Part I

Axioms of AQFT

Every physical theory starts with a set of physical ideas, for relativistic quantum theory we want this set to be the union of the quantum ideas and relativity ideas. In order to do this one starts with the very basic structural requirement of quantum theories i.e., a von Neumann algebra describing the effects of the quantum theory. We are interested in putting constraints on this set of effects coming from the Einstein-Weyl causality. The standard reference to the subject are Haag's book, [1], and Araki's book, [2].

1 | Operational Quantum Theory

The notion of 'state' is defined in terms of the preparation procedure. A preparation procedure is characterized by the kind of system it prepares. The other important thing is the existence of a measuring instrument that is capable of undergoing changes upon their interaction. The observable change is called an 'effect'. Any measurement can be interpreted as a combination of yes-no measurements. These yes-no instruments can be used to build any general instrument. Suppose we have such an instrument, label its registration procedure by R. If the experiment is conducted a lot of times, we get a relative frequency of occurrence of 'yes'. To every preparation procedure ρ and registration procedure R_i there exists a probability $\mu(\rho, R_i)$ of occurrence of 'yes' associated with the pair.

$$(\rho, R_i) \longrightarrow \mu(\rho|R_i).$$

The numbers $\mu(\rho|R_i)$ are called operational statistics. Two completely different preparation procedures may give the same probabilities for all experiments R. Such preparation procedures must be considered equivalent. Such preparation procedures are called operationally equivalent preparations. A precursor to the notion of a state of the system is an equivalence class of preparations procedures yielding the same result. They are called ensembles.

Denote the class of ensembles by S and the class of effects by E. The maps of interest to us are the following,

$$S \times E \xrightarrow{\mu} [0,1].$$

There may be two experiments that give the same probabilities for every ensemble. Such apparatuses must be considered equivalent. They are called operationally equivalent effects. An effect is the equivalence class of apparatuses yielding the same result. In general, a registration procedure R for an experiment will have outcomes $\{R_i\}_{i\in I}$. For an outcome, R_i of the registration procedure R, denotes the corresponding equivalence class of measurement procedures by E_{R_i}. Each outcome R_i of the registration procedure corresponds to a functional E_{R_i} called the effect of R_i that acts on the ensemble of the system to yield the corresponding probability.

$$E_{R_i} : \rho \mapsto E_{R_i}(\rho) = \mu(\rho|R_i).$$

Maps of interest to us will be those that assign to each of its outcomes R_i its associated effect E_{R_i}. Since each ensemble fixes a probability distribution we have,

$$\mu_\rho : R_i \mapsto \mu_\rho(R_i) = \mu(\rho|R_i).$$

The above-given map μ_ρ is determined by the instrument and the registration procedure. If we can form mixtures and the set of ensembles is convex we call the theory convex operational theory. Since a mixture of ensembles corresponds to a convex combination of probabilities each functional E_{R_i} preserves the convex structure. Since two preparations giving the same result on every effect represent the same ensemble and two measurement procedures that can't distinguish ensemble represent the same effect, ensembles and effects are mutually separating. A generalized probabilistic theory or a GPT for short is an association of a convex state space and effect vectors to a given system, such that the states and effects are uniquely determined by the probabilities they produce. This is known as the principle of tomography.

One takes an operational theory and 'quotients' with operational equivalences to obtain a GPT. Denote by $\mathcal{S}$ the set of maps, $f : E \longrightarrow \mathbb{R}$ such that $f(X) = \sum_i \alpha_i \mu(\rho_i|X)$ and denote by $\mathcal{E}$ the set of maps, $g : S \longrightarrow \mathbb{R}$ such that $g(\rho) = \sum_i \beta_i \mu(\rho|R_i)$ where ρ_i and R_i are ensembles and effects respectively and $\alpha_i, \beta_i \in \mathbb{R}$. Clearly $\mathcal{S}$ and $\mathcal{E}$ are real vector spaces. We can embed ensembles inside $\mathcal{S}$ with the map,

$$\rho \longmapsto \mu_\rho,$$

and similarly embed effects inside $\mathcal{E}$ with the map,

$$R_i \longmapsto E_{R_i}.$$

The bilinear map $\langle\cdot|\cdot\rangle : \mathcal{S} \times \mathcal{E} \to \mathbb{R}$ which coincides with μ is then uniquely determined.

