Complex Analysis

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Part I

One Variable

1 | Holomorphic & Meromorphic Functions

1.1 | Holomorphic Functions

Main aim of this section is to show holomorphic functions are analytic. Topics for these notes include, holomorphic functions, analytic functions, integration along paths, Cauchy's theorem, Cauchy integral formula, analytic continuation, Liouville theorem, fundamental theorem of algebra, the maximum modulus principle, Morera's theorem, Weierstrass & Montel's theorem.

We are interested in differentiating functions defined on some open set $\Omega \subset \mathbb{C}$[1]. The topology on $\mathbb{C}$ comes from the standard euclidean norm on $\mathbb{C} = \mathbb{R}^2$. A function f defined on Ω is complex differentiable at a point $z \in \Omega$ if it's differentiable and the differential is complex linear i.e., there exists a complex linear function $df(z)$ such that,

$$\frac{f(z+h) - f(z)}{h} \xrightarrow[h \to 0]{} df(z)(h), \tag{1D}$$

The function f can be approximated infinitesimally by a complex linear function. If f is complex differentiable on Ω it's said to be holomorphic on Ω and the derivative is defined as,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

If f is holomorphic on all of $\mathbb{C}$ it's called entire.

If the function f is thought of as a map from Ω as an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, then f can be written as, $f = u + iv$. The condition of complex differentiability of f at $z \in \Omega$ can then be split into two requirements, real differentiable and complex linearity of the differential. This means that the differential is a real linear map,

$$df(z) : \mathbb{R}^2 \to \mathbb{R}^2.$$

The matrix form of $df(z)$ is called the Jacobian and in terms of partial derivatives in the standard basis, it is given by,

$$J_f(z) = (\partial_j f_i(z))_{i,j}$$

A real linear map T is complex linear if $T(i) = iT(1)$. This condition puts the required constraint. So the Jacobian matrix $J_f(z)$ will be of the form,

$$J_f(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{bmatrix} \partial_x u(z) & \partial_y u(z) \\ \partial_x v(z) & \partial_y v(z) \end{bmatrix}. \tag{2D}$$

[1] Note that whenever we talk about Ω, we assume it to be open subset of $\mathbb{C}$, and whenever we say $h \in \mathbb{C}$ is small we mean $\|h\|$ is small.

This is called the Cauchy-Riemann equation.[2]

The set of all holomorphic functions on Ω is denoted by $\mathcal{H}(\Omega)$. If f, and g are two complex differentiable functions on Ω and $\lambda \in \mathbb{C}$ then it follows directly from definition that, $f+g$, $f \cdot g$, and $\lambda \cdot f$ are also complex differentiable. So, $\mathcal{H}(\Omega)$ is an algebra over $\mathbb{C}$.

If f is a complex differentiable function on Ω and g is a complex differentiable function on V then the composition map $g \circ f$ is a complex differentiable function on Ω and,

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

which is the chain rule.

Clearly, all polynomials are holomorphic functions. We can generalise this to all power series within their disk of convergence. A formal power series is a series

$$\sum_{n \geq 1} a_n z^n$$

the set of all formal power series is denoted by $\mathbb{C}[z]$. Abel's theorem says that every power series has a radius of convergence, i.e., there exists $R > 0$ such that for any z in the ball of radius R the power series converges. The radius of convergence is given by,

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

A function $f : \Omega \to \mathbb{C}$ is said to be analytic if f is represented by a convergent power series expansion on a neighborhood around every point Ω.

The set of all analytic functions on Ω is denoted by $\mathcal{A}(\Omega)$. Clearly this set is an algebra. The main goal of this part is to prove that holomorphic functions and analytic functions are the same.

Theorem 1.1.1.

$$\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega).$$

Proof

Let $f \in \mathcal{A}(\Omega)$, without loss of generality assume the power series expansion of f is

$$f(z) = \sum_{n \geq 0} a_n z^n$$

for all $|z| < R$. Write, $f(z+h) = f(z) + g(z)h + r(z,h)$, where $g(z) = \sum_{n \geq 1} n a_n z^{n-1}$. By the radius of convergence formula, the radius of convergence of $g(z)$ is the same as $f(z)$. We have,

$$r(z,h) = \sum_{n \geq 0} a_n B_n(z,h),$$

[2] Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can express $z \in \mathbb{C}$ as a vector $(x,y) \in \mathbb{R}^2$, with $z = x+iy$ and $dz = dx+idy$, $d\bar{z} = dx - idy$. For $f \in C^\infty(U)$, as a function on $\mathbb{R}^2$ we have, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$. By writing $dx = \frac{dz+d\bar{z}}{2}$ and $dy = \frac{dz-d\bar{z}}{2i}$, we get,

$$df = \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)}_{:= \frac{\partial f}{\partial z}} dz + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}_{:= \frac{\partial f}{\partial \bar{z}}} d\bar{z}$$

So, $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$. f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ or $df = \frac{\partial f}{\partial z}dz$.

where $B_n(z,h) = (z+h)^n - z^n - nz^{n-1}h$ and $B_0 = B_1 = 0$. Let $|z| + |h| < r_1 < r_2 < R$, we have, $|a_n| \le M/r_2^n$ for some $M < \infty$. So we have,

$$\begin{aligned}|B_n(z,h)| \le |h|^2 \textstyle\sum_{k=0}^{n-2} \binom{n}{k+2} |h|^k |z|^{n-2-k} &\le |h|^2 \textstyle\sum_{k=0}^{n-2} n^2 \binom{n-2}{k} |h|^k |z|^{n-2-k} \\ &= |h|^2 n^2 (|z| + |h|)^{n-2}.\end{aligned}$$

Hence,

$$|r(z,h)| \le \textstyle\sum_{n\ge 2} \frac{M}{r_2^n} n^2 (|z|+|h|)^{n-2} \le \frac{|h|^2 M}{r_2^2} \sum_{n\ge 0} (n+2)^2 \left(\frac{r_1}{r_2}\right)^n.$$

or, for some constant C, we have,

$$|r(z,h)| \le C|h|^2.$$

or that $g(z) = f'(z)$ for $|z| < R$ because for $h \to 0$ we have $f(z+h) - f(z) \approx g(z)h$. So f is holomorphic. □

1.1.1 | Path Integrals; Cauchy's theorem

To prove the other inclusion, i.e., $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$ we need more tools and complex integration. Let $\eta : [a,b] \to \mathbb{C}$ be a smooth curve. Let f be a continuous function defined atleast on the compact image $\eta([a,b])$. The path integral of f along η is defined by,

$$\int_\eta f(z)dz = \int_{[a,b]} f(\eta(t))\eta'(t)dt$$

since $(f \circ \eta) \cdot \eta'$ is continuous on $[a,b]$ the integral is well-defined. For a piecewise C^1-path the integral along $\eta = \eta_1 + \cdots + \eta_n$ is defined by, $\int_\eta f(z)dz = \sum_{i=1}^n \int_{\eta_i} f(z)dz$.

Suppose we have a reparametrization of the interval $[a,b]$, given by the C^1-map $\varphi : [a',b'] \to [a,b]$ then we have,

$$\int_{\eta\circ\varphi} f(z)dz = \int_{[a',b']} f(\eta(\varphi(t)))\eta'(\varphi(t))\varphi'(t)dt = \int_{[a,b]} f(\eta(s))\eta'(s)ds = \int_\eta f(z)dz.$$

where $s = \varphi(t)$, $ds = \varphi'(t)dt$. Hence the path integral is invariant under reparametrization. Length of a path $\eta : [a,b] \to \mathbb{C}$ is defined by,

$$L(\eta) = \int_{[a,b]} |\eta'(t)|dt.$$

Below we list some immediate properties of the path integral,

If η is a path then, f, g in the domain containing η and $a, b \in \mathbb{C}$,

$$\int_\eta (af + bg)dz = a\int_\eta f dz + b\int_\eta g dz.$$

If the path η_1 starts at the end point of η_2 then,

$$\int_{\eta_1+\eta_2} f dz = \int_{\eta_1} f dz + \int_{\eta_2} f dz.$$

The reverse path can be written as $\widetilde{\eta}(s) = \eta(a+b-s)$, then by changing the variable, we have,

$$\int_{\widetilde{\eta}} f dz = \int_\eta f(\eta(a+b-s))\eta'(a+b-s)(-1)ds = -\int_\eta f dz.$$

If $\eta([a,b]) \subseteq \Omega$ and $g \in \mathcal{H}(\Omega)$ with continuous derivative g', we have,

$$\int_{g\circ\eta} f dz = \int_{\eta} f(g(z))g'(z)dz.$$

For all $f \in \mathcal{H}(\Omega)$, $|\int_\eta f dz| \leq \int_{[a,b]} |f(\eta(t))||\eta'(t)|dt \leq \sup_{z\in(\eta([a,b]))} |f(z)| \int_{[a,b]} |\eta'(t)|dt = \|f\|_\eta L(\eta)$. and this can be extended to piecewise C^1 paths. So we have,

$$\left| \int_\eta f dz \right| \leq L(\eta)\|f\|_\eta.$$

where $\|f\|_\eta = \sup_{z\in\eta([a,b])} |f(z)|$. Using this we can show that if f_n converges to f uniformly in $\eta([a,b])$ then, $\lim_n \int_\eta f_n dz = \int_\eta f dz$.

Let $f \in \mathcal{H}(\Omega)$ with continuous derivative, for a smooth curve η, we have, $\frac{df(\eta(t))}{dt} = f'(\eta(t))\eta'(t)$ So, we have,

$$\int_\eta f' dz = \int_\eta f'(\eta(t))\eta'(t)dt = \int_{[a,b]} \frac{df(\eta(t))}{dt} dt = f(\eta(t))\big|_a^b = f(\eta(b)) - f(\eta(a)).$$

If f has an anti-derivative F, i.e., $F' = f$ then,

$$\int_\eta f(z)dz = F(\eta(b)) - F(\eta(a)).$$

So, if f has an anti-derivative in Ω then the path integral is independent of the path, and only depends on the end points. Let η_r be the circle $\{|z| = r\}$ for some positive number r, with the counter clockwise orientation, i.e., $\eta_r(t) = re^{it}$. Then,

$$\int_{\eta_r} z^n dz = \int_{[0,2\pi]} r^n e^{int} rie^{it} dt$$

For $n \neq -1$ the function $f(z) = z^n$ has the primitive $F(z) = \frac{z^{n+1}}{(n+1)}$, and $F(\eta_r(2\pi)) - F(\eta_r(0)) = 0$. So whenever $\int_{\eta_r} z^n dz = 0$ whenever $n \neq -1$. For $n = -1$ the integral becomes,

$$\int_{[0,2\pi]} r^{-1}e^{-it} rie^{it} dt = \int_{[0,2\pi]} i dt = 2\pi i$$

Theorem 1.1.2. (Cauchy-Goursat theorem) *$f \in \mathcal{H}(\Omega)$, if η is a loop in Ω that can be deformed to a point in Ω from within Ω, then,*

$$\int_\eta f(z)dz = 0$$

Proof

Let U be such that $\partial U = \eta$. By setting $u = f dz$, we have, $du = df \wedge dz$. By Stokes theorem, $\int_{\partial U} f dz = \int_U du$, which gives us,

$$\int_{\partial U} f dz = \int_U \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} \right) \wedge dz$$

which will be zero for holomorphic functions, by Cauchy-Riemann equations. □

The Cauchy-Goursat theorem also tells us that for homotopic loops, and $f \in \mathcal{H}(\Omega)$, $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$, when the homotopy lies entirely inside Ω. This is because we can triangulate the homotopy, and $\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \sum_j \int_{\eta_j} f(z)dz = 0$, where the sum is over a finite collection of small loops.

