Differential Geometry
via Sheaves

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The goal here is to define topological spaces on which we can do calculus. We make them locally look like Euclidean spaces and import calculus from the them. I will follow Ramanan's approach to differential geometry via sheaves, as he develops in the book [?].

1.1 | Sheaf of Differentiable Functions

Our starting point is a topological space $\mathcal{M}$ that's Hausdorff and admits a countable base for the topology. This condition is to make sure there are no pathological spaces we should be worried about. The Hausdorff condition makes the points distinguishable by the topology itself. The countable basis allows us to do analysis. On this topological space we want a differentiable structure, i.e., the structure that allows us to do calculus.

The differentiable structure allows us to define differentiable functions. We expect differentiable functions to have some form of local nature, similar to continuous functions. The notion of a sheaf axiomatizes this 'local nature'.

1.1.1 | Sheaves

Given a topological space $\mathcal{M}$, a sheaf is a way of describing a class of objects on $\mathcal{M}$ that have a local nature. To motivate the definition, consider the set of continuous functions on the space $\mathcal{M}$. Denote by CU the set of real-valued continuous functions on U. Then every function, $f \in CU$ has the following local properties,

If $V \subset U$ then f restricted to V is a continuous map, $f|_V : V \to \mathbb{R}$. The map, $f \mapsto f|_V$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_V)|_W = f|_W.$$

This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

where $\mathcal{O}(\mathcal{M})$ are open sets of $\mathcal{M}$ and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(\mathcal{M})^{\mathrm{op}}$ is the dual category of $\mathcal{O}(\mathcal{M})$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(\mathcal{M})^{\mathrm{op}}$ we get restriction morphism in **Sets**, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_V$.

This captures the property of 'local' objects. The objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

where morphisms in $\mathcal{O}(\mathcal{M})$ are inclusion maps and **Sets** has a class of morphisms called restriction maps $|_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that, $|_{VW} \circ |_{UV} = |_{UW}$.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions. Let $U = \bigcup_{i\in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i\cap U_j} = f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i\in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i\cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i\cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i\cap U_j} = f|_{U_i}|_{U_i\cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$\begin{tikzcd} CU \arrow[r, dashed, "e"] & \prod\limits_i CU_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j} C(U_i \cap U_j). \end{tikzcd}$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

A sheaf of sets $\mathcal{F}$ on a topological space $\mathcal{M}$ is a functor,

$$\mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

such that each open covering $U = \bigcup_{i\in I} U_i$ of an open set U of $\mathcal{M}$ yields an equalizer diagram.

$$\begin{tikzcd} \mathcal{F}U \arrow[r, dashed, "e"] & \prod\limits_i \mathcal{F}U_i \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod\limits_{i,j} \mathcal{F}(U_i \cap U_j). \end{tikzcd}$$

We start with what we expect from the 'differentiable' functions. The differentiable functions are continuous functions and hence satisfy the locality requirements and should form a sheaf. The sheaf of 'differentiable functions' is our starting point.

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}.$$

Since each differentiable function is expected to be a continuous function as well we have, $\mathcal{A}^{\mathcal{M}}(U) \subseteq C^{\mathcal{M}}(U)$, where $C^{\mathcal{M}}$ is the sheaf of continuous functions, $C^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets}$, on $\mathcal{M}$ i.e., $\mathcal{A}^{\mathcal{M}}$ is a subsheaf of $C^{\mathcal{M}}$.

1.1.1.1 | Differentiable Manifolds

Let $\mathcal{F}_n$ be the sheaf of differentiable functions on the Euclidean space $\mathbb{R}^n$. The 'locally looks like Euclidean space' means that the sheaf $\mathcal{A}^{\mathcal{M}}$ locally looks like differentiable functions over a Euclidean space.

A differentiable manifold is a Hausdorff, second countable topological space $\mathcal{M}$ together with a sheaf,

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

of subalgebras of $C^{\mathcal{M}}$ such that for any $x \in \mathcal{M}$ there is an open neighborhood $x \in U$ with a homeomorphism $U \cong_{\varphi} V \subseteq \mathbb{R}^n$, such that

$$(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{F}_n(V),$$

where $(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{A}^{\mathcal{M}}(\varphi^{-1}(V))$. This is easier to see in the diagram,

$$V \xrightarrow{\varphi^{-1}} U \xrightarrow{\mathcal{A}^{\mathcal{M}}} \mathcal{A}^{\mathcal{M}}U.$$

The pair $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ is called a differentiable manifold. The homeomorphisms φs are called coordinate charts and $\mathcal{A}^{\mathcal{M}}$ is called the differentiable structure. We will assume that each $\mathcal{A}^{\mathcal{M}}U$ to be an $\mathbb{R}$-algebra. The homeomorphisms transfer the smoothness on euclidean space to the manifold. The sections of $\mathcal{A}^{\mathcal{M}}U$ are called differentiable functions on U, and we can do calculus on them.

