

THE SIX FUNCTORS OF GROTHENDIECK

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INTRODUCTION

These notes provide an abstract introduction to sheaf theory and sheaf cohomology. The goal is to understand Grothendieck's six operations, abstractly.¹ The abstract machinery divides the difficulty into multiple smaller steps, so the difficulty of the theorems is not felt. I have tried to provide intuitive motivation for the categorical definitions. The purpose of these notes is to prepare the reader for Kashiwara-Schapira's books.

The notion of a pre-sheaf axiomatizes 'local nature' of certain mathematical objects. Given a topological space X , a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X . Denote by CU the set of real-valued continuous functions on an open set U . If $V \subset U$ then f restricted to V is a continuous map, $f|_{UV} : V \rightarrow \mathbb{R}$. The map, $f \mapsto f|_{UV}$ is a function $CU \rightarrow CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_{UV})|_{VW} = f|_{UW}.$$

So, this is functorial in nature, i.e., $|_{VW} \circ |_{UV} = |_{UW}$. This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\text{op}} \rightarrow \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \rightarrow U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\text{op}}$ is the opposite category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\text{op}}$ the functor assigns a morphism, the restriction morphism in \mathbf{Sets} , $\{U \supset V\} \mapsto \{CU \rightarrow CV\}$ given by $f \mapsto f|_{UV}$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}, \quad (\text{pre-sheaf})$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps $U \subset V$ and the corresponding maps in \mathbf{Sets} are called restriction maps, $|_{UV} : \mathcal{F}U \rightarrow \mathcal{F}V$.

Denote by $\text{PSh}(X)$ the collection of all pre-sheaves over a topological space X . Each pre-sheaf is a functor from $\mathcal{O}(X)^{\text{op}}$ to \mathbf{Sets} can be considered an object and the natural transformation between the two pre-sheaves as morphism between these objects. So, the category of pre-sheaves is the functor category,

$$\text{PSh}(X) = \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}}$$

So, to understand pre-sheaves, we could use tools for studying functor categories. The goal of the first half of this document will be to develop the tools needed to study functor categories, and hence pre-sheaves.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions. Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

¹This is in the spirit of Fosco's comment on mathoverflow that 'If it's not possible to derive these statements from purely Kan-extensional arguments, then sheaves do belong to Algebraic Geometry'.

So, what we have is an I -indexed family of functions $(f_i)_{i \in I} \in \prod_i CU_i$, and two maps

$$p(\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and that's what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j|_{U_i \cap U_j}} = f|_{U_i|_{U_i \cap U_j}}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$.

$$CU \xrightarrow{e} \prod_i CU_i \xrightleftharpoons[p]{p} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q .

A sheaf of sets \mathcal{F} on a topological space X is a functor,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i \in I} U_i$ of an open set U of X yields an equalizer diagram.

$$\mathcal{F}U \xrightarrow{e} \prod_i \mathcal{F}U_i \xrightleftharpoons[p]{p} \prod_{i,j} \mathcal{F}(U_i \cap U_j). \quad (\text{collation property})$$

The category of sheaves, $\text{Sh}(X)$, is a subcategory of the functor category,

$$\text{Sh}(X) \hookrightarrow \text{PSh}(X) = \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}},$$

which is the category consisting of all functors from $\mathcal{O}(X)^{\text{op}}$ to \mathbf{Sets} , that satisfy the [collation](#). The obstruction of going from local to global is carefully studied via cohomology. The goal of this document is to develop the tools needed to study cohomology of sheaves.

The starting point is to develop the necessary tools from category theory to study these objects. The main concepts of category theory is the representables-limits-adjoints triad. While Kashiwara-Schapira uses a lot of abstract categorical tools, their approach still contains lots of difficult proofs. Our goal is to introduce a bit more abstract tools than Kashiwara-Schapira, but significantly reduce the difficulty of proofs. I feel the ‘derived’ concepts can be introduced much more neatly and intuitively with slightly more abstraction than the approach in Kashiwara-Schapira. Category theory allows us to formalize patterns, when a collection of theorems indicate the existence of such a pattern. Category theory itself is useless to find such patterns. However, when the patterns are noted, and formalized already, abstract categorical way of stating and proving the theorems can be very helpful. The goal of these notes is to understand and formalize already noticed patterns.

The use of category theory in mathematics by Grothendieck starts with the philosophical idea that an object can be studied by its relation to other objects. Heuristically, if we think of categories as describing information about certain type of mathematical objects, we would like to study this information contained in each object by its relation to information contained in other objects. The category of sets plays an important role in this, because the starting point is the collection of all relations an object has with other objects. So, in this sense, the information a category describes can be represented in the category of sets. So, the starting point in Grothendieck’s categorical approach to mathematics is to assume that the information contained in the categories of interest to us can be represented by sets in a ‘Grothendieck universe’. We will not be getting into the subtleties of Grothendieck universes here.

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1 | CATEGORY OF PRE-SHEAVES

Category theory develops very general tools, which can be applied to many mathematical phenomena. The use of category theory in sheaf theory was started by Grothendieck based on the following philosophical position that an object is determined by its relation to other objects. The Yoneda lemma makes this precise and exploitable. Yoneda lemma embeds a given category inside the category of functors from the given category to the category of sets, via the functor, ‘maps to the given object’ or ‘maps from the given object’. This allows us to utilize the nice properties of the target category in our case the category of sets.

A set is a collection of ‘elements’. A category \mathcal{A} is more sophisticated, it possesses ‘objects’ similar to how sets possess elements, but for each pair of objects, X and Y in \mathcal{A} , there is a set of relations between X and Y , called morphisms, denoted by $\text{Hom}_{\mathcal{A}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects. Studying objects by their relations to other objects could be called the Yoneda-Grothendieck philosophy.

A functor \mathcal{F} between two categories \mathcal{A} and \mathcal{B} consists of a mapping of objects of \mathcal{A} to objects of \mathcal{B} , $X \mapsto \mathcal{F}X$ together with a map of the set of homomorphisms,

$$\text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y).$$

the image of $f \in \text{Hom}_{\mathcal{A}}(X, Y)$ denoted by $\mathcal{F}(f)$. That takes identity to identity and respects composition¹ i.e.,

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from \mathcal{A} to \mathcal{B} can be thought of as a covariant functor from \mathcal{A}^{op} to \mathcal{B} . A functor \mathcal{F} is faithful if the map $\text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$ is injective for all X, Y . It’s full if the map is surjective. If it’s a bijection the functor is called fully faithful.

A natural transformation κ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \rightarrow \mathcal{B}$, denoted by,

$$\kappa : \mathcal{F} \Rightarrow \mathcal{G},$$

is a collection of mappings κ_X for every $X \in \mathcal{A}$, such that for all $f : X \rightarrow Y$, the diagram,

$$\begin{array}{ccc} X & \mathcal{F}X & \xrightarrow{\kappa_X} \mathcal{G}X \\ \downarrow f & \mathcal{F}(f) \downarrow & \downarrow \mathcal{G}(f) \\ Y & \mathcal{F}Y & \xrightarrow{\kappa_Y} \mathcal{G}Y \end{array} \quad (\text{natural transformation})$$

¹The composition $f \circ g$ assumes they are composable.

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors \mathcal{F} and \mathcal{G} is denoted by,

$$\text{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

We say two functors \mathcal{F} and \mathcal{G} are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $\mathcal{F} \cong \mathcal{G}$. The collections of all functors from \mathcal{A} to \mathcal{B} together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{B}^{\mathcal{A}}$. The nice thing about functor category $\mathcal{B}^{\mathcal{A}}$ is that it inherits many of the useful properties of the category \mathcal{B} .

Equivalence of two categories can be thought of as giving two complementary description of same mathematical object. We can compare two categories \mathcal{A} and \mathcal{B} via the functors between them. The starting point is the functor category $\mathcal{B}^{\mathcal{A}}$.

A functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of categories if there is a functor $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$ such that

$$\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}, \text{ and } \mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}},$$

where the identity functor $\mathbb{1}_{\mathcal{A}}$ sends objects of \mathcal{A} to the same objects, and morphisms to the same morphisms. \mathcal{G} is called quasi-inverse functor. In such a case, \mathcal{A} and \mathcal{B} are said to be equivalent.

Quotient categories can be defined when we have an equivalence relation on the collection of morphisms. The objects remain the same, and the hom-sets get quotiented.

HEURISTICS

The idea of category theory is to study objects with similar structure together. The information groups contain are similar, so instead of study the information contained in a group, we can instead study the collection of objects that have such group like information.

The information a category can describe is the total of the information of all the objects it can describe. So, each category consists of ‘objects’. Each object in the category would correspond to some part of information the category describes. Suppose two objects are related, this means the information the two objects contains is related. So, there is less independent information between the categories. An isomorphism between two objects means that the information the two objects contains is identical, and hence can be treated as the same. So, the more objects a category has, the more information the category has, and similarly, the more relations the category has, or the more connected the category is, the less information the category has.

1.1 | REPRESENTABLE FUNCTORS

Each category describes the type of information it can describe. Given two categories, functors between them relate how the information in the two categories are related. The fascinating thing is that, much of the information of interest to us can be represented by the information contained in the category of sets. The idea is to study information contained in an object by its relationship to other objects in the category. The category of sets enters the picture via the ‘set of all relations’. Representable functors and Yoneda lemma make this precise.

Each $\text{Hom}_{\mathcal{A}}(X, Y)$ tells us about all the relations the object X has with other object Y . The thing we should be studying is the functor $h_X = \text{Hom}_{\mathcal{A}}(X, -)$ and $h^X = \text{Hom}_{\mathcal{A}}(-, X)$.

These are called hom functors.

$$\begin{aligned} h^X : \mathcal{A}^{\text{op}} &\rightarrow \mathbf{Sets} \\ Y &\mapsto \text{Hom}_{\mathcal{A}}(Y, X). \end{aligned}$$

which maps each morphism $f : Y \rightarrow Z$ to a morphism of hom sets given by the composition,

$$Y \xrightarrow{f} Z \xrightarrow{g} X$$

We will denote this by,

$$\begin{aligned} h^X(f) : h^X(Y) &\rightarrow h^X(Z) \\ g &\mapsto g \circ f. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\text{Hom}_{\mathcal{A}}(Y, X)$ s are all sets. Such categories are called locally small categories.

A contravariant functor $\mathcal{F} : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \quad (\text{representable})$$

in such a case, \mathcal{F} is said to be represented by the object X . We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \rightarrow \mathbf{Sets}$, this will be $\mathcal{G} \cong h_X$. Where \cong stands for natural isomorphism.

1.1.1 | YONEDA EMBEDDING

Yoneda embedding and representable functors allow us to use the nice properties (ability to take limits) of the category of sets to study more complex categories that are not so nice. We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\text{Hom}_{\mathcal{A}}(-, X)$ and $\text{Hom}_{\mathcal{A}}(X, -)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{A}$ and the functor $h^X \in \mathbf{Sets}^{\mathcal{A}^{\text{op}}}$.

THEOREM 1.1.1. (YONEDA LEMMA) *For a functor $\mathcal{F} : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Sets}$ and any $X \in \mathcal{A}$, there is a natural bijection,*

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) \cong \mathcal{F}X \quad (\text{Yoneda})$$

such that $\kappa \in \text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) \leftrightarrow \kappa_X(\mathbb{1}_X) \in \mathcal{F}X$.

PROOF

In the [natural transformation](#) diagram, replace \mathcal{F} by h^X , and \mathcal{G} by \mathcal{F} . $\kappa_X : h^X X \rightarrow \mathcal{F}X$. Now, $h^X X = \text{Hom}_{\mathcal{A}}(X, X)$, which contains $\mathbb{1}_X$. Using this we construct a map,

$$\begin{aligned} \mu : \text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) &\rightarrow \mathcal{F}X \\ \kappa &\mapsto \kappa_X(\mathbb{1}_X). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $Y \in \mathcal{A}$. For any $f : Y \rightarrow X$, we have,

$$\begin{array}{ccccc} X & & h^X X & \xrightarrow{\kappa_X} & \mathcal{F}X & & \mathbb{1}_X & \xrightarrow{\kappa_X} & \mu(\kappa) \\ f \uparrow & & h^X(f) \downarrow & & \downarrow \mathcal{F}(f) & & \downarrow & & \downarrow \\ Y & & h^X Y & \xrightarrow{\kappa_Y} & \mathcal{F}Y & & f & \xrightarrow{\kappa_Y} & \kappa_Y(f) \end{array}$$

Hence $\kappa_Y(f) = \mathcal{F}(f)(\mu(\kappa))$, or the action of κ_Y is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_Y(f) = \varphi_Y(f)$ for all $Y \in \mathcal{A}$, so it's injective.

For surjectivity we have to show that for all sets $x \in \mathcal{F}X$, there exists a natural transformation φ such that $\varphi_X(\mathbb{1}_X) = x$. For $x \in \mathcal{F}X$, and $f : Y \rightarrow X$, construct the map,

$$\begin{aligned}\varphi : h^X &\rightarrow \mathcal{F} \\ f &\mapsto \mathcal{F}(f)(x).\end{aligned}$$

this satisfies the requirement that $\varphi_X(\mathbb{1}_X) = x$, because clearly, $\mathbb{1}_X \mapsto \mathcal{F}(\mathbb{1}_X)(x) = \mathbb{1}_x(x) = x$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{array}{ccc} Y & h^X Y & \xrightarrow{\varphi_Y} \mathcal{F}Y \\ g \uparrow & h^X(g) \downarrow & \downarrow \mathcal{F}(g) \\ Z & h^X Z & \xrightarrow{\varphi_Z} \mathcal{F}Z \end{array}$$

commutes for all $Y, Z \in \mathcal{A}$, $g \in \text{Hom}_{\mathcal{A}}(Z, Y)$. For $f : Y \rightarrow X$, by definition of φ ,

$$\mathcal{F}(g) \circ (\varphi_Y(f)) = \mathcal{F}(g) \circ \mathcal{F}(f)(x)$$

which by functoriality of \mathcal{F} is $= \mathcal{F}(f \circ g)(x)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_Z \circ (h_X(g)(f)) = \varphi_Z(h_X(f \circ g))$$

which again by definition of φ is $= \mathcal{F}(f \circ g)(x)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, \mathcal{F}) \rightarrow \mathcal{F}X$ is a bijection. \square

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof covariant version is exactly the same, just have to reverse the arrows on the category \mathcal{A} . The Yoneda lemma gives us an embedding of the category \mathcal{A} inside the functor category $\mathbf{Sets}^{\mathcal{A}^{\text{op}}}$, given by,

$$X \mapsto h^X.$$

This embedding is called the Yoneda embedding $h^{(-)} : \mathcal{A} \rightarrow \mathbf{Sets}^{\mathcal{A}^{\text{op}}}$, which sends an object $X \in \mathcal{A}$ to the sets of morphisms $\text{Hom}_{\mathcal{A}}(-, X)$. These functors are fully faithful by Yoneda lemma, because by replacing the functor \mathcal{F} by h^Y we have,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h^X, h^Y) \cong h^Y(X) = \text{Hom}_{\mathcal{A}}(X, Y). \quad (\text{weak Yoneda})$$

Similarly for the covariant embedding, in which case this will be $\text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(h_X, h_Y) \cong \text{Hom}_{\mathcal{A}}(Y, X)$.

Given a contravariant functor, $\mathcal{F} : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Sets}$, the [Yoneda](#) tells us that we can think of the action of \mathcal{F} on the element X as natural transformations to the hom functor h^X in the functor category. So, every functor \mathcal{F} can be extended and be thought of as a representable functor,

$$\begin{aligned}h^{\mathcal{F}} : (\mathbf{Sets}^{\mathcal{A}^{\text{op}}})^{\text{op}} &\rightarrow \mathbf{Sets} \\ \mathcal{G} &\mapsto \text{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\text{op}}}}(\mathcal{G}, \mathcal{F})\end{aligned}$$

where elements $X \in \mathcal{A}$ are to be interpreted as the elements $h^X \in \mathbf{Sets}^{\mathcal{A}^{\text{op}}}$. The following frequently used corollary of Yoneda lemma allows us to compare objects of locally small category by their hom-functors, and hence using the properties of the category of sets.

LEMMA 1.1.2. (YONEDA PRINCIPLE)

$$h^X \cong h^Y \Rightarrow X \cong Y. \quad (\text{Yoneda principle})$$

Note that, [Yoneda](#) associates to each set in $\mathcal{F}X$ a natural transformation between h^X and \mathcal{F} . If the functor \mathcal{F} is representable, i.e., there exists $Y \in \mathcal{A}$ such that there exists a natural isomorphism,

$$\mathcal{F} \xrightarrow[\alpha]{\cong} h^Y$$

Let $\mu(\alpha)$ be the corresponding element in $\mathcal{F}Y = \text{Hom}_{\mathcal{A}}(Y, Y)$. The pair $(Y, \mu(\alpha))$ is called a universal object for \mathcal{F} . It's such that for any other object $Z \in \mathcal{A}$, and each $g \in \mathcal{F}X = \text{Hom}_{\mathcal{A}}(X, Y)$ there exists a unique morphism $f : X \rightarrow Y$ such that,

$$\mathcal{F}(f)(\mu(\alpha)) = g.$$

DIGRESSION: ENRICHED CATEGORIES

The hom-sets in practice usually are richer than merely being sets, they come equipped with additional structure. Enriched categories are categories, where the hom-sets have additional structure. Many of the constructions we can make in **Sets** can also be done in many enriched categories. Here will informally discuss the minimal necessary stuff from the theory of enriched categories so that the reader doesn't feel out of place.

An enriched category is a category in which the hom-sets come equipped with additional structure, that is, the hom-sets are objects in some *enriching* category, usually denoted by \mathcal{V} . \mathcal{V} is called the base for enrichment. This already requires the category \mathcal{V} to have some special properties.

Given any two composable morphisms $f \in \text{Hom}_{\mathcal{A}}(X, Y)$, and $g \in \text{Hom}_{\mathcal{A}}(Y, Z)$ in the category \mathcal{A} , we can consider the composition of the morphisms,

$$g \circ f \in \text{Hom}_{\mathcal{A}}(X, Z)$$

If \mathcal{A} is a \mathcal{V} enriched category, then $f, g, g \circ f \in \mathcal{V}$. So the notion of composition of morphisms in the category \mathcal{A} ,

$$\text{Hom}_{\mathcal{A}}(Y, Z) \times \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{A}}(X, Z)$$

should correspond to a notion of 'composition of objects' or a product in the enriching category,

$$\odot : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$$

There should be unit object corresponding to the identity morphisms, denoted by $\mathbb{1}_{\mathcal{V}}$ such that multiplication by the unit object leaves every object unchanged. This can be formalized by saying there exist natural transformations such that,

$$\mathbb{1}_{\mathcal{V}} \odot V \cong V \odot \mathbb{1}_{\mathcal{V}} \cong V \quad (\text{identity})$$

for all $V \in \mathcal{V}$. Since composition of morphisms is associative, we want \odot to also be associative. This can be formalized by saying there exist natural transformations such that,

$$U \odot (V \odot W) \cong (U \odot V) \odot W \quad (\text{associativity})$$

These properties can also be formalized in terms of a commutative diagram but we will skip that. A category \mathcal{V} with a ‘product’ \odot with [identity](#), and [associativity](#) is called a monoidal category. \odot is called the monoidal product. If in addition the product is such that $U \odot V \cong V \odot U$ it’s called a symmetric monoidal category.

A category \mathcal{A} is said to be enriched by a monoidal category \mathcal{V} if the hom-sets belong to \mathcal{V} and the composition corresponds to the monoidal product \odot . We will denote the hom-sets of a \mathcal{V} -enriched category \mathcal{A} by,

$$\text{Hom}_{\mathcal{A}}^{\mathcal{V}}(X, Y)$$

for all $X, Y \in \mathcal{A}$. When there is no confusion we will drop the superscript \mathcal{V} , or sometimes $\mathcal{A}^{\mathcal{V}}(X, Y)$

A closed monoidal category is monoidal category \mathcal{V} where the functors $- \odot V : \mathcal{V} \rightarrow \mathcal{V}$ admits a right adjoint denoted by $\mathcal{H}om^{\mathcal{V}}(V, -)$. The family of right adjoints assemble in a unique way to give a bifunctor,

$$\mathcal{H}om^{\mathcal{V}} : \mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}.$$

such that,

$$\text{Hom}_{\mathcal{V}}^{\mathcal{V}}(U \times V, W) \cong \text{Hom}_{\mathcal{V}}^{\mathcal{V}}(U, \mathcal{H}om^{\mathcal{V}}(V, W))$$

for all $U, V, W \in \mathcal{V}$. $\mathcal{H}om^{\mathcal{V}}$ is called the internal hom. The internal homs act as the product \odot , and hence, \mathcal{V} together with $\mathcal{H}om^{\mathcal{V}}$ is an enriched category over itself. A category is called cartesian closed if it’s locally small, that is, it’s enriched by the category of sets, and if \odot is the cartesian product in **Sets**.

