Part II

The Fourier Transform & Pontrjagin Duality

We will assume that $\mathcal{G}$ is a locally compact abelian group. Some immediate facts are that the left and right translations coincide, the modular homomorphism is trivial, and convolution is commutative.

$$f * g(x) = \int_{\mathcal{G}} f(xy^{-1})g(y)d\lambda(y) = g * f(x).$$

Involution is given by

$$f^*(x) = \overline{f(x^{-1})}.$$

By Schur's lemma, every irreducible representation will be one dimensional. The dual group, and Fourier transforms of locally compact abelian groups are very similar to the techniques developed in the Gelfand-Naimark theory of commutative C^*-algebras. This subsection contains some of the big theorems in Fourier analysis, including Bochner's theorem, Fourier inversion, Pontrjagin duality, etc.

1 | The Dual Group

If π is an irreducible unitary representation of a locally compact abelian group $\mathcal{G}$ then $\mathcal{H}_\pi \cong \mathbb{C}$. In such a case, there exists $\chi(x)$ for every $x \in \mathcal{G}$ such that

$$\pi(x)(z) = \chi(x)z$$

and χ is a continuous homomorphism of $\mathcal{G}$ into the circle group $\mathbb{T}$, by unitarity of $\chi(x)$. Such homomorphisms are called characters of $\mathcal{G}$. The collection of all characters of $\mathcal{G}$ is denoted by $\widehat{\mathcal{G}}$. For a character χ, we will denote $\chi(x)$ by $\langle x, \chi \rangle$. A unitary representations of $\mathcal{G}$ determines a $*$-homomorphism of $L^1(\mathcal{G})$ into the operators on the representation space $\mathcal{H}_\pi$, which in our case is $\mathbb{C}$. This representation is given by

$$\chi(f) = \int_{\mathcal{G}} \langle x, \chi \rangle f(x) d\lambda(x).$$

By identifying the bounded linear maps $\mathcal{B}(\mathbb{C})$ with $\mathbb{C}$, this action determines a multiplicative functional on $L^1(\mathcal{G})$,

$$f \mapsto \chi(f).$$

We will use the notation from Gelfand theory for the spectrum of an algebra, $\sigma(\mathcal{A})$ will denote the set of all non-zero algebra homomorphisms from a Banach algebra $\mathcal{A}$ to $\mathbb{C}$. The Banach algebra of interest to us is the group algebra $L^1(\mathcal{G})$.

Let φ be a linear functional on $L^1(\mathcal{G})$, by the duality $L^\infty(\Omega) \cong L^1(\Omega)'$, on φ there must exist some $\chi \in L^\infty(\mathcal{G})$ such that,

$$\varphi(f) = \int_{\mathcal{G}} f(x)\chi(x)d\lambda(x)$$

for all $f \in L^1(\mathcal{G})$. Suppose φ is a multiplicative linear functional, that is, $\varphi \in \sigma(L^1(\mathcal{G}))$, then we have, for any $f, g \in L^1(\mathcal{G})$,

$$\begin{aligned}\int_{\mathcal{G}} \big[\varphi(f)\chi(x)\big] g(x)d\lambda(x) &= \varphi(f)\int_{\mathcal{G}} g(x)\chi(x)d\lambda(x) = \varphi(f)\varphi(g) = \varphi(f * g)\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} \chi(y)f(yx^{-1})g(x)d\lambda(x)d\lambda(y) = \int_{\mathcal{G}} \big[\varphi(L_x f)\big] g(x)dx.\end{aligned}$$

Since this holds for every $g \in L^1(\mathcal{G})$, the square bracketed terms on both sides must be the same almost everywhere. That is to say,

$$\varphi(f)\chi(x) = \varphi(L_x f)$$

almost everywhere. Choose a function $f \in L^1(\mathcal{G})$ such that $\varphi(f) \neq 0$, then we can define,

$$\chi(x) \coloneqq \varphi(L_x f)/\varphi(f).$$

For any $x, y \in \mathcal{G}$, by the above computation, we have,

$$\chi(xy)\varphi(f) = \varphi(L_{xy}f) = \varphi(L_x L_y f) = \chi(x)\chi(y)\varphi(f).$$

hence

$$\chi(xy) = \chi(x)\chi(y).$$

Therfore, χ is a group homomorphism from $\mathcal{G}$ to the circle group $\mathbb{T}$. So, every character gives rise to a multiplicative functional on $L^1(\mathcal{G})$ and conversely, every multiplicative functional on $L^1(\mathcal{G})$ corresponds to a character.