Theorem 1.1.3. (Cauchy Integral Formula) *Let $B_r(z_0) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. Then,*

$$f(z) = \frac{1}{2\pi i}\int_\eta \frac{f(w)}{w-z}dw,$$

where $\eta(t) = z_0 + re^{it}$ and for all $z \in B_r(z_0)$.

Proof

Assume $z_0 = 0$. For any $z \in B_r(0)$, from $B_r(0)$ remove a small ball centered around z, i.e,. let $U_\epsilon = B_r(0) \backslash B_\epsilon(z)$.

$$w \mapsto \tfrac{f(w)}{(w-z)}$$

is a holomorphic function on U_ϵ and Cauchy-Goursat theorem is applicable. The boundary circles of U_ϵ are homotopic relative to $\Omega\backslash\{z\}$.

$$0 = \tfrac{1}{2\pi i}\int_{\partial U_\epsilon} \tfrac{f(w)}{z-w}dw = \tfrac{1}{2\pi i}\int_{\partial B_r} \tfrac{f(w)}{z-w}dw - \tfrac{1}{2\pi i}\underbrace{\int_{\partial B_\epsilon(z)}\left(\tfrac{f(w)-f(z)}{z-w}\right)dw}_{\leq ML(\partial B_\epsilon(z))=2\pi M\epsilon} - \tfrac{f(z)}{2\pi i}\underbrace{\int_{\partial B_\epsilon(z)} \tfrac{1}{z-w}dw}_{\int_\gamma z^{-1}dz=2\pi i}$$

The second term can be made arbitrarily small. Hence we have,

$$f(z) = \frac{1}{2\pi i}\int_\eta \frac{f(w)}{w-z}dw.$$

□

Note that in general case, $\frac{f(w)}{w-z}dw$ gets replaced by $\frac{f(w)}{(z-w)}dw + \frac{\partial f(w)}{d\bar{z}}\frac{dz\wedge d\bar{z}}{(z-w)}$ which for the case of holomorphic functions is the same.

Theorem 1.1.4.

$$\mathcal{A}(\Omega) = \mathcal{H}(\Omega).$$

Proof

For a small rectangle around z,

$$f(z) = \frac{1}{2\pi i}\int_{\partial R} \frac{f(w)}{w-z}dw.$$

Whenever $|z-z_0| < |w-z_0|$, we have, $\frac{1}{w-z} = \frac{1}{(w-z_0)}\left(1 - \frac{z-z_0}{w-z_0}\right)^{-1} = \frac{1}{(w-z_0)}\sum_{n\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^n$. So we have,

$$f(z) = \frac{1}{2\pi i}\int_{\partial R} f(w)\sum_{n\geq 0}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw.$$

So,

$$f(z) = \textstyle\sum_{n\geq 0} a_n(z-z_0)^n$$

where,

$$a_n = \frac{1}{2\pi i}\int_{\partial R}\frac{f(w)}{(w-z_0)^{n+1}}dw$$

So, $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$. □

An immediate consequence of the above is that every holomorphic function is infinitely differentiable. Every holomorphic function has a Taylor series expansion,

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

with $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw$, for some small circle γ around z_0. If $|f(z)| \leq M$ on Ω, by taking modulus on both sides we have,

$$|f^{(n)}(z)| \leq \frac{Mn!}{\text{dist}(z, \partial\Omega)^n}$$

An immediate corollary of this is the Liouville's theorem, where $\Omega = B_R(0)$ where $R \to \infty$ and hence $\text{dist}(z, \partial\Omega) \to \infty$ and we have, $f^{(k)} \equiv 0$ for all k

Corollary 1.1.5. (Liouville's Theorem) *Bounded entire functions are constant.* □

A corollary of Liouville's theorem is the fundamental theorem of algebra which says that every polynomial of positive degree, $P \in \mathbb{C}[z]$ has a complex zero i.e., $\exists z \in \mathbb{C}$ such that $P(z) = 0$ and has exactly as many zeros as the degree.

If $P(z)$ has no complex zeros, the function $f(z) = 1/P(z)$ is an entire function. If $P(z) = \sum_{i=0}^n a_i z^i$ with $a_n \neq 0$ we have,

$$|P(z)| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i||z|^i \geq \frac{1}{2}|a_n|R^n$$

for all $|z| = R$ with large enough R. Hence $f(z) = 1/P(z)$ is bounded and hence must be constant by Liouville's theorem. Hence there must be $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$. Every polynomial can be factored,

$$P(z) = a_n \prod_{i=1}^n (z - \alpha_i).$$

Another consequence of this is the principle of analytic continuation. Suppose $f \in \mathcal{H}(\Omega)$, Ω be a connected open set. If $f|_U \equiv 0$ for $U \subset \Omega$. Let $E_n = \{z \mid f^{(n)}(z) = 0\}$, and let

$$E = \bigcap_{n\geq 0} E_n$$

Since f is a continuous function E_n is a closed set, and hence E is a closed set. Since $f \in \mathcal{H}(\Omega)$ it can be locally written as a power series, $f(z) = \sum_{n\geq 0} a_n z^n$, So, if $f|_U \equiv 0$ we have $a_n = 0$ in a neighborhood, and hence E must be open. So E is both open and closed, and since it's non empty, as $U \subset E$ it must be Ω, and hence $f \equiv 0$ on Ω.

Theorem 1.1.6. (Maximum Modulus Principle) *$f \in \mathcal{H}(\Omega)$, $f \in C(\overline{\Omega})$, Ω bounded, then,*

$$|f(z)| \leq \max_{w\in\partial\Omega} |f(w)|.$$

Sketch of Proof

The first step is to prove the same for small ϵ neighborhoods. Suppose there exists a maximum modulus and attained at z_0, let U be an ϵ neighborhood of z_0. Let $\gamma_\rho = \rho e^{it}$ be the circle around z_0 of radius $\rho < \epsilon$. By Cauchy integral formula,

$$f(z_0) = \tfrac{1}{2\pi i}\int_{\gamma_\rho} \tfrac{f(z)}{z-z_0}dz = \tfrac{1}{2\pi i}\int_{[0,2\pi]} \tfrac{f(z_0+\rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it}dt = \tfrac{1}{2\pi i}\int_{[0,2\pi]} f(z_0+\rho e^{it})dt.$$

Taking modulus we get,

$$|f(z_0)| \leq \frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0+\rho e^{it})|dt.$$

Since the maximum modulus is attained for z_0, we have, $|f(z_0)| \leq |f(z)|$ for all $z \in U$. Hence we have,

$$\frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0+\rho e^{it})|dt \leq \frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0)|dt = |f(z_0)|.$$

So we have,

$$0 = \frac{1}{2\pi i}\int_{[0,2\pi]} \underbrace{(|f(z_0)| - |f(z_0+\rho e^{it})|)}_{\geq 0} dt$$

Hence we have $|f(z_0)| = |f(z)|$ for all $z \in \gamma_\rho$. Since ρ was arbitrary the equality holds for all $z \in U$. Now for the general case, let $z_0 \in D$ for which $|f(z_0)| \geq |f(z)|$ for all $z \in D$. For any $w \in D$ consider a path joining z_0 and w, the ϵ neighborhoods cover the line, and by compactness of the path, only finitely many such ϵ-neighborhoods are required. For each of these ϵ-neighborhoods, the function f is constant, and hence we will get that $|f(w)| = |f(z_0)|$ □

Theorem 1.1.7. (Open Mapping Theorem) *Let Ω be connected open set. $f \in \mathcal{H}(\Omega)$. If f is not a constant map, then f is an open map from Ω to $\mathbb{C}$ i.e., $f(U)$ is open for all U open.*

Sketch of Proof

Without loss of generality assume that $f(x) = 0$.[3] Let $B_\epsilon(x)$ be a neighborhood of x contained in Ω such that $f(z) \neq 0$ for $z \in \overline{B_\epsilon(x)}$. This happens because in a neighborhood U we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(z-x)^n$$

and $a_0 = 0$, and hence we can write it as $(z-x)^k \sum_{n=k}^{\infty} a_n(z-x)^n$, so in a small enough neighborhood V, we will have, $\{f(z) = 0 \mid z \in V\} = \{x\}$.

Let $\delta = \inf\{|f(z)| \ : \ |z-x| \leq \epsilon\}$ i.e., the smallest value taken by f at the boundary of $B_\epsilon(x)$. Now we claim that if $w \notin f(B_\epsilon)$ then it belongs to a closed set. Hence showing that $f(B_\epsilon(x))$ is open in $\mathbb{C}$. To show this, consider the function $\varphi(z) = 1/(f(z)-w) \in \mathcal{H}(\Omega)$. By Maximum modulus principle, it takes a maximum value at the boundary and hence,

$$\tfrac{1}{|w|} = |\varphi(x)| \leq \sup_{|z-x|=\epsilon} |\varphi(x)| = \tfrac{1}{\inf_{|z-x|=\epsilon}\{|f(z)-w|\}}.$$

But now we have, $|f(z)-w| \geq |f(z)| - |w| \geq \delta - |w|$. Which gives us $|w| \geq \frac{1}{2}\delta$. □

[3] Otherwise consider $f - f(x)$.

Morera's theorem provides a sort of converse to Cauchy's theorem. We now have the necessary tools for it's proof.

Theorem 1.1.8. (Morera's Theorem) *$f \in C(\Omega)$, if for every rectangle, $R \subset \Omega$,*

$$\int_{\partial R} f dz = 0,$$

then $f \in \mathcal{H}(\Omega)$.

Sketch of Proof

The idea is to find a local holomorphic primitive F of f, i.e., $F' = f$, and then by infinite differentiability of holomorphic functions we would have proved that f is itself holomorphic.

We will prove for the case when Ω is a disc of radius r around the origin. Let $\gamma(t) = tz$ be the straight line joining 0 and z. Set,

$$F(z) := \int_0^z f(w)dw = z \int_{[0,1]} f(tz)dt$$

for all $|z| < r$. For small h, it follows that

$$F(z+h) - F(z) = \int_z^{z+h} f(w)dw$$

here we used the assumption about integral being zero for rectangles in the region Ω, and did Riemann integral type trick for the line joining z and $z+h$. The line segment is parametrized as $z + th$ with $t \in [0,1]$. We obtain,

$$\tfrac{F(z+h)-F(z)}{h} = \int_{[0,1]} f(z+th)dt \xrightarrow[h \to 0]{} f(z)$$

Hence $F \in \mathcal{H}(\Omega)$ and $F' = f$, and hence $f \in \mathcal{H}(\Omega)$. □

Theorem 1.1.9. (Branch of Logarithm) *Let Ω be simply connected, Let $f \in \mathcal{H}(\Omega)$ such that $f \neq 0$ on Ω. Then there exists $g \in \mathcal{H}(\Omega)$ with $e^{g(z)} = f(z)$. g is unique upto constant addition by $2\pi n$.*

Proof

Consider the function $\frac{f'}{f} \in \mathcal{H}(\Omega)$. It's in $\mathcal{H}(\Omega)$ because f doesn't vanish on Ω. Since any two curves with same end points in a simply connected space are homotopic, we have a well defined integral,

$$g(z) := \textstyle\int_{z_0}^{z} \frac{f'(w)}{f(w)} dw.$$

that doesn't depend on the path from z_0 to z. Now,

$$\frac{g(z+h) - g(z)}{h} = \int_{[0,1]} \frac{f'(z+th)}{f(z+th)} dt \to \frac{f'(z)}{f(z)} \quad \text{as } h \to 0.$$

So, $g \in \mathcal{H}(\Omega)$ and $(fe^g)' = 0$ which means that $e^g = cf$ which is means $e^{g-k} = f$ for some constant k, unique upto addition by $2\pi i n$. □

Let f be a continuous function on Ω, and γ be a piecewise differentiable curve in Ω, we can define a function g on $\mathbb{C}\backslash\mathrm{Im}\gamma$ by,

$$g(w) = \int_\gamma \frac{f(z)}{z-w}dw.$$

For fixed w, and for small $h \in \mathbb{C}$, we have,

$$\frac{(g(w+h)-g(w))}{h} = \int_\gamma f(z)\left(\frac{1}{z-w-h} - \frac{1}{z-w}\right) \cdot \frac{1}{h}dz$$

as $h \to 0$, we have, $\left(\frac{1}{z-w-h} - \frac{1}{z-w}\right) \cdot \frac{1}{h} \to \frac{1}{(z-w)^2}$ so we have,

$$g'(w) = \int_\gamma \frac{f(z)}{(z-w)^2}dz.$$

In particular $g \in \mathcal{H}(\Omega)$.

