Part II

Exterior Derivative & Integration

Our goal is doing integration on manifolds. In order to do that we must understand what integrands should look like. A differential form is a section of the cotangent sheaf. From the tangency pairing, the cotangent vector sends tangent vectors to real numbers, or 'measure the length' of the tangent vector. A differential form provides us a way to measure 'length' of tangent vectors at each point in the manifold. We want to extend this to 'areas' and 'volumes'. Functions such as oriented area, volume, etc. carry a lot of geometric information about the manifold. Given two vectors we can assign to them the oriented area of the parallelogram with the vectors as its sides. When the two vectors are the same we expect the area to be zero. Instead of studying the geometric information contained in one such multilinear map, we will consider all of them.

1 | Differential Forms

The multilinear maps of interest to us are alternating multilinear maps, alternating multilinear maps also carry with them information about orientation. Orientation makes the order of vectors important. To study such multilinear maps we can restrict ourselves to the study of exterior powers instead of studying the much larger tensor product. So the starting point is the cotangent pre-sheaf.

$$\mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends each open set U to $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ which we will denote by $\mathcal{C}U$. We can consider the exterior algebra of this cotangent pre-sheaf.

$$\bigwedge^k \mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathrm{Sets},$$

which sends U to $\wedge^k \mathcal{C}U$. The elements of the stalks of this pre-sheaf will be called k-forms. We can now bundle these stalks and consider the sheaf of sections of this bundle.

$$(\bigwedge^k \mathcal{C})^{\mathrm{Sh}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

This is a sheaf of vector spaces over $\mathbb{R}$ usually denoted by Ω^k. We can similarly consider the exterior algebra, which consists of the direct sum of all the exterior powers.

$$\Omega^\bullet = \bigoplus_{i=0}^{\infty} \Omega^k$$

Note that $\Omega^k = 0$ for $k > n$ where n is the dimension of the manifold and $\Omega^0 = \mathcal{A}^{\mathcal{M}}$. A differential k-form is a section of sheafification of kth exterior power of cotangent pre-sheaf. Equivalently it's an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields.

1.1 | Exterior Product

We want to be able to multiply two lengths and find out area. This is the idea of exterior product. Given two differential forms which intuitively measure some sort of length, we want to define an 'oriented area'. Let ω and κ be two differential forms. These give us a map,

$$\tau \mapsto (\omega(\tau), \kappa(\tau))$$

for each tangent vector τ. Now, we can define $(\omega\wedge\kappa)(\tau_1, \tau_2)$ to be the area of the parallelogram with sides $(\omega(\tau_1), \kappa(\tau_1))$ and $(\omega_1(\tau_2), \omega_2(\tau_2))$. Now, the area of the parallelogram is given by,

$$(\omega \wedge\ \kappa)(\tau_1, \tau_2) = \begin{vmatrix} \omega(\tau_1) & \kappa(\tau_1) \\ \omega(\tau_2) & \kappa(\tau_2) \end{vmatrix}.$$

This is called the exterior product of the differential forms ω and κ. We can generalize this to more general volumes. Let ω be a differential k-form and κ be differential l-form. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$(\omega \wedge \kappa)(\tau_1, \dots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)})\kappa(\tau_{\sigma(k+1)}, \dots, \tau_{\sigma(k+l)}),$$

where σ is a partition[1] of the set $\{1, \dots k+l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ where $\mathrm{sgn}(\sigma)$ is the sign of the partition. Note that $\mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$. This is a bilinear map. The exterior algebra with the above product is a $\mathbb{Z}$-graded algebra. We can now list the basic properties of the exterior product.

