Harmonic Analysis for AQFT

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1 | Locally Compact Groups

By a locally compact group we mean a topological group that is locally compact and Hausdorff. Let $\mathcal{G}$ be a locally compact group. Let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on $\mathcal{G}$. Let $C_c^+(\mathcal{G})$ denote the compactly supported continnuous functions that are always non-negative. Note that $C_c(\mathcal{G})$ is the linear span of $C_c^+(\mathcal{G})$.

1.1 | The Modular Function

We start analysis of locally compact groups by constructing invariant measures on $\mathcal{G}$, which allows us to ignore some of the irrelevant data contained in the topology. The data that remains, heuristically is the data as seen by the invariant measure.

1.1.1 | The Haar Measure

A left Haar measure on $\mathcal{G}$ is a non-zero Radon measure λ, such that for all $x \in \mathcal{G}$,

$$\lambda(xE) = \lambda(E).$$

where E is a Borel subset of $\mathcal{G}$ and $x \in \mathcal{G}$. Radon measures are measures that respect the topology of the underlying space. In case of locally compact spaces the Riesz representation theorem tells us that Radon measures are in bijection with linear functionals on the space of compactly supported continuous functions, given by $f \mapsto \int f d\lambda$, so the invariance translated to this setting means,

$$\int L_x f d\lambda = \int f d\lambda.$$

where $(L_x f)(y) = f(x^{-1}y)$. Haar measures are very useful, and they allow much of the analysis that was possible with Lebesgue measures on $\mathbb{R}^n$.

Theorem 1.1.1. *Every locally compact group has a unique left Haar measure.*

Proof

Existence of Haar measures on locally compact groups is a very important tool for analysis of locally compact groups. We need to construct a functional on the space of compactly supported continuous functions that is translation invariant. All that we have to work with is the set of continuous functions on the locally commpact group. Compactly supported functions are well behaved, so, we can try to exploit their nice properties.

Let f, g be two compactly supported positive functions. Note that the support of each of these functions can be covered by finitely many open sets. We can relate the two functions using this. We can dominate the function f by a finite linear combination of scaled translations of g. Note that this is possible because f attains a maximum, and g is a non-zero positive function. So, there exists some $x \in \mathcal{G}$ such that $g(x) > 0$, and we can find a number λ such that $\lambda g(x) > \|f\|_{\text{sup}}$. By continuity of translation we can find a neighborhood around x where this holds. We can now transfer via translation L_y all the problematic parts to this neighborhood. Since the support of f is compact, we can cover it with finitely many translations of this neighborhood, hence we can make sense of a sum. So, there exists a finite collection of real numbers λ_i and translations L_{x_i} such that,

$$f \leq \textstyle\sum_i \lambda_i (L_{x_i}(g)).$$

Consider the set of all finite collections of numbers $\{\lambda_i\}$ such that $f \leq \sum_{i=1}^n \lambda_i (L_{x_i}(g))$. This is nonempty due to the above reason. Denote this set by $\mathcal{D}(f : g)$. The intuitive meaning here is that $\sum_i \lambda_i (L_{x_i} g)$ is an approximation of f in terms of translations of the function g. Using this we can define the 'best' approximation based on how little scaling we have to do with the translations of g. This defines some sort of index of the function f with respect to g.

Since each element of this set is a finite collection of real numbers, we can add them up. So, we have a 'sum' function on $\mathcal{D}(f : g)$,

$$\begin{aligned} \textstyle\sum : \mathcal{D}(f : g) &\to \mathbb{R} \\ \{\lambda_i\} &\mapsto \textstyle\sum_i \lambda_i. \end{aligned}$$

Using this function, we can define a new object,

$$(f : g) := \inf_{\{\lambda_i\} \in \mathcal{D}(f:g)} \big(\textstyle\sum_i \lambda_i \big).$$

We will call this the 'approximation index' of the function f with respect to g. This approximation index allows us to reduce the problem of constructing a special linear functional on the space of compactly supported continuous functions to doing some stuff with some 'reference function'. This will be made precise below. Note that this is where the 'locally compact' part has been important. The Riesz representation tells us that the construction of Radon measures is equivalent to constructing some special linear functionals on compactly supported continuous functions. Then the compactness of the supports of the functions ensure that the sets $\mathcal{D}(f : g)$ are non-empty for each compactly supported functions f and g.

Note that the translation L_y is a linear map on the space of continuous functions, so we have,

$$L_{y^{-1}} \big(\textstyle\sum_i \lambda_i (L_{x_i} g) \big) = \textstyle\sum_i \lambda_i (L_{y^{-1} x_i} g).$$

Since the approximation index is an infimum, the approximation index of any translation L_y of the function f with respect to g will be the same. So, we have,

$$(f : g) = (L_y f : g). \tag{invariance}$$

Scaling the functions f by λ just scales the sets $\mathcal{D}(f : g)$, so we have,

$$(\lambda f : g) = \lambda (f : g). \tag{scaling}$$

If $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(h : g)$, then the disjoint union, $\{\lambda_i\} \coprod \{\mu_j\} \in \mathcal{D}(f + h : g)$. So, we have,

$$(f + h : g) \leq (f : g) + (h : g). \tag{subadditivity}$$

Note that if this was additive, we would be done, we would have constructed a linear functional that is invariant under left translation. It is however possible choose g such that the inequality gets closer to an equality. The key lies in how well we can approximate the given function f, in terms of linear combinations of translations of g. So, the problem is now choosing an appropriate g. We will need some further properties of the functionals $(f : g)$ to do this.

If $f \leq h$ then any upper approximation of h by translations of g is also an upper approximation of f by translations of g. Hence we have,

$$f \leq h \Rightarrow (f : g) \leq (h : g). \qquad \text{(monotonicity)}$$

Let f attain a maximum at x, that is $f(x) = \|f\|_{\text{sup}}$ and g attain a maximum at y, that is, $g(y) = \|g\|_{\text{sup}}$. Then the translation $L_{yx^{-1}}(f)$ attains a maximum at y, by $(f : g) = (L_{yx^{-1}} f : g)$, we obtain,

$$(f : g) \geq \|f\|_{\text{sup}} / \|g\|_{\text{sup}}. \qquad \text{(boundedness)}$$

Suppose $f \leq \sum_i \lambda_i (L_{x_i} g)$ and $g \leq \sum_j \mu_j (L_{y_j} h)$, then by linearity of translations, we have, $f \leq \sum_{i,j} \lambda_i \mu_j (L_{x_i y_j} h)$. That is to say, $\{\lambda_i \mu_j\} \in \mathcal{D}(f : h)$ whenever $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(g : h)$. So, we have,

$$(f : h) \leq (f : g)(g : h). \qquad \text{(product)}$$

Now we choose a 'reference function' F, and define a normalized index,

$$I_g(f) := (f : g)/(F : g).$$

This gives us a functional $I_g : C_c^+(\mathcal{G}) \to \mathbb{R}$, that is left invariant, subadditive and monotone. By, $(F : g) \leq (F : f)(f : g)$ and $(f : g) \leq (f : F)(F : g)$ we have,

$$f_F \equiv (F : f)^{-1} \leq I_g(f) \leq (f : F) \equiv f^F \qquad \text{(rescaling)}$$

Now the problem is to choose an appropriate g such that (subadditivity) becomes an equality. In order to choose such a g, we have to understand what makes the approximations $(f : g)$ not 'close' to the function f. The function is determined by its value at each point. So, given a number, we can scale it appropriately at each point to get f. What we are trying to do is to approximate the function f, by scaling a bunch of functions on some open sets. So, smaller open sets means better approximation. So, to make the approximation closer to the actual value of f, we could use g that has smaller support.

Let f_i, f_j be two compactly supported continuous functions. Let g be a positive compactly supported continuous function such that $g \equiv 1$ on the support of the functions $(f_i + f_j)$. Let $\delta > 0$ be a number (will be specified a bit later). Then we can consider the function,

$$h = f_i + f_j + \delta \cdot g$$

Using this we can construct new functions,

$$h_i = f_i / h, \quad h_j = f_j / h.$$

This implies that $h_i \in \mathcal{C}_c^+(\mathcal{G})$, and $h_i \equiv 0$ whenever $f_i \equiv 0$. By continuity of h_i, for every $\delta > 0$ we can find a small enough neighborhood V of $1 \in \mathcal{G}$ such that

$$|h_i(x) - h_i(y)| \leq \delta$$

whenever $x, y \in V$. For any function φ with support in V, consider an approximate of h by finitely many translations of φ, $h \le \sum_k \lambda_k L_{x_k}\varphi$, then we have,

$$f_i(x) = h(x)h_i(x) \le \textstyle\sum_j \lambda_k L_{x_k}\varphi(x)h_i(x) \le \sum_k \lambda_k \varphi(x_k^{-1}x)h_i(x) \le \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k)+\delta)$$

whenever $x, x_k \in V$. Since $h_i + h_j \le 1$, we get,

$$(f_i : \varphi) + (f_j : \varphi) \le \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k)+\delta) + \sum_k \lambda_k \varphi(x_k^{-1}x)(h_j(x_k)+\delta) \le (1+2\delta)\sum_k \lambda_k.$$

Hence, by taking the infimum of all such $\sum_k \lambda_k$, we get,

$$\begin{aligned} I_\varphi(f_i) + I_\varphi(f_j) \le (1+2\delta)I_\varphi(h) &\le (1+2\delta)(I_\varphi(f_i) + I_\varphi(f_j) + \delta I_\varphi(g)) \\ &= (I_\varphi(f_i + f_j)) + \delta(2I_\varphi(f_i+f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g). \end{aligned}$$

The last step is by (subadditivity) and (scaling). Then by (rescaling) we can choose δ small enough such that the last part is less than ϵ, $\delta(2I_\varphi(f_i+f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g) < \epsilon$. Hence we can choose a function φ with a small support such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \le I_\varphi(f_i + f_j) + \epsilon.$$

The choice of δ determines the support of the function used for approximation. This can be chosen such that $I_\varphi(f_i) + I_\varphi(f_j)$ is as close to $I_\varphi(f_i + f_j)$ as we want. However this choice of δ depends on the functions f_i and f_j.

Each function $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ gives us a map $f \mapsto I_\varphi(f) \in \mathbb{R}$. The goal is to find a functional that is linear. For each φ the value of $I_\varphi(f)$ lies in the interval $X_f \equiv [f_F, f^F]$. So, we can embed all such functionals inside the compact Hausdorff space,

$$X = \prod_{f \in \mathcal{C}_c^+(\mathcal{G})} X_f.$$

This is compact Hausdorff space because it is a product of compact Hausdorff spaces. It consists of all real valued functions from the space of compactly supported functions whose value at f lies in the interval X_f.

Consider for each neighborhood V of $1 \in \mathcal{G}$, the compact subsets $K(V)$ of X, consisting of closures of the collection of all functionals of the form I_φ with support of φ inside V.

$$X = \bigcup_{1 \in V} K(V)$$

Clearly, $V \subseteq W$ implies $K(V) \subseteq K(W)$. So, these sets satisfy the 'finite intersection property' $\bigcap_{i=1}^n K(V_i) \supset K(\bigcap_{i=1}^n V_i)$. This is a strict subset because there always exist functions with domain that is larger than $\bigcap_i V_i$. Since X is compact, the intersection $\bigcap_i K(V_i)$ is nonempty.[1] So there exists some point I in X that lies in all of the $K(V)$'s.

Every neighborhood of I contains I_φ's. So, for any collection of functions $\{f_i\}_{i=1}^n \subset \mathcal{C}_c^+(\mathcal{G})$ and $\epsilon > 0$, there exists some $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ such that $|I(f_i) - I_\varphi(f_i)| < \epsilon$, by definition of neighborhoods in product topology.

[1] This is an alternate characterization of compactness. The equivalence of the two definitions is basically rewriting the compactness definition of open sets and unions in terms of closed sets and intersections using De Morgan is laws.

This implies that I is left translation invariant $L_y I = I$, and commutes with scaling. Since by adjusting the support for all $\epsilon > 0$, we can find a function φ such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

This implies that, $I(f_i) + I(f_j) \leq I(f_i + f_j) + \epsilon$ for all ϵ, and since I belongs to every $K(V)$, ϵ can be arbitrarily small, hence,

$$I(f_i) + I(f_j) = I(f_i + f_j).$$

This has a unique extension to $\mathcal{C}_c(\mathcal{G})$, because any function can be written as $f = g - h$ with $g, h \in \mathcal{C}_c^+(\mathcal{G})$, and if $f = g' - h'$, then we have $g + h' = g' + h$ and hence, $I(g) + I(h') = I(g') + I(h)$, or, $I(f) = I(g) - I(h)$ is well defined. This linear functional corresponds to a left Haar measure.