Theorem 1.1.

$$\sigma(L^1(\mathcal{G})) \cong \widehat{\mathcal{G}}.$$

With the pointwise multiplication $(\chi_1 \cdot \chi_2)(x) \coloneqq \chi_1(x)\chi_2(x)$ and pointwise inverse $\chi^{-1}(x) = \big(\chi(x)\big)^{-1}$, the set $\widehat{\mathcal{G}}$ is an abelian group. It is called the dual group of $\mathcal{G}$. The following is a useful computational tool,

$$\langle x, \chi^{-1}\rangle = \langle x^{-1}, \chi\rangle = \overline{\langle x, \chi\rangle}.$$

Since $\widehat{\mathcal{G}}$ is identified with the spectrum of $L^1(\mathcal{G})$, we can introduce the appropriate topology on $\widehat{\mathcal{G}}$ from $L^\infty(\mathcal{G})$, which is the weak* topology since we expect characters to be 'close' to each other if their evaluations are 'close'. This topology coincides on $\widehat{\mathcal{G}}$ with the topology of compact convergence, see §3.3 [2]. The set of all homomorphism from $L^1(\mathcal{G})$ to $\mathbb{C}$ is the set $\widehat{\mathcal{G}} \cup \{0\}$. By Alaoglu's theorem, $\widehat{\mathcal{G}} \cup \{0\}$ is compact, thus, $\widehat{\mathcal{G}}$ is locally compact.

Theorem 1.2. *$\widehat{\mathcal{G}}$ is a locally compact abelian group.*

Since $\widehat{\mathcal{G}}$ and $\sigma(L^1(\mathcal{G}))$ are identified via isomorphism, consider the composition with the inverse, which associates with the character χ the functional, $f \mapsto \overline{\chi(f)} = \chi^{-1}(f)$.

Recall that the Gelfand transform Γ on a Banach $*$-algebra $\mathcal{A}$ is a map

$$\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$$

which sends $A \in \mathcal{A}$ to a continuous function on the space of characters, given by evaluation. In our case, $\mathcal{A} \equiv L^1(\mathcal{G})$ and $C(\sigma(\mathcal{A})) = C(\widehat{\mathcal{G}})$. The Gelfand transformation on the Banach $*$-algebra $L^1(\mathcal{G})$ is called the Fourier transform on $\mathcal{G}$. The Fourier transform on $\mathcal{G}$ is then the map

$$\mathcal{F} : f \mapsto \mathcal{F}f := \widehat{f}$$

whose action on the characters of $\mathcal{G}$ is given by

$$\mathcal{F}f(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi\rangle} f(x) d\lambda(x). \qquad \text{(Fourier transform)}$$

Note that this assigns to each $f \in L^1(\mathcal{G})$ a continuous bounded function $\mathcal{F}(f)$ on the space of characters. Since characters are homomorphisms, we have,

$$\begin{aligned}
\mathcal{F}(f * g)(\chi) &= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle x, \chi\rangle} f(xy^{-1}) g(y) d\lambda(y) d\lambda(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle xy, \chi\rangle} f(x) g(y) d\lambda(x) d\lambda(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle y, \chi\rangle}\, \overline{\langle x, \chi\rangle} f(x) g(y) d\lambda(x) d\lambda(y) = \mathcal{F}f(\chi)\mathcal{F}g(\chi).
\end{aligned}$$

So, Fourier transform is an algebra homomorphism. Similarly,

$$\mathcal{F}(f^*)(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi\rangle}\, \overline{f(x^{-1})} d\lambda(x) = \int_{\mathcal{G}} \langle x, \chi\rangle\, \overline{f(x)} d\lambda(x) = \overline{\int_{\mathcal{G}} \overline{\langle x, \chi\rangle} f(x) d\lambda(x)} = \overline{\mathcal{F}(f)(\chi)}.$$

So, the Fourier transform is a $*$-homomorphism. The norm of the Fourier transform is bounded by

$$\|\mathcal{F}(f)\|_{\sup} = \sup_{\chi \in \widehat{\mathcal{G}}} \big|\mathcal{F}(f)(\chi)\big| \leq |\sup_{\mathcal{G}} \langle x, \chi\rangle| \int_{\mathcal{G}} \big|f(x)\big| d\lambda(x) \leq \|f\|_1,$$

which shows that

$$\mathcal{F} : f \mapsto \mathcal{F}f$$

is a norm decreasing $*$-homomorphism from $L^1(\mathcal{G})$ to $\mathcal{C}_b(\widehat{\mathcal{G}})$, the continuous bounded functions with $\|\cdot\|_\infty$ norm. The action of left translation is given by

$$\widehat{L_y f}(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi\rangle} f(y^{-1}x) d\lambda(x) = \int_{\mathcal{G}} \overline{\langle yx, \chi\rangle} f(x) d\lambda(x) = \overline{\langle y, \chi\rangle} \widehat{f}(\chi).$$