Consider,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \kappa(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)})\omega(\tau_{\sigma(k+1)}, \dots, \tau_{\sigma(k+l)}),$$

We can act this by a permutation, γ, such that, $(\gamma(1), \gamma(2), \dots, \gamma(k+l)) = (k+1, \dots, k+l, 1, \dots, k)$. The sign of this permutation is, $\mathrm{sgn}(\gamma) = (-1)^{kl}$. So we have,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = (-1)^{kl}(\omega \wedge \kappa)(\tau_1, \dots, \tau_{k+l}).$$

Some basic combinatorics argument shows us that,

$$(\omega \wedge \kappa) \wedge \xi = \omega \wedge (\kappa \wedge \xi)$$

At each point x, every cotangent vector can be written in terms of local coordinates φ as, $[f] = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(x)\right] dx_i$. where dx_i is the equivalence class corresponding to the function $\varphi_i(x) - x_i$. Since we expect differential forms to be smooth sections of the cotangent sheaf, every differential form can be written as,

$$\omega = \sum_{i=1}^{n} a_i dx_i.$$

where $a_i \in \mathcal{A}^{\mathcal{M}}$. Similarly, differential k-forms can be written as,

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1, \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

[1] a partition is a permutation such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+l)$.

where each $a_{i_1,\dots i_k} \in \mathcal{A}^{\mathcal{M}}$.

Using the tangency pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0},$$

we can pair a differential form and a vector field pointwise which measures the length of X using the differential form ω at each point. This is called a contraction or interior product of ω with X. Denoted by

$$\iota_X \omega \coloneqq \langle \omega, X \rangle$$

This can also be extended to differential k-forms.

$$\iota_X \omega(X_1, \dots, X_{k-1}) \coloneqq \omega(X, X_1, \dots, X_{k-1}).$$

We can now start listing down the algebraic properties of the contraction. For exterior product of differential forms is given by,

$$\iota_X(\omega \wedge \kappa) = (\iota_X \omega) \wedge \kappa + (-1)^p \omega \wedge (\iota_X \kappa) \,.$$

From the anti-symmetry of differential forms, we have, $\omega(X, Y, \dots) = -\omega(Y, X, \dots)$. So,

$$\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$$

The contraction provides a map,

$$\iota_X : \Omega^k \to \Omega^{k-1}.$$

For a differential k-form, and vector fields $X_1 \dots X_k$, we denote the evaluation by,

$$\langle \omega, (X_1 \dots X_k) \rangle = \omega(X_1 \dots X_k).$$

1.2 | Exterior Differentiation

Differential forms define at each point, the notion of length, area, volume, etc. Now we want to do calculus with them i.e., differentiate and integrate stuff. A k-form provides some sort of k-volume on tangent spaces. They are more complicated and Lie derivative doesn't describe how they change correctly. We want to define a notion of differentiation that captures all the ways in which it changes.

For a function $f \in \mathcal{A}^{\mathcal{M}}$, the differential is the flow of the 0-volume.

$$f : \mathcal{M} \to \mathbb{R}$$

gives us the map $df(x)$ of equivalence classes of curves, $\tau_h \mapsto \tau_{f \circ h}$. This is a map from $T_x\mathcal{M}$ to $\mathbb{R}$. Hence $df(x)$ is an element in the stalk of the cotangent pre-sheaf $\mathcal{C}$. Since this depends smoothly on the point x, it's a differential form i.e., $df \in \Omega^1$. Note here that this is the reason why the equivalence classes of functions $\varphi_i(x) - x_i$ were written as dx_i.

We now motivate the definition of 'exterior derivative' of differential forms. For a differential form ω, and vector fields X and Y, we have the pairings $\langle \omega, X \rangle$ and $\langle \omega, y \rangle$. Each of the pairings are differentiable functions on $\mathcal{M}$ i.e.,

$$\langle \omega, X \rangle, \langle \omega, Y \rangle \in \mathcal{A}^{\mathcal{M}}.$$

We are interested in understanding how the function $\langle \omega, X \rangle$ changes along another vector field Y, and $\langle \omega, Y \rangle$ changes along X. The change of $\langle \omega, X \rangle$ along Y is given by the new pairing, $L_Y(\langle \omega, X \rangle) = \langle d(\langle \omega, X \rangle), Y \rangle$, so the difference,

$$L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle)$$

is a differential 2-form. We want to take into account all the changes that are happening, so we should also take into account how X changes with respect to Y, as measured by the differential form ω which is $\omega(L_X(Y))$. So, we define the exterior derivative as,

$$(d\omega)(X, Y) = L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle) - \omega(L_X(Y)).$$