Theorem 1.1.2. *Left invariant Haar measure is unique upto scaling.*

Proof

Suppose λ and μ are two left Haar measures on $\mathcal{G}$. To prove uniqueness we have to show that the ratio $\int f d\lambda / \int f d\mu$ is the same for all compactly supported functions f. So, it is sufficient to show for any $\epsilon > 0$,

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| < \epsilon$$

Let f be a compactly supported function, let $K_f = \operatorname{supp}(f)$. Since f is a continuous function, for every $\epsilon > 0$ there exists a neighborhood U_x of x such that,

$$|f(z) - f(x)| < \tfrac{1}{2}\epsilon.$$

for any $z \in U_x$. We can now translate this neighborhood to the identity $L_{x^{-1}} U_x$, and hence we have for every ϵ, a neighborhood of 1 such that $|f(yx) - f(x)| < \frac{1}{2}\epsilon$ for all y in the neighborhood. Choose a symmetric neighborhood V_x of 1 such that $V_x V_x \subset L_{x^{-1}} U_x$. The sets xV_x cover the compact set $K_f = \operatorname{supp}(f)$, and hence there must exist a finite subcover, $K \subseteq \bigcup_{i=1}^n x_i V_{x_i}$.

Consider the neighborhood of 1, $V = \bigcap_{i=1}^n V_{x_i}$. Let $y \in V$. If $x \in K_f$ then there exists some i such that $x_i^{-1} x \in V_{x_i}$, so that $xy = x_i(x_i^{-1} x) \in U_{x_i}$. Then,

$$|f(xy) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon.$$

If both x and yx are not in K then $f(x) = f(yx) = 0$. Hence if f is a compact supported function, then it's left uniformly continuous, that is $\|L_y f - f\|_{\sup} \to 0$ uniformly.

Let $f, g \in \mathcal{C}_c^+(\mathcal{G})$, let V_0 be a symmetric, compact neighborhood of 1, and let $V_f = V_0 K_f \cup K_f V_0$, and $V_g = V_0 K_g \cup K_g V_0$. V_f and V_g are compact sets. For any $y \in V_0$, we have continuous functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto g(xy) - g(yx)$ with supports V_f and V_g respectively. Note that their value at 1 is zero.

Hence for any $\epsilon > 0$, there exists a neighborhood $V \subset V_0$ of 1 such that,

$$|f(xy) - f(yx)| < \epsilon, \;\; |g(xy) - g(yx)| < \epsilon$$

Pick a nice function h such that $h(x) = h(x^{-1})$, with support in V. Then,

$$\int h d\mu \int f d\lambda = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(x) d\lambda(x) d\mu(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(yx) d\lambda(x) d\mu(y).$$

Using $h(x) = h(x^{-1})$ we get,

$$\begin{aligned}
\int h d\lambda \int f d\mu &= \int_{\mathcal{G}} \int_{\mathcal{G}} h(x) f(y) d\lambda(x) d\mu(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}x) f(y) d\lambda(x) d\mu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(x^{-1}y) f(y) d\mu(y) d\lambda(x) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(yx) d\mu(y) d\lambda(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) f(xy) d\lambda(x) d\mu(y).
\end{aligned}$$

Therefore,

$$\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| = \left| \int_{\mathcal{G}} \int_{\mathcal{G}} h(y) \big[f(xy) - f(yx) \big] d\lambda(x) d\mu(y). \right| \leq \epsilon \lambda(V_f) \int h d\mu.$$

Similarly,

$$\left| \int h d\lambda \int f d\mu - \int h d\mu \int f d\lambda \right| =\leq \epsilon \lambda(V_f) \int h d\mu.$$

Dividing these inequalities by $\int h d\mu \int f d\mu$ and $\int h d\mu \int g d\mu$ and adding them, we get,

$$\left| \frac{\int f d\lambda}{\int f d\mu} - \frac{\int g d\lambda}{\int g d\mu} \right| < \epsilon \left[\frac{\lambda(V_f)}{\int f d\mu} + \frac{\lambda(V_g)}{\int g d\mu} \right].$$

Since ϵ is arbitrary, the difference must be zero, and hence λ and μ are unique upto a constant multiple. $\square$

If $\mathcal{G}$ is non-commutative, the left Haar measures allows us to quantify the 'amount of non-commutativity'. For each $x \in \mathcal{G}$, define, $\lambda_x(E) = \lambda(Ex)$. This is again a left Haar measure, and by uniqueness of Haar measures, there exists a number $\Delta(x) > 0$ such that,

$$\lambda_x = \Delta(x) \lambda.$$

The number $\Delta(x)$ is independent of the choice of λ by the uniqueness Haar measure. So, $x \mapsto \Delta(x)$ is a well defined function called the modular function. For any $x, y \in \mathcal{G}$, and $E \subset \mathcal{G}$, we have, $\lambda_{xy}(E) = \lambda(Exy) = \Delta(y)\Delta(x)\lambda(E)$, so,

$$xy \mapsto \Delta(x)\Delta(y),$$

or, Δ is a group homomorphism from the group $\mathcal{G}$ to the multiplicative group of positive numbers denoted by $\mathbb{R}_{\times}$. If χ_E is the characteristic function on E, then we have, $\chi_E(xy) = \chi_{Ey^{-1}}(x)$. So,

$$\int_{\mathcal{G}} \chi_E(xy) d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1}) \int_{\mathcal{G}} \chi_E(x) d\lambda(x).$$

Since a general function $f \in L^1(\mathcal{G})$ can be approximated by simple functions, we have,

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda.$$

or equivalently, this can be abbreviated as,

$$d\lambda(xy) = \Delta(y) d\lambda(x). \tag{right translation}$$

Since the map $y \mapsto R_y f$ is uniformly continuous, the map $y \mapsto \int R_y f d\lambda$ is a continuous map. So, Δ is continuous. So, the homomorphism Δ also respects the topological structure.

To each left Haar measure, we can associate a right Haar measure, $\rho(E) = \lambda(E^{-1})$. The modular modular function relates the two measures. If $f \in \mathcal{C}_c(\mathcal{G})$, then we have,

$$\int_{\mathcal{G}} R_y f(x)\Delta(x^{-1}) d\lambda(x) = \Delta(y) \int_{\mathcal{G}} f(xy)\Delta(xy^{-1}) d\lambda(x) = \int_{\mathcal{G}} f(x)\Delta(x^{-1}) d\lambda(x).$$

In the last step we used (right translation). Thus the functional $f \mapsto \int f(x)\Delta(x^{-1})d\lambda(x)$ is right invariant, and hence its associated Radon measure is a right Haar measure. Since ρ is a right Haar measure it must vary from the Radon measure associated to this functional by a constant, say κ. On symmetric neighborhoods, the left and right Haar measures are equal. If the constant κ is not 1, then by continuity of Δ, we can choose a small enough symmetric neighborhood of $1 \in \mathcal{G}$ such that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|\kappa - 1|$ on that neighborhood.

$$|\kappa - 1|\lambda(U) = |\kappa\rho(U) - \lambda(U)| = \left| \int_U [\Delta(x^{-1}) - 1] d\lambda(x) \right| \leq \tfrac{1}{2}|\kappa - 1|\lambda(U).$$

which can only happen if $\kappa = 1$. This gives us,

$$d\rho(x) = \Delta(x^{-1}) d\lambda(x)$$

This can be restated in a more convinient way as,

$$d\lambda(x^{-1}) = \Delta(x^{-1}) d\lambda(x). \tag{inversion}$$

If $\Delta \equiv 1$, in such a case the left Haar measures do not measure any non-commutativity, such a locally compact group $\mathcal{G}$ is called unimodular. Clearly, abelian groups are unimodular. For discrete groups, every element of the group has same measure, and hence whether we left translate or right translate, it would still have the same measure. So, discrete groups are also unimodular. Since Δ is a continuous homomorphism, it takes compact groups to compact subgroups of $\mathbb{R}_\times$, which can only be $\{1\}$. So, compact groups are unimodular, and if $K \subset \mathcal{G}$ is a compact subgroup, then $\Delta|_K \equiv 1$.

Let $[\mathcal{G}, \mathcal{G}]$ be the smallest closed subgroup of $\mathcal{G}$ containing all elements of the form, $[x, y] = xyx^{-1}y^{-1}$, called the commutator subgroup of $\mathcal{G}$. It is a normal subgroups since, $z[x, y]z^{-1} = zxyx^{-1}y^{-1}z^{-1} = [zxz^{-1}, zyz^{-1}]$. Since Δ is a group homomorphism, we have,

$$\Delta([x, y]) = [\Delta(x), \Delta(y)]$$

since $\mathbb{R}_\times$ is abelian, we have $[\Delta(x), \Delta(y)] = 1$. So, by the isomorphism theorem of groups, the group homomorphism $\Delta : \mathcal{G} \to \mathbb{R}_\times$ must factor through $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$. This implies that if $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ is compact, then $\mathcal{G}$ is unimodular. This is intuitively obvious, the modular function was built to measure the non-commutativity and this just means that, $[\mathcal{G}, \mathcal{G}]$ does not contain any information about the non-commutativity contained in the group $\mathcal{G}$. Intuitively, the important thing to note about the modular function is that the left Haar measures can only measure non-commutativity of 'large' groups in some sense.

1.2 | Convolution & Involution of Functions

Given any two Radon measures μ, ν on $\mathcal{G}$, we can define a functional,

$$\mu * \nu : f \mapsto \int_{\mathcal{G}} \int_{\mathcal{G}} f(xy) d\mu(x) d\mu(y).$$

for $f \in C_c(\mathcal{G})$. Clearly this is a linear map, and satisfies, $|(\mu * \nu)(f)| \leq \|f\|_{\sup} \|\mu\| \, \|\nu\|$, so it is a bounded linear functional, and hence defines a measure. This new measure is called the convolution of μ and ν, we may write,

$$\int f d(\mu * \nu) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(xy) d\mu(x) d\nu(y).$$

The order of the variables is important, the order of integration is not. If σ is some other measure, Fubini's rule guarantees that convolution is an associative operation,

$$\begin{aligned} \int f d[\mu * (\nu * \sigma)] &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(xy) d\mu(x) d(\nu * \sigma)(y) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\mathcal{G}} f(xyz) d\mu(x) d\nu(y) d\sigma(z) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(yz) d(\mu * \nu)(y) d\sigma(z) \\ &= \int f d[(\mu * \nu) * \sigma]. \end{aligned}$$

If $\mathcal{G}$ is abelian, $f(xy) = f(yx)$ and by definition, the convolution will be commutative. Conversely, consider the delta measures, $\int f d(\delta_x * \delta_y) = \int \int f(uv) d\delta_x(u) d\delta_y(v) = f(xy) = \int f d\delta_{xy}$. So, if convolutions are commutative, we would have $\delta_{xy} = \delta_x * \delta_y = \delta_y * \delta_x = \delta_{yx}$. Or, equivalently $xy = yx$. So, $\mathcal{G}$ is abelian if and only if convolutions commute.

Convolutions define a product structure on the Radon measures on $\mathcal{G}$. The estimate, $\|\mu * \nu\| \leq \|\mu\| \, \|\nu\|$ turns the collection of all Radon measures $\mathcal{M}(\mathcal{G})$ on $\mathcal{G}$ into a Banach algebra, called the measure algebra on $\mathcal{G}$. The delta measure at identity, δ_1 determines the identity on this algebra.

Consider the operation on $\mathcal{M}(\mathcal{G})$ defined by,

$$\mu^*(E) = \overline{\mu(E^{-1})}.$$

If $\mu, \nu \in \mathcal{M}(\mathcal{G})$ then we have,

$$\begin{aligned} \int f d(\mu * \nu)^* &= \int_{\mathcal{G}} f(x^{-1}) d\overline{(\mu * \nu)} = \int_{\mathcal{G}} \int_{\mathcal{G}} f((xy)^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(y^{-1} x^{-1}) d\overline{\mu}(x) d\overline{\nu}(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(yx) d\mu^*(x) d\nu^*(y) = \int f d(\nu^* * \mu^*). \end{aligned}$$

So, $(\mu * \nu)^* = \nu^* * \mu^*$. The operation $\mu \mapsto \mu^*$ is an involution operation.

