Part III

Introduction to Sheaves

Sheaf theory makes precise the process of gluing together locally defined properties of topological spaces or more generally 'sites'. In these notes we study the basics of the theory of sheaves.

1 | Sheaves and Presheaves

The first step is to axiomatize this local nature we expect from the space. Given a topological space X, a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X. Denote by CU the set of real-valued continuous functions on U. Then every function, $f \in CU$ has the following local properties,

If $V \subset U$ then f restricted to V is a continuous map, $f|_V : V \to \mathbb{R}$. The map, $f \mapsto f|_V$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_V)|_W = f|_W.$$

This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\mathrm{op}}$ is the dual category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\mathrm{op}}$ we get restriction morphism in **Sets**, $\{V \subset U\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_V$.

This captures the property of 'local' objects. The mathematical objects that have this property are called pre-sheaves.

Definition 1.1. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps and **Sets** has a class of morphisms called restriction maps $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that, $\mathrm{res}_{VW} \circ \mathrm{res}_{UV} = \mathrm{res}_{UW}$.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following property of continuous functions, Let $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$\begin{tikzcd} CU \arrow[r, dashed, "e"] & \displaystyle\prod_i CU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \displaystyle\prod_{i,j} C(U_i \cap U_j). \end{tikzcd}$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

Definition 1.2. A sheaf of sets $\mathcal{F}$ on a topological space X is a functor, $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to$ **Sets**, such that each open covering $U = \bigcup_{i \in I} U_i$ of an open set U of X yields an equalizer diagram.

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \displaystyle\prod_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \displaystyle\prod_{i,j} \mathcal{F}(U_i \cap U_j). \end{tikzcd}$$

where the maps $e, p,$ and q are the unique maps such that the diagram,

$$\begin{tikzcd} & \mathcal{F}U_i \arrow[rr, "\mathcal{F}(U_i \cap U_j \subset U_i)"] & & \mathcal{F}(U_i \cap U_j) \\ \mathcal{F}U \arrow[ur] \arrow[r, dashed, "e"] \arrow[dr] & \displaystyle\prod_i \mathcal{F}U_i \arrow[u] \arrow[d] \arrow[rr, shift left, "p"] \arrow[rr, shift right, "q"'] & & \displaystyle\prod_{i,j} \mathcal{F}(U_i \cap U_j) \arrow[u] \arrow[d] \\ & \mathcal{F}U_j \arrow[rr, "\mathcal{F}(U_i \cap U_j \subset U_j)"] & & \mathcal{F}(U_i \cap U_j) \end{tikzcd}$$

commutes for all $i, j \in I$. The vertical maps are projections of the products.

The sets $\mathcal{F}$ usually come with additional structure. The sheaf of continuous functions is a sheaf of algebras over $\mathbb{R}$ or the sheaf of module over the ring $\mathbb{R}$. In this case, we have,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}U \arrow[r, dashed, "e"] & \displaystyle\prod_i \mathcal{F}U_i \arrow[r, "p-q"] & \displaystyle\prod_{i,j} \mathcal{F}(U_i \cap U_j). \end{tikzcd}$$

This is the same as the above equalizer diagram but the extra algebraic structure lets us write it this way. It gives us a left exact sequence i.e., $\mathrm{Im}(e) = \mathrm{Ker}(p - q)$. This structure will be later useful in defining sheaf cohomology.

Denote by $\mathrm{PSh}(X)$ the collection of all pre-sheaves over a topological space X. Each sheaf which is a functor from $\mathcal{O}(X)^{\mathrm{op}}$ to **Sets** can be considered an object and the natural transformation between the two pre-sheaves as morphism between these objects.

$$\mathcal{O}(X)^{\mathrm{op}} \overset{\mathcal{F}}{\underset{\mathcal{G}}{\rightrightarrows}} \mathbf{Sets}.$$

$\mathrm{PSh}(X)$ will denote the category of all sheaves of sets on X. A morphism between pre-sheaves $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation of functors. $\mathrm{PSh}(X)$ is a subcategory of the functor category,

$$\mathrm{PSh}(X) \rightarrowtail \widehat{\mathcal{O}(X)} = \mathbf{Sets}^{\mathcal{O}(X)^{\mathrm{op}}},$$

which is the category consisting of all functors from $\mathcal{O}(X)^{\mathrm{op}}$ to **Sets** where the functors are objects and the natural transformations are the morphisms.

