Part Ib

The Fourier Transform & Pontrjagin Duality

We will assume that $\mathcal{G}$ is a locally compact abelian group. Some immediate facts are that the left and right translations coincide, the modular homomorphism is trivial, and convolution is commutative.

$$f * g(x) = \int f(xy^{-1})g(y)d\lambda(y) = g * f(x).$$

Involution is given by,

$$f^*(x) = \overline{f(x^{-1})}.$$

By Schur's lemma, every irreducible representation will be one dimensional.

1 | The Fourier Transform

Dual group, and Fourier transforms of locally compact abelian groups are very similar to the techniques developed in the Gelfand-Naimark theory of commutative C^*-algebras.

1.1 | The Dual Group

If π is an irreducible unitary representation of a locally compact abelian group $\mathcal{G}$ then $\mathcal{H}_\pi \cong \mathbb{C}$. In such a case, there exists $\chi(x)$ for every $x \in \mathcal{G}$ such that

$$\pi(x)(z) = \chi(x)z$$

where χ is a continuous homomorphism of $\mathcal{G}$ into the circle group $\mathbb{T}$. By unitarity of $\chi(x)$ we have, $|\chi(x)| = 1$. So, $\chi(x)$ satisfies, $\chi(xy) = \chi(x)\chi(y)$. Such homomorphisms are called characters of $\mathcal{G}$. The collection of all characters of $\mathcal{G}$ is denoted by $\widehat{\mathcal{G}}$. For a character χ, we will denote $\chi(x)$ by $\langle x, \chi \rangle$. Unitary representations of $\mathcal{G}$ determines a $*$-homomorphism of $L^1(\mathcal{G})$ on the representation space $\mathcal{H}_\pi$, which in our case is $\mathbb{C}$. This representation is given by,

$$\chi(f) = \int_{x \in \mathcal{G}} \langle x, \chi \rangle f(x) d\lambda(x).$$

By identifying the bounded linear maps on $\mathcal{B}(\mathbb{C})$ with $\mathbb{C}$, this action determines a multiplicative functional on $L^1(\mathcal{G})$,

$$f \mapsto \chi(f)$$

We will use the notation from Gelfand theory for spectrum of an algebra, $\sigma(\mathcal{A})$ will denote the set of all non-zero algebra homomorphisms from a Banach algebra $\mathcal{A}$ to $\mathbb{C}$. The Banach algebra of interest to us is the group algebra $L^1(\mathcal{G})$.

Let φ be a linear functional on $L^1(\mathcal{G})$, by the Riesz representation theorem, on φ there must exist some $\chi \in L^\infty(\mathcal{G})$ such that,

$$\varphi(f) = \int_{x\in\mathcal{G}} f(x)\chi(x)d\lambda(x)$$

for all $f \in L^1(\mathcal{G})$. Suppose φ is a multiplicative linear functional, that is, $\varphi \in \sigma(L^1(\mathcal{G}))$, then we have, for any $f, g \in L^1(\mathcal{G})$,

$$\begin{aligned}\int_{x\in\mathcal{G}} \big[\varphi(f)\chi(x)\big] g(x)d\lambda(x) &= \varphi(f)\int_{x\in\mathcal{G}} g(x)\chi(x)d\lambda(x) = \varphi(f)\varphi(g) = \varphi(f * g)\\ &= \iint \chi(y)f(yx^{-1})g(x)d\lambda(x)d\lambda(y) = \int_{x\in\mathcal{G}} \big[\varphi(L_x f)\big] g(x)dx.\end{aligned}$$

Since this holds for every $g \in L^1(\mathcal{G})$, the square bracketed terms on both sides must be the same almost everywhere. That is to say,

$$\varphi(f)\chi(x) = \varphi(L_x f)$$

almost everywhere. Choose a function $f \in L^1(\mathcal{G})$ such that $\varphi(f) \neq 0$, then we can define,

$$\chi(x) \coloneqq \varphi(L_x f)/\varphi(f).$$

For any $x, y \in \mathcal{G}$, by the above computation, we have,

$$\chi(xy)\varphi(f) = \varphi(L_{xy}f) = \varphi(L_x L_y f) = \chi(x)\chi(y)\varphi(f).$$

So, we have,

$$\chi(xy) = \chi(x)\chi(y).$$

So, χ is a group homomorphism from $\mathcal{G}$ to the circle group $\mathbb{T}$. So, every character gives rise to a multiplicative functional on $L^1(\mathcal{G})$ and conversely, every multiplicative functional on $L^1(\mathcal{G})$ corresponds to a character.