So the Fourier transformation of the left translation L_y of a function f acts as a multiplication of the Fourier transformation of the function by the function $x \mapsto \overline{\langle y, x\rangle}$,

$$\widehat{L_y f}(\chi) = \overline{\langle y, \chi\rangle} \widehat{f}(\chi).$$

Similarly, for any character Ξ,

$$(\Xi f)^{\wedge}(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi \rangle} \langle x, \Xi \rangle f(x) d\lambda(x) \widehat{f}(\Xi^{-1}\chi) = L_{\Xi}\widehat{f}(\chi).$$

The range $\mathcal{F}(L^1(\mathcal{G}))$ is a selfadjoint subalgebra that separates points in $\widehat{\mathcal{G}}$, and by the Stone-Weierstrass theorem, it is a dense subspace of $C_0(\widehat{\mathcal{G}})$, the continuous functions vanishing at infinity.

Theorem 1.3. *$\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\widehat{\mathcal{G}})$.*

The Fourier transform $\mathcal{F}f$ of a function f is usually denoted by $\widehat{f}$. The Fourier-Stieltjes transform of a measure $\mu \in \mathcal{M}(\mathcal{G})$ is the bounded continuous function $\widehat{\mu}$ on $\widehat{\mathcal{G}}$ defined by,

$$\widehat{\mu}(\chi) = \int_{\mathcal{G}} \overline{\langle x, \chi \rangle} d\mu(x).$$

The Fourier transform of convolution of measures μ and ν is given by

$$\begin{aligned}
\widehat{\mu * \nu}(\chi) &= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle xy, \chi \rangle} d\mu(x) d\nu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{\langle x, \chi \rangle}\, \overline{\langle y, \chi \rangle} d\mu(x) d\nu(y) = \widehat{\mu}(\chi)\widehat{\nu}(\chi).
\end{aligned}$$

Similarly, the Fourier-Stieltjes transform on $\mathcal{M}(\widehat{\mathcal{G}})$ is defined as,

$$\omega_\mu(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi \rangle d\mu(\chi)$$

Suppose $\mu \in \mathcal{M}(\widehat{\mathcal{G}})$, then it defines a linear functional on $L^1(\mathcal{G})$ via

$$f \mapsto \mu(f) := \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} f(x) \langle x, \chi \rangle d\mu(\chi) d\lambda(x).$$

This defines a function of positive type, because for $f \in L^1(\mathcal{G})$,

$$\begin{aligned}
\mu(f^* * f) &= \int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} \overline{f(y)} f(x) \langle y^{-1}x, \chi \rangle d\mu(\chi) d\lambda(x) d\lambda(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} \overline{f(y)}\, \overline{\langle y, \chi \rangle}\, f(x) \langle x, \chi \rangle d\mu(\chi) d\lambda(x) d\lambda(y) \\
&= \int_{\widehat{\mathcal{G}}} \overline{\big(\widehat{\overline{f}}(\chi)\big)} \big(\widehat{\overline{f}}(\chi)\big) d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f^*}(\chi^{-1})\big|^2 d\mu(\chi) \\
&= \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(\chi^{-1})\big|^2 d\mu(\chi) \geq 0.
\end{aligned}$$

Bochner's theorem says the converse is also true, that is every function of positive type is due to a positive measure as above.

Theorem 1.4. (Bochner's Theorem) *If $\omega \in \mathcal{S}(\mathcal{G})$, then there exists a unique positive $\mu \in \mathcal{M}(\widehat{\mathcal{G}})$, such that $\omega = \omega_\mu$.*

Proof

The idea is to construct a continuous linear functional on $C_0(\widehat{\mathcal{G}})$ and apply then Riesz-Markov theorem, which says that for every continuous functional κ on $C_0(\widehat{\mathcal{G}})$ there exists a finite Borel measure μ on $\widehat{\mathcal{G}}$ such that,

$$\kappa(f) = \int_{\widehat{\mathcal{G}}} f(x) d\mu(\chi).$$

Let ω be a function of positive type. We consider the linear functional $\widehat{f} \to \int \omega f$. We need to show that this extends to a continuous functional on $C_0(\widehat{\mathcal{G}})$. Since functions of positive type give rise to positive Hermitian forms, one could use the Cauchy-Schwarz inequality to show boundedness, and hence continuity of the functional. Assume without loss of generality that $\omega(1) = 1$ and consider the corresponding positive Hermitian form,

$$\langle f, g \rangle_\omega = \int_{\mathcal{G}} \int_{\mathcal{G}} \overline{g(y)} f(x) \omega(y^{-1}x) d\lambda(x) d\lambda(y) = \int \omega(g^* * f).$$