Let $f : X \to Y$ be a continuous function of spaces then each pre-sheaf $\mathcal{F}$ on X yields a pre-sheaf $f_*\mathcal{F}$ on Y given by $(f_*\mathcal{F})V = \mathcal{F}(f^{-1}V)$ where $V \in \mathcal{O}(Y)$. It's the composite functor,

$$\mathcal{O}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathbf{Sets}$$

$f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$ under f. The map f_* is a functor,

$$f_* : \mathrm{PSh}(X) \to \mathrm{PSh}(Y).$$

This functor also satisfies $(fg)_* = f_*g_*$, to each continuous map f in the category of topological space we have an associated map in the category of pre-sheaves $\mathrm{PSh}(f) = f_*$. So to each topological space X we have an associated object $\mathrm{PSh}(X)$ and to each continuous map f we have a morphism f_*. This makes PSh a functor on the category of topological spaces. We will denote the category of sheaves on a topological space X by $\mathrm{Sh}(X)$. It's a full subcategory of $\mathrm{PSh}(X)$

$$\mathrm{Sh}(X) \rightarrowtail \mathrm{PSh}(X).$$

1.1 | Stalks and Sheafification

What we want to study is the behavior of a function in a neighborhood of a point. The starting point is the notion of direct limit. A directed system within a category $\mathcal{C}$ is a set of objects $\{C_i\}_{i\in I}$, where I has a preorder $\leq$, together with morphisms, $f_{ij} : C_i \to C_j$ such that $f_{ii} = \mathrm{Id}_{C_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$.

A direct limit of a directed system in a category $\mathcal{C}$ is an object C together with morphisms $\varphi_i : C_i \to C$ with the universal property described by the following diagram,

$$\begin{tikzcd}
 & C_i \arrow[d, "f_{ij}"] \arrow[ddl, bend right, "\tau"'] \arrow[ddr, bend left, "\varphi_i"] & \\
 & C_j \arrow[dl] \arrow[dr, "\varphi_j"] & \\
L & & C \arrow[ll, dashed, "\exists! t"]
\end{tikzcd}$$

All the categories of interest to us, such as the category of modules over some ring possess direct limits. We will not prove this fact. The direct limit as above will be denoted,

$$C = \varinjlim_{i\in I} C_i$$

Inclusion is a preorder on the collection of open sets given by

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For a pre-sheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in **Sets** given by $\{\mathcal{F}U\}_{U \in \mathcal{D}}$. We will focus on this particular directed system.

Definition 1.3. The stalk $\mathcal{F}_x$ of a pre-sheaf $\mathcal{F}$ at x is the direct limit of the directed system $\{\mathcal{F}U_i\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}U.$$

Stalks are functors,

$$\begin{aligned} \mathrm{Stalk}_x : \mathrm{PSh}(X) &\to \mathbf{Sets} \\ \mathcal{F} &\mapsto \mathcal{F}_x. \end{aligned}$$

The elements of $\mathcal{F}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{F}U$ then we denote it by $\mathrm{germ}_x f$. $\mathrm{germ}_x : \mathcal{F}U \to \mathcal{F}_x$, is a homomorphism of the respective category for each U.

If $f, g \in \mathcal{F}U$ such that $\mathrm{germ}_x f = \mathrm{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset U$ such that $f|_{U_x} = g|_{U_x}$. The neighborhoods U_x is an open cover of U and if $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$ is a sheaf then,

$$\mathcal{F}U \to \prod_{x \in U} \mathcal{F}U_x,$$

is an injective map and hence we have $f = g$ on U.