The small \mathcal{V} -categories themselves form a category. A \mathcal{V} -functor is a functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ between \mathcal{V} -enriched categories such that the morphisms,

$$\text{Hom}_{\mathcal{A}}^{\mathcal{V}}(X, Y) \xrightarrow{\mathcal{F}_{X,Y}} \text{Hom}_{\mathcal{B}}^{\mathcal{V}}(\mathcal{F}X, \mathcal{F}Y)$$

commute with the \odot operation and the identity. Similarly a \mathcal{V} -natural transformation between a pair of \mathcal{V} -functors \mathcal{F} and \mathcal{G} consist of natural transformation between \mathcal{F} and \mathcal{G} such that it commutes with composition by the natural transformations in [identity](#) and [associativity](#). The collection of all \mathcal{V} -natural transformations between \mathcal{F} and \mathcal{G} will be denoted by,

$$\text{Hom}_{\mathcal{A}^{\mathcal{B}}}^{\mathcal{V}}(\mathcal{F}, \mathcal{G})$$

We will now state the enriched Yoneda lemma without proof.

THEOREM 1.1.3. (ENRICHED YONEDA LEMMA) *\mathcal{A} be a small \mathcal{V} -category, then, for all \mathcal{V} -functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{V}$,*

$$\text{Hom}_{\mathcal{A}^{\mathcal{V}}}^{\mathcal{V}}(\text{Hom}_{\mathcal{A}}^{\mathcal{V}}(X, -), \mathcal{F}) \cong \mathcal{F}X.$$

□

The ideas of the proofs are the same as those for the generic ones but with more verification (verifying the natural transformations preserve the enriching structure and so on). Similarly we will have a notion of limits and colimits for the enriched categories.

1.2 | ADJOINT SITUATIONS

Categories are compared by means of functors, and functors themselves are compared via natural transformations. Equivalence of categories allows us to basically think of the two categories as the same thing. This is however too restrictive. The relaxation of the notion of equivalence gives us the notion of adjoint.

The philosophy of adjoint functors is the following; when we want to study an object in mathematics, belonging to some weird category, we can take it, via a functor to some well understood category. But now this new category will not have the same meaning to the objects as the original category. So we would like a functor to get back to the original category. This functor is the adjoint functor.

An adjunction from \mathcal{A} to \mathcal{B} is a pair of functors,

$$\mathcal{A} \begin{matrix} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}} \end{matrix} \mathcal{B},$$

such that there is a natural isomorphisms of bifunctors $(X, Y) \mapsto \text{Hom}_{\mathcal{A}}(X, \mathcal{G}Y)$ and $(X, Y) \mapsto \text{Hom}_{\mathcal{B}}(\mathcal{F}X, Y)$, i.e.,

$$\text{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \text{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \quad (\text{adjoint})$$

for all $X \in \mathcal{A}, Y \in \mathcal{B}$. Denote by $\mathcal{F} \dashv \mathcal{G}$. Since composition of natural isomorphisms is also a natural isomorphism if \mathcal{F} has two adjoints \mathcal{G} and $\hat{\mathcal{G}}$, then we have,

$$\text{Hom}_{\mathcal{A}}(X, \mathcal{G}Y) \cong \text{Hom}_{\mathcal{B}}(\mathcal{F}X, Y) \cong \text{Hom}_{\mathcal{A}}(X, \hat{\mathcal{G}}Y).$$

So, by Yoneda principle, adjoints if they exist are unique upto isomorphism. Consider the following two adjoint situations,

$$\mathcal{A} \begin{matrix} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}} \end{matrix} \mathcal{B} \begin{matrix} \xrightarrow{\mathcal{H}} \\ \xleftarrow{\mathcal{K}} \end{matrix} \mathcal{C},$$

By definition, we have for all $X \in \mathcal{A}$ and $Y \in \mathcal{C}$, $\text{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{K}Y) \cong \text{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{K}Y) \cong \text{Hom}_{\mathcal{C}}(\mathcal{H} \circ \mathcal{F}X, Y)$, hence,

$$\mathcal{F} \circ \mathcal{H} \dashv \mathcal{G} \circ \mathcal{K}$$

When we are working with locally small categories, we can exploit the properties of the category of sets. We can look at adjoints from a functor category perspective, and representable functors.

Given a functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$, for each $X \in \mathcal{B}$, we have the composite functor,

$$\begin{aligned} \hat{\mathcal{F}}(X) &:= h^X \circ \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathbf{Sets} \\ A &\mapsto \text{Hom}_{\mathcal{B}}(\mathcal{F}A, X). \end{aligned}$$

So, $\hat{\mathcal{F}}$ is a functor to the functor category,

$$\hat{\mathcal{F}} : \mathcal{B} \rightarrow \mathbf{Sets}^{\mathcal{A}^{\text{op}}},$$

which sends $X \mapsto \hat{\mathcal{F}}(X)$. For each morphism $f : X \rightarrow Y$ in \mathcal{B} , the functor $\hat{\mathcal{F}}$ associates a morphism in the functor category, i.e., a natural transformation, each $g : \mathcal{F}A \rightarrow X$,

$$\hat{\mathcal{F}}(f) : g \mapsto f \circ g$$

So, $\hat{\mathcal{F}}(f \circ h) = \hat{\mathcal{F}}(f) \circ \hat{\mathcal{F}}(h)$, i.e., it's a covariant functor.

LEMMA 1.2.1. $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ admits a left adjoint iff for all $X \in \mathcal{B}$,

$$\hat{\mathcal{F}}(X) : \mathcal{A} \mapsto \text{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$$

is representable.

PROOF

\Leftarrow Suppose $\hat{\mathcal{F}}(X)$ is representable for all $X \in \mathcal{B}$, then, $\exists \mathcal{G}X \in \mathcal{A}$ with, $\hat{\mathcal{F}}(X) \cong h^{\mathcal{G}X}$, i.e.,

$$\hat{\mathcal{F}}(X)(A) \cong \text{Hom}_{\mathcal{A}}(A, \mathcal{G}X)$$

We have to make sure this is functorial, i.e., $X \mapsto \mathcal{G}X$ is a functor from \mathcal{B} to \mathcal{A} . So, now we have to show that for each morphism $f : X \rightarrow Y$ there exists a morphism in the functor category, a natural transformation of functors, $\mathcal{G}f : \hat{\mathcal{F}}(X) \rightarrow \hat{\mathcal{F}}(Y)$, defined to be the maps that makes the following diagram commute.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(A, \mathcal{G}X) & \longrightarrow & \hat{\mathcal{F}}(X)(A) \\ \mathcal{G}(f) \circ \downarrow & & \downarrow \hat{\mathcal{F}}(f) \\ \text{Hom}_{\mathcal{A}}(A, \mathcal{G}Y) & \longrightarrow & \hat{\mathcal{F}}(Y)(A) \end{array}$$

This also satisfies the composition needs by construction. By Yoneda lemma, this determines the functor \mathcal{G} uniquely upto isomorphism.

\Rightarrow The other direction is obvious and follows directly from the definition of adjoint, i.e., if there exists a left adjoint $\mathcal{G} \dashv \mathcal{F}$ each functor $A \mapsto \text{Hom}_{\mathcal{B}}(\mathcal{F}A, X)$ is representable with representative $\mathcal{G}X$. \square

1.2.1 | ADJOINTS AS REFLECTIONS

The notion of reflection of a functor provides a bit more intuitive meaning of what adjoints are doing. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. We want to associate to each object $B \in \mathcal{B}$ and object $R_B \in \mathcal{A}$ such that $\mathcal{F}R_B$ is the best estimate of B in \mathcal{A} . In categorical terms, the estimation is done with morphisms. So, the ‘best estimate’ is a morphism,

$$\kappa_B : B \rightarrow \mathcal{F}R_B$$

such that for any other $A \in \mathcal{A}$, with an estimation $\varkappa : B \rightarrow \mathcal{F}A$ factors uniquely through R_B . R_B together with the morphism κ_B is called the reflection of B along \mathcal{F} . Visualised by the diagram,

$$\begin{array}{ccc} R_B & \mathcal{F}R_B & \xleftarrow{\kappa_B} B \\ \exists! f \downarrow & \mathcal{F}(f) \downarrow & \swarrow \varkappa \\ A & \mathcal{F}A & \end{array} \quad (\text{reflection})$$

that’s to say there exists a unique morphism $f : R_B \rightarrow A$ such that,

$$\mathcal{F}(f) \circ \kappa_B = \varkappa.$$

Intuitively κ_B is a better estimate than \varkappa . We can’t have two best estimates κ_B and κ'_B because in that case we have two maps $f : R_B \rightarrow R'_B$ and $f' : R'_B \rightarrow R_B$, such that,

$$\mathcal{F}(f) \circ \kappa_B = \kappa'_B, \quad \mathcal{F}(f') \circ \kappa'_B = \kappa_B$$

So we get,

$$\mathcal{F}(f \circ f') \circ \kappa'_B = \kappa'_B,$$

By uniqueness this means $f \circ f' = \mathbb{1}_{R'_B}$, so any two reflections are isomorphic.

LEMMA 1.2.2. $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$, suppose reflection exists for each $B \in \mathcal{B}$, then there exists a unique functor $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$ such that $\mathcal{G}B = R_B$ and a natural transformation κ such that,

$$\kappa_B : B \rightarrow \mathcal{F} \circ \mathcal{G}B.$$

PROOF

The existence of reflection to each object in \mathcal{B} gives us an associated object for each object in \mathcal{B} , we want to understand what happens to the morphisms.

Let $f : X \rightarrow Y$ be a morphism in \mathcal{B} . Then we have,

$$\begin{array}{ccccc} R_X & & \mathcal{F}R_X & \xleftarrow{\kappa_X} & X \\ \exists! f \downarrow & & \mathcal{F}(f) \downarrow & & \downarrow g \\ R_Y & & \mathcal{F}R_Y & \xleftarrow{\kappa_Y} & Y \end{array}$$

In this diagram, $\kappa_Y \circ g : X \rightarrow \mathcal{F}R_Y$ is an estimate, and hence there must exist a morphism $f : R_X \rightarrow R_Y$ such that $\mathcal{F}(f) \circ \kappa_X = \kappa_Y \circ g$. So, we define,

$$\mathcal{G}(g) := f.$$

By construction this makes κ a natural transformation which is determined by the components κ_X . By exploiting uniqueness we can show that this is functorial, that is $\mathcal{G}(g \circ h) = \mathcal{G}(g) \circ \mathcal{G}(h)$. \square

If the functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ has a reflection, by definition, to each morphism $\varkappa : X \rightarrow \mathcal{F}A$ there exists a unique morphism $f : \mathcal{G}X \rightarrow A$. Conversely, any map f uniquely determines \varkappa by,

$$\mathcal{F}(f) \circ \kappa_X = \varkappa.$$

Which means we have an isomorphism of sets,

$$\text{Hom}_{\mathcal{A}}(\mathcal{G}X, A) \cong \text{Hom}_{\mathcal{B}}(X, \mathcal{F}A).$$

Since this holds for every X and A it is the adjoint condition. So, adjoint and reflection are the same functor. Note that in the construction of the ‘reflection functor’ we assumed every object in \mathcal{B} has a reflection. The adjoint functor theorems try to simplify this condition, under additional constraints.

1.3 | LIMITS & COLIMITS

The notion of limits and colimits is very important as they allow us to construct new objects and functors. They are also very closely related to adjoint functors. To heuristically motivate, limits is the categorical notion of ‘closest’ object to or from a system. Here, we intend to find an object that’s nearest to a system. The notion of nearness must come from the morphisms, so the idea is to put the starting category in some other category, where the morphisms

provide some sort of ‘categorical distance’, and then use this notion of distance of the target category to describe the ‘limit’.

Let \mathcal{I} and \mathcal{A} be two categories. An inductive system in \mathcal{A} indexed by \mathcal{I} is a functor,

$$\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}.$$

Intuitively, the limit of the system is an object in \mathcal{A} that is ‘closest’ to the system.

This can be formalised using the functor category as follows; Attach to each object $X \in \mathcal{A}$ the constant functor $\Delta X : \mathcal{I} \rightarrow \mathcal{A}$ that sends everything in \mathcal{I} to X , and each morphism in \mathcal{I} to the identity on X . A relation between an object X and the system \mathcal{F} is a natural transformation between ΔX and \mathcal{F} . Such a natural transformation is called a cone. The collection of all such cones is the set of all natural transformations,

$$\begin{aligned} C_{\mathcal{F}} : \mathcal{A}^{\text{op}} &\rightarrow \mathbf{Sets} \\ X &\mapsto \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}). \end{aligned}$$

It’s a contravariant functor from \mathcal{A} to \mathbf{Sets} . If the functor $C_{\mathcal{F}}$ is representable, there exists an object $L_{\mathcal{F}} \in \mathcal{A}$ such that,

$$C_{\mathcal{F}} \cong h^{L_{\mathcal{F}}}.$$

So, in such case $C_{\mathcal{F}}(X) \cong \text{Hom}_{\mathcal{A}}(X, L_{\mathcal{F}})$. The representative $L_{\mathcal{F}}$ if it exists is called the limit of the system \mathcal{F} , and is denoted by $\varprojlim \mathcal{F} := L_{\mathcal{F}}$ i.e.,

$$\text{Hom}_{\mathcal{A}^{\mathcal{I}}}(\Delta X, \mathcal{F}) \cong \text{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \quad (\text{limit})$$

and hence every cone must factor through $L_{\mathcal{F}}$. Intuitively the limit is the ‘closest’ object to the system. The notion of closeness must come from morphisms, so if there exists any other object with morphisms to the system, then it must be ‘farther’ than the limit, or in terms of morphisms there must exist a morphism between this object and the limit, and hence the morphisms to the system must factor through the limit.

This means that for all objects $X \in \mathcal{A}$ and all family of morphisms $f_I : X \rightarrow \mathcal{F}I$, in \mathcal{A} such that for all $f \in \text{Hom}_{\mathcal{I}}(I, J)$, with $f_J = f_I \circ \mathcal{F}(f)$ factors uniquely through $\varprojlim \mathcal{F}$.

A projective system in \mathcal{A} indexed by \mathcal{I} is a functor,

$$\mathcal{G} : \mathcal{I}^{\text{op}} \rightarrow \mathcal{A}.$$

Similar to the inductive system, for projective system $\mathcal{G} : \mathcal{I}^{\text{op}} \rightarrow \mathcal{A}$, we study the collection of cocones, i.e.,

$$\begin{aligned} C^{\mathcal{G}} : \mathcal{A} &\rightarrow \mathbf{Sets} \\ X &\mapsto \text{Hom}_{\mathcal{A}^{\mathcal{I}^{\text{op}}}}(\mathcal{G}, \Delta X). \end{aligned}$$

If it's representable with representative $L^{\mathcal{G}}$,

$$C^{\mathcal{G}} \cong h_{L^{\mathcal{G}}}.$$

Denote the representative by $\varinjlim \mathcal{G} := L^{\mathcal{G}}$ is called the colimit or the limit of the projective system. If the colimit exists, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{Top}}}(\mathcal{G}, \Delta X) \cong \mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, X) \quad (\text{colimit})$$

Projective limits can be written in terms of universal property as,

$$\begin{array}{ccc} \mathcal{G}I & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}J \\ & \searrow \varphi_I \quad \swarrow \varphi_J & \\ & \varinjlim \mathcal{G} & \\ & \searrow f_I \quad \swarrow f_J & \\ & X & \end{array}$$

Note that if \mathcal{I} admits initial object 0, then the limit $\varprojlim \mathcal{F}$ corresponds to the object $\mathcal{F}(0)$. Similarly for colimit, with terminal object.

1.3.1 | (Co)LIMIT CALCULUS

A category \mathcal{A} is cocomplete with respect to \mathcal{I} if for all inductive systems indexed by \mathcal{I} , the colimit exists, if \mathcal{I} is not explicitly said, then it means that \mathcal{A} is cocomplete with respect to all small categories. \mathcal{A} is complete with respect to \mathcal{I} if it has all limits for all projective systems indexed by \mathcal{I} . \mathcal{A} is called bicomplete if it's both complete and cocomplete.

THEOREM 1.3.1. \mathcal{A} is complete $\Rightarrow \mathcal{A}^{\mathcal{K}}$ is complete.

PROOF

Given an inductive system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}^{\mathcal{K}}$, the goal is to construct a new functor $\varprojlim \mathcal{F}$ in $\mathcal{A}^{\mathcal{I}}$ such that,

$$\mathrm{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

where $\Delta^{\mathcal{K}}$ is the constant functor in $\mathcal{A}^{\mathcal{K}}$ which sends $A \in \mathcal{A}$ to the constant functor $\Delta^{\mathcal{K}}A$. This is then by definition the [limit](#) of the system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}^{\mathcal{K}}$.

In order to do this we have to describe where $\varprojlim \mathcal{F}$ sends elements of \mathcal{I} . The construction involves argument wise assignment of objects in \mathcal{A} for each object in \mathcal{I} , and then showing the functoriality, that is verify it respects composition of morphisms.

CONSTRUCTION. By definition of [limit](#), for every $I \in \mathcal{I}$, we have,

$$\mathrm{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}I, \mathcal{F}I) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}I, \varprojlim(\mathcal{F}I)) \quad (\text{isomorphism})$$

here $\mathcal{F}I : \mathcal{K} \rightarrow \mathcal{A}$ is a fixed functor, an inductive system and $\varprojlim(\mathcal{F}I)$ its inductive limit. So we define the associated object by,

$$\varprojlim \mathcal{F}(I) := \varprojlim(\mathcal{F}I).$$

FUNCTORIALITY.

$$\text{Hom}_{\mathcal{A}^{\mathcal{K}}}(\Delta^{\mathcal{K}}\mathcal{H}(\cdot), \mathcal{F}(\cdot)) : \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \mathbf{Sets}$$

is a bifunctor, so for each natural transformation $\kappa : \Delta^{\mathcal{K}}\mathcal{H} \Rightarrow \mathcal{F}$,

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathcal{F}} & \mathcal{A}^{\mathcal{K}} \\ & \uparrow \kappa & \nearrow \Delta^{\mathcal{K}} \\ & \mathcal{A} & \end{array}$$

and morphism $f : I \rightarrow J$, we have maps,

$$\begin{array}{ccccc} I & & \Delta^{\mathcal{K}}\mathcal{H}I & \xrightarrow{\kappa_I} & \mathcal{F}I \\ f \downarrow & & \Delta^{\mathcal{K}}\mathcal{H}(f) \downarrow & & \downarrow \mathcal{F}(f) \\ J & & \Delta^{\mathcal{K}}\mathcal{H}J & \xrightarrow{\kappa_J} & \mathcal{F}J \end{array}$$

The commutative square gives us, $\kappa_J \circ \Delta^{\mathcal{K}}\mathcal{H}(f) = \mathcal{F}(f) \circ \kappa_I$, and the [isomorphism](#) of sets gives us a morphisms, $\hat{\kappa}_I$ and $\hat{\kappa}_J$ such that,

$$\hat{\kappa}_I \circ \mathcal{H}(f) = \varprojlim(\mathcal{F}(f)) \circ \hat{\kappa}_J.$$

So, $\hat{\kappa} : \mathcal{H} \rightarrow \varprojlim \mathcal{F}(\cdot)$ is a natural transformation. So we get,

$$\text{Hom}_{(\mathcal{A}^{\mathcal{K}})^{\mathcal{I}}}(\Delta^{\mathcal{K}}\mathcal{H}, \mathcal{F}) \cong \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(\mathcal{H}, \varprojlim \mathcal{F})$$

Since this natural transformation is defined using the natural transformation κ it will satisfy the required compositions. So we have, $\varprojlim \mathcal{F}(f \circ g) = (\varprojlim \mathcal{F}(f)) \circ (\varprojlim \mathcal{F}(g))$. \square

This gives us an adjoint situation, where colimit is left-adjoint to the constant functor and the constant functor is left-adjoint to the limit,

$$\varinjlim \dashv \Delta \dashv \varprojlim \quad (\text{limit-diagonal adjointness})$$

THEOREM 1.3.2.

$$\text{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \text{Hom}_{\mathcal{A}}(A, \mathcal{F}),$$

$$\text{Hom}_{\mathcal{A}}(\varinjlim \mathcal{G}, A) \cong \varinjlim \text{Hom}_{\mathcal{A}}(\mathcal{G}, A).$$

PROOF

The idea is to study an appropriate functor category, get hom-set isomorphisms and then apply Yoneda principle. So, we have to show for each set $X \in \mathbf{Sets}$, we have an isomorphism of sets,

$$\text{Hom}_{\mathbf{Sets}}(X, \text{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F})) \cong \text{Hom}_{\mathbf{Sets}}(X, \varprojlim \text{Hom}_{\mathcal{A}}(A, \mathcal{F}))$$

Using the [limit-diagonal adjointness](#), this reduces to showing $\text{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \text{Hom}_{\mathcal{A}}(A, \mathcal{F}))$ and $\text{Hom}_{\mathbf{Sets}}(X, \text{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$ are isomorphic. Δ s are in the appropriate categories.

Let $\kappa : \Delta X \rightarrow \text{Hom}_{\mathcal{A}}(A, \mathcal{F})$ be a natural transformation, then κ is determined by its components

$$\kappa_I : (\Delta X)I \rightarrow \text{Hom}_{\mathcal{A}}(A, \mathcal{F}I).$$

Each $(\Delta X)I$ is a set, and hence the maps κ_I is itself determined by its action on the elements of the set X , So, for each $x \in X$, $\kappa_I(x)$ is a morphism in $\text{Hom}_{\mathcal{A}}(A, \mathcal{F}I)$.

