Part I

Category of Pre-Sheaves

This chapter is an abstract categorical study of pre-sheaves. Category theory develops very general tools, which can be applied to many mathematical phenomena. The use of category theory in sheaf theory was started by Grothendieck based on the following philosophical position,

Yoneda-Grothendieck. An object is determined by its relation to other objects.

The Yoneda lemma makes this precise and exploitable. Yoneda lemma embeds a given category inside the category of functors from the given category to the category of sets, via the functor, 'maps to the given object' or 'maps from the given object'. This allows us to utilize the nice properties of the target category in our case the category of sets.

1 | Category of Functors

A set is a collection of 'elements'. A category $\mathcal{A}$ is more sophisticated, it possesses 'objects' similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{A}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects. Studying objects by their relations to other objects could be called the Yoneda-Grothendieck philosophy.

A functor $\mathcal{F}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of a mapping of objects of $\mathcal{A}$ to objects of $\mathcal{B}$, $X \mapsto \mathcal{F}X$ together with a map of the set of homomorphisms,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y).$$

the image of $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ denoted by $\mathcal{F}(f)$. That takes identity to identity and respects composition[1] i.e.,

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ can be thought of as a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. A functor $\mathcal{F}$ is faithful

[1]The composition $f \circ g$ assumes they are composable.

if the map $\mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$ is injective for all X, Y. It's full if the map is surjective. If it's a bijection the functor is called fully faithful.

A natural transformation κ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, denoted by,

$$\kappa : \mathcal{F} \Rightarrow \mathcal{G},$$

is a collection of mappings κ_X for every $X \in \mathcal{A}$, such that for all $f : X \to Y$, the diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & \mathcal{F}X \arrow[r, "\kappa_X"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}X \arrow[d, "\mathcal{G}(f)"] \\ Y & \mathcal{F}Y \arrow[r, "\kappa_Y"'] & \mathcal{G}Y \end{tikzcd} \qquad \text{(natural transformation)}$$

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors $\mathcal{F}$ and $\mathcal{G}$ is denoted by,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

We say two functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $\mathcal{F} \cong \mathcal{G}$. The collections of all functors from $\mathcal{A}$ to $\mathcal{B}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{B}^{\mathcal{A}}$. The nice thing about functor category $\mathcal{B}^{\mathcal{A}}$ is that it inherits many of the useful properties of the category $\mathcal{B}$.

Equivalence of two categories can be thought of as giving two complementary description of same mathematical object. We can compare two categories $\mathcal{A}$ and $\mathcal{B}$ via the functors between them. The starting point is the functor category $\mathcal{B}^{\mathcal{A}}$.

A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if there is a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that

$$\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}, \quad \text{and} \quad \mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}},$$

where the identity functor $\mathbb{1}_{\mathcal{A}}$ sends objects of $\mathcal{A}$ to the same objects, and morphisms to the same morphisms. $\mathcal{G}$ is called quasi-inverse functor. In such a case, $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent. Quotient categories can be defined when we have an equivalence relation on the collection of morphisms. The objects remain the same, and the hom-sets get quotiented.

2 | Representable Functors

Each $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ tells us about all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{A}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{A}}(-, X)$. These are called hom functors.

$$\begin{aligned} h^X : \ & \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X). \end{aligned}$$

which maps each morphism $f : Y \to Z$ to a morphism of hom sets given by the composition,

$$Y \xrightarrow{f} Z \xrightarrow{g} X$$

We will denote this by,

$$\begin{aligned} h^X(f): \ h^X(Y) &\to h^X(Z) \\ g &\mapsto g \circ f. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{A}}(Y, X)$s are all sets. Such categories are called locally small categories.

A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be $\mathcal{G} \cong h_X$. Where $\cong$ stands for natural isomorphism.

2.1 | Yoneda Embedding

Yoneda embedding and representable functors allow us to use the nice properties (ability to take limits) of the category of sets to study more complex categories that are not so nice. We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\mathrm{Hom}_{\mathcal{A}}(-, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, -)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{A}$ and the functor $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$.

Theorem 2.1. (Yoneda Lemma) *For a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{A}$, there is a natural bijection,*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \cong \mathcal{F}X \tag{Yoneda}$$

such that $\kappa \in \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \leftrightarrow \kappa_X(\mathbb{1}_X) \in \mathcal{F}X$.

Proof

In the natural transformation diagram, replace $\mathcal{F}$ by h^X, and $\mathcal{G}$ by $\mathcal{F}$. $\kappa_X : h^X X \to \mathcal{F}X$. Now, $h^X X = \mathrm{Hom}_{\mathcal{A}}(X, X)$, which contains $\mathbb{1}_X$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) &\to \mathcal{F}X \\ \kappa &\mapsto \kappa_X(\mathbb{1}_X). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $Y \in \mathcal{A}$. For any $f : Y \to X$, we have,

$$\begin{tikzcd} X & h^X X \arrow[r, "\kappa_X"] \arrow[d, "h^X(f)"'] & \mathcal{F}X \arrow[d, "\mathcal{F}(f)"] & \mathbb{1}_X \arrow[r, mapsto, "\kappa_X"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ Y \arrow[u, "f"] & h^X Y \arrow[r, "\kappa_Y"'] & \mathcal{F}Y & f \arrow[r, mapsto, "\kappa_Y"'] & \kappa_Y(f) \end{tikzcd}$$

Hence $\kappa_Y(f) = \mathcal{F}(f)(\mu(\kappa))$, or the action of κ_Y is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_Y(f) = \varphi_Y(f)$ for all $Y \in \mathcal{A}$, so it's injective.

For surjectivity we have to show that for all sets $x \in \mathcal{F}X$, there exists a natural transformation φ such that $\varphi_X(\mathbb{1}_X) = x$. For $x \in \mathcal{F}X$, and $f : Y \to X$, construct the map,

$$\begin{aligned} \varphi : h^X &\to \mathcal{F} \\ f &\mapsto \mathcal{F}(f)(x). \end{aligned}$$

this satisfies the requirement that $\varphi_X(\mathbb{1}_X) = x$, because clearly, $\mathbb{1}_X \mapsto \mathcal{F}(\mathbb{1}_X)(x) = \mathbb{1}_x(x) = x$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} Y & h^X Y \arrow[r, "\varphi_Y"] \arrow[d, "h^X(g)"'] & \mathcal{F}Y \arrow[d, "\mathcal{F}(g)"] \\ Z \arrow[u, "g"] & h^X Z \arrow[r, "\varphi_Z"'] & \mathcal{F}Z \end{tikzcd}$$

commutes for all $Y, Z \in \mathcal{A}$, $g \in \mathrm{Hom}_{\mathcal{A}}(Z, Y)$. For $f : Y \to X$, by definition of φ,

$$\mathcal{F}(g) \circ (\varphi_Y(f)) = \mathcal{F}(g) \circ \mathcal{F}(f)(x)$$

which by functoriality of $\mathcal{F}$ is $= \mathcal{F}(f \circ g)(x)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_Z \circ (h_X(g)(f)) = \varphi_Z(h_X(f \circ g))$$

which again by definition of φ is $= \mathcal{F}(f \circ g)(x)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \to \mathcal{F}X$ is a bijection. $\square$

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof covariant version is exactly the same, just have to reverse the arrows on the category $\mathcal{A}$. The Yoneda lemma gives us an embedding of the category $\mathcal{A}$ inside the functor category $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, given by,

$$X \mapsto h^X.$$

This embedding is called the Yoneda embedding $h^{(-)} : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, which sends an object $X \in \mathcal{A}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{A}}(-, X)$. These functors are fully faithful by Yoneda lemma, because by replacing the functor $\mathcal{F}$ by h^Y we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, h^Y) \cong h^Y(X) = \mathrm{Hom}_{\mathcal{A}}(X, Y). \tag{weak Yoneda}$$

Similarly for the covariant embedding, in which case this will be $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{A}}(Y, X)$.

Given a contravariant functor, $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$, the Yoneda tells us that we can think of the action of $\mathcal{F}$ on the element X as natural transformations to the hom functor h^X in the functor category. So, every functor $\mathcal{F}$ can extended and be thought of as a representable functor,

$$\begin{aligned} h^{\mathcal{F}} : (\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{G} &\mapsto \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(\mathcal{G}, \mathcal{F}) \end{aligned}$$

where elements $X \in \mathcal{A}$ are to be interpreted as the elements $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$. The following frequently used corollary of Yoneda lemma allows us to compare objects of locally small category by their hom-functors, and hence using the properties of the category of sets.

Lemma 2.2. (Yoneda Principle)

$$h^X \cong h^Y \Rightarrow X \cong Y. \tag{Yoneda principle}$$

$\square$

Note that, Yoneda associates to each set in $\mathcal{F}X$ a natural transformation between h^X and $\mathcal{F}$. If the functor $\mathcal{F}$ is representable, i.e., there exists $Y \in \mathcal{A}$ such that there exists a natural isomorphism,

$$\mathcal{F} \xrightarrow[\alpha]{\cong} h^Y$$

Let $\mu(\alpha)$ be the corresponding element in $\mathcal{F}Y = \mathrm{Hom}_{\mathcal{A}}(Y, Y)$. The pair $(Y, \mu(\alpha))$ is called a universal object for $\mathcal{F}$. It's such that for any other object $Z \in \mathcal{A}$, and each $g \in \mathcal{F}X = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ there exists a unique morphism $f : X \to Y$ such that,

$$\mathcal{F}(f)(\mu(\alpha)) = g.$$

Digression: Enriched Categories

The hom-sets in practice usually are richer than merely being sets, they come equipped with additional structure. Enriched categories are categories, where the hom-sets have additional structure. Many of the constructions we can make in **Sets** can also be done in many enriched categories. Here will informally discuss the minimal necessary stuff from the theory of enriched categories so that the reader doesn't feel out of place.