It's important to note here that the notion of effect is an abstract concept. A measuring instrument is only a representative of the equivalence class corresponding to the effect. Such a measurement apparatus need not exist. Similarly for ensembles. These abstract notions apply to any generalized probabilistic theories or GPTs for short.

The effects of interest to us are identified with projection operators in a von Neumann algebras and states of interest to us are normal states on these von Neumann algebras. The minimal structure a quantum theory should have is this. So, a quantum system is a pair, $(\mathcal{A}, \omega)$, where $\mathcal{A}$ is a von Neumann algebra and ω is a normal state on $\mathcal{A}$, together with a group of automorphisms α_g on $\mathcal{A}$ of the symmetries of the system. The effects of the quantum system are given by the projection operators in $\mathcal{A}$.

Postulate. (Effects)

$$\mathcal{E} \equiv \mathcal{P}(\mathcal{A})$$

It's physically acceptable to assume the measurement scales to be separable, i.e., they have a countable dense subset, for example, $\mathbb{R}$ is a separable measurement scale. So we expect the observables to have the same property. If the algebra of observables is $\mathcal{A}$ and A is a self-adjoint operator corresponding to some observable then we should expect the projections correpsonding to values of the observable to be of at most countable cardinality.

A von Neumann algebra $\mathcal{A}$ is σ-finite if all collections of mutually orthogonal projections have at most a countable cardinality. von Neumann algebras on separable Hilbert spaces are always σ-finite. We will assume von Neumann algebras in our discussion to be σ-finite.

A quantum mechanical observable is a map of the form,

$$E_A:\ \Sigma_A \to \mathcal{P}(\mathcal{A}),$$

where Σ_A is a Boolean lattice, and the map E_A is a projection valued Boolean algebra homomorphism. These correspond to the collection of effects that can be measured by a typical measurement instruments. Usually in physical experiments, the statements that can be made are of the type 'the value of the observable lies in some set ϵ_i of real numbers'. To accommodate the fact that the measurement scale is composed of real numbers, we identify Σ_A with the Borel sets of $\mathbb{R}$. The quantum observables are analogous to classical random variables, namely, that of a projection valued measure,

$$E_A:\ \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{A}).$$

A spectral measure is a projection operator-valued function E defined on the sets of $\mathbb{R}$ such that, $E(\mathbb{R}) = I$ and $E(\sqcup_i \epsilon_i) = \sum_i E(\epsilon_i)$, where ϵ_is are disjoint Borel sets of $\mathbb{R}$. The spectral theorem says that every self-adjoint operator A corresponds to a spectral measure E_A such that,

$$A = \int \lambda \, dE_A(\lambda),$$

and conversely, every spectral measure corresponds to a self-adjoint operator. In the finite-dimensional case this reduces to $A = \sum_i \lambda_i E_i$ where E_is are projections onto eigenspaces of λ_is. The characteristic feature of quantum theory is that the product structure on effects is a non-commutative.

The mathematical representatives of the physical states for the quantum case are the maps, $\omega : \mathcal{E} \to [0,1]$, such that $\omega(0) = 0$, $\omega(E^{\perp}) = 1 - \omega(E)$ and $\omega(\vee_i E_i) = \sum_i \omega(E_i)$ for mutually orthogonal E_i. For an observable with the associated self-adjoint operator A, the map

$$\mu^A = \omega \circ E_A :\ \Sigma_A = \mathcal{B}(\mathbb{R}) \to [0,1],$$

So, in quantum theory the physical states correspond to normal states which are ultraweakly continuous positive linear functionals of norm one on the von Neumann algebra $\mathcal{A}$.