Given a left Haar measure λ, each function $f \in L^1(\mathcal{G})$ can be identified with the measure $f(x) d\lambda(x) \in \mathcal{M}(\mathcal{G})$. The convolution operation on $\mathcal{M}(\mathcal{G})$ gives a convolution operation on $L^1(\mathcal{G})$ given by,

$$f * g(x) = \int_{\mathcal{G}} f(y) g(y^{-1} x) d\lambda(y).$$

With some manipulation and using (right translation), this is the same as,

$$f * g(x) = \int_{\mathcal{G}} f(xy^{-1}) g(y) \Delta(y^{-1}) d\lambda(y). \tag{convolution}$$

The involution on $\mathcal{M}(\mathcal{G})$, restricted to $L^1(\mathcal{G})$ is defined by the relation,

$$f^*(x) d\lambda(x) = \overline{f(x^{-1})} d\lambda(x^{-1}).$$

Which after some manipulations, and using (inversion), gives,

$$f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}. \tag{involution}$$

$L^1(\mathcal{G})$ with the convolution product and involution is a Banach algebra, called the L^1 group algebra of $\mathcal{G}$. These structures also can be defined for $L^p(\mathcal{G})$, for $p \in [1, \infty)$. $L^p(\mathcal{G})$ consists of functions such that $|f|^p$ is integrable. In such a case we set,

$$\|f\|_p = \left(\int_{\mathcal{G}} |f|^p d\lambda(x) \right)^{1/p}$$

1.3 | Representation Theory

The action of groups of interest in quantum theory are as unitary operators, so, the objects of interest to us are unitary representations of locally compact groups. The physically important topology on operator algebras is the weak operator topology. The weak topology and strong topology coincide on unitary operators, so either topology works fine for us.[2]

A unitary representation $(\mathcal{H}_\pi, \pi)$ of a locally compact group $\mathcal{G}$ is a strongly continuous group homomorphism

$$\pi : \mathcal{G} \to U(\mathcal{H}_\pi),$$

where $\mathcal{H}_\pi$ is some Hilbert space, and $U(\mathcal{H}_\pi)$ is the group of unitary operators on $\mathcal{H}_\pi$. The dimension of the Hilbert space $\mathcal{H}_\pi$ is called the degree of the representation π. When there is no confusion, the representation will just be denoted by π. If π is an injection of $\mathcal{G}$ into $U(\mathcal{H}_\pi)$ it is called a faithful representation.

Let $(\mathcal{H}_\pi, \pi)$ be a representation of $\mathcal{G}$, a subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ is said to be invariant under π if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in \mathcal{G}$. If $\mathcal{M}$ is closed and $P_{\mathcal{M}}$ is the orthogonal projection onto the closed subspace $\mathcal{M}$ the invariance implies that,

$$P_{\mathcal{M}} \pi(x) = \pi(x) P_{\mathcal{M}},$$

for all $x \in \mathcal{G}$. We can define a new representation $(\mathcal{M}, \pi^{\mathcal{M}})$ by defining,

$$\pi^{\mathcal{M}}(x) = P_{\mathcal{M}} \pi(x) P_{\mathcal{M}},$$

It is called a subrepresentation of $(\mathcal{H}_\pi, \pi)$. This procedure of going to subrepresentation gives a decomposition of π. If $\mathcal{M}$ is invariant under π then so is $\mathcal{M}^\perp$. This gives us a second subrepresentation. Now the original Hilbert space $\mathcal{H}$ can be written as a direct sum, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and each operator $\pi(x)$ then decomposes as a direct sum $\pi(x) = \pi^{\mathcal{M}}(x) \oplus \pi^{\mathcal{M}^\perp}(x)$. So the representation can be written as $(\mathcal{H}_\pi, \pi) = (\mathcal{M}, \pi^{\mathcal{M}}) \oplus (\mathcal{M}^\perp, \pi^{\mathcal{M}^\perp})$.

[2]Suppose the net of unitary operators $\{T_\alpha\}$ converge to T strongly, then for any $|\varphi\rangle \in \mathcal{H}$, $\|(T_\alpha - T)\varphi\|^2 = \|T_\alpha \varphi\|^2 - 2\mathrm{Re}\langle T_\alpha \varphi | T\varphi\rangle + \|T\varphi\|^2 = 2\|\varphi\|^2 - 2\mathrm{Re}\langle T_\alpha\varphi | T\varphi\rangle$ So, the $\{T_\alpha\}$ converges to T in the strong topology only if the last term converges to $2\|T\varphi\|^2$.

Given a family of representations $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha\in I}$ of $\mathcal{G}$ the direct sum of the representations $\{\pi_\alpha\}$ is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha\in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form $\varphi = \{\varphi_\alpha\}_{\alpha\in I}$ such that $\lim_F[\sum_{\alpha\in F} \|\varphi_\alpha\|^2] < \infty$ where F is a finite subset of I. The purpose of this definition is so that the norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha\in I} \pi_\alpha,$$

is called direct sum of representations $\{(\mathcal{H}_\alpha, \pi_\alpha)\}_{\alpha\in I}$, denoted by, $\sum_{\alpha\in I}^{\oplus}\{(\mathcal{H}_\alpha, \pi_\alpha)\}$. A representation is trivial if $\pi(x) = 1$ for every $x \in \mathcal{G}$. These are uninteresting representations. A representation can however have a trivial part.

A vector φ in a Hilbert space $\mathcal{H}_\pi$ is called cyclic for $\mathcal{A}$ if $\{\pi(x)|\varphi\rangle\}_{x\in\mathcal{G}}$ is dense in $\mathcal{H}_\pi$. A cyclic representation of $\mathcal{G}$ is a triple $(\mathcal{H}, \pi, |\varphi\rangle)$ where $(\mathcal{H}_\pi, \pi)$ is a representation of $\mathcal{G}$ and φ is a cyclic vector for $\pi(\mathcal{G})$.

Let $(\mathcal{H}, \pi)$ be a nondegenerate representation of $\mathcal{A}$. Take a maximal family of nonzero vectors $|\{\Omega_\alpha\rangle\}_{\alpha\in I}$ in $\mathcal{H}$ such that,

$$\langle \pi(A)\Omega_\alpha | \pi(B)\Omega_\beta\rangle = 0,$$

for all $A, B \in \mathcal{A}$ and $\alpha \neq \beta$. Define, $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}}_{A\in\mathcal{A}}$. This is an invariant subspace of $\mathcal{H}$. Define $\pi_\alpha = P_{\mathcal{H}_\alpha}\pi P_{\mathcal{H}_\alpha}$ where $P_{\mathcal{H}_\alpha}$ is projection onto $\mathcal{H}_\alpha$. Then by construction the $\mathcal{H}_\alpha$ are mutually orthogonal and hence the representation $(\mathcal{H}, \pi)$ is of the form,

$$\mathcal{H} = \bigoplus_{\alpha\in I}\{(\mathcal{H}_\alpha, \pi_\alpha)\}$$

By Zorn is lemma, every non-degenerate representation can be written as a direct sum of a family of cyclic subrepresentations. If it has no nontrivial closed invariant subspaces the representation $(\mathcal{H}_\pi, \pi)$ of $\mathcal{G}$ is called irreducible.

Given two representations π_i and π_j of $\mathcal{G}$, an intertwining operator for π_i and π_j is a bounded linear operator,

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

such that for every $x \in \mathcal{G}$, $T\pi_i(x) = \pi_j(x)T$. The collection of all intertwiners between π_i and π_j is denoted by $\mathcal{C}(\pi_i, \pi_j)$. If $\mathcal{C}(\pi_i, \pi_j)$ contains a unitary operator, the two representations are called unitarily equivalent.

The collection $\mathcal{C}(\pi)$ of bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(x)$ for all $x \in \mathcal{G}$ is called the commutant of π is an algebra. Since product operation is weakly continuous, $\mathcal{C}(\pi)$ is weakly closed. Since $T^*\pi(x) = [\pi(x^{-1})T]^* = [T\pi(x^{-1})]^* = \pi(x)T^*$, it is also closed under taking adjoints. Thus $\mathcal{C}(\pi)$ is a von Neumann algebra.

If π is reducible, $\mathcal{C}(\pi)$ contains non-trivial projections. Conversely, if $\mathcal{C}(\pi)$ contains bounded operators that are not multiples of identity, say T, then we can split it into the canonical sum of two self-adjoint operators. Since these are linear combinations of the operators T and T^* which belong to $\mathcal{C}(\pi)$, they also belong to $\mathcal{C}(\pi)$. If an operator commutes with the self-adjoint operator, it also commutes with its spectral projections. So, $\pi(x)$ must also commute with the spectral projections of some self-adjoint operator. So, the subspace corresponding to the projection operator is an invariant subspace.

Let π_i and π_j be two unitary representations of $\mathcal{G}$. Then for every intertwining operator

$$T : \mathcal{H}_{\pi_i} \to \mathcal{H}_{\pi_j}$$

for π_i and π_j, the adjoint of the operator $T^* : \mathcal{H}_{\pi_j} \to \mathcal{H}_{\pi_i}$ is an intertwining operator for π_j and π_i because,

$$T^*\pi_j(x) = [\pi_j(x^{-1})T]^* = [T\pi_i(x^{-1})]^* = \pi_i(x)T^*.$$

So, $T^*T \in \mathcal{C}(\pi_i)$ and $TT^* \in \mathcal{C}(\pi_j)$. If π_i and π_j are two irreducible unitary representations of $\mathcal{G}$, then both of these should be a multiple of the identity, that is, $T^*T = \kappa I$ or equivalently, $\kappa^{-1/2}T$ must be a unitary operator or T is zero. So, two irreducible representations π_i and π_j are equivalent if $\mathcal{C}(\pi_i, \pi_j)$ is one dimensional. These facts about irreducibility of representations are summarized in Schur's lemma, which we now state.

Theorem 1.3.1. (Schur's Lemma) *If π_i and π_j are irreducible representations of $\mathcal{G}$ then either they are equivalent representations, or they have no intertwiners.*

If $\mathcal{G}$ is abelian, then for every representation π of $\mathcal{G}$, the operators $\pi(x)$ commutes with all operators in $\pi(\mathcal{G})$, so $\pi(x) \in \mathcal{C}(\pi)$ for every $x \in \mathcal{G}$. If π is irreducible, then we should have $\pi(x) = \kappa_x I$, so every irreducible representation of a locally compact abelian group is one dimensional. Although this looks very surprising, it is because spaces of operators on a vector space contain a lot more information than groups. This is an important corollary of Schur's lemma.

Corollary 1.3.2. *If $\mathcal{G}$ is abelian and π is irreducible, then $\mathcal{H}_\pi \cong \mathbb{C}$.*

The unitary representations of the group $\mathcal{G}$ are closely related to the $*$-representations of the algebra of integrable functions $L^1(\mathcal{G})$. Once we have fixed a left Haar measure λ, let $f \in L^1(\mathcal{G})$, then for any unitary representation π of the group $\mathcal{G}$, we can define an operator $\pi(f)$ on the Hilbert space $\mathcal{H}_\pi$ by,

$$\langle \pi(f)\varphi | \varkappa \rangle = \int_{\mathcal{G}} f(x) \langle \pi(x)\varphi | \varkappa \rangle d\lambda(x).$$

Or, more compactly,

$$\pi(f) = \int_{\mathcal{G}} f(x)\pi(x) d\lambda(x),$$

in the sense of the strong operator topology. This satisfies,

$$|\langle \pi(f)\varphi | \varkappa \rangle| \leq \|f\|_1 \|\varphi\| \|\varkappa\|.$$

So, $\pi(f)$ is a bounded linear operator on $\mathcal{H}_\pi$ with the norm $\|\pi(f)\| \leq \|f\|_1$. For any two $f, g \in L^1(\mathcal{G})$,

$$\begin{aligned}\pi(f * g) &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(y)g(y^{-1}x)\pi(x) d\lambda(y) d\lambda(x) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(y)g(x)\pi(yx) d\lambda(x) d\lambda(y) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(y)g(x)\pi(y)\pi(x) d\lambda(x) d\lambda(y) = \pi(f)\pi(g).\end{aligned}$$

So, we obtain an algebra homomorphism. Moreover,

$$\pi(f^*) = \int_{\mathcal{G}} \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)d\lambda(x) = \int_{\mathcal{G}} \overline{f(x)}\pi(x^{-1})d\lambda(x) = \int_{\mathcal{G}} [f(x)\pi(x)]^* d\lambda(x) = \pi(f)^*,$$

hence, π is a $*$-homomorphism. The left translation by x of the function f is given by,

$$\begin{aligned}\pi(L_x f) &= \int_{\mathcal{G}} f(x^{-1}y)\pi(y)d\lambda(y) = \int_{\mathcal{G}} f(y)\pi(xy)d\lambda(y) \\ &= \int_{\mathcal{G}} f(y)\pi(x)\pi(y)d\lambda(y) = \pi(x)\pi(f).\end{aligned}$$