An enriched category is a category in which the hom-sets come equipped with additional structure, that is, the hom-sets are objects in some *enriching* category, usually denoted by $\mathcal{V}$. $\mathcal{V}$ is called the base for enrichment. This already requires the category $\mathcal{V}$ to have some special properties.

Given any two composable morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, and $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$ in the category $\mathcal{A}$, we can consider the composition of the morphisms,

$$g \circ f \in \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

If $\mathcal{A}$ is a $\mathcal{V}$ enriched category, then $f, g, g \circ f \in \mathcal{V}$. So the notion of composition of morphisms in the category $\mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

should correspond to a notion of 'composition of objects' or a product in the enriching category,

$$\odot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

There should be unit object corresponding to the identity morphisms, denoted by $\mathbb{1}_{\mathcal{V}}$ such that multiplication by the unit object leaves every object unchanged. This can be formalized by saying there exist natural transformations such that,

$$\mathbb{1}_{\mathcal{V}} \odot V \cong V \odot \mathbb{1}_{\mathcal{V}} \cong V \tag{identity}$$

for all $V \in \mathcal{V}$. Since composition of morphisms is associative, we want $\odot$ to also be associative. This can be formalized by saying there exist natural transformations such that,

$$U \odot (V \odot W) \cong (U \odot V) \odot W \tag{associativity}$$

These properties can also be formalized in terms of a commutative diagram but we will skip that. A category $\mathcal{V}$ with a 'product' $\odot$ with identity, and associativity is called a monoidal category. $\odot$ is called the monoidal product. If in addition the product is such that $U \odot V \cong V \odot U$ it's called a symmetric monoidal category.

A category $\mathcal{A}$ is said to be enriched by a monoidal category $\mathcal{V}$ if the hom-sets belong to $\mathcal{V}$ and the composition corresponds to the monoidal product $\odot$. We will denote the hom-sets of a $\mathcal{V}$-enriched category $\mathcal{A}$ by,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X,Y)$$

for all $X, Y \in \mathcal{A}$. When there is no confusion we will drop the superscript $\mathcal{V}$, or sometimes $\mathcal{A}^{\mathcal{V}}(X,Y)$

A closed monoidal category is monoidal category $\mathcal{V}$ where the functors $- \odot V : \mathcal{V} \to \mathcal{V}$ admits a right adjoint denoted by $\mathcal{H}om^{\mathcal{V}}(V, -)$. The family of right adjoints assemble in a unique way to give a bifunctor,

$$\mathcal{H}om^{\mathcal{V}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}.$$

such that,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(U \times V, W) \cong \mathrm{Hom}^{\mathcal{V}}_{\mathcal{V}}(U, \mathcal{H}om^{\mathcal{V}}(V, W))$$

for all $U, V, W \in \mathcal{V}$. $\mathcal{H}om^{\mathcal{V}}$ is called the internal hom. The internal homs act as the product $\odot$, and hence, $\mathcal{V}$ together with $\mathcal{H}om^{\mathcal{V}}$ is an enriched category over itself. A category is called cartesian closed if it's locally small, that is, it's enriched by the category of sets, and if $\odot$ is the cartesian product in **Sets**.

The small $\mathcal{V}$-categories themselves form a category. A $\mathcal{V}$-functor is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between $\mathcal{V}$-enriched categories such that the morphisms,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X,Y) \xrightarrow{\mathcal{F}_{X,Y}} \mathrm{Hom}^{\mathcal{V}}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$$

commute with the $\odot$ operation and the identity. Similarly a $\mathcal{V}$-natural transformation between a pair of $\mathcal{V}$-functors $\mathcal{F}$ and $\mathcal{G}$ consist of natural transformation between $\mathcal{F}$ and $\mathcal{G}$ such that it commutes with composition by the natural transformations in identity and associativity. The collection of all $\mathcal{V}$-natural transformations between $\mathcal{F}$ and $\mathcal{G}$ will be denoted by,

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}^{\mathcal{B}}}(\mathcal{F}, \mathcal{G})$$

We will now state the enriched Yoneda lemma without proof.

Theorem 2.3. (Enriched Yoneda Lemma) *$\mathcal{A}$ be a small $\mathcal{V}$-category, then, for all $\mathcal{V}$-functors $\mathcal{F} : \mathcal{A} \to \mathcal{V}$,*

$$\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}^{\mathcal{V}}}(\mathrm{Hom}^{\mathcal{V}}_{\mathcal{A}}(X, -), \mathcal{F}) \cong \mathcal{F}X.$$

□

The ideas of the proofs are the same as those discussed the generic ones but with more verification (verifying the natural transformations preserve the enriching structure and so on.). Similarly we will have a notion of limits and colimits for the enriched categories.

2.2 | Representable Constructions

The Yoneda-Grothendieck philosophy now has a precise formulation; the properties of a category can be thought of as representability properties in functor category. A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to$ **Sets** is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \qquad \text{(representable)}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be

$$\mathcal{G} \cong h_X$$

where $\cong$ stands for natural isomorphism. We will now see this in the following examples.

2.2.1 | (Co)Products

Let $\mathcal{A}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{A}$ indexed by a set I, then we can consider the contravariant functor,

$$\mathcal{G} : Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $\mathcal{G}(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, P)$. This is called the product, denoted by, $\prod_{i\in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
 & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"'] \arrow[dr, "f_j"] & \\
X_i & \prod_{i\in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I. So, the property that the category $\mathcal{A}$ has products, is translated into a statement that certain functor is representable.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $\mathcal{F}(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i\in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\
 & Y &
\end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

The categorical notions of product and coproduct correspond to the arithematic operations such as multiplication and addition. We can similarly talk about exponentiation. In the category of sets, **Sets**, for $X, Z \in \mathcal{A}$, Z^X is the function set consisting of all functions $h : X \to Z$. Here we have the bijection,

$$\mathrm{Hom}_{\mathbf{Sets}}(Y \times X, Z) \to \mathrm{Hom}_{\mathbf{Sets}}(Y, Z^X).$$

for a function, $f : Y \times X \to Z$, this map sends each $y \in Y$ to the function $f(y, -) \in Z^X$. Conversely given a function $f' : Y \to Z^X$, we can define a map $f(y, x) = f'(y)(x)$. So,

$$\mathrm{Hom}_{\mathbf{Sets}}(Y \times X, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(Y, Z^X)$$

or equivalently, $(-)^X$ is the right adjoint of $(-) \times X$. By setting $Y = 1$, we obtain,

$$Z^X \cong \mathrm{Hom}_{\mathbf{Sets}}(1, Z^X) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, Z).$$

Exponentiation can be representably defined in terms of products. Suppose $\mathcal{A}$ has products, it has exponentiation if for every Y, Z, the functor,

$$X \mapsto \mathrm{Hom}_{\mathcal{A}}(Y \times X, Z)$$

is representable.

2.2.2 | (Co)Kernel

For sets, the kernel of two maps s, t is defined as the set $\ker(s, t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{A}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"] \arrow[drr, bend left, "j"] & & \\
\ker(f,g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object Z is called a zero object if for any object A, there exists a unique morphism $Z \to A$ and a unique morphism $A \to Z$. It's unique upto isomorphism and denoted by 0. Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \to \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{A}}(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\left(\mathrm{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(Z, X)\right).$$

This can be visualized by the diagram,

$$\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f,g) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd}
Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

2.2.3 | Pullback or Fibered Product

Let $\mathcal{I}$ be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \to Z$ then for functors $\mathcal{F} : \mathcal{I} \to \mathcal{A}$, pullback $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$\mathcal{F}Y \xrightarrow{f} \mathcal{F}X \xleftarrow{g} \mathcal{F}Z$$

in a category $\mathcal{A}$ is the commutative square with vertex $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ such that any other commutative square factors through it, i.e.,

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_P"] \arrow[ddr, bend right, "g_P"'] & & \\
& \mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}Z \arrow[d, "g"] \\
& \mathcal{F}Y \arrow[r, "f"'] & \mathcal{F}X
\end{tikzcd}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ as above,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{F}Z \arrow[d, dashed] \arrow[ddr, bend left, "f_P"] & \\
\mathcal{F}Y \arrow[r, dashed] \arrow[drr, bend right, "g_P"'] & \mathcal{F}Y \coprod_{\mathcal{F}X} \mathcal{F}Z \arrow[dr, dashed, "\exists!"] & \\
& & C
\end{tikzcd}$$

3 | Adjoint Situations

Categories are compared by means of functors, and functors themselves are compared via natural transformations. Equivalence of categories allows us to basically think of the two categories as the same thing. This is however too restrictive. The relaxation of the notion of equivalence gives us the notion of adjoint.

The philosophy of adjoint functors is the following; when we want to study an object in mathematics, belonging to some weird category, we can take it, via a functor to some well understood category. But now this new category will not have the same meaning to the objects as the original category. So we would like a functor to get back to the original category. This functor is the adjoint functor.

