Part III

Geometric Morphisms of Sheaves

In these notes we study certain adjoint pairs that don't need the language of derived categories. These can be called internal functors or external depending on whether we stay within that category of sheaves over a topological space or we leave the category. These are four of Grothendieck's 'six operations'.

1 | Direct and Inverse Image Sheaves

Continuous maps $X \to Y$ gives rise to an adjoint pair of functors $\mathrm{Sh}(X) \rightleftharpoons \mathrm{Sh}(Y)$ called direct image, and inverse image, these are external operations, i.e., we are moving to a different topological space. A topological space X determines a category $\mathrm{PSh}(X)$ of sheaves on X. A continuous map of spaces $f : X \to Y$ will induce functors in both directions, forward and backward, on the associated category of pre-sheaves $\mathrm{PSh}(X)$ and $\mathrm{PSh}(Y)$.

Let $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ be a sheaf on X. Using this sheaf we can construct a sheaf on Y as follows, the sheaf assigns to each open set $U \subset X$ a set $\mathcal{F}U$, we can associate the same set $\mathcal{F}U$ to the open set whose image under f was U. The continuous function f gives us a functor of categories of open sets,

$$f^{-1} : \mathcal{O}(Y)^{\mathrm{op}} \to \mathcal{O}(X)^{\mathrm{op}}.$$

This gives rise to a new sheaf, the induced sheaf $f_*\mathcal{F}$ on Y, defined as the composition of functors,

$$\mathcal{O}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}.$$

Defined for each open set V of Y, i.e., $V \in \mathcal{O}(Y)$ by,

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

This is defined because $f^{-1}V$ is an open set by definition of continuous functions. $f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$ under f. This gives us a functor,

$$f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y).$$

This respects the composition of functions, i.e.,

$$(fg)_* = f_*g_*$$

So, if we set $\mathrm{PSh}(f) = f_*$, then PSh is a functor from the category of topological spaces to the category of sheaves. The direct image functor f_* has a left exact, left adjoint f^*. The construction of this left adjoint is considerably more complex.

1.1 | Inverse Image Sheaves via Étale Space

To construct inverse image, we make use of pullbacks. Let $E \to Y$ be a bundle over Y, we can then pullback E along a function $f : X \to Y$ giving us the bundle, $f^*E \to X$,

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

f^* is a functor $f^* : \mathbf{Bund}\ Y \to \mathbf{Bund}\ X$.

Suppose E is an étale bundle over Y, i.e., around each point $e \in E$ has a neighborhood U_e that is homeomorphic to its image $\pi(U_e)$. By definition of pullback, the space f^*E consists of points which we can label using points of X and E by $\langle x, e\rangle$ such that $fx = \pi e$, i.e., the pullback is the universal equalizer of the two maps.

By definition of étale spaces, there is a neighborhood U_e of e that's mapped homeomorphically to its image. Using this image we can construct an open neighborhood of x, by taking $f^{-1}(\pi(U_e))$. This is possible because by definition of pullback we have $fx = \pi e$. So, $\langle f^{-1}(\pi(U_e)), U_e\rangle$ is an open neighborhood of $\langle x, r\rangle$ that is mapped homeomorphically onto $f^{-1}(\pi(U_e))$ of X. Hence f^*E is étale.

If we start with a sheaf $\mathcal{F}$ on Y, a point in the pullback $f^*\mathcal{F}$ is of the form, $\langle x, \operatorname{germ}_{f(x)} s\rangle$ where $s \in \mathcal{F}V$ is an element of the sheaf $\mathcal{F}$.

Lemma 1.1. *Pullback of an étale space under a continuous map is étale.* □

This gives us a map of sheaves,

$$\mathrm{Sh}(Y) \xrightarrow{\mathcal{E}} \mathbf{Etale}\, Y \xrightarrow{f^*} \mathbf{Etale}\, X \xrightarrow{\Gamma} \mathrm{Sh}(X).$$

Here the first map is the bundling of the stalks of a sheaf to an étale space, and the last map is taking the sheaf of sections of the bundle. The composition gives us a functor of sheaves, which we denote again by f^*,

$$\mathrm{Sh}(Y) \xrightarrow{f^*} \mathrm{Sh}(X).$$

Theorem 1.2.

$$\mathrm{Sh}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Sh}(Y), \qquad f^* \dashv f_*.$$

Sketch of Proof

So, what we need to prove is that, $\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*F, G) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(F, f_*G)$, which for the sake of brevity we will denote by,

$$\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Sh}(Y)(F, f_*G).$$

Since we have an equivalence between the category of étale spaces and category of sheaves over X, we have that $\mathrm{Sh}(X)(f^*F, G) \cong \mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$, where $\mathrm{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$ is the collection of morphisms between étale spaces $\mathcal{E}f^*F$ and $\mathcal{E}G$.

