Part V

Representations

We are interested in representing an abstract C*-algebra as operators on Hilbert space. The important part is the GNS construction, that allows us to construct a representation of an abstract C*-algebra using a state acting on it. The tools in this part will be very important in quantum field theory.

1 | Representations of C*-algebras

A $*$-homomorphism between two C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a mapping

$$\pi : A \to \pi(A),$$

such that $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$, $\pi(AB) = \pi(A)\pi(B)$ and $\pi(A^*) = (\pi(A))^*$. It's an algebra homomorphism which also preserves the $*$-operation.

Given a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$, we have, $1 = \pi(AA^{-1}) = \pi(A)\pi(A^{-1})$ so, π maps invertible elements to invertible elements, and $\pi(A^{-1}) = \pi(A)^{-1}$. Hence we have,

$$\sigma(\pi(A)) \subset \sigma(A).$$

This immediately tells us that, for self-adjoint operators, $\|\pi(A)\| = \rho(\pi(A)) \leq \rho(A) = \|A\|$. Since A^*A is self-adjoint, we have,

$$\|\pi(A)\|^2 = \|\pi(A^*)\pi(A)\| = \|\pi(A^*A)\| \leq \|A^*A\| = \|A\|^2.$$

A representation $(\mathcal{H}, \pi)$ of a unital C*-algebra $\mathcal{A}$ is a $*$-homomorphism,

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

which is unital, i.e., $\pi(1) = 1$ for some Hilbert space $\mathcal{H}$. Two representation $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of an algebra $\mathcal{A}$ are said to be equivalent if there exists a unitary operation $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_1(A) = U\pi_2(A)U^*,$$

for all $A \in \mathcal{A}$.

If the π is an isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$ it's called a faithful representation. Suppose we have a faithful representation of $\mathcal{A}$ then by injectivity we have, $\ker \pi = \{0\}$. There exists π^{-1} from range of π into $\mathcal{A}$.

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\| \leq \|A\|.$$

So whenever π is a faithful representation, then for every $A \in \mathcal{A}$,

$$\|A\| = \|\pi(A)\|.$$

If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$, a subspace $\mathcal{H}_1$ of $\mathcal{H}$ is said to be invariant under π if $\pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1$ for all $A \in \mathcal{A}$. If $\mathcal{H}_1$ is closed and $P_{\mathcal{H}_1}$ is the orthogonal projection with range $\mathcal{H}_1$ then the invariance implies,

$$P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1} = \pi(A)P_{\mathcal{H}_1}.$$

for all $A \in \mathcal{A}$. Hence,

$$\begin{aligned}
\pi(A)P_{\mathcal{H}_1} &= (P_{\mathcal{H}_1}\pi(A^*)P_{\mathcal{H}_1})^* \\
&= (\pi(A^*)P_{\mathcal{H}_1})^* \\
&= P_{\mathcal{H}_1}\pi(A).
\end{aligned}$$

for all $A \in \mathcal{A}$. Hence $\mathcal{H}_1$ is invariant under π if and only if, $\pi(A)P_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(A)$ for all $A \in \mathcal{A}$. If we define π_1 by,

$$\pi_1(A) = P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1},$$

then $(\mathcal{H}_1, \pi_1)$ is a representation of $\mathcal{A}$. It's called a subrepresentation of $(\mathcal{H}, \pi)$. This procedure of going to subrepresentation gives a decomposition of π. If $\mathcal{H}_1$ is invariant under π then so is $\mathcal{H}_1^\perp$. Setting $\mathcal{H}_2 = \mathcal{H}_1^\perp$ one can define a second subrepresentation. Now the original Hilbert space $\mathcal{H}$ can be written as a direct sum, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and each operator $\pi(A)$ then decomposes as a direct sum $\pi(A) = \pi_1(A) \oplus \pi_2(A)$. So the representation can be written as $(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2)$.