An adjuntion from $\mathcal{A}$ to $\mathcal{B}$ is a pair of functors,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B},$$

such that there is a natural isomorphisms of bifunctors $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y)$ and $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \tag{adjoint}$$

for all $X \in \mathcal{A}, Y \in \mathcal{B}$. Denote by $\mathcal{F} \dashv \mathcal{G}$. Since composition of natural isomorphisms is also a natural isomorphism if $\mathcal{F}$ has two adjoints $\mathcal{G}$ and $\widehat{\mathcal{G}}$, then we have,

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, \widehat{\mathcal{G}}Y).$$

So, by Yoneda principle, adjoints if they exist are unique upto isomorphism. Consider the following two adjoint situations,

$$\mathcal{A} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{B} \underset{\mathcal{K}}{\overset{\mathcal{H}}{\rightleftarrows}} \mathcal{C},$$

By definition, we have for all $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{K}Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{H} \circ \mathcal{F}X, Y)$, hence,

$$\mathcal{F} \circ \mathcal{H} \dashv \mathcal{G} \circ \mathcal{K}$$

When we are working with locally small categories, we can exploit the properties of the category of sets. We can look at adjoints from a functor category perspective, and representable functors.

Given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, for each $X \in \mathcal{B}$, we have the composite functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) &:= h^X \circ \mathcal{F} : \mathcal{A} \to \mathcal{B} \to \mathbf{Sets} \\ & \qquad A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

So, $\widehat{\mathcal{F}}$ is a functor to the functor category,

$$\widehat{\mathcal{F}} : \mathcal{B} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}},$$

which sends $X \mapsto \widehat{\mathcal{F}}(X)$. For each morphism $f : X \to Y$ in $\mathcal{B}$, the functor $\widehat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \to X$,

$$\widehat{\mathcal{F}}(f) : g \mapsto f \circ g$$

So, $\widehat{\mathcal{F}}(f \circ h) = \widehat{\mathcal{F}}(f) \circ \widehat{\mathcal{F}}(h)$, i.e., it's a covariant functor.

Lemma 3.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$ admits a left adjoint iff for all $X \in \mathcal{B}$,*

$$\widehat{\mathcal{F}}(X) : A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$$

is representable.

Proof

$\Leftarrow$ Suppose $\widehat{\mathcal{F}}(X)$ is representable for all $X \in \mathcal{B}$, then, $\exists\ \mathcal{G}X \in \mathcal{A}$ with, $\widehat{\mathcal{F}}(X) \cong h^{\mathcal{G}X}$, i.e.,

$$\widehat{\mathcal{F}}(X)(A) \cong \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X)$$

We have to make sure this is functorial, i.e., $X \mapsto \mathcal{G}X$ is a functor from $\mathcal{B}$ to $\mathcal{A}$. So, now we have to show that for each morphism $f : X \to Y$ there exists a morphism in the functor category, a natural transformation of functors, $\mathcal{G}f : \widehat{\mathcal{F}}(X) \to \widehat{\mathcal{F}}(Y)$, defined to be the maps that makes the following diagram commute.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}X) \arrow[r] \arrow[d, "\mathcal{G}(f)\circ"'] & \widehat{F}(X)(A) \arrow[d, "\widehat{\mathcal{F}}(f)"] \\ \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}Y) \arrow[r] & \widehat{F}(Y)(A) \end{tikzcd}$$

This also satisfies the composition needs by construction. By Yoneda lemma, this determines the functor $\mathcal{G}$ uniquely upto isomorphism.

$\Rightarrow$ The other direction is obvious and follows directly from the definition of adjoint, i.e., if there exists a left adjoint $\mathcal{G} \dashv \mathcal{F}$ each functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ is representable with representative $\mathcal{G}X$. $\square$

3.1 | Adjoints as Reflections

The notion of reflection of a functor provides a bit more intuitive meaning of what adjoints are doing. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a functor. We want to associate to each object $B \in \mathcal{B}$ and object $R_B \in \mathcal{A}$ such that $\mathcal{F}R_B$ is the best estimate of B in $\mathcal{A}$. In categorical terms, the estimation is done with morphisms. So, the 'best estimate' is a morphism,

$$\kappa_B : B \to \mathcal{F}R_B$$

such that for any other $A \in \mathcal{A}$, with an estimation $\varkappa : B \to \mathcal{F}A$ factors uniquely through R_B. R_B together with the morphism κ_B is called the reflection of B along $\mathcal{F}$. Visualised by the diagram,

$$\begin{tikzcd} R_B \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_B \arrow[d, "\mathcal{F}(f)"'] & B \arrow[l, "\kappa_B"'] \arrow[dl, "\varkappa"] \\ A & \mathcal{F}A & \end{tikzcd} \tag{reflection}$$

that's to say there exists a unique morphism $f : R_B \to A$ such that,

$$\mathcal{F}(f) \circ \kappa_B = \varkappa.$$

Intuitively κ_B is a better estimate than $\varkappa$. We can't have two best estimates κ_B and κ'_B because in that case we have two maps $f : R_B \to R'_B$ and $f' : R'_B \to R_B$, such that,

$$\mathcal{F}(f) \circ \kappa_B = \kappa'_B, \quad \mathcal{F}(f') \circ \kappa'_B = \kappa_B$$

So we get,

$$\mathcal{F}(f \circ f') \circ \kappa'_B = \kappa'_B,$$

By uniqueness this means $f \circ f' = \mathbb{1}_{R'_B}$, so any two reflections are isomorphic.

Lemma 3.2. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$, suppose reflection exists for each $B \in \mathcal{B}$, then there exists a unique functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{G}B = R_B$ and a natural transformation κ such that,*

$$\kappa_B : B \to \mathcal{F} \circ \mathcal{G}B.$$

Proof

The existence of reflection to each object in $\mathcal{B}$ gives us an associated object for each object in $\mathcal{B}$, we want to understand what happens to the morphisms.

Let $f : X \to Y$ be a morphism in $\mathcal{B}$. Then we have,

$$\begin{tikzcd} R_X \arrow[d, dashed, "\exists! f"'] & \mathcal{F}R_X \arrow[d, "\mathcal{F}(f)"'] & X \arrow[l, "\kappa_X"'] \arrow[d, "g"] \\ R_Y & \mathcal{F}R_Y & Y \arrow[l, "\kappa_Y"] \end{tikzcd}$$

In this diagram, $\kappa_Y \circ g : X \to \mathcal{F}R_Y$ is an estimate, and hence there must exist a morphism $f : R_X \to R_Y$ such that $\mathcal{F}(f) \circ \kappa_X = \kappa_Y \circ g$. So, we define,

$$\mathcal{G}(g) \coloneqq f.$$

By construction this makes κ a natural transformation which is determined by the components κ_X. By exploiting uniqueness we can show that this is functorial, that is $\mathcal{G}(g \circ h) = \mathcal{G}(g) \circ \mathcal{G}(h)$. $\square$

If the functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a reflection, by definition, to each morphism $\varkappa : X \to \mathcal{F}A$ there exists a unique morphism $f : \mathcal{G}X \to A$. Conversely, any map f uniquely determines $\varkappa$ by,

$$\mathcal{F}(f) \circ \kappa_X = \varkappa.$$

Which means we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, A) \cong \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}A).$$

Since this holds for every X and A it is the adjoint condition. So, adjoint and reflection are the same functor. Note that in the construction of the 'reflection functor' we assumed every object in $\mathcal{B}$ has a reflection. The adjoint functor theorems try to simplify this condition, under additional constraints.

4 | (Co)Limits

The notion of limits and colimits is very important as they allow us to construct new objects and functors. They are also very closely related to adjoint functors. To heuristacally motivate, limits is the categorical notion of 'closest' object to or from a system. Here, we intend to find an object that's nearest to a category. The notion of nearness comes from the morphisms, so the idea is to put the starting category in some other category, where the morphisms provide some sort of 'categorical distance', and then use this notion of distance of the target category to describe the 'limit'.

Let $\mathcal{I}$ and $\mathcal{A}$ be two categories. An inductive system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{F} : \mathcal{I} \to \mathcal{A}.$$

Intuitively, the limit of a system is an object in $\mathcal{A}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $X \in \mathcal{A}$ the constant functor $\Delta X : \mathcal{I} \to \mathcal{A}$ that sends everything in $\mathcal{I}$ to X, and each morphism in $\mathcal{I}$ to the identity on X. A relation between an object X and the system $\mathcal{F}$ is a natural transformation between ΔX and $\mathcal{F}$. Such a natural transformation is called a cone. The collection of all such cones is the set of all natural transformations,

$$\begin{aligned} C_{\mathcal{F}} : \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}). \end{aligned}$$

It's a contravariant functor from $\mathcal{A}$ to **Sets**. If the functor $C_{\mathcal{F}}$ is representable, there exists an object $L_{\mathcal{F}} \in \mathcal{A}$ such that,

$$C_{\mathcal{F}} \cong h^{L_{\mathcal{F}}}.$$

So, in such case $C_{\mathcal{F}}(X) \cong \mathrm{Hom}_{\mathcal{A}}(X, L_{\mathcal{F}})$. The representative $L_{\mathcal{F}}$ if it exists is called the limit of the system $\mathcal{F}$, and is denoted by $\varprojlim \mathcal{F} := L_{\mathcal{F}}$ i.e.,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \tag{limit}$$

and hence every cone must factor through $L_{\mathcal{F}}$. Intuitively the limit is the 'closest' object to the system. The notion of closeness must come from morphisms, so if there exists any other object with morphisms to the system, then it must be 'farther' than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

This means that for all objects $X \in \mathcal{A}$ and all family of morphisms $f_I : X \to \mathcal{F}I$, in $\mathcal{A}$

such that for all $f \in \mathrm{Hom}_{\mathcal{I}}(I, J)$, with $f_J = f_I \circ \mathcal{F}(f)$ factors uniquely through $\varprojlim \mathcal{F}$.