Theorem 1.1.10. (Weierstrass) *$\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$, $\{f_n\} \to f$ uniformly, then $f \in \mathcal{H}(\Omega)$ and $\{f_n'\} \to f'$ uniformly.*

Proof

Let $B_r(z_0)$ be a ball around $z_0 \in \Omega$ with radius r, such that $\overline{B_r(z_0)} \subset \Omega$, for $|w| < \rho < r$, and $\gamma_r = re^{2\pi i t}$, we have by Cauchy's integral formula,

$$f_n(w) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-w}dw.$$

which by assumption converges uniformly to the continuous function f,

$$f(w) = \lim_{n\to\infty} f_n(w) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-w}dw = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dw.$$

This function must hence be holomorphic in $B_r(z_0)$, and since the choice of z_0 was arbitrary we have that $f \in \mathcal{H}(\Omega)$. Since, $f_n'(w) = \int_\gamma \frac{f_n(z)}{(z-w)^2}dz$ and $\frac{1}{|z-w|^2} \leq \frac{1}{(r-\rho)^2}$ for $|z| = r$, the limit $\lim_{n\to\infty} f_n'(w)$ exists uniformly for $|w| \leq \rho$ and hence,

$$f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}dz$$

for any compact set $K \subset \Omega$, we can cover K with balls of the above type, and from this we can pick out a finite cover, from this a maximum bound can be picked hence proving the uniform convergence of the above limit. □

Theorem 1.1.11. (Montel) *Let $\mathcal{F} \subset \mathcal{H}(\Omega)$ such that for any compact set $K \subset \Omega$ there exists $M_K > 0$ such that for all $f \in \mathcal{F}$,*

$$|f(z)| \leq M_K.$$

Then for any sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}_{k\geq 1} \subset \{f_n\}_{n\geq 1}$ which converges uniformly on every compact subset of Ω. The limit is given by the Weierstrass theorem.

Proof

The proof is now a simple application of Arzela-Ascoli's theorem in analysis which says that given a uniformly bounded family of functions, there exists a subsequence that converges. Now, this convergent subsequence converges to a holomorphic function by Weierstrass' theorem. □

1.2 | Meromorphic Functions

We can use the tools developed about holomorphic functions to study certain functions that are holomorphic except at a few points. We can study such functions by studying the functions around the 'singularities'. The important tool to study such functions is the Laurent series expansion. With analytic functions we generalised polynomials to power series, with meromorphic functions, our goal is to similarly generalise rational functions.

1.2.1 | Singularities & Residues

We are interested in functions that are holomorphic in a neighborhood, except at some isolated points. These are similar to rational functions of the form $1/P(z)$. The zeros of the $P(z)$ are the problematic parts. To study such functions we study the behavior of the function in an annulus around the point where it's holomorphic.

Let Ω be the annulus $\rho_1 < |z| < \rho_2$, then for any function $f \in \mathcal{H}(\Omega)$, and a loop, $\gamma_r = re^{2\pi it}$, for $t \in [0,1]$, we have,

$$\int_{\gamma_r} f dz = \int_{[0,1]} f(re^{2\pi it})(2\pi i)re^{2\pi it}dt = 2\pi i \int_{[0,1]} g(re^{2\pi it})dt$$

where $g(z) = zf(z)$. So,

$$\tfrac{d}{dr}\int_{\gamma_r} f dz = 2\pi i \int_{[0,1]} g'(re^{2\pi it}) \cdot e^{2\pi it}dt = r^{-1}\int_{[0,1]} \tfrac{d}{dt}g(re^{2\pi it})dt = r^{-1}[g(r) - g(r)] = 0.$$

So the integral, $\int_{\gamma_r} f dz$ is independent of $\rho_1 < r < \rho_2$. For any $w \in \Omega$, define a holomorphic function $g \in \mathcal{H}(\Omega)$ by,

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \in \Omega, w \neq z \\ f'(w) & z = w \end{cases}$$

$$\int_{\gamma_r} \tfrac{f(z)-f(w)}{z-w}dz = \int_{\gamma_r} \tfrac{f(z)}{z-w}dz - \int_{\gamma_r} \tfrac{f(w)}{z-w}dz.$$

The second term equals $2\pi i f(w)$ if $|w| < r$ and is zero for $|w| > r$. For all $w \in \Omega$, we can find r_1, r_2 with $\rho_1 < r_1 < |w| < r_2 < \rho_2$, by the independence of $\int_{\gamma_r} g dz$ on r for all $\rho_1 < r < \rho_2$, we get,

$$f(w) = \frac{1}{2\pi i}\left[\int_{\gamma_{r_2}} \frac{f(z)}{z-w}dz - \int_{\gamma_{r_1}} \frac{f(z)}{z-w}dz\right]$$

we will exploit this formula to study these functions with singularities. Whenever its not problematic, we will assume $w = 0$.

Theorem 1.2.1. (Laurent Series) *Let $f \in \mathcal{H}(\Omega)$, where Ω is the annulus with $\rho_1 < |z| < \rho_2$. Then f can be uniquely written as,*

$$f(z) = \textstyle\sum_{n=-\infty}^{\infty} a_n z^n.$$

The series converges uniformly and absolutely for any compact set in Ω.

Proof

Similar to the proof of showing holomorphic functions are analytic, the proof involves expanding $1/(z-w)$. Let a_n be defined by,

$$a_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}dz$$

For $|w| < |z| = r_2$, we have, $1/(z-w) = \sum_{n=0}^{\infty} w^n/z^{n+1}$ and for $|w| > |z| = r_1$ we have, $1/(z-w) = -\sum_{m=0}^{\infty} z^m/w^{m+1} = -\sum_{n=-\infty}^{-1} w^n/z^{n-1}$, where $n = -m-1$. This gives us,

$$\frac{1}{2\pi i}\int_{\gamma_{r_2}} \frac{f(z)}{(z-w)}dz = \sum_{n=0}^{\infty} a_n w^n \text{ and } \frac{1}{2\pi i}\int_{\gamma_{r_1}} \frac{f(z)}{(z-w)}dz = -\sum_{n=-\infty}^{-1} a_n w^n.$$

Since, $f(w) = \frac{1}{2\pi i}\left[\int_{\gamma_{r_2}} \frac{f(z)}{z-w}dz - \int_{\gamma_{r_1}} \frac{f(z)}{z-w}dz\right]$, we have,

$$f(w) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

convergence follows from the convergence of $\sum_{n=-\infty}^{0} a_n z^n$ and $\sum_{n=0}^{\infty} a_n z^n$. For the uniqueness, let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, we can determine c_n, by the uniform convergence of $\sum_{n=-\infty}^{\infty} c_n z^n$, consider the integral

$$\int_{[0,1]} f(re^{2\pi it})e^{2\pi mt}dt = \sum_{n=-\infty}^{\infty} c_n \int_{[0,1]} r^n e^{2\pi i(n-m)t}dt = c_m r^m.$$

or we can write $c_n = r^{-n}\int_{[0,1]} f(re^{2\pi it})e^{2\pi imt}dt = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}dz$ which is the same as a_n. $\square$

Theorem 1.2.2. (Riemann Extension Theorem) *Let Ω be a disc of radius ρ around 0, and let $f \in \mathcal{H}(\Omega^*)$, $\Omega^* = \Omega\backslash\{0\}$. If*

$$zf(z) \to 0$$

as $z \to 0$, then there exists $F \in \mathcal{H}(\Omega)$ such that $F|_{\Omega^*} = f$.

Proof

For $w \in \Omega^*$ we have,

$$f(w) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where $a_n = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}dz$ for $\gamma_r \subset \Omega$. Let $M(r) = \sup_{|z|=r}|f(z)|$, by assumption we have, $rM(r) \to 0$ as $r \to 0$. So we have,

$$|a_n| = \left|\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^{n+1}}dz\right| = \left|\int_{[0,1]} f(re^{2\pi it})r^{-n}e^{-2\pi int}dt\right| \le r^{-n}M(r).$$

for $n \le -1$ the term $r^{-n-1}\cdot rM(r) \to 0$ as $r \to 0$. Since a_n is independent of r, a_n must be identically zero. Thus we have,

$$f(w) = \sum_{n=0}^{\infty} a_n z^n.$$

By Weirstrass theorem, $\{\sum_{n=0}^{m} a_n z^n\}_{m\ge 0} \subset \mathcal{H}(\Omega)$ converges to $\sum_{n=0}^{\infty} a_n z^n$ and hence $F := \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\Omega) \in \mathcal{H}(\Omega)$ with $F|_{\Omega^*} = f$. $\square$

1.2.2 | Meromorphic Functions

A rational function is a function of the form, $\frac{Q(z)}{P(z)}$, where $Q(z)$ and $P(z)$ are polynomials. Analytic functions were a generalization of polynomial functions, and we considered all power series. Now our goal is to study functions of the form

$$f(z) = \frac{g(z)}{h(z)}$$

where h and g are analytic functions, i.e., they can be locally written as power series.

A function f on Ω is meromorphic if it's holomorphic on Ω except at a finite number of points $\mathcal{E}$, such that around each point in $\mathcal{E}$, there exists a small disc $D \subset \Omega$ such that

$$f \cdot h|_D = g|_D.$$

with h and g being holomorphic functions on D.

Lemma 1.2.3. *Let Ω is a disc around 0, and let $f \in \mathcal{H}(\Omega^*)$, let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be its Laurent expansion at 0. f is meromorphic on D iff there exists an integer N such that $c_n = 0$ for $n < -N$.*

Proof

Suppose f is meromorphic on Ω, let B_ρ be a disc around 0 for which there exists two holomorphic functions $g, h \in \mathcal{H}(B_\rho)$ such that

$$f \cdot h|_{B_\rho} = g|_{B_\rho}.$$

Since $h \in \mathcal{H}(B_\rho)$ we can write it as, $h(z) = \sum_{n=0}^{\infty} h_n z^n$, let $N = \inf\{n \,|\, h_n \neq 0\}$. Now $h(z)$ can be written as, $h(z) = \sum_{n=0}^{\infty} h_n z^n = z^N \varphi(z)$, so by definition of N we have that $\varphi(0) = h_N$. So there exists some neighborhood U of 0 for which $\varphi(z) \neq 0$ and hence $g/\varphi \in \mathcal{H}(V)$.

If $g(z) = \sum_{n=0}^{\infty} g_n z^n$ is the power series expansion of g then we have,

$$f(z) = \textstyle\sum_{n=0}^{\infty} g_n z^{n-N}$$

By uniqueness of Laurent expansion we have that $a_n = g_{n+N}$.

The converse is much simpler, given $f(z) = \sum_{n=-N}^{\infty} a_n z^n$, we can write it as,

$$\underbrace{(z)^N}_{h(z)} f(z) = \underbrace{\textstyle\sum_{n=0}^{\infty} a_{n-N} z^n}_{g(z)}.$$

$\square$

Theorem 1.2.4. *Let $f \in \mathcal{H}(\Omega \backslash \mathcal{E})$. f is meromorphic on Ω iff for every $z_0 \in \mathcal{E}$, there exists a neighborhood U of z_0 with $U \cap \mathcal{E} = \{z_0\}$ such that*

$f|_{U\backslash\{z_0\}}$ is bounded, or, $|f(z)| \to \infty$ as $z \to z_0$.

