From the anti-symmetry of differential forms, we have, $\omega(X, Y, \dots) = -\omega(Y, X, \dots)$. So,

$$\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$$

The contraction provides a map,

$$\iota_X : \Omega^k \to \Omega^{k-1}.$$

For a differential k-form, and vector fields $X_1 \dots X_k$, we denote the evaluation by,

$$\langle \omega, (X_1 \dots X_k) \rangle = \omega(X_1 \dots X_k).$$

8.2.2 | Exterior Differentiation

Differential forms define at each point, the notion of length, area, volume, etc. Now we want to do calculus with them i.e., differentiate and integrate stuff. A k-form provides some sort of k-volume on tangent spaces. They are more complicated and Lie derivative doesn't describe how they change correctly. We want to define a notion of differentiation that captures all the ways in which it changes.

For a function $f \in \mathcal{A}^{\mathcal{M}}$, the differential is the flow of the 0-volume.

$$f : \mathcal{M} \to \mathbb{R}$$

gives us the map $df(x)$ of equivalence classes of curves, $\tau_h \mapsto \tau_{f \circ h}$. This is a map from $T_x\mathcal{M}$ to $\mathbb{R}$. Hence $df(x)$ is an element in the stalk of the cotangent pre-sheaf $\mathcal{C}$. Covectors can be thought of as assigning to each tangent vector its 'length'. Since this depends smoothly on the point x, it's a differential form i.e., $df \in \Omega^1$. Note here that this is the reason why the equivalence classes of functions $\varphi_i(x) - x_i$ were written as dx_i.

The definition of exterior derivative is not very intuitive. We try to motivate the definition of 'exterior derivative' of differential forms below. Although this motivation is not sufficient to characterize the definition, it can help understand what's happening.

For a differential form ω, and vector fields X and Y, we have the pairings $\langle \omega, X \rangle$ and $\langle \omega, y \rangle$. Each of the pairings are differentiable functions on $\mathcal{M}$ i.e.,

$$\langle \omega, X \rangle, \langle \omega, Y \rangle \in \mathcal{A}^{\mathcal{M}}.$$

We are interested in understanding how the function $\langle \omega, X \rangle$ changes along another vector field Y, and $\langle \omega, Y \rangle$ changes along X. The change of $\langle \omega, X \rangle$ along Y is given by the new pairing, $L_Y(\langle \omega, X \rangle) = \langle d(\langle \omega, X \rangle), Y \rangle$, so the difference,

$$L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle)$$

is a differential 2-form. We want to take into account all the changes that are happening, so we should also take into account how X changes with respect to Y, as measured by the differential form ω which is $\omega(L_X(Y))$. So, we define the exterior derivative as,

$$(d\omega)(X, Y) = L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle) - \omega(L_X(Y)).$$

So, we have,

$$(d\omega)(X, Y) = X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \omega([X, Y]).$$

The negative signs are used so that this is a differential form. It's not yet clear why we have to take $\omega(L_X(Y))$ and not $\omega(L_Y(X))$, but we will not try to motivate that here. For a differential k-form, the exterior derivative is defined as,

$$\begin{aligned}(d\omega)(X_1 \dots X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1 \dots \widehat{X_i} \dots X_{k+1})) \\ &+ \sum_{1 \le j < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_{k+1}).\end{aligned} \tag{exterior derivative}$$

The explicit form of the exterior derivative in terms of local coordinates, is given by $d\omega = \sum \partial a_i / \partial x_i dx_i \wedge dx_I$, where $\omega = \sum a_i dx_I$. So, the exterior derivative is a map,

$$d : \Omega^k \to \Omega^{k+1}$$

We can now start listing all the properties of the exterior derivative. From the definition, it follows that the exterior derivative is linear,

$$d(\lambda\omega + \mu\kappa) = \lambda d(\omega) + \mu d(\kappa).$$

For a vector field X, we have,

$$\begin{aligned}(d\iota_X + \iota_X d)(\omega)(X_1 \dots X_k) = &\textstyle\sum(-1)^{i+1} X_i(\iota_X \omega(X_1 \dots \widehat{X_i} \dots X_k)) \\ &+ \textstyle\sum(-1)^{i+j}(\iota_X\omega)([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) \\ &+ (d\omega)(X, X_1 \dots \widehat{X_i} \dots X_j \dots X_k)\end{aligned}$$

which on expanding gives,

$$= X(\omega(X_1 \dots X_k)) + \textstyle\sum(-1)^i \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) = (L_X(\omega))(X_1 \dots X_k)$$