Clearly the Euclidean space is a differentiable manifold. For an open set $U \subset \mathcal{M}$ the pair $(U, \mathcal{A}^{\mathcal{M}}|_U)$ is a differentiable manifold. If $\cup_i U_i$ is an open cover of $\mathcal{M}$ then $(U_i, \mathcal{A}^{\mathcal{M}}|_{U_i})$ are open manifolds. Let $U_i \cong_{\varphi_i} V_i$ and U_i and U_j intersect, let $\varphi_i(U_i \cap U_j) = V_{ij}$ and $\varphi_j(U_i \cap U_j) = V_{ji}$ then,

$$V_{ij} \cong_{\varphi_j \circ \varphi_i^{-1}} V_{ji}.$$

So, a collection of differentiable manifolds $(U_i, \mathcal{A}_i)$ can be glued together to form a differentiable manifold if the homeomorphisms $\varphi_j \circ \varphi_i^{-1}$ map the restriction $\mathcal{A}_i|_{U_i \cap U_j}$ to $\mathcal{A}_j|_{U_i \cap U_j}$ i.e., differentiable maps are mapped to differentiable maps. This means that $\varphi_j \circ \varphi_i^{-1}$ is differentiable for every i, j.

$\mathcal{M}$ may be obtained by taking all the open sets U_i and pasting $U_{ij} \subset U_i$ to $U_{ji} \subset U_j$ together by the transition functions.

$$\coprod_{i,j} U_i \cap U_j \underset{q}{\overset{p}{\rightrightarrows}} \coprod_i U_i \xrightarrow{c} \mathcal{M}.$$

The map c sends all the points $x \in U_i$ to the same point $x \in \mathcal{M}$. c is the coequalizer of p and q in the category **Top** of topological spaces. This is parallel to the definition of sheaf.

A continuous map f of a differentiable manifold $\mathcal{M}$ into a differentiable manifold $\mathcal{N}$,

$$f : \mathcal{M} \to \mathcal{N},$$

is said to be differentiable if it locally maps differentiable functions to differentiable functions, i.e., for all $x \in \mathcal{M}$ if g is a differentiable function in a neighborhood U of $f(x)$ then $g \circ f$ is differentiable function on $f^{-1}(U)$. If $g \in \mathcal{A}^{\mathcal{N}}(U)$ then $g \circ f \in \mathcal{A}^{\mathcal{M}}(f^{-1}(U))$. Hence to each differentiable maps there is a homomorphism of the sheaf $\mathcal{A}_{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$ given by the map,

$$g \mapsto g \circ f.$$

This is the map

$$C^{\mathcal{N}} \to f_*(C^{\mathcal{M}}),$$

of sheaves on $\mathcal{N}$ which sends the subsheaf $\mathcal{A}^{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$. Differentiable manifolds together with morphisms like this is called the category of smooth manifolds. f_* is called the structure homomorphism associated to f. A differentiable map $f : \mathcal{M} \to \mathcal{N}$ of differentiable manifolds is called a diffeomorphism if there is a differentiable inverse.

1.1.2 | Stalks, Étale Spaces & Sheafification

What we want to study is the behavior of a function in a neighborhood of a point. The starting point is the notion of direct limit. A directed system within a category $\mathcal{C}$ is a set of objects $\{C_i\}_{i\in I}$, where I has a preorder $\leq$, together with morphisms, $f_{ij} : C_i \to C_j$ such that $f_{ii} = \mathbb{1}_{C_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$.

A direct limit of a directed system in a category $\mathcal{C}$ is an object C together with morphisms $\varphi_i : C_i \to C$ with the universal property described by the following diagram,

$$
\begin{tikzcd}
C_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr] & & C_j \arrow[dl, "\varphi_j"'] \arrow[ddl] \\
& C \arrow[d, dashed] & \\
& L &
\end{tikzcd}
$$

All the categories of interest to us (abelian categories), such as the category of modules over some ring possess direct limits also called colimit or inductive limit. We will not prove this fact in this part. The direct limit as above will be denoted,

$$C = \varinjlim_{i\in I} C_i$$

Inclusion is a preorder on the collection of open sets given by

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For a pre-sheaf $\mathcal{F} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in $\mathbf{Sets}$ given by $\{\mathcal{F}U\}_{U\in\mathcal{D}}$. We will focus on this particular directed system.