$$\begin{tikzcd}
& X \arrow[d, dashed] \arrow[ddl, "f_I"'] \arrow[ddr, "f_J"] & \\
& \varprojlim \mathcal{F} \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_J"'] & \\
\mathcal{F}I \arrow[rr, "\mathcal{F}(f)"'] & & \mathcal{F}J
\end{tikzcd}$$

A projective system in $\mathcal{A}$ indexed by $\mathcal{I}$ is a functor,

$$\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}.$$

Similar to the inductive system, for projective system $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^{\mathcal{G}} &: \mathcal{A} \to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X). \end{aligned}$$

If it's representable with representative $L^{\mathcal{G}}$,

$$C^{\mathcal{G}} \cong h_{L^{\mathcal{G}}}.$$

Denote the representative by $\varinjlim \mathcal{G} := L^{\mathcal{G}}$ is called the colimit or the limit of the projective system. If the colimit exists, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}^{\mathrm{op}}}}(\mathcal{G}, \Delta X) \cong \mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, X) \tag{colimit}$$

Projective limits can be written in terms of universal property as,

$$\begin{tikzcd}
\mathcal{G}I \arrow[rr, "\mathcal{G}(f)"] \arrow[dr, "\varphi_I"] \arrow[ddr, "f_I"'] & & \mathcal{G}J \arrow[dl, "\varphi_J"'] \arrow[ddl, "f_J"] \\
& \varinjlim \mathcal{G} \arrow[d, dashed] & \\
& X &
\end{tikzcd}$$

Note that if $\mathcal{I}$ admits initial object 0, then the limit $\varprojlim \mathcal{F}$ corresponds to the object $\mathcal{F}(0)$. Similarly for colimit, with terminal object.

4.1 | (Co)limit Calculus

A category $\mathcal{A}$ is cocomplete with respect to $\mathcal{I}$ if for all inductive systems indexed by $\mathcal{I}$, the colimit exists, if $\mathcal{I}$ is not explicitly said, then it means that $\mathcal{A}$ is cocomplete with respect to all small categories. $\mathcal{A}$ is complete with respect to $\mathcal{I}$ if it has all limits for all projective systems indexed by $\mathcal{I}$. $\mathcal{A}$ is called bicomplete if it's both complete and cocomplete.

Theorem 4.1. *$\mathcal{A}$ is complete $\Rightarrow$ $\mathcal{A}^{\mathcal{K}}$ is complete.*

Proof

Given an inductive system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$, the goal is to construct a new functor $\varprojlim \mathcal{F}$ in $\mathcal{A}^{\mathcal{I}}$ such that,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

where $\Delta^{\mathcal{K}}$ is the constant functor in $\mathcal{A}^{\mathcal{K}}$ which sends $A \in \mathcal{A}$ to the constant functor $\Delta^{\mathcal{K}}A$. This is then by definition the limit of the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}^{\mathcal{K}}$.

In order to do this we have to describe where $\varprojlim \mathcal{F}$ sends elements of $\mathcal{I}$. The construction involves argument wise assignment of objects in $\mathcal{A}$ for each object in $\mathcal{I}$, and then showing the functoriality, that is verify it respects composition of morphisms.

Construction. By definition of limit, for every $I \in \mathcal{I}$, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}I, \mathcal{F}I) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}I, \varprojlim(\mathcal{F}I)) \qquad \text{(isomorphism)}$$

here $\mathcal{F}I : \mathcal{K} \to \mathcal{A}$ is a fixed functor, an inductive system and $\varprojlim(\mathcal{F}I)$ its inductive limit. So we define the associated object by,

$$\varprojlim \mathcal{F}(I) \coloneqq \varprojlim(\mathcal{F}I).$$

Functoriality.

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}(\cdot), \mathcal{F}(\cdot)) : \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathbf{Sets}$$

is a bifunctor, so for each natural transformation $\kappa : \Delta^{\mathcal{K}}\mathcal{H} \Rightarrow \mathcal{F}$,

$$\begin{tikzcd}
\mathcal{I} \arrow[rr, bend left, "\mathcal{F}"] \arrow[dr, "\mathcal{H}"'] & \arrow[d, phantom, "\Uparrow\kappa"] & \mathcal{A}^{\mathcal{K}} \\
& \mathcal{A} \arrow[ur, "\Delta^{\mathcal{K}}"'] &
\end{tikzcd}$$

and morphism $f : I \to J$, we have maps,

$$\begin{tikzcd}
I \arrow[d, "f"'] & \Delta^{\mathcal{K}}\mathcal{H}I \arrow[r, "\kappa_I"] \arrow[d, "\Delta^{\mathcal{K}}\mathcal{H}(f)"'] & \mathcal{F}I \arrow[d, "\mathcal{F}(f)"] \\
J & \Delta^{\mathcal{K}}\mathcal{H}J \arrow[r, "\kappa_J"'] & \mathcal{F}J
\end{tikzcd}$$

The commutative square gives us, $\kappa_J \circ \Delta^{\mathcal{K}}\mathcal{H}(f) = \mathcal{F}(f) \circ \kappa_I$, and the isomorphism of sets gives us a morphisms, $\widehat{\kappa}_I$ and $\widehat{\kappa}_J$ such that,

$$\widehat{\kappa}_I \circ \mathcal{H}(f) = \varprojlim(\mathcal{F}(f)) \circ \widehat{\kappa}_J.$$

So, $\widehat{\kappa} : \mathcal{H} \to \varprojlim \mathcal{F}(\cdot)$ is a natural transformation. So we get,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

Since this natural transformation is defined using the natural transformation κ it will satisfy the required compositions. So we have, $\varprojlim \mathcal{F}(f \circ g) = (\varprojlim \mathcal{F}(f)) \circ (\varprojlim \mathcal{F}(g))$. $\square$

This gives us an adjoint situation, where colimit is left-adjoint to the constant functor and the constant functor is left-adjoint to the limit,

$$\varinjlim \dashv \Delta \dashv \varprojlim \qquad \text{(limit-diagonal adjointness)}$$

Theorem 4.2.

$$\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}),$$
$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, A) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, A).$$

Proof

The idea is to study an appropriate functor category, get hom-set isomorphisms and then apply Yoneda principle. So, we have to show for each set $X \in \mathbf{Sets}$, we have an isomorphism of sets,

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$$

Using the limit-diagonal adjointness, this reduces to showing $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}))$ and $\mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$ are isomorphic. Δs are in the appropriate categories.

Let $\kappa : \Delta X \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$ be a natural transformation, then κ is determined by its components

$$\kappa_I : (\Delta X)I \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I).$$

Each $(\Delta X)I$ is a set, and hence the maps κ_I is itself determined by its action on the elements of the set X, So, for each $x \in X$, $\kappa_I(x)$ is a morphism in $\mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F}I)$.

If we think of A as the constant functor, we can define using $\varphi_x(\cdot) := \kappa_{(\cdot)}(x)$ defines an element $\varphi \in \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$. This is a bijection and hence,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \mathrm{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$$

Applying the limit-diagonal adjointness again to this and the Yoneda principle, we get that $\mathrm{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{F})$. The other isomorphism is also similar. □

So, hom-functor takes limits to colimits in the first argument and takes limits to limits in the second argument. This can now be used to prove many things easily.

Intuitively, limits take us to the 'last object' in a diagram and colimits take us to the 'first object' in a diagram. Here 'most' is quantified in terms of morphisms, and first and last are quantified by the direction of the morphisms. So, if the indexing category is a product category, then we can think of limits as trying to find the bottom right corner of the rectangle diagram, and similarly the colimit is the top left corner. So, whether we go to right most first and then go to the bottom or whether go to the bottom first and then go to the right shouldn't change which object we reach after doing this. We now formalize this.

Theorem 4.3. (Fubini for limits)

$$\varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathcal{F} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F} \cong \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}$$

Proof

This can be proved using the relation between the constant functors. Let $\Delta^{\mathcal{I} \times \mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I} \times \mathcal{J}}$, $\Delta^{\mathcal{I}} : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ and $\Delta^{\mathcal{J}} : \mathcal{A} \to \mathcal{A}^{\mathcal{J}}$ be the constant functors in the appropriate functor category. Then we have, $\Delta^{\mathcal{I} \times \mathcal{J}} = \Delta^{\mathcal{I}} \Delta^{\mathcal{J}}$. This gives us,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}) &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I} \times \mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}} \Delta^{\mathcal{J}} A, \mathcal{F}) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F}). \end{aligned}$$

By Yoneda principle we have the required isomorphism. Since $\mathcal{I} \times \mathcal{J} \cong \mathcal{J} \times \mathcal{I}$, the other isomorphism also follows. $\square$

Lemma 4.4. *Right adjoints preserve limits.*

Proof

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ has a left adjoint say $\mathcal{G} : \mathcal{B} \to \mathcal{A}$, and suppose $\mathcal{H} : \mathcal{I} \to \mathcal{A}$ is an inductive system with a limit $\varprojlim \mathcal{H}$, we must prove that $\mathcal{F}(\varprojlim \mathcal{H})$ is the limit of the inductive system $\mathcal{F} \circ \mathcal{H}$.

$\mathcal{F}\mathcal{H} : \mathcal{I} \to \mathcal{B}$ is an inductive system in $\mathcal{B}$ indexed by $\mathcal{I}$. For all $X \in \mathcal{B}$,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F} \varprojlim \mathcal{H}) &\cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \varprojlim \mathcal{H}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}X, \mathcal{H}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{F}\mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathcal{B}}(X, \varprojlim \mathcal{F}\mathcal{H})
\end{aligned}$$

By Yoneda principle, we have, $\mathcal{F}(\varprojlim \mathcal{H}) \cong \varprojlim \mathcal{F} \circ \mathcal{H}$. $\square$

Right adjoints preserve limits, can be remembered by the acronym, RAPL. Under additional conditions on the category $\mathcal{A}$, the converse holds, these theorems are called adjoint functor theorems.