This is called Cartan formula, and can be written compactly as,

$$d\iota_X + \iota_X d = L_X \qquad \text{(Cartan formula)}$$

Let ω be a differential k-form and κ be a differential l-form, it can be checked that,

$$d(\omega \wedge \kappa) = (d\omega) \wedge \kappa + (-1)^k \omega \wedge (d\kappa)$$

Such maps are called derivations of odd type. It can be showed that[3],

$$d \circ d = 0.$$

The last property is very important and allows us to study the topological properties of the manifolds by studying the differential forms. The proofs of all these are very simple in local coordinates.

8.2.3 | Invariant Forms vs. Invariant Measures

To motivate and make this relation precise, we start translation invariant measures on Euclidean space. The translational invariance makes sure that we can determine the measure of any measurable set if we know its value for some model set, say, a cube, because cubes generate the Borel σ-algebra, and the translational invariance allows us to measure any scaled copy of the cube.

8.2.3.1 | Motivating Example, $\mathbb{R}^n$

The translation invariant measures are determined uniquely upto a constant multiplication, and for Euclidean space it corresponds to the Lebesgue measure, upto scalar multiplication. So, for the vector field $X = \partial_v$ with the flow given by translations by $\varkappa_t = tv$, the Lie derivative $L_{\partial_v}\nu$ of the Lebesgue measure ν along ∂_v is given by,

$$L_{\partial_v}\nu(f) = \nu\big(\lim_{t\to 0} \tfrac{f\circ\varkappa_t^{-1} - f}{t}\big) = \nu\big(\tfrac{f-f}{t}\big) = 0.$$

Here we used the translational invariance. So, it exists and equals 0. Since we have,

$$(L_{hX}\mu)(f) = (L_X\mu)(hf) + X(h)\mu(f)$$

[3] Proof of this using the coordinate expression is very simple and only involves using the fact that $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$. In fact all computations are easier done in local coordinates.

The Lie derivative along any vector field $\sum_i h_i \partial_i$ exists and equals,

$$(L_{(\sum_i h_i \partial_i)}\nu)(f) = \textstyle\sum_i \partial_i h_i \nu(f).$$

Hence Lebesgue measures are differentiable measures. Denote the set of all Lebesgue measures on $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. Lebesgue measures only vary by a scalar multiple, and are determined hence by their action on cubes. Each Lebesgue measure ν determines a map,

$$\begin{aligned}\widehat{\nu}:\ \textstyle\prod^n \mathbb{R}^n &\to \mathbb{R}.\\ (v_i)_{i=1}^n &\mapsto \nu([v_i]_{i=1}^n),\end{aligned}$$

where $[v_i]_{i=1}^n$ is the n-dimensional cube in $\mathbb{R}^n$ determined by the vectors $\{v_i\}$. $\nu([v_i]_{i=1}^n)$ is non-zero only if $\{v_i\}$ forms a basis of $\mathbb{R}^n$. Because otherwise, $[v_i]_{i=1}^n$ is a measure zero set, they are $< n$ dimensional sheets. The translation invariant, additivity, and continuity guarantee that,

$$\nu([v_1, \dots rv_i, \dots v_n]) = r\nu([v_i]_{i=1}^n).$$

So the map, $(v_i) \mapsto \nu([v_i])$ is multilinear in v_i, and since if any two v_is are equal we should have the measure to be zero, it's an alternating multilinear map and must factor through $\wedge^n \mathbb{R}^n$.

$$\begin{tikzcd} \prod^n \mathbb{R}^n \arrow[r, "i"] \arrow[dr, "\widehat{\nu}"'] & \bigwedge^n \mathbb{R}^n \arrow[d, "\exists!\ e_\nu"] \\ & \mathbb{R} \end{tikzcd}$$

