Part IV

Covering spaces & Applications

In these notes we start looking at the étale space of sheaf of holomorphic functions. The étale space of holomorphic functions has special properties that allow us to use the techniques of covering spaces and elementary homotopy theory.

1 | Recalling Elementary Homotopy Theory

Homotopy theory we study spaces that are equivalent upto deformation. The way we relate two spaces is through functions between them. We have to understand what is meant by deformation.

Let X and Y be two spaces two maps from X to Y are said to be homotopic if there exist continuous family of functions that deforms the first function to the other. Let $f, g : X \to Y$ be two functions, the above sentence means that there exists a continuous function $F : X \times I \to Y$ such that,

$$F|_{X\times\{0\}} = f, \quad F|_{X\times\{1\}} = g. \tag{1H}$$

In such a case $f \sim g$ and F is called the homotopy between f and g. The set of all equivalence classes of continuous functions between the spaces X and Y are denoted by $[X, Y]$.

We say two spaces X and Y are homotopy equivalent if we can deform the space X to Y and reverse deform to get back X. This is equivalent to saying, there exist functions $f : X \to Y$ and $g : Y \to X$ such that their compositions are homotopic to the identity function.

$$f \circ g \sim \mathbb{I}_Y, \quad g \circ f \sim \mathbb{I}_X. \tag{2H}$$

Now, the space X can be probed using paths and loops. The idea is to first embed I inside X. A path in a topological space X is a continuous map from the interval I to X,

$$\eta : I \to X.$$

Two paths η_1 and η_2 are equivalent if they are homotopic. Denote the equivalence class of η by $[\eta]$. The set of all loops in X will be denoted by $[I, X]$. On this we can give a group structure. The inverse of a loop η is defined by,

$$\tilde{\eta}(t) = \eta(1 - t)$$

We can define a notion of product on these equivalence classes using the concatenation operation. The equivalence class of this concatenation will provide the product operation on the equivalence class of all paths.

$$[\eta_1][\eta_2] = [\eta_1 \star \eta_2].$$

For continuity of the concatenation we require the ending point of the first path is the same as the starting point of the second path. The constant map is the identity loop. With this product structure the set of equivalence classes of all loops at a point $x_0 \in X$ is a group,

$$\Pi_1(X, x_0) = [I, X]/\sim$$

This group is called the fundamental group of X at x_0.

Let $f : X \to Y$ be a continuous function. Then to each path η of X we have a path in Y given by the composition $f \circ \eta$,

$$I \xrightarrow{\eta} X \xrightarrow{f} Y$$

Each continuous map induces a homomorphism of the fundamental groups $\Pi_1(X, x_0)$ to $\Pi_1(Y, f(x_0))$. We will forget about the base point for the sake of notational simplicity.

$$\Pi_1(X) \xrightarrow{f_*} \Pi_1(Y)$$

Homeomorphisms of spaces give rise to isomorphism of fundamental groups. A space X is called simply connected if $\Pi_1(X, x_0) = \{1\}$. In such a case any two paths with the same end points are homotopic as we can form a loop using these paths which is in turn homotopic to the constant map.

Let X and X' be two topological spaces, be a local homeomorphism $\pi : X' \to X$ is called a covering space if each $x \in X$ has a connected neighborhood V such every connected component of $\pi^{-1}(V)$ is homeomorphic to V.

Suppose $\pi : X' \to X$ be a covering space, we have for each $x \in X$ a neighborhood V or x such that,

$$\pi^{-1}(V) = \coprod_{i \in I} V_i,$$

such that $\pi|_{V_i} : V_i \cong V$. Such a neighborhood is called an evenly covered neighborhood. Let $x \in X$ and let $|\pi^{-1}(x)|$ be the number of elements in $\pi^{-1}(x)$. Let

$$\mathcal{P}_{|\pi^{-1}(x)|} = \{y \in X \mid |\pi^{-1}(y)| = |\pi^{-1}(x)|\},$$

be the set of all points in X which have $|\pi^{-1}(x)|$ inverse images. This set is open because of local homeomorphism and since the complement is also open the set $\mathcal{P}_{|\pi^{-1}(x)|}$ is also closed. Since $x \in \mathcal{P}_{|\pi^{-1}(x)|}$, it's nonempty.