4.2 | (Co)end Calculus

Similar to how natural transformations relate functors between two categories, dinatural transformations relate bifunctors. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors. Bifunctors are functors when one of the arguments are fixed. We will denote $\mathcal{F}^X$ for the contravariant functor $\mathcal{F}(\cdot, X)$, and $\mathcal{F}_X$ for the covariant functor $\mathcal{F}(X, \cdot)$.

Each morphism $f : X \to Y$ gives rise to the following morphisms in $\mathcal{B}$,

$$\begin{tikzcd}
X \arrow[dd, "f"'] & & \mathcal{F}(X,X) \arrow[dl, "\mathcal{F}^X(f)"'] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\
& \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \\
Y & & \mathcal{F}(Y,Y) & \mathcal{G}(Y,Y) \arrow[ur, "\mathcal{G}^Y(f)"'] &
\end{tikzcd}$$

A dinatural transformation $\kappa : \mathcal{F} \to \mathcal{G}$ consists of a family of morphisms,

$$\kappa_X : \mathcal{F}(X, X) \to \mathcal{G}(X, X)$$

such that for any $f : X \to Y$ the following commutes,

$$\begin{tikzcd}
X \arrow[dd, "f"'] & & \mathcal{F}(X,X) \arrow[dl, "\mathcal{F}^X(f)"'] \arrow[r, "\kappa_X"] & \mathcal{G}(X,X) \arrow[dr, "\mathcal{G}_X(f)"] & \\
& \mathcal{F}(Y,X) \arrow[dr, "\mathcal{F}_Y(f)"'] & & & \mathcal{G}(X,Y) \\
J & & \mathcal{F}(Y,Y) \arrow[r, "\kappa_Y"] & \mathcal{G}(Y,Y) \arrow[ur, "\mathcal{G}^Y(f)"'] &
\end{tikzcd}$$

Similar to the case of limits and cones, we can describe 'doubly indexed limits'. Suppose we are given a system $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$, intuitively, the end of the system is an object in $\mathcal{B}$ that is 'closest' to the system.

This can be formalised using the functor category as follows; Attach to each object $E \in \mathcal{B}$ the constant bifunctor $\Delta E : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ that sends everything in $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ to E, and each morphism in $\mathcal{B}$ to the identity on E. A relation between an object X and the system $\mathcal{F}$ is a dinatural transformation δ between ΔE and $\mathcal{F}$. Represented by the diagram,

$$\begin{tikzcd}
X \arrow[dd, "f"'] & & & \mathcal{F}(X,X) \arrow[dr, "\mathcal{F}^X(f)"] & \\
& & E \arrow[ur, "\delta^E_X"] \arrow[dr, "\delta^E_Y"'] & & \mathcal{F}(X,Y) \\
Y & & & \mathcal{F}(Y,Y) \arrow[ur, "\mathcal{F}_Y(f)"'] &
\end{tikzcd}$$

Such a dinatural transformation is called a wedge. The collection of all such wedges is the set of all such dinatural transformations,

$$\begin{aligned} W_{\mathcal{F}} : \mathcal{B} &\to \mathbf{Sets} \\ E &\mapsto \mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}). \end{aligned}$$

The end of the system $\mathcal{F}$ is intuitively the object which is closest to the system. A relation between an object $E \in \mathcal{B}$ and the system $\mathcal{F}$ consists of a dinatural transformation from the constant bifunctor ΔE to $\mathcal{F}$. So, the 'closest' would be such that any other dinatural transformation should factor through the 'closest' one.

The end of a system $\mathcal{F}$, is an object $\int_{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\omega : \Delta \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$ such that every other wedge factors through it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(E, \int_{\mathcal{A}} \mathcal{F}) \tag{end}$$

Also denoted by $\int_{A \in \mathcal{A}} \mathcal{F}A$. Expressed in a commutative diagram by,

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\kappa_X"] \arrow[ddr, bend right, "\kappa_Y"'] & & \\
& \int_{\mathcal{A}} \mathcal{F} \arrow[r, dashed, "\omega_X"] \arrow[d, dashed, "\omega_Y"'] & \mathcal{F}(X,X) \arrow[d, "\mathcal{F}_X(f)"] \\
& \mathcal{F}(Y,Y) \arrow[r, "\mathcal{F}^Y(f)"'] & \mathcal{F}(X,Y)
\end{tikzcd}$$

The integral notation corresponds to the intuition that the equalizer given in $\int_{\mathcal{A}}$ can be thought of as an averaging operation on the functor where we run through the objects of $\mathcal{A}$.

The coend of a system $\mathcal{F}$, is an object $\int^{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\sigma : \mathcal{F} \to \Delta \int^{\mathcal{A}} \mathcal{F}$, such that every other cowedge factors from it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}}}(\mathcal{F}, \Delta E) \cong \mathrm{Hom}_{\mathcal{B}}(\int^{\mathcal{A}} \mathcal{F}, E) \tag{coend}$$

Expressed in a commutative diagram by,

$$
\begin{tikzcd}
\mathcal{F}(Y,X) \arrow[r, "\mathcal{F}_Y(f)"] \arrow[d, "\mathcal{F}^X(f)"'] & \mathcal{F}(Y,Y) \arrow[d, dashed] \arrow[ddr, bend left, "\kappa_Y"] & \\
\mathcal{F}(X,X) \arrow[r, dashed] \arrow[drr, bend right, "\kappa_X"'] & \displaystyle\int^{\mathcal{A}} \mathcal{F} \arrow[dr, dashed, "\exists!"] & \\
 & & E
\end{tikzcd}
$$

Functoriality. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be two bifunctors and let $\eta : \mathcal{F} \to \mathcal{G}$ be a dimatural transformation. Any connection from an object E to the system $\mathcal{F}$ gives rise to a connection from the object to the system $\mathcal{G}$, given by the composition.

In particular we have the composition for the connection $\omega : \int_{\mathcal{A}} \mathcal{F} \to \mathcal{F}$. So, this connection must factor through $\int_{\mathcal{A}} \mathcal{G}$, and hence we have a map, whose components are given by the composition,

$$
\begin{tikzcd}
\displaystyle\int_{\mathcal{A}} \mathcal{F} \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\eta_X \circ \omega_X"] \arrow[ddr, bend right, "\eta_Y \circ \omega_Y"'] & & \\
 & \displaystyle\int_{\mathcal{A}} \mathcal{G} \arrow[r, dashed, "\varphi_X"] \arrow[d, dashed, "\varphi_Y"'] & \mathcal{G}(X,X) \arrow[d, "\mathcal{G}_X(f)"] \\
 & \mathcal{G}(Y,Y) \arrow[r, "\mathcal{G}^Y(f)"'] & \mathcal{G}(X,Y)
\end{tikzcd}
$$

This is functorial since the $\int_{\mathcal{A}} \eta : \int_{\mathcal{A}} \mathcal{F} \to \int_{\mathcal{A}} \mathcal{G}$ are defined component wise and so the composition will be component wise. So $\int$ is functorial, that is, $\int_{\mathcal{A}} (\eta \circ \kappa) = \int_{\mathcal{A}} \eta \circ \int_{\mathcal{A}} \kappa$.

Fubini Rule. Given a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}$, the end in the first two coordinates, gives rise to a functor,

$$\int_{\mathcal{A}} \mathcal{F} : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{C}.$$

The end of this functor is then,

$$\int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F} \in \mathcal{C}$$

With some tedious checking it turns out that,

$$\int_{\mathcal{A} \times \mathcal{B}} \mathcal{F} \cong \int_{\mathcal{A}} \int_{\mathcal{B}} \mathcal{F} \cong \int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F}. \qquad \text{(Fubini for ends)}$$

4.2.1 | (Co)ends as (Co)limits

The motivation for the definition of ends/coends was very similar to that of limits/colimits. So we should expect the two concepts to be closely related. To intuitively motivate the relation between ends and limits, we have to intuitively understand how ends describe average of a system and limits describe closeness to the system. Limits only take into account the relation an object has 'with' the system, that is, a natural transformation from constant functor to the system. In case of ends, the relation 'between' the objects of the system is important, this is encoded in the dinatural transformation. So, the 'average' should also be expected to be some sort of limit taken over morphisms between objects of the indexing category. This is what makes it average over the relations between objects of the system.

We can think of ends and coends as limits and colimits. The first step is then to associate to each bifunctor system a functor, and turn the ends/coends of bifunctor systems into limits/colimits of systems. This involves estalishing an equivalence between the category of bifunctors and an appropriate category of functors. Consider a bifunctor

$$\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$$

attaches to pairs of objects in $\mathcal{A}$ objects in $\mathcal{B}$ such that the relation between the pair of objects is preserved. So, we could think of the functor as assigning to each morphism between pairs fo objects in $\mathcal{A}$ an object of $\mathcal{B}$. So, we can start with a new category where objects are morphisms of the category $\mathcal{A}$.