Postulate. (States) *The physical states correspond to normal states on $\mathcal{A}$.*

It's important to note that the notion of effects and states are abstract mathematical objects used to describe the structure of observed phenomenon. An instrument can act as a representative of the equivalence class of an effect. Such an instrument might not exist. For example, it doesn't make sense to talk about measuring instruments in the early universe. Similarly for preparation instruments.

After performing a measurement, an experimenter notices an effect. Accordingly the information contained in the state is updated and the state 'collapses'. This is a quantum mechanical event corresponding to the observed effect.

2 | Axioms of AQFT

The starting point of AQFT is the observation that every measurement is performed in some finite region of space and in a limited time period. We can associate to the observable that particular space-time region. Assuming these regions of space-time are open subsets of the space-time X, we have a map that assigns to each open set $\mathcal{O}$ of the space-time the algebra of observables $\mathcal{A}(\mathcal{O})$ that are measured in the region.

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}).$$

This is called a net of observable algebras. This has property similar to restriction property of presheaves. In case of presheaves, we had a restriction map which mapped functions on larger domains to give functions in smaller domains contained in the larger domain. The situation here is opposite to the presheaf situation.

If an observable can be measured in a space-time region $\mathcal{O}_1$, then it can be measured in any space-time region $\mathcal{O}_2$ containing $\mathcal{O}_1$. If $\mathcal{O}_1 \subset \mathcal{O}_2$ we have an associated co-restriction map, such that, for every observable $A \in \mathcal{A}(\mathcal{O}_1)$ we have the co-restriction $A|^{\mathcal{O}_2} \in \mathcal{A}(\mathcal{O}_2)$. The notation $|^{\mathcal{O}_2}$ is the co-restriction to $\mathcal{O}_2$. The map, $A \mapsto A|^{\mathcal{O}_2}$ is a function $\mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$.

If $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3$ are nested open sets then the co-restriction is transitive.

$$(A|^{\mathcal{O}_2})|^{\mathcal{O}_3} = A|^{\mathcal{O}_3}.$$

This can be summarised by saying the assignment $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is a functor,

$$\mathcal{A} : \mathcal{O}(X) \to \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $\mathcal{O}_1 \to \mathcal{O}_2$ are inclusions $\mathcal{O}_1 \subset \mathcal{O}_2$. To each such inclusion morphism in $\mathcal{O}(X)$ we get restriction morphism in **Sets**, $\{\mathcal{O}_1 \subset \mathcal{O}_2\} \mapsto \{\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)\}$, given by $A \mapsto A|^{\mathcal{O}_2}$.

This is a co-presheaf. A co-presheaf on a category $\mathcal{C}$ is a sheaf on the opposite category $\mathcal{C}^{\mathrm{op}}$. The co-restriction map are required to be inclusion maps coming from the subalgebra inclusion so we will forget about the notation $|^{\mathcal{O}_2}$ and denote $A|^{\mathcal{O}_2}$ by A. This is captured in by the following monotone property,

Postulate (Monotone Property). *If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.*

The algebra $\mathcal{A}(\mathcal{O})$s are called local algebras. They are the algebras of observables 'local' to that region. The closure of the total algebra consisting of all the observables is a directed co-limit of the local algebras. The closure of this co-limit is denoted $\mathcal{A}$, and is called the quasi-local algebra. If we assume the observables in all the regions combined generate the whole algebra of observables, and if $\{\mathcal{O}\}$ is an open cover of the space-time by bounded open sets then,

$$\overline{\vee_{\mathcal{O}}\mathcal{A}(\mathcal{O})} = \mathcal{A}.$$

The algebras $\mathcal{A}(\mathcal{O})$ are called local algebras. Since double cones form a basis for the topology of Minkowski space-time these would be nice choice for $\mathcal{O}$s.