Applying the Cauchy-Schwarz inequality gives

$$\left| \int \omega(g * f) \right|^2 \leq \int \omega(f^* * f) \int \omega(g^* * g)$$

for all $f, g \in L^1(\mathcal{G})$. If $\{\psi_U\}$ is an approximate identity, then by definition, we have, $\lim_{U \to 1} \|\psi_U^* * f - f\|_1 = 0$, so we have, $\langle f, \psi_U \rangle_\omega \to \int \omega f$, and $\langle \psi_U, \psi_U \rangle_\omega = |\int \psi_U d\lambda(x)|^2 \to \omega(1) = 1$.

From the above inequality, we now obtain

$$\left| \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x) \right|^2 \leq \langle f, f \rangle_\omega.$$

We may use this to construct a continuous linear functional on $\mathcal{F}(L^1(\mathcal{G}))$. For any $f \in L^1(\mathcal{G})$, construct a self-adjoint element, $h = f^* * f$. We can now apply the spectral radius formula and the Gelfand-Naimark theorem implying

$$\lim_{n \to \infty} \|h^{2^n}\|_1^{1/2^{n+1}} = \|\widehat{h}\|_\infty^{1/2} = \||\widehat{f}|^2\|_\infty^{1/2} = \|\widehat{f}\|_\infty.$$

The above inequality gives successively,

$$\left| \int \omega f \right| \leq \left| \int \omega h \right|^{1/2} \leq \left| \int \omega(h * h) \right|^{1/4} \leq \cdots \leq \left| \int \omega(h * \cdots * h) \right|^{1/2^{n+1}} \leq \|h^{2^n}\|_1^{1/2^{n+1}} \to \|\widehat{f}\|_\infty.$$

So, this induces a continuous linear functional on $\mathcal{F}(L^1(\mathcal{G}))$,

$$\widehat{f} \mapsto \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x).$$

Since $\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\mathcal{G})$, the linear functional can be extended to $C_0(\mathcal{G})$. By the Riesz-Markov representation theorem, there exists a measure $\widehat{\mu}_\omega \in \mathcal{M}(\widehat{\mathcal{G}})$ such that,

$$\widehat{f} \mapsto \int_{\mathcal{G}} \omega(x) f(x) d\lambda(x) = \int_{\widehat{\mathcal{G}}} \widehat{f}(\chi) d\widehat{\mu}_\omega(\chi).$$

Expanding $\widehat{f}$ we get

$$\int \omega f = \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} f(x)\langle x, \chi^{-1}\rangle d\widehat{\mu}_\omega(\chi) d\lambda(x).$$

This implies $\omega(x) = \int \langle x, \chi\rangle d\mu_\omega(\chi)$ where $d\mu_\omega(\chi) = d\widehat{\mu}_\omega(\chi^{-1})$. Since $1 = \omega(1) = \mu_\omega(\widehat{\mathcal{G}}) \leq \|\mu_\omega\| \leq 1$, we must have, $\|\mu_\omega\| = \mu_\omega(\widehat{\mathcal{G}})$. Positivity follows from ω being positive. $\square$

The Bochner space is defined to be the set $B(\mathcal{G}) = \{\omega_\mu \mid \mu \in \mathcal{M}(\widehat{\mathcal{G}})\}$. Bochner's theorem says that $B(\mathcal{G}) = [\mathcal{S}(\mathcal{G})]$, which is the span of $\mathcal{S}(\mathcal{G})$. Define conversely, for each $\omega \in \mathcal{S}(\mathcal{G})$, a measure μ_ω that corresponds to ω, as described above. Bochner's theorem can be restated establishing

$$B(\mathcal{G}) \to \mathcal{M}(\widehat{\mathcal{G}})$$

Bochner's theorem now allows us to relate functions on $\mathcal{G}$ and functions on $\widehat{\mathcal{G}}$. Before describing these relations, we define,

$$B^p(\mathcal{G}) \coloneqq B(\mathcal{G}) \cap L^p(\mathcal{G}).$$

We will focus our attention on L^1 and L^2 functions. For the case of L^p functions, see [2]. The Fourier inversion theorems I & II relate the L^1 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$. The Plancherel theorem relates the L^2 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