That is,

$$\pi(L_x f) = \pi(x)\pi(f)$$

Similarly, for the right translation,

$$\begin{aligned}\Delta(x^{-1})\pi(R_{x^{-1}}f) &= \Delta(x^{-1})\int_{\mathcal{G}} f(yx^{-1})\pi(y)d\lambda(y) = \int_{\mathcal{G}} f(y)\pi(xy)d\lambda(y) \\ &= \int_{\mathcal{G}} f(y)\pi(y)\pi(x)d\lambda(y) = \pi(f)\pi(x).\end{aligned}$$

That is,

$$\Delta(x^{-1})\pi(R_{x^{-1}}f) = \pi(f)\pi(x).$$

This will be reversed if we had chosen a right Haar measure to define integration. So, the map $f \mapsto \pi(f)$ is a $*$-homomorphism from $L^1(\mathcal{G})$ to $\mathcal{B}(\mathcal{H}_\pi)$. Since π is strongly continuous, we can find a neighborhood V of 1, such that for all $x \in V$,

$$\|\pi(x)\varphi - \varphi\| < \|\varphi\|.$$

Now consider the function, $f = |V|^{-1}\chi_V$, we get,

$$\|\pi(f)\varphi - \varphi\| = |V|^{-1}\left\| \int_V [\pi(x)\varphi - \varphi]d\lambda(x)\right\| < \|\varphi\|$$

So, $\|\pi(x)\varphi\| \neq 0$, or $\pi(x)\varphi \neq 0$. Hence,

$$\pi : f \mapsto \pi(f)$$

is a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on $\mathcal{H}_\pi$. So, a representation of the locally compact group $\mathcal{G}$ gives rise to a representation of integrable functions on the same Hilbert space. Conversely, a non-degenerate $*$-representation of $L^1(\mathcal{G})$ on a Hilbert space $\mathcal{H}_\pi$ gives rise to a unique unitary representation of $\mathcal{G}$. The idea is to use the functions f approaching the delta function at x, to get an operator for $\pi(x)$. See §3.2, [2].

Theorem 1.3.3. *The von Neumann algebras generated by $\pi(\mathcal{G})$ and $\pi(L^1(\mathcal{G}))$ are identical.*

Sketch of Proof

The idea is to approximate functions in $L^1(\mathcal{G})$ by compactly supported functions, and then approximate the representatives of these compactly supported functions using the representatives of the group elements.

If $g \in \mathcal{C}_c(\mathcal{G})$, we can divide up the support of g into a finite partition, say $E = \{E_i\}$. Now we can approximate the function g, using the value g at some point $x_i \in E_i$, by,

$$\Sigma_E = \sum_i g(x_i)\pi(x_i)|E_i|$$

Given any $\epsilon > 0$, by compactness of the support of g, and continuity of the map $x \mapsto g(x)\pi(x)\varphi$, for each φ, there exists a partition $E = \{E_i\}$ of the support of g such that, $\|g(x)\pi(x)\varphi - g(y)\pi(y)\varphi\| < \epsilon$, whenever $x, y \in E_j$ for some j. Thus,

$$\|\Sigma_E\varphi - \pi(g)\varphi\| < \epsilon|\cup_i E_i|.$$

Thus every neighborhood of $\pi(g)$ with respect to the strong topology, contains sums Σ_E.

Now every function $f \in L^1(\mathcal{G})$ is the L^1 limit of functions in $\mathcal{C}_c(\mathcal{G})$. So, by continuity of π, $\pi(f)$ is a norm limit of operators in $\pi(\mathcal{C}_c(\mathcal{G}))$. So, the von Neumann algebra generated by $\pi(L^1(\mathcal{G}))$ is contained in the von Neumann algebra generated by $\pi(\mathcal{G})$. Conversely, each $\pi(x)$ is the strong limit of $\pi(L_x\phi_U)$, as $U \to \{1\}$. Where ϕ_U is an 'approximate identity'. See [2] §2.5 for approximate identities, and §3.2 for a complete proof. □

1.3.1 | Functions of Positive Type

In the C^* algebra case representations and states are closely related to each other, we need to find analogues of states for group algebras. These are provided by functions of positive type. We can do similar constructions with group algebras that we did with C^*-algebras.

Let $L^\infty(\mathcal{G})$ denote the collection of all classes of bounded measurable functions on $\mathcal{G}$. With the sup-norm, $\|f\|_\infty := \sup_{x\in\mathcal{G}} |f(x)|$, $L^\infty(\mathcal{G})$ is a Banach space.

Every function $\omega \in L^\infty(\mathcal{G})$ defines a linear functional on $L^1(\mathcal{G})$ by

$$f \mapsto \widehat{\omega}(f) := \int_{\mathcal{G}} f(x)\omega(x)d\lambda(x)$$

This is bounded because,

$$|\widehat{\omega}(f)| = \left|\int_{\mathcal{G}} f(x)\omega(x)d\lambda(x)\right| \le \|\omega\|_\infty \int_{\mathcal{G}} \left|f(x)\right|d\lambda(x)\Bigg| = \|\omega\|_\infty\|f\|_1.$$

So, $\widehat{\omega}$ is a bounded linear functional on the group algebra $L^1(\mathcal{G})$. A function $\omega \in L^\infty(\mathcal{G})$ is said to be a function of positive type if for any function $f \in L^1(\mathcal{G})$,

$$\widehat{\omega}(f^* * f) = \int (f^* * f)\omega \ge 0.$$

Expanding the integral, by (convolution) and (involution), we have,

$$\begin{aligned}\int (f^* * f)\omega &= \int_{\mathcal{G}}\int_{\mathcal{G}} \Delta(y^{-1})\overline{f(y^{-1})}f(y^{-1}x)\omega(x)d\lambda(y)d\lambda(x)\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} \overline{f(y)}f(yx)\omega(x)d\lambda(x)d\lambda(y) = \int_{\mathcal{G}}\int_{\mathcal{G}} f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).\end{aligned}$$

We can restate that a function $\omega \in L^\infty(\mathcal{G})$ is of positive type if,

$$\omega(f^* * f) = \int_{\mathcal{G}}\int_{\mathcal{G}} f(x)\overline{f(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y) \ge 0 \qquad \text{(positive type)}$$

Let ω be a function of positive type, consider the function complex conjugate, $\overline{\omega}$,

$$\int (f^* * f)\overline{\omega} = \int_{\mathcal{G}}\int_{\mathcal{G}} \overline{f(y)} f(yx)\overline{\omega(x)} d\lambda(y) d\lambda(x) = \overline{\int [(\overline{f})^* * \overline{f}]\omega} \geq 0$$

So, if ω is a function of positive type, then so is $\overline{\omega}$. The set of all continuous functions of positive type on $\mathcal{G}$ is denoted by $\mathcal{S}(\mathcal{G})$.

Given a representation π of $\mathcal{G}$, every vector $\varphi \in \mathcal{H}_\pi$ defines a function on the group,

$$\Phi(x) = \langle \pi(x)\varphi | \varphi \rangle.$$

By the continuity requirements of the representation, this is a continuous function on $\mathcal{G}$. $\Phi(y^{-1}x) = \langle \pi(y^{-1}x)\varphi|\varphi\rangle = \langle \pi(y^{-1})\pi(x)\varphi|\varphi\rangle = \langle \pi(x)\varphi|\pi(y)\varphi\rangle$. So for every $f \in L^1(\mathcal{G})$,

$$\begin{aligned}
\hat{\Phi}(f^* * f) &= \int_{\mathcal{G}}\int_{\mathcal{G}} f(x)\overline{f(y)}\Phi(y^{-1}x) d\lambda(x) d\lambda(y) \\
&= \int_{\mathcal{G}}\int_{\mathcal{G}} \langle f(x)\pi(x)\varphi | f(y)\pi(y)\varphi \rangle d\lambda(x) d\lambda(y) = \|\pi(f)\varphi\| \geq 0,
\end{aligned}$$

i.e., $\Phi \in \mathcal{S}(\mathcal{G})$. Consider the space $L^2(\mathcal{G})$ of all square integrable functions on $\mathcal{G}$, consisting of functions g such that $\int |g|^2 d\lambda < \infty$. Equipped with the inner product,

$$\langle h, g \rangle = \int_{\mathcal{G}} \overline{h(x)} g(x) d\lambda(x),$$

$L^2(\mathcal{G})$ is a Hilbert space. On this space, we have a representation of the group $\mathcal{G}$, given by,

$$[\pi_L(x)g](y) = L_x g(y) = g(x^{-1}y).$$

The left invariance of the left Haar measure ensures that this is a unitary representation. It is called the left regular representation of $\mathcal{G}$ on $L^2(\mathcal{G})$. The right regular representation is similarly defined. For any $g \in L^2(\mathcal{G})$, let $\tilde{g}(x) = \overline{g(x^{-1})}$, then we have,

$$\langle \pi_L(x)g|g\rangle = \int_{\mathcal{G}} g(x^{-1}y)\overline{g(y)} d\lambda(y) = \overline{g * \tilde{g}(x)}.$$

Since every function of the form $\langle \pi(x)\varphi|\varphi\rangle$ arising from a representation π is a function of positive type, the above defined function is one too. Since conjugates of functions of positive type are also functions of positive type, we have,

$$g * \tilde{g} \in \mathcal{S}(\mathcal{G}).$$

In particular, $\mathcal{S}(\mathcal{G})$ is non-empty. So, for each $g \in L^2(\mathcal{G})$, $g * \tilde{g}$ defines a function of positive type. In particular, functions of positive type do indeed exist.

1.3.2 | Construction of Representations

Every unitary representations gives rise to functions of positive type. The converse is also true, that is every function of positive type arises from a unitary representation. The key ingredient to proving this is to use functions of positive type to construct Hilbert spaces, much like GNS construction in the case of C^*-algebras.

The goal is to define an inner product using the function of positive type on $L^1(\mathcal{G})$ and consider the completion, similar to how it is done in the GNS construction. Given a function ω of positive type, define a hermitian form on $L^1(\mathcal{G})$ by,

$$\langle f, g\rangle_\omega = \int (g^* * f)\omega = \int_{\mathcal{G}}\int_{\mathcal{G}} f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

By Fubini's is theorem this satisfies,

$$|\langle f, g\rangle_\omega| \leq \|\omega\|_\infty \|f\|_1 \|g\|_1. \tag{bounded}$$

The Hermitian form is positive semi-definite since ω satifies (positive type). So, $\langle , \rangle_\omega$ defines a positive definite or semi-definite Hermitian form on $L^1(\mathcal{G})$. It satisfies the Cauchy-Schwarz inequality,

$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle_\omega \langle g, g\rangle_\omega.$$

Consider the ideal $\mathcal{J}_\omega = \{f \mid \langle f, f\rangle_\omega = 0\}$. By the above inequality, $f \in \mathcal{J}_\omega$ if and only if $\langle f, g\rangle_\omega = 0$ for all $g \in L^1(\mathcal{G})$. So, $\langle , \rangle_\omega$ defines an inner product on the quotient space $L^1(\mathcal{G})/\mathcal{J}_\omega$. The image of $f \in L^1(\mathcal{G})$ in $L^1(\mathcal{G})/\mathcal{J}_\omega$ is denoted by $[f]$. The inner product on $L^1(\mathcal{G})/\mathcal{J}_\omega$ is given by,

$$\langle [f], [g]\rangle = \langle f, g\rangle_\omega$$

Denote the completion of this inner product space by $\mathcal{H}_\omega$. We have a natural action of the group $\mathcal{G}$ on the Hilbert space $\mathcal{H}_\omega$ given by the left translation.

$$\pi_\omega(x)[f] = [L_x f].$$

This action is a unitary operator, because

$$\begin{aligned}\langle L_x f, L_x g\rangle_\omega &= \int_{\mathcal{G}}\int_{\mathcal{G}} f(x^{-1}y)\overline{g(x^{-1}z)}\omega(z^{-1}y)d\lambda(y)d\lambda(z)\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} f(y)\overline{g(z)}\omega((xz)^{-1}(xy))d\lambda(y)d\lambda(z) = \langle f, g\rangle_\omega.\end{aligned}$$

In the last step, we used the left invariance of the Haar measure λ. This also tells us that the action keeps $\mathcal{J}_\omega$ invariant, and hence L_x gives rise to a unitary operator on $\mathcal{H}_\omega$. We also have an action of $L^1(\mathcal{G})$ on $\mathcal{H}_\omega$ given by,

$$\pi_\omega(g)[f] = [g * f].$$

Theorem 1.3.4. *There exists $\Omega \in \mathcal{H}_\omega$ such that for all $f \in L^1(\mathcal{G})$, $\pi_\omega(f)|\Omega\rangle = [f]$, and $\omega(x) = \langle \pi_\omega(x)\Omega, \Omega\rangle$ almost everywhere.*

Sketch of Proof

We want to think of ω as a limit of $L^1(\mathcal{G})$ functions. The idea is that if $\psi \in L^1(\mathcal{G})$ and $\omega \in L^\infty(\mathcal{G})$ then $\psi * \omega \in L^1(\mathcal{G})$[3] and consider the product of ω with an approximate identity to obtain the required sequence. $\widehat{f}_i(f) \to \widehat{\omega}(f)$ because $\psi_{U_i} * \omega \to \omega$ in appropriate topologies. Now all we are left with is some routine checks, which can be found in §3.3 [2]. □

[3] this is where almost everywhere is used.