So, we have,

$$(d\omega)(X, Y) = X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \omega([X, Y]).$$

The negative signs are used so that this is a differential form. It's not yet clear why we have to take $\omega(L_X(Y))$ and not $\omega(L_Y(X))$ yet, this will become clearer in the uniqueness result in the next subsection. For a differential k-form, the exterior derivative is defined as,

$$\begin{aligned}(d\omega)(X_1 \dots X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1 \dots \widehat{X_i} \dots X_{k+1})) \\ &+ \sum_{1 \le j < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_{k+1}).\end{aligned} \tag{Exterior derivative}$$

The explicit form of the exterior derivative in terms of local coordinates, is given by $d\omega = \sum \partial a_i / \partial x_i dx_i \wedge dx_I$, where $\omega = \sum a_i dx_I$. So, the exterior derivative is a map,

$$d : \Omega^k \to \Omega^{k+1}$$

We can now start listing all the properties of the exterior derivative. From the definition, it follows that the exterior derivative is linear,

$$d(\lambda \omega + \mu \kappa) = \lambda d(\omega) + \mu d(\kappa).$$

For a vector field X, we have,

$$\begin{aligned}(d\iota_X + \iota_X d)(\omega)(X_1 \dots X_k) = &\sum (-1)^{i+1} X_i(\iota_X \omega(X_1 \dots \widehat{X_i} \dots X_k)) \\ &+ \sum (-1)^{i+j} (\iota_X \omega)([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) \\ &+ (d\omega)(X, X_1 \dots \widehat{X_i} \dots X_j \dots X_k)\end{aligned}$$

which on expanding gives,

$$= X(\omega(X_1 \dots X_k)) + \sum (-1)^i \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) = (L_X(\omega))(X_1 \dots X_k)$$

This is called Cartan formula, and can be written compactly as,

$$d\iota_X + \iota_X d = L_X$$

Let ω be a differential k-form and κ be a differential l-form, it can be checked that,

$$d(\omega \wedge \kappa) = (d\omega) \wedge \kappa + (-1)^k \omega \wedge (d\kappa)$$

Such maps are called derivations of odd type. It can be showed that[2],

$$d \circ d = 0.$$

The last property is very important and allows us to study the topological properties of the manifolds by studying the differential forms. The proofs of all these are very simple in local coordinates.

2 | Integration on Manifolds

To develop calculus, we need the ability to integrate on manifolds. As it turns out integration on manifolds is very closely related to differential forms. So, to start with we need a measure on the manifold. Since we expect the measure to respect the topology of the manifold, it should be a Borel measure. By Riesz duality theorem, measure on a locally compact Hausdorff space can be identified with a linear functional,

$$\mu : \mathcal{C}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{C}_c^{\mathcal{M}}$ with $f \geq 0$. Where $\mathcal{C}_c^{\mathcal{M}}$ are all compactly supported continuous functions. The positivity condition makes the functional continuous.

Differentiable functions with compact support form a self-adjoint subalgebra of this algebra of compactly supported continuous functions, and contain constants, and separate points of the space $\mathcal{M}$. Hence by Stone-Weierstrass theorem, the algebra $\mathcal{A}_c^{\mathcal{M}}$ of compactly supported differentiable functions is dense in $\mathcal{C}_c^{\mathcal{M}}$. Since by Hahn-Banach theorem, linear functionals on subalgebras uniquely extend to the whole algebra, we can study functionals,

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{A}_c^{\mathcal{M}}$ with $f \geq 0$.