Definition 1.1.1. The stalk $\mathcal{F}_x$ of a pre-sheaf $\mathcal{F}$ at x is the direct limit of the directed system $\{\mathcal{F}U_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a directed set of open neighborhoods of x.

$$
\begin{tikzcd}
& \mathcal{F}U \arrow[d] \arrow[ddl, "\tau_U"'] \arrow[ddr, "\mathrm{germ}_x"] & \\
& \mathcal{F}V \arrow[dl] \arrow[dr, "\mathrm{germ}_x"] & \\
L & & \mathcal{F}_x \arrow[ll, dashed, "\exists! t"]
\end{tikzcd}
$$

i.e., for every $\tau_U : \mathcal{F}U \to L$ there exists a unique t as in the above diagram.

$$\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}U.$$

Stalks are functors,

$$\begin{aligned} \mathrm{Stalk}_x : \mathrm{PSh}(X) &\to \mathbf{Sets} \\ \mathcal{F} &\mapsto \mathcal{F}_x. \end{aligned}$$

The elements of $\mathcal{F}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{F}U$ then we denote it by $\text{germ}_x f$. $\text{germ}_x : \mathcal{F}U \to \mathcal{F}_x$, is a homomorphism of the respective category for each U.

If $f, g \in \mathcal{F}U$ such that $\text{germ}_x f = \text{germ}_x g$ for all $x \in U$ then it means that there exists some $U_x \subset U$ such that $f|_{U_x} = g|_{U_x}$. The neighborhoods U_x is an open cover of U and if $\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Sets}$ is a sheaf then,

$$\mathcal{F}U \to \prod_{x \in U} \mathcal{F}U_x,$$

is an injective map and hence we have $f = g$ on U. 'Bundle' the various sets $\mathcal{F}_x$ into a disjoint union,

$$\mathcal{E}\mathcal{F} = \coprod_x \mathcal{F}_x,$$

and define the map, $\pi : \mathcal{E}\mathcal{F} \to X$ that sends each $\text{germ}_x f$ to the point x. Each $f \in \mathcal{F}U$ determines a function $\hat{f} : U \to \mathcal{E}\mathcal{F}$ given by,

$$\hat{f} : x \mapsto \text{germ}_x f$$

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{E}\mathcal{F}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{E}\mathcal{F}$. This topology makes both π and $\hat{f}$ continuous by construction. Each point $\text{germ}_x f$ in $\mathcal{E}\mathcal{F}$ has an open neighborhood $\hat{f}(U)$. π restricted to $\hat{f} : U \to \hat{f}(U)$, is a homeomorphism. The space $\mathcal{E}\mathcal{F}$ together with the topology just defined is called the étale space of $\mathcal{F}$.

So we get a functor

$$\mathcal{E} : \text{PSh}(X) \to \mathbf{Top},$$

which assigns to each pre-sheaf $\mathcal{F}$ of X a topological space $\mathcal{E}\mathcal{F}$. $\pi : \mathcal{E}\mathcal{F} \to X$ is a bundle. For a given pre-sheaf $\mathcal{F}$, consider the collection of sections of the bundle $\mathcal{E}\mathcal{F}$, denoted $\Gamma\mathcal{E}\mathcal{F}$. A section is a continuous map $\hat{s} : X \to \mathcal{E}\mathcal{F}$ such that $\pi \circ \hat{s} = Id$.

Note that a bundle over an object X in a category $\mathcal{C}$ is simply an object E of $\mathcal{C}$ equipped with a morphism p in $\mathcal{C}$ from E to X.

$$p : E \to X.$$

In our case, the category $\mathcal{C}$ is the category of topological spaces **Top**.

The collection of sections is a pre-sheaf over X because, it assigns to each open subset $U \subset X$ the corresponding set of sections over U and we have the obvious restriction map, i.e., restriction of the continuous map to the smaller domain. It's also a sheaf because s_i are sections of U_i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section s defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{E}\mathcal{F}$ is a sheaf over X.

$$\Gamma\mathcal{E}\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Sets},$$

that assigns to each $U \in \mathcal{O}(X)$ the set $\Gamma\mathcal{E}\mathcal{F}(U) = \coprod_{x \in U} \mathcal{F}_x$. For each open subset $U \subset X$ there is a function,

$$\eta_U : \mathcal{F}U \to \Gamma\mathcal{E}\mathcal{F}(U),$$
$$f \mapsto \hat{f}.$$

The natural transformation of functors, $\eta : \mathcal{F} \mapsto \Gamma\mathcal{E}\mathcal{F}$, maps pre-sheaves to sheaves. It's called sheafification of $\mathcal{E}$. We will denote the sheafification $\Gamma\mathcal{E}\mathcal{F}$ of $\mathcal{F}$, by $\mathcal{F}^{\text{Sh}}$.