Theorem 1.1. *If $\pi : X' \to X$ is a covering then $\deg(\pi) = |\pi^{-1}(x)|$ is a constant.* □

Since covering spaces are locally homeomorphic we can lift a function $f : Y \to X$ to a function $\tilde{f} : Y \to X'$ using the local homeomorphism. $\tilde{f} : Y \to X'$ is called a lift of $f : Y \to X$ if the diagram,

$$\begin{tikzcd} & X' \arrow[d, "\pi"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

commutes or $f = \tilde{f} \circ \pi$.

Let $\pi : X' \to X$ be a covering space and X' be path connected. Let $\eta : I \to X$ be a path in X. The space X can be written as $X = \bigcup_i U_i$, where U_is are evenly covered neighborhoods. The interval I can be partitioned $I = \cup_i I_i$ such that the image of each part I_i under η belongs entirely in one of U_i. (Note that this requires the Lebesgue lemma) The initial point $\eta(0) \in X$ belongs to some neighborhood U_i we can lift the path using the local homeomorphism to get a lift of the path and continue from the end of this part. Without using the Lebesgue lemma this can be proved via induction.

Similarly we can lift homotopes to the covering space. So if two paths are homotopic in the base space then their lifts are also homotopic if they have the same initial point.

Theorem 1.2. (Path Lifting) *Every path $\eta : I \to X$ has a unique lift $\tilde{\eta} : I \to X'$ with $\tilde{\eta}(0) = x'$, every homotopy $F : I \times I \to X$ has a unique lift $\tilde{F} : I \times I \to X'$.* □

If $\pi : X' \to X$ is a covering space then $\Pi_1(X')$ is a subgroup of $\Pi_1(X)$ and the degree of the map is given by,

$$\deg(\pi) = |\Pi_1(X) : \Pi_1(X')|.$$

the proof is routine well-definedness checks and injective verification. If $\pi : X' \to X$ is a local homeomorphism with the curve lifting property then we can lift every path, and hence to each element in the fundamental group we get an element. $\pi : X' \to X$ is a covering map if and only if it has the curve lifting property.

Let $\pi : X' \to X$ be a covering space and let $f : Y \to X$ be a continuous map with Y connected. Let $\tilde{f}_1, \tilde{f}_2$ be two lifts of f. Let

$$\mathcal{F} = \{y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\}$$

This is an open set of Y because around each $\tilde{f}_1(y)$ we can choose a uniformly covered neighborhood U such that $U \cong \pi(U)$. Since f is continuous $f^{-1}(U)$ is open with $x \in f^{-1}(U)$ where $\tilde{f}_1|_U = \tilde{f}_2|_U$.

Consider $Y \backslash \mathcal{F}$. Suppose Y is Hausdorff, let $y \in Y \backslash \mathcal{F}$ i.e., $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ choose disjoint neighborhoods $\tilde{U}_1, \tilde{U}_2$ of $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ respectively such that

$$\pi|_{\tilde{U}_i} : U_i \cong U \subset X.$$

Then $V = f^{-1}(U) \subseteq Y \backslash \mathcal{F}$ or i.e., $Y \backslash \mathcal{F}$ is open. So, $\mathcal{F}$ is both open and closed. If there exists some $y \in Y$ for which $\tilde{f}_1(y) = \tilde{f}_2(y)$ then the lifts are the same.

$$\begin{tikzcd} & X' \arrow[d, "\pi"] \\ Y \arrow[ur, bend left, "\tilde{f}_2"] \arrow[ur, "\tilde{f}_1"'] \arrow[r, "f"] & X \end{tikzcd}$$