Let $f : X \to Y$ be a morphism in $\mathcal{A}$, the bifunctor assigns to the pair X, Y the object $\mathcal{F}(X, Y)$, so instead we could assign to the morphism f, the object $\mathcal{F}(\mathrm{src}(f), \mathrm{tgt}(f))$. Where src is the source object of f, and $\mathrm{tgt}(f)$ is the target. To make sure the bifunctoriality transfers to functoriality of this new association we have to define the morphisms suitably. For two every morphism $f, g \in \mathrm{Hom}_{\mathcal{A}}$ define a morphism between them to be a pair of morphisms,

$$\begin{tikzcd} X \arrow[d, "f"'] & \widehat{X} \arrow[l] \arrow[d, "g"] \\ Y \arrow[r] & \widehat{Y}. \end{tikzcd} \tag{bimorph}$$

Note that the reverse direction of the connection from $\widehat{X}$ to X is necessary to make the association a functor. Because the bifunctor is contravariant in the first argument, with the second argument fixed. So with this, we have a new functor,

$$\widehat{\mathcal{F}} : \widehat{\mathcal{A}} \to \mathcal{B}$$

where $\widehat{\mathcal{A}}$ is the category consisting of morphisms of $\mathcal{A}$ as objects and the above rule for morphisms, bimorph. So the functor,

$$\mathcal{B}^{\mathcal{A}^{\mathrm{op}} \times \mathcal{A}} \to \mathcal{B}^{\widehat{\mathcal{A}}}, \quad \mathcal{F} \mapsto \widehat{\mathcal{F}},$$

is an equivalence of categories, and must respect initial/terminal objects. Since limits/colimits and ends/coends are initial/terminal objects in the respective categories we have,

$$\int_{\mathcal{A}} \mathcal{F} \cong \varprojlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}, \quad \int^{\mathcal{A}} \mathcal{F} \cong \varinjlim_{\widehat{\mathcal{A}}} \widehat{\mathcal{F}}. \tag{(co)ends as (co)limits}$$

For a detailed proof, see [?]. This immediately leads to the following observation that if $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ preserves all limits then it preserves all the ends that exist. If $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$ is a system, and $\mathcal{G}$ is a continuous functor then

$$\mathcal{G}\Big(\int_{\mathcal{A}} \mathcal{F}\Big) \cong \int_{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

If $\mathcal{G}$ is a contravariant functor and cocontinuous, then,

$$\mathcal{G}\Big(\int_{\mathcal{A}} \mathcal{F}\Big) \cong \int^{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

An immediate corollary is that Hom functors preserve ends in the second argument and maps ends to coends in the first argument.

Corollary 4.5. (Hom-(Co)End relations)

$$\mathrm{Hom}_{\mathcal{B}}(\int_{\mathcal{A}} \mathcal{F}, A) \cong \int^{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, A)$$

$$\mathrm{Hom}_{\mathcal{B}}(A, \int_{\mathcal{A}} \mathcal{F}) \cong \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(A, \mathcal{F}).$$

□

Theorem 4.6.

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A). \qquad \text{(natural transformations as ends)}$$

Proof

We have to show that $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$ satisfies the universal property of ends. That's, any collection of morphisms from an object $E \in \mathbf{Sets}$ to the system $\mathcal{H} \equiv \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-), \mathcal{G}(-))$ factors through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$.

Suppose we have a wedge $\tau : E \to \mathcal{H}$, then for each $A \in \mathcal{A}$ we have a morphisms $\tau_A : E \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$. Since E is a set this is a set map, which maps each $x \in E$ to a morphism between the objects $\mathcal{F}A$ and $\mathcal{G}A$. That's to say $\tau_A(x) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$.

For a morphism $f : A \to B$, the wedge condition says that,

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\tau_A"] \arrow[ddr, bend right, "\tau_B"'] & & \\
& \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \arrow[r, dashed, "\omega_A"] \arrow[d, dashed, "\omega_B"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \arrow[d, "\circ\mathcal{F}(f)"] \\
& \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}B, \mathcal{G}B) \arrow[r, "\mathcal{G}(f)\circ"'] & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}B)
\end{tikzcd}$$

So, we have,

$$\mathcal{G}(f) \circ \tau_A(x) = \tau_B(x) \circ \mathcal{F}(f).$$

This means that $\tau_{(-)}(x)$ is a natural transformation,

$$\begin{tikzcd}
A \arrow[d, "f"] & \mathcal{F}A \arrow[r, "\tau_A(x)"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}A \arrow[d, "\mathcal{G}(f)"] \\
B & \mathcal{F}B \arrow[r, "\tau_B(x)"'] & \mathcal{G}B
\end{tikzcd}$$

So, it must factor through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$. This is precisely the universal property of ends. So,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A).$$

□

The collection of all natural transformations from $\mathcal{F}$ to $\mathcal{G}$ can be thought of as taking an average of elements of $\mathcal{A}$ as 'measured' by the functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$.

4.3 | Density theorem for Pre-Sheaves

We now have the necessary tools to prove the famous result about the category of pre-sheaves, $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Every pre-sheaf can be canonically presented as a colimit of representable functors. For the sake of familiarity we will assume that $\mathcal{C} = \mathcal{O}(X)$.

Theorem 4.7. (Co-Yoneda/Density Formula) $\mathcal{F} \in \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$,

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} \mathcal{F}U \times \mathrm{Hom}_{\mathcal{C}}(V, U) \cong \int_{U \in \mathcal{C}} (\mathcal{F}U)^{\mathrm{Hom}_{\mathcal{C}}(U,V)}.$$

Proof

By Yoneda applied to the functor

$$\mathcal{H} : V \mapsto \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W),$$

we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^V, \mathcal{H}) \cong \mathcal{H}V.$$

The following chain of isomorphisms lets us sneak in Yoneda principle,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W) &\cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}\left(h^V, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}(-), W)\right) \\
&\cong \int_{U \in \mathcal{C}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U, \mathrm{Hom}_{\mathbf{Sets}}(\mathcal{F}U, W)\right) \\
&\cong \int_{U \in \mathcal{C}} \mathrm{Hom}_{\mathbf{Sets}}\left(h^V U \times \mathcal{F}U, W\right) \\
&\cong \mathrm{Hom}_{\mathbf{Sets}}\left(\int^{U \in \mathcal{C}} h^V U \times \mathcal{F}U, W\right)
\end{aligned}$$

Here, in the second step we used 4.6, and in the third step we used hom-tensor adjointness in **Sets**. The last step follows from hom-sets taking coends to ends in the first argument. So, we have by Yoneda principle,

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} h^V U \times \mathcal{F}U$$

This holds when the pre-sheaf is to any category that's cartesian closed, and the product are replaced with the correct tensor product in the target category. In case of modules this corresponds to the internal hom-tensor product adjointness. □

When $\mathcal{C}$ is the category of open sets of a space X, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(U, V)$ is nonempty, only when $V \subseteq U$, in which case it is a singleton set, with the inclusion map as the only map. If we consider continuous functions as the pre-sheaf, and $V \subset U$, any 'continuous function' belonging to U will give rise to a 'continuous function' in V. Intuitively, we want to include all such functions and it will be a limit in this sense.

For the enriched case, the product will be replaced by tensor product $\odot$ in the enriching category $\mathcal{V}$. Limits will be replaced by weighted limits and using these we get the enriched density formula,

Theorem 4.8. (Enriched Density Formula) *Let $\mathcal{F} \in \mathcal{V}^{\mathcal{A}^{\mathrm{op}}}$, then,*

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} h^V U \odot \mathcal{F}U$$

□

For proofs, and details, see [?],[?] and the references therein.

4.4 | Weighted (Co)limits

Weighted (co)limits are an extreme generalization of standard (co)limits. When one deals with enriched categories, the hom-sets come equipped with additional structure, and it provides some convinient tools for studying various other categorical objects.

Given a system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$, the standard (co)limits describe an object that's closest to or from the system, where closest is quantified in terms of morphisms. In terms of (co)cones this means that any (co)cone must factor through the (co)limits. Note here that for each $I \in \mathcal{I}$, the (co)cones provide a single morphism to or from an object. Weighted (co)limits generalize this. Here, we consider 'weighted' (co)cones, which consist of assigning to each FI a collection of morphisms to or from a vertex. To formalize this we will start by looking at the standard (co)limits again.

Cones from an object $X \in \mathcal{A}$ to the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ are natural transformations from the constant functor $\Delta X : \mathcal{I} \to \mathcal{A}$ to the functor $\mathcal{F}$. So, each cone is a functor, from the constant functor to the terminal object in sets (singleton set), to the functor $\mathrm{Hom}_{\mathcal{A}}(X, F-)$. So, the limit of the system can be defined up to isomorphism by isomorphism,

$$\mathrm{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(1, \mathrm{Hom}_{\mathcal{A}}(X, F-)).$$

The terminal functor gives equal weightage to each FI. Instead of assigning one morphism from X to each FI we can instead assign a collection of morphisms, and study these complicated cones. This can be formalized functorially by defining a 'weight' functor,

$$W : \mathcal{I} \to \mathbf{Sets}$$

Given a diagram of functors,

$$\mathcal{A} \xleftarrow{\mathcal{F}} \mathcal{I} \xrightarrow{W} \mathbf{Sets}$$

The weighted limit of the system $\mathcal{F}$ by W denoted $\varprojlim^W \mathcal{F}$ is defined by the isomorphism,

$$\mathrm{Hom}_{\mathcal{A}}(X, \varprojlim^W \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(W, \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}-)). \tag{weighted limit}$$

The following chain of isomorphisms gives us a formula for weighted limits,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(W, \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}-)) &\cong \int_{I \in \mathcal{I}} \mathrm{Hom}_{\mathbf{Sets}}(WI, \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}I)) \\
&\cong \int_{I \in \mathcal{I}} \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}I^{WI}) \\
&\cong \mathrm{Hom}_{\mathcal{A}}\big(X, \int_{I \in \mathcal{I}} \mathcal{F}I^{WI}\big).
\end{aligned}$$