The above theorem says that the information about the action of ω is determined by its infinitesimal behavior around the identity and $\Omega = [\{\psi_{U_i}\}]$. So, every function of positive type agrees with continuous functions locally almost everywhere. So, there is a correspondence between cyclic representations and continuous functions of positive type.

Given a function of positive type $\omega \in \mathcal{S}(\mathcal{G})$, there exists some representation π_ω such that

$$|\omega(x)| = |\langle \pi_\omega(x)\Omega, \Omega\rangle| \leq \|\Omega\|^2 = \omega(1)$$

Similarly,

$$\omega(x^{-1}) = \langle \pi_\omega(x^{-1})\Omega, \Omega\rangle = \langle \Omega, \pi_\omega(x)\Omega\rangle = \overline{\omega(x)}$$

Suppose π_i and π_j are two cyclic representations of $\mathcal{G}$ with cyclic vectors Ω_i and Ω_j and such that $\langle \pi_i(x)\Omega_i, \Omega_i\rangle = \omega(x) = \langle \pi_j(x)\Omega_j, \Omega_j\rangle$, then it follows that $T\pi_i(x)\Omega_i = \pi_j(x)\Omega_j$ extends to a linear isometry on the spans, and by continuity to a unitary map between the Hilbert spaces $\mathcal{H}_{\pi_i}$ and $\mathcal{H}_{\pi_j}$. So, $\pi_i(x)T = T\pi_j(x)$ for all $x \in \mathcal{G}$. So, any two cyclic representations are unitarily equivalent. So, every cyclic representation π such that $\omega(x) = \langle \pi(x)\Omega, \Omega\rangle$ is unitarily equivalent to the one we constructed above.

1.3.3 | States on Locally Compact Groups

Let $\mathcal{S}(\mathcal{G})$ be the set of all continuous functions of positive type, then by linearity of integration, we have for every finite set $\{\omega_i\}_{i\in\mathcal{I}} \subset \mathcal{S}(\mathcal{G})$ and $\lambda_i \in [0,1]$ with $\sum_i \lambda_i = 1$,

$$\int (f^* * f)\big(\textstyle\sum_i \lambda_i\omega_i\big) = \displaystyle\sum_{i\in\mathcal{I}} \lambda_i \int (f^* * f)\omega_i \geq 0.$$

for all $f \in L^1(\mathcal{G})$. So, the set $\mathcal{S}(\mathcal{G})$ is a convex set, and since scaling of functions of positive type by positive numbers also gives a function of positive type it is a convex cone. Some special subsets of $\mathcal{S}(\mathcal{G})$ can be singled out,

$$\mathcal{S}_1 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty = 1\}, \quad \mathcal{S}_0 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty \leq 1\}.$$

Let $\mathcal{E}(\mathcal{S}_i)$ be the extreme points of $\mathcal{S}_i$.

Theorem 1.3.5. *Let $\omega \in \mathcal{S}(\mathcal{G})$, then π_ω is irreducible iff $\omega \in \mathcal{E}(\mathcal{S}_1)$.*

Proof

Suppose π_ω is a reducible representation of $\mathcal{G}$, say, $\mathcal{H}_\omega = \mathcal{M} \oplus \mathcal{M}^\perp$ where $\mathcal{M}$ is nontrivial and invariant under π_ω. Let Ω be a cyclic vector for the representation π_ω, then Ω cannot belong to either $\mathcal{M}$ or $\mathcal{M}^\perp$, because otherwise $\pi_\omega(\mathcal{G})\Omega$ can only belong to either $\mathcal{M}$ or $\mathcal{M}^\perp$. But, if Ω is a cyclic vector, then $\pi_\omega(\mathcal{G})\Omega$ must be dense in $\mathcal{H}_\omega$.

So, $\Omega = \alpha\varphi + \beta\varkappa$ where φ and $\varkappa$ are unit vectors in $\mathcal{M}$ and $\mathcal{M}^\perp$ respectively. Let $\omega_\varphi(x) = \langle \pi_\omega(x)\varphi|\varphi\rangle$ and $\omega_\varkappa(x) = \langle \pi_\omega(x)\varkappa|\varkappa\rangle$. Then $\omega_\varphi, \omega_\varkappa \in \mathcal{S}(\mathcal{G})$. We have

$$\begin{aligned}\omega_\varphi(x) - \omega_\varkappa(x) &= \langle \pi_\omega(x)\varphi|\varphi\rangle - \langle \pi_\omega(x)\varkappa|\varkappa\rangle \\ &= \langle \pi_\omega(x)\Omega|\alpha^{-1}\varphi - \beta^{-1}\varkappa\rangle,\end{aligned}$$

as can be seen by standard calculations using $\Omega = \alpha\varphi + \beta\varkappa$ and $\langle\varphi, \varkappa\rangle = 0$. Since Ω is cyclic, and $\alpha^{-1}\varphi - \beta^{-1}\varkappa$ is non-zero, there must exist some $x \in \mathcal{G}$ such that $\langle \pi_\omega(x)\Omega|\alpha^{-1}\varphi - \beta^{-1}\varkappa\rangle$ is non-zero. So, in particular for some $x \in \mathcal{G}$,

$$\omega_\varphi(x) - \omega_\varkappa(x) \neq 0,$$

in particular $\omega_\varphi \neq \omega_\varkappa$ as continuous functions on $\mathcal{G}$. Since $\omega_\varphi(1) = \omega_\varkappa(1) = 1$ they are linearly independent, because if not, we would have $\omega_\varphi = \delta\omega_\varkappa$, in which case we should have $\omega_\varphi(1) = \delta\omega_\varkappa(1) = \delta$, hence $\omega_\varphi = \omega_\varkappa$. We arrive at

$$\omega(x) = \langle \pi_\omega(x)\Omega|\Omega\rangle = \alpha^2\langle \pi_\omega\varphi|\varphi\rangle + \beta^2\langle \pi_\omega\varkappa|\varkappa\rangle = \alpha^2\omega_\varphi(x) + \beta^2\omega_\varkappa(x).$$

therefore ω is not an extreme point in $\mathcal{S}(\mathcal{G})$.

Suppose π_ω is irreducible, and suppose ω is not an extreme point of $\mathcal{S}(\mathcal{G})$. Then there exist ω_φ and $\omega_\varkappa$ in $\mathcal{S}(\mathcal{G})$ such that $\omega = \omega_\varphi + \omega_\varkappa$. For any $f, g \in L^1(\mathcal{G})$, we have,

$$\langle f, f\rangle_{\omega_\varphi} = \langle f, f\rangle_\omega - \langle f, f\rangle_{\omega_\varkappa} \leq \langle f, f\rangle_\omega.$$

So,

$$|\langle f, g\rangle_{\omega_\varphi}| \leq \langle f, f\rangle_{\omega_\varphi}^{1/2}\langle g, g\rangle_{\omega_\varphi}^{1/2} \leq \langle f, f\rangle_\omega^{1/2}\langle g, g\rangle_\omega^{1/2}.$$

This gives us a bounded Hermitian form on $\mathcal{H}_\omega$, $(f, g) \to \langle f, g\rangle_{\omega_\varphi}$. So by the Riesz lemma, there exists a bounded self-adjoint operator T on $\mathcal{H}_\omega$ such that,

$$\langle f, g\rangle_{\omega_\varphi} = \langle T[f], [g]\rangle_\omega.$$

By the left action of $\mathcal{G}$ on $L^1(\mathcal{G})$, we have,

$$\begin{aligned}\langle T\pi_\omega(x)[f], [g]\rangle_\omega = \langle T[L_x f], [g]\rangle_\omega &= \langle L_x f, g\rangle_{\omega_\varphi} = \langle f, L_{x^{-1}} g\rangle_{\omega_\varphi} \\ &= \langle T[f], \pi(x^{-1})[g]\rangle_{\omega_\varphi} = \langle \pi_\omega(x)T[f], [g]\rangle_\omega.\end{aligned}$$

So, $T \in \mathcal{C}(\pi_\omega)$. By Schur's lemma, this means that $T = c\mathbb{I}$ or that $\langle f, g\rangle_{\omega_\varphi} = c\langle f, g\rangle_\omega$. Since

$$\langle f, g\rangle_\omega = \int (g^* * f)\omega = \int_\mathcal{G}\int_\mathcal{G} f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y),$$

we must have $\omega_\varphi = c\omega$, or that ω is an extreme point of $\mathcal{S}(\mathcal{G})$. $\square$

If ω_i is a net of functions of positive types that converges weakly to ω, then the weak* limit ω is also a function of positive type.[4] So, $\mathcal{S}_0$ is the weak* closed unit ball in $L^\infty(\mathcal{G})$. Now, by Alaoglu is theorem, $\mathcal{S}_0$ is weak* compact and by the Krein-Milman theorem it is the weak* closure of its extreme points.

The constant zero-function, which maps every element of the group to zero is a function of positive type by definition. Since $\|\omega\|_\infty = \omega(1)$, we have that $\|0\|_\infty = \omega(1) = 0$. This implies that the constant zero-function cannot be a mixture of other functions of positive type. Suppose ω be an extreme point in $\mathcal{S}_1$, and suppose it is a mixture of two other functions of positive type, that is, $\omega = \mu_i\omega_i + \mu_j\omega_j$ where $\mu_i + \mu_j = 1$. Then $\omega(1) = \mu_i\omega_i(1) + \mu_j\omega_j(1)$, this can only happen if $\omega_i(1) = \omega_j(1) = 1$. Which means that if $\omega \in \mathcal{S}_1$, then it can only be a mixture of other ω_i's in $\mathcal{S}_1$. Let $\omega \in \mathcal{S}(\mathcal{G})$, then we can construct the state $\omega/\omega(1) \in \mathcal{S}_1$. So, we can always write ω as a mixture,

$$\omega = (1 - \omega(1))0 + (\omega(1))\omega/\omega(1),$$

therefore we ahve shown

$$\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}.$$

[4]recall that weak* limit implies that $\widehat{\omega}_i(f) \to \widehat{\omega}(f)$ for all f.

Theorem 1.3.6. *The convex hull of $\mathcal{E}(\mathcal{S}_1)$ is weak* dense in $\mathcal{S}_1$.*

Proof

We have to show that every function of positive type, ω of norm 1, is a weak* limit of mixtures of functions in $\mathcal{E}(\mathcal{S}_1)$. Note that by Alaoglu's theorem $\mathcal{S}_0$ is a compact set in the weak* topology, in particular it is weak* closed by the Krein-Milman theorem. There exists a net of functions of positive type $\{\omega_i\}$ in the convex hull of $\mathcal{E}(\mathcal{S}_0)$ such that $\omega_i \to \omega$ weakly. We must ensure that if $\omega \in \mathcal{S}_1$ then these ω_i's can themselves be described as mixtures $\omega_i = \sum_j \mu_j \psi_j$ such that each $\psi_j \in \mathcal{S}_1$.