Theorem 1.3. (Uniqueness) *Let Y be connected, and X, X', Y be Hausdorff. If $\tilde{f}_1, \tilde{f}_2$ are two lifts of $f : Y \to X$ such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$. Then $\tilde{f}_1 = \tilde{f}_2$.* □

Theorem 1.4. (Lifting Criterion)

2 | Sheaf of Holomorphic Functions

The sheaf of holomorphic functions has some special properties. As we will show, the étale space of the sheaf of holomorphic functions is a Hausdorff space and hence the uniqueness theorems in the previous section are applicable. This allows us to apply theorems of algebraic topology to the study of holomorphic functions.

To each open set $U \subseteq X$ corresponds the associated collection of holomorphic functions $\mathcal{H}U$. Each $\mathcal{H}U$ has an algebra structure. Consider the sheaf of holomorphic functions,

$$\mathcal{H} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that for each open covering $U = \bigcup_{i \in I} U_i$ of an open set U of X comes with an equilizer diagram,

$$\mathcal{H}U \overset{e}{\dashrightarrow} \prod_i \mathcal{H}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{H}(U_i \cap U_j).$$

where p and q are the maps,

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

The collection of open sets has a preorder given by, $V \geq U$ if $V \subset U$. For any directed collection $\mathcal{D}$ of open sets we have a directed system in **Sets** given by $\{\mathcal{H}U\}_{U \in \mathcal{D}}$. The stalk $\mathcal{H}_x$ of the sheaf $\mathcal{H}$ at x is the direct limit of the directed system $\{\mathcal{H}U_i\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{H}_x = \varinjlim_{x \in U} \mathcal{H}U.$$

stalks are functors, $\mathrm{Stalk}_x : \mathrm{PSh}(X) \to \mathbf{Sets}$ with, $\mathcal{H} \mapsto \mathcal{H}_x$. The elements of $\mathcal{H}_x$ are called germs of holomorphic functions at x.

$$\begin{aligned} \mathrm{germ}_x : \mathcal{H}(U) &\to \mathcal{H}_x \\ f &\mapsto \mathrm{germ}_x(f). \end{aligned}$$

$\mathrm{germ}_x : \mathcal{H}U \to \mathcal{H}_x$, is a homomorphism of the respective category for each U. If $f, g \in \mathcal{H}U$ such that $\mathrm{germ}_x f = \mathrm{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset U$ such that $f|_{U_x} = g|_{U_x}$ i.e., they have the same power series expansion around x.

Bundle the sets $\mathcal{F}_x$ into a disjoint union,

$$\mathcal{E}\mathcal{F} = \coprod_x \mathcal{F}_x,$$

and define the map, $\pi : \mathcal{E}\mathcal{F} \to X$ that sends each $\mathrm{germ}_x f$ to the point x.

Each $f \in \mathcal{F}U$ determines a function from $\tilde{f} : U \to \mathcal{E}\mathcal{F}$ that maps

$$\tilde{f} : x \mapsto \mathrm{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{F}$ by taking as base of open sets all the image sets $\tilde{f}(U) \subset \mathcal{E}\mathcal{F}$. This topology makes both π and $\tilde{f}$ continuous by construction. Each point $\mathrm{germ}_x f$ in $\mathcal{E}\mathcal{F}$ has an open neighborhood $\tilde{f}(U)$. π restricted to $\tilde{f} : U \to \tilde{f}(U)$, is a homeomorphism. The space $\mathcal{E}\mathcal{F}$ together with the topology just defined is called the étale space of $\mathcal{F}$.