If we think of A as the constant functor, we can define using $\varphi_x(\cdot) := \kappa_{(\cdot)}(x)$ defines an element $\varphi \in \text{Hom}_{\mathbf{Sets}}(X, \text{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$. This is a bijection and hence,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(\Delta X, \text{Hom}_{\mathcal{A}}(A, \mathcal{F})) \cong \text{Hom}_{\mathbf{Sets}}(X, \text{Hom}_{\mathcal{A}}(\Delta A, \mathcal{F}))$$

Applying the [limit-diagonal adjointness](#) again to this and the Yoneda principle, we get that $\text{Hom}_{\mathcal{A}}(A, \varprojlim \mathcal{F}) \cong \varprojlim \text{Hom}_{\mathcal{A}}(A, \mathcal{F})$. The other isomorphism is also similar. \square

So, hom-functor takes limits to colimits in the first argument and takes limits to limits in the second argument. This can now be used to prove many things easily.

Intuitively, limits take us to the ‘last object’ in a diagram and colimits take us to the ‘first object’ in a diagram. Here ‘most’ is quantified in terms of morphisms, and first and last are quantified by the direction of the morphisms. So, if the indexing category is a product category, then we can think of limits as trying to find the bottom right corner of the rectangle diagram, and similarly the colimit is the top left corner. So, whether we go to right most first and then go to the bottom or whether go to the bottom first and then go to the right shouldn’t change which object we reach after doing this. We now formalize this.

THEOREM 1.3.3. (FUBINI FOR LIMITS)

$$\varinjlim_{\mathcal{I}} \varinjlim_{\mathcal{J}} \mathcal{F} \cong \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F} \cong \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}$$

PROOF

This can be proved using the relation between the constant functors. Let $\Delta^{\mathcal{I} \times \mathcal{J}} : \mathcal{A} \rightarrow \mathcal{A}^{\mathcal{I} \times \mathcal{J}}$, $\Delta^{\mathcal{I}} : \mathcal{A} \rightarrow \mathcal{A}^{\mathcal{I}}$ and $\Delta^{\mathcal{J}} : \mathcal{A} \rightarrow \mathcal{A}^{\mathcal{J}}$ be the constant functors in the appropriate functor category. Then we have, $\Delta^{\mathcal{I} \times \mathcal{J}} = \Delta^{\mathcal{I}} \Delta^{\mathcal{J}}$. This gives us,

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{I} \times \mathcal{J}} \mathcal{F}) &\cong \text{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I} \times \mathcal{J}} A, \mathcal{F}) \\ &\cong \text{Hom}_{\mathcal{A}}(\Delta^{\mathcal{I}} \Delta^{\mathcal{J}} A, \mathcal{F}) \\ &\cong \text{Hom}_{\mathcal{A}}(A, \varinjlim_{\mathcal{J}} \varinjlim_{\mathcal{I}} \mathcal{F}). \end{aligned}$$

By Yoneda principle we have the required isomorphism. Since $\mathcal{I} \times \mathcal{J} \cong \mathcal{J} \times \mathcal{I}$, the other isomorphism also follows. \square

LEMMA 1.3.4. Right adjoints preserve limits.

PROOF

Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ has a left adjoint say $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$, and suppose $\mathcal{H} : \mathcal{I} \rightarrow \mathcal{A}$ is an inductive system with a limit $\varprojlim \mathcal{H}$, we must prove that $\mathcal{F}(\varprojlim \mathcal{H})$ is the limit of the inductive system $\mathcal{F} \circ \mathcal{H}$.

$\mathcal{F}\mathcal{H} : \mathcal{I} \rightarrow \mathcal{B}$ is an inductive system in \mathcal{B} indexed by \mathcal{I} . For all $X \in \mathcal{B}$,

$$\begin{aligned} \text{Hom}_{\mathcal{B}}(X, \mathcal{F} \varprojlim \mathcal{H}) &\cong \text{Hom}_{\mathcal{A}}(\mathcal{G}X, \varprojlim \mathcal{H}) \\ &\cong \varprojlim \text{Hom}_{\mathcal{A}}(\mathcal{G}X, \mathcal{H}) \\ &\cong \varprojlim \text{Hom}_{\mathcal{B}}(X, \mathcal{F}\mathcal{H}) \\ &\cong \text{Hom}_{\mathcal{B}}(X, \varprojlim \mathcal{F}\mathcal{H}) \end{aligned}$$

By Yoneda principle, we have, $\mathcal{F}(\varprojlim \mathcal{H}) \cong \varprojlim \mathcal{F} \circ \mathcal{H}$. \square

Right adjoints preserve limits, can be remembered by the acronym, RAPL. Under additional conditions on the category \mathcal{A} , the converse holds, these theorems are called adjoint functor theorems.

1.3.2 | (Co)END CALCULUS

Similar to how natural transformations relate functors between two categories, dinatural transformations relate bifunctors. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{B}$ be two bifunctors. Bifunctors are functors when one of the arguments are fixed. We will denote \mathcal{F}^X for the contravariant functor $\mathcal{F}(\cdot, X)$, and \mathcal{F}_X for the covariant functor $\mathcal{F}(X, \cdot)$.

Each morphism $f : X \rightarrow Y$ gives rise to the following morphisms in \mathcal{B} ,

$$\begin{array}{ccccc} X & & \mathcal{F}(X, X) & & \mathcal{G}(X, X) \\ \downarrow f & \swarrow \mathcal{F}^X(f) & & \searrow \mathcal{G}_X(f) & \\ & \mathcal{F}(Y, X) & & & \mathcal{G}(X, Y) \\ & \searrow \mathcal{F}_Y(f) & & \swarrow \mathcal{G}^Y(f) & \\ Y & & \mathcal{F}(Y, Y) & & \mathcal{G}(Y, Y) \end{array}$$

A dinatural transformation $\kappa : \mathcal{F} \rightarrow \mathcal{G}$ consists of a family of morphisms,

$$\kappa_X : \mathcal{F}(X, X) \rightarrow \mathcal{G}(X, X)$$

such that for any $f : X \rightarrow Y$ the following commutes,

$$\begin{array}{ccccc} X & & \mathcal{F}(X, X) & \xrightarrow{\kappa_X} & \mathcal{G}(X, X) \\ \downarrow f & \swarrow \mathcal{F}^X(f) & & \searrow \mathcal{G}_X(f) & \\ & \mathcal{F}(Y, X) & & & \mathcal{G}(X, Y) \\ & \searrow \mathcal{F}_Y(f) & & \swarrow \mathcal{G}^Y(f) & \\ Y & & \mathcal{F}(Y, Y) & \xrightarrow{\kappa_Y} & \mathcal{G}(Y, Y) \end{array}$$

Similar to the case of limits and cones, we can describe ‘doubly indexed limits’. Suppose we are given a system $\mathcal{F} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{B}$, intuitively, the end of the system is an object in \mathcal{B} that is ‘closest’ to the system.

This can be formalised using the functor category as follows; Attach to each object $E \in \mathcal{B}$ the constant bifunctor $\Delta E : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{B}$ that sends everything in $\mathcal{A}^{\text{op}} \times \mathcal{A}$ to E , and each morphism in \mathcal{B} to the identity on E . A relation between an object X and the system \mathcal{F} is a dinatural transformation δ between ΔE and \mathcal{F} . Represented by the diagram,

$$\begin{array}{ccccc} X & & \mathcal{F}(X, X) & & \\ \downarrow f & \nearrow \delta_X^E & \searrow \mathcal{F}^X(f) & & \\ E & & & & \mathcal{F}(X, Y) \\ & \searrow \delta_Y^E & \nearrow \mathcal{F}_Y(f) & & \\ Y & & \mathcal{F}(Y, Y) & & \end{array}$$

Such a dinatural transformation is called a wedge. The collection of all such wedges is the set of all such dinatural transformations,

$$\begin{aligned} W_{\mathcal{F}} : \mathcal{B} &\rightarrow \mathbf{Sets} \\ E &\mapsto \text{DNat}_{\mathcal{B}^{\mathcal{A}^{\text{op}} \times \mathcal{A}}}(\Delta E, \mathcal{F}). \end{aligned}$$

The end of the system \mathcal{F} is intuitively the object which is closest to the system. A relation between an object $E \in \mathcal{B}$ and the system \mathcal{F} consists of a dinatural transformation from the constant bifunctor ΔE to \mathcal{F} . So, the ‘closest’ would be such that any other dinatural transformation should factor through the ‘closest’ one.

The end of a system \mathcal{F} , is an object $\int_{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\omega : \Delta \int_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{F}$ such that every other wedge factors through it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathrm{op}} \times \mathcal{A}}(\Delta E, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{B}}(E, \int_{\mathcal{A}} \mathcal{F}) \quad (\text{end})$$

Also denoted by $\int_{A \in \mathcal{A}} \mathcal{F} A$. Expressed in a commutative diagram by,

$$\begin{array}{ccc} E & \xrightarrow{\kappa_X} & \int_{\mathcal{A}} \mathcal{F} \\ \downarrow \kappa_Y & \searrow \exists! & \downarrow \omega_X \\ \mathcal{F}(Y, Y) & \xrightarrow{\mathcal{F}^Y(f)} & \mathcal{F}(X, X) \\ & & \downarrow \mathcal{F}_X(f) \\ & & \mathcal{F}(X, Y) \end{array}$$

The integral notation corresponds to the intuition that the equalizer given in $\int_{\mathcal{A}}$ can be thought of as an averaging operation on the functor where we run through the objects of \mathcal{A} .

The coend of a system \mathcal{F} , is an object $\int^{\mathcal{A}} \mathcal{F} \in \mathcal{B}$, together with a dinatural transformation $\sigma : \mathcal{F} \rightarrow \Delta \int^{\mathcal{A}} \mathcal{F}$, such that every other cowedge factors from it.

$$\mathrm{DNat}_{\mathcal{B}^{\mathrm{op}} \times \mathcal{A}}(\mathcal{F}, \Delta E) \cong \mathrm{Hom}_{\mathcal{B}}(\int^{\mathcal{A}} \mathcal{F}, E) \quad (\text{coend})$$

Expressed in a commutative diagram by,

$$\begin{array}{ccc} \mathcal{F}(Y, X) & \xrightarrow{\mathcal{F}_Y(f)} & \mathcal{F}(Y, Y) \\ \downarrow \mathcal{F}^X(f) & & \downarrow \\ \mathcal{F}(X, X) & \dashrightarrow & \int^{\mathcal{A}} \mathcal{F} \\ & \searrow \kappa_X & \downarrow \exists! \\ & & E \end{array}$$

FUNCTORIALITY. Let $\mathcal{F}, \mathcal{G} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \rightarrow \mathcal{B}$ be two bifunctors and let $\eta : \mathcal{F} \rightarrow \mathcal{G}$ be a dinatural transformation. Any connection from an object E to the system \mathcal{F} gives rise to a connection from the object to the system \mathcal{G} , given by the composition.

In particular we have the composition for the connection $\omega : \int_{\mathcal{A}} \mathcal{F} \rightarrow \mathcal{F}$. So, this connection must factor through $\int_{\mathcal{A}} \mathcal{G}$, and hence we have a map, whose components are given by

the composition,

$$\begin{array}{ccc}
\int_{\mathcal{A}} \mathcal{F} & \xrightarrow{\eta_X \circ \omega_X} & \mathcal{G}(X, X) \\
\downarrow \exists! & \searrow \varphi_X & \downarrow \mathcal{G}_X(f) \\
\int_{\mathcal{A}} \mathcal{G} & \xrightarrow{\varphi_X} & \mathcal{G}(X, X) \\
\downarrow \varphi_Y & & \downarrow \mathcal{G}_X(f) \\
\mathcal{G}(Y, Y) & \xrightarrow{\mathcal{G}^Y(f)} & \mathcal{G}(X, Y)
\end{array}$$

$\eta_Y \circ \omega_Y$ (curved arrow from $\int_{\mathcal{A}} \mathcal{F}$ to $\mathcal{G}(Y, Y)$)

This is functorial since the $\int_{\mathcal{A}} \eta : \int_{\mathcal{A}} \mathcal{F} \rightarrow \int_{\mathcal{A}} \mathcal{G}$ are defined component wise and so the composition will be component wise. So \int is functorial, that is, $\int_{\mathcal{A}} (\eta \circ \kappa) = \int_{\mathcal{A}} \eta \circ \int_{\mathcal{A}} \kappa$.

FUBINI RULE. Given a functor $\mathcal{F} : \mathcal{A}^{\text{op}} \times \mathcal{A} \times \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{C}$, the end in the first two coordinates, gives rise to a functor,

$$\int_{\mathcal{A}} \mathcal{F} : \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{C}.$$

The end of this functor is then,

$$\int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F} \in \mathcal{C}$$

With some tedious checking it turns out that,

$$\int_{\mathcal{A} \times \mathcal{B}} \mathcal{F} \cong \int_{\mathcal{A}} \int_{\mathcal{B}} \mathcal{F} \cong \int_{\mathcal{B}} \int_{\mathcal{A}} \mathcal{F}. \quad (\text{Fubini for ends})$$

1.3.2.1 | (Co)ENDS AS (Co)LIMITS

The motivation for the definition of ends/coends was very similar to that of limits/colimits. So we should expect the two concepts to be closely related. To intuitively motivate the relation between ends and limits, we have to intuitively understand how ends describe average of a system and limits describe closeness to the system. Limits only take into account the relation an object has ‘with’ the system, that is, a natural transformation from constant functor to the system. In case of ends, the relation ‘between’ the objects of the system is important, this is encoded in the dinatural transformation. So, the ‘average’ should also be expected to be some sort of limit taken over morphisms between objects of the indexing category. This is what makes it average over the relations between objects of the system.

We can think of ends and coends as limits and colimits. The first step is then to associate to each bifunctor system a functor, and turn the ends/coends of bifunctor systems into limits/colimits of systems. This involves establishing an equivalence between the category of bifunctors and an appropriate category of functors. Consider a bifunctor

$$\mathcal{F} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{B}$$

attaches to pairs of objects in \mathcal{A} objects in \mathcal{B} such that the relation between the pair of objects is preserved. So, we could think of the functor as assigning to each morphism between pairs of objects in \mathcal{A} an object of \mathcal{B} . So, we can start with a new category where objects are morphisms of the category \mathcal{A} .

Let $f : X \rightarrow Y$ be a morphism in \mathcal{A} , the bifunctor assigns to the pair X, Y the object $\mathcal{F}(X, Y)$, so instead we could assign to the morphism f , the object $\mathcal{F}(\text{src}(f), \text{tgt}(f))$. Where

src is the source object of f , and $\text{tgt}(f)$ is the target. To make sure the bifactoriality transfers to functoriality of this new association we have to define the morphisms suitably. For two every morphism $f, g \in \text{Hom}_{\mathcal{A}}$ define a morphism between them to be a pair of morphisms,

$$\begin{array}{ccc} X & \longleftarrow & \hat{X} \\ f \downarrow & & \downarrow g \\ Y & \longrightarrow & \hat{Y}. \end{array} \quad (\text{bimorph})$$

Note that the reverse direction of the connection from \hat{X} to X is necessary to make the association a functor. Because the bifunctor is contravariant in the first argument, with the second argument fixed. So with this, we have a new functor,

$$\hat{\mathcal{F}} : \hat{\mathcal{A}} \rightarrow \mathcal{B}$$

where $\hat{\mathcal{A}}$ is the category consisting of morphisms of \mathcal{A} as objects and the above rule for morphisms, [bimorph](#). So the functor,

$$\mathcal{B}^{\mathcal{A}^{\text{op}} \times \mathcal{A}} \rightarrow \mathcal{B}^{\hat{\mathcal{A}}}, \quad \mathcal{F} \mapsto \hat{\mathcal{F}},$$

is an equivalence of categories, and must respect initial/terminal objects. Since limits/colimits and ends/coends are initial/terminal objects in the respective categories we have,

$$\int_{\mathcal{A}} \mathcal{F} \cong \varprojlim_{\hat{\mathcal{A}}} \hat{\mathcal{F}}, \quad \int^{\mathcal{A}} \mathcal{F} \cong \varinjlim_{\hat{\mathcal{A}}} \hat{\mathcal{F}}. \quad ((\text{co})\text{ends as } (\text{co})\text{limits})$$

For a detailed proof, see [2]. This immediately leads to the following observation that if $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ preserves all limits then it preserves all the ends that exist. If $\mathcal{F} : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{C}$ is a system, and \mathcal{G} is a continuous functor then

$$\mathcal{G}(\int_{\mathcal{A}} \mathcal{F}) \cong \int_{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

If \mathcal{G} is a contravariant functor and cocontinuous, then,

$$\mathcal{G}(\int^{\mathcal{A}} \mathcal{F}) \cong \int^{\mathcal{A}} \mathcal{G}\mathcal{F}.$$

An immediate corollary is that Hom functors preserve ends in the second argument and maps ends to coends in the first argument.

COROLLARY 1.3.5. (HOM-(CO)END RELATIONS)

$$\text{Hom}_{\mathcal{B}}(\int_{\mathcal{A}} \mathcal{F}, A) \cong \int^{\mathcal{A}} \text{Hom}_{\mathcal{B}}(\mathcal{F}, A)$$

$$\text{Hom}_{\mathcal{B}}(A, \int_{\mathcal{A}} \mathcal{F}) \cong \int_{\mathcal{A}} \text{Hom}_{\mathcal{B}}(A, \mathcal{F}).$$

□

THEOREM 1.3.6. (NATURAL TRANSFORMATIONS AS ENDS)

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A).$$

PROOF

We have to show that $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$ satisfies the universal property of ends. That's, any collection of morphisms from an object $E \in \mathbf{Sets}$ to the system $\mathcal{H} \equiv \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(-), \mathcal{G}(-))$ factors through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$.

Suppose we have a wedge $\tau : E \rightarrow \mathcal{H}$, then for each $A \in \mathcal{A}$ we have a morphisms $\tau_A : E \rightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$. Since E is a set this is a set map, which maps each $x \in E$ to a morphism between the objects $\mathcal{F}A$ and $\mathcal{G}A$. That's to say $\tau_A(x) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$.

For a morphism $f : A \rightarrow B$, the wedge condition says that,

$$\begin{array}{ccc} E & \xrightarrow{\tau_A} & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \\ \searrow \tau_B & \swarrow \exists! & \downarrow \omega_A \\ & \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) & \xrightarrow{\omega_A} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A) \\ & \downarrow \omega_B & \downarrow \circ \mathcal{F}(f) \\ & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}B, \mathcal{G}B) & \xrightarrow{\mathcal{G}(f) \circ} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}B) \end{array}$$

So, we have,

$$\mathcal{G}(f) \circ \tau_A(x) = \tau_B(x) \circ \mathcal{F}(f).$$

This means that $\tau_{(-)}(x)$ is a natural transformation,

$$\begin{array}{ccccc} A & & \mathcal{F}A & \xrightarrow{\tau_A(x)} & \mathcal{G}A \\ \downarrow f & & \mathcal{F}(f) \downarrow & & \downarrow \mathcal{G}(f) \\ B & & \mathcal{F}B & \xrightarrow{\tau_B(x)} & \mathcal{G}B \end{array}$$

So, it must factor through $\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G})$. This is precisely the universal property of ends. So,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \cong \int_{\mathcal{A}} \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A).$$

□

The collection of all natural transformations from \mathcal{F} to \mathcal{G} can be thought of as taking an average of elements of \mathcal{A} as ‘measured’ by the functor $A \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}A, \mathcal{G}A)$.

1.3.3 | DENSITY THEOREM FOR PRE-SHEAVES

We now have the necessary tools to prove the famous result about the category of pre-sheaves, $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$. Every pre-sheaf can be canonically presented as a colimit of representable functors. For the sake of familiarity we will assume that $\mathcal{C} = \mathcal{O}(X)$.

THEOREM 1.3.7. (CO-YONEDA/DENSITY FORMULA) $\mathcal{F} \in \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$,

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} \mathcal{F}U \times \text{Hom}_{\mathcal{C}}(V, U) \cong \int_{U \in \mathcal{C}} (\mathcal{F}U)^{\text{Hom}_{\mathcal{C}}(U, V)}.$$

PROOF

By [Yoneda](#) applied to the functor

$$\mathcal{H} : V \mapsto \text{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W),$$

we have,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(h^V, \mathcal{H}) \cong \mathcal{H}V.$$

The following chain of isomorphisms lets us sneak in Yoneda principle,

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}}(\mathcal{F}V, W) &\cong \text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(h^V, \text{Hom}_{\mathbf{Sets}}(\mathcal{F}(\cdot), W)) \\ &\cong \int_{U \in \mathcal{C}} \text{Hom}_{\mathbf{Sets}}(h^V U, \text{Hom}_{\mathbf{Sets}}(\mathcal{F}U, W)) \\ &\cong \int_{U \in \mathcal{C}} \text{Hom}_{\mathbf{Sets}}(h^V U \times \mathcal{F}U, W) \\ &\cong \text{Hom}_{\mathbf{Sets}}\left(\int^{U \in \mathcal{C}} h^V U \times \mathcal{F}U, W\right) \end{aligned}$$

Here, in the second step we used [1.3.6](#), and in the third step we used hom-tensor adjointness in **Sets**. The last step follows from hom-sets taking coends to ends in the first argument. So, we have by Yoneda principle,

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} h^V U \times \mathcal{F}U$$

This holds when the pre-sheaf is to any category that's cartesian closed, and the product are replaced with the correct tensor product in the target category. In case of modules this corresponds to the internal hom-tensor product adjointness. \square

When \mathcal{C} is the category of open sets of a space X , $\text{Hom}_{\mathcal{C}^{\text{op}}}(U, V)$ is nonempty, only when $V \subseteq U$, in which case it is a singleton set, with the inclusion map as the only map. If we consider continuous functions as the pre-sheaf, and $V \subset U$, any ‘continuous function’ belonging to U will give rise to a ‘continuous function’ in V . Intuitively, we want to include all such functions and it will be a limit in this sense.