Given a family of representations $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ of $\mathcal{A}$ the direct sum of representations $\mathcal{H}_\alpha$ is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form $\varphi = \{\varphi_\alpha\}_{\alpha \in I}$ such that $\lim_F[\sum_{\alpha \in F} \|\varphi_\alpha\|^2] < \infty$ where F is a finite subset of I. The purpose of this definition is so that norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha \in I} \pi_\alpha,$$

is called direct sum of representations $\{(\mathcal{H}_\alpha, \pi_\alpha)\}_{\alpha \in I}$, denoted by, $\sum_{\alpha \in I}^{\oplus}\{(\mathcal{H}_\alpha, \pi_\alpha)\}$. The operators $\pi(A)$ on $\mathcal{H}$ are bounded because $\|\pi_\alpha(A)\| \leq \|A\|$ for each $\alpha \in I$.

A representation is trivial if $\pi(A) = 0$ for every $A \in \mathcal{A}$. These are uninteresting representations. A representation can however have a trivial part.

$$\mathcal{D} = \{\varphi \in \mathcal{H} \mid \pi(A)\varphi = 0 \, \forall A \in \mathcal{A}\}.$$

It follows that $\pi_{\mathcal{D}} = P_{\mathcal{D}}\pi P_{\mathcal{D}} = 0$ where $P_{\mathcal{D}}$ is the projection onto the subspace $\mathcal{D}$. A representation $(\mathcal{H}, \pi)$ is non degenerate if $\mathcal{D} = \{0\}$.

A vector $|\Omega\rangle$ in a Hilbert space $\mathcal{H}$ is called cyclic for $\mathcal{A}$ if $\{A|\Omega\rangle\}_{A \in \mathcal{A}}$ is dense in $\mathcal{H}$. A cyclic representation of $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, |\Omega\rangle)$ where $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and $|\Omega\rangle$ is a cyclic for $\pi(\mathcal{A})$.

Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of $\mathcal{A}$. Take a maximal family of nonzero vectors $|\{\Omega_\alpha\rangle\}_{\alpha \in I}$ in $\mathcal{H}$ such that,

$$\langle \pi(A)\Omega_\alpha | \pi(B)\Omega_\beta \rangle = 0,$$

for all $A, B \in \mathcal{A}$ and $\alpha \neq \beta$. Define, $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}}_{A \in \mathcal{A}}$. This is an invariant subspace of $\mathcal{H}$. Define $\pi_\alpha = P_{\mathcal{H}_\alpha} \pi P_{\mathcal{H}_\alpha}$ where $P_{\mathcal{H}_\alpha}$ is projection onto $\mathcal{H}_\alpha$. Then by construction each $\mathcal{H}_\alpha$ are mutually orthogonal and hence the representation $(\mathcal{H}, \pi)$ is of the form,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \{(\mathcal{H}_\alpha, \pi_\alpha)\}$$

So every nondegenerate representation can be written as a direct sum of a family of cyclic sub-representations. If no invariant subspaces the representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ is called irreducible. If $\mathcal{A}$ be a self-adjoint algebra of operators on Hilbert space $\mathcal{H}$,

1.1 | Positive Linear Functionals and States

Denote the dual space of $\mathcal{A}$ i.e., the set of all continuous linear functionals over $\mathcal{A}$ by $\mathcal{A}^\#$. The norm of a functional f over $\mathcal{A}$ is defined by, $\|f\| = \sup_{\|A\|=1}\{|f(A)|\}$. A linear functional ω over the algebra $\mathcal{A}$ is called positive if,

$$\omega(A^*A) \geq 0$$

for all $A \in \mathcal{A}$. A positive linear functional over $\mathcal{A}$ with $\|\omega\| = 1$ is called a state. The state is called faithful if $\omega(A^*A) = 0$ implies $A = 0$.

If ω_1 and ω_2 are two states then clearly,

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2,$$

is also a state for all $\lambda \in [0, 1]$. The set of states is a convex subset of $\mathcal{A}^\#$.