Here we used the 4.6 in the first step, hom-tensor adjointness in the next step and 4.5 in the last step. So we have,

$$\varprojlim^W \mathcal{F} \cong \int_{I \in \mathcal{I}} \mathcal{F}I^{WI}. \qquad \text{(weighted limit as ends)}$$

Similarly, for a weight $W : \mathcal{I}^{op} \to \mathbf{Sets}$, the weighted colimits are defined by the isomorphism of sets,

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim^W \mathcal{F}, X) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}}(W, \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}-, X)). \qquad \text{(weighted colimit)}$$

The following chain of isomorphisms gives us a formula for weighted limits,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}}(W, \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}-, X)) &\cong \int_{I \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{Sets}}(WI, \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}I, X)) \\
&\cong \int_{I \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(WI \times \mathcal{F}I, X) \\
&\cong \mathrm{Hom}_{\mathcal{A}}\big(\int^{I \in \mathcal{I}^{\mathrm{op}}} \mathcal{F}I \times WI, X\big).
\end{aligned}$$

Here we used the 4.6 in the first step, hom-tensor adjointness in the next step and 4.5 in the last step. So we have,

$$\varinjlim^W \mathcal{F} \cong \int^{I \in \mathcal{I}^{\mathrm{op}}} \mathcal{F}I \times WI. \qquad \text{(weighted colimit as coends)}$$

Intuitively, the weight functor assigns to each object $I \in \mathcal{I}$ a weight WI, and the weighted limit of the system is the object $\varprojlim^W \mathcal{F}$ for which $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}I)$ is 'closest' to the set WI. This is similar to weights in other branches of mathematics, where one assigns to objects certain real numbers, and the order relation on $\mathbb{R}$ provides the notion of 'more' weight. In general case, the real numbers is replaced by the category of sets and order relation is replaced by any function of sets.

These can be genrealised to the enriched category setting, in which case, $\varprojlim^W \mathcal{F}$ is usually denoted by $\{W, \mathcal{F}\}$ and $\varinjlim^W \mathcal{F}$ is usually denoted by $W \odot \mathcal{F}$.

4.4.1 | Functor Tensor Product

5 | Kan Extension of Functors

Given two functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$, the goal of Kan extension is to find a new functor $\mathcal{K}$ such that the manipulation done by composite functor $\mathcal{K} \circ \mathcal{F}$ is closest to the manipulation done by the functor $\mathcal{G}$. The obstruction to $\mathcal{K} \circ \mathcal{F}$ being the same as $\mathcal{G}$ comes from $\mathcal{F}$ losing information that $\mathcal{G}$ preserves.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd}
 & \mathcal{B} \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C}
\end{tikzcd}$$

The left Kan extension of $\mathcal{G}$ along $\mathcal{F}$ is a functor is a functor $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G} : \mathcal{B} \to \mathcal{C}$ such that there exists a natural transformation $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F}$, such that any other extension $\mathcal{K}$ with natural transformation $\mathcal{G} \to \mathcal{K} \circ \mathcal{F}$ factors through $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$, i.e., $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F}$.

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathcal{K}"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[rr, phantom, bend left=20, "\Uparrow"] & & \mathcal{C}
\end{tikzcd}$$

So, in terms of a commutative diagram a Kan extension is the functor that makes the diagram as close to commutative as possible.

The left Kan extension $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ can be visualised by the diagram,

$$\mathcal{G} \Longrightarrow (\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{K} \circ \mathcal{F} \Longrightarrow \cdots$$

This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{K} \circ \mathcal{F}$ there exists a natural transformation from $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ to $\mathcal{K}$, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}, \mathcal{K})$$

Intuitively the left Kan extension is the left most functor to the functor in the above visualisation. Here 'most' is quantified in terms of natural transformations. The left Kan extension is the left most (as explained above) extension of the functor $\mathcal{G}$ with respect to $\mathcal{F}$. We should expect the construction of the 'left most' functor to be related to taking colimits in an appropriate category. If $\mathcal{C}$ is cocomplete, the functor category to $\mathcal{C}$ will also be cocomplete, and $\mathcal{A}$ has some 'nice properties' then we could think of these as a system in the functor category and intuitively, the left 'most' should exist.

The right Kan extension is defined similarly, and only the direction of the natural transformation is changed from $\mathcal{K} \circ \mathcal{F}$ to $\mathcal{G}$.

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathcal{H}"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[rr, phantom, bend left=20, "\Downarrow"] & & \mathcal{C}
\end{tikzcd}$$

The right Kan extension is denoted by $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$. It's the right most functor in the following visualization,

$$\cdots \Longrightarrow \mathcal{H} \circ \mathcal{F} \Longrightarrow (\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}) \circ \mathcal{F} \Longrightarrow \mathcal{G}$$

Similar to left Kan extensions, we should expect the construction of the 'right most' functor to be related to taking limits in an appropriate category. This means that for every natural transformation from $\mathcal{G}$ to a functor $\mathcal{H} \circ \mathcal{F}$ there exists a natural transformation from $\mathcal{H}$ to $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{H}\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathcal{H}, \mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}).$$

Kan extension are functors in the appropriate functor category.

Lemma 5.1. *$\forall \mathcal{G}$, $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$, $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$ exist, then,*

$$\mathrm{Lan}_{\mathcal{F}} \dashv \circ\mathcal{F} \dashv \mathrm{Ran}_{\mathcal{F}}\,.$$

The proof of this adjointness with precomposition has already been described, and directly follows from definition of Kan extensions, for this to make sense we only need the fact that $\mathrm{Lan}_{\mathcal{F}}$ and $\mathrm{Ran}_{\mathcal{F}}$ are functors, i.e., $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ and $\mathrm{Ran}_{\mathcal{F}}\,\mathcal{G}$ exist $\forall \mathcal{G}$, and satisfy some composition rules, which it will due to the definition.

If $\mathcal{C}$ is a locally small category, a Kan extension is said to be pointwise Kan extension if it is preserved by representable functors $\mathrm{Hom}_{\mathcal{C}}(C, -)$ for all $C \in \mathcal{C}$. It's called absolute if any functor $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ preserves the Kan extension.

Lemma 5.2. *If $\mathcal{L}$ is a left adjoints, then,*

$$\mathrm{Lan}_{\mathcal{F}}(\mathcal{L} \circ \mathcal{G}) \cong \mathcal{L} \circ \mathrm{Lan}_{\mathcal{F}}\, \mathcal{G}.$$

Proof

Suppose $\mathcal{G}$ has a left Kan extension along $\mathcal{F}$,

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}"] \arrow[drr, bend left, "\mathcal{K}"] & & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & \Uparrow & \mathcal{C} \arrow[r, shift left, "\mathcal{L}"] & \mathcal{D} \arrow[l, shift left, "\mathcal{R}"]
\end{tikzcd}$$

Let $\mathcal{L} : \mathcal{C} \to \mathcal{D}$ be a functor that's left adjoint, i.e., there exists a functor $\mathcal{R} : \mathcal{D} \to \mathcal{C}$ to which $\mathcal{L}$ is a left adjoint, then we have,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}^{\mathcal{B}}}(\mathcal{L}\,\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}, \mathcal{K}) &\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}, \mathcal{R}\mathcal{K}) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, (\mathcal{R}\mathcal{K})\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{A}}}(\mathcal{L}\mathcal{G}, \mathcal{K}\mathcal{F})
\end{aligned}$$

So, by definition of Kan extension, we have that $\mathcal{L}\,\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G} \cong \mathrm{Lan}_{\mathcal{F}}(\mathcal{L}\mathcal{G})$. □

What's happening is that left adjoints preserve colimits, and this guarantees the existence of the Kan extenstion for the composition. So, when the required colimits exist, the notion of closest makes sense. The closest functor to $\mathcal{L}$ is $\mathcal{L}$, so once we know the Kan extension exists, it must be the one described above.

5.1 | Kan Extensions as Coends

Certain additional constraints on the starting categories guarantees the existence of Kan extensions. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ be two functors.

$$\begin{tikzcd}
 & \mathcal{B} & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] & & \mathcal{C}
\end{tikzcd}$$

Theorem 5.3. (Existence/Coend Formula)

$$\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}B \cong \int^{A \in \mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, B) \odot \mathcal{G}A$$

whenever $\mathcal{A}$ is small, $\mathcal{B}$ is locally small, $\mathcal{C}$ is cocomplete.

Proof

Note that the local smallness of $\mathcal{B}$, or being enriched by the category of sets is needed for the existence of tensor product $\odot$, which is needed for the existence of extensions, this condition maybe replaced with an appropriate enriched category with a tensor product. Once they exist, the cocompleteness is needed for the existence of the limit, and to take the limits, we need the starting category to be small. So the conditions we require are to be expected.