Theorem 1.1.1. *If the pre-sheaf $\mathcal{F}$ is a sheaf, then η is an isomorphism. $\mathcal{F} \cong \mathcal{F}^{\mathrm{Sh}}$.*

Sketch of proof

The injectivity part is simple, we have to show $\hat{f} = \hat{g}$ implies $f = g$. This is true because if $\hat{f} = \hat{g}$ then $\operatorname{germ}_x f = \operatorname{germ}_x g$ for every $x \in U$. So for each $x \in U$ we have a neighborhood U_x for which $f|_{U_x} = g|_{U_x}$. Since $\mathcal{F}$ is a sheaf the collation property implies the uniqueness and we have $f = g$.

For surjectivity, we have to construct a function $f \in \mathcal{F}U$ for every continuous section h of $\mathcal{E}\mathcal{F}$. Since h is a section, we have for each $x \in U$ a germ $\operatorname{germ}_x f_x \in \mathcal{E}\mathcal{F}$ such that,

$$h(x) = \operatorname{germ}_x f_x,$$

where $f_x \in \mathcal{F}U_x$. Now since h is continuous and $\hat{f}_x(U_x)$ is an open set, so there must exist open set $V_x \subset U_x$ such that $h(V_x) \subset \hat{f}_x(U_x)$ i.e., $h = \hat{f}_x$ on V_x. Now we have to verify these functions agree on intersections. This is true because they give rise to the same germs. Then by collation property there exists a function f such that $f|_{V_x} = f_x$. $\square$

Note that the above proof also establishes an isomorphism between $\mathcal{F}_x$ and $\mathcal{F}_x^{\mathrm{Sh}}$ for all pre-sheaves. The stalkwise isomorphism holds for pre-sheaves. Sheaves are exactly the pre-sheaves that tie its stalks into a bundle. This stalkwise isomorphisms also guarantees that the sheafification is a universal solution i.e., $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, then it factors through $\mathcal{F}^{\mathrm{Sh}}$, this follows from our construction, where our starting point was stalks.

The identification of sheaves with the sheaves of sections of a bundle suggests that a sheaf $\mathcal{F}$ on X can be replaced by the corresponding bundle $\pi : \mathcal{E}\mathcal{F} \to X$, and that this bundle is always a local homeomorphism. In this section, we show that the opposite is also true. Every 'étale bundle' can be interpreted as a sheaf.

Definition 1.1.2. A bundle $\pi : E \to X$ is said to be étale if π is a local homeomorphism i.e., to each $e \in \mathcal{E}$ there exists an open set $e \in V$ such that $\pi(V) \subset X$ is open and $\pi|_V$ is a homeomorphism.

Étale spaces of a pre-sheaf over X is clearly an étale bundle. The projection $\pi : X \times \mathbb{R} \to X$ is not a étale map because open sets in $X \times \mathbb{R}$ are of type $U \times V$ and this can never be homeomorphic to an open neighborhood of X. Similarly, vector bundles are not étale. Note that the definition of étale space is different from that of covering space, a covering space is a map $p : C \to X$ such that each point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ can be written as the disjoint union of homeomorphic open sets of C. Étale spaces generalize covering spaces. Every covering space is an étale space. Both étale spaces and covering spaces of topological manifolds are of same dimension as the base space.

A morphism between two bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ is a map φ_{12} such that the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi_{12}"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

The collection of all bundles over X with the above notion of morphism is a category. Denote by **Bund** X the category of all bundles over X. Denote by **Etale** X the collection of all étale bundles over X. **Etale** X is a full subcategory of **Bund** X.

In the previous subsection, we associated to each sheaf $\mathcal{F}$ a bundle $\mathcal{E}\mathcal{F}$

$$\mathcal{E} : \mathcal{F} \mapsto \mathcal{E}\mathcal{F},$$

and the sheaf of sections of this bundle $\Gamma\mathcal{E}\mathcal{F} = \mathcal{F}^{\mathrm{Sh}}$ was identified with the sheaf itself. Now we are interested in is associating to each étale bundle $\mathcal{E}$ over X a sheaf. If $p : Y \to X$ is a bundle then $\Gamma : Y \to \Gamma Y$, maps the bundle Y to the sheaf of sections of Y. Associate to this sheaf the corresponding étale space $\mathcal{E}\Gamma Y$.