$\mathcal{H} : U \to \mathcal{H}(U)$. The sets $\mathcal{H}(U)$ have additional structure i.e., they are algebras. Two holomorphic functions $f, g : U \to \mathbb{C}$ have the same 'germ' at a point $c \in U$ if they have their power series expansion around c are the same. Let be a pre-sheaf on X. Let $f \in \mathcal{H}U$ and $g \in \mathcal{H}V$. We say f and g have the same germ at c if there exists some open set $W \subset U \cap V$ with $c \in W$ such that $f|_W = g|_W$. This relation is an equivalence relation. Denote by the equivalence class of functions with a representative element f that have the same germ at c as f by, $\mathrm{germ}_c f$. Let

$$\mathcal{H}_c = \{\mathrm{germ}_c f \mid f \in \mathcal{H}U,\ c \in U, U \in \mathcal{O}(X)\}.$$

be the set of all germs at c called stalk of $\mathcal{H}$ at c. Taking 'germ at c' is a functor, $\mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}} \to \mathbf{Sets}$, which maps the pre-sheaf $\mathcal{H}$ to the stalk $\mathcal{H}_c$. Combine the various sets $\mathcal{H}_c$ into a disjoint union,

$$\mathcal{E}_{\mathcal{H}} = \coprod_z \mathcal{H}_z,$$

and define the map, $\pi : \mathcal{E}_{\mathcal{H}} \to X$ that sends each $\mathrm{germ}_z f$ to the point z. Each $f \in \mathcal{H}U$ determines a function from $\tilde{f} : U \to \mathcal{E}_{\mathcal{H}}$ that maps $z \mapsto \mathrm{germ}_z f$ for $z \in U$. By using these 'sections', we can put a topology on $\mathcal{E}_{\mathcal{H}}$ by taking as base of open sets all the image sets $\tilde{f}(U) \subset \mathcal{E}_{\mathcal{H}}$. This topology makes both π and $\tilde{f}$ continuous. Each point $\mathrm{germ}_z f$ in $\mathcal{E}_{\mathcal{H}}$ has an open neighborhood $\tilde{f}(U)$. π restricted to

$$\tilde{f} : U \to \tilde{f}(U),$$

is a homeomorphism. Hence, for the case of pre-sheaf of holomorphic functions, $\mathcal{E}_{\mathcal{H}}$ is a 2-dimensional manifold. The map, $\mathcal{H} \mapsto \mathcal{E}_{\mathcal{H}}$ is a functor from pre-sheaves to bundles. The $\pi : \mathcal{E}_{\mathcal{H}} \to X$ is a local homeomorphism. $\mathcal{E}_{\mathcal{H}}$ for a general pre-sheaf is not Hausdorff. As an example consider the pre-sheaf $\mathcal{P}$ of continuous real valued functions on the real line. The two functions from $\mathbb{R}$ to $\mathbb{R}$ given by, $g(x) = 0\ \forall x \in \mathbb{R},\ \&\ f(x) = x^2$ for $x > 0$ and $f(x) = 0$ for $x < 0$ represent the same germ, $\mathrm{germ}_t f$ for $t < 0$. So every neighborhood of $\mathrm{germ}_0 f$ intersects with every neighborhood of $\mathrm{germ}_0 g$. So the two points $\mathrm{germ}_0 g$ and $\mathrm{germ}_0 f$ cannot be separated by disjoint open sets in $\mathcal{E}_{\mathcal{P}}$.

But for the case of holomorphic functions if two functions represent the same germ then they have the same power series expansion in a neighborhood. So for the pre-sheaf of holomorphic functions $\mathcal{H}$, the space $\mathcal{E}_{\mathcal{H}}$ is Hausdorff.

Given a pre-sheaf $\mathcal{H} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, the topological space $\mathcal{E}_{\mathcal{H}}$ together with the map

$$\pi : \mathcal{E}_{\mathcal{H}} \to X$$

is a bundle over X. Let $\Gamma\mathcal{E}_{\mathcal{H}}$ denote the sections of the bundle $\mathcal{E}_{\mathcal{H}} \to X$. For each open set $U \in \mathcal{O}(X)$ there is a function,

$$\eta_U : \mathcal{H}(U) \to \Gamma\mathcal{E}_{\mathcal{H}}$$

3 | Sheaf of Holomorphic Functions

References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer Science+Business Media, LLC, 2000