Let $f, g \in B^1(\mathcal{G})$, for any $h \in L^1(\mathcal{G})$, we have,

$$\begin{aligned}
\int_{\widehat{\mathcal{G}}} \widehat{h}(\chi) d\mu_f(\chi) &= \int_{\widehat{\mathcal{G}}} \left[\int_{\mathcal{G}} \overline{\langle x, \chi\rangle} h(x) d\lambda(x) \right] d\mu_f(\chi) \\
&= \int_{\mathcal{G}} h(x) \left[\int_{\widehat{\mathcal{G}}} \langle x^{-1}, \chi\rangle d\mu_f(\chi) \right] d\lambda(x) \\
&= \int_{\mathcal{G}} h(x) f(x^{-1}) d\lambda(x) = \big(h * f\big)(1).
\end{aligned}$$

Replacing h by $h * g$ or $h * f$ and f by g, and by commutativity of convolution in locally compact abelian groups we get,

$$\int \widehat{h}\widehat{g} d\mu_f = \big((h * g) * f\big)(1) = \big((h * f) * g\big)(\mathbb{1}) = \int \widehat{h}\widehat{f} d\mu_g.$$

Hence it follows that,

$$\widehat{g} d\mu_f = \widehat{f} d\mu_g,$$

on $\mathcal{F}(L^1(\mathcal{G}))$. Since $\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\widehat{\mathcal{G}})$, it also holds on $C_0(\mathcal{G})$.

Theorem 1.5. (Fourier Inversion Theorem I) *If $f \in B^1(\mathcal{G})$ then $\widehat{f} \in L^1(\widehat{\mathcal{G}})$ and,*

$$f(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \widehat{f}(\chi) d\mu(\chi),$$

where μ denotes the suitably normalized Haar measure on $\widehat{\mathcal{G}}$.

Proof

The idea again is to construct a linear functional on $\mathcal{C}_c(\widehat{\mathcal{G}})$ using the function f and apply the Riesz representation theorem and think of this linear functional as an integral. So, the first step is to develop some tools to construct positive functionals on compactly supported continuous functions.

Let $K \subseteq \widehat{\mathcal{G}}$ be a compact set. Pick a $h \in \mathcal{C}_c(\mathcal{G})$ such that $\int_{\mathcal{G}} h(x)d\lambda(x) = 1$. Then,

$$\begin{aligned}
\widehat{h^* * h}(\chi) &= \int_{\mathcal{G}} \overline{\langle x, \chi\rangle}\left[\int_{\mathcal{G}} \overline{h(yx^{-1})}h(y)d\lambda(y)\right]d\lambda(x) \\
&= \int_{\mathcal{G}}\int_{\mathcal{G}} \langle yx^{-1}, \chi\rangle\overline{\langle y, \chi\rangle h(yx^{-1})}h(y)d\lambda(x)d\lambda(y) \\
&= \int_{\mathcal{G}}\int_{\mathcal{G}} \langle z, \chi\rangle\ \overline{\langle y, \chi\rangle}\ \overline{h(z)}h(y)d\lambda(z)d\lambda(y) \\
&= \left[\int_{\mathcal{G}} \langle z, \chi\rangle\overline{h(z)}d\lambda(z)\right]\left[\int_{\mathcal{G}} \overline{\langle y, \chi\rangle}h(y)d\lambda(y)\right] = |\widehat{h}|^2(\chi).
\end{aligned}$$

So, we have,

$$\widehat{h^* * h} = |\widehat{h}|^2.$$

hence $\widehat{h^* * h} \geq 0$, and $\widehat{h^* * h}(\mathbb{1}_{\widehat{\mathcal{G}}}) = 1$. By continuity, there exists a neighborhood V of $1_{\widehat{\mathcal{G}}}$, such that $\widehat{h^* * h} > 0$. We can cover K with finitely many of such functions, with translations. Consider the function,

$$\widehat{f}(\chi) = \sum_i L_{\chi_i}\big(\widehat{h^* * h}\big)(\chi).$$

This is the Fourier transform of the function, $f(x) = (\sum_i \chi_i(x)) \cdot (h^* * h)(x)$. By construction, this is such that $\widehat{f} > 0$ everywhere on K. So, for every compact set $K \subseteq \widehat{\mathcal{G}}$, there is a function $f \in \mathcal{S}(\mathcal{G})$, such that $\widehat{f} > 0$ on K.