1.2 | Gelfand-Naimark-Segal Construction

If ω is a positive linear functional over $\mathcal{A}$ then we can define a sesquilinear form, $\varrho(B, A) = \omega(B^*A)$. i.e, $\varrho(\mu A, \lambda B) = \overline{\mu}\lambda\varrho(A, B)$, and $\varrho(A, B) = \overline{\varrho(B, A)}$. Since ω a positive linear functional we have,

$$\varrho(\lambda A - B, \lambda A - B) \geq 0.$$

On expanding it, we obtain,

$$|\lambda|^2\varrho(A, A) - \overline{\lambda}\varrho(A, B) - \lambda\varrho(B, A) + \varrho(B, B) \geq 0$$

By letting $\lambda = \varrho(A, B)/\varrho(A, A)$ we obtain, $0 \leq [|\varrho(A, B)|^2/\varrho(A, A)^2]\varrho(A, A) - 2[|\varrho(A, B)|^2/\varrho(A, A)] + \varrho(B, B)$. This gives us,

$$|\varrho(A, B)|^2 \leq B_\omega(A, A)B_\omega(B, B).$$

If ω is a positive linear functional then it satisfies the Cauchy-Schwarz inequality,

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B).$$

If ω_1 and ω_2 are two positive linear functionals we write $\omega_1 \geq \omega_2$ if $\omega_1 - \omega_2$ is positive. This gives an order on positive linear functionals. If ω_1 and ω_2 are two states over $\mathcal{A}$ and $0 < \lambda < 1$ then $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ is also a state such that $\omega \geq \lambda\omega_1$ and $\omega \geq (1 - \lambda)\omega_2$. The set of all states is a convex subset of $\mathcal{A}^\#$ and we will denote it by $S(\mathcal{A})$. The extreme points of this convex set are called pure states. They are such that $\omega > \lambda\omega_1$ iff $\omega_1 = \omega$.

Given a closed two-sided ideal $\mathcal{J} \subseteq \mathcal{A}$, the quotient algebra is defined by,

$$\mathcal{A}_\mathcal{J} = \mathcal{A}/\mathcal{J} = \{[A] = A + J \ \ | \ \ J \in \mathcal{J}\}$$

with the norm, $\|[A]\| = \inf_{J\in\mathcal{J}}\{\|A+J\|\}$ the algebra $\mathcal{A}_{\mathcal{J}}$ is a C^*-algebra.

The Gelfand-Naimark-Segal theorem constructs for a given C^*-algebra $\mathcal{A}$ and a state ω a representation of the algebra of observables $\mathcal{A}$ on the set of bounded operators $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$.

Theorem 1.1. (Gelfand-Naimark-Segal) *Let ω be a state on a unital C^*-algebra $\mathcal{A}$ then there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, |\Omega\rangle)$ of unit norm such that*

$$\omega(A) = \langle\Omega|\pi_\omega(A)\Omega\rangle$$

for all $A \in \mathcal{A}$. The representation is unique in the sense that if $(\mathcal{H}, \pi, |\Omega_\varphi\rangle)$ is a cyclic representation such that, $\varphi(A) = \langle\Omega_\varphi|\pi(A)\Omega_\varphi\rangle$ then there exists a unique unitary operator $U : \mathcal{H} \to \mathcal{H}_\omega$, such that,

$$\pi_\omega(A) = U\pi(A)U^*,$$

and $U|\Omega_\varphi\rangle = |\Omega_\omega\rangle$.

Proof

Given a state ω on $\mathcal{A}$ one considers the set in $\mathcal{A}$ defined by,

$$\mathcal{J}_\omega = \{A \mid \omega(A^*A) = 0\}.$$

By Cauchy-Schwarz inequality whenever $A \in \mathcal{J}_\omega$ for any $B \in \mathcal{A}$ we have,

$$|\omega((BA)^*BA)|^2 = |\omega(C^*A)|^2 \leq \omega(C^*C)\omega(A^*A) = 0,$$

where $C = B^*BA$. So, $BA \in \mathcal{J}_\omega$. So $\mathcal{J}_\omega$ is an ideal. Factorizing $\mathcal{A}$ by $\mathcal{J}_\omega$ an inner product is introduced on the quotient space $\mathcal{A}_{\mathcal{J}_\omega}$ defined by,