So we have, $\nu([v_i]_{i=1}^n) = e_\nu(\wedge_{i=1}^n v_i)$. So, to each Lebesgue measure on $\mathbb{R}^n$ we have an associated differential n-form. Equivalently, we get a map from the space of invariant measures into the one-dimensional space $(\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}$.

$$\nu \mapsto e_\nu$$

The wedge product however is order sensitive, and the measure is not. So we should have,

$$e_\nu(\wedge^n v_i) = \pm\nu([v_i]_{i=1}^n).$$

This can be interpreted in the following sense, consider $(\wedge^n \mathbb{R}^n)^\vee \backslash \{0\} \cong \mathbb{R}\backslash\{0\}$, which has two connected components. The assignment of the sign 1 (respectively -1) in the definition is based upton whether $v_1 \wedge \cdots \wedge v_n$ belongs to the chosen component (or not). The choice of a connected component is equivalent to chosing a basis for $(\wedge^n \mathbb{R}^n)^\vee$. Such a choice of basis in $(\wedge^n \mathbb{R}^n)^\vee$ is called a volume element of $\mathbb{R}^n$, and such a volume element fixes the sign of Lebesgue measures. The map,

$$E:\ \mathcal{L}(\mathbb{R}^n) \to (\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}, \quad \nu \mapsto e_\nu.$$

is an isomorphism of the space of Lebesgue measures $\mathcal{L}(\mathbb{R}^n)$ and the space of differential n-forms $(\wedge^n \mathbb{R}^n)^\vee$, and this isomorphism depends on the choice of basis for $(\wedge^n \mathbb{R}^n)^\vee$, and the two isomorphisms differ by constant multiple (-1). The choice of basis is called the orientation of the vector space $\mathbb{R}^n$. A Euclidean space has two orientations. corresponding to the choice. In this case,

$$\mathcal{K}^{\mathbb{R}^n} = \Omega^n \mathbb{R}^n = \mathcal{A}^{\mathbb{R}^n} \otimes_{\mathbb{R}} (\Lambda^n \mathbb{R}^n)^\vee$$

and similarly we can tensor the space of invariant measures with $\mathcal{A}^{\mathbb{R}^n}$ of differentiable functions on $\mathbb{R}^n$, this is a subsheaf of the differentiable measures, $\mathcal{B}^{\mathbb{R}^n}$ consisting of measures of the form $f \cdot \nu$ for $f \in \mathcal{A}^{\mathbb{R}^n}$ and $\nu \in \mathcal{L}(\mathbb{R}^n)$. So the isomorphism above yields an isomorphism of these sheaves. The elements of $\mathcal{K}^{\mathbb{R}^n}$ are called volume forms.

8.3 | Sheaf of Densities

Although invariant measures and differential n-forms are closely related in Euclidean space, it might not be the case in general differentiable manifold. The manifold might have twists which might make such an association that patches up nicely impossible. We want to study homomorphism from the sheaf of volume forms $\mathcal{K}^{\mathcal{M}}$ to the sheaf invariant measures in $\mathcal{B}^{\mathcal{M}}$.

8.3.1 | Orientation Sheaf

We now have two pre-sheaves, $\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}} \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ of $\mathcal{A}^{\mathcal{M}}$-modules, for any $U \subset \mathcal{M}$, consider the new pre-sheaves, the restructions, $\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U \in \mathrm{PSh}(U, \mathcal{A}^{\mathcal{M}}|_U)$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A}^{\mathcal{M}}|_U)}(\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{tikzcd} U \arrow[d, "|_V"] & \mathcal{K}^{\mathcal{M}}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{K}^{\mathcal{M}}(|_V)"'] & \mathcal{B}^{\mathcal{M}}U \arrow[d, "\mathcal{B}^{\mathcal{M}}(|_V)"] \\ V & \mathcal{K}^{\mathcal{M}}V \arrow[r, "\kappa_V"'] & \mathcal{B}^{\mathcal{M}}V. \end{tikzcd}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom of $\mathcal{K}^{\mathcal{M}}$ and $\mathcal{B}^{\mathcal{M}}$, denoted by,

$$\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}) \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}}).$$