Theorem 1.1.2. *For any space X we have an equivalence of categories,*

$$\mathrm{Sh}(X) \rightleftarrows \mathbf{Etale}\, X \rightarrowtail \mathbf{Bund}\, X$$

Sketch of Proof

Our aim is now to define a natural transformation of bundles, $\epsilon : \mathcal{E}\Gamma Y \mapsto Y$, and show that if the bundle $p : Y \to X$ is étale then ϵ is an isomorphism.

The étale space $\mathcal{E}\Gamma Y$ consists of elements of the form $\hat{s}(x)$ for some point $x \in X$ and some section $s : U \to Y$. Define ϵ as follows,

$$\epsilon(\hat{s}(x)) = s(x).$$

Note that this definition is independent of the choice of s because if t is some other representative of the same germ $\hat{s}(x)$ at x then $s = t$ in some neighborhood, so it would mean $s(x) = t(x)$. When the bundle is étale we need to show there exists an inverse to ϵ. Suppose $p : Y \to X$ is étale, to each point $y \in Y$ with $p(y) = x$ there is a neighborhood U of x and a section $s : U \to Y$ such that $s(x) = y$. Define the inverse θ to ϵ as,

$$\theta : y \mapsto \hat{s}(x).$$

This is well defined and is the inverse of ϵ. □

1.2 | Tangent and Cotangent Bundles

What we want to do is give a linear approximation of a manifold at each point. In order to do this, we use curves passing through the point, and linearize them, and then study them.

Around each point $x \in \mathcal{M}$, consider all the smooth functions $f \in \mathcal{A}^{\mathcal{M}}U$, i.e., $f : U \to \mathbb{R}$ defined in some open neighborhood U of $x \in \mathcal{M}$. For each smooth path $h : \mathbb{R} \to U$ which passes through x with $h(0) = x$ we can define a smooth map,

$$f \circ h : \mathbb{R} \to \mathbb{R},$$

which has a first derivative at 0. This gives us a pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0}, \qquad \text{(tangency pairing)}$$

To remove redundant information, we define the equivalences $f \equiv f'$ at x if $\langle f, h \rangle_x = \langle f', h \rangle_x$ for all h and $h \equiv h'$ at x if $\langle f, h \rangle_x = \langle f, h' \rangle_x$ for all f. Under addition and scalar multiplication of functions, this set of all equivalence classes of functions f forms a real vector space, denoted T^x. Each function f in the neighborhood of x determines a vector $[f]$.

The sheaf of differentiable functions has more algebraic structure. It's a sheaf of algebras over $\mathbb{R}$ or the sheaf of module over the ring $\mathbb{R}$. The category of modules over some ring possess direct limits. Inclusion is a preorder on the collection of open sets given by,

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For the pre-sheaf $\mathcal{A}^{\mathcal{M}} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in **Sets** given by $\{\mathcal{A}^{\mathcal{M}}U\}_{U\in\mathcal{D}}$. We will focus on this particular directed system.

The stalk $\mathcal{A}^{\mathcal{M}}_x$ of a pre-sheaf $\mathcal{A}^{\mathcal{M}}$ at x is the direct limit of the directed system $\{\mathcal{A}^{\mathcal{M}}U_i\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a directed set of open neighborhoods of x.

$$\begin{tikzcd}
 & \mathcal{F}U \arrow[d] \arrow[ddl, "\tau_U"'] \arrow[ddr, "\mathrm{germ}_x"] & \\
 & \mathcal{F}V \arrow[dl] \arrow[dr, "\mathrm{germ}_x"] & \\
L & & \mathcal{F}_x \arrow[ll, dashed, "\exists! t"]
\end{tikzcd}$$

i.e., for every $\tau_U : \mathcal{F}U \to L$ there exists a unique t as in the above diagram.

$$\mathcal{A}^{\mathcal{M}}_x = \varinjlim_{x\in U} \mathcal{A}^{\mathcal{M}}U.$$

Stalks are functors,

$$\begin{aligned}\mathrm{Stalk}_x : \mathrm{PSh}(\mathcal{M}) &\to \mathbf{Sets}\\ \mathcal{A}^{\mathcal{M}} &\mapsto \mathcal{A}^{\mathcal{M}}_x.\end{aligned}$$

The elements of $\mathcal{A}^{\mathcal{M}}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{A}^{\mathcal{M}}U$ then we denote it by $\mathrm{germ}_x f$. Note that $\mathcal{A}^{\mathcal{M}}_x$ is an algebra, and in particular a ring. $\mathrm{germ}_x : \mathcal{A}^{\mathcal{M}}U \to \mathcal{A}^{\mathcal{M}}_x$, is a homomorphism of the respective category for each U.