Combine the various sets $\mathcal{F}_x$ into a disjoint union,

$$\mathcal{E}\mathcal{F} = \coprod_x \mathcal{F}_x,$$

and define the map, $\pi : \mathcal{E}\mathcal{F} \to X$ that sends each $\mathrm{germ}_x f$ to the point x. Each $f \in \mathcal{F}U$ determines a function $\hat{f} : U \to \mathcal{E}\mathcal{F}$ given by,

$$\hat{f} : x \mapsto \mathrm{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{F}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{E}\mathcal{F}$. This topology makes both π and $\hat{f}$ continuous by construction. Each point $\mathrm{germ}_x f$ in $\mathcal{E}\mathcal{F}$ has an open neighborhood $\hat{f}(U)$. π restricted to $\hat{f} : U \to \hat{f}(U)$, is a homeomorphism. The space $\mathcal{E}\mathcal{F}$ together with the topology just defined is called the étale space of $\mathcal{F}$.

So we get a functor

$$\mathcal{E} : \mathrm{PSh}(X) \to \mathbf{Top},$$

which assigns to each pre-sheaf $\mathcal{F}$ of X a topological space $\mathcal{E}\mathcal{F}$. $\pi : \mathcal{E}\mathcal{F} \to X$ is a bundle. For a given pre-sheaf $\mathcal{F}$, consider the collection of sections of the bundle $\mathcal{E}\mathcal{F}$, denoted $\Gamma\mathcal{E}\mathcal{F}$. A section is a continuous map $\hat{s} : X \to \mathcal{E}\mathcal{F}$ such that $\pi \circ \hat{s} = Id$.

Note that a bundle over an object X in a category $\mathcal{C}$ is simply an object E of $\mathcal{C}$ equipped with a morphism p in $\mathcal{C}$ from E to X.

$$p : E \to X.$$

In our case, the category $\mathcal{C}$ is the category of topological spaces **Top**.

The collection of sections is a pre-sheaf over X because, it assigns to each open subset $U \subset X$ the corresponding set of sections over U and we have the obvious restriction map, i.e., restriction of the continuous map to the smaller domain. It's also a sheaf because s_i are sections of U_i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section s defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{EF}$ is a sheaf over X.

$$\Gamma\mathcal{EF} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets},$$

that assigns to each $U \in \mathcal{O}(X)$ the set $\Gamma\mathcal{EF}(U) = \coprod_{x \in U} \mathcal{F}_x$. For each open subset $U \subset X$ there is a function,

$$\eta_U : \mathcal{F}U \to \Gamma\mathcal{EF}(U),$$
$$f \mapsto \hat{f}.$$

The natural transformation of functors, $\eta : \mathcal{F} \mapsto \Gamma\mathcal{EF}$, maps pre-sheaves to sheaves. It's called sheafification of $\mathcal{E}$.

Theorem 1.1. *If the pre-sheaf $\mathcal{F}$ is a sheaf, then η is an isomorphism. $\mathcal{F} \cong \Gamma\mathcal{EF}$.*

Sketch of proof

The injectivity part is simple, we have to show $\hat{f} = \hat{g}$ implies $f = g$. This is true because if $\hat{f} = \hat{g}$ then $\operatorname{germ}_x f = \operatorname{germ}_x g$ for every $x \in U$. So for each $x \in U$ we have a neighborhood U_x for which $f|_{U_x} = g|_{U_x}$. Since $\mathcal{F}$ is a sheaf the collation property implies the uniqueness and we have $f = g$.

For surjectivity, we have to construct a function $f \in \mathcal{F}U$ for every continuous section h of $\mathcal{EF}$. Since h is a section, we have for each $x \in U$ a germ $\operatorname{germ}_x f_x \in \mathcal{EF}$ such that,

$$h(x) = \operatorname*{germ}_x f_x,$$

where $f_x \in \mathcal{F}U_x$. Now since h is continuous and $\hat{f}_x(U_x)$ is an open set, so there must exist open set $V_x \subset U_x$ such that $h(V_x) \subset \hat{f}_x(U_x)$ i.e., $h = \hat{f}_x$ on V_x. Now we have to verify these functions agree on intersections. This is true because they give rise to the same germs. Then by collation property there exists a function f such that $f|_{V_x} = f_x$. $\square$

Note that the above proof also establishes an isomorphism between $\mathcal{F}_x$ and $\Gamma\mathcal{EF}_x$ for all pre-sheaves. The stalkwise isomorphism holds for pre-sheaves. Sheaves are exactly the pre-sheaves that tie its stalks into a bundle. This stalkwise isomorphisms also guarantees that the sheafification is a universal solution i.e., $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, then it factors through $\Gamma\mathcal{EF}$, this follows from our construction, where our starting point was stalks.