Since $\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}$, we can write each ω_i as $\mu_1\psi_1 + \cdots + \mu_{n_i}\psi_{n_i} + \mu_{n_i+1}0$ where each $\psi_j \in \mathcal{E}(\mathcal{S}_1)$. We can now remove the 0 component of the mixture. We can consider the new net,

$$\omega_i' = \frac{1}{\omega_i(1)} \sum_{j=1}^{n_i} \mu_j \psi_j$$

Each of these $\omega_i' \in \mathcal{S}_1$, by construction, and hence ω_i' is in the convex hull of $\mathcal{E}(\mathcal{S}_1)$ and $\omega = \lim_i \omega_i'$. $\square$

If $g \in \mathcal{C}_c(\mathcal{G})$ then $g \in L^2(\mathcal{G})$ and hence $g * \tilde{g} \in \mathcal{S}(\mathcal{G})$. Now we can apply polarization identity, to say that for any $f, g \in \mathcal{C}_c(\mathcal{G})$,

$$f * \tilde{g} \in \mathcal{S}(\mathcal{G})$$

Using approximate identities it can be showed that $\{f * g \mid f, g \in \mathcal{C}_c(\mathcal{G})\}$ is dense in $\mathcal{C}_c(\mathcal{G})$. So, the linear span of $\mathcal{C}_c(\mathcal{G}) \cap \mathcal{S}(\mathcal{G})$ contains all functions of the form $f * g$ for $f, g \in \mathcal{C}_c(\mathcal{G})$. $[\mathcal{C}_c(\mathcal{G}) \cap \mathcal{S}(\mathcal{G})]$ is dense in $\mathcal{C}_c(\mathcal{G})$ in the uniform norm. So, every compactly supported function can be approximated by a functions of positive type that are compactly supported.

Theorem 1.3.7. (Gelfand-Raikov) *Irreducible representations of $\mathcal{G}$ separate the points on $\mathcal{G}$.*

Idea of Proof

We have to show that for any $x \neq y$ there exists an irreducible representation π such that $\pi(x) \neq \pi(y)$. The idea is to use the fact that $\mathcal{G}$ is locally compact and Hausdorff to find a compactly supported function. One can choose a function such that the support does not include one of the points. This function can be written as a linear combination of functions of positive type. This linear combination can again be written as a convex combination of extreme points. Since these extreme points correspond to irreducible representations we would have the required irreducible representation that separates points on $\mathcal{G}$. The full proof requires a careful discussion regarding topology of compact convergence and its equivalence with the weak* topology on $\mathcal{E}(\mathcal{S}_1)$. This can be found in [2].

2 | Fourier Transform & The Pontrjagin Duality

We will assume that $\mathcal{G}$ is a locally compact abelian group. Some immediate facts are that the left and right translations coincide, the modular homomorphism is trivial, and convolution is commutative.

$$(f * g)(x) = \int_{\mathcal{G}} f(xy^{-1})g(y)d\lambda(y) = g * f(x).$$

Involution is given by

$$f^*(x) = \overline{f(x^{-1})}.$$

By Schur's lemma, every irreducible representation will be one dimensional. The dual group, and Fourier transforms of locally compact abelian groups are very similar to the techniques developed in the Gelfand-Naimark theory of commutative C^*-algebras. This subsection contains some of the big theorems in Fourier analysis, including Bochner's theorem, Fourier inversion, Pontrjagin duality, etc.

2.1 | The Dual Group

If π is an irreducible unitary representation of a locally compact abelian group $\mathcal{G}$ then $\mathcal{H}_\pi \cong \mathbb{C}$. In such a case, there exists $\chi(x)$ for every $x \in \mathcal{G}$ such that

$$\pi(x)(z) = \chi(x)z$$

and χ is a continuous homomorphism of $\mathcal{G}$ into the circle group $\mathbb{T}$, by unitarity of $\chi(x)$. Such homomorphisms are called characters of $\mathcal{G}$. The collection of all characters of $\mathcal{G}$ is denoted by $\widehat{\mathcal{G}}$. For a character χ, we will denote $\chi(x)$ by $\langle x, \chi \rangle$. A unitary representations of $\mathcal{G}$ determines a $*$-homomorphism of $L^1(\mathcal{G})$ into the operators on the representation space $\mathcal{H}_\pi$, which in our case is $\mathbb{C}$. This representation is given by

$$\chi(f) = \int_{\mathcal{G}} \langle x, \chi \rangle f(x) d\lambda(x).$$

By identifying the bounded linear maps $\mathcal{B}(\mathbb{C})$ with $\mathbb{C}$, this action determines a multiplicative functional on $L^1(\mathcal{G})$,

$$f \mapsto \chi(f).$$

We will use the notation from Gelfand theory for the spectrum of an algebra, $\sigma(\mathcal{A})$ will denote the set of all non-zero algebra homomorphisms from a Banach algebra $\mathcal{A}$ to $\mathbb{C}$. The Banach algebra of interest to us is the group algebra $L^1(\mathcal{G})$.

Let φ be a linear functional on $L^1(\mathcal{G})$, by the duality $L^\infty(\Omega) \cong L^1(\Omega)'$, on φ there must exist some $\chi \in L^\infty(\mathcal{G})$ such that,

$$\varphi(f) = \int_{\mathcal{G}} f(x)\chi(x)d\lambda(x)$$

for all $f \in L^1(\mathcal{G})$. Suppose φ is a multiplicative linear functional, that is, $\varphi \in \sigma(L^1(\mathcal{G}))$, then we have, for any $f, g \in L^1(\mathcal{G})$,

$$\int_{\mathcal{G}} \big[\varphi(f)\chi(x)\big] g(x) d\lambda(x) = \varphi(f) \int_{\mathcal{G}} g(x)\chi(x) d\lambda(x) = \varphi(f)\varphi(g) = \varphi(f * g)$$
$$= \int_{\mathcal{G}} \int_{\mathcal{G}} \chi(y) f(yx^{-1}) g(x) d\lambda(x) d\lambda(y) = \int_{\mathcal{G}} \big[\varphi(L_x f)\big] g(x) dx.$$

Since this holds for every $g \in L^1(\mathcal{G})$, the square bracketed terms on both sides must be the same almost everywhere. That is to say,

$$\varphi(f)\chi(x) = \varphi(L_x f)$$

almost everywhere. Choose a function $f \in L^1(\mathcal{G})$ such that $\varphi(f) \neq 0$, then we can define,

$$\chi(x) \coloneqq \varphi(L_x f)/\varphi(f).$$

For any $x, y \in \mathcal{G}$, by the above computation, we have,

$$\chi(xy)\varphi(f) = \varphi(L_{xy} f) = \varphi(L_x L_y f) = \chi(x)\chi(y)\varphi(f).$$

hence

$$\chi(xy) = \chi(x)\chi(y).$$

Therfore, χ is a group homomorphism from $\mathcal{G}$ to the circle group $\mathbb{T}$. So, every character gives rise to a multiplicative functional on $L^1(\mathcal{G})$ and conversely, every multiplicative functional on $L^1(\mathcal{G})$ corresponds to a character.

Theorem 2.1.1.

$$\sigma(L^1(\mathcal{G})) \cong \widehat{\mathcal{G}}.$$

With the pointwise multiplication $(\chi_1 \cdot \chi_2)(x) \coloneqq \chi_1(x)\chi_2(x)$ and pointwise inverse $\chi^{-1}(x) = \big(\chi(x)\big)^{-1}$, the set $\widehat{\mathcal{G}}$ is an abelian group. It is called the dual group of $\mathcal{G}$. The following is a useful computational tool,

$$\langle x, \chi^{-1} \rangle = \langle x^{-1}, \chi \rangle = \overline{\langle x, \chi \rangle}.$$

Since $\widehat{\mathcal{G}}$ is identified with the spectrum of $L^1(\mathcal{G})$, we can introduce the appropriate topology on $\widehat{\mathcal{G}}$ from $L^\infty(\mathcal{G})$, which is the weak* topology since we expect characters to be 'close' to each other if their evaluations are 'close'. This topology coincides on $\widehat{\mathcal{G}}$ with the topology of compact convergence, see §3.3 [2]. The set of all homomorphism from $L^1(\mathcal{G})$ to $\mathbb{C}$ is the set $\widehat{\mathcal{G}} \cup \{0\}$. By Alaoglu's theorem, $\widehat{\mathcal{G}} \cup \{0\}$ is compact, thus, $\widehat{\mathcal{G}}$ is locally compact.

Theorem 2.1.2. *$\widehat{\mathcal{G}}$ is a locally compact abelian group.*

2.1.1 | The Fourier Transform

Since $\widehat{\mathcal{G}}$ and $\sigma(L^1(\mathcal{G}))$ are identified via isomorphism, consider the composition with the inverse, which associates with the character χ the functional, $f \mapsto \overline{\chi(f)} = \chi^{-1}(f)$.

Recall that the Gelfand transform Γ on a Banach $*$-algebra $\mathcal{A}$ is a map

$$\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$$

which sends $A \in \mathcal{A}$ to a continuous function on the space of characters, given by evaluation. In our case, $\mathcal{A} \equiv L^1(\mathcal{G})$ and $C(\sigma(\mathcal{A})) = C(\widehat{\mathcal{G}})$. The Gelfand transformation on the Banach $*$-algebra $L^1(\mathcal{G})$ is called the Fourier transform on $\mathcal{G}$. The Fourier transform on $\mathcal{G}$ is then the map

$$\mathcal{F} : f \mapsto \mathcal{F}f := \widehat{f}$$

whose action on the characters of $\mathcal{G}$ is given by

$$\mathcal{F}f(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi \rangle} f(x) d\lambda(x). \tag{Fourier transform}$$

Note that this assigns to each $f \in L^1(\mathcal{G})$ a continuous bounded function $\mathcal{F}(f)$ on the space of characters. Since characters are homomorphisms, we have,

$$\begin{aligned}
\mathcal{F}(f * g)(\chi) &= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle x, \chi \rangle} f(xy^{-1}) g(y) d\lambda(y) d\lambda(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle xy, \chi \rangle} f(x) g(y) d\lambda(x) d\lambda(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle y, \chi \rangle}\, \overline{\langle x, \chi \rangle} f(x) g(y) d\lambda(x) d\lambda(y) = \mathcal{F}f(\chi)\mathcal{F}g(\chi).
\end{aligned}$$

So, Fourier transform is an algebra homomorphism. Similarly,

$$\mathcal{F}(f^*)(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi \rangle}\, \overline{f(x^{-1})} d\lambda(x) = \int_{\mathcal{G}} \langle x, \chi \rangle\, \overline{f(x)} d\lambda(x) = \overline{\int_{\mathcal{G}} \overline{\langle x, \chi \rangle} f(x) d\lambda(x)} = \overline{\mathcal{F}(f)(\chi)}.$$

So, the Fourier transform is a $*$-homomorphism. The norm of the Fourier transform is bounded by

$$\|\mathcal{F}(f)\|_{\sup} = \sup_{\chi \in \widehat{\mathcal{G}}} \big|\mathcal{F}(f)(\chi)\big| \le |\sup_{\mathcal{G}} \langle x, \chi \rangle| \int_{\mathcal{G}} \big|f(x)\big| d\lambda(x) \le \|f\|_1,$$

which shows that

$$\mathcal{F} : f \mapsto \mathcal{F}f$$

is a norm decreasing $*$-homomorphism from $L^1(\mathcal{G})$ to $\mathcal{C}_b(\widehat{\mathcal{G}})$, the continuous bounded functions with $\|\cdot\|_\infty$ norm. The action of left translation is given by

$$\widehat{L_y f}(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi \rangle} f(y^{-1}x) d\lambda(x) = \int_{\mathcal{G}} \overline{\langle yx, \chi \rangle} f(x) d\lambda(x) = \overline{\langle y, \chi \rangle} \widehat{f}(\chi).$$

So the Fourier transformation of the left translation L_y of a function f acts as a multiplication of the Fourier transformation of the function by the function $x \mapsto \overline{\langle y, x \rangle}$,

$$\widehat{L_y f}(\chi) = \overline{\langle y, \chi \rangle} \widehat{f}(\chi).$$

Similarly, for any character Ξ,

$$(\Xi f)^\wedge(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi\rangle}\langle x, \Xi\rangle f(x)d\lambda(x)\widehat{f}(\Xi^{-1}\chi) = L_\Xi \widehat{f}(\chi).$$

The range $\mathcal{F}(L^1(\mathcal{G}))$ is a selfadjoint subalgebra that separates points in $\widehat{\mathcal{G}}$, and by the Stone-Weierstrass theorem, it is a dense subspace of $C_0(\widehat{\mathcal{G}})$, the continuous functions vanishing at infinity.