Proof

If $f|_{U\setminus\{z_0\}}$ is bounded then by Riemann extension theorem, there exists a holomorphic function $g \in \mathcal{H}(U)$ such that $f|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}$, and hence we have,

$$1 \cdot f|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}.$$

If $|f(z)| \to \infty$ as $z \to z_0$ then by continuity there exists some disc B_ρ around 0 for which $|f(z)| \geq 1$, or $1/|f(z)|$ is bounded. So, applying Riemann extension to $1/f(z)$, we have, $1 \cdot \frac{1}{f(z)} = g(z)$, or

$$f \cdot g|_{B_\rho} = 1|_{B_\rho}.$$

Hence f is meromorphic.

For the other side, let f be a meromorphic function, let U be a neighborhood of $z_0 \in \mathcal{E}$ such that $U \cap \mathcal{E} = \{z_0\}$. Since f is meromorphic, we have,

$$hf|_{U\setminus\{z_0\}} = g|_{U\setminus\{z_0\}}.$$

h, g holomorphic on U. So we can write them as $h(z) = \sum_{n=0}^{\infty} h_n(z-z_0)^n = (z-z_0)^k\varphi(z)$ and $g(z) = \sum_{n=0}^{\infty} g_n(z-z_0)^n = (z-z_0)^l\varkappa(z)$, with $\varphi(z), \varkappa(z) \neq 0$. So, we have,

$$f(z) = (z-z_0)^{k-l}\varphi(z)/\varkappa(z)$$

if $k \geq l$, then f is bounded otherwise $|f(z)| \to \infty$ as $z \to z_0$. $\square$

Let f be a meromorphic function on an open set Ω, a point z_0 is said to be a pole of f if $|f(z)| \to \infty$ as $z \to z_0$. If it's not a pole then it can be extended to a holomorphic function by Riemann extension theorem.

For a meromorphic function f on Ω, let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series expansion at z_0. The order of the meromorphic function f at z_0 is defined as

$$\mathrm{ord}_{z_0}(f) \coloneqq \inf\{n \ : \ c_n \neq 0\}.$$

If $f \equiv 0$ at z_0 then we set $\mathrm{ord}_{z_0} \coloneqq \infty$. For the meromorphic function $1/(z-z_0)^n$, the order is n. This is an easy way to remember the order. It's also the smallest $(z-z_0)^n$ we have to multiply to remove the singularity. Clearly, z_0 is a pole of f if and only if the $\mathrm{ord}_{z_0}(f) < 0$. If the $\mathrm{ord}_{z_0}(f) = -1$ it's called a simple pole, example is $1/z$.

If f is a holomorphic function, then it has a zero at z_0 if and only if $\mathrm{ord}_{z_0}(f) > 0$. An example is $(z-z_0)^n$. Here $\mathrm{ord}_{z_0} > n$. This is called the order of zero at z_0. f is holomorphic at z_0 and $f(z_0) \neq 0$ if and only if $\mathrm{ord}_{z_0}(f) = 0$. If the $\mathrm{ord}_{z_0}(f) = 1$ it's called a simple zero, an example is z.

Some basic properties of order can be quickly checked. Let f, g be meromorphic on Ω,

$$\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$$

For $\lambda \in \mathbb{C}$,

$$\mathrm{ord}_{z_0}(\lambda f) = \mathrm{ord}_{z_0}(f)$$

For the sums,

$$\mathrm{ord}_{z_0}(f+g) \geq \min(\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)).$$

A pole is called essential if there are infinitely many $a_n \neq 0$ for $n < 0$, i.e., f is not a meromorphic function.

2 | Sheaf of Holomorphic Functions

In these notes we start looking at the étale space of sheaf of holomorphic functions. The étale space of holomorphic functions has special properties that allow us to use the techniques of covering spaces and elementary homotopy theory. We will assume some basic knowledge of homotopy and fundamental groups.

2.1 | Covering Spaces

Covering spaces have the same local topological properties as the base space but different global topological properties. Covering spaces can have less global constraints compared to base spaces (covering spaces with least constraints are called universal covers), and this allows the functions to them to have a bit more freedom to be weird compared to functions to the base space. This allows us to study certain properties of functions that wouldn't have been possible within the base space. This structure is again lost as we go down to the base space.

Definition 2.1.1. A local homeomorphism $\pi : C \to X$ is called a covering space if each $x \in X$ has a connected neighborhood U_x such every,

$$\pi^{-1}(U_x) = \coprod_{i \in I} \widehat{U}_{x_i},$$

such that, $\pi(\widehat{U}_{x_i}) \cong U_x$.

X is called the base space and C the cover. The neighborhoods U_x are called evenly covered neighborhoods.

Theorem 2.1.1. $\deg(\pi) \coloneqq |\pi^{-1}(x)|$ *is a constant for all* $x \in X$.

Proof

Let $x \in X$ and let $|\pi^{-1}(x)|$ be the number of elements in $\pi^{-1}(x)$. Let

$$\mathcal{P}_{|\pi^{-1}(x)|} = \{y \in X \mid |\pi^{-1}(y)| = |\pi^{-1}(x)|\},$$

This set is open because of local homeomorphism and since the complement is also open the set $\mathcal{P}_{|\pi^{-1}(x)|}$ is also closed. Since $x \in \mathcal{P}_{|\pi^{-1}(x)|}$, it's nonempty. $\square$

Since covering spaces are locally homeomorphic we can lift a function $f : Y \to X$ to a function $\widehat{f} : Y \to C$ using the local homeomorphism. These lifts allow us to study certain properties of the function f that were not possible within the base space due to constraints coming from its topology.

Definition 2.1.2. $\widehat{f} : Y \to C$ is called a lift of $f : Y \to X$ if $f = \widehat{f} \circ \pi$.

The local homeomorphism aspect of covering spaces allows us to lift paths and homotopies. The goal is to divide up the path into smaller paths such that each of these individual path belongs entirely to some evenly covered neighborhood and then lift each of them. Similarly for the homotopy, we can divide up the square of homotopy into smaller squares such that each smaller square belongs entirely to some evenly covered neighborhood and lift them.

Theorem 2.1.2. *Every path and homotopy of paths in X can be lifted to paths and homotopies in C.*

Sketch of Proof

Let $\pi : C \to X$ be a covering space and C be path connected. Let $\eta : I \to X$ be a path in X. The space X can be written as

$$X = \bigcup_{i \in J} U_i,$$

where U_is are evenly covered neighborhoods. The interval I can be finitely partitioned $I = \cup_{i \in K} I_i$ such that the image of each part I_i under η belongs entirely in one of U_i.[1] The initial point $\eta(0) \in X$ belongs to some neighborhood U_i we can lift the path using the local homeomorphism to get a lift of the path and continue from the end of this part. Without using the Lebesgue lemma this can be proved via induction.

Similarly, we can lift homotopies to the covering space. So if two paths are homotopic in the base space then their lifts are also homotopic if they have the same initial point. □

Every continuous map between spaces gives a group homomorphism between their fundamental groups, with the paths given by the composition of the path with the continuous map. If $f : Y \to X$ is a continuous map, then the induced homomorphism of fundamental groups is denoted by f_*. If $\pi : C \to X$ is a covering space then $\Pi_1(C)$ is a subgroup of $\Pi_1(X)$ and the degree of the map is given by,

$$\deg(\pi) = |\Pi_1(X) : \Pi_1(C)|.$$

the proof is routine well-definedness checks and injective verification. If $\pi : C \to X$ is a local homeomorphism with the curve lifting property then we can lift every path, and hence to each element in the fundamental group we get an element. $\pi : C \to X$ is a covering map if and only if it has the curve lifting property.

Theorem 2.1.3. *Let Y be connected, and X, Y, and C be Hausdorff. Let $\pi : E \to X$ be a covering space. If $\widehat{f_1}, \widehat{f_2}$ are two lifts of a continuous map $f : Y \to X$ such that $\widehat{f_1}(y) = \widehat{f_2}(y)$ for some $y \in Y$. Then $\widehat{f_1} = \widehat{f_2}$.*

$$\begin{tikzcd} & C \arrow[d, "\pi"] \\ Y \arrow[ur, bend left, "\widehat{f_2}"] \arrow[ur, bend right, "\widehat{f_1}"'] \arrow[r, "f"'] & X \end{tikzcd}$$

[1] this requires the Lebesgue lemma,

Sketch of Proof

Let $\widehat{f_1}, \widehat{f_2}$ be two lifts of f. Let

$$\mathcal{F} = \{y \mid \widehat{f_1}(y) = \widehat{f_2}(y)\}$$

This is an open set of Y because around each $\widehat{f_1}(y)$ we can choose a uniformly covered neighborhood U such that $U \cong \pi(U)$. Since f is continuous $f^{-1}(U)$ is open with $x \in f^{-1}(U)$ where $\widehat{f_1}|_U = \widehat{f_2}|_U$.

Consider $Y \backslash \mathcal{F}$. Since C is Hausdorff, let $y \in Y \backslash \mathcal{F}$ i.e., $\widehat{f_1}(y) \neq \widehat{f_2}(y)$ choose disjoint neighborhoods $\widehat{U}_1, \widehat{U}_2$ of $\widehat{f_1}(y)$ and $\widehat{f_2}(y)$ respectively such that

$$\pi|_{\widehat{U}_i} : U_i \cong U \subset X.$$

Then $V = f^{-1}(U) \subseteq Y \backslash \mathcal{F}$ or i.e., $Y \backslash \mathcal{F}$ is open. So, $\mathcal{F}$ is both open and closed. If there exists some $y \in Y$ for which $\widehat{f_1}(y) = \widehat{f_2}(y)$ then the lifts are the same. $\square$

Theorem 2.1.4. (Lifting Criterion) *Let $\pi : C \to X$ be a covering space with $\pi(c_0) = b_0$, let $f : Y \to X$ be a continuous map with Y a connected, locally path connected space, with $f(y_0) = x_0$, then, there exists a lift $\widehat{f} : Y \to C$ with $\widehat{f}(y_0) = c_0$ iff $f_*(\Pi_1(Y, y_0)) \leq \pi_*(\Pi_1(C, c_0))$.*

Sketch of Proof

If there exists a lift $\widehat{f} : Y \to C$ then we have $f = \pi \circ \widehat{f}$ and from this we have,

$$f_*(\Pi_1(Y)) = (\pi \circ \widehat{f})_*(\Pi_1(Y)) = \pi_*(\underbrace{\tilde{f}_*(\Pi_1(Y))}_{\subseteq \Pi_1(C)}) \subseteq \pi_*(\Pi_1(C)).$$

For the otherside we construct lifts using paths, as the space is assumed to be locally path connected and connected. For $y \in Y$, take any path from y_0 to y, say γ. Lift the path $f \circ \gamma$, $\widehat{f \circ \gamma}$ and define the lift of f to be,

$$\widehat{f}(y) := \widehat{f \circ \gamma}(1).$$

The fact that $f_*(\Pi_1(Y, y_0)) \leq \pi_*(\Pi_1(C, c_0))$ is used to prove the welldefinedness of this definition, i.e., the definition doesn't depend on the choice of path, and continuity uses the covering space aspect. $\square$

2.2 | Sheaf of Holomorphic Functions

The sheaf of holomorphic functions has some special properties. As we will show, the étale space of the sheaf of holomorphic functions is a Hausdorff space and hence the uniqueness theorems in the previous section are applicable. This allows us to apply theorems of algebraic topology to the study of holomorphic functions.