The proof is again find a chain of isomorphisms so we can sneak in Yoneda principle. Firstly, by the end formula for natural transformations, we have,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \int_{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}\mathcal{F}A\big)$$

Applying Yoneda to the functor $\mathcal{H} : X \mapsto \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}X\big)$, we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}\mathcal{F}A\big) = \mathcal{H}(\mathcal{F}A)$$

Applying the end formula for natural transformations again, we get,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}(h^{\mathcal{F}A}B, \mathcal{H}B)$$

This gives us the following double 'integral',

$$\int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\Big(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}B\big)\Big)$$

By using the hom-tensor adjointness we get,

$$\begin{aligned}\int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathbf{Sets}}\Big(h^{\mathcal{F}A}B, \mathrm{Hom}_{\mathcal{C}}\big(\mathcal{G}A, \mathcal{K}B\big)\Big) &\cong \int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\Big(h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\Big)\\ &\cong \int_{B\in\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}\Big(\int_{A\in\mathcal{A}} h^{\mathcal{F}A}B \odot \mathcal{G}A, \mathcal{K}B\Big)\\ &\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}\Big(\int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, -) \odot \mathcal{G}A, \mathcal{K}\Big)\end{aligned}$$

Here, we used the Fubini rule for ends, and hom-(co)end relations in the first step, and end formula for natural transformations in the last step. Now Yoneda principle proves the theorem. $\square$

Using weighted colimit as coends we get,

$$\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}B \cong \int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, B) \odot \mathcal{G}A \cong \varinjlim{}^{\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}-, B)}\,\mathcal{F}.$$

Assume the conditions of coend formula are satisfied. For each functor $\mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, the left Kan extension along $\mathcal{F}$ is the functor, $\mathrm{Lan}_{\mathcal{F}}(-) : \mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{B}}$, $\mathcal{G} \mapsto \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$. Intuitively we expect 'nearest' composed with 'nearest' to be the 'nearest'. So we will have,

$$\mathrm{Lan}_{\mathcal{F}\circ\mathcal{E}}\,\mathcal{G} = \mathrm{Lan}_{\mathcal{F}}(\mathrm{Lan}_{\mathcal{E}}\,\mathcal{G}).$$

The proof is a simple application of the coend formula and some end-coend calculus.

If a Kan extension at each point $B \in \mathcal{B}$ can be written as a limit such as the one above, then it's always pointwise Kan extension, as Hom-functor preserves limits. Conversely, if $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ is pointwise, then for each $C \in \mathcal{C}$ we have, $\mathrm{Lan}_{\mathcal{F}}(h_C \circ \mathcal{G}) \cong h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$.

$$
\begin{tikzcd}
 & \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}"] \arrow[drr, bend left, "h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}"] & & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathcal{G}"'] \arrow[ur, phantom, shift left=6, "\Uparrow"] & & \mathcal{C} \arrow[r, "h_C"'] & \mathbf{Sets}
\end{tikzcd}
$$

Applying Yoneda to the functor $\mathcal{H} \coloneqq h_C \circ \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ we get,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{B}}}(h^B, \mathcal{H}) \cong \mathcal{H}B = \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}B)$$

Since **Sets** is cocomplete, we can express the Kan extension as a coend formula. Now, by Yoneda principle this means that the Kan extension $\mathrm{Lan}_{\mathcal{F}}\,\mathcal{G}$ can itself be written as a coend.

5.2 | All Concepts are Kan Extensions

Any concept that talks about some notion of nearest/closest/best should be expected to be related to Kan extensions. Kan extensions is a very general construction which unites limits, adjoints, and other constructions. Many of the categorical constructions involve limits or adjunctions. Limits correspond to nearest objects to/from a system. Adjoints when viewed as reflections also say that the associated objects are the nearest to the original object when transformed back. So both of these concepts should be expected to be related to Kan extensions.

5.2.1 | (Co)Limits as Kan Extensions

Limits are the nearest object to a system. We can now think of objects in a category $\mathcal{A}$ as functors from a terminal category to the category $\mathcal{A}$. Here the terminal category is a category with one object $*$ and one morphism, the identity morphism on the object $\mathbb{1}_*$. This category will be denoted by $\mathbb{1}$. Now each object A in $\mathcal{A}$ can be thought of as a functor $\widehat{A} : \mathbb{1} \to \mathcal{A}$. Since we want the 'nearest' object to a system, we want a nearest approximation of the system by a functor that corresponds to such an object. This is a Kan extension situation.

The terminal category $\mathbb{1}$ is the unique object in the category of categories, such that there exist only one functor from any category $\mathcal{I}$ to $\mathbb{1}$, which sends everything to the only object in the category, and every morphism to the only morphism in $\mathbb{1}$. Denote this functor by $\mathcal{T}$. Then we have the following Kan extension problem,

$$
\begin{tikzcd}
 & \mathbb{1} \arrow[dr, dashed, "\mathcal{H}"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{I} \arrow[ur, "\mathcal{T}"] \arrow[rr, "\mathcal{F}"'] & \, & \mathcal{A}
\end{tikzcd}
$$

For each functor $\mathcal{H} : \mathbb{1} \to \mathcal{A}$ corresponds to an object H in the category $\mathcal{A}$, and $\mathcal{H} \circ \mathcal{T}$ represents the constant system which maps everything in the indexing category $\mathcal{I}$ to a constant object. The natural transformations from $\mathcal{H} \circ \mathcal{T}$ to $\mathcal{F}$ represent the cones, from the object H to the system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$.

The limit of a system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ is the right Kan extension of $\mathcal{F}$ along $\mathcal{T}$,

$$\varprojlim \mathcal{F} = \mathrm{Ran}_{\mathcal{T}}\,\mathcal{F}(*).$$

Similarly the colimit of a system is the object that's nearest from the system. So, the colimit will be the left Kan extension of the system along $\mathcal{T}$.

$$\varinjlim \mathcal{F} \cong \mathrm{Lan}_{\mathcal{T}} \mathcal{F}(*).$$

5.2.2 | Adjoint Functors as Kan Extensions

Adjoint functors when viewed as reflections §3.1, bring with them some notion of 'nearness'. So adjoints should be expected to be related to Kan extensions. Consider an adjoint situation $\mathcal{F} \dashv \mathcal{G}$,

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathcal{F}"] & \mathcal{B} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd},$$

If we think of objects in $\mathcal{A}$ as having some information, and after performing $\mathcal{F}$ we lose information in the category $\mathcal{A}$, then the functor $\mathcal{G}$ is such that $\mathcal{G}\mathcal{F}$ is the 'best' recovery of the original information about $\mathcal{A}$. So, $\mathcal{G}\mathcal{F}$ is the 'nearest' to the identity functor $\mathbb{1}_{\mathcal{A}}$, 'along' $\mathcal{F}$. This is a natural transformation,

$$\mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}\,.$$

The nearest means that any other 'recovery' will have to factor through this that's to say, $\cdots \Rightarrow \mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}$. Similarly, the left adjoint is the 'nearest' from the identity functor on $\mathcal{B}$, so we have,

$$\mathbb{1}_{\mathcal{A}} \Rightarrow \mathcal{F}\mathcal{G}.$$

So, we get the following Kan extension way of thinking about adjoints.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}} \cong \mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A} \arrow[dr, dashed, "\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F}"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{B} \arrow[ur, "\mathcal{G}"] \arrow[rr, "\mathbb{1}_{\mathcal{B}}"'] & {} & \mathcal{B} \end{tikzcd}$$

So, the right adjoint $\mathcal{G}$ is the left Kan extension of $\mathbb{1}_{\mathcal{A}}$ along $\mathcal{F}$. Similarly, the left adjoint $\mathcal{F}$ is the right Kan extension of $\mathbb{1}_{\mathcal{B}}$ along $\mathcal{G}$. An adjoint situation can be expressed as the following pair of Kan extensions,

$$\begin{tikzcd} & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & & \mathcal{B} \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \arrow[ur, "\mathcal{F}"'] \arrow[u, phantom, "\Uparrow"] & \end{tikzcd} = \begin{tikzcd} & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] & & \mathcal{B} \\ \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \arrow[ur, "\mathcal{F}"'] \arrow[ul, Rightarrow] & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & {} & \mathcal{B} \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & \\ & \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] \arrow[u, phantom, "\Uparrow"] & {} & \mathcal{A} \end{tikzcd} = \begin{tikzcd} \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & & \mathcal{B} \arrow[dr, "\mathcal{G}"] & \\ & \mathcal{A} \arrow[ur, Rightarrow] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & & \mathcal{A} \end{tikzcd}$$

such that any other extension $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ factors through the Kan extension,

$$\cdots \Longrightarrow \mathcal{H} \circ \mathcal{G} \Longrightarrow (\mathrm{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}) \circ \mathcal{G} \Longrightarrow \mathbb{1}_{\mathcal{B}}$$

visualised by the diagram,

$$
\begin{tikzcd}
\mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \\
& \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} \arrow[u, phantom, "\Uparrow"] & \mathcal{A} \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"'] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \\
& \mathcal{A} \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}
$$

Similarly, any extension $\mathcal{K}$ of $\mathcal{F}$ along $\mathbb{1}_{\mathcal{A}}$ must factor through the adjoint $\mathcal{G}$,

$$\mathbb{1}_{\mathcal{A}} \Longrightarrow (\operatorname{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}) \circ \mathcal{F} \Longrightarrow \mathcal{K} \circ \mathcal{F} \Longrightarrow \cdots$$

visualised by the diagram,

$$
\begin{tikzcd}
& \mathcal{B} \arrow[rr, "\mathbb{1}_{\mathcal{B}}"] \arrow[dr, "\mathcal{G}"] \arrow[d, phantom, "\Uparrow"] & {} \arrow[d, phantom, "\Uparrow"] & \mathcal{B} \arrow[dr, "\mathcal{K}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{B} \arrow[dr, "\mathcal{K}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{A} \arrow[ur, "\mathcal{F}"] \arrow[rr, "\mathbb{1}_{\mathcal{A}}"'] & {} & \mathcal{A}
\end{tikzcd}
$$

Lemma 5.4.

$$\operatorname{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F} \dashv \mathcal{G} \cong \operatorname{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}} .$$

□

So, $\mathcal{G}$ has a left adjoint if and only if $\operatorname{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}$ exists. Similarly, $\mathcal{F}$ has a right adjoint if and only if $\operatorname{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}$ exists. The adjoint functor theorems usually are some conditions for the existence of these Kan extensions. When the categories are small, we can express the Kan extension as a coend. So in such cases, adjoints do exist.

5.3 | Nerve Realization

Formula Sheet