For the enriched case, the product will be replaced by tensor product \odot in the enriching category \mathcal{V} . Limits will be replaced by weighted limits and using these we get the enriched density formula,

THEOREM 1.3.8. (ENRICHED DENSITY FORMULA) *Let $\mathcal{F} \in \mathcal{V}^{\mathcal{A}^{\text{op}}}$, then,*

$$\mathcal{F}V \cong \int^{U \in \mathcal{C}} h^V U \odot \mathcal{F}U$$

\square

For proofs, and details, see [\[2\],\[3\]](#) and the references therein.

1.3.4 | WEIGHTED (Co)LIMITS

Weighted (co)limits are an extreme generalization of standard (co)limits. When one deals with enriched categories, the hom-sets come equipped with additional structure, and it provides some convenient tools for studying various other categorical objects.

Given a system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$, the standard (co)limits describe an object that's closest to or from the system, where closest is quantified in terms of morphisms. In terms of (co)cones this means that any (co)cone must factor through the (co)limits. Note here that for each $I \in \mathcal{I}$, the (co)cones provide a single morphism to or from an object. Weighted (co)limits generalize this. Here, we consider 'weighted' (co)cones, which consist of assigning to each FI a collection of morphisms to or from a vertex. To formalize this we will start by looking at the standard (co)limits again.

Cones from an object $X \in \mathcal{A}$ to the system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$ are natural transformations from the constant functor $\Delta X : \mathcal{I} \rightarrow \mathcal{A}$ to the functor \mathcal{F} . So, each cone is a functor, from the constant functor to the terminal object in sets (singleton set), to the functor $\text{Hom}_{\mathcal{A}}(X, F-)$. So, the limit of the system can be defined up to isomorphism by isomorphism,

$$\text{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{F}) \cong \text{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(1, \text{Hom}_{\mathcal{A}}(X, F-)).$$

The terminal functor gives equal weightage to each FI . Instead of assigning one morphism from X to each FI we can instead assign a collection of morphisms, and study these complicated cones. This can be formalized functorially by defining a 'weight' functor,

$$W : \mathcal{I} \rightarrow \mathbf{Sets}$$

Given a diagram of functors,

$$\mathbf{Sets} \xleftarrow{W} \mathcal{I} \xrightarrow{\mathcal{F}} \mathcal{A}.$$

The weighted limit of the system \mathcal{F} , weighted by W , denoted $\varprojlim^W \mathcal{F}$ is defined by the isomorphism,

$$\text{Hom}_{\mathcal{A}}(X, \varprojlim^W \mathcal{F}) \cong \text{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(W, \text{Hom}_{\mathcal{A}}(X, \mathcal{F}-)). \quad (\text{weighted limit})$$

The following chain of isomorphisms gives us a formula for weighted limits,

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(W, \text{Hom}_{\mathcal{A}}(X, \mathcal{F}-)) &\cong \int_{I \in \mathcal{I}} \text{Hom}_{\mathbf{Sets}}(WI, \text{Hom}_{\mathcal{A}}(X, FI)) \\ &\cong \int_{I \in \mathcal{I}} \text{Hom}_{\mathcal{A}}(X, FI^{WI}) \\ &\cong \text{Hom}_{\mathcal{A}}\left(X, \int_{I \in \mathcal{I}} FI^{WI}\right). \end{aligned}$$

Here we used the 1.3.6 in the first step, hom-tensor adjointness in the next step and 1.3.5 in the last step. So we have,

$$\varprojlim^W \mathcal{F} \cong \int_{I \in \mathcal{I}} FI^{WI}. \quad (\text{weighted limit as ends})$$

Similarly, for a weight $W : \mathcal{I}^{op} \rightarrow \mathbf{Sets}$, the weighted colimits are defined by the isomorphism of sets,

$$\text{Hom}_{\mathcal{A}}(\varinjlim^W \mathcal{F}, X) \cong \text{Hom}_{\mathbf{Sets}^{\mathcal{I}^{op}}}(W, \text{Hom}_{\mathcal{A}}(\mathcal{F}-, X)). \quad (\text{weighted colimit})$$

The following chain of isomorphisms gives us a formula for weighted limits,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}^{\mathrm{op}}}}(W, \mathrm{Hom}_{\mathcal{A}}(\mathcal{F} -, X)) &\cong \int_{I \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{Sets}}(WI, \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}I, X)) \\ &\cong \int_{I \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{A}}(WI \times \mathcal{F}I, X) \\ &\cong \mathrm{Hom}_{\mathcal{A}}\left(\int_{I \in \mathcal{I}^{\mathrm{op}}} \mathcal{F}I \times WI, X\right). \end{aligned}$$

Here we used the 1.3.6 in the first step, hom-tensor adjointness in the next step and 1.3.5 in the last step. So we have,

$$\varinjlim^W \mathcal{F} \cong \int_{I \in \mathcal{I}^{\mathrm{op}}} \mathcal{F}I \times WI. \quad (\text{weighted colimit as coends})$$

Intuitively, the weight functor assigns to each object $I \in \mathcal{I}$ a weight WI , and the weighted limit of the system is the object $\varinjlim^W \mathcal{F}$ for which $\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}I)$ is ‘closest’ to the set WI . This is similar to weights in other branches of mathematics, where one assigns to objects certain real numbers, and the order relation on \mathbb{R} provides the notion of ‘more’ weight. In general case, the real numbers is replaced by the category of sets and order relation is replaced by any function of sets.

1.4 | KAN EXTENSION OF FUNCTORS

Given two functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{C}$, the goal of Kan extension is to find a new functor \mathcal{K} such that the manipulation done by composite functor $\mathcal{K} \circ \mathcal{F}$ is closest to the manipulation done by the functor \mathcal{G} . The obstruction to $\mathcal{K} \circ \mathcal{F}$ being the same as \mathcal{G} comes from \mathcal{F} losing information that \mathcal{G} preserves.

Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{C}$ be two functors.

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow \mathcal{F} & \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} \end{array}$$

The left Kan extension of \mathcal{G} along \mathcal{F} is a functor $\mathrm{Lan}_{\mathcal{F}} \mathcal{G} : \mathcal{B} \rightarrow \mathcal{C}$ such that there exists a natural transformation $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F}$, such that any other extension \mathcal{K} with natural transformation $\mathcal{G} \rightarrow \mathcal{K} \circ \mathcal{F}$ factors through $\mathrm{Lan}_{\mathcal{F}} \mathcal{G}$, i.e., $\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F}$.

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow \mathcal{F} & \nearrow \mathcal{K} \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} \end{array}$$

So, in terms of a commutative diagram a Kan extension is the functor that makes the diagram as close to commutative as possible.

The left Kan extension $\mathrm{Lan}_{\mathcal{F}} \mathcal{G}$ can be visualised by the diagram,

$$\mathcal{G} \Rightarrow (\mathrm{Lan}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F} \Rightarrow \dots$$

This means that for every natural transformation from \mathcal{G} to a functor $\mathcal{K} \circ \mathcal{F}$ there exists a natural transformation from $\mathrm{Lan}_{\mathcal{F}} \mathcal{G}$ to \mathcal{K} , i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{G}, \mathcal{K} \circ \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathrm{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K})$$

Intuitively the left Kan extension is the left most functor to the functor in the above visualization. Here ‘most’ is quantified in terms of natural transformations. The left Kan extension is the left most (as explained above) extension of the functor \mathcal{G} with respect to \mathcal{F} . We should expect the construction of the ‘left most’ functor to be related to taking colimits in an appropriate category. If \mathcal{C} is cocomplete, the functor category to \mathcal{C} will also be cocomplete, and \mathcal{A} has some ‘nice properties’ then we could think of these as a system in the functor category and intuitively, the left ‘most’ should exist.

The right Kan extension is defined similarly, and only the direction of the natural transformation is changed from $\mathcal{K} \circ \mathcal{F}$ to \mathcal{G} .

$$\begin{array}{ccc} & \mathcal{B} & \\ \mathcal{F} \nearrow & \Downarrow & \dashrightarrow \mathcal{H} \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} \end{array}$$

The right Kan extension is denoted by $\text{Ran}_{\mathcal{F}} \mathcal{G}$. It’s the right most functor in the following visualization,

$$\dots \Rightarrow \mathcal{H} \circ \mathcal{F} \Rightarrow (\text{Ran}_{\mathcal{F}} \mathcal{G}) \circ \mathcal{F} \Rightarrow \mathcal{G}$$

Similar to left Kan extensions, we should expect the construction of the ‘right most’ functor to be related to taking limits in an appropriate category. This means that for every natural transformation from \mathcal{G} to a functor $\mathcal{H} \circ \mathcal{F}$ there exists a natural transformation from \mathcal{H} to $\text{Ran}_{\mathcal{F}} \mathcal{G}$, i.e.,

$$\text{Hom}_{\mathcal{C}^{\mathcal{A}}}(\mathcal{H}\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{C}^{\mathcal{B}}}(\mathcal{H}, \text{Ran}_{\mathcal{F}} \mathcal{G}).$$

Kan extension are functors in the appropriate functor category.

LEMMA 1.4.1. $\forall \mathcal{G}$, $\text{Lan}_{\mathcal{F}} \mathcal{G}$, $\text{Ran}_{\mathcal{F}} \mathcal{G}$ exist, then,

$$\text{Lan}_{\mathcal{F}} \dashv \circ \mathcal{F} \dashv \text{Ran}_{\mathcal{F}}.$$

The proof of this adjointness with precomposition has already been described, and directly follows from definition of Kan extensions, for this to make sense we only need the fact that $\text{Lan}_{\mathcal{F}}$ and $\text{Ran}_{\mathcal{F}}$ are functors, i.e., $\text{Lan}_{\mathcal{F}} \mathcal{G}$ and $\text{Ran}_{\mathcal{F}} \mathcal{G}$ exist $\forall \mathcal{G}$, and satisfy some composition rules, which it will due to the definition.

If \mathcal{C} is a locally small category, a Kan extension is said to be pointwise Kan extension if it is preserved by representable functors $\text{Hom}_{\mathcal{C}}(C, -)$ for all $C \in \mathcal{C}$. It’s called absolute if any functor $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{D}$ preserves the Kan extension.

LEMMA 1.4.2. If \mathcal{L} is a left adjoints, then,

$$\text{Lan}_{\mathcal{F}}(\mathcal{L} \circ \mathcal{G}) \cong \mathcal{L} \circ \text{Lan}_{\mathcal{F}} \mathcal{G}.$$

PROOF

Suppose \mathcal{G} has a left Kan extension along \mathcal{F} ,

$$\begin{array}{ccccc} & \mathcal{B} & & & \\ \mathcal{F} \nearrow & \uparrow & \searrow & \xrightarrow{\kappa} & \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} & \xleftarrow{\mathcal{L}} & \mathcal{D} \\ & \mathcal{G} & & \mathcal{R} & \end{array}$$

Let $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that's left adjoint, i.e., there exists a functor $\mathcal{R} : \mathcal{D} \rightarrow \mathcal{C}$ to which \mathcal{L} is a left adjoint, then we have,

$$\begin{aligned} \text{Hom}_{\mathcal{D}\mathcal{B}}(\mathcal{L} \text{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{K}) &\cong \text{Hom}_{\mathcal{C}\mathcal{B}}(\text{Lan}_{\mathcal{F}} \mathcal{G}, \mathcal{R}\mathcal{K}) \\ &\cong \text{Hom}_{\mathcal{C}\mathcal{A}}(\mathcal{G}, (\mathcal{R}\mathcal{K})\mathcal{F}) \cong \text{Hom}_{\mathcal{D}\mathcal{A}}(\mathcal{L}\mathcal{G}, \mathcal{K}\mathcal{F}) \end{aligned}$$

So, by definition of Kan extension, we have that $\mathcal{L} \text{Lan}_{\mathcal{F}} \mathcal{G} \cong \text{Lan}_{\mathcal{F}}(\mathcal{L}\mathcal{G})$. \square

What's happening is that left adjoints preserve colimits, and this guarantees the existence of the Kan extension for the composition. So, when the required colimits exist, the notion of closest makes sense. The closest functor to \mathcal{L} is \mathcal{L} , so once we know the Kan extension exists, it must be the one described above.

1.4.1 | KAN EXTENSIONS AS COENDS

Certain additional constraints on the starting categories guarantees the existence of Kan extensions. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{C}$ be two functors.

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow \mathcal{F} & \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} \end{array}$$

THEOREM 1.4.3. (EXISTENCE/COEND FORMULA)

$$\text{Lan}_{\mathcal{F}} \mathcal{G} \cong \int^{A \in \mathcal{A}} \text{Hom}_{\mathcal{B}}(\mathcal{F}A, B) \odot \mathcal{G}A$$

whenever \mathcal{A} is small, \mathcal{B} is locally small, \mathcal{C} is cocomplete.

PROOF

Note that the local smallness of \mathcal{B} , or being enriched by the category of sets is needed for the existence of tensor product \odot , which is needed for the existence of extensions, this condition maybe replaced with an appropriate enriched category with a tensor product. Once they exist, the cocompleteness is needed for the existence of the limit, and to take the limits, we need the starting category to be small. So the conditions we require are to be expected.

The proof is again to find a chain of isomorphisms so we can sneak in [Yoneda principle](#). Firstly, by the end formula for natural transformations, we have,

$$\text{Hom}_{\mathcal{C}\mathcal{A}}(\mathcal{G}, \mathcal{K}\mathcal{F}) \cong \int_{A \in \mathcal{A}} \text{Hom}_{\mathcal{C}}(\mathcal{G}A, \mathcal{K}\mathcal{F}A)$$

Applying [Yoneda](#) to the functor $\mathcal{H} : X \mapsto \text{Hom}_{\mathcal{C}}(\mathcal{G}A, \mathcal{K}X)$, we have,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \text{Hom}_{\mathcal{C}}(\mathcal{G}A, \mathcal{K}\mathcal{F}A) = \mathcal{H}(\mathcal{F}A)$$

Applying the end formula for natural transformations again, we get,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{C}^{\text{op}}}}(h^{\mathcal{F}A}, \mathcal{H}) \cong \int_{B \in \mathcal{B}} \text{Hom}_{\mathbf{Sets}}(h^{\mathcal{F}A}B, \mathcal{H}B)$$

This gives us the following double ‘integral’,

$$\int_{A \in \mathcal{A}} \int_{B \in \mathcal{B}} \text{Hom}_{\mathbf{Sets}} \left(h^{\mathcal{F}A} B, \text{Hom}_{\mathcal{C}} (\mathcal{G}A, \mathcal{K}B) \right)$$

By using the hom-tensor adjointness we get,

$$\begin{aligned} \int_{A \in \mathcal{A}} \int_{B \in \mathcal{B}} \text{Hom}_{\mathbf{Sets}} \left(h^{\mathcal{F}A} B, \text{Hom}_{\mathcal{C}} (\mathcal{G}A, \mathcal{K}B) \right) &\cong \int_{A \in \mathcal{A}} \int_{B \in \mathcal{B}} \text{Hom}_{\mathcal{C}} \left(h^{\mathcal{F}A} B \odot \mathcal{G}A, \mathcal{K}B \right) \\ &\cong \int_{B \in \mathcal{B}} \text{Hom}_{\mathcal{C}} \left(\int_{A \in \mathcal{A}} h^{\mathcal{F}A} B \odot \mathcal{G}A, \mathcal{K}B \right) \\ &\cong \text{Hom}_{\mathcal{C}^{\mathcal{A}}} \left(\int^{A \in \mathcal{A}} \text{Hom}_{\mathcal{B}} (\mathcal{F}A, -) \odot \mathcal{G}A, \mathcal{K} \right) \end{aligned}$$

Here, we used the Fubini rule for ends, and hom-(co)end relations in the first step, and end formula for natural transformations in the last step. Now [Yoneda principle](#) proves the theorem. \square

Using [weighted colimit as coends](#) we get,

$$\text{Lan}_{\mathcal{F}} \mathcal{G}B \cong \int^{A \in \mathcal{A}} \text{Hom}_{\mathcal{B}} (\mathcal{F}A, B) \odot \mathcal{G}A \cong \varinjlim^{\text{Hom}_{\mathcal{B}}(\mathcal{F}-, B)} \mathcal{G}.$$

Assume the conditions of coend formula are satisfied. For each functor $\mathcal{F} \in \mathcal{B}^{\mathcal{A}}$, the left Kan extension along \mathcal{F} is the functor, $\text{Lan}_{\mathcal{F}}(-) : \mathcal{C}^{\mathcal{A}} \rightarrow \mathcal{C}^{\mathcal{B}}$, $\mathcal{G} \mapsto \text{Lan}_{\mathcal{F}} \mathcal{G}$. Intuitively we expect ‘nearest’ composed with ‘nearest’ to be the ‘nearest’. So we will have,

$$\text{Lan}_{\mathcal{F} \circ \mathcal{E}} \mathcal{G} = \text{Lan}_{\mathcal{F}} (\text{Lan}_{\mathcal{E}} \mathcal{G}).$$

The proof is a simple application of the coend formula and some end-coend calculus.

If a Kan extension at each point $B \in \mathcal{B}$ can be written as a limit such as the one above, then it’s always pointwise Kan extension, as Hom-functor preserves limits. Conversely, if $\text{Lan}_{\mathcal{F}} \mathcal{G}$ is pointwise, then for each $C \in \mathcal{C}$ we have, $\text{Lan}_{\mathcal{F}} (h_C \circ \mathcal{G}) \cong h_C \circ \text{Lan}_{\mathcal{F}} \mathcal{G}$.

$$\begin{array}{ccccc} & & \mathcal{B} & & \\ & \nearrow \mathcal{F} & \uparrow \text{Lan}_{\mathcal{F}} \mathcal{G} & \searrow h_C \circ \text{Lan}_{\mathcal{F}} \mathcal{G} & \\ \mathcal{A} & \xrightarrow{\mathcal{G}} & \mathcal{C} & \xrightarrow{h_C} & \mathbf{Sets} \end{array}$$

Applying [Yoneda](#) to the functor $\mathcal{H} := h_C \circ \text{Lan}_{\mathcal{F}} \mathcal{G}$ we get,

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{B}}} (h^B, \mathcal{H}) \cong \mathcal{H}B = \text{Hom}_{\mathcal{C}} (C, \text{Lan}_{\mathcal{F}} \mathcal{G}B)$$

Since \mathbf{Sets} is cocomplete, we can express the Kan extension as a coend formula. Now, by Yoneda principle this means that the Kan extension $\text{Lan}_{\mathcal{F}} \mathcal{G}$ can itself be written as a coend.

1.4.2 | ALL CONCEPTS ARE KAN EXTENSIONS

Any concept that talks about some notion of nearest/closest/best should be expected to be related to Kan extensions. Kan extensions is a very general construction which unites limits, adjoints, and other constructions. Many of the categorical constructions involve limits or adjunctions. Limits correspond to nearest objects to/from a system. Adjoints when viewed as reflections also say that the associated objects are the nearest to the original object when transformed back. So both of these concepts should be expected to be related to Kan extensions.

1.4.2.1 | (Co)LIMITS AS KAN EXTENSIONS

Limits are the nearest object to a system. We can now think of objects in a category \mathcal{A} as functors from a terminal category to the category \mathcal{A} . Here the terminal category is a category with one object $*$ and one morphism, the identity morphism on the object $\mathbb{1}_*$. This category will be denoted by $\mathbb{1}$. Now each object A in \mathcal{A} can be thought of as a functor $\hat{A} : \mathbb{1} \rightarrow \mathcal{A}$. Since we want the ‘nearest’ object to a system, we want a nearest approximation of the system by a functor that corresponds to such an object. This is a Kan extension situation.

The terminal category $\mathbb{1}$ is the unique object in the category of categories, such that there exist only one functor from any category \mathcal{I} to $\mathbb{1}$, which sends everything to the only object in the category, and every morphism to the only morphism in $\mathbb{1}$. Denote this functor by \mathcal{T} . Then we have the following Kan extension problem,

$$\begin{array}{ccc} & & \mathbb{1} \\ & \nearrow \mathcal{T} & \dashrightarrow \mathcal{H} \\ \mathcal{I} & \xrightarrow{\mathcal{F}} & \mathcal{A} \end{array}$$

For each functor $\mathcal{H} : \mathbb{1} \rightarrow \mathcal{A}$ corresponds to an object H in the category \mathcal{A} , and $\mathcal{H} \circ \mathcal{T}$ represents the constant system which maps everything in the indexing category \mathcal{I} to a constant object. The natural transformations from $\mathcal{H} \circ \mathcal{T}$ to \mathcal{F} represent the cones, from the object H to the system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$.