To each open set $\Omega \subseteq \mathbb{C}$ corresponds the associated collection of holomorphic functions $\mathcal{H}\Omega$. Each $\mathcal{H}\Omega$ has an algebra structure. Consider the sheaf of holomorphic functions,

$$\mathcal{H} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

such that for each open covering $\Omega = \bigcup_{i\in I} U_i$ of an open set Ω of $\mathbb{C}$ comes with an equalizer diagram,

$$\mathcal{H}\Omega \overset{e}{\dashrightarrow} \prod_i \mathcal{H}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{H}(U_i \cap U_j).$$

where p and q are the maps,

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i\cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i\cap U_j}.$$

The collection of open sets has a preorder given by, $V \geq U$ if $V \subset U$. For any directed collection $\mathcal{D}$ of open sets we have a directed system in **Sets** given by $\{\mathcal{H}U\}_{U\in\mathcal{D}}$. The stalk $\mathcal{H}_x$ of the sheaf $\mathcal{H}$ at x is the direct limit of the directed system $\{\mathcal{H}U_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{H}_x = \varinjlim_{x\in U} \mathcal{H}U.$$

stalks are functors, $\mathrm{Stalk}_x : \mathrm{PSh}(X) \to \mathbf{Sets}$ with, $\mathcal{H} \mapsto \mathcal{H}_x$. The elements of $\mathcal{H}_x$ are called germs of holomorphic functions at x.

$$\begin{aligned} \mathrm{germ}_x : \mathcal{H}\Omega &\to \mathcal{H}_x \\ f &\mapsto \mathrm{germ}_x(f). \end{aligned}$$

$\mathrm{germ}_x : \mathcal{H}\Omega \to \mathcal{H}_x$, is a homomorphism of the respective category for each Ω. If $f, g \in \mathcal{H}\Omega$ such that $\mathrm{germ}_x f = \mathrm{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset \Omega$ such that $f|_{U_x} = g|_{U_x}$ i.e., they have the same power series expansion around x.

Let $\mathrm{germ}_x f$, $\mathrm{germ}_x g \in \mathcal{H}_x$. Let f and g be the corresponding representatives with domains U_f and U_g respectively. We can define $\mathrm{germ}_x f + \mathrm{germ}_x g$ as the germ defined by the function $\mathrm{germ}_x(f+g)$ with representative $f+g$ on the domain $U_f \cap U_g$.

$$\mathrm{germ}_x f + \mathrm{germ}_x g = \mathrm{germ}_x(f+g).$$

Similarly, we can define multiplication by,

$$\mathrm{germ}_x f \cdot \mathrm{germ}_x g = \mathrm{germ}_x(f\cdot g).$$

$\mathcal{H}_x$ is a commutative ring with the above notion of addition, multiplication. The ring also contains the unit element given by the constant function. So $\mathcal{H}_x$ is an unital commutative ring. We can also define scaling operation as follows, for every $\lambda \in \mathbb{C}$, $\lambda \cdot \mathrm{germ}_x f = \mathrm{germ}_x(\lambda \cdot f)$. With these operations, the set $\mathcal{H}_x$ is a complex vector space.

Consider the set $\mathcal{I}_x$ of all non-units, i.e., non invertible elements in $\mathcal{H}_x$. Suppose $\mathrm{germ}_x f \in \mathcal{I}_x$ and suppose $f(x) \neq 0$ then there exists some neighborhood U of x such that $f|_U \neq 0$. Then $1/f$ on the neighborhood U is a function such that

$$\mathrm{germ}_x f \cdot \mathrm{germ}_x(1/f) = 1,$$

or $\mathrm{germ}_x f$ is a unit, contradicting the assumption. Conversely, if $\mathrm{germ}_x f$ is a unit element then there exists some g on some neighborhood U_g such that $\mathrm{germ}_x f\, \mathrm{germ}_x g = 1$. This means that $f(z) \cdot g(z) = 1$ on some neighborhood of x, or $f(x) \neq 0$. So the set of non-units is given,

$$\mathcal{I}_x = \{\mathrm{germ}_x f \mid f(x) = 0\} \subset \mathcal{H}_x.$$

Since $(f \cdot g)(x) = f(x)g(x) = 0$ whenever $f(x) = 0$ we have, $\mathrm{germ}_x f \cdot \mathrm{germ}_x g \in \mathcal{I}_x$ for all $\mathrm{germ}_x g$, whenever $\mathrm{germ}_x f \in \mathcal{I}_x$. The set $\mathcal{I}_x$ is an ideal in $\mathcal{H}_x$. Let $\mathcal{I}$ be any proper ideal, it can't contain any units because otherwise it $1 \in \mathcal{I}$ which in turn means that $\mathcal{I} = \mathcal{H}_x$ or that $\mathcal{I}$ is not a proper ideal. Since $\mathcal{I}_x$ is the collection of all non-units it must be a maximal ideal.

Theorem 2.2.1.

$$\mathcal{H}_x/\mathcal{I}_x \cong \mathbb{C}.$$

Proof

The evaluation map,

$$\mathrm{germ}_x f \mapsto f(x),$$

is a homomorphism of $\mathcal{H}_x$ onto $\mathbb{C}$ with kernel $\mathcal{I}_x$. Hence by isomorphism theorem, we have, $\mathcal{H}_x/\mathcal{I}_x = \mathbb{C}$. □

'Bundle' the sets $\mathcal{H}_x$ into a disjoint union and define the map,

$$\mathcal{E}\mathcal{H} = \coprod_{x \in X} \mathcal{H}_x \xrightarrow{\pi} X,$$

that sends each $\mathrm{germ}_x f$ to the point x. Each $f \in \mathcal{H}U$ determines a function $\widehat{f} : U \to \mathcal{E}\mathcal{H}$ that maps

$$\widehat{f} : x \mapsto \mathrm{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{H}$ by taking as base of open sets all the image sets $\widehat{f}(U) \subset \mathcal{E}\mathcal{H}$. This topology makes both π and $\widehat{f}$ continuous by construction.

Theorem 2.2.2. *$\pi : \mathcal{E}\mathcal{H} \to X$ is a local homeomorphism.*

Proof

Each point $\mathrm{germ}_x f$ in $\mathcal{E}\mathcal{H}$ has an open neighborhood $\widehat{f}(U)$. π restricted $\widehat{f}(U)$ is a homeomorphism of $\widehat{f}(U)$ and U. So $\pi : \mathcal{E}\mathcal{H} \to X$ is a local homeomorphism. □

The disjoint union $\coprod_{x \in X} \mathcal{H}_x$ together with the topology just described is the étale space of holomorphic functions on $\mathbb{C}$. The local homeomorphism above makes the étale space of holomorphic functions a two-dimensional manifold.

Theorem 2.2.3. *$\mathcal{E}\mathcal{H}$ is a Hausdorff topological space.*

Proof

Let $\mathrm{germ}_x f \in \mathcal{H}_x$, $\mathrm{germ}_y g \in \mathcal{H}_y$ such that $\mathrm{germ}_x f \neq \mathrm{germ}_y g$. If $x \neq y$, find disjoint neighborhoods U_f and U_g for the representatives of $\mathrm{germ}_x f$ and $\mathrm{germ}_y g$. Then we have $\mathrm{germ}_x f \in \widehat{f}(U_f)$ and $\mathrm{germ}_y g \in \widehat{g}(U_g)$ are disjoint neighborhoods. So whenever $\mathrm{germ}_x f \neq \mathrm{germ}_y g$ we can find disjoint open neighborhoods around them.

If $x = y$, then $\mathrm{germ}_x f \neq \mathrm{germ}_y\, g$ only if they have different power series expansion around x i.e., there exists a neighborhood U such that $f|_U \neq g|_U$. So

$$\widehat{f}(U) \cap \widehat{g}(U) = \varnothing,$$

because otherwise if there existed some $\mathrm{germ}_z\, h \in \widehat{f}(U) \cap \widehat{g}(U)$ it means $\mathrm{germ}_z\, f = \mathrm{germ}_z\, g$ which is a contradiction by choice of U.

So, whenever $\mathrm{germ}_x\, f \neq \mathrm{germ}_y\, g$, we can find disjoint open sets U, V such that $\mathrm{germ}_x\, f \in U$ and $\mathrm{germ}_y\, g \in V$. In other words, $\mathcal{EH}$ is Hausdorff. □

The complex derivative induces a map on the étale space of holomorphic functions. Let $\mathrm{germ}_x\, f \in \mathcal{H}_x$ with the representative function f with a domain U. Define the derivative d as the map,

$$\begin{aligned} d : \mathcal{EH} &\longrightarrow \mathcal{EH} \\ \mathrm{germ}_x\, f &\mapsto \mathrm{germ}_x(f') \coloneqq d\,\mathrm{germ}_x\, f. \end{aligned}$$

where f' is the complex derivative of the function f.

Theorem 2.2.4. *$d : \mathcal{EH} \to \mathcal{EH}$ is a covering space.* □

Proof

Let $\mathrm{germ}_x\, f \in \mathcal{H}_x$ and let f be a representative of $\mathrm{germ}_x\, f$ with domain U. Let F be a primitive of f, i.e., $F' = f$ for some $B \subset U$. Clearly,

$$d\widehat{(F + c)}(U) = \widehat{f}(U),$$

or $\widehat{(F + c)}(U) \subset d^{-1}(\widehat{f}(U))$ for all $c \in \mathbb{C}$.

If $d\widehat{g}(z) = \widehat{f}(z)$, then $\mathrm{germ}_z\, g' = \mathrm{germ}_x\, f$ or $g' = f$ in a neighborhood of z, so, $d/dz(g - F) = 0$ in the neighborhood or $g = F + c$ in the neighborhood. So we have for some neighborhood U,

$$d^{-1}\widehat{f}(U) = \coprod_{c \in \mathbb{C}} \widehat{(F + c)}(U).$$

Each $\widehat{(F + c)}(U)$ maps injectively onto $\widehat{f}(U)$, since d is a map from $\mathcal{EH}$ to $\mathcal{EH}$ it's hence a homeomorphism. So, $d : \mathcal{EH} \to \mathcal{EH}$ is a covering map. □

$\mathcal{EH}$ is a Hausdorff space and the map $d : \mathcal{EH} \to \mathcal{EH}$ is a covering map, so theorem 2.1.4 about uniqueness of lifts is applicable here. Consider a curve $\eta : I \to \Omega$, for every holomorphic function $f \in \mathcal{H}\Omega$ the curve induces a map,

$$\Gamma : I \to \mathbb{C} \to \mathcal{EH}$$

given by,

$$\begin{aligned} \Gamma :\ \ [0{,}1] &\to \mathcal{EH} \\ t &\mapsto \mathrm{germ}_{\eta(t)}\, f \in \mathcal{H}_{\eta(t)} \end{aligned}$$

So a primitive of f along η is the lifting of the function Γ with respect to $d : \mathcal{EH} \to \mathcal{EH}$,

$$\begin{tikzcd} & \mathcal{EH} \arrow[d, "d"] \\ I \arrow[ur, "\widehat{\Gamma}"] \arrow[r, "\Gamma"'] & \mathcal{EH} \end{tikzcd}$$

For fixed initial value or any common value, the uniqueness of 2.1.4 applies. The primitive of f along a curve η as defined above, $F = \widehat{\Gamma} : I \to \mathcal{EH}$. To each germ $\mathrm{germ}_{\eta(t)} f$ we have an association $\widehat{\Gamma}(t) \in \mathcal{EH}$ such that $d(\widehat{\Gamma}(t)) = \mathrm{germ}_{\eta(t)} f$.

Let $t \in I$ and let U_t be a neighborhood around $\eta(t)$ and h be such that,

$$h(\eta(t)) = \int_{[0,t]} f(\eta(s))\eta'(s)ds.$$

Let $F(t) = \mathrm{germ}_{\eta(t)} h$, by definition of d we have, $dF(t) = \mathrm{germ}_{\eta(t)} h' = \mathrm{germ}_{\eta(t)} f$. It can then be verified that F is a continuous, and hence a lift of f along η. This will work for any piecewise smooth continuous curve.