So, by a Borel measure on a differentiable manifold $\mathcal{M}$ we mean a linear form μ on the vector space $\mathcal{A}_c^{\mathcal{M}}$ of differentiable functions with a compact support on $\mathcal{M}$ which satisfies certain continuity requirement i.e., for a sequence of compactly supported differentiable functions, $\{f_i\}$, with support contained in the compact set K, if $\sup\{|f_i|\} \to 0$ as $i \to \infty$ then $\mu(f_i) \to 0$.

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

The scalar $\mu(f)$ is denoted by $\int f d\mu$. On the space of compactly supported differentiable functions, we can define the sup norm making it into a Banach space. We can then start doing functional analysis. The measures under our consideration will be continuous linear functionals on $\mathcal{A}_c^{\mathcal{M}}$.

2.1 | The Sheaf of Differentiable Measures

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map. For any function $g \in \mathcal{A}_c^{\mathcal{N}}$, the composition, $f \circ \varkappa \in \mathcal{A}_c^{\mathcal{M}}$. This is because the manifold is Hausdorff, and hence the inverse image of compact set is compact. The image measure can then defined by,

$$(\varkappa^*(\mu))(f) = \mu(f \circ \varkappa).$$

[2]Proof of this using the coordinate expression is very simple and only involves using the fact that $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$. In fact all computations are easier done in local coordinates.

The continuity and linearity follow from continuity of differentiable functions.

Now with this composition, we can start defining the Lie derivative. Let $\varkappa_t$ be the flow of a vector field X, The Lie derivative of a measure μ with respect to the vector field X is the functional,

$$f \mapsto \lim_{t\to 0} \tfrac{((\varkappa_t^{-1})^*(\mu))(f)-\mu(f)}{t} = \mu\big(\lim_{t\to 0} \tfrac{f\circ\varkappa_t^{-1}-f}{t}\big).$$

In the last step, we used the fact that the measure is continuous, and took the limit inside. So we have,

$$L_X(\mu)(f) = -\mu(X(f)). \tag{Lie derivative}$$

This will be our notion of differentiation of measures. Multiple differentiations will be defined as multiple iterations of the Lie derivative of the measure. We say a Borel measure μ is indefinitely differentiable if the k times differentiations is a Borel measure for all k.

We now start looking at the collection of all indefinitely differentiable measures. Let $U \subseteq V$, then we have the natural inclusion of compactly supported functions functions $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$, by setting the functions to be equal to zero outside U.

Let $\mathcal{B}^{\mathcal{M}}U$ be the set of all differentiable measures on U, we have used $\mathcal{B}$ here for Borel. The inclusion $U \subset V$ gives rise to a restriction map of differentiable measures $\mu \mapsto \mu|_V$. The action of $\mu|_U$ on $\mathcal{A}_c^{\mathcal{M}}U$, is given by the action of μ on $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$. So,

$$\mathcal{B}^{\mathcal{M}} : U \mapsto \mathcal{B}^{\mathcal{M}}U$$

is a presheaf. Let $\{U_i\}$ be a locally finite family of open sets and μ_i be Borel measure on them. Suppose $\mu_i|_{U_i\cap U_j} = \mu_i|_{U_i\cap U_j}$ for all i, j, then we can define a measure μ on $U = \cup\{U_i\}$ by multiplying any function $f \in \mathcal{A}_c^{\mathcal{M}}U$ with a partition of unity associated with $\{U_i\}$, and then define,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f).$$

Since $\{U_i\}$ is locally finite, the sum is welldefined. Now, suppose $f \in \mathcal{A}_c^{\mathcal{M}}U_i$, then $\mu(f) = \sum_i \mu_i(\varphi_i f)$, since the support of f is contained in U_i, we have for every $U_i \cap U_j$, $\mu_i|_{U_i\cap U_j} = \mu_j|_{U_i\cap U_j}$ and hence we have,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f) = \sum_i \mu_j(\varphi_i f)$$

By linearity of measures this is

$$= \mu_j((\textstyle\sum_i \varphi_i)f) = \mu_j(f).$$

Hence the collation property holds, i.e., there exists an equilizer map e such that,

$$\mathcal{B}^{\mathcal{M}}U \overset{e}{\dashrightarrow} \prod_i \mathcal{B}^{\mathcal{M}}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(U_i \cap U_j).$$