Theorem 2.1.3. *$\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\widehat{\mathcal{G}})$.*

The Fourier transform $\mathcal{F}f$ of a function f is usually denoted by $\widehat{f}$. The Fourier-Stieltjes transform of a measure $\mu \in \mathcal{M}(\mathcal{G})$ is the bounded continuous function $\widehat{\mu}$ on $\widehat{\mathcal{G}}$ defined by,

$$\widehat{\mu}(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi\rangle} d\mu(x).$$

The Fourier transform of convolution of measures μ and ν is given by

$$\begin{aligned}
\widehat{\mu * \nu}(\chi) &= \int_{\mathcal{G}}\int_{\mathcal{G}} \overline{\langle xy, \chi\rangle} d\mu(x)d\nu(y) \\
&= \int_{\mathcal{G}}\int_{\mathcal{G}} \overline{\langle x, \chi\rangle}\,\overline{\langle y, \chi\rangle} d\mu(x)d\nu(y) = \widehat{\mu}(\chi)\widehat{\nu}(\chi).
\end{aligned}$$

Similarly, the Fourier-Stieltjes transform on $\mathcal{M}(\widehat{\mathcal{G}})$ is defined as,

$$\omega_\mu(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle d\mu(\chi)$$

Suppose $\mu \in \mathcal{M}(\widehat{\mathcal{G}})$, then it defines a linear functional on $L^1(\mathcal{G})$ via

$$f \mapsto \mu(f) := \int_{\mathcal{G}}\int_{\widehat{\mathcal{G}}} f(x)\langle x, \chi\rangle d\mu(\chi)d\lambda(x).$$

This defines a function of positive type, because for $f \in L^1(\mathcal{G})$,

$$\begin{aligned}
\mu(f^* * f) &= \int_{\mathcal{G}}\int_{\mathcal{G}}\int_{\widehat{\mathcal{G}}} \overline{f(y)} f(x)\langle y^{-1}x, \chi\rangle d\mu(\chi)d\lambda(x)d\lambda(y) \\
&= \int_{\mathcal{G}}\int_{\mathcal{G}}\int_{\widehat{\mathcal{G}}} \overline{f(y)}\,\overline{\langle y, \chi\rangle}\, f(x)\langle x, \chi\rangle d\mu(\chi)d\lambda(x)d\lambda(y) \\
&= \int_{\widehat{\mathcal{G}}} \overline{\big(\widehat{\widehat{f}}(\chi)\big)}\big(\widehat{\widehat{f}}(\chi)\big)d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f^*}(\chi^{-1})\big|^2 d\mu(\chi) \\
&= \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(\chi^{-1})\big|^2 d\mu(\chi) \geq 0.
\end{aligned}$$

Bochner's theorem says the converse is also true, that is every function of positive type is due to a positive measure as above.

Theorem 2.1.4. (Bochner's Theorem) *If $\omega \in \mathcal{S}(\mathcal{G})$, then there exists a unique positive $\mu \in \mathcal{M}(\widehat{\mathcal{G}})$, such that $\omega = \omega_\mu$.*

Proof

The idea is to construct a continuous linear functional on $C_0(\widehat{\mathcal{G}})$ and apply then Riesz-Markov theorem, which says that for every continuous functional κ on $C_0(\widehat{\mathcal{G}})$ there exists a finite Borel measure μ on $\widehat{\mathcal{G}}$ such that,

$$\kappa(f) = \int_{\widehat{\mathcal{G}}} f(x) d\mu(\chi).$$

Let ω be a function of positive type. We consider the linear functional $\widehat{f} \to \int \omega f$. We need to show that this extends to a continuous functional on $C_0(\widehat{\mathcal{G}})$. Since functions of positive type give rise to positive Hermitian forms, one could use the Cauchy-Schwarz inequality to show boundedness, and hence continuity of the functional. Assume without loss of generality that $\omega(1) = 1$ and consider the corresponding positive Hermitian form,

$$\langle f, g\rangle_\omega = \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{g(y)} f(x) \omega(y^{-1}x) d\lambda(x) d\lambda(y) = \int \omega(g^* * f).$$

Applying the Cauchy-Schwarz inequality gives

$$\left| \int \omega(g * f) \right|^2 \leq \int \omega(f^* * f) \int \omega(g^* * g)$$

for all $f, g \in L^1(\mathcal{G})$. If $\{\psi_U\}$ is an approximate identity, then by definition, we have, $\lim_{U\to 1} \|\psi_U^* * f - f\|_1 = 0$, so we have, $\langle f, \psi_U\rangle_\omega \to \int \omega f$, and $\langle \psi_U, \psi_U\rangle_\omega = |\int \psi_U d\lambda(x)|^2 \to \omega(1) = 1$.

From the above inequality, we now obtain

$$\left| \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x) \right|^2 \leq \langle f, f\rangle_\omega.$$

We may use this to construct a continuous linear functional on $\mathcal{F}(L^1(\mathcal{G}))$. For any $f \in L^1(\mathcal{G})$, construct a self-adjoint element, $h = f^* * f$. We can now apply the spectral radius formula and the Gelfand-Naimark theorem implying

$$\lim_{n\to\infty} \|h^{2^n}\|_1^{1/2^{n+1}} = \|\widehat{h}\|_\infty^{1/2} = \||\widehat{f}|^2\|_\infty^{1/2} = \|\widehat{f}\|_\infty.$$

The above inequality gives successively,

$$\left| \int \omega f \right| \leq \left| \int \omega h \right|^{1/2} \leq \left| \int \omega(h*h) \right|^{1/4} \leq \cdots \leq \left| \int \omega(h*\cdots*h) \right|^{1/2^{n+1}} \leq \|h^{2^n}\|_1^{1/2^{n+1}} \to \|\widehat{f}\|_\infty.$$

So, this induces a continuous linear functional on $\mathcal{F}(L^1(\mathcal{G}))$,

$$\widehat{f} \mapsto \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x).$$

Since $\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\mathcal{G})$, the linear functional can be extended to $C_0(\mathcal{G})$. By the Riesz-Markov representation theorem, there exists a measure $\widehat{\mu}_\omega \in \mathcal{M}(\widehat{\mathcal{G}})$ such that,

$$\widehat{f} \mapsto \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x) = \int_{\widehat{\mathcal{G}}} \widehat{f}(\chi) d\widehat{\mu}_\omega(\chi).$$

Expanding $\widehat{f}$ we get

$$\int \omega f = \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} f(x)\langle x, \chi^{-1}\rangle d\widehat{\mu}_\omega(\chi) d\lambda(x).$$

This implies $\omega(x) = \int \langle x, \chi\rangle d\mu_\omega(\chi)$ where $d\mu_\omega(\chi) = d\widehat{\mu}_\omega(\chi^{-1})$. Since $1 = \omega(1) = \mu_\omega(\widehat{\mathcal{G}}) \leq \|\mu_\omega\| \leq 1$, we must have, $\|\mu_\omega\| = \mu_\omega(\widehat{\mathcal{G}})$. Positivity follows from ω being positive. $\square$

The Bochner space is defined to be the set $B(\mathcal{G}) = \{\omega_\mu \mid \mu \in \mathcal{M}(\widehat{\mathcal{G}})\}$. Bochner's theorem says that $B(\mathcal{G}) = [\mathcal{S}(\mathcal{G})]$, which is the span of $\mathcal{S}(\mathcal{G})$. Define conversely, for each $\omega \in \mathcal{S}(\mathcal{G})$, a measure μ_ω that corresponds to ω, as described above. Bochner's theorem can be restated establishing

$$B(\mathcal{G}) \to \mathcal{M}(\widehat{\mathcal{G}})$$

Bochner's theorem now allows us to relate functions on $\mathcal{G}$ and functions on $\widehat{\mathcal{G}}$. Before describing these relations, we define,

$$B^p(\mathcal{G}) \coloneqq B(\mathcal{G}) \cap L^p(\mathcal{G}).$$

We will focus our attention on L^1 and L^2 functions. For the case of L^p functions, see [2]. The Fourier inversion theorems I & II relate the L^1 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$. The Plancherel theorem relates the L^2 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

Let $f, g \in B^1(\mathcal{G})$, for any $h \in L^1(\mathcal{G})$, we have,

$$\begin{aligned}
\int_{\widehat{\mathcal{G}}} \widehat{h}(\chi) d\mu_f(\chi) &= \int_{\widehat{\mathcal{G}}} \left[\int_{\mathcal{G}} \overline{\langle x, \chi\rangle} h(x) d\lambda(x) \right] d\mu_f(\chi) \\
&= \int_{\mathcal{G}} h(x) \left[\int_{\widehat{\mathcal{G}}} \langle x^{-1}, \chi\rangle d\mu_f(\chi) \right] d\lambda(x) \\
&= \int_{\mathcal{G}} h(x) f(x^{-1}) d\lambda(x) = \big(h * f\big)(1).
\end{aligned}$$

Replacing h by $h * g$ or $h * f$ and f by g, and by commutativity of convolution in locally compact abelian groups we get,

$$\int \widehat{h}\widehat{g} d\mu_f = \big((h * g) * f\big)(1) = \big((h * f) * g\big)(\mathbb{1}) = \int \widehat{h}\widehat{f} d\mu_g.$$

Hence it follows that,

$$\widehat{g} d\mu_f = \widehat{f} d\mu_g,$$

on $\mathcal{F}(L^1(\mathcal{G}))$. Since $\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\widehat{\mathcal{G}})$, it also holds on $C_0(\mathcal{G})$.

Theorem 2.1.5. (Fourier Inversion Theorem I) *If $f \in B^1(\mathcal{G})$ then $\widehat{f} \in L^1(\widehat{\mathcal{G}})$ and,*

$$f(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \widehat{f}(\chi) d\mu(\chi),$$

where μ denotes the suitably normalized Haar measure on $\widehat{\mathcal{G}}$.

Proof

The idea again is to construct a linear functional on $\mathcal{C}_c(\widehat{\mathcal{G}})$ using the function f and apply the Riesz representation theorem and think of this linear functional as an integral. So, the first step is to develop some tools to construct positive functionals on compactly supported continuous functions.

Let $K \subseteq \widehat{\mathcal{G}}$ be a compact set. Pick a $h \in \mathcal{C}_c(\mathcal{G})$ such that $\int_{\mathcal{G}} h(x)d\lambda(x) = 1$. Then,

$$\begin{aligned}
\widehat{h^* * h}(\chi) &= \int_{\mathcal{G}} \overline{\langle x, \chi\rangle} \left[\int_{\mathcal{G}} \overline{h(yx^{-1})} h(y) d\lambda(y) \right] d\lambda(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \langle yx^{-1}, \chi\rangle \overline{\langle y, \chi\rangle h(yx^{-1})} h(y) d\lambda(x) d\lambda(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \langle z, \chi\rangle \, \overline{\langle y, \chi\rangle} \, \overline{h(z)} h(y) d\lambda(z) d\lambda(y) \\
&= \left[\int_{\mathcal{G}} \langle z, \chi\rangle \overline{h(z)} d\lambda(z) \right] \left[\int_{\mathcal{G}} \overline{\langle y, \chi\rangle} h(y) d\lambda(y) \right] = |\widehat{h}|^2(\chi).
\end{aligned}$$

So, we have,

$$\widehat{h^* * h} = |\widehat{h}|^2.$$

hence $\widehat{h^* * h} \geq 0$, and $\widehat{h^* * h}(\mathbb{1}_{\widehat{\mathcal{G}}}) = 1$. By continuity, there exists a neighborhood V of $1_{\widehat{\mathcal{G}}}$, such that $\widehat{h^* * h} > 0$. We can cover K with finitely many of such functions, with translations. Consider the function,

$$\widehat{f}(\chi) = \sum_i L_{\chi_i} \big(\widehat{h^* * h}\big)(\chi).$$

This is the Fourier transform of the function, $f(x) = (\sum_i \chi_i(x)) \cdot (h^* * h)(x)$. By construction, this is such that $\widehat{f} > 0$ everywhere on K. So, for every compact set $K \subseteq \widehat{\mathcal{G}}$, there is a function $f \in \mathcal{S}(\mathcal{G})$, such that $\widehat{f} > 0$ on K.