Let $K(\mathcal{E}f^*F, \mathcal{E}G)$ be the set of functions $k : \mathcal{E}f^*F \to \mathcal{E}G$ over X.

1.2 | Inverse Image Sheaves via Kan Extension

2 | Category of Abelian Sheaves

Sheaves encountered often in geometry are abelian sheaves. Let X be a topological space, and consider continuous functions $f : X \to \mathbb{R}$ on the topological space X. It forms a sheaf,

$$C : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}.$$

Continuous functions can be added, substracted, multiplied and scaled to obtain new continuous functions. For each open set $U \in \mathcal{O}(X)$, the collection of all continuous functions on it CU forms an $\mathbb{R}$-algebra. Each CU are $\mathbb{R}$-module objects in the category **Sets**.

Since we can encounter lot of sheaves which take values in an abelian category, it's justified to give them special attention.[1] Let $\mathcal{A}$ be an abelian category, an abelian pre-sheaf on a topological space X is a functor $\mathcal{F}$,

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathcal{A}, \tag{pre-sheaf}$$

The category of all pre-sheaves on a topological space X with values in $\mathcal{A}$ is denoted by $\mathrm{PSh}(X, \mathcal{A})$. The functor category,

$$\mathrm{PSh}(X, \mathcal{A}) \coloneqq \mathcal{A}^{\mathcal{O}(X)^{\mathrm{op}}}.$$

admits small limits since the abelian category $\mathcal{A}$ admits small limits , small colimits, and the small filtered limits are exact and limits in functor category are computed pointwise. Once we have small limits we can start doing all sorts of things like take pullbacks, pushforward, products, coproducts, etc.

A pre-sheaf is a sheaf if the local associations $\mathcal{F}U_i$ are restrictions of global association. So, given an open covering $U = \bigcup_{i \in I} U_i$, if $f_i \in \mathcal{F}U_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it should mean that there exists a section $f \in \mathcal{F}U$ such that $f_i = f|_{U_i}$. The maps $f_i \in \mathcal{F}U_i$ and $f_j \in \mathcal{F}U_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from $\mathcal{F}U$ to $\prod_i \mathcal{F}U_i$ such that $pe = qe$. $\mathcal{F}U \to \prod_i \mathcal{F}U_i$

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j). \tag{collation}$$

This is the collation property. For general sites, the intersection $\cap$ will be replaced by the fibered product $\prod_{i,j}$ and the open covers are replaced by covers on sites, injective maps replaced by monic maps and so on. We are interested in exploiting the target category now, i.e., $\mathcal{A}$.

[1] Abelian categories are categories where we can do homological algebra, i.e., kernels, cokernels, images, coimages, direct sums, products, etc exist. The important result we need about them is that if $\mathcal{A}$ is an abelian category then so is the functor category $\mathcal{A}^{\mathcal{C}}$ for any category $\mathcal{C}$. For a discussion on abelian categories see [2].

2.1 | Internal $\mathcal{H}om$

The operations of interest to us now are within a category of pre-sheaves $\mathrm{PSh}(X, \mathcal{A})$ or sheaves $\mathrm{Sh}(X, \mathcal{A})$ on a given topological space X or site. Since we are not going to use or do anything special with the underlying topological space or site, we will for the sake of simplicity assume it to be a topological space X.