Sometimes also written as $(\mathcal{B}^{\mathcal{M}})^{\mathcal{K}^{\mathcal{M}}}$. We will now show that the internal hom is also a sheaf, i.e., it satisfies the collation property,

$$\begin{tikzcd} \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U \arrow[r, dashed, "e"] & \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i \cap U_j). \end{tikzcd}$$

To show that $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U$ and at $\prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 8.3.1. *If $\mathcal{B}^{\mathcal{M}}$ is a sheaf then so is $\mathcal{H}\mathrm{om}(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i\in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{K}, \mathcal{B}^{\mathcal{M}})U$, and $U_i \subset U$, we have,

$$\begin{tikzcd} U \arrow[d, "|_{U_i}"] & \mathcal{K}^{\mathcal{M}}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{K}^{\mathcal{M}}(|_{U_i})"'] & \mathcal{B}^{\mathcal{M}}U \arrow[d, "\mathcal{B}^{\mathcal{M}}(|_{U_i})"] \\ U_i & \mathcal{K}^{\mathcal{M}}U_i \arrow[r, "\kappa_{U_i}"'] & \mathcal{B}^{\mathcal{M}}U_i. \end{tikzcd}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{K}^{\mathcal{M}} U_i$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \cap V} = 0.$$

Now, $\{V \cap U_i\}$ is a cover of V, and $\mathcal{B}^{\mathcal{M}} V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \cap U_j} = \kappa_j|_{U_i \cap U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{B}^{\mathcal{M}}$ is a sheaf to patch these natural transformations.

Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{K}^{\mathcal{M}} V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

here the first map comes from the natural transformation, $\mathcal{K} V \ni f \mapsto \kappa_i(f|_{V \prod U_i})$. Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, this must uniquely factor through $\mathcal{B}^{\mathcal{M}} V$, by definition of equaliser. Hence, we have,

$$\mathcal{K}^{\mathcal{M}} V \xrightarrow{\exists!} \mathcal{B}^{\mathcal{M}} V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ defines the patched up element that equalizes the diagram, and hence the internal hom is a sheaf whenever $\mathcal{B}^{\mathcal{M}}$ is a sheaf. $\square$

The natural transformations $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ are $\mathcal{A}^{\mathcal{M}}$-module homomorphisms. In particular, $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf of $\mathbb{R}$-modules. However not all of these are important to us. We are interested in those which map invariant volume forms to invariant measures. Since the information about the measure being invariant has to do with Lie derivatives, we just have to preserve that structure, i.e, performing Lie derivation before the homomorphism should be the same as taking Lie derivation after the homomorphism.

Hence, the natural transformation of interest to us should preserve the Lie derivative. $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is said to be flat if

$$\kappa L_X = L_X \kappa. \tag{flat}$$

The collection of all flat homomorphisms is denoted by $OR_{\mathcal{M}}$.

Flat homomorphisms take invariant forms to invariant measures. The set of all flat homomorphisms $\mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules and is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$. Flat homomorphisms do exist when $\mathcal{M} = \mathbb{R}^n$, as described in 8.2.3.1.

Proposition 8.3.2. *If $\mathcal{M}$ is connected, κ, φ flat, then there exists $\lambda \in \mathbb{R}$ such that $\kappa = \lambda\varphi$.*

Proof

It's enough to describe the action on invariant forms. Let ω be an invariant form, i.e., in local coordinates, $L_{\partial_v}\omega = 0$. Since κ, φ are flat homomorphisms, $\kappa(\omega)$ and $\varphi(\omega)$ are invariant measures, because $\kappa L_X = L_X\kappa$, $\varphi L_X = L_X\varphi$, and κ, φ are homomorphisms,

$$L_{\partial_v}(\kappa(\omega)) = L_{\partial_v}(\varphi(\omega)) = 0.$$

Locally, these invariant measures are Lebesgue measures and hence must vary by a constant multiple. On intersections, this constant is preserved. Since the manifold is connected, there can't be any abrupt change to this constant multiple. So,

$$\kappa(\omega) = \lambda(\varphi(\omega))$$

for some $\lambda \in \mathbb{R}$. $\square$

The pre-sheaf of flat homomorphisms,

$$\begin{aligned} OR_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\text{op}} &\to {}_{\mathbb{R}}\text{Mod} \\ U &\mapsto OR_{\mathcal{M}}(U), \end{aligned}$$

where $OR_{\mathcal{M}}(U)$ is the collection of all flat homomorphisms, $\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules. $OR_{\mathcal{M}}$ is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$,

$$OR_{\mathcal{M}} \rightarrowtail \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$$