Since F is a primitive, any other primitive would be of the form $F + c$ and hence,

$$F(1)(\eta(1)) - F(0)(\eta(0)) = \int_{[0,1]} f(\eta(s))\eta'(s)ds. \tag{I}$$

If $f \in \mathcal{H}\Omega$, and $\eta : I \to \Omega$ is a continuous curve, define,

$$\int_\eta f dz = F(1)(\eta(1)) - F(0)(\eta(0))$$

where $F : I \to \mathcal{EH}$ is a primitive of f along η.

2.2.1 | Cauchy's Theorem & Monodromy

The covering space and Hausdorff properties of the étale space of holomorphic functions lets us apply theorems of covering spaces and homotopy theory. In particular, we can lift curves, homotopies.

Let $f \in \mathcal{H}\Omega$ and let η_1 and η_2 be two curves in Ω. Suppose they are homotopic with homotopy H, then for each $s \in I$ we have a curve H_s. Let Γ_s be the map,

$$\begin{aligned} \Gamma_s : \ & [0,1] \to \mathcal{EH} \\ & t \mapsto \mathrm{germ}_{H_s(t)} f, \end{aligned}$$

then Γ_s is a homotopy between Γ_0 and Γ_1 with fixed end points. Let $\widetilde{\Gamma_s}$ be the homotopy lift. We have, $\widetilde{\Gamma_0}(0) = \widetilde{\Gamma_1}(0)$ and similarly $\widetilde{\Gamma_0}(1) = \widetilde{\Gamma_1}(1)$.

Theorem 2.2.5. (Homotopy form of Cauchy's theorem) *Let $f \in \mathcal{H}\Omega$, suppose $\eta_1 \simeq \eta_2$ be two curves in Ω, then,*

$$\int_{\eta_1} f(z)dz = \int_{\eta_2} f(z)dz.$$

If Ω is simply connected and η is a loop then,

$$\int_\eta f(z)dz = 0.$$

□

The lifting properties can be applied to the local homeomorphism, $\pi : \mathcal{EH} \to \mathbb{C}$, if $\text{germ}_a f \in \mathcal{H}_a$ and $\text{germ}_a f$ can be continued analytically along $\eta_s = H_s$ then analytic continuation along η_1 and η_2 yield same germ at b.

2.3 | Winding number

The starting point is to study how many times a loop winds around a point. Finding this is however not possible within the base space. To do this we will lift the loop to the universal cover and then find a way to study the number of times it winds the point. This is similar to calculating the fundamental group of circle.

The starting point is to find a universal covering space. Consider the map,

$$\exp : z \mapsto e^z.$$

This is a local homeomorphism of $\mathbb{C}$ and $\mathbb{C}^*$ and around each point z the neighborhoods $B_\epsilon(z)$ with $\epsilon < 2\pi$ we have a local homeomorphism,

$$B_\epsilon(z) \cong e^{B_\epsilon(z)}.$$

So the map $z \mapsto e^z$ and hence the maps $z \mapsto x + e^z$ are covering maps as we have around each point a connected neighborhood thats homeomorphic to its image. Since the covering space is simply connected, it's the universal cover. Now we can start lifting paths, homotopies to the covering space. We will denote by $\mathbb{C}^*$ the complex plane without the point x.

Let $\eta : I \to \mathbb{C}$ be a loop or closed curve in $\mathbb{C}$ and let $x \in \mathbb{C}$ that's not in the image of η. So η is a loop in the base space and we can lift this loop to $\mathbb{C}$ using the above covering map $p : z \mapsto x + e^z$. Let $\widehat{\eta}_1$ and $\widehat{\eta}_2$ be two lifts then, $p \circ \widehat{\eta}_1 = p \circ \widehat{\eta}_2 = \eta$, so we have,

$$x + e^{\widehat{\eta}_1(0)} = x + e^{\widehat{\eta}_2(0)} = \eta(0)$$

So, $\widehat{\eta}_1(0) = \widehat{\eta}_2(0) + 2\pi ik$ for some $k \in \mathbb{Z}$. By uniqueness, of lifts, we have, $\widehat{\eta}_1 = \widehat{\eta}_2 + 2\pi ik$. So, the difference, $\widehat{\eta}(1) - \widehat{\eta}(0)$ is well defined, i.e., it's independent of the lift. Since η is a loop we have, $\eta(0) = \eta(1)$, and hence we have, $x + e^{\widehat{\eta}}(0) = x + e^{\widehat{\eta}(1)}$ or

$$\widehat{\eta}(0) = \widehat{\eta}(1) + 2\pi in(\eta, x)$$

for some integer $n(\eta, x)$. The winding number of η with respect to x is defined to be,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)].$$

Note that this depends on the covering map, and hence on the point x. Every time the curve η winds a circle the argument changes by 2π, since the starting point and the end point of the curve are the same the radius is the same for both start and end. Hence $1/2\pi i[\widehat{\eta}(1) - \widehat{\eta}(0)]$ represents the number of times the loop winds around the point x.

Lemma 2.3.1. *If η is a loop in $\mathbb{C}$ then $x \mapsto n(\eta, x)$ is locally constant on $\mathbb{C} \backslash \{Im(\eta)\}$ i.e., it's constant on each connected component.*

Proof

The proof goes by showing the map $x \mapsto n(\eta, x)$ is continuous map to $\mathbb{C}$. Since $n(\eta, x) \in \mathbb{Z}$, it has to be constant on every connected component. To compute the winding number $n(\eta, x)$ we have to lift the curve η to $\widehat{\eta}$ with respect to the covering map $z \mapsto e^z$, and compute $\widehat{\eta}(1) - \widehat{\eta}(0)$.

For every loop η in $\mathbb{C}$, we have to define new curves,

$$\eta_a(t) = \eta(t) - a$$

is a loop in $\mathbb{C}^*$. Let $B_\epsilon(w) = \{|w - x| < \epsilon\}$ be a disc that's entirely inside $\mathbb{C} \backslash \{Im(\eta)\}$. This now gives us a map,

$$(t, x) \mapsto \eta(t) - x$$

This is a continuous map from $I \times B_\epsilon(w)$ to $\mathbb{C}^*$. $I \times B_\epsilon(w)$ is simply connected, i.e., the fundamental group is zero and hence by lifting criterion, every continuous map can be lifted, to a continuous map,

$$\lambda : I \times B_\epsilon(w) \to \mathbb{C}$$

such that

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z \mapsto e^z"] \\ I \times B_\epsilon(w) \arrow[ur, "\lambda"] \arrow[r, "\eta(t)-x"'] & \mathbb{C}^* \end{tikzcd}$$

If $\widehat{\eta_x}$ is a lift of η_x with respect to the map, $z \mapsto e^z$, and in terms of λ it's $\widehat{\eta_x} = \lambda(\cdot, x)$. Hence the index is given by,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta_x}(1) - \widehat{\eta_x}(0)] = \frac{1}{2\pi i}[\lambda(1, x) - \lambda(0, x)].$$

So, the map $x \mapsto n(\eta, x)$ is a continuous map of $B_\epsilon(w)$ to $\mathbb{C}$. □

Now we need a way to compute the winding number exploiting the structure of complex numbers. We will use the structure of holomorphic functions to compute index of loops. To begin this, we construct a loop in $\mathcal{EH}$ using the loop η. For each point, z in we will associate a germ in $\mathcal{EH}$ and the compose with the loop η.

Theorem 2.3.2.

$$n(\eta, x) = \frac{1}{2\pi i} \int_\eta \frac{dz}{z - x}.$$

Proof

Consider the map

$$f_x(z) = 1/z - x.$$

This is the derivative of the logarithm function which when composed with the covering map $z \mapsto x + e^z$ yields identity. So, we can use this to construct lifts. Consider the function,

$$\nu : z \mapsto \text{germ}_z\, f_x$$

The composition of this map with η gives us a loop in $\mathcal{EH}$.

$$\Gamma = \nu \circ \eta : I \to \mathcal{EH}.$$

Our goal is to lift this map using the covering space $d : \mathcal{EH} \to \mathcal{EH}$ and use the primitive to construct a lift of the loop η, and thus relate the integral along η of the function $1/z - x$ to index.

Let $\widehat{\Gamma}$ be the lift of Γ with respect to the covering map $d : \mathcal{EH} \to \mathcal{EH}$. This associates to each $t \in I$, the germ $\widehat{\Gamma}\eta(t) \in \mathcal{H}_{\eta(t)}$. Let F with domain $\widehat{U}_{\eta(t)}$ be the representative of the germ $\widehat{\Gamma}(\eta(t))$. By definition of the derivative map d we have,

$$F'(z) = 1/(z - x).$$

So, we have, $F'(z)(z - x) - 1 = 0$. Multiplying by $e^{-F(z)}$ we have, $\frac{d}{dz}\left[(z - x)e^{-F}\right] = (1 - F'(z)(z - x))e^{-F(z)} = 0$. So, the term $(z - x)e^{-F(z)}$ is locally constant. We will use this function to construct a lift of the loop η. Consider the valuation map,

$$\begin{aligned} \Xi : I &\longrightarrow \mathcal{EH} \longrightarrow \mathbb{C} \\ t &\mapsto \widehat{\Gamma}(t)(\eta(t)) \end{aligned}$$

which evaluates the germ $\widehat{\Gamma}(t)$ at $\eta(t)$. Since $(z - x)e^{-F(z)}$ is locally constant on $U_{\eta(t)}$ with value α, so, the map, $t \mapsto (\eta(t) - x)e^{-\Xi(t)}$ is constant because it's local constant and I is compact. Let c be such that $e^c = \alpha$. Consider the map,

$$\widehat{\eta} : t \mapsto \Xi(t) + c$$

We claim that this is a lift of η. This is a simple check and we have to verify that $x + e^{\Xi(t)+c} = \eta(t)$. $x + e^{\Xi(t)+c} = x + e^{\Xi(t)}e^c = x + \alpha e^{\Xi(t)} = x + (\eta(t) - x)e^{-\Xi(t)}e^{\Xi(t)} = x + \eta(t) - x = \eta(t)$, i.e., $p \circ \widehat{\eta} = \eta$ or $\widehat{\eta}$ is a lift of η. So,

$$n(\eta, x) = \frac{1}{2\pi i}[\widehat{\eta}(1) - \widehat{\eta}(0)] = \frac{1}{2\pi i}\int_\eta \frac{1}{(z - x)}dz$$

as $\int_\eta f(z)dz = F(1)(\eta(1)) - F(0)(\eta(0))$ for F primitive of f along η. □

We can now start listing some immediate conseuences of this new relation between the line integral $\int_\eta \frac{dz}{z-x}$ and the winding number of the curve η with respect to x.