Consider two pre-sheaves, $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(X, \mathcal{A})$, for any $U \subset X$, consider the new pre-sheaves, the restructions, $\mathcal{F}|_U, \mathcal{G}|_U \in \mathrm{PSh}(U, \mathcal{A})$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A})}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{tikzcd} U \arrow[d, "|_V"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_V)"'] & \mathcal{G}U \arrow[d, "G(|_V)"] \\ V & \mathcal{F}V \arrow[r, "\kappa_V"'] & \mathcal{G}V. \end{tikzcd}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom, denoted by,

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(X, \mathcal{A}).$$

Sometimes also written as $\mathcal{G}^{\mathcal{F}}$. We will now show that the collation property holds for the $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, and hence $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, i.e., for any cover $\{U_i\}$ of U, the following is an exact sequence,

$$0 \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{G})U \overset{e}{\dashrightarrow} \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i \textstyle\prod_{i,j} U_j).$$

To show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{F}, \mathcal{G})U$ and at $\prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 2.1. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then so is $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i\in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})U$, and $U_i \subset U$, we have,

$$\begin{tikzcd} U \arrow[d, "|_{U_i}"] & \mathcal{F}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{F}(|_{U_i})"'] & \mathcal{G}U \arrow[d, "G(|_{U_i})"] \\ U_i & \mathcal{F}U_i \arrow[r, "\kappa_{U_i}"'] & \mathcal{G}U_i. \end{tikzcd}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{F}(U_i)$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \prod V} = 0.$$

Now, $\{W \prod U_i\}$ is a cover of W, and $\mathcal{G}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \prod_{i,j} U_j} = \kappa_j|_{U_i \prod_{i,j} U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{G}$ is a sheaf to patch these natural transformations.

Since $\mathcal{G}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{F}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

here the first map comes from the natural transformation, $\mathcal{F}V \ni f \mapsto \kappa_i(f|_{V \prod U_i})$. Since $\mathcal{G}$ is a sheaf, this must uniquely factor through $\mathcal{G}V$, by definition of equaliser. Hence, we have,

$$\mathcal{F}V \xrightarrow{\exists!} \mathcal{G}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \textstyle\prod U_i) \underset{q}{\overset{p}{\rightrightarrows}} \displaystyle\prod_{i,j} \mathcal{G}(V \textstyle\prod (U_i \prod_{i,j} U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ defines the patched up element that equalizes the diagram. $\square$

Note here that $\mathcal{F}$ need not be a sheaf, the above proposition also holds if $\mathcal{F}$ is a pre-sheaf, and $\mathcal{G}$ is a sheaf.

2.1.1 | Hom-Tensor Adjointness

Note here that the natural transformations κ respect the abelian structure. The exponentiation category $\mathcal{G}^{\mathcal{F}}$ has more structure than the standard exponentiation in **Sets**. Let $\mathcal{R}$ be a sheaf of commutative rings.

$$\begin{aligned} \mathcal{R} \,:\, \mathcal{O}(X)^{\mathrm{op}} &\to \text{ CRings} \\ U &\mapsto \mathcal{R}U. \end{aligned}$$

Then we can consider the pre-sheaves which take values in the category ${}_{\mathcal{R}}\mathrm{Mod}$ of $\mathcal{R}$-modules, i.e., U gets mapped to $\mathcal{R}U$-modules. We will denote such pre-sheaves by $\mathrm{PSh}(X, \mathcal{R})$. It naturally inherits a tensor product from the module structure. So, for two pre-sheaves $\mathcal{F}$ and $\mathcal{G}$, we can construct the tensor product pre-sheaf,

$$\begin{aligned} \mathcal{F} \widehat{\otimes}_{\mathcal{R}} \mathcal{G} \,:\, \mathcal{O}(X)^{\mathrm{op}} &\to {}_{\mathcal{R}}\mathrm{Mod} \\ U &\mapsto \mathcal{F}U \otimes_{\mathcal{R}U} \mathcal{G}U \end{aligned}$$

We will denote the sheafification of this pre-sheaf by,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} \in \mathrm{Sh}(X, \mathcal{R}).$$

This is a bifunctor,

$$\cdot \otimes_{\mathcal{R}} \cdot : \ \mathrm{Sh}(X, \mathcal{R}) \times \mathrm{Sh}(X, \mathcal{R}) \to \mathrm{Sh}(X, \mathcal{R}).$$

2.2 | Tensor Product via Adjoint Functor Theorem

References

[1] S Mac Lane, L Moerdijk, Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer, 1992

[2] S Gelfand, Y Manin, Methods of Homological Algebra Springer, 2003

[3] M Kashiwara, P Schapira, Categories and Sheaves, Springer, 2000