If $h \in \mathcal{C}_c(\widehat{\mathcal{G}})$, there exists a function $f \in \mathcal{S}(\mathcal{G})$ which is strictly positive on the support of h. Using this function, define a functional, $I_f : \mathcal{C}_c(\widehat{\mathcal{G}}) \to \mathbb{C}$ by,

$$I_f(h) = \int_{\widehat{\mathcal{G}}} \left[\frac{h(\chi)}{\widehat{f}(\chi)} \right] d\mu_f(\chi).$$

The strict positivity is used here, so that $[h/\widehat{f}]$ is well behaved. Now for any other such function g, we have,

$$\begin{aligned}
I_f(h) &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{f}(\chi)\big] d\mu_f(\chi) = \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\widehat{f}(\chi)\big] \widehat{g}(\chi) d\mu_f(\chi) \\
&= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\widehat{f}(\chi)\big] \widehat{f}(\chi) d\mu_g(\chi) = \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\big] d\mu_g(\chi) = I_g(h).
\end{aligned}$$

So, the value of $I_f(h)$ depends only on h, so we will forget the subscript and write it as $I(h)$. Clearly, this is a linear functional. Since $d\mu_f$ is a positive measure, $I(h) \geq 0$ for all $h \geq 0$. We obtained a non-trivial positive linear functional on $\mathcal{C}_c(\widehat{\mathcal{G}})$, in particular,

$$I(\widehat{g}h) = \int \big[h/\widehat{f}\big] \widehat{g} d\mu_f = \int h d\mu_g.$$

We now need to show that this functional is well behaved, that is we have to show that it is invariant under translations. For any $\Xi \in \widehat{\mathcal{G}}$, we have,

$$\int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle d\mu_f(\Xi\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \Xi^{-1}\chi\rangle d\mu_f(\Xi^{-1}\Xi\chi) = \int_{\widehat{\mathcal{G}}} \overline{\langle x, \Xi\rangle} \langle x, \chi\rangle d\mu_f(\chi) = \overline{\langle x, \Xi\rangle} f(x) = \big(\Xi f\big)(x)$$

So, we have, $d\mu_f(\Xi\chi) = d\mu_{\overline{\Xi} f}(\chi)$. Recall also that

$$\big(\overline{\Xi} f\big)^{\wedge}(\chi) = \widehat{f}(\Xi\chi).$$

Now, for the invariance, if $f > 0$ on the union of the support of h and $L_{\Xi}h$, then we have,

$$\begin{aligned} I(L_{\Xi}h) = \int_{\widehat{\mathcal{G}}} \big[h(\Xi^{-1}\chi)/\widehat{f}(\chi)\big] d\mu_f(\chi) &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{f}(\Xi\chi)\big] d\mu_f(\Xi\chi) \\ &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{\overline{\Xi} f}(\chi)\big] d\mu_{\overline{\Xi} f}(\chi) = I(h). \end{aligned}$$

So, I is translation invariant. By the Riesz representation theorem, this can be written as an integral, $I(h) = \int_{\widehat{\mathcal{G}}} h(\chi) d\mu(\chi)$, where μ is the dual measure of λ. If $f \in B^1(\mathcal{G})$ and $h \in \mathcal{C}_c(\widehat{\mathcal{G}})$ then

$$\int_{\widehat{\mathcal{G}}} h(\chi)\widehat{f}(\chi) d\mu(\chi) = I(h\widehat{f}) = \int_{\widehat{\mathcal{G}}} h(\chi) d\mu_f(\chi).$$

hence $\widehat{f}(\chi)d\mu(\chi) = d\mu_f(\chi)$. So, $\widehat{f} \in L^1(\widehat{\mathcal{G}})$ and $f(x) = \int \langle x, \chi\rangle \widehat{f}(\chi) d\mu(\chi)$. $\square$

The Fourier inversion theorem II requires Pontrjagin duality, which we will postpone until next subsection. The Plancherel theorem relates the L^2 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

Theorem 2.1.6. (Plancherel Theorem) *The Fourier transform is extendable to a unitary isomorphism between $L^2(\mathcal{G})$ and $L^2(\widehat{\mathcal{G}})$, in particular it is an isometry,*

$$\|f\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} \big|f(x)\big| d\lambda(x) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(x)\big| d\mu(\chi) = \|\widehat{f}\|^2_{L^2(\widehat{\mathcal{G}})}.$$

Proof

Let $f \in L^2(\mathcal{G}) \cap L^1(\mathcal{G})$, then $f * f^* \in L^1(\mathcal{G}) \cap \mathcal{S}(\mathcal{G}) \subset B^1(\mathcal{G})$. The element $f * f^*$ is self-adjoint and hence we have,

$$\widehat{f * f^*} = |\widehat{f}|^2.$$

By the Fourier inversion theorem,

$$\int_{\mathcal{G}} |f|^2(x) d\lambda(x) = f * f^*(1) = \int_{\widehat{\mathcal{G}}} \big(\widehat{f * f^*}\big)(\chi) d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(\chi)\big|^2 d\mu(\chi).$$

So, the Fourier transform is an isometry in L^2-norm, and extends to an isometry $L^2(\mathcal{G}) \to L^2(\widehat{\mathcal{G}})$ and we have for each $f \in L^2(\mathcal{G})$, $\widehat{f} \in L^2(\widehat{\mathcal{G}})$. So, the Fourier transform is injective.

For surjectivity, we will show that if a function $h \in L^2(\widehat{\mathcal{G}})$ is orthogonal to $\mathcal{F}(L^2(\mathcal{G}))$ then it is zero almost everywhere. This implies that if h has no preimage, then it is zero almost everywhere. Let $\langle , \rangle_{L^2(\widehat{\mathcal{G}})}$ be the inner product on $L^2(\widehat{\mathcal{G}})$ given by,

$$\langle h, g\rangle_{L^2(\widehat{\mathcal{G}})} = \int_{\widehat{\mathcal{G}}} \overline{g(\chi)} h(\chi) d\mu(\chi).$$

Let h be a function orthogonal to $\mathcal{F}(L^2(\mathcal{G}))$, then for all $f \in L^2(\mathcal{G})$ and $x \in \mathcal{G}$,

$$\langle h, \widehat{L_x f}\rangle_{L^2(\widehat{\mathcal{G}})} = 0$$

Expanding the inner product we get,

$$0 = \int_{\widehat{\mathcal{G}}} \overline{\widehat{L_x f}(\chi)} h(\chi) d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \overline{\widehat{f}(\chi)} h(\chi) d\mu(\chi).$$

Since $h\overline{\widehat{f}} \in L^1(\widehat{\mathcal{G}})$ and Fourier-Stieltjes transform is injective, it follows that $h\overline{\widehat{f}} = 0$ almost everywhere. $\square$

2.1.2 | Pontrjagin Duality

Pontrjagin dual is a functor on the category of locally compact abelian groups. The Pontrjagin duality theorem says that the double dual functor is naturally isomorphic to the identity functor. So, there is a correspondence between elements of $\mathcal{G}$ and characters on $\widehat{\mathcal{G}}$. Consider for each $x \in \mathcal{G}$, a character defined by,

$$\langle \chi, \Phi(x) \rangle = \langle x, \chi \rangle.$$

Clearly this is a group homomorphism from $\mathcal{G}$ to $\widehat{\widehat{\mathcal{G}}}$. The goal of this subsection is to show that this is an isomorphism.

Theorem 2.1.7. (Pontrjagin Duality) $\mathcal{G} \overset{\Phi}{\cong} \widehat{\widehat{\mathcal{G}}}$.

Proof

Injectivity follows from Gelfand-Raikov theorem which says that characters separate points on $\mathcal{G}$, and hence if $\Phi(x) = \Phi(y)$ then $\langle x, \chi \rangle = \langle y, \chi \rangle$.

2.2 | Stone's Theorem

Recall that similar to how we define measures, we consider a measure space (Ω, Σ) consisting of a set Ω, together with a σ-algebra $\Sigma = \mathcal{B}(\Omega)$. A $\mathcal{H}$-projection valued measure on (Ω, Σ) or spectral measure is a map,

$$E : \Sigma \to \mathcal{B}(\mathcal{H}).$$

such that,

$$E(\epsilon) = E(\epsilon)^2 = E(\epsilon)^*$$

for all $\epsilon \in \Sigma$,

$$E(\varnothing) = 0, \ \ E(\Omega) = \mathbb{1}$$

for intersection of two sets ϵ, ϵ',

$$E(\epsilon \cap \epsilon') = E(\epsilon)E(\epsilon')$$

If ϵ_i are disjoint then the disjoint union strongly converges,

$$E(\coprod_i \epsilon_i) = \sum_i E(\epsilon_i)$$

Given a spectral measure $E : \Sigma \to \mathcal{B}(\mathcal{H})$, for each $\varphi, \varkappa \in \mathcal{H}$, one can construct ordinary complex measures,

$$E_{\varphi,\varkappa}(\epsilon) = \langle E(\epsilon)\varphi | \varkappa \rangle.$$

this turns out to be a measure, because the above requirements force it. This is a 'measure valued inner product', $(\varphi, \varkappa) \mapsto E_{\varphi,\varkappa}$. $\|E_{\varphi,\varphi}\| = E_{\varphi,\varphi}(\Omega) = \|\varphi\|^2$. A measure is called regular if any measurable set can be approximated from below by compact sets, and from above by open sets. These are the well behaved measures. For any function $f \in B((\Omega, \Sigma))$, for any $\varphi, \varkappa$ with $\|\varphi\|^2 = \|\varkappa\|^2 = 1$, we have by polarization,

$$\left| \int f dE_{\varphi,\varkappa} \right| \leq \tfrac{1}{4}\|f\|_\infty \left[\|\varphi + \varkappa\|^2 + \|\varphi - \varkappa\|^2 + \|\varphi + i\varkappa\|^2 + \|\varphi - i\varkappa\|^2 \right] \leq 4\|f\|_\infty.$$

So it is bounded, and hence defines a bounded operator T, such that,

$$\langle T\varphi|\varkappa\rangle = \int_\Omega f dE_{\varphi,\varkappa}.$$

We will hence denote T by,

$$T = \int_\Omega f dE.$$

The map $f \mapsto \int f dE$ is linear and $|\int f dE| \leq 4\|f\|_\infty$, and hence continuous. Gelfand-Naimark theorem, combined with Riesz representation theorem gives us the spectral theorem,

Theorem 2.2.1. (Spectral Theorem) *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be commutative C^*-algebra, and let $\Omega = \sigma(\mathcal{A})$, then there exists a unique spectral measure E on Ω such that*

$$A = \int \widehat{A} dE.$$

where $\widehat{A}$ is the Gelfand transform of A. If S commutes with all $A \in \mathcal{A}$ then S commutes with all $E(\epsilon)$, for Borel set $\epsilon \subset \Omega = \sigma(\mathcal{A})$.

Note that if $\mathcal{A}$ is a commutative Banach $*$-algebra and π is a $*$-representation of $\mathcal{A}$ on $\mathcal{H}$, then we can consider the norm closure $\mathcal{B}$ of $\pi(\mathcal{A})$ which is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. If the identity $1 \in \mathcal{B}(\mathcal{H})$ belongs to the closure, we can directly apply the spectral theorem. We will have,

$$T = \int_\Omega \widehat{T} dE_{\pi(T)}$$

for all $T \in \mathcal{B}$. The spectral measure can be pulled back, $E_A(\epsilon) = E_{\pi(A)}(\pi^{-1}(\epsilon))$. If $\mathcal{B}$ does not contain identity then the first step would be to unitisation it and then apply spectral theorem. We now have an important corollary of this. See [2] for details.

Theorem 2.2.2. (Stone's Theorem) *Let $\pi : \mathcal{G} \to U(\mathcal{H})$ be a unitary representation of a locally compact abelian group $\mathcal{G}$, then there exists a unique regular $\mathcal{H}_\pi$-spectral measure E_A on $\widehat{\mathcal{G}}$ such that,*

$$\pi(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle dE_A(\chi)$$

$$\pi(f) = \int_{\widehat{\mathcal{G}}} \chi(f) dE_A(\chi)$$

Proof

$\widehat{\mathcal{G}}$ was identified with $\sigma(L^1(\mathcal{G}))$, so, we can apply the Spectral theorem to the commutative Banach $*$-algebra $L^1(\mathcal{G})$, and hence there exists a unique $\mathcal{H}_\pi$-spectral measure such that,

$$\pi(f) = \int_{\widehat{\mathcal{G}}} \chi(f) dE(\chi)$$

for all $f \in L^1(\mathcal{G})$. Using this and a left translation L_x of an approximate identity $\{\psi_U\}$, $\pi(x)$ is the strong limit of $\pi(L_x\psi_U)$. So, we have,

$$\pi(L_x\psi_U) = \int_{\widehat{\mathcal{G}}} \chi(L_x\psi_U) dE(\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \chi(\psi_U) dE(\chi).$$

Since each $|\chi(\psi_U)| \leq 1$ and $\chi(\psi_U) \to 1$ for every χ, $\pi(L_x\psi_U)$ converges to $\int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle dE(\chi) = \pi(x)$. $\square$

[1] T Tao, Haar Measure and the Peter-Weyl Theorem. https://terrytao.wordpress.com/2011/09/27/254a-notes-3-haar-measure-and-the-peter-weyl-theorem/

[2] G Folland, A Course in Abstract Harmonic Analysis. CRC Press, 2015