Lemma 2.3.3. *Let η_1, η_2 be two homotopic loops in $\mathbb{C}^*$, then, $n(\eta_1, 0) = n(\eta_2, x)$*

Proof

This is a simple application of the Cauchy's theorem for the map $z \mapsto (z)^{-1}$ which is holomorphic on $\mathbb{C}^*$. □

Lemma 2.3.4. *Let U be the unique connected component of $\mathbb{C}\backslash Im(\eta)$ that's unbounded, then, $n(\eta, x) = 0$ for all $x \in U$.*

Proof

Since for any loop η in a simply connected region U and $f \in \mathcal{H}U$, $\int_\eta f dz = 0$, we have,

$$\int_\eta \frac{dz}{z - x} = 0$$

for $x \in U$. Let $B_R(w)$ be the disc around w such that $Im(\eta)$ lies inside $B_R(w)$. Then for any $x \notin B_R(w)$, x lies 'outside' η. So, η can be collapsed in $B_R(w)$. So, $\int_\eta \frac{dz}{z-x} = 0$. Hence, $n(\eta, x) = 0$. Since it's constant on connected components, it must be zero for all of U. □

Lemma 2.3.5. *Let $\Omega \subset \mathbb{C}$ and $x, y \in \mathbb{C}\backslash\Omega$, then, there exists $f \in \mathcal{H}(\Omega)$ such that,*

$$e^{f(z)} = \frac{z-x}{z-y}, \quad z \in \Omega.$$

Sketch of Proof

Let $g(z) = \frac{z-x}{z-y}$, we will show that g'/g has a primitive and this will determine an $f \in \mathcal{H}(\Omega)$ with the required property.

$$g'/g = \frac{(x-y)/(z-y)^2}{(z-x)/(z-y)} = \frac{x-y}{(z-x)(z-y)} = \frac{1}{(z-x)} - \frac{1}{(z-y)}.$$

So, we have,

$$\int_\eta \left(\frac{1}{(z-x)} - \frac{1}{(z-y)}\right) dz = 2\pi i[n(\eta, x) - n(\eta, y)]$$

Since the winding number is constant on connected components, we have that $\int_\eta g'/g dz = 0$ for all η. Hence by Morera's theorem there exists a primitive h such that, $h' = g'/g$. Now we have,

$$e^{-h(z)}[-h'(z)g(z) - g'(z)] = 0$$

or,

$$g = \alpha e^h$$

Or, $g = e^{h+c}$ for some appropriate c. $f(z) = h(z) + c$ is the required function. □

Note that if the loop goes around a point x only once, the winding number will be 1. To prove this, we will have to create a small loop around the point and create a new loop, and reduce working with the harder outer loop to working with this small loop. We will however not prove this, as it's a bit irritating to track all the paths.

2.4 | Residue Theorem

Suppose $f \in \mathcal{H}(\Omega\backslash E)$ where E is a discrete set. Let the Laurent series expansion of f at x be $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$. The residue of f at $x \in E$ is defined to be,

$$\text{res}_f(x) = a_{-1}.$$

The principal part of f at x is defined to be,

$$g(z) = \sum_{-\infty}^{-1} a_n(z-x)^n.$$

The principal part is holomorphic on $\Omega\backslash\{x\}$. The residue has the nice property that,

$$f(z) - \frac{\text{res}_f(x)}{(z-x)}$$

has a primitive on some small annulus around x, given by, $\sum_{n\neq -1} a_n(z-a)^{n+1}/(n+1)$. Conversely, whenever there is a holomorphic function g such that, $g'(z) = f(z) - \frac{\alpha}{(z-x)}$, we have that,

$$0 = \int_\eta g'(z)dz = \int_\eta f(z)dz - \underbrace{\int_\eta \frac{\alpha}{(z-x)}dz}_{2\pi i\alpha}.$$

Lemma 2.4.1. (Residue Theorem) *$\Omega \subset \mathbb{C}$, E be a discrete set in Ω. Let η be a loop in $\Omega\backslash E$ that's is homotopic to a point as a curve in Ω. Then for any $f \in \mathcal{H}(\Omega\backslash E)$, $\{x \in E \ : \ n(\eta, x) \neq 0\}$ is finite and,*

$$\frac{1}{2\pi i}\int_\eta f(z)dz = \sum_{x\in E} \operatorname{res}_f(x)n(\eta, x). \tag{Residue}$$

Proof

Let $H : I \times I \to \Omega$ be the homotopy of η to the constant map. Let $K = H(I \times I)$, be the compact image of the homotopy. Since K is compact, $K \cap E$ is a finite set. If $x \notin K$, then η is homotopic to a point in $\mathbb{C}\backslash\{x\}$ and hence $n(\eta, x) = 0$. So $n(\eta, x) \neq 0$ for only finitely many points $x \in E$.

Let g_i be the principle part of f at $x_i \in K \cap E$. Then the function $f - \sum_i g_i$ is holomorphic on an open set U that contains the compact set K. So,

$$\int_\eta (f - \textstyle\sum_i g_i)dz = 0$$

or, $\int_\eta f dz = \int_\eta \sum_i g_i dz$. Now we will compute $\int_\eta \sum_i g_i dz$. Each $g_i(z) = \sum_{-\infty}^{-1} a_n(z - x_i)^n$. In this every term except $a_{-1}(z - x_i)^{-1}$ has a primitive, and hence the integral becomes,

$$\int_\eta g_i dz = 2\pi i a_{-1} \int_\eta \frac{dz}{z - x_i} = 2\pi i \operatorname{res}_f(x_i)n(\eta, x_i).$$

Hence we have,

$$\frac{1}{2\pi i}\int_\eta f(z)dz = \textstyle\sum_{x\in E} \operatorname{res}_f(x)n(\eta, x).$$

□

Let f be a meromorphic function on Ω. In a small neighborhood U, we can write $f(z)$ as,

$$f(z) = (z - x)^k g(z)$$

where $g \in \mathcal{H}(U)$. Where k is the order of the pole at x. Suppose $f(x) \neq 0$, the f'/f is a meromorphic function and, $\frac{f'(z)}{f(z)} = \frac{k}{z-x} + \frac{g'(z)}{g(z)}$. g'/g is holomorphic at x, and hence we have that

$$\operatorname{res}_{f'/f} x = \operatorname{ord}_x(f).$$

Lemma 2.4.2. (Global Cauchy's Formula) *$f \in \mathcal{H}(\Omega)$, $x \in \Omega\backslash\{x\}$, then*

$$n(\eta, x)f(x) = \frac{1}{2\pi i}\int_\eta \frac{f(z)}{z - x}dz.$$

Proof

Since $f \in \mathcal{H}(\Omega)$, $g(z) = f(z)/(z-x) \in \mathcal{H}(\Omega\backslash\{x\})$ with residue, $\operatorname{res}_g x = f(x)$. Hence we have by Residue theorem,

$$n(\eta, x)f(x) = \frac{1}{2\pi i}\int_\eta \frac{f(z)}{z-x}dz.$$

$\square$.

Lemma 2.4.3. (Argument Principle) *f be meromorphic on Ω, Z_f be zeros of f. P_f be the poles of f. Then assuming the poles and zeros are not in the image of a loop η,*

$$n(f\circ\eta, 0) = \frac{1}{2\pi i}\int_{f\circ\eta}\frac{1}{z}dz = \frac{1}{2\pi i}\int_\eta \frac{f'}{f}dz = \sum_{x\in Z_f\cup P_f} \operatorname{ord}_x(f)n(\eta, x).$$

Proof

This directly follows from the Residue theorem and the fact that, $\operatorname{res}_{f'/f} x = \operatorname{ord}_x f$. $\square$

A holomorphic function $f : \Omega \to \Omega'$ is called an analytic isomorphism of Ω onto Ω' if there exists a holomorphic function $g : \Omega' \to \Omega$ such that $g\circ f$ and $f \circ g$ are identity maps on Ω and Ω' respectively. If $\Omega = \Omega'$ it's called an analytic automorphism.

3 | Runge, Mittag-Leffler, Weierstrass

Runge's theorem says that if K doesn't have hole inside it, then every function analytic in a neighborhood of K can be approxmated on K by holomorphic functions on Ω. In case there is a hole, there could be some meromorphic function with the pole inside this hole, and it'll not be possible to approximate such functions with holomorphic functions on Ω.

The standard theory for detecting holes in spaces is cohomology and homology in algebraic topology, and hence we should expect Runge's theorem to have some deep relation to homology and cohomology.

3.1 | Runge's Theorem

Using the Taylor expansion, every analytic function on a disc can be approximated uniformly by polynomials in z on any smaller disc. Every entire function can be approximated by polynomials uniformly on every compact set. Runge's theorem generalizes this.

Let $\Omega \subseteq \mathbb{C}$ and let $K \subset \Omega$ be a compact subset. For any continuous function $f \in C(K)$, we define the norm of the function f on K by,

$$|f|_K = \sup_{z \in K} |f(z)|.$$

With this norm $C(K)$ is a Banach space. Using this we can define a topology on $\mathcal{H}(\Omega)$ by taking as neighborhoods the sets,

$$B_{\epsilon,K}(f) = \{g \in \mathcal{H}(\Omega) \mid |f - g|_K < \epsilon\}$$

With this topology, a sequence $\{f_n\}$ converges in $\mathcal{H}(\Omega)$ if and only if $\{f_n\}$ converges uniformly on any compact set in Ω. The topology is called compact open topology. Let $\mathcal{O}(K)$ denote the space of all continuous functions on K, that are restrictions of holomorphic functions on some bigger domain containing K, i.e., for all $f \in \mathcal{O}(K)$ there exists $U \supset K$, and $g \in \mathcal{H}(U)$ such that $g|_K = f$. Hence $\mathcal{O}(K) \subset C(K)$.

We now have the restriction map,

$$\rho : \mathcal{H}(\Omega) \to \mathcal{O}(K), \qquad \rho(f) = f|_K.$$

Runge's theorem says that the image of $\mathcal{H}(\Omega)$ under ρ is dense in $\mathcal{O}(K)$ iff K has no holes. Before going further, we note a few topological definitions, an open set $U \subset \Omega$ is said to be relatively compact if the closure $\overline{U}$ is compact. The boundary of an open set U with respect to Ω, $\partial_\Omega U$, is defined to be the set of all points z such that every neighborhood of z intersects with U and $\Omega \backslash U$.

Theorem 3.1.1. (Runge Approximation theorem) *Let $K \subset \Omega$ be a compact subset. Then the following conditions on Ω and K are equivalent.*

1. *$\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$.*
2. *No connected component of $\Omega \backslash K$ is relatively compact in Ω.*
3. *For every $z \in \Omega \backslash K$, there is a function $f \in \mathcal{H}(\Omega)$ such that,*

$$|f(z)| > |f|_K.$$

Sketch of Proof

The proof is a bit difficult, even though the ideas involved are simple and intuitive. We will only sketch the proof, and tell what is happening intuitively. This should make it less scary. For a detailed proof along these lines follow [2].

Note first that *2* is saying that K has no holes. Observe that *2* and *3* are the same statements. The maximum modulus principle allows us to say that K has no holes in terms of holomorphic functions. So, we just have to show *1* and *2* are equivalent.

1 $\Rightarrow$ *2*., Suppose *1* holds, i.e., every function in $\mathcal{O}(K)$ can be approximated by elements of $\rho(\mathcal{H}(\Omega))$. Now suppose *2* doesn't hold, i.e., the compact subset has a connected hole O, then we can choose any point inside O, w, and consider the function,

$$f(z) = 1/(z-w).$$

This belongs to $\mathcal{O}(K)$. But this cannot be approximated by functions in $\rho(\mathcal{H}(\Omega))$.

Suppose it can be approximated by holomorphic functions on Ω, then there exists, $\{f_n\} \subset \mathcal{H}(\Omega)$ such that $f_n|_K \to f|_K$ uniformly. Since each f_n are holomorphic, so is $f_n - f_m$, and by maximum modulus principle,

$$|f_n(z) - f_m(z)| \leq \sup_{z \in \overline{O}} |f_n(z) - f_m(z)| = \sup_{z \in \partial O} |f_n(z) - f_m(z)| \leq |f_n - f_m|_K$$

Since $|f_n - f_m|_K \to 0$, we have $f_n(z) - f_m(z) \to 0$ for all $z \in O$. Hence, $\{f_n|_O\} \to g \in \mathcal{H}(O)$. But we know that $f(z)(z-w) = 1$ and hence we must have $f_n(z)(z-w) \to 1$ uniformly on the boundary of the hole O, and hence on O. This means $g(z)(z-w) = 1$, which cannot be a holomorphic function. Hence there must not exist any connected holes O of K.