The limit of a system $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$ is the right Kan extension of \mathcal{F} along \mathcal{T} ,

$$\varprojlim \mathcal{F} = \text{Ran}_{\mathcal{T}} \mathcal{F}(*).$$

Similarly the colimit of a system is the object that’s nearest from the system. So, the colimit will be the left Kan extension of the system along \mathcal{T} .

$$\varinjlim \mathcal{F} \cong \text{Lan}_{\mathcal{T}} \mathcal{F}(*).$$

1.4.2.2 | ADJOINTS AS KAN EXTENSIONS

Adjoint functors when viewed as reflections §1.2.1, bring with them some notion of ‘nearness’. So adjoints should be expected to be related to Kan extensions. Consider an adjoint situation $\mathcal{F} \dashv \mathcal{G}$,

$$\mathcal{A} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}} \end{array} \mathcal{B},$$

If we think of objects in \mathcal{A} as having some information, and after performing \mathcal{F} we lose information in the category \mathcal{A} , then the functor \mathcal{G} is such that $\mathcal{G}\mathcal{F}$ is the ‘best’ recovery of the original information about \mathcal{A} . So, $\mathcal{G}\mathcal{F}$ is the ‘nearest’ to the identity functor $\mathbb{1}_{\mathcal{A}}$, ‘along’ \mathcal{F} . This is a natural transformation,

$$\mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}.$$

The nearest means that any other ‘recovery’ will have to factor through this that’s to say, $\cdots \Rightarrow \mathcal{G}\mathcal{F} \Rightarrow \mathbb{1}_{\mathcal{A}}$. Similarly, the left adjoint is the ‘nearest’ from the identity functor on \mathcal{B} , so we have,

$$\mathbb{1}_{\mathcal{A}} \Rightarrow \mathcal{F}\mathcal{G}.$$

So, we get the following Kan extension way of thinking about adjoints.

$$\begin{array}{ccc}
 & \mathcal{B} & \\
 \mathcal{F} \nearrow & \uparrow \mathbb{1}_{\mathcal{A}} & \searrow \text{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}} \cong \mathcal{G} \\
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A}
 \end{array}
 \quad
 \begin{array}{ccc}
 & \mathcal{A} & \\
 \mathcal{G} \nearrow & \downarrow \mathbb{1}_{\mathcal{B}} & \searrow \text{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F} \\
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B}
 \end{array}$$

So, the right adjoint \mathcal{G} is the left Kan extension of $\mathbb{1}_{\mathcal{A}}$ along \mathcal{F} . Similarly, the left adjoint \mathcal{F} is the right Kan extension of $\mathbb{1}_{\mathcal{B}}$ along \mathcal{G} . An adjoint situation can be expressed as the following pair of Kan extensions,

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A} \\
 \mathcal{F} \nearrow & \uparrow \mathbb{1}_{\mathcal{A}} & \searrow \mathcal{G} \\
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A} \\
 \mathcal{F} \nearrow & \uparrow \mathbb{1}_{\mathcal{A}} & \searrow \mathcal{G} \\
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B} \\
 \mathcal{G} \nearrow & \uparrow \mathbb{1}_{\mathcal{B}} & \searrow \mathcal{F} \\
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B} \\
 \mathcal{G} \nearrow & \uparrow \mathbb{1}_{\mathcal{B}} & \searrow \mathcal{F} \\
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A}
 \end{array}$$

such that any other extension $\mathcal{H} : \mathcal{A} \rightarrow \mathcal{B}$ factors through the Kan extension,

$$\dots \Rightarrow \mathcal{H} \circ \mathcal{G} \Rightarrow (\text{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}) \circ \mathcal{G} \Rightarrow \mathbb{1}_{\mathcal{B}}$$

visualised by the diagram,

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B} \\
 \mathcal{G} \nearrow & \uparrow \mathbb{1}_{\mathcal{B}} & \searrow \mathcal{H} \\
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B} \\
 \mathcal{G} \nearrow & \uparrow \mathbb{1}_{\mathcal{B}} & \searrow \mathcal{H} \\
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A}
 \end{array}$$

Similarly, any extension \mathcal{K} of \mathcal{F} along $\mathbb{1}_{\mathcal{A}}$ must factor through the adjoint \mathcal{G} ,

$$\mathbb{1}_{\mathcal{A}} \Rightarrow (\text{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}) \circ \mathcal{F} \Rightarrow \mathcal{K} \circ \mathcal{F} \Rightarrow \dots$$

visualised by the diagram,

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A} \\
 \mathcal{F} \nearrow & \uparrow \mathbb{1}_{\mathcal{A}} & \searrow \mathcal{K} \\
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B}
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\mathbb{1}_{\mathcal{A}}} & \mathcal{A} \\
 \mathcal{F} \nearrow & \uparrow \mathbb{1}_{\mathcal{A}} & \searrow \mathcal{K} \\
 \mathcal{B} & \xrightarrow{\mathbb{1}_{\mathcal{B}}} & \mathcal{B}
 \end{array}$$

LEMMA 1.4.4.

$$\text{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}} \cong \mathcal{F} \dashv \mathcal{G} \cong \text{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}.$$

□

So, \mathcal{G} has a left adjoint if and only if $\text{Ran}_{\mathcal{G}} \mathbb{1}_{\mathcal{B}}$ exists. Similarly, \mathcal{F} has a right adjoint if and only if $\text{Lan}_{\mathcal{F}} \mathbb{1}_{\mathcal{A}}$ exists. The adjoint functor theorems usually are some conditions for the existence of these Kan extensions. When the categories are small, we can express the Kan extension as a coend. So in such cases, adjoints do exist.

SUMMARY

2 | SITES & SHEAVES ON SITES

Grothendieck topology formalizes the intuitive property of locality in spaces. The goal is to model spaces where it is possible to talk about ‘regions’ of the space, and properties of the space that are region dependent. This allows us to formalize all questions involving ‘where’. The goal of this chapter is to follow Grothendieck’s approach to the formalization of locality. Before focusing on Grothendieck topology we will briefly discuss heuristically what we mean by locality and a few alternate paths to formalizing locality. The heuristic will also fix the contents and also provide motivation for the contents of the chapter. The guiding philosophy for the previous chapter was that the properties of mathematical objects can be studied by the objects’ relation to other objects. In our case, the property of interest is that of locality. Since we want to focus on the property that is relevant to us, it suffices to study the relationship with relevant objects.

Suppose a space with locality property can be modeled by a category, \mathcal{L} , we can study the properties of the category \mathcal{L} by its relations to other categories. One approach is to study \mathcal{L} by studying functors to \mathcal{L} , from a certain ‘model category’. As is the case in most classical branches of mathematics, the idea is to use the properties of sets to formalize the property of interest. The starting point in this approach is the category of sets. The idea is that the subsets of a set can be used to model regions of a geometric object. So, the starting point is a collection of subsets of a set. Let $\mathcal{T}_{\mathcal{L}}$ denote the collection of subsets of a set that can be used to model locality on the category \mathcal{L} . The model category in this case is the category with objects given by $\mathcal{T}_{\mathcal{L}}$ and subsets inclusions as the relations between the objects.

This means that there exists a subset in $\mathcal{T}_{\mathcal{L}}$ for each region of the space \mathcal{L} , and since we expect the region relations can be described by subset inclusions, we also have an association for each morphism in $\mathcal{T}_{\mathcal{L}}$ a region relations in \mathcal{L} . This means that we have a functor,

$$\mathcal{T} : \mathcal{T}_{\mathcal{L}} \rightarrow \mathcal{L}.$$

If we assume regions of any space can be described by some special collection of subsets of a set and every relation between regions of the space corresponds to inclusion relation of subsets then \mathcal{T} is equivalence of categories. This is the topological approach to formalising locality.

Grothendieck topologies formalise locality directly, and describe the structure the category \mathcal{L} should have. Categories that can model the intuitive notion of locality are called sites. The properties of spaces that respect locality of the space are called sheaves on the space.

2.1 | LOCALITY: GROTHENDIECK SITES

Grothendieck’s approach to formalising locality is more direct. The starting point is a category expected to have properties that correspond to locality. The category is then required to have additional structures that correspond to our heuristic intuition regarding regions of spaces. This approach is not very different from the traditional approach in mathematics.

This is similar to say, the definition of a vector space where one starts with a set, and requires additional structures on it that allow us to model the intuition we have about straight lines. In the case of Grothendieck topology and sites, the starting point is a category, and we impose additional constraints on this category.

Grothendieck's approach to formalising locality is less 'categorical' than the classical approach to formalising locality as it makes very less use of functor category tools. The problem with the classical approach was the the model category we chose was not good enough to model locality in full generality. The approach via toposes however do use functor category methods, and instead of studying the category itself, we study functors from this category to some relevant category. We will not discuss toposes in this chapter.

2.1.1 | GROTHENDIECK TOPOLOGY

We are interested in describing a category on which allows us to make sense of regions of a space. Heuristically speaking, what we are trying to do is describe a category on which the heuristics of locality and regions of that space can be represented. Let \mathcal{L} be such a category corresponding to a space. The idea now is to model the regions of the space using objects of the category \mathcal{L} . The goal now is to impose conditions on the objects and morphisms of the category \mathcal{L} that correspond to our intuition about regions and relation between regions.

2.1.1.1 | COVERING FAMILIES

Suppose we have two objects corresponding to two regions of the space. The relations between the regions will correspond to some morphism between the corresponding objects.

Heuristically, a cover of a region U should be a collection of regions which together contain U . A cover of a region U consists of a collection of morphisms to U ,

$$\mathcal{U} = \{U_i \rightarrow U\}_{i \in \mathcal{I}}. \quad (\text{covering})$$

We immediately have some requirements for coverings now. Any region U must be its own cover. Hence the one element collection $\{U \subseteq U\}$ consisting of isomorphism of U must be a cover of U . If we had a covering $\{U_i \rightarrow U\}_{i \in \mathcal{I}}$ of U , and for each U_i a covering $\{U_{ij} \rightarrow U_i\}_{j \in \mathcal{I}_i}$, these smaller regions together must cover U . So, in terms of morphisms the collection of compositions,

$$\hat{\mathcal{U}} = \{U_{ij} \rightarrow U\}_{i,j} \quad (\text{locality})$$

must be a covering. The collection of all coverings of U is denoted by $\text{Cov}(U)$.

Now we need formalise the notion of a 'common subregion'. Given two regions, U_i and U_j of \mathcal{C} we are interested in describing the 'common region' categorically. If U is a region containing the subregions U_i and U_j , that is to say if we have two morphisms φ_i, φ_j , into the region U ,

$$U_i \xrightarrow{\varphi_i} U \xleftarrow{\varphi_j} U_j,$$

and for the common region, denoted $U_i \cap_U U_j$, we have two more morphisms from the 'common region', $U_i \cap_U U_j \rightarrow U_i$ and $U_i \cap_U U_j \rightarrow U_j$. So this gives us the following commutative square,

$$\begin{array}{ccc} U_i \cap_U U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow \varphi_i \\ U_j & \xrightarrow{\varphi_j} & U \end{array} \quad (\text{commonality})$$

Since this region is the common region, any other subregions common to both U_i and U_j must be a subregion of this. So, the above square must be the universal square, i.e., $U_i \cap_U U_j$

must be the pullback of the maps $U_i \rightarrow U \leftarrow U_j$. The existence of such pullbacks formalises the heuristic notion of ‘common regions’.

Given a covering $\mathcal{U} = \{U_i \rightarrow U\}_{i \in \mathcal{I}}$ and the inclusion $V \subseteq U$, then we can use the pullback and the cover $\{U_i \rightarrow U\}_{i \in \mathcal{I}}$ to construct a covering of V , by taking pullbacks $V \cap_U U_i$. Using the maps $U_i \xrightarrow{\varphi_i} U \leftarrow V$, we can construct the pullback,

$$\begin{array}{ccc} V \cap_U U_i & \xrightarrow{\hat{\varphi}_i} & V \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{\varphi_i} & U \end{array}.$$

So, the covering of V can be constructed with the morphisms $\hat{\varphi}_i$, is given by,

$$\mathcal{V} = \{U_i \cap V \xrightarrow{\hat{\varphi}_i} V\}_{i \in \mathcal{I}} \quad (\text{base change})$$

A category \mathcal{C} with these properties is called a site. The map, $U \mapsto \text{Cov}(U)$ is called a Grothendieck topology on \mathcal{C} . Topological spaces are a very obvious examples of sites.

The subject of this essay is to make sense of the notion of infinitesimal on sites. Topological spaces are much nicer to work with because they are built out of points, and hence they contain all small limits. This might not be the case for sites.

2.2 | LOCAL PROPERTIES: SHEAVES

There are many ways to study properties of the space \mathcal{C} . The Yoneda-Grothendieck approach to studying mathematical objects is to study the object by its relations to other objects. We can similarly study the space \mathcal{C} by studying functors from the space \mathcal{C} .

In the standard topology approach we study by considering a set, and defining a topology on this set, and putting extra structures on this set is added via sheaves. The notion of a pre-sheaf axiomatizes ‘local nature’ of certain mathematical objects. Given a topological space X , a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X . Denote by CU the set of real-valued continuous functions on an open set U . If $V \subset U$ then f restricted to V is a continuous map, $f|_{UV} : V \rightarrow \mathbb{R}$. The map, $f \mapsto f|_{UV}$ is a function $CU \rightarrow CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_{UV})|_{VW} = f|_{UW}.$$

So, this is functorial in nature, i.e., $|_{VW} \circ |_{UV} = |_{UW}$. This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\text{op}} \rightarrow \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \rightarrow U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\text{op}}$ is the opposite category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\text{op}}$ the functor assigns a morphism, the restriction morphism in \mathbf{Sets} , $\{U \supset V\} \mapsto \{CU \rightarrow CV\}$ given by $f \mapsto f|_{UV}$.

This captures the property of ‘local’ objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}, \quad (\text{pre-sheaf})$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps $U \subset V$ and the corresponding maps in **Sets** are called restriction maps, $|_{UV} : \mathcal{F}U \rightarrow \mathcal{F}V$.

Denote by $\mathbf{PSh}(X)$ the collection of all pre-sheaves over a topological space X . Each pre-sheaf is a functor from $\mathcal{O}(X)^{\text{op}}$ to **Sets** can be considered an object and the natural transformation between the two pre-sheaves as morphism between these objects. So, the category of pre-sheaves is the functor category,

$$\mathbf{PSh}(X) = \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}}$$

So, to understand pre-sheaves, we could use tools for studying functor categories. The goal of the first half of this document will be to develop the tools needed to study functor categories, and hence pre-sheaves.

We now need some way to extend structures defined ‘locally’ to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following ‘collation’ property of continuous functions. Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I -indexed family of functions $(f_i)_{i \in I} \in \prod_i CU_i$, and two maps

$$p(\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and that what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$.

$$CU \xrightarrow{e} \prod_i CU_i \xrightleftharpoons[p]{p} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global.

Sheaves describe information that is local. They come equipped with a notion of where. So and so property holds in so and so region.

such as a differentiable structure in differential geometry. The purpose of a differentiable structure on a topological space is to meaningfully to approximate the differentiable functions by linear maps ‘infinitesimally’.

Alternatively, one can think of the requirement of d

The only topological properties we needed in describing a sheaf on a topological space X , in the introduction were that of intersection and covering. We can isolate these properties, reformulate them categorically. This gives us the notion of a Grothendieck topology, a category with a Grothendieck topology is a sites.

To motivate the definition, consider a topological space X . We use the topology to construct a category $\mathcal{O}(X)$ in which the open sets are the objects, and the morphisms $V \rightarrow U$ are inclusion maps $V \subset U$. So, the hom sets, if non-empty contain only one element, the inclusion map between the open sets. The space X is the final object of the category $\mathcal{O}(X)$.

Let $\{U_i\}$ be a covering of an open set $U \in \mathcal{O}(X)$. This first means that $U_i \subseteq U$, i.e., there exists an inclusion of U_i to U . So, the covering can be thought of as a collection of morphisms,

$$\mathcal{U} = \{U_i \xrightarrow{i} U\}_{i \in I} \quad (\text{covering})$$

For any open set U the trivial inclusion $U \subseteq U$ must be a covering of U ,

$$\mathcal{U} = \{U \subseteq U\} \quad (\text{isomorphism})$$

If we had a covering $\{U_i\}$ of U , and for each U_i a covering $\{U_{ij}\}$, these smaller open sets together must cover U . So, in terms of morphisms, the first covering is the collection of inclusions, $\mathcal{U}_i = \{ij : U_{ij} \rightarrow U_i\}_{j \in \mathcal{I}_i}$, and the second covering is the collection of inclusions $\mathcal{U} = \{i : U_i \rightarrow U\}_{i \in \mathcal{I}}$. We obtain that the composition,

$$\hat{\mathcal{U}} = \{U_{ij} \xrightarrow{ioij} U\}_{(i,j) \in \prod_i i \times \mathcal{I}_i} \quad (\text{locality})$$

must be a covering.

Now to we need generalize the notion of an interesection. Given two open sets, U_i and U_j of $\mathcal{O}(X)$ we are interested in describing the intersection categorically. The open sets U_i and U_j come equipped with an inclusion maps i, j , into the space X ,

$$U_i \xrightarrow{i} X \xleftarrow{j} U_j,$$

and for the intersection, we have two more morphisms from the intersection, $U_i \cap U_j \rightarrow U_i$ and $U_i \cap U_j \rightarrow U_j$. So this gives us the following commutative square,

$$\begin{array}{ccc} U_i \cap U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow i \\ U_j & \xrightarrow{j} & X \end{array} \quad (\text{intersection})$$

Intersection is the largest set, containing elements of both U_i and U_j , this amounts to saying, the above square must be the universal square, i.e., $U_i \cap U_j$ must be the pullback of the maps $U_i \rightarrow X \leftarrow U_j$.

Given a covering $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ and the inclusion $V \subseteq U$, then we can use the intersection and the cover $\{U_i\}_{i \in \mathcal{I}}$ to construct a covering of V , by taking intersections $V \cap U_i$. Using the maps $U_i \xrightarrow{i} U \leftarrow V$, we can construct the pullback,

$$\begin{array}{ccc} V \cap U_i & \xrightarrow{\hat{i}} & V \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{i} & U \end{array} \quad .$$

So, the covering of V can be constructed with the morphisms \hat{i} , is given by,

$$\mathcal{V} = \{U_i \cap V \xrightarrow{\hat{i}} V\}_{i \in \mathcal{I}} \quad (\text{base change})$$

These are easily generalized to any category \mathcal{C} by replacing the inclusion maps in topological spaces with a collection of morphisms in the category.

A Grothendieck topology on a category \mathcal{C} is a set Cov of families of morphisms $\text{Cov}(U)$,

$$U \mapsto \text{Cov}(U)$$

such that the covering of an object U in the category \mathcal{C} , the collection of inclusion maps, in ??, is replaced with a collection of morphisms $\{U_i \rightarrow U\}_{i \in \mathcal{I}}$, the trivial inclusion in ?? is

replaced by an isomorphism of objects. Intersection of open sets, in ?? is replaced by the pullback or fibered product of two maps. An inclusion morphism, $V \rightarrow U$ in ?? is replaced with the collection of pullback morphisms $\{V \prod U_i \rightarrow V\}$.

A category with a Grothendieck topology is called a site. Let $\mathcal{O}(X)$ be a category equipped with a Grothendieck topology \mathcal{T}_X . A pre-sheaf of sets on a site $\mathcal{O}(X)$ is a contravariant functor,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}.$$

For the sake of simplicity we will denote the maps $\mathcal{F}(i)$ by $|_{U_i}$ like in the case of pre-sheaves on topological spaces. The pre-sheaf \mathcal{F} is called separated if given a covering $\{U_i \rightarrow U\}$, and two sections $f, g \in \mathcal{F}U$, whose pullbacks to each $\mathcal{F}U_i$ coincide, it follows that $f = g$. That is for all covers $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(U)$,

$$\forall i, f|_{U_i} = g|_{U_i} \Rightarrow f = g \quad (\text{separated})$$

Some examples of pre-sheaves that are not separated can be found [here](#).

A pre-sheaf is a sheaf if the sections can be patched up. That's to say, given a covering $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(U)$ and a set of elements $f_i \in \mathcal{F}U_i$ which coincide on the 'intersection' i.e., if f_i and f_j coincide on the fibered product,

$$\begin{array}{ccc} U_i \prod_{i,j} U_j & \longrightarrow & U_i \\ \downarrow & & \downarrow i \\ U_j & \xrightarrow{j} & U \end{array}$$

That is to say if $f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j}$, then there must exist unique section $f \in \mathcal{F}U$ such that $f|_{U_i} = f_i$.

$$f_i|_{U_i \prod_{i,j} U_j} = f_j|_{U_i \prod_{i,j} U_j} \Rightarrow \exists f \in \mathcal{F}U, f|_{U_i} = f_i.$$

This can be traslated into saying that there exists the following equalizer.

$$\mathcal{F}U \dashrightarrow \prod_i \mathcal{F}U_i \xrightleftharpoons[q]{p} \prod_{i,j} \mathcal{F}(U_i \prod_{i,j} U_j). \quad (\text{collation})$$

where,

$$p(\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

This is an equalizer similar to the case of sheaves on topological spaces. This is also called gluing or patching. The meaning of separated is that there is at most one way to patch up the pre-sheaf. Here the big \prod s are product of sets, and small $\prod_{i,j}$ s are fibered products in the category $\mathcal{O}(X)$. In the classical case of a topological space we have $U_i \prod_{i,i} U_i = U_i \cap U_i = U_i$, so the two possible pullbacks from $U_i \prod_{i,i} U_i$ coincide; but if the map $U_i \rightarrow U$ is not injective, then the two projections $U_i \prod_{i,i} U_i \rightarrow U_i$ will be different.