Since the partition of unity is a differentiable map, the linear map μ is also continuous, and hence is a Borel measure. The differentiable measures on a differentiable manifold $\mathcal{M}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

If μ is a measure and $h \in \mathcal{A}^{\mathcal{M}}$, then the map,

$$f \mapsto \mu(hf)$$

is a linear form on $\mathcal{A}_c^{\mathcal{M}}$ and satisfies the continuity requirement i.e., if $\sup\{|hf_n|\}$ tends to zero then so does $\mu(hf_n)$. We will denote this measure by $h\mu$. Together with this notion of multiplication, $\mathcal{B}^{\mathcal{M}}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

Using $(L_X\mu)(f) = -\mu(Xf)$ the Lie derivative of this new measure $h\mu$ is then given by,

$$(L_{hX}\mu)(f) = -\mu(hX(f))$$

by using $X(hf) = fX(h) + hX(f)$ this is

$$= -\mu(X(hf) - fX(h)) = -\mu(X(hf)) + X(h)\mu(f)$$

So, we have,

$$L_X(h\mu) = h \cdot L_X(\mu) + L_X(h) \cdot \mu.$$

Similarly by expanding we get that,

$$L_{[X,Y]}(\mu) = L_X L_Y(\mu) - L_Y L_X(\mu).$$

So, the behavior of differentiable measures under Lie derivative is similar to that of differential forms. We are interested in measures that are locally translation invariant. Let V be a vector space, and μ be a measure. It's said to be translation invariant if $\varkappa_t^*\mu = \mu$ for $\varkappa_t = tv$ or equivalently, $L_{\partial_v}\mu = 0$ for all v.

2.2 | Relation to Differential Forms

To motivate the relation between differential forms and measures we will look at the standard Euclidean space $\mathcal{V}$. The translational invariance makes sure that we can determine the measure of any measurable set if we know its value for, say, a cube, because cubes generate the Borel σ-algebra, and the translational invariance allows us to measure any scaled copy of the cube. The translation invariant measures are determined uniquely upto a constant multiplication, and for Euclidean space it corresponds to the Lebesgue measure, upto scalar multiplication. So, the measure μ determines a map,

$$\begin{aligned} \mu: \ \textstyle\prod^n \mathcal{V} &\to \mathbb{R}. \\ (v_i) &\mapsto \mu([v_i]), \end{aligned}$$

where $[v_i]$ is the n-dimensional cube in V determined by the vectors $\{v_i\}$, an invariant measure μ is uniquely determined by its value for the cube $[v_i]$, $\mu([v_i])$. $\mu([v_i])$ is non-zero only if $\{v_i\}$ forms a basis of V. For integer r, we have $[v_1, \dots rv_i, \dots v_n] = [v_1 \dots v_i \dots v_n] \coprod [v_1 \dots (r-1)v_i, \dots v_n]$, hence it follows by induction that $\mu([v_1, \dots rv_i, \dots v_n]) = r\mu([v_i])$. It can be showed that it's $\mu([v_i])$ is multilinear in v_i, Now each of the cube can be divided up into k smaller cubes along v_i, i.e., $[v_1, \dots v_i, \dots v_n] = \coprod_{i=1}^k [v_1, \dots (\frac{i}{k})v_i, \dots v_n]$, and by invariance of the measure each of these smaller cubes are of same measure. Hence, $\mu([v_1, \dots rv_i, \dots v_n]) = r\mu([v_i])$, holds for rational values and by continuity of μ it must also hold for real values. So the map, $(v_i) \mapsto \mu([v_i])$ is multilinear in v_i, and since if any two v_is are equal we should have the measure to be zero, it's an alternating multilinear map and must factor through $\Lambda^n\mathcal{V}$.