$$\langle[A]|[B]\rangle := \omega(A^*B).$$

where $[A]$ and $[B]$ denote the equivalence classes determined by A and B respectively. The new vector space is completed by adding all the Cauchy sequences and we denote the Hilbert space by $\mathcal{H}_\omega$. On this Hilbert space we have the representation of the algebra $\mathcal{A}$,

$$\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega),$$

defined by,

$$\pi_\omega(A)[B] \equiv [AB].$$

Let $[I] = |\Omega_\omega\rangle$. The expectation of any observable can then be written as,

$$\omega(A) = \langle\Omega_\omega|A\Omega_\omega\rangle.$$

A state on the algebra can be represented as a vector in some Hilbert space. A vector $\varphi \in \mathcal{H}$ is said to be cyclic for $\mathcal{A}$ if the closure of $\mathcal{A}\varphi$ is same as $\mathcal{H}$. A vector φ is separating for $\mathcal{A}$ if $A\varphi = 0$ implies $A = 0$ for all $A \in \mathcal{A}$. The vector $|\Omega_\omega\rangle$ is cyclic for $\mathcal{A}$.

For every state ω on an algebra $\mathcal{A}$ there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$.

$$\omega(A) = \langle\Omega_\omega|\pi_\omega(A)\Omega_\omega\rangle, \quad \forall\, A \in \mathcal{A}.$$

If there is another cyclic representation $(\mathcal{H}, \pi, |\Omega\rangle)$ then define a map, $U\pi(A)|\Omega\rangle = \pi_\omega(A)|\Omega_\omega\rangle$. This is an isometry with an inverse, hence it extends to a unitary map. □

The representation is faithful if the state if faithful. That's probably where the name faithful state comes from.

Theorem 1.2. *Let $A \in \mathcal{A}$ be a self-adjoint element. Then there exists a cyclic representation $(\mathcal{H}, \pi, |\Omega_\varphi\rangle)$ of $\mathcal{A}$ such that*

$$\|\pi(A)\| = \|A\|$$

Proof

The norm of a self-adjoint operator is the same as it's spectral radius,

$$\|A\| = \rho(A) = \sup_{\lambda \in \sigma(A)} \{|\lambda|\}$$

Let λ be this maxima, using this we can define a functional on the algebra generated by A and identity. Defined by,

$$\varphi_0 : \alpha A + \beta 1 \mapsto \alpha\lambda + \beta$$

It also maps $\varphi_0(1) = 1$. So the linear functional is also a state. Now by Hahn-Banach theorem this can be extended to a state φ on $\mathcal{A}$ with $\varphi(1) = \varphi_0(1) = 1 = \|\varphi\|$. The GNS representation for this state satisfy,

$$\|A\| = |\varphi_0(A)| = |\varphi(A)| = |\langle\Omega_\varphi|\pi_\varphi(A)\Omega_\varphi\rangle \leq \|\pi_\varphi(A)\|.$$

$\square$

Now, to each element A_i, we have a representation such that $\|\pi_i(A_i)\| = A_i$. Using these representations we can form a direct sum representation. Let $\{A_i\}_{i\in I}$ be a dense set in $\mathcal{A}$, For each $i \in I$ we have a representation $(\mathcal{H}_i, \pi_i)_{i\in I}$ such that $\|\pi_i(A_i)\| = \|A_i\|$ because $\|\pi_i(A_i^* A_i)\| = \|A_i^* A_i\|$ and C* identity. Thus the direct sum will be such that,

$$\|\pi(A)\| = \|A\|$$

for all $A \in \mathcal{A}$. If $\mathcal{A}$ is separable, I can be assumed to be countable set, and hence we can assume the representation $(\mathcal{H}, \pi)$ to be separable representation.

References

[1] V S Sunder, Functional Analysis: Spectral Theory, Birkhauser Advanced Texts, 1991