2.2.1 | COVERING SIEVES

It's useful to get a functorial way of expressing inclusion of open sets. Subset inclusions are special relation between open sets of a topological space. A functorial description of such relations will be useful for the case of sites. A general category usually contains a lot more relations between objects. These morphisms don't necessarily correspond to subset relation. So, we would like to throw out these extra morphisms. The objects of interest to us are

subfunctors to hom-functor. Given a category \mathcal{C} , and an object $U \in \mathcal{C}$, a sieve on the object U is the subfunctor of the hom-functor, $\text{Hom}_{\mathcal{C}}(-, U)$. The notion of sieve formalizes what is meant by allowed relations.

Given two objects in a category $U, V \in \mathcal{C}$, the allowed morphisms from V to U should be a subset of all morphisms between V and U . If $S^U(V)$ represents the allowed morphisms from V to U , we have,

$$S^U(V) \subseteq \text{Hom}_{\mathcal{C}}(V, U).$$

Let this subset inclusion be denoted by $\kappa_V : S^U(V) \rightarrow \text{Hom}_{\mathcal{C}}(V, U)$. Let $V \subseteq U$ and W be some other open set which is related to V . Then topologically it's irrelevant how W is related to V . The composite relation would still be a subset relation. Let $\coprod S^U$ be the union of all $S^U(V)$, corresponding to all the allowed relations to U .

$$g \in \coprod S^U \Rightarrow g \circ f \in \coprod S^U$$

whenever the composition makes sense. This means that S^U is functorial.

$$\begin{array}{ccccc} W & & S^U W & \xrightarrow{\kappa_W} & h^U(W) \\ \downarrow f & & S^U(f) \downarrow & & \downarrow \circ f \\ V & & S^U V & \xrightarrow{\kappa_V} & h^U(V). \end{array}$$

A sieve S^U on U is a subfunctor of the hom-functor h^U , denoted by $S^U \subseteq h^U$. This means that there is a natural transformation from S^U to h^U whose components are monic.

If W is some other object in \mathcal{C} with a morphism to V , if f represents a subset relation between V and U that is $f \in S^U(V)$ and $g \in h^U(W)$ is any morphism, then we should have, the composition would still yield a subset. Topologically it's irrelevant how W is related to V . So,

2.2.2 | SHEAFIFICATION FUNCTOR

Sheafification introduces extra structure on the pre-sheaf using the structure of the underlying topological space or site. In particular, the covers of any open set forms a category which is cofiltered. Sheafification utilizes this structure to construct a sheaf. This section will be informal, for a more formal discussion, see [?].

The category of sheaves $\text{Sh}(X)$ over a site X is a full subcategory of the category of pre-sheaves $\text{PSh}(X)$ over the site X ,

$$\text{Sh}(X) \hookrightarrow \text{PSh}(X)$$

A sheaf is a contravariant functor, $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}$, together with the [collation](#) property. We have the forgetful functor from sheaves to presheaves that forgets about the patching properties of the sheaf. We can forget about the collation property, and this forgetful functor associates with the sheaf the underlying pre-sheaf. Denote this forgetful functor by,

$$\iota_X : \text{Sh}(X) \rightarrow \text{PSh}(X).$$

The left-adjoint to this functor is the sheafification functor,

$$\text{Sh} : \text{PSh}(X) := \mathbf{Sets}^{\mathcal{O}(X)^{\text{op}}} \rightarrow \text{Sh}(X).$$

We want to generalize sheafification we did for the category of open sets on a topological space to sites. We should be careful, and all the steps in the process should be carefully motivated. Although we will still work withing the category of open sets corresponding to a topological space, we will do it in the more general language. We will then state this construction as a theorem without making things too formal.

2.2.2.1 | SHEAFIFICATION ON TOPOLOGICAL SPACES

ETALE SPACES

2.2.2.2 | FILTRANT (Co)LIMITS

Filtrant categories or filtered categories generalize the notion of a directed set, and hence the category of open sets of a topological space. This allows us to talk about restriction or corestriction on these general categories. Intuitively, a filtrant category should be a category such that for every two objects, there exists a larger object containing both. Largeness is to be expressed in terms of morphisms.

Categorically, this can be formalized by saying for every two objects V and W , there exists an object U together with morphisms from the two objects.

$$V \rightarrow U \leftarrow W. \quad (\text{container})$$

In case of directed sets, the maps described above are inclusion maps. In the categorical counterpart, there can be multiple maps $V \rightrightarrows W$ describing different ways in which W is ‘larger’ than V . Intuitively, we want there to be an object that’s larger in a common way. This means that for any two morphisms $V \rightrightarrows W$ there exists an morphism $h : W \rightarrow U$, such that $h \circ f = h \circ g$. Visualised by the diagram,

$$\begin{array}{ccc} V & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & W \\ & \searrow & \downarrow h \\ & & U \end{array} \quad (\text{directedness})$$

This does not have to be coequaliser. A filtrant category \mathcal{C} is a non-empty category that has [container](#) and [directedness](#) properties. It’s cofiltrant if \mathcal{C}^{op} is filtrant.

The category of open sets of a topological space is an example of a filtrant category. Given any two open sets, there exists a larger open set containing them, since between any two objects, there can only be one map, the [directedness](#) condition is trivially satisfied. The topological space X itself will be the largest open set.

LEMMA 2.2.1. *\mathcal{C} is a small filtrant category iff for any finite diagram, there exists a cocone.*

PROOF

Suppose every finite diagram has a cocone, to show that it is filtrant, we have to consider appropriate finite diagrams show that it is non-empty and satisfies [container](#) and [directedness](#). For non-emptiness, can be shown by choosing the empty diagram. For [container](#), we can consider a discrete diagram, which consists of two objects V, W , with no morphisms between them, in such a case, cocone corresponds to an object U such that, there are morphisms, $V \rightarrow U$ and $W \rightarrow U$. So, we get that, $V \rightarrow U \leftarrow W$. Similarly, for [directedness](#) consider

the diagram, $V \rightrightarrows W$, it should have a cocone with vertex U , is precisely the [directedness](#) condition.

Conversely, if the small category \mathcal{C} is filtering, we have to show for every finite diagram there exists a cocone. The proof is by induction on the number of arrows, the [container](#) condition gives for each two objects, a larger object, and [directedness](#) guarantees a common larger object for any finite collection of objects. If there are no arrows at all, then the diagram consists of a discrete collection of objects $\{V_i\}_{i \in N}$, and repeated application of [container](#) gives us a cocone.

Suppose $\{V_i\}_{i \in N}$ is a finite system, with $f_{ij} : V_i \rightarrow V_j$ being the arrows in the system, repeated application of [container](#) gives us a large object W , with arrows $g_i : V_i \rightarrow W$, however they might not respect compositions, that is,

$$g_i \neq g_j \circ f_{ij}$$

So, the larger object need not form a cocone. In this case, we can find a larger object $g : W \rightarrow U$ by [directedness](#), such that the composition $g \circ g_i = g \circ g_j \circ f_{ij}$. This will satisfy the necessary cocone condition. \square

This means that if \mathcal{C} is a small filtrant category, for any finite collection of objects there exist an object that's ‘larger’ all the objects in the collection. If a category has a final object it will always be filtrant.

2.2.2.3 | SHEAFIFICATION ON SITES: THE $+$ FUNCTOR

The problem with pre-sheaves is that they might not be separated and even if they are separated, might not patch up, so the first step is to divide up the covers into smaller and smaller refinements, separate it out and manually patch it up. This is the same as bundling up stalks like we did for the case of topological spaces.

SEPARATION

Let $U \in \mathcal{O}(X)$, and

$$\mathcal{U} = \{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(U)$$

be a covering of U . By definition of a pre-sheaf we have the association, $U_i \mapsto \mathcal{F}U_i$. The set,

$$\text{Eq}(\mathcal{U}, \mathcal{F}) = \left\{ (f_i)_{i \in I} \in \prod_i \mathcal{F}U_i \mid f_i|_{U_i \cap_{i,j} U_j} = f_j|_{U_i \cap_{i,j} U_j} \right\},$$

consists of all tuples of ‘functions’ that can be patched up whenever there is an intersection, i.e., they must agree on the ‘intersection’ $U_i \cap_{i,j} U_j$. So we are throwing out all the functions that cannot be patched up. But now it might happen that things are patchable on refinements but not on bigger open sets. So we must refine.

To motivate refinement, consider the example of a topological space, X . Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of an open set $U \subset X$. A refinement of $\mathcal{V} = \{V_j\}_{j \in J}$ of the open cover \mathcal{U} is a set of open subsets $V_j \subset X$ which is itself an open cover, and is such that each for each $j \in J$, there exists an $i \in I$ such that $V_j \subset U_i$. Hence there exists a map $f : J \rightarrow I$ such that,

$$\begin{array}{ccc} V_j & \xrightarrow{f} & U_{f(j)} \\ & \searrow & \swarrow \\ & U & \end{array} .$$

Now, by going to refinements, the intersections are smaller, the elements of the pre-sheaves have ‘less space’ to disagree on. So, the refinements give us a map,

$$\mathrm{Eq}(\mathcal{U}, \mathcal{F}) \rightarrow \mathrm{Eq}(\mathcal{V}, \mathcal{F}).$$

By taking limit over such refinements, we include the ‘essense’ of all functions in the pre-sheaf \mathcal{F} similar to how in case of topological space, the stalks of the pre-sheaf was the same as the stalks of the sheafification. So if $\mathrm{Cov}(U)$ is the category of all coverings of U , with refinements as morphisms, then we have a functor,

$$\begin{aligned} \mathrm{Eq}_{\mathcal{F}}U : \mathrm{Cov}(U)^{\mathrm{op}} &\rightarrow \mathbf{Sets} \\ \mathcal{U} &\mapsto \mathrm{Eq}(\mathcal{U}, \mathcal{F}). \end{aligned}$$

Since the category of pre-sheaves \mathbf{Sets} has limits, we can take the limit of this functor. Now using this, we can associate to each pre-sheaf \mathcal{F} , a new pre-sheaf with,

$$\mathcal{F}^+U := \varinjlim_{\mathrm{Cov}(U)} \mathrm{Eq}_{\mathcal{F}}U$$

The construction so far can be stated as follows,

LEMMA 2.2.2. *If \mathcal{F} is a pre-sheaf, then \mathcal{F}^+ is a separated pre-sheaf.* □

Note here that if \mathcal{F} is a sheaf then the equalisers already exist, i.e., we don’t have to remove elements or take restrictions of elements that don’t patch up. So in such a case $\mathcal{F}^+ = \mathcal{F}$.

PATCH-UP

Once we have a separated pre-sheaf, it means that there is only one way it can be patched up. Or the patch up is unique. So now we have to patch up \mathcal{F}^+ .

use the functor \mathcal{F} to construct a functor on the category of covers of U for any $U \subseteq X$. Firstly we already have for each $U_i \in \text{Cov}(U)$ the association $\mathcal{F}U_i$ coming from the functor $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}$.

So, for every inclusion $U_i \subseteq U_j$ we have to construct a new restriction map,

THEOREM 2.2.3. ι_X admits a left-adjoint $^{\text{Sh}}$, i.e.,

$$\text{Hom}_{\text{PSh}(X)}(\mathcal{F}, \iota_X \mathcal{G}) \cong \text{Hom}_{\text{Sh}(X)}(\mathcal{F}^{\text{Sh}}, \mathcal{G}).$$

¹

Note that we didn't use any properties of the category \mathcal{A} here, we only used the properties of the initial category to add structure.

2.3 | TOPOSES IN GEOMETRY

Heuristically speaking, the goal of topos theory is to capture the information about locality on topological spaces, via local objects on such spaces. This is a standard theme in Grothendieck's philosophy to mathematics. Locality allows us to describe the notion of where, such as where two functions are equal. The idea is that the notion of locality on spaces can be studied in terms of the local objects on such spaces. That is to say toposes describes the notion of locality by axiomatising the properties carried by the local objects on spaces that have a notion of locality. Although local objects corresponds to pre-sheaves, they don't patch up on the whole space, or more heuristically speaking, they don't hold the space together. So, the appropriate category that takes into account the expected properties of locality is the category of sheaves. Note here that the target category is very important. It's only possible to represent the notion of locality if the target category can describe such an information. The safest category to use for this case is the category of sets.

The category with the local nature is a site. Instead of staying in this category, the goal is to study it via some special functor categories. Although toposes are also of interest in foundations of mathematics and logic, we are not concerned about those topics here.

¹When we are working with general sites, the cover $\text{Cov}(U)$ is cofiltered i.e., has a notion similar to directed set, and hence the adjoint functor, sheafification is possible. We will not discuss the filtered category or sheafification for general sites here.

3 | ABELIAN SHEAVES & THE SIX FUNCTOR FORMALISM

Many of the properties of the category of pre-sheaves depend on the target category, to say this more abstractly, the properties of the functor category $\mathcal{A}^{\mathcal{B}}$ are very closely related to the properties of the category \mathcal{A} . The goal of this chapter is to discuss abelian categories, which have some of the nice properties that allow us to do some of the operations that can be performed in the category **Sets** of sets. Many of the interesting pre-sheaves take values in such abelian categories.

3.1 | ABELIAN CATEGORIES

The abelian categories are the categories in which it makes sense to talk about exactness of sequences of morphisms. Homological algebra is the study of obstruction to the exactness of sequences. The primary tools for the study of such obstruction is the theory of derived categories and derived functors.

3.1.1 | REPRESENTABLE CONSTRUCTIONS

The Yoneda-Grothendieck philosophy now has a precise formulation; the properties of a category can be thought of as representability properties in functor category. A contravariant functor $\mathcal{F} : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \quad (\text{representable})$$

in such a case, \mathcal{F} is said to be represented by the object X . We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \rightarrow \mathbf{Sets}$, this will be

$$\mathcal{G} \cong h_X$$

where \cong stands for natural isomorphism. We will now see this in the following examples.

3.1.1.1 | (Co)PRODUCTS

Let \mathcal{A} be a category and consider a family $\{X_i\}_{i \in I}$ of objects of \mathcal{A} indexed by a set I , then we can consider the contravariant functor,

$$\mathcal{G} : Y \mapsto \prod_{i \in I} \text{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $\mathcal{G}(Y) = \text{Hom}_{\mathcal{A}}(Y, P)$. This

is called the product, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

$$\text{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i) = \prod_{i \in I} \text{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \rightarrow X_i$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{array}{ccccc} & & Y & & \\ & f_i \swarrow & \downarrow \exists! h & \searrow f_j & \\ X_i & \xleftarrow{\pi_i} & \prod_{i \in I} X_i & \xrightarrow{\pi_j} & X_j \end{array}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I . So, the property that the category \mathcal{A} has products, is translated into a statement that certain functor is representable.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i \in I} \text{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $\mathcal{F}(Y) = \text{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\prod_{i \in I} X_i$ and by definition we have,

$$\text{Hom}_{\mathcal{A}}(\prod_{i \in I} X_i, Y) = \prod_{i \in I} \text{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \rightarrow Y$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{array}{ccccc} X_j & \xrightarrow{\epsilon_j} & \prod_{i \in I} X_i & \xleftarrow{\epsilon_i} & X_i \\ & f_j \searrow & \downarrow h & \swarrow f_i & \\ & & Y & & \end{array}$$

In algebra, for modules, etc. the coproduct is denoted by \oplus , and is called direct sum. It follows directly from definition that,

$$\text{Hom}_{\mathcal{A}}(\prod_{i \in I} X_i, Y) = \text{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

The categorical notions of product and coproduct correspond to the arithmetic operations such as multiplication and addition. We can similarly talk about exponentiation. In the category of sets, **Sets**, for $X, Z \in \mathcal{A}$, Z^X is the function set consisting of all functions $h : X \rightarrow Z$. Here we have the bijection,

$$\text{Hom}_{\mathbf{Sets}}(Y \times X, Z) \rightarrow \text{Hom}_{\mathbf{Sets}}(Y, Z^X).$$

for a function, $f : Y \times X \rightarrow Z$, this map sends each $y \in Y$ to the function $f(y, -) \in Z^X$. Conversely given a function $f' : Y \rightarrow Z^X$, we can define a map $f(y, x) = f'(y)(x)$. So,

$$\text{Hom}_{\mathbf{Sets}}(Y \times X, Z) \cong \text{Hom}_{\mathbf{Sets}}(Y, Z^X)$$

or equivalently, $(-)^X$ is the right adjoint of $(-) \times X$. By setting $Y = 1$, we obtain,

$$Z^X \cong \text{Hom}_{\mathbf{Sets}}(1, Z^X) \cong \text{Hom}_{\mathbf{Sets}}(X, Z).$$

Exponentiation can be representably defined in terms of products. Suppose \mathcal{A} has products, it has exponentiation if for every Y, Z , the functor,

$$X \mapsto \text{Hom}_{\mathcal{A}}(Y \times X, Z)$$

is representable.

3.1.1.2 | (Co)KERNEL

For sets, the kernel of two maps s, t is defined as the set $\ker(s, t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker(\text{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \text{Hom}_{\mathcal{A}}(X, Z)).$$

This is a covariant functor from the category \mathcal{A} to \mathbf{Sets} . Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g .

This isomorphism can be translated as follows, given an object X and morphisms $i : X \rightarrow Y$ and $j : X \rightarrow Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{array}{ccccc} X & & & & \\ \downarrow \exists! & \searrow i & & \searrow j & \\ \ker(f, g) & \xrightarrow{e} & Y & \xrightleftharpoons[g]{f} & Z \end{array}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e., an object that's both initial and terminal. An object Z is called a zero object if for any object A , there exists a unique morphism $Z \rightarrow A$ and a unique morphism $A \rightarrow Z$. It's unique upto isomorphism and denoted by 0 . Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \rightarrow 0 \rightarrow B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \rightarrow \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \rightarrow Z$ is a morphism $\iota : \ker(f) \rightarrow Y$ such that $f \circ \iota = 0_{\ker(f), Z}$ and any other morphism $i : X \rightarrow Y$ with $f \circ i = 0_{X, Z}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{array}{ccccc} X & & & & \\ \downarrow e & \searrow i & & & \\ \ker(f) & \xrightarrow{\iota} & Y & \xrightarrow{f} & Z \end{array}$$

Here we have not written the zero morphism from X to Z . Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\text{Hom}_{\mathcal{A}}(Y, X) \rightarrow \text{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker(\text{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \text{Hom}_{\mathcal{A}}(Z, X)).$$

This can be visualized by the diagram,

$$\begin{array}{ccccc}
 Y & \xrightarrow{f} & Z & \xrightarrow{\iota} & \text{coker}(f, g) \\
 & \searrow g & & \searrow k & \downarrow e \\
 & & & & X \\
 & & \searrow l & & \\
 & & & &
 \end{array}$$

The cokernel of a morphism f is a morphism $\iota : X \rightarrow \text{coker}(f)$ with $\iota \circ f = 0_{A, \text{coker}(f)}$, and for any morphism $k : B \rightarrow L$ with $k \circ f = 0_{A, L}$ will factor uniquely through $\text{coker}(f)$.

$$\begin{array}{ccccc}
 Y & \xrightarrow{f} & Z & \xrightarrow{\iota} & \text{coker}(f) \\
 & & & \searrow k & \downarrow e \\
 & & & & X
 \end{array}$$

3.1.1.3 | PULLBACK OR FIBERED PRODUCT

Let \mathcal{I} be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \rightarrow Z$ then for functors $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$, pullback $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$\mathcal{F}Y \xrightarrow{f} \mathcal{F}X \xleftarrow{g} \mathcal{F}Z$$

in a category \mathcal{A} is the commutative square with vertex $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ such that any other commutative square factors through it, i.e.,

$$\begin{array}{ccccc}
 P & & & & \\
 \swarrow \text{---} \exists! & & f_P & & \\
 & \mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z & \dashrightarrow & \mathcal{F}Z & \\
 \searrow g_P & \downarrow & & \downarrow g & \\
 & \mathcal{F}Y & \xrightarrow{f} & \mathcal{F}X &
 \end{array}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $\mathcal{G} : \mathcal{I}^{\text{op}} \rightarrow \mathcal{A}$ as above,

$$\begin{array}{ccccc}
 \mathcal{F}X & \xrightarrow{f} & \mathcal{F}Z & & \\
 g \downarrow & & \downarrow & \searrow f_P & \\
 \mathcal{F}Y & \dashrightarrow & \mathcal{F}Y \amalg_{\mathcal{F}X} \mathcal{F}Z & & \\
 & \searrow g_P & & \searrow \exists! & \\
 & & & & C
 \end{array}$$

3.2 | CATEGORY OF COMPLEXES

Given an additive category \mathcal{A} , a differential object in \mathcal{A} is a sequence,

$$A^\bullet \equiv \dots \xrightarrow{\partial_{i-2}^A} A^{i-1} \xrightarrow{\partial_{i-1}^A} A^i \xrightarrow{\partial_i^A} A^{i+1} \xrightarrow{\partial_{i+1}^A} \dots$$

where each $A^i \in \mathcal{A}$ and homomorphisms $\partial_i : A^i \rightarrow A^{i+1}$ are morphisms in the category \mathcal{A} . The morphisms between two such differential objects A^\bullet and B^\bullet , $A^\bullet \xrightarrow{u} B^\bullet$ consists of morphisms $u^i : A^i \rightarrow B^i$ such that the following commutative diagram commutes,

$$\begin{array}{ccccccc} A^\bullet & & \dots & \xrightarrow{\partial_{i-2}^A} & A^{i-1} & \xrightarrow{\partial_{i-1}^A} & A^i & \xrightarrow{\partial_i^A} & A^{i+1} & \xrightarrow{\partial_{i+1}^A} & \dots \\ \downarrow u & & & & \downarrow u^{i-1} & & \downarrow u^i & & \downarrow u^{i+1} & & \\ B^\bullet & & \dots & \xrightarrow{\partial_{i-2}^B} & B^{i-1} & \xrightarrow{\partial_{i-1}^B} & B^i & \xrightarrow{\partial_i^B} & B^{i+1} & \xrightarrow{\partial_{i+1}^B} & \dots \end{array}$$

The set of morphisms $A^\bullet \rightarrow B^\bullet$ is denoted by $\text{Hom}(A^\bullet, B^\bullet)$. The category with differential objects as objects and $\text{Hom}(A^\bullet, B^\bullet)$ is the functor category $\mathcal{A}^{\mathbb{Z}}$.