If $h \in \mathcal{C}_c(\widehat{\mathcal{G}})$, there exists a function $f \in \mathcal{S}(\mathcal{G})$ which is strictly positive on the support of h. Using this function, define a functional, $I_f : \mathcal{C}_c(\widehat{\mathcal{G}}) \to \mathbb{C}$ by,

$$I_f(h) = \int_{\widehat{\mathcal{G}}} \left[\frac{h(\chi)}{\widehat{f}(\chi)}\right]d\mu_f(\chi).$$

The strict positivity is used here, so that $[h/\widehat{f}]$ is well behaved. Now for any other such function g, we have,

$$\begin{aligned}
I_f(h) &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{f}(\chi)\big]d\mu_f(\chi) = \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\widehat{f}(\chi)\big]\widehat{g}(\chi)d\mu_f(\chi) \\
&= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\widehat{f}(\chi)\big]\widehat{f}(\chi)d\mu_g(\chi) = \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{g}(\chi)\big]d\mu_g(\chi) = I_g(h).
\end{aligned}$$

So, the value of $I_f(h)$ depends only on h, so we will forget the subscript and write it as $I(h)$. Clearly, this is a linear functional. Since $d\mu_f$ is a positive measure, $I(h) \geq 0$ for all $h \geq 0$. We obtained a non-trivial positive linear functional on $\mathcal{C}_c(\widehat{\mathcal{G}})$, in particular,

$$I(\widehat{g}h) = \int \big[h/\widehat{f}\big]\widehat{g}d\mu_f = \int hd\mu_g.$$

We now need to show that this functional is well behaved, that is we have to show that it is invariant under translations. For any $\Xi \in \widehat{\mathcal{G}}$, we have,

$$\int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle d\mu_f(\Xi\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \Xi^{-1}\chi\rangle d\mu_f(\Xi^{-1}\Xi\chi) = \int_{\widehat{\mathcal{G}}} \overline{\langle x, \Xi\rangle}\langle x, \chi\rangle d\mu_f(\chi) = \overline{\langle x, \Xi\rangle}f(x) = \big(\Xi f\big)(x)$$

So, we have, $d\mu_f(\Xi\chi) = d\mu_{\overline{\Xi}f}(\chi)$. Recall also that

$$\big(\overline{\Xi}f\big)^{\wedge}(\chi) = \widehat{f}(\Xi\chi).$$

Now, for the invariance, if $f > 0$ on the union of the support of h and $L_{\Xi}h$, then we have,

$$\begin{aligned} I(L_{\Xi}h) = \int_{\widehat{\mathcal{G}}} \big[h(\Xi^{-1}\chi)/\widehat{f}(\chi)\big] d\mu_f(\chi) &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{f}(\Xi\chi)\big] d\mu_f(\Xi\chi) \\ &= \int_{\widehat{\mathcal{G}}} \big[h(\chi)/\widehat{\overline{\Xi}f}(\chi)\big] d\mu_{\overline{\Xi}f}(\chi) = I(h). \end{aligned}$$

So, I is translation invariant. By the Riesz representation theorem, this can be written as an integral, $I(h) = \int_{\widehat{\mathcal{G}}} h(\chi) d\mu(\chi)$, where μ is the dual measure of λ. If $f \in B^1(\mathcal{G})$ and $h \in \mathcal{C}_c(\widehat{\mathcal{G}})$ then

$$\int_{\widehat{\mathcal{G}}} h(\chi)\widehat{f}(\chi) d\mu(\chi) = I(h\widehat{f}) = \int_{\widehat{\mathcal{G}}} h(\chi) d\mu_f(\chi).$$

hence $\widehat{f}(\chi)d\mu(\chi) = d\mu_f(\chi)$. So, $\widehat{f} \in L^1(\widehat{\mathcal{G}})$ and $f(x) = \int \langle x, \chi\rangle \widehat{f}(\chi) d\mu(\chi)$. $\square$

The Fourier inversion theorem II requires Pontrjagin duality, which we will postpone until next subsection. The Plancherel theorem relates the L^2 functions on $\mathcal{G}$ and $\widehat{\mathcal{G}}$.

Theorem 1.6. (Plancherel Theorem) *The Fourier transform is extendable to a unitary isomorphism between $L^2(\mathcal{G})$ and $L^2(\widehat{\mathcal{G}})$, in particular it is an isometry,*

$$\|f\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} \big|f(x)\big| d\lambda(x) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(x)\big| d\mu(\chi) = \|\widehat{f}\|^2_{L^2(\widehat{\mathcal{G}})}.$$