An ideal $\mathcal{I}^{\mathcal{M}}_x \subset \mathcal{A}^{\mathcal{M}}_x$ is a subalgebra such that if $\mathrm{germ}_x f \in \mathcal{I}^{\mathcal{M}}_x$ then $\mathrm{germ}_x f\, \mathrm{germ}_x g \in \mathcal{I}^{\mathcal{M}}_x$ for all $\mathrm{germ}_x g \in \mathcal{A}^{\mathcal{M}}_x$. A proper ideal cannot contain the identity because that would mean the whole algebra is contained in the ideal. For each $\mathrm{germ}_x f \in \mathcal{A}^{\mathcal{M}}_x$, the evaluation map, $\mathrm{germ}_x f \to f(x)$, gives us an algebra homomorphism,

$$\text{ß} : \mathcal{A}^{\mathcal{M}}_x \to \mathbb{R}.$$

The kernel of this evaluation map $\mathcal{I}^{\mathcal{M}}_x = \ker(\text{ß})$ is an ideal of $\mathcal{A}^{\mathcal{M}}_x$, consisting of all germs that vanish at x, i.e., $f(x) = 0$ and hence $f(x)g(x) = 0$ for all $g \in \mathcal{A}^{\mathcal{M}}$. Hence,

$$\mathcal{A}^{\mathcal{M}}_x / \mathcal{I}^{\mathcal{M}}_x \cong \mathbb{R}.$$

Evaluation can hence be thought of as taking quotient with the maximal ideal $\mathcal{I}^{\mathcal{M}}_x$. This is also the only maximal ideal, because all other functions have local inverse, because if a function f is non-zero in a small neighborhood it has an inverse, defined by, $\mathrm{germ}_x(1/f)$, and hence this would mean the constant function belongs to the ideal which means it's not proper ideal. So, no other proper ideal can exist.

Going back to the tangency pairing, the set of equivalence classes of paths h are called tangent vectors at x denoted by $T_x\mathcal{M}$. Each smooth path through x has a tangent vector denoted by τ_h. Using the pairing, we get a pairing of the equivalence classes.

$$\langle [f], \tau_h \rangle = \langle f, h \rangle_x.$$

The tangent vector τ_h determines a linear map, $D_{\tau_h} : T^x \to \mathbb{R}$ given by the action,

$$D_{\tau_h}([f]) = \langle [f], \tau_h \rangle.$$

We would like to understand what T^x is. The set $T_x\mathcal{M}$ of all tangent vectors at x spans the set of all linear maps $T^x \to \mathbb{R}$. That's to say, $T_x\mathcal{M}$ is the dual space of T^x, and hence is itself a vector space. We will hence denote T^x by $T_x^\vee\mathcal{M}$ or $\mathrm{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R})$.

1.2.1 | Tangent Sheaf

The derivative of a product, in the tangency pairing, the map $D = D_{\tau_h}$ satisfies the following product rule,

$$D(fg) = f(x)D(g)(x) + g(x)D(f)(x). \tag{Leibniz}$$

for all $f, g \in \mathcal{A}^\mathcal{M}$. This is called the Leibniz property, and all the maps D with the Leibniz property are called derivations. Conversely, every derivation there is a corresponding curve h such that $D_{\tau_h} = D$.[1] Note that we used the local homomorphism of $\mathcal{A}^\mathcal{M}$ and $\mathcal{F}_n$ here. For a local chart (U, φ), we have, $h \mapsto (\varphi \circ h)'(0)$ which is a well-defined bijection, of T_x and $\mathbb{R}^n$ with the inverse, $h_a(t) := \varphi^{-1}(\varphi(x) + ta)$. The linear maps,

$$D : \mathcal{A}_x^\mathcal{M} \to \mathbb{R},$$

with the above Leibniz property at x are called derivations, denoted by $T_x\mathcal{M}$.