2 $\Rightarrow$ *1*., Firstly we note that,

$$\rho(\mathcal{H}(\Omega)) \subset \mathcal{O}(K) \subset C(K).$$

If a subspace W is dense in V then the action of a continuous functional is determined by it's action on the subspace W. Hahn-Banach theorem allows us to extend bounded linear functionals defined on subspaces to the whole space, the extension is unique if the subspace is dense. So, $\rho(\mathcal{H}(K))$ is dense in $\mathcal{O}(K)$ if and only if for every continuous linear form λ on $C(K)$, whenever $\lambda|_{\rho(\mathcal{H}(\Omega))} = 0$ implies $\lambda|_{\mathcal{O}(K)} = 0$. Continuous linear functional λ on functions on locally compact spaces correspond to measures μ. So, to show $\rho(\mathcal{H}(\Omega))$ is dense in $\mathcal{O}(K)$, we have to show that if for a measure $\mu(f) = 0$ for all $f \in \rho(\mathcal{H}(\Omega))$ then $\mu(g) = 0$ for all $g \in \mathcal{O}(K)$.

What we are trying to do is, if a function f has poles outside K, then we can push it to infinity, so that the function can be approximated by elements in $\rho(\mathcal{H}(\Omega))$. It's enough to show that $f(z,w) = (z-w)^{-1} \in \mathcal{O}(K)$, for $w \in \mathbb{C} \backslash K$, can be approximated by elements in $\rho(\mathcal{H}(\Omega))$, i.e., $f(z,w) \in \overline{\rho(\mathcal{H}(\Omega))}$. Because then, by taking products, we can approximate functions with higher order poles outside K.

Let λ be a continuous linear functional on $C(K)$ that vanishes on $\rho(\mathcal{H}(\Omega))$. Consider the function $f(z) = (z-w)^{-1}$ for $w \in \mathbb{C} \backslash K$. If $|w| > \sup_{z \in K} |z|$, then we have,

$$f(z,w) = -\sum_{i=0}^{\infty} \frac{z^n}{w^{n+1}}, \quad \forall\, z \in K.$$

Since λ vanishes on $\rho(\mathcal{H}(\Omega))$, λ vanishes for each term of the series, by continuity of λ we can take the summation inside. By continuity of λ,

$$\lambda\big(\lim_{h \to 0} \tfrac{f(z,w+h) - f(z,w)}{h}\big) = \lim_{h \to 0} \tfrac{\lambda(f(z,w+h)) - \lambda(f(z,w))}{h}.$$

So, $\lambda(f(-,w))$ is holomorphic on K with, $\frac{\partial(\lambda(f(-,w)))}{\partial w} = \lambda\big(\frac{\partial(f(-,w))}{\partial w}\big)$.

Suppose $f(z,w) \notin \overline{\rho(\mathcal{H}(\Omega))}$, consider the bounded linear functional defined on the span of $f(z,w)$ and $\rho(\mathcal{H}(\Omega))$,

$$\varphi\big(g(-) + t f(-,w)\big) \coloneqq t.$$

this bounded linear functional is such that $\varphi|_{\rho(\mathcal{H}(\Omega))} \equiv 0$, and not zero for $f(z,w)$. By Hahn-Banach theorem this can be extended to $C(K)$. But this is a contradiction because from our assumption any continuous functional λ with $\lambda|_{\rho(\mathcal{H}(\Omega))} \equiv 0$ must also vanish for $\mathcal{O}(K)$. But this functional isn't vanishing for $f(z,w) \in \mathcal{O}(K)$. Hence the assumption

$$f(z,w) \notin \overline{\rho(\mathcal{H}(\Omega))},$$

must be false. $\square$

Let $K \subset \Omega$ be a compact set. We define,

$$K_{\mathcal{H}(\Omega)} \coloneqq \{z \in \Omega \mid |f(z)| \leq |f|_K \text{ for all } f \in \mathcal{H}(\Omega)\}.$$

This set is called the holomorphic convex hull of K. This includes all points in K.

Intuitively this would be K if we fill up all the interior holes, because on this filled up compact space, by maximum modulus principle, the maxima can only be attained on the boundary. Intuitively, we will have for each point on the outer boundary a holomorphic functions that attain maxima at the point. The Runge's approximation is applicable to this filled up compact set.

3.1.1 | Homology form of Cauchy's Theorem

As we stated before, Runge's theorem relates to holes inside a compact set K, and naturally we should expect this to be related to homology and cohomology. We give here a version of Cauchy's theorem using Runge's theorem.

Let Ω be a connected open set in $\mathbb{C}$, two loops η_1, η_2 are homologous if they are boundary of the same surface. A loop $\eta : I \to \Omega$ in Ω, is said to be homologous to zero, denoted $\eta \sim_{\Omega} 0$, if there exists a surface whose boundary is η. We can now use complex analysis to understand when this is possible. Firstly we cannot have a surface whose boundary is the loop if there is an obstruction in Ω, i.e., a hole 'inside' the loop η. We can make this precise in terms of winding number.

For a closed curve η and $x \in \mathbb{C}$, the index $n(\eta, x)$ is zero if the point x is outside the curve, i.e., the curve doesn't wind around any point outside. So, the set of all points for which the winding number is nonzero must be inside Ω. So, we define that $\eta \sim_\Omega 0$ if the set,

$$S = \{x \in \mathbb{C} \backslash \mathrm{Im}(\eta) \mid n(\eta, x) \neq 0\} \subset \Omega$$

Lemma 3.1.2. *If η is homotopic to a point in Ω then $\eta \sim_\Omega 0$.*

Proof

We have to show if η is homotopic to a point in Ω then the set $S = \{x \mid n(\eta, x) \neq 0\}$ is contained in Ω. Suppose not, i.e., suppose there exists some $x \in \mathbb{C}$ not in Ω such that $n(\eta, x) \neq 0$. We have to arrive at a contradiction if η is homotopic to a point.

Let $x \in \mathbb{C}$ that's not in Ω. Since η is homotopic to a point in $\Omega \backslash \{x\}$. For homotopic loops η_1, η_2, we must have,

$$n(\eta_1, x) = n(\eta_2, x)$$

By applying Cauchy's theorem for the function $z \mapsto (z - x)^{-1}$ and monodromy theorem. Hence we have that $n(\eta, x) = 0$, or that it cannot be nonzero. □

The converse is not true, there can be loops homologous to zero but not homotopic to a point. For example, look up Pochhammer contour. This makes the homology form of Cauchy's theorem below more general than the homotopy form.

Theorem 3.1.3. (Cauchy's theorem; Homology Form) *Let Ω be connected open set. $\eta : I \to \Omega$ be a loop. If $\eta \sim_\Omega 0$ then $\forall f \in \mathcal{H}(\Omega)$,*

$$\int_\eta f(z)dz = 0.$$

Proof

Firstly we will sketch why any loop is homotopic to a piecewise differentiable curve. Since the loop η (compact set) is inside Ω, we can choose an $\epsilon > 0$ such that $B_\epsilon(\eta(t)) \subset \Omega$ for all $t \in I$, i.e., there is an ϵ width patch around η entirely contained in Ω. Now we can partition the interval $I = [0, t_1] \cup \cdots \cup [t_n, 1]$ such that each $[t_i, t_{i+1}]$ is inside one of the ϵ balls, and then connect the end points by a straight line which is a differentiable curve. So, we obtained a piecewise differentiable curve. Let this curve be Γ. Now we have a homotopy between the curves,

$$F(t, s) = (1 - s)\eta(t) + s\Gamma(t).$$

By the homotopy form of Cauchy's theorem we have reduced calculating the integral for η to calculating the integral for a piecewise differentiable curve. So, the index,

$$n(\eta, x) = n(\Gamma, x) = \int_\Gamma (z - x)^{-1} dz.$$

So, we have $\Gamma \sim_\Omega 0$. We now have to show that $\int_\Gamma f(z)dz = 0$ for $f \in \mathcal{H}(\Omega)$.

For any $f \in \mathcal{H}(\Omega)$, we can find, by Runge's theorem, a sequence of rational functions with poles outside Ω that converge uniformly to f on compact sets (in our case $\mathrm{Im}(\eta)$). Then,

$$\lim_{n\to\infty}\int_\Gamma R_n dz = \lim_{n\to\infty}\int_{[0,1]} R_n(\eta(t))\eta'(t)dt = \int_\Gamma f(z)dz.$$

By Residue theorem,

$$\tfrac{1}{2\pi i}\int_\Gamma R_n dz = \sum_{x\in E_n} n(\Gamma,x)\operatorname*{res}_{R_n}(x)$$

where E_n is the set of poles of R_n. Since $x \notin \Omega$, $n(\Gamma, x) = 0$, and hence the integral is zero. Thus $\int_\Gamma f dz = \lim_{n\to\infty}\int_\Gamma R_n dz$. □

This type of uniform convergence is very useful trick. Suppose $\Omega \subset \mathbb{C}$ such that $\mathbb{C}\backslash\Omega$ has no compact connected components, i.e., has no holes, then Runge's theorem is applicable. For any function $f \in \mathcal{H}(\Omega)$, there is a sequence $\{f_n\} \subset \mathcal{H}(\mathbb{C})$ such that $f_n \to f$ uniformly on any compact subset of Ω. For any loop η in Ω,

$$\int_\eta f dz = \lim_{n\to\infty}\int_\eta f_n dz = 0$$

By Morera's theorem, f must have a primitive.

3.1.2 | Mittag-Leffler Theorem

Mittag-Leffler theorem allows us to construct functions that have prescribed poles. Given a discrete set E of points and for each $x \in E$ a holomorphic function $f_x \in \mathcal{H}(\mathbb{C}_x^*)$, i.e., holomorphic for $\mathbb{C}\backslash\{x\}$, then one can construct a function f such that $f - f_x$ is holomorphic at x for all $x \in E$.

If the collection of points E is a finite set, then this is a triviality, we can just add these functions $\sum_{x\in E} f_x$ and it works. The problem now is adding up an infinite collection and making sure it doesn't blow up. This is possible due to Runge's theorem.

Theorem 3.1.4. (Mittag-Leffler) *$E \subset \Omega$ discrete. Let $f_x \in \mathcal{H}(\mathbb{C}_x^*)$. Then there exists $f \in \mathcal{H}(\Omega\backslash E)$ such that $f - f_x$ holomorphic at $x \in E$.*

Proof

The idea is to use the Runge's approximation what restricted to compact subsets. Let $K \subset \Omega$ be a compact, and let $\widehat{K}$ be its holomorphic convex hull. There exists a sequence of holomorphically convex compact sets $\{K_i\}_{i\geq 1}$ with $K_i \subset \mathrm{int}(K_{i+1})$ such that,

$$\bigcup_{i\geq 1} K_i = \Omega$$

Since each $\widehat{K} \cap E$ is a finite set, we can define the sum,

$$g_i := \textstyle\sum_{x\in E\cap K_i} f_x$$

Now to use Runge's theorem, consider the functions $h_i = g_{i+1} - g_i = \sum_{x\in E\cap(K_{i+1}\backslash K_i)} f_x$. The poles of this lie outside K_i, and hence $g_{i+1} - g_i \in \mathcal{O}(K_i)$. So, now we can use Runge's theorem. Since $\mathcal{O}(K_i)$ can be approximated with holomorphic functions upto any amount of closeness, there exists a function $h_i \in \mathcal{H}(\Omega)$, such that,

$$\|(g_{i+1} - g_i) - h_i\| \leq \tfrac{1}{2^i}$$

The purpose of taking $1/2^i$ is because sum of these terms is well behaved, i.e., for the convergence issues. So, now we can add up. Define,

$$f \coloneqq \textstyle\sum_{i\geq 1} \big((g_{i+1} - g_i) - h_i\big)$$

Now this function converges uniformly on Ω. The convergence is by design, we used the $1/2^i$, so that the norm of the sum, will be less than the sum $\sum_i 1/2^i$. Hence this is the required function. $\square$

[1] W Schlag, A Course in Complex Analysis and Riemann Surfaces, AMS, 2010

[2] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000