Proof

Let $f \in L^2(\mathcal{G}) \cap L^1(\mathcal{G})$, then $f * f^* \in L^1(\mathcal{G}) \cap \mathcal{S}(\mathcal{G}) \subset B^1(\mathcal{G})$. The element $f * f^*$ is self-adjoint and hence we have,

$$\widehat{f * f^*} = |\widehat{f}|^2.$$

By the Fourier inversion theorem,

$$\int_{\mathcal{G}} |f|^2(x) d\lambda(x) = f * f^*(1) = \int_{\widehat{\mathcal{G}}} \big(\widehat{f * f^*}\big)(\chi) d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \big|\widehat{f}(\chi)\big|^2 d\mu(\chi).$$

So, the Fourier transform is an isometry in L^2-norm, and extends to an isometry $L^2(\mathcal{G}) \to L^2(\widehat{\mathcal{G}})$ and we have for each $f \in L^2(\mathcal{G})$, $\widehat{f} \in L^2(\widehat{\mathcal{G}})$. So, the Fourier transform is injective.

For surjectivity, we will show that if a function $h \in L^2(\widehat{\mathcal{G}})$ is orthogonal to $\mathcal{F}(L^2(\mathcal{G}))$ then it is zero almost everywhere. This implies that if h has no preimage, then it is zero almost everywhere. Let $\langle , \rangle_{L^2(\widehat{\mathcal{G}})}$ be the inner product on $L^2(\widehat{\mathcal{G}})$ given by,

$$\langle h, g\rangle_{L^2(\widehat{\mathcal{G}})} = \int_{\widehat{\mathcal{G}}} \overline{g(\chi)} h(\chi) d\mu(\chi).$$

Let h be a function orthogonal to $\mathcal{F}(L^2(\mathcal{G}))$, then for all $f \in L^2(\mathcal{G})$ and $x \in \mathcal{G}$,

$$\langle h, \widehat{L_x f}\rangle_{L^2(\widehat{\mathcal{G}})} = 0$$

Expanding the inner product we get,

$$0 = \int_{\widehat{\mathcal{G}}} \overline{\widehat{L_x f}(\chi)} h(\chi) d\mu(\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \overline{\widehat{f}(\chi)} h(\chi) d\mu(\chi).$$

Since $h\overline{\widehat{f}} \in L^1(\widehat{\mathcal{G}})$ and Fourier-Stieltjes transform is injective, it follows that $h\overline{\widehat{f}} = 0$ almost everywhere. $\square$

1.2 | Pontrjagin Duality

Pontrjagin dual is a functor on the category of locally compact abelian groups. The Pontrjagin duality theorem says that the double dual functor is naturally isomorphic to the identity functor. So, there is a correspondence between elements of $\mathcal{G}$ and characters on $\widehat{\mathcal{G}}$. Consider for each $x \in \mathcal{G}$, a character defined by,

$$\langle \chi, \Phi(x) \rangle = \langle x, \chi \rangle.$$

Clearly this is a group homomorphism from $\mathcal{G}$ to $\widehat{\widehat{\mathcal{G}}}$. The goal of this subsection is to show that this is an isomorphism.

Theorem 1.7. (Pontrjagin Duality) $\mathcal{G} \overset{\Phi}{\cong} \widehat{\widehat{\mathcal{G}}}$.

Proof

Injectivity follows from Gelfand-Raikov theorem which says that characters separate points on $\mathcal{G}$, and hence if $\Phi(x) = \Phi(y)$ then $\langle x, \chi \rangle = \langle y, \chi \rangle$.

2 | Decomposition of Unitary Representations

Recall that similar to how we define measures, we consider a measure space (Ω, Σ) consisting of a set Ω, together with a σ-algebra $\Sigma = \mathcal{B}(\Omega)$. A $\mathcal{H}$-projection valued measure on (Ω, Σ) or spectral measure is a map,

$$E : \Sigma \to \mathcal{B}(\mathcal{H}).$$

such that,

$$E(\epsilon) = E(\epsilon)^2 = E(\epsilon)^*$$

for all $\epsilon \in \Sigma$,

$$E(\varnothing) = 0, \;\; E(\Omega) = \mathbb{1}$$

for intersection of two sets ϵ, ϵ',

$$E(\epsilon \cap \epsilon') = E(\epsilon)E(\epsilon')$$

If ϵ_i are disjoint then the disjoint union strongly converges,

$$E(\coprod_i \epsilon_i) = \sum_i E(\epsilon_i)$$

Given a spectral measure $E : \Sigma \to \mathcal{B}(\mathcal{H})$, for each $\varphi, \varkappa \in \mathcal{H}$, one can construct ordinary complex measures,

$$E_{\varphi,\varkappa}(\epsilon) = \langle E(\epsilon)\varphi|\varkappa\rangle.$$