3.2.0.1 | COHOMOLOGY FUNCTORS

A sequence of morphisms in \mathcal{A} ,

$$A^\bullet \equiv \dots \xrightarrow{\partial_{i-2}^A} A^{i-1} \xrightarrow{\partial_{i-1}^A} A^i \xrightarrow{\partial_i^A} A^{i+1} \xrightarrow{\partial_{i+1}^A} \dots$$

is called complex if for all i ,

$$\partial_i^A \circ \partial_{i-1}^A = 0,$$

The category of complexes over an abelian category \mathcal{A} is an abelian subcategory of $\mathcal{A}^{\mathbb{Z}}$. Denoted by $\mathcal{C}(\mathcal{A})$. Since \mathcal{A} is an abelian category every morphism $A^{i-1} \rightarrow A^i$ splits as,

$$\ker \partial^{i-1} \longrightarrow A^{i-1} \longrightarrow \Im \partial^{i-1} \longrightarrow A^i \longrightarrow \text{coker } \partial^{i-1}$$

The sequence of morphisms can then be written as,

$$\begin{array}{ccccccc} & & \ker \partial_{i-1}^A & & \ker \partial_i^A & & \ker \partial_{i+1}^A \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & A^{i-1} & \longrightarrow & \Im \partial_{i-1}^A & \longrightarrow & A^i & \longrightarrow & \Im \partial_i^A & \longrightarrow & A_{i+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \text{coker } \partial_{i-1}^A & & \text{coker } \partial_{i-1}^A & & \text{coker } \partial_i^A & & \text{coker } \partial_i^A & & \text{coker } \partial_{i+1}^A \end{array}$$

Then by the universal property of \ker and coker , we get the following connections,

$$\begin{array}{ccccccc} & & \ker \partial_{i-1}^A & & \ker \partial_i^A & & \ker \partial_{i+1}^A \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & A^{i-1} & \longrightarrow & \Im \partial_{i-1}^A & \longrightarrow & A^i & \longrightarrow & \Im \partial_i^A & \longrightarrow & A_{i+1} & \longrightarrow & \dots \quad (\text{split}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \text{coker } \partial_{i-1}^A & & \text{coker } \partial_{i-1}^A & & \text{coker } \partial_i^A & & \text{coker } \partial_i^A & & \text{coker } \partial_{i+1}^A \end{array}$$

(Dashed arrows: f_i from $\ker \partial_{i-1}^A$ to $\Im \partial_{i-1}^A$, c_i from $\text{coker } \partial_{i-1}^A$ to $\Im \partial_i^A$, and c_i from $\text{coker } \partial_i^A$ to $\Im \partial_{i+1}^A$)

The complex A^\bullet is called exact if the morphisms are such that,

$$\text{Im } \partial_{i-1}^A \cong \ker \partial_i^A.$$

Given a complex A^\bullet , the cohomology is the ‘objects’ by which sequence fails to be exact. If the complex A^\bullet is not exact at the node A_i , then the $\ker \partial_i^A$ is ‘larger’ than $\text{Im } \partial_{i-1}^A$. The

extra stuff in $\ker \partial_i^A$ that's not in $\Im \partial_{i-1}^A$ is the cokernel of the morphism f_i . So, we have by universal property of kernels, there exists a morphism,

$$\text{Im } \partial_{i-1}^A \xrightarrow{f_i} \ker \partial_i^A \longrightarrow \text{coker } f_i := \overline{H}^i(A^\bullet).$$

The extra stuff in $\ker \partial_i^A$ that's not in the image of ∂_{i-1}^A is also the kernel of $\text{coker } \partial_{i-1}^A$. By universal property of cokernels, there exists a morphism,

$$\underline{H}^i(A^\bullet) = \ker c_i \longrightarrow \text{coker } \partial_{i-1}^A \xrightarrow{c_i} \Im \partial_i^A.$$

This is the object we want and it's this part which is making the complex not exact.

$$\begin{array}{ccccccc} & & \ker \partial_{i-1}^A & & \ker \partial_i^A & \dashrightarrow & \overline{H}^i(A^\bullet) & & \ker \partial_{i+1}^A & & \\ & \swarrow & \downarrow & \swarrow f_i & \downarrow & & & \swarrow & \downarrow & & \\ \dots & \longrightarrow & A^{i-1} & \longrightarrow & \Im \partial_{i-1}^A & \longrightarrow & A^i & \longrightarrow & \Im \partial_i^A & \longrightarrow & A_{i+1} & \longrightarrow & \dots \\ & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ & & \text{coker } \partial_{i-1}^A & \xrightarrow{\underline{H}^i(A^\bullet)} & \text{coker } \partial_{i-1}^A & \xrightarrow{c_i} & \text{coker } \partial_i^A & & \text{coker } \partial_i^A & & \end{array}$$

In this diagram, the sequences,

$$\ker \partial_i^A \rightarrow A^i \rightarrow \text{coker } \partial_{i-1}^A$$

need not be exact. It's this non-exactness we are trying to capture. Let the composite morphism $\ker \partial_i^A \rightarrow A^i \rightarrow \text{coker } \partial_{i-1}^A$ be h^i , the cohomology can be defined as the image of the morphism h^i ,

$$\ker \partial_i^A \rightarrow H^i(A^\bullet) := \Im h^i \rightarrow \text{coker } \partial_{i-1}^A$$

So, $H^i(A^\bullet)$ is the replacement of A^i in [split](#) that turns it into an exact sequence. The three definitions are equivalent, see [4] for the proof where they give two more equivalent definitions.

So, $\Im h_i = \ker c_i = \text{coker } f_i$.

So, $H^i(A^\bullet)$ is well defined and we get an exact sequence of morphisms,

$$\begin{array}{ccccc} & & \ker \partial_i^A & & \\ & \swarrow f_i & \downarrow & \swarrow & \\ \text{Im } \partial_{i-1}^A & & H^i(A^\bullet) & & \Im \partial_i^A \\ & & \downarrow & \swarrow c_i & \\ & & \text{coker } \partial_{i-1}^A & & \end{array}$$

The new exact sequence we have is,

$$\begin{array}{ccccccc} & & \ker \partial_i^A & & \ker \partial_i^A & & \ker \partial_{i+1}^A & & \\ & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\ \dots & \longrightarrow & H^{i-1}(A^\bullet) & \longrightarrow & \text{Im } \partial_{i-1}^A & \longrightarrow & H^i(A^\bullet) & \longrightarrow & \Im \partial_i^A & \longrightarrow & H^{i+1}(A^\bullet) & \longrightarrow & \dots \\ & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \\ & & \text{coker } \partial_{i-1}^A & \xrightarrow{\quad} & \text{coker } \partial_{i-1}^A & \xrightarrow{\quad} & \text{coker } \partial_i^A & & \text{coker } \partial_i^A & & \end{array}$$

By replacing the A^i s with $H^i(A^\bullet)$ we get an exact sequence. This gives us a collection of functors from $\mathcal{C}(\mathcal{A})$ to \mathcal{A} , called the cohomology functors,

$$H^i : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}, \quad H^i(A^\bullet) = \ker \partial_i^A / \text{Im } \partial_{i-1}^A.$$

A morphism of complexes,

$$\begin{array}{ccccccc} A^\bullet & & \cdots & \xrightarrow{\partial_{i-2}^A} & A^{i-1} & \xrightarrow{\partial_{i-1}^A} & A^i & \xrightarrow{\partial_i^A} & A^{i+1} & \xrightarrow{\partial_{i+1}^A} & \cdots \\ \downarrow f & & & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\ B^\bullet & & \cdots & \xrightarrow{\partial_{i-2}^B} & B^{i-1} & \xrightarrow{\partial_{i-1}^B} & B^i & \xrightarrow{\partial_i^B} & B^{i+1} & \xrightarrow{\partial_{i+1}^B} & \cdots \end{array}$$

is called a quasi-isomorphism (\cong_Q) if the induced morphism at cohomology,

$$H^i(f) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$$

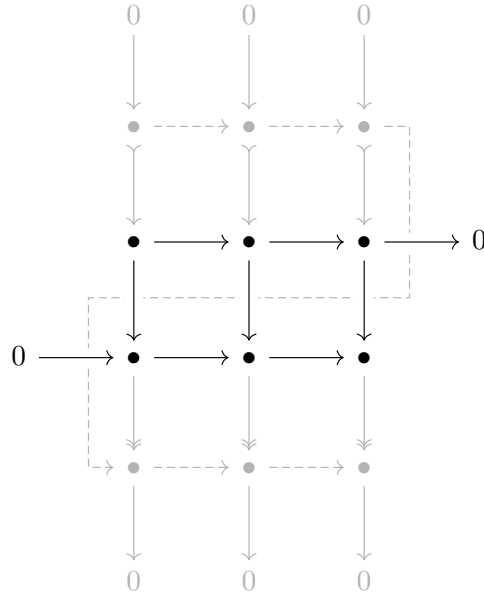
is an isomorphism for all i . We want to treat quasi-isomorphic objects as isomorphic objects. Quasi-isomorphisms manipulate the complexes so that the obstruction to exactness is preserved. So the collection of quasi-isomorphic complexes have same obstructions and could be treated to be the same. For the sake of simplicity, I will stop writing the bullets for the remainder of this note.

3.3 | DIAGRAM CHASING

Diagram chases usually lead to simple connection between distant objects in a complex. In this chapter we discuss some of the diagram chase lemmas. The main goal here is to prove the Salamander lemma due to [7], which provides a more visual understanding of diagram chases and the standard diagram chase lemmas become corollaries.

3.3.1 | SALAMANDER LEMMA

3.3.1.1 | SNAKE LEMMA



THEOREM 3.3.1. (SNAKE LEMMA) *\mathcal{A} abelian, if in ??, the solid exact rows and columns are exact then there exists a sequence of dashed arrows which is exact.*

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \ker f_{i-1} & \dashrightarrow & \ker f_i & \dashrightarrow & \ker f_{i+1} & \dashrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & A^{i-1} & \xrightarrow{\partial_A^{i-1}} & A^i & \xrightarrow{\partial_A^i} & A^{i+1} & \xrightarrow{\quad} 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & f_{i-1} & \dashrightarrow & f_i & \dashrightarrow & f_{i+1} & \dashrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & B^{i-1} & \xrightarrow{\partial_B^{i-1}} & B^i & \xrightarrow{\partial_B^i} & B^{i+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{coker } f_{i-1} & \longrightarrow & \text{coker } f_i & \longrightarrow & \text{coker } f_{i+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The starting point is a double complex in an abelian category \mathcal{A} ,

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \vdots & & \vdots & & \vdots &
 \end{array}$$

where each \bullet is an object in \mathcal{A} . Focusing on what happens around an object A in this double complex, consider the following morphisms to and from the object,

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
 \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \longrightarrow & \bullet & \longrightarrow & A & \longrightarrow & \bullet & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow &
 \end{array}$$

3.3.2 | COROLLARIES OF SALAMANDER LEMMA

LEMMA 3.3.2. (FIVE LEMMA) *Consider two exact sequences A^\bullet and B^\bullet ,*

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow & A^{i+2} & \longrightarrow & A^{i+3} & \longrightarrow & \dots \\
 & & \downarrow u_i & & \downarrow u_{i+1} & & \downarrow u_{i+2} & & \downarrow u_{i+3} & & \\
 \dots & \longrightarrow & B^i & \longrightarrow & B^{i+1} & \longrightarrow & B^{i+2} & \longrightarrow & B^{i+3} & \longrightarrow & \dots
 \end{array}$$

3.4 | INDIZATION OF CATEGORIES

An ind-object in a category \mathcal{A} is a functor $\in \mathbf{Sets}^{\mathcal{A}}$ which is a limit of a system

3.4.1 | IND OBJECTS

3.5 | LOCALIZATION OF CATEGORIES

Localization is the process of adding all the formal inverses to a collection of morphisms. Note that the objects remain the same, but there will be more relation between these objects. Let \mathcal{Q} denote a collection of morphisms in \mathcal{C} , the aim of localization is to construct a new category $\mathcal{C}[\mathcal{Q}^{-1}]$ and a functor,

$$\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{Q}^{-1}]$$

which sends morphisms belonging to \mathcal{Q} to isomorphisms in $\mathcal{C}[\mathcal{Q}^{-1}]$. This construction being ‘universal’ i.e., for any other category \mathcal{D} with a functor $\widehat{\mathcal{L}}_{\mathcal{Q}} : \mathcal{C} \rightarrow \mathcal{D}$, which sends morphisms in \mathcal{Q} to isomorphisms gets factored through $\mathcal{L}_{\mathcal{Q}}$, that’s to say there exists a functor

$$\mathcal{H} : \mathcal{C}[\mathcal{Q}^{-1}] \rightarrow \mathcal{D}$$

such that $\widehat{\mathcal{L}}_{\mathcal{Q}} = \mathcal{H} \circ \mathcal{L}_{\mathcal{Q}}$, so we have the following commutative diagram of functors,

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\widehat{\mathcal{L}}_{\mathcal{Q}}} & \mathcal{D} \\
 \mathcal{L}_{\mathcal{Q}} \downarrow & \nearrow \mathcal{H} & \\
 \mathcal{C}[\mathcal{Q}^{-1}] & &
 \end{array}$$

The extra things \mathcal{D} has that’s not already in $\mathcal{C}[\mathcal{Q}^{-1}]$ should not related to the localization process, i.e adding inverses to the morphisms in \mathcal{Q} . This can be formalized by saying precomposition of functors with $\mathcal{L}_{\mathcal{Q}}$ is an isomorphism of the respective natural transformations. If we have two functors $\mathcal{F}, \mathcal{G} : \mathcal{C}[\mathcal{Q}^{-1}] \rightarrow \mathcal{D}$, then,

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}[\mathcal{Q}^{-1}]}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\mathcal{F} \circ \mathcal{L}_{\mathcal{Q}}, \mathcal{G} \circ \mathcal{L}_{\mathcal{Q}}).$$

Or equivalently, the functor,

$$\circ \mathcal{L}_{\mathcal{Q}} : \mathcal{C}[\mathcal{Q}^{-1}]^{\mathcal{D}} \rightarrow \mathcal{C}^{\mathcal{D}}.$$

is fully faithful. $\mathcal{C}[\mathcal{Q}^{-1}]$ together with the functor $\mathcal{L}_{\mathcal{Q}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{Q}^{-1}]$ is called the localization of \mathcal{C} with \mathcal{Q} .

Localization, if it exists is unique upto equivalence of categories. The problem now is to understand what constraint on \mathcal{Q} that guarantees the existence of localization of \mathcal{C} with respect to \mathcal{Q} . Every category \mathcal{D} that has the needed inverses to \mathcal{Q} will be such that, there exists a functor from the localization $\mathcal{C}[\mathcal{Q}^{-1}]$ to it,

$$\mathcal{C}[\mathcal{Q}^{-1}] \xrightarrow{\mathcal{H}} \mathcal{D}$$

So the localization is the ‘left most’ category that satisfies certain properties, contains formal inverses. We should expect some colimit type thing happening here in the category of categories.

SKETCH OF CONSTRUCTION

The goal is to add the ‘inverses’ and turn it into a category. Let \mathcal{Q}^{-1} be the set in \mathcal{C}^{op} corresponding to the collection of morphisms \mathcal{Q} . Note here that we are assuming the hom-sets are small sets. The new category should include these extra morphisms.

Consider first the graph with objects of \mathcal{C} as vertices, and the arrows of the graph consists of morphisms in \mathcal{C} together with the collection \mathcal{Q}^{-1} . So, the new collection of morphisms is given by, $\text{Hom}_{\mathcal{C}} \amalg \mathcal{Q}^{-1}$. With concatenation as composition this is a category, denoted by $\mathcal{FC}[\mathcal{Q}^{-1}]$. The identity morphisms given by the empty path from and to the same vertex. To turn this into a category we need we have to define equivalences that make the compositions $f \circ f^{-1}$ into identities for all $f \in \mathcal{Q}$.

So the localization of \mathcal{C} with \mathcal{Q} is,

$$\mathcal{C}[\mathcal{Q}^{-1}] := \mathcal{FC}(\mathcal{Q}^{-1}) / \sim$$

where the quotient consists of same objects and morphisms are quotiented by the above equivalence. This quotient is the colimit we needed. \square

For \mathcal{Q} with special properties more direct formulas for the hom-sets of $\mathcal{C}[\mathcal{Q}^{-1}]$ can be obtained.

3.6 | DERIVED CATEGORIES

The goal of cohomology is to measure the obstruction to exactness. So, if a morphism of complexes keeps the obstruction to exactness the same, then we have not lost the information about the obstruction by going in between such complexes.

A morphism of complexes,

$$\begin{array}{ccccccc}
 A^\bullet & & \dots & \xrightarrow{\partial_{i-2}^A} & A^{i-1} & \xrightarrow{\partial_{i-1}^A} & A^i & \xrightarrow{\partial_i^A} & A^{i+1} & \xrightarrow{\partial_{i+1}^A} & \dots \\
 \downarrow f & & & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\
 B^\bullet & & \dots & \xrightarrow{\partial_{i-2}^B} & B^{i-1} & \xrightarrow{\partial_{i-1}^B} & B^i & \xrightarrow{\partial_i^B} & B^{i+1} & \xrightarrow{\partial_{i+1}^B} & \dots
 \end{array}$$

is a quasi-isomorphism ($\cong_{\mathcal{Q}}$) if the induced morphism at cohomology, $H^i(f) : H^i(A^\bullet) \rightarrow H^i(B^\bullet)$ is an isomorphism for all i . We want to treat quasi-isomorphic objects as isomorphic objects. A derived category is a category in which such quasi-isomorphic complexes are treated as the same object, i.e., quasi-isomorphisms are isomorphisms.

3.6.1 | LOCALIZATION WITH QUASI-ISOMORPHISMS

The localization of the category of complexes with the collection of quasi-isomorphisms is called the derived category.

$$\mathcal{D}(\mathcal{A}) := \mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$$

where \mathcal{Q} is the collection of quasi-isomorphisms. Our goal is to obtain a more direct formula for localization of the category of complexes with quasi-isomorphisms. We want to get an explicit description of localization for the case of quasi-isomorphisms. This can be done with the special properties the collection of quasi-isomorphisms comes equipped with. Composition of morphisms that preserve cohomology also preserve cohomology, so composition of quasi-isomorphisms is also a quasi-isomorphism. Identity morphisms on the complexes preserve cohomology, so identity morphism should be a quasi-isomorphism.

Let $f : A \rightarrow B$ be a morphism, and suppose $q : B \rightarrow C$ is a quasi-isomorphism, then it's invertible in the localized category. So, $\mathcal{L}(q)$ is an isomorphism, so we could just invert and get a map $A \rightarrow C$.

$$\begin{array}{ccc}
 & & \mathcal{L}(C) \\
 & \nearrow & \downarrow \cong \\
 \mathcal{L}(A) & \longrightarrow & \mathcal{L}(B)
 \end{array}$$

So, in the original category, there should exist a quasi-isomorphism, $A \xrightarrow{\cong_{\mathcal{Q}}} D$ such that the following diagram commutes,

$$\begin{array}{ccc}
 D & \dashrightarrow & C \\
 \downarrow \cong_{\mathcal{Q}} & & \downarrow \cong_{\mathcal{Q}} \\
 A & \longrightarrow & B
 \end{array} \quad (\text{extension})$$

Similarly for the arrows reversed, for the same reason. Suppose we have two morphisms, $f, g : B \rightarrow C$ and a quasi-isomorphism $\tau : C \rightarrow D$ such that $\tau \circ f = \tau \circ g$ then it means that f and g manipulate the information about cohomology similarly. So, what τ did was rearrange

the relevant information to make the two morphisms equal. It should also be possible to rearrange this information before manipulation by f and g . So, there should exist a quasi-isomorphism $\sigma : A \rightarrow B$ such that $f \circ \sigma = g \circ \sigma$.

$$A \dashrightarrow_{\cong_Q} B \Longrightarrow C \dashrightarrow_{\cong_Q} D. \quad (\text{symmetry})$$

A collection of morphisms $\mathcal{Q} \subset \text{Hom}_{\mathcal{C}(\mathcal{A})}$ is called a right multiplicative system if it contains every isomorphism in $\text{Hom}_{\mathcal{C}(\mathcal{A})}$, closed under composition, and satisfies [extension](#) and [symmetry](#).