The quasi-local algebra $\mathcal{A}$ is the starting point of AQFT. On this algebra we would like to impose the additional constraint coming from the Einstein-Weyl causality.

2.1 | Einstein-Weyl Causality

Space-time will be assumed to be flat in this section. Let $\mathcal{G}$ be the group of automorphisms of space-time that preserve Einstein-Weyl causal order on space-time. Depending on whether a space-time event can influence some other space-time event we have a partial order on the space-time events or points in space-time.[1]

Let g be the Minkowski metric, we write $x \prec y$ if $g(y-x) \geq 0$. An automorphism of space-time T is a causal automorphism if

$$x \prec y \iff Tx \prec Ty.$$

This automatically means T^{-1} is also such a map. Let $\mathcal{G}$ be the group of all causal automorphisms. Zeeman in [3], has shown that the group of automorphisms that preserve Einstein-Weyl causal order on the Minkowski space is the orthochronous Poincaré group and dilatations. Much of the proof involves proving light rays are mapped to light rays, and parallel light rays to parallel light rays. [4] has a detailed proof.

We are interested in understanding what happens to the observables upon the action of the causal automorphisms. If an observable in a region $\mathcal{O}$ is transformed by a proper orthochronous Poincaré transformation g we obtain an observable in the region $g\mathcal{O}$. So we should expect to each such causal automorphism g an algebra automorphism α_g such that,

$$\alpha_g \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(g\mathcal{O}).$$

But we expect the same for g^{-1}. So we have the reverse inclusion as well.

$$\alpha_{g^{-1}} \mathcal{A}(g\mathcal{O}) \subset \mathcal{A}(g^{-1}g\mathcal{O}) = \mathcal{A}(\mathcal{O}).$$

This amounts to the physical idea that one can repeat experiments at different places and at different times.

Postulate (Covariance). *For $g \in \mathcal{P}_+^\uparrow$ we have $\alpha_g \mathcal{A}(\mathcal{O}) = \mathcal{A}(g\mathcal{O})$.*

This gives us the action of the Poincaré group on the algebra of observables. This however doesn't capture the Einstein-Weyl causality completely. The Einstein-Weyl causality states that space-like separated regions cannot influence each other. In our case we have a net of algebras. The Einstein-Weyl causality in terms of algebra means that the effects belonging to space-like separated regions shouldn't be related, i.e., the effects in these regions should commute with each other.

If two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated means that quantum events in the region $\mathcal{O}_1$, i.e., updating the state with operators belonging to $\mathcal{A}(\mathcal{O}_1)$ and quantum events in the region $\mathcal{O}_2$ do not interfere with each other. If we perform measurement in both regions $\mathcal{O}_1$ and $\mathcal{O}_2$ we should get same result as if they were performed separately. The measurements values can be known together. So, the operators in $\mathcal{A}(\mathcal{O}_1)$ must commute with operators in $\mathcal{A}(\mathcal{O}_2)$.

[1] Note that we should not confuse between a space-time event and a quantum mechanical event. This is a syntax-sematics correspondence issue. In quantum setting an event is the process of updating the state, in relativity setting its points in space-time. The notion of effects however are related to space-time events.

Postulate (Locality). *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then the operators in $\mathcal{A}(\mathcal{O}_1)$ must commute with operators in $\mathcal{A}(\mathcal{O}_2)$. We denote this by,*

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0.$$

This is a very important postulate of AQFT. As we will see later this has very interesting consequences, mainly due to the work of H J Borchers, H-W Wiesbrock, B Shroer, R Brunetti, R Longo, D Guido, D Buchholz, S J Summers and their collaborators. In classical relativity formulated by Einstein and other, one started with a set of events, i.e., space-time points, then the relation between these points was provided by the property of light i.e., Einstein-Weyl causality. In case of quantum theory we start with the algebra of observables, the properties of light is carried by the subalgebra relations.

A | The Poincaré Group

References

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[4] G Naber, The Geometry of Minkowski Spacetime. Springer, 2012