this turns out to be a measure, because the above requirements force it. This is a 'measure valued inner product', $(\varphi, \varkappa) \mapsto E_{\varphi,\varkappa}$. $\|E_{\varphi,\varphi}\| = E_{\varphi,\varphi}(\Omega) = \|\varphi\|^2$. A measure is called regular if any measurable set can be approximated from below by compact sets, and from above by open sets. These are the well behaved measures. For any function $f \in B((\Omega, \Sigma))$, for any $\varphi, \varkappa$ with $\|\varphi\|^2 = \|\varkappa\|^2 = 1$, we have by polarization,

$$\left| \int f dE_{\varphi,\varkappa} \right| \leq \tfrac{1}{4}\|f\|_\infty \left[\|\varphi + \varkappa\|^2 + \|\varphi - \varkappa\|^2 + \|\varphi + i\varkappa\|^2 + \|\varphi - i\varkappa\|^2\right] \leq 4\|f\|_\infty.$$

So it is bounded, and hence defines a bounded operator T, such that,

$$\langle T\varphi|\varkappa\rangle = \int_\Omega f dE_{\varphi,\varkappa}.$$

We will hence denote T by,

$$T = \int_\Omega f dE.$$

The map $f \mapsto \int f dE$ is linear and $|\int f dE| \leq 4\|f\|_\infty$, and hence continuous. Gelfand-Naimark theorem, combined with Riesz representation theorem gives us the spectral theorem,

Theorem 2.1. (Spectral Theorem) *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be commutative C^*-algebra, and let $\Omega = \sigma(\mathcal{A})$, then there exists a unique spectral measure E on Ω such that*

$$A = \int \widehat{A} dE.$$

where $\widehat{A}$ is the Gelfand transform of A. If S commutes with all $A \in \mathcal{A}$ then S commutes with all $E(\epsilon)$, for Borel set $\epsilon \subset \Omega = \sigma(\mathcal{A})$.

Note that if $\mathcal{A}$ is a commutative Banach $*$-algebra and π is a $*$-representation of $\mathcal{A}$ on $\mathcal{H}$, then we can consider the norm closure $\mathcal{B}$ of $\pi(\mathcal{A})$ which is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. If the identity $1 \in \mathcal{B}(\mathcal{H})$ belongs to the closure, we can directly apply the spectral theorem. We will have,

$$T = \int_\Omega \widehat{T} dE_{\pi(T)}$$

for all $T \in \mathcal{B}$. The spectral measure can be pulled back, $E_A(\epsilon) = E_{\pi(A)}(\pi^{-1}(\epsilon))$. If $\mathcal{B}$ does not contain identity then the first step would be to unitisation it and then apply spectral theorem. We now have an important corollary of this. See [2] for details.

Theorem 2.2. (Stone's Theorem) *Let $\pi : \mathcal{G} \to U(\mathcal{H})$ be a unitary representation of a locally compact abelian group $\mathcal{G}$, then there exists a unique regular $\mathcal{H}_\pi$-spectral measure E_A on $\widehat{\mathcal{G}}$ such that,*

$$\pi(x) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle dE_A(\chi)$$

$$\pi(f) = \int_{\widehat{\mathcal{G}}} \chi(f) dE_A(\chi)$$

Proof

$\widehat{\mathcal{G}}$ was identified with $\sigma(L^1(\mathcal{G}))$, so, we can apply the Spectral theorem to the commutative Banach $*$-algebra $L^1(\mathcal{G})$, and hence there exists a unique $\mathcal{H}_\pi$-spectral measure such that,

$$\pi(f) = \int_{\widehat{\mathcal{G}}} \chi(f) dE(\chi)$$

for all $f \in L^1(\mathcal{G})$. Using this and a left translation L_x of an approximate identity $\{\psi_U\}$, $\pi(x)$ is the strong limit of $\pi(L_x\psi_U)$. So, we have,

$$\pi(L_x\psi_U) = \int_{\widehat{\mathcal{G}}} \chi(L_x\psi_U) dE(\chi) = \int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle \chi(\psi_U) dE(\chi).$$

Since each $|\chi(\psi_U)| \leq 1$ and $\chi(\psi_U) \to 1$ for every χ, $\pi(L_x\psi_U)$ converges to $\int_{\widehat{\mathcal{G}}} \langle x, \chi\rangle dE(\chi) = \pi(x)$. $\square$

References

[1] T Tao, Haar Measure and the Peter-Weyl Theorem. https://terrytao.wordpress.com/2011/09/27/254a-notes-3-haar-measure-and-the-peter-weyl-theorem/

[2] G Folland, A Course in Abstract Harmonic Analysis. CRC Press, 2015