The explicit construction of the localization of a right multiplicative system is as follows, $\mathcal{C}(\mathcal{A})[\mathcal{Q}^{-1}]$ consists of the same objects as $\mathcal{C}(\mathcal{A})$. For morphisms, we should think of objects that are quasi-isomorphic as the same object. So, the morphisms in the derived category will be represented by pairs of morphisms in the original category of complexes, visualised by the diagram,

$$\begin{array}{c} \mathcal{L}(C) \xrightarrow{f} \mathcal{L}(B) \\ \downarrow \cong \\ \mathcal{L}(A) \end{array}$$

The important part is the transformation described by the map f . Consider two morphisms between $\mathcal{L}(A)$ and $\mathcal{L}(B)$,

$$\begin{array}{ccc} \mathcal{L}(C) & \longrightarrow & \mathcal{L}(B) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{L}(A) & & \mathcal{L}(A) \end{array}$$

Qualitatively, these two morphisms represent the same morphism if they transform same quasi-isomorphic information. So the non-quasi-isomorphic maps are the important parts. Two morphisms are equivalent if their non-quasi-isomorphic representative factor through the same morphism. Visualised by the diagram,

$$\begin{array}{ccccc} & & C & & \\ & \swarrow \cong_Q & \downarrow & \searrow & \\ A & \dashrightarrow_{\cong_Q} & E & \dashrightarrow & B \\ & \nwarrow \cong_Q & \uparrow & \swarrow & \\ & & D & & \end{array} \quad (\text{derived equivalence})$$

By the symmetry of the diagram this is clearly an equivalence relation.

For the composition of morphisms in the derived category, for two morphisms, $\mathcal{L}(A) \rightarrow \mathcal{L}(B)$ and $\mathcal{L}(B) \rightarrow \mathcal{L}(C)$ the composition of the morphisms corresponds to the following representative in the original category of complexes,

$$\begin{array}{ccccc} & & E & & \\ & \swarrow \cong_Q & \downarrow & \searrow & \\ & C & & D & \\ \swarrow \cong_Q & & \searrow & \swarrow \cong_Q & \searrow \\ A & & B & & C \end{array} \quad (\text{derived composition})$$

Note that the above square is possible due to [extension](#) property. The category with same objects as the category of complexes, with morphisms described by the equivalence classes of morphisms in the category of complexes as described in [derived equivalence](#), together with composition as described in [derived composition](#) is the derived category $\mathcal{D}(\mathcal{A})$.

For proof that this is indeed the localization of $\mathcal{C}(\mathcal{A})$ with \mathcal{Q} see, [1]. What needs to be checked is that any other functor which takes quasi-isomorphisms to isomorphisms must factor through this category.

3.6.2 | STRUCTURE OF DERIVED CATEGORY

To the category of complexes we added a bunch of new morphisms to obtain the derived category. So we cannot expect these newly added morphisms to have kernels and cokernels. The derived category of an abelian category will not in general be an abelian category. It also doesn't have to be a locally small category as we have added new morphisms to the hom-sets.

Each object $A \in \mathcal{A}$ defines a complex,

$$A \equiv \cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

where the A is at the 0th position. So, we have an embedding of the category \mathcal{A} in $\mathcal{C}(\mathcal{A})$ given by,

$$\mathcal{A} \rightarrow \mathcal{C}(\mathcal{A}), \quad A \mapsto A_{\bullet}.$$

By definition of zero objects, there can only be one map 0 to and from any object. This implies that the functor $\mathcal{A} \rightarrow \mathcal{C}(\mathcal{A})$ is fully faithful. That's to say,

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A_{\bullet}, B_{\bullet})$$

So, \mathcal{A} can be considered as a full subcategory of $\mathcal{C}(\mathcal{A})$, denote the subcategory by $\iota(\mathcal{A})$.

The main point of doing all the localization stuff was to throw out unnecessary information and focus the study on the information preserved by cohomology functors H^i . The cohomology functors transform quasi-isomorphisms to isomorphisms, by definition of quasi-isomorphism. Going from $\mathcal{C}(\mathcal{A})$ to $\mathcal{D}(\mathcal{A})$, we have added new morphisms, making the objects more 'connected', so more objects will be isomorphic.

A complex A^{\bullet} is said to be H^0 -complex if all the cohomological information is contained in the zeroth position, that is to say, $H^i(A^{\bullet}) = 0$ for all i except at 0. Denote the subcategory of all H^0 -complexes by $H^0(\mathcal{A})$.

LEMMA 3.6.1.

$$\mathcal{A} \xrightleftharpoons[H^0]{\mathcal{L} \circ \iota} H^0(\mathcal{A}) \twoheadrightarrow \mathcal{D}(\mathcal{A}).$$

is an equivalence of categories.

PROOF

The goal is to identify each H^0 -complex A^{\bullet} with $H^0(A^{\bullet})$. This gives us a functor $H^0 : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{A}$.

3.6.2.1 | TRIANGULATED CATEGORIES

3.7 | DERIVED FUNCTORS

Functors from one category to other transform the information about the objects and morphisms between them. The information we are interested in here is regarding the obstruction to exactness, in complexes. For this to make sense, we have to restrict ourselves to functors that take objects from a category where we can talk about complexes, and exactness, to categories where we can talk about complexes and exactness. So, the functors will be assumed to be additive functors between abelian categories.

3.7.1 | VIA KAN EXTENSIONS

Given a functor what we want is a new functor that takes quasi-isomorphisms to isomorphisms. This can be formulated cleanly, via Kan extensions. So the idea is to find a functor that is along the localization functor, so by making it pass through the derived category the quasi-isomorphisms are transformed to isomorphisms and then any functor from the derived category will have to map isomorphisms to isomorphisms, by functoriality. Let

$$\mathcal{L} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}),$$

be the localization of $\mathcal{C}(\mathcal{A})$ with \mathcal{Q} . Let $\mathcal{F} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}$ be an arbitrary functor. A right derived functor of \mathcal{F} is a functor $R\mathcal{F}$ together with a natural transformation,

$$\begin{array}{ccc} & \mathcal{D}(\mathcal{A}) & \\ \mathcal{L} \nearrow & \uparrow & \searrow R\mathcal{F} \\ \mathcal{C}(\mathcal{A}) & \xrightarrow{\mathcal{F}} & \mathcal{D} \end{array}$$

such that every other functor $\mathcal{K} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}$ factors through $R\mathcal{F}$, i.e., there exists a natural transformation from $R\mathcal{F} \Rightarrow \mathcal{K}$.

$$\begin{array}{ccc} & \mathcal{D}(\mathcal{A}) & \\ \mathcal{L} \nearrow & \uparrow & \searrow R\mathcal{F} \\ \mathcal{C}(\mathcal{A}) & \xrightarrow{\mathcal{F}} & \mathcal{D} \end{array} \quad \begin{array}{c} \text{---} \kappa \text{---} \\ \text{---} \end{array}$$

This means that $R\mathcal{F}$ is exactly the left Kan extension of \mathcal{F} along the localization functor \mathcal{L} . $R\mathcal{F}$ is called absolute right derived functor if for every functor $\mathcal{H} : \mathcal{D} \rightarrow \mathcal{E}$ the right derived functor of the composition $\mathcal{H} \circ \mathcal{F}$ is given by $\mathcal{H} \circ R\mathcal{F}$.

Note that existence of derived functors is not guaranteed here. The starting category that is $\mathcal{C}(\mathcal{A})$ need not be small, and $\mathcal{D}(\mathcal{A})$ need not be locally small. So even though the target category is cocomplete, other requirements of coend formula are not met.

THEOREM 3.7.1.

COROLLARY 3.7.2. (QUILLEN ADJUNCTION THEOREM)

3.7.2 | RESOLUTIONS

3.7.3 | SPECTRAL SEQUENCES

3.7.4 | DERIVED FUNCTORS AS KAN EXTENSIONS

4 | THE SIX FUNCTORS

In these notes we study certain adjoint pairs that don't need the language of derived categories. These can be called internal functors or external depending on whether we stay within that category of sheaves over a topological space or we leave the category. These are four of Grothendieck's 'six operations'.

4.1 | DIRECT AND INVERSE IMAGE SHEAVES

Continuous maps $X \rightarrow Y$ gives rise to an adjoint pair of functors $\text{Sh}(X) \rightleftharpoons \text{Sh}(Y)$ called direct image, and inverse image, these are external operations, i.e., we are moving to a different topological space. A topological space X determines a category $\text{PSh}(X)$ of sheaves on X . A continuous map of spaces $f : X \rightarrow Y$ will induce functors in both directions, forward and backward, on the associated category of pre-sheaves $\text{PSh}(X)$ and $\text{PSh}(Y)$.

Let $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}$ be a sheaf on X . Using this sheaf we can construct a sheaf on Y as follows, the sheaf assigns to each open set $U \subset X$ a set $\mathcal{F}U$, we can associate the same set $\mathcal{F}U$ to the open set whose image under f was U . The continuous function f gives us a functor of categories of open sets,

$$f^{-1} : \mathcal{O}(Y)^{\text{op}} \rightarrow \mathcal{O}(X)^{\text{op}}.$$

This gives rise to a new sheaf, the induced sheaf $f_*\mathcal{F}$ on Y , defined as the composition of functors,

$$\mathcal{O}(Y)^{\text{op}} \xrightarrow{f^{-1}} \mathcal{O}(X)^{\text{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}.$$

Defined for each open set V of Y , i.e., $V \in \mathcal{O}(Y)$ by,

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V).$$

This is defined because $f^{-1}V$ is an open set by definition of continuous functions. $f_*\mathcal{F}$ is called the direct image of \mathcal{F} under f . This gives us a functor,

$$f_* : \text{PSh}(X) \rightarrow \text{PSh}(Y).$$

This respects the composition of functions, i.e.,

$$(fg)_* = f_*g_*$$

So, if we set $\text{PSh}(f) = f_*$, then PSh is a functor from the category of topological spaces to the category of sheaves. The direct image functor f_* has a left exact, left adjoint f^* . The construction of this left adjoint is considerably more complex.

4.1.1 | INVERSE IMAGE SHEAVES VIA ÉTALE SPACE

To construct inverse image, we make use of pullbacks. Let $E \rightarrow Y$ be a bundle over Y , we can then pullback E along a function $f : X \rightarrow Y$ giving us the bundle, $f^*E \rightarrow X$,

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

f^* is a functor $f^* : \mathbf{Bund} Y \rightarrow \mathbf{Bund} X$.

Suppose E is an étale bundle over Y , i.e., around each point $e \in E$ has a neighborhood U_e that is homeomorphic to its image $\pi(U_e)$. By definition of pullback, the space f^*E consists of points which we can label using points of X and E by $\langle x, e \rangle$ such that $fx = \pi e$, i.e., the pullback is the universal equalizer of the two maps.

By definition of étale spaces, there is a neighborhood U_e of e that's mapped homeomorphically to its image. Using this image we can construct an open neighborhood of x , by taking $f^{-1}(\pi(U_e))$. This is possible because by definition of pullback we have $fx = \pi e$. So, $\langle f^{-1}(\pi(U_e)), U_e \rangle$ is an open neighborhood of $\langle x, e \rangle$ that is mapped homeomorphically onto $f^{-1}(\pi(U_e))$ of X . Hence f^*E is étale.

If we start with a sheaf \mathcal{F} on Y , a point in the pullback $f^*\mathcal{F}$ is of the form, $\langle x, \text{germ}_{f(x)} s \rangle$ where $s \in \mathcal{F}V$ is an element of the sheaf \mathcal{F} .

LEMMA 4.1.1. *Pullback of an étale space under a continuous map is étale.* □

This gives us a map of sheaves,

$$\mathbf{Sh}(Y) \xrightarrow{\mathcal{E}} \mathbf{Etale} Y \xrightarrow{f^*} \mathbf{Etale} X \xrightarrow{\Gamma} \mathbf{Sh}(X).$$

Here the first map is the bundling of the stalks of a sheaf to an étale space, and the last map is taking the sheaf of sections of the bundle. The composition gives us a functor of sheaves, which we denote again by f^* ,

$$\mathbf{Sh}(Y) \xrightarrow{f^*} \mathbf{Sh}(X).$$

THEOREM 4.1.2.

$$\mathbf{Sh}(X) \xrightleftharpoons[f^*]{f_*} \mathbf{Sh}(Y), \quad f^* \dashv f_*.$$

SKETCH OF PROOF

So, what we need to prove is that, $\text{Hom}_{\mathbf{Sh}(X)}(f^*F, G) \cong \text{Hom}_{\mathbf{Sh}(Y)}(F, f_*G)$, which for the sake of brevity we will denote by,

$$\mathbf{Sh}(X)(f^*F, G) \cong \mathbf{Sh}(Y)(F, f_*G).$$

Since we have an equivalence between the category of étale spaces and category of sheaves over X , we have that $\mathbf{Sh}(X)(f^*F, G) \cong \text{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$, where $\text{Et}_X(\mathcal{E}f^*F, \mathcal{E}G)$ is the collection of morphisms between étale spaces $\mathcal{E}f^*F$ and $\mathcal{E}G$.

Let $K(\mathcal{E}f^*F, \mathcal{E}G)$ be the set of functions $k : \mathcal{E}f^*F \rightarrow \mathcal{E}G$ over X .

4.1.2 | VIA KAN EXTENSION

4.2 | CATEGORY OF ABELIAN SHEAVES

Sheaves encountered often in geometry are abelian sheaves. Let X be a topological space, and consider continuous functions $f : X \rightarrow \mathbb{R}$ on the topological space X . It forms a sheaf,

$$C : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Sets}.$$

Continuous functions can be added, subtracted, multiplied and scaled to obtain new continuous functions. For each open set $U \in \mathcal{O}(X)$, the collection of all continuous functions on it CU forms an \mathbb{R} -algebra. Each CU are \mathbb{R} -module objects in the category \mathbf{Sets} .

Since we can encounter lot of sheaves which take values in an abelian category, it's justified to give them special attention.¹ Let \mathcal{A} be an abelian category, an abelian pre-sheaf on a topological space X is a functor \mathcal{F} ,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \rightarrow \mathcal{A}, \quad (\text{pre-sheaf})$$

The category of all pre-sheaves on a topological space X with values in \mathcal{A} is denoted by $\text{PSh}(X, \mathcal{A})$. The functor category,

$$\text{PSh}(X, \mathcal{A}) := \mathcal{A}^{\mathcal{O}(X)^{\text{op}}}.$$

admits small limits since the abelian category \mathcal{A} admits small limits, small colimits, and the small filtered limits are exact and limits in functor category are computed pointwise. Once we have small limits we can start doing all sorts of things like take pullbacks, pushforward, products, coproducts, etc.

A pre-sheaf is a sheaf if the local associations $\mathcal{F}U_i$ are restrictions of global association. So, given an open covering $U = \bigcup_{i \in I} U_i$, if $f_i \in \mathcal{F}U_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it should mean that there exists a section $f \in \mathcal{F}U$ such that $f_i = f|_{U_i}$. The maps $f_i \in \mathcal{F}U_i$ and $f_j \in \mathcal{F}U_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I -indexed family of functions $(f_i)_{i \in I} \in \prod_i \mathcal{F}U_i$, and two maps

$$p(\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \quad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and that's what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j|_{U_i \cap U_j}} = f|_{U_i|_{U_i \cap U_j}}$ which means that there is a map e from $\mathcal{F}U$ to $\prod_i \mathcal{F}U_i$ such that $pe = qe$. $\mathcal{F}U \rightarrow \prod_i \mathcal{F}U_i$

$$\mathcal{F}U \xrightarrow{e} \prod_i \mathcal{F}U_i \xrightleftharpoons[q]{p} \prod_{i,j} \mathcal{F}(U_i \cap U_j). \quad (\text{collation})$$

This is the collation property. For general sites, the intersection \cap will be replaced by the fibered product $\prod_{i,j}$ and the open covers are replaced by covers on sites, injective maps replaced by monic maps and so on. We are interested in exploiting the target category now, i.e., \mathcal{A} .

¹Abelian categories are categories where we can do homological algebra, i.e., kernels, cokernels, images, coimages, direct sums, products, etc exist. The important result we need about them is that if \mathcal{A} is an abelian category then so is the functor category $\mathcal{A}^{\mathcal{C}}$ for any category \mathcal{C} . For a discussion on abelian categories see [1].

4.2.1 | INTERNAL $\mathcal{H}om$

The operations of interest to us now are within a category of pre-sheaves $\mathbf{PSh}(X, \mathcal{A})$ or sheaves $\mathbf{Sh}(X, \mathcal{A})$ on a given topological space X or site. Since we are not going to use or do anything special with the underlying topological space or site, we will for the sake of simplicity assume it to be a topological space X .

Consider two pre-sheaves, $\mathcal{F}, \mathcal{G} \in \mathbf{PSh}(X, \mathcal{A})$, for any $U \subset X$, consider the new pre-sheaves, the restrictions, $\mathcal{F}|_U, \mathcal{G}|_U \in \mathbf{PSh}(U, \mathcal{A})$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathbf{Hom}_{\mathbf{PSh}(U, \mathcal{A})}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{array}{ccccc} U & & \mathcal{F}U & \xrightarrow{\kappa_U} & \mathcal{G}U \\ \downarrow |_V & & \downarrow \mathcal{F}(|_V) & & \downarrow \mathcal{G}(|_V) \\ V & & \mathcal{F}V & \xrightarrow{\kappa_V} & \mathcal{G}V. \end{array}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom, denoted by,

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in \mathbf{PSh}(X, \mathcal{A}).$$

Sometimes also written as $\mathcal{G}^{\mathcal{F}}$. We will now show that the [collation](#) property holds for the $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, and hence $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, i.e., for any cover $\{U_i\}$ of U , the following is an exact sequence,

$$0 \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{G})U \xrightarrow{e} \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i) \xrightarrow[p]{q} \prod_{i,j} \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i \prod_{i,j} U_j).$$

To show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{F}, \mathcal{G})U$ and at $\prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q .

PROPOSITION 4.2.1. *If \mathcal{F} and \mathcal{G} are sheaves, then so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.*

PROOF

First, we have to show that e is injective. Let $\{U_i\}_{i \in I}$ be a cover of U . For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})U$, and $U_i \subset U$, we have,

$$\begin{array}{ccccc} U & & \mathcal{F}U & \xrightarrow{\kappa_U} & \mathcal{G}U \\ \downarrow |_{U_i} & & \downarrow \mathcal{F}(|_{U_i}) & & \downarrow \mathcal{G}(|_{U_i}) \\ U_i & & \mathcal{F}U_i & \xrightarrow{\kappa_{U_i}} & \mathcal{G}U_i. \end{array}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{F}(U_i)$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \cap V} = 0.$$

Now, $\{W \cap U_i\}$ is a cover of W , and $\mathcal{G}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q , i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{F}, \mathcal{G})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \cap U_j} = \kappa_j|_{U_i \cap U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that \mathcal{G} is a sheaf to patch these natural transformations.

Since \mathcal{G} is a sheaf, we have for all $V \subset U$,

$$\mathcal{F}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \cap U_i) \xrightarrow[p]{q} \prod_{i,j} \mathcal{G}(V \cap (U_i \cap U_j)).$$

here the first map comes from the natural transformation, $\mathcal{F}V \ni f \mapsto \kappa_i(f|_{V \cap U_i})$. Since \mathcal{G} is a sheaf, this must uniquely factor through $\mathcal{G}V$, by definition of equaliser. Hence, we have,

$$\mathcal{F}V \xrightarrow{\exists!} \mathcal{G}V \xrightarrow{\kappa_V} \prod_i \mathcal{G}(V \cap U_i) \xrightarrow[p]{q} \prod_{i,j} \mathcal{G}(V \cap (U_i \cap U_j)).$$

Let this unique map be κ_V , then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ defines the patched up element that equalizes the diagram. \square

Note here that \mathcal{F} need not be a sheaf, the above proposition also holds if \mathcal{F} is a pre-sheaf, and \mathcal{G} is a sheaf.

4.2.1.1 | HOM-TENSOR ADJOINTNESS

Note here that the natural transformations κ respect the abelian structure. The exponentiation category $\mathcal{G}^{\mathcal{F}}$ has more structure than the standard exponentiation in **Sets**. Let \mathcal{R} be a sheaf of commutative rings.

$$\begin{aligned} \mathcal{R} : \mathcal{O}(X)^{\text{op}} &\rightarrow \mathbf{CRings} \\ U &\mapsto \mathcal{R}U. \end{aligned}$$

Then we can consider the pre-sheaves which take values in the category $\mathcal{R}\text{Mod}$ of \mathcal{R} -modules, i.e., U gets mapped to $\mathcal{R}U$ -modules. We will denote such pre-sheaves by $\text{PSh}(X, \mathcal{R})$. It naturally inherits a tensor product from the module structure. So, for two pre-sheaves \mathcal{F} and \mathcal{G} , we can construct the tensor product pre-sheaf,

$$\begin{aligned} \mathcal{F} \hat{\otimes}_{\mathcal{R}} \mathcal{G} : \mathcal{O}(X)^{\text{op}} &\rightarrow \mathcal{R}\text{Mod} \\ U &\mapsto \mathcal{F}U \otimes_{\mathcal{R}U} \mathcal{G}U \end{aligned}$$

We will denote the sheafification of this pre-sheaf by,

$$\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G} \in \text{Sh}(X, \mathcal{R}).$$

This is a bifunctor,

$$\cdot \otimes_{\mathcal{R}} \cdot : \text{Sh}(X, \mathcal{R}) \times \text{Sh}(X, \mathcal{R}) \rightarrow \text{Sh}(X, \mathcal{R}).$$

4.2.2 | TENSOR PRODUCT AS A COEND

4.3 | VERDIER DUALITY

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