The equivalence relation, $f \equiv f'$ iff $\langle f, h \rangle_x = \langle f', h \rangle_x$, for all h, uniquely determine the map D_{τ_h}. The equivalence classes are called cotangent vectors at x. By plugging in the constant 1, it can be checked that D annihilates constant functions.

$$D(\lambda) = 0 \quad \forall \lambda \in \mathbb{R}.$$

Hence all functions that differ by constant belong to the same equivalence class. Hence, for every $\mathrm{germ}_x\, f \in \mathcal{A}_x^\mathcal{M}$, we can consider the functions $\mathrm{germ}_x(f - f(x))$. These functions vanish at x, the action of D on the ideal $\mathcal{I}_x^\mathcal{M}$ is sufficient to describe the map D. So, we have a surjection of the ideal $\mathcal{I}_x^\mathcal{M}$ to the set of equivalence classes.

$$\mathcal{I}_x^\mathcal{M} \twoheadrightarrow T_x^\vee\mathcal{M}.$$

Now we have to remove all the redundant information from $\mathcal{I}_x^\mathcal{M}$. The kernel of the map is the ideal $(\mathcal{I}_x^\mathcal{M})^2 = \{\sum_{i,j} g_i f_j \ : \ f_i, g_j \in \mathcal{I}_x^\mathcal{M}\}$. D annihilates every element of this set because,$D(fg) = f(x)D(g) + g(x)D(f)$ and since $f, g \in \mathcal{I}_x$ this will be zero. Hence, we can quotient it out of $\mathcal{I}_x^\mathcal{M}$ and we have,

$$T_x^\vee\mathcal{M} \cong \mathcal{I}_x^\mathcal{M}/(\mathcal{I}_x^\mathcal{M})^2,$$

[1] The idea is to express it in local charts, and this should be of the form $\sum_i h_i \partial/\partial x_i$, and using this define a curve $h : t \mapsto \varphi^{-1}(t(h_i x_i))$. This works.

or that the equivalence class for the function f only contains the first order information of f. From the definition of a differentiable manifold around each x, there is a neighborhood U, such that,

$$\mathcal{A}^{\mathcal{M}}(\varphi^{-1}(U)) = \mathcal{F}_n(V)$$

for the local chart, φ. Since $\mathcal{F}_n$ consist of smooth functions on $\mathbb{R}^n$, we can describe them in terms of their Taylor expansion. If we denote the local coordinates by x_i, we have,

$$f(y) = f(\varphi(x)) + \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i}(\varphi(x))\right](\varphi_i(x) - x_i) + \sum_{i,j} \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi(x))\right](\varphi_i(x) - x_i)(\varphi_j(x) - x_j) + \cdots$$

Since the elements should be in $\mathcal{I}_x$, we have, $f(x) = 0$, and since we are quotienting out by $\mathcal{I}_x^2$, the higher order terms will go. The equivalence classes $[\varphi_i(x) - x_i]$ form a basis for the cotangent space. We denote them by dx_i.

The tangent space then is,

$$T_x\mathcal{M} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2, \mathbb{R}).$$

In local coordinates, the dual basis for the equivalence classes $[\varphi_i(x) - x_i]$ will be the equivalence classes $\partial/\partial x_i$. However we want to understand the structure of tangent spaces from a sheaf theoretic perspective.

For any derivation, $D \in T_x\mathcal{M}$, and $h \in \mathcal{A}^{\mathcal{M}}$,

$$h(x)D(fg) = h(x)f(x)D(g) + h(x)g(x)D(f).$$

If $D, D' \in T_x\mathcal{M}$, then their sum $D + D' \in T_x\mathcal{M}$. $hD \in T_x\mathcal{M}$. So, $T_x\mathcal{M}$ is an $\mathcal{A}_x^{\mathcal{M}}$-module. Using these derivations we can define the tangent sheaf.

Define $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ to be the set of all derivations. That is to say $D \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, if for all $f, g \in \mathcal{A}^{\mathcal{M}}U$,

$$D(fg) = fD(g) + gD(f).$$

Such operators $D : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}$ are called first order linear homogeneous differential operators. If $D, D' \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, then we can define a new operator $[D, D']$ defined by,

$$[D, D'](f) = D(D'(f)) - D'(D(f)). \tag{Lie bracket}$$

The geometric meaning of Lie bracket will become clear later. The tangent sheaf is the sheaf,

$$\begin{aligned} \mathcal{T} = \mathcal{D}(\mathcal{A}^{\mathcal{M}}) :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathcal{A}^{\mathcal{M}}}\mathrm{Mod} \\ U &\mapsto \mathcal{D}(\mathcal{A}^{\mathcal{M}}U). \end{aligned}$$

It is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

$\{f_i\}_{i=1}^n \mapsto \sum_i f_i \frac{\partial}{\partial x_i}$ is an isomorphism of modules $(\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$ and $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ for a chart (φ, U). Such sheaves of modules are called locally free modules. The tangent sheaf consists of the sections of the tangent bundle where the tangent bundle $T\mathcal{M}$ is the disjoint union,

$$T\mathcal{M} = \coprod_{x \in \mathcal{M}} T_x\mathcal{M}.$$

These sections correspond to vector fields. The stalks of this sheaf consist of germs of vector fields.