1.2 | Sheaves as Étale Spaces

The identification of sheaves with the sheaves of sections of a bundle suggests that a sheaf $\mathcal{F}$ on X can be replaced by the corresponding bundle $\pi : \mathcal{E}\mathcal{F} \to X$, and that this bundle is always a local homeomorphism. In this section, we show that the opposite is also true. Every 'étale bundle' can be interpreted as a sheaf.

Definition 1.4. A bundle $\pi : E \to X$ is said to be étale if π is a local homeomorphism i.e., to each $e \in \mathcal{E}$ there exists an open set $e \in V$ such that $\pi(V) \subset X$ is open and $\pi|_V$ is a homeomorphism.

Étale spaces of a pre-sheaf over X is clearly an étale bundle. The projection $\pi : X \times \mathbb{R} \to X$ is not a étale map because open sets in $X \times \mathbb{R}$ are of type $U \times V$ and this can never be homeomorphic to an open neighborhood of X. Similarly, vector bundles are not étale. Note that the definition of étale space is different from that of covering space, a covering space is a map $p : C \to X$ such that each point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ can be written as the disjoint union of homeomorphic open sets of C. Étale spaces generalize covering spaces. Every covering space is an étale space. Both étale spaces and covering spaces of topological manifolds are of same dimension as the base space.

A morphism between two bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ is a map φ_{12} such that the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi_{12}"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

The collection of all bundles over X with the above notion of morphism is a category. Denote by **Bund** X the category of all bundles over X. Denote by **Etale** X the collection of all étale bundles over X. **Etale** X is a full subcategory of **Bund** X.

In the previous subsection, we associated to each sheaf $\mathcal{F}$ a bundle $\mathcal{E}\mathcal{F}$

$$\mathcal{E} : \mathcal{F} \mapsto \mathcal{E}\mathcal{F},$$

and the sheaf of sections of this bundle $\Gamma\mathcal{E}\mathcal{F}$ was identified with the sheaf itself. Now we are interested in is associating to each étale bundle $\mathcal{E}$ over X a sheaf. If $p : Y \to X$ is a bundle then $\Gamma : Y \to \Gamma Y$, maps the bundle Y to the sheaf of sections of Y. Associate to this sheaf the corresponding étale space $\mathcal{E}\Gamma Y$.

Theorem 1.2. *For any space X we have an equivalence of categories,*

$$\mathrm{Sh}(X) \rightleftarrows \mathbf{Etale}\, X \rightarrowtail \mathbf{Bund}\, X$$

Sketch of Proof

Our aim is now to define a natural transformation of bundles, $\epsilon : \mathcal{E}\Gamma Y \mapsto Y$, and show that if the bundle $p : Y \to X$ is étale then ϵ is an isomorphism.

The étale space $\mathcal{E}\Gamma Y$ consists of elements of the form $\hat{s}(x)$ for some point $x \in X$ and some section $s : U \to Y$. Define ϵ as follows,

$$\epsilon(\hat{s}(x)) = s(x).$$

Note that this definition is independent of the choice of s because if t is some other representative of the same germ $\hat{s}(x)$ at x then $s = t$ in some neighborhood, so it would mean $s(x) = t(x)$. When the bundle is étale we need to show there exists an inverse to ϵ. Suppose $p : Y \to X$ is étale, to each point $y \in Y$ with $p(y) = x$ there is a neighborhood U of x and a section $s : U \to Y$ such that $s(x) = y$. Define the inverse θ to ϵ as,

$$\theta : y \mapsto \hat{s}(x).$$

This is well defined and is the inverse of ϵ. □

We will usually be working with sheaves that have additional algebraic structures. The sheaf of continuous functions over a topological space will be a sheaf of abelian algebras for example. The suitable category for such sheaves is called an abelian category denoted by **Ab**. An abelian category is a category that has kernels, cokernels, direct sums, etc. i.e., if $\alpha : A \to B$ is a morphism in the category then $\ker \alpha$ is also an object in the category and so on for other properties. To avoid formality and details we will assume them to be abelian algebras.

References

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