Since in local coordinates each flat homomorphism varies from a standard flat homomorphism by a constant real multiple, it's a locally constant sheaf of $\mathbb{R}$-modules of rank 1.

The existence of global sections of $OR_{\mathcal{M}}$, i.e., global flat homomorphisms depends geometrically on whether the manifold has 'twists' or not. To intuitively motivate this twisting, consider for example, the case of a mobius strip. We can choose local coordinates around a point $x \in \mathcal{M}$, if there existed a flat homomorphism, then it should patch up nicely with local restrictions, however, when we go around the strip and return, the order of the basis we had chosen will be reversed. So, although we have local flat homomorphisms that agree on intersections, there doesn't exist a global patch up of them. The only possible flat homomorphism is the trivial homomorphism, which sends everything to zero. This twisting can hence be made precise in terms of the orientation sheaf.

Since flat homomorphisms vary by constant multiples, we can just consider equivalence classes. Choose in a local chart a choice of ordered basis, this determines locally, a standard flat homomorphism

$$\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$$

Now, we can consider all flat homomorphisms that vary by an integral multiple of κ, i.e., flat homomorphisms of the type $\lambda \cdot \kappa$ with $\lambda \in \mathbb{Z}$. This collection is a locally constant sheaf, denoted by $OZ_{\mathcal{M}}$, and the sheaf is called the local system of 'twisted integers'.

A connected manifold $\mathcal{M}$ is called oriented if this is the constant sheaf. Equivalently, we say $\mathcal{M}$ is oriented if the étale space of the sheaf of twisted integers, $OZ_{\mathcal{M}}$, is $\mathcal{M} \times \mathbb{Z}$. In such a case, there are two trivializations, and each of which is called an orientation on $\mathcal{M}$. Clearly,

$$OR_{\mathcal{M}} = OZ_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathbb{R}$$

The existence of a flat homomorphisms means that at each point in the manifold, we can associate an invariant measure, and these invariant measures patch up nicely. Assuming a flat homomorphism $\kappa : \mathcal{K}^{\mathcal{M}} \to \mathcal{B}^{\mathcal{M}}$ exists, the sheaf of densities is the image $\mathcal{S}_{\mathcal{M}} := \kappa(\mathcal{K}^{\mathcal{M}}) \subseteq \mathcal{B}^{\mathcal{M}}$.

$$\begin{aligned} \mathcal{S}_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathbb{R}}\mathrm{Mod} \\ U &\mapsto \kappa(\mathcal{K}^{\mathcal{M}}|_U), \end{aligned}$$

where $\kappa(\mathcal{K}^{\mathcal{M}}|_U) = \{\kappa(\omega) \mid \omega \in \mathcal{K}^{\mathcal{M}}|_U\}$. Since flat homomorphisms vary only by a constant multiple, no information is lost by choosing a flat homomorphism.

On $\mathbb{R}^n$, $\mathcal{S}_{\mathbb{R}^n}$ consists of all measures of the form, $f d\mu$ where μ is the Lebesgue measure on $\mathbb{R}^n$. So, in the case of a differential manifold, by local isomorphism of the sheaf of differentiable functions, in any coordinate system, these measures can be expressed as $f d\mu$.

8.3.2 | Pullback of Sheaf of Densities

Now that we have the sheaf of densities, we can study the behavior of the sheaf under diffeomorphisms. We are interested in how the elements of the sheaf change, and get a local formula for this change in terms of coordinates. So, it's good enough to restrict to diffeomorphisms from domains in $\mathbb{R}^n$.

Theorem 8.3.3. (Change of Variable Formula)

8.3.3 | Adjoint of Differential Operators

8.3.3.1 | Stokes Theorem

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