$$\begin{tikzcd} \prod^n \mathcal{V} \arrow[r, "i"] \arrow[dr, "\mu"'] & \bigwedge^n \mathcal{V} \arrow[d, "\exists!\ \tilde{\mu}"] \\ & \mathbb{R} \end{tikzcd}$$

So, $\mu([v_i]) = \tilde{\mu}(\wedge_{i=1}^n v_i)$. Equivalently, we get a map from the space of invariant measures into the one-dimensional space $(\wedge^n\mathcal{V})^\vee = \Omega^n\mathcal{V}$.

$$\mu \mapsto \tilde{\mu}$$

The wedge product however is order sensitive, and the measure is not. So we should have,

$$\tilde{\mu}(\wedge^n v_i) = \pm\mu([v_i]).$$

This can be interpreted in the following sense, consider $\wedge^n \mathcal{V}\backslash\{0\}$, which has two connected components. The assignmemt of the sign 1 (respectively -1) in the definition if $v_1 \wedge \cdots \wedge v_n$ belongs to the chosen component (or not). The choice of a connected component is equivalent to chosing a basis for $\wedge^n \mathcal{V}^\vee$. Such a choice of basis in $\wedge^n \mathcal{V}^\vee$ is called a volume element of $\mathcal{V}$. The map,

$$\begin{aligned} H : \mathcal{L} &\to \Omega^n(\mathcal{V}) \\ \mu &\mapsto \tilde{\mu}. \end{aligned}$$

is an isomorphism of the space of invariant measures $\mathcal{L}$ and the space of n-differential forms $\Omega^n \mathcal{V}$ depending on the choice of basis for $\Lambda^n \mathcal{V}$. The choice of basis is called the orientation of the vector space $\mathcal{V}$. They vary by a sign for different orientation. In this case, $\Omega^n \mathcal{V} = \mathcal{A}^{\mathcal{V}} \otimes_{\mathbb{R}} \Lambda^n(\mathcal{V})^\vee$ and similarly we can tensor the space of invariant measures with $\mathcal{A}^{\mathcal{V}}$ of differentiable functions on $\mathcal{V}$, this is a subsheaf of $\mathcal{B}^{\mathcal{V}}$ consisting of measures of the form $f \cdot \mu$ for $f \in \mathcal{A}^{\mathcal{V}}$ and $\mu \in \mathcal{L}$. So the isomorphism above yields an isomorphism of these sheaves.

Coming back to differentiable manifolds, we can say that the sheaf of differentiable measures $\mathcal{B}^{\mathcal{M}}$ is closely related to the sheaf of n-differentials. We want a homomorphism from the sheaf of differential n forms to invariant measures inside $\mathcal{B}^{\mathcal{M}}$. Since the information about the measure being invariant has to do with Lie derivatives, we just have to preserve that structure, i.e, performing Lie derivation before the homomorphism should be the same as taking Lie derivation after the homomorphism.

Definition 2.1. An $\mathcal{A}^{\mathcal{M}}$-homomorphism α of sheaves, $\alpha : \Omega^n \to \mathcal{B}^{\mathcal{M}}$ is said to be flat if it commutes with Lie derivatives with respect to all differentiable vector fields, i.e.,

$$\alpha L_X = L_X \alpha.$$

Flat homomorphisms take invariant forms to invariant measures. So, any two non-zero flat homomorphisms of Ω^n into $\mathcal{B}^{\mathcal{M}}$ differ by a non-zero scalar factor. The above relation of $\mu \to \tilde{\mu}$ guarantees the existence of such homomorphisms when $\mathcal{M} = \mathbb{R}^n$, once the choice of orientation is made. This might not be possible in general case, because although we might construct flat homomorphisms locally, in charts, they depend on the choice of orientation, locally. Such a choice depends on the local coordinate system and the orientations might be negatives of each other.

Given two flat homomorphisms, take invariant form ω to invariant measures. Since invariant measures are uniquely determined upto a scalar factor, two flat homomorphisms must differ by a non-zero scalar factor at each point.

2.2.1 | Stokes Theorem

References

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