Theorem 1.1.

$$\sigma(L^1(\mathcal{G})) \cong \widehat{\mathcal{G}}.$$

With the pointwise multiplication $\chi_i \cdot \chi_j(x) \coloneqq \chi_i(x)\chi_j(x)$ and pointwise inverse $\chi^{-1}(x) = \big(\chi(x)\big)^{-1}$, the set $\widehat{\mathcal{G}}$ is an abelian group. It's called the dual group of $\mathcal{G}$. The following is a useful computational tool,

$$\langle x, \chi^{-1}\rangle = \langle x^{-1}, \chi\rangle = \overline{\langle x, \chi\rangle}.$$

Since $\widehat{\mathcal{G}}$ is identified with spectrum of $L^1(\mathcal{G})$, we can introduce the appropriate topology $\widehat{\mathcal{G}}$ from $L^\infty(\mathcal{G})$, which is the weak* topology since we expect characters to be 'close' to each other if their evalutations are 'close'. This topology coincides with the topology of compact convergence on $L^\infty(\mathcal{G})$, see §3.3 [2]. The set of all homomorphism from $L^1(\mathcal{G})$ to $\mathbb{C}$ is the set $\widehat{\mathcal{G}} \cup \{0\}$. By Alaoglu's theorem $\widehat{\mathcal{G}} \cup \{0\}$ is compact, or, $\widehat{\mathcal{G}}$ is locally compact.

Theorem 1.2. *$\widehat{\mathcal{G}}$ is a locally compact abelian group.*

1.2 | The Fourier Transform

Since $\widehat{\mathcal{G}}$ and $\sigma(L^1(\mathcal{G}))$ are identified, with the isomorphism, consider the composition with the inverse, which associates with the character χ the functional, $f \mapsto \overline{\chi}(f) = \chi^{-1}(f)$.

Recall that the Gelfand transform Γ on a Banach $*$-algebra $\mathcal{A}$, is a map,

$$\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$$

which sends an operator A to a continuous function on the space of characters, given by evaluation. In our case, $\mathcal{A} \equiv L^1(\mathcal{G})$ and $C(\sigma(\mathcal{A})) = C(\widehat{\mathcal{G}})$. The Gelfand transformation on the Banach $*$-algebra $L^1(\mathcal{G})$ is called the Fourier transform on $\mathcal{G}$. The Fourier transform on $\mathcal{G}$ is then the map,

$$f \mapsto \mathcal{F}f$$

whose action on characters of $\mathcal{G}$ is given by,

$$\mathcal{F}f(\chi) = \int_{x\in\mathcal{G}} \overline{\langle x, \chi\rangle} f(x) d\lambda(x).$$

Note that this assigns to each $f \in L^1(\mathcal{G})$ a continuous function $\mathcal{F}(f)$ on the space of characters. Since characters are homomorphisms, we have,

$$\begin{aligned}
\mathcal{F}(f * g)(\chi) &= \iint_{x,y\in\mathcal{G}} \overline{\langle x, \chi\rangle} f(xy^{-1}) g(y) d\lambda(y) d\lambda(x) \\
&= \iint_{x,y\in\mathcal{G}} \overline{\langle xy, \chi\rangle} f(x) g(y) d\lambda(y) d\lambda(x) \\
&= \iint_{x,y\in\mathcal{G}} \overline{\langle y, \chi\rangle} \cdot \overline{\langle x, \chi\rangle} f(x) g(y) d\lambda(y) d\lambda(x) = \mathcal{F}f(\chi)\mathcal{F}g(\chi).
\end{aligned}$$

So, Fourier transform is an algebra homomorphism. Similarly,

$$\mathcal{F}(f^*)(\chi) = \int_{x\in\mathcal{G}} \overline{\langle x, \chi\rangle f(x^{-1})} d\lambda(x) = \int_{x\in\mathcal{G}} \langle x, \chi\rangle \overline{f(x)} d\lambda(x) = \overline{\int_{x\in\mathcal{G}} \overline{\langle x, \chi\rangle} f(x) d\lambda(x)} = \overline{\mathcal{F}(f)(\chi)}.$$

So, the Fourier transform is a $*$-homomorphism. The norm of the Fourier transform is,

$$\|\mathcal{F}(f)\|_{\sup} = \sup_{\chi\in\widehat{\mathcal{G}}} \left|\mathcal{F}(f)(\chi)\right| \leq |\sup_{x\in\mathcal{G}} \langle x, \chi\rangle| \left| \int f(x) d\lambda(x) \right| \leq \|f\|_1.$$

So,

$$\mathcal{F} : f \mapsto \mathcal{F}f$$

is a norm descreasing $*$-homomorphism. $\mathcal{F}(L^1(\mathcal{G}))$ is a selfadjoint subalgebra that separates points in $\widehat{\mathcal{G}}$, and by Stone-Weierstrass theorem, it is dense subspace of $C_0(\widehat{\mathcal{G}})$, consisting of continuous functions vanishing at infinity.

2 | The Pontrjagin Duality

References

[1] T Tao, Haar Measure and the Peter-Weyl Theorem. https://terrytao.wordpress.com/2011/09/27/254a-notes-3-haar-measure-and-the-peter-weyl-theorem/

[2] G Folland, A Course in Abstract Harmonic Analysis. CRC Press, 2015