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) = \varinjlim_{x \in U} \mathcal{D}(\mathcal{A}^{\mathcal{M}}U).$$

So, $\operatorname{germ}_x D : \mathcal{A}_x^{\mathcal{M}} \to \mathcal{A}_x^{\mathcal{M}}$. The evaluation of the germs at the point x should give us vectors of the tangent space and they do. We evaluate

$$\mathcal{A}_x^{\mathcal{M}} \xrightarrow{\operatorname{germ}_x D} \mathcal{A}_x^{\mathcal{M}} \xrightarrow{\text{ß}} \mathbb{R}.$$

So, the composition of the derivation with this evaluation map corresponds to a derivation at x which are tangent vectors at x. The evaluation map gave us an isomorphism, $\mathcal{A}_x^{\mathcal{M}}/\mathcal{I}_x^{\mathcal{M}} \cong \mathbb{R}$. For locally free sheaves, for every $x \in \mathcal{M}$, there exists a neighborhood U such that $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) = (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$. So, we have,

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) \cong \mathbb{R}^n.$$

In this sense $\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$ is the space of valuation of the sections at x. This is the same as the tangent space at x.

$$T_x\mathcal{M} = \mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$$

We have a projection from the tangent bundle to the base space,

$$\pi : T\mathcal{M} \to \mathcal{M}$$

which sends $T_x\mathcal{M} \mapsto x$. Since $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$. So, the set $\pi^{-1}(U)$ can be given a differentiable structure of a product. These can be patched up to get a differentiable structure on $T\mathcal{M}$. This topology is Hausdorff and has a countable basis because locally it's a product of Hausdorff spaces with countable basis.[2] Smooth sections of this bundle are called vector fields.

1.2.2 | Cotangent Sheaf

Similarly we can consider the cotangent pre-sheaf,

$$\begin{aligned} \mathcal{C} = \mathcal{I}^{\mathcal{M}}/(\mathcal{I}^{\mathcal{M}})^2 :\ \mathcal{O}(\mathcal{M})^{\text{op}} &\to\ {}_{\mathcal{A}^{\mathcal{M}}}\text{Mod} \\ U &\mapsto \mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2 \end{aligned}$$

where $\mathcal{I}^{\mathcal{M}}U$ is the maximal ideal of $\mathcal{A}^{\mathcal{M}}U$. This might not be a sheaf however. So we might have to sheafify such pre-sheafs. The stalks of this pre-sheaf are given by,

$$\text{Stalk}_x : \mathcal{C} \mapsto \mathcal{C}_x = \mathcal{I}_x^{\mathcal{M}}/(\mathcal{I}_x^{\mathcal{M}})^2.$$

We can now 'bundle' these stalks together,

$$\mathcal{E}\mathcal{C} = \coprod_x \mathcal{C}_x,$$

and define the map, $\pi : \mathcal{E}\mathcal{C} \to \mathcal{M}$ that sends each $\operatorname{germ}_x f$ to the point x. Each $f \in \mathcal{C}U$ determines a function $\hat{f} : U \to \mathcal{E}\mathcal{C}$ given by,

$$\hat{f} : x \mapsto \operatorname{germ}_x f$$

[2]Note that in the case of Etale space, the properties of individual elements of the sheaf are used to get a topology, in the case of tangent bundle we used the properties of the sheaf itself to get a topology. We first quotiented the stalks with the maximal ideal of the ring of functions and then bundled them, and didn't care about the properties of the individual elements for the topology.

for $x \in U$. By using these 'sections', we can put a topology on $\mathcal{EF}$ by taking as base of open sets all the image sets $\hat{f}(U) \subset \mathcal{EF}$. This topology makes both π and $\hat{f}$ continuous by construction.

For the pre-sheaf $\mathcal{C}$, consider the collection of sections of the bundle $\mathcal{EC}$, denoted $\Gamma\mathcal{EC}$, i.e., is a continuous map $\hat{s} : \mathcal{M} \to \mathcal{EC}$ such that $\pi \circ \hat{s} = Id$. This collection of sections is a pre-sheaf over $\mathcal{M}$ because, it assigns to each open subset $U \subset \mathcal{M}$ the corresponding set of sections over U and we have the obvious restriction map, i.e., restriction of the continuous map to the smaller domain. It's also a sheaf because s_i are sections of U_i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a continuous section s defined by $s|_{U_i} = s_i$. It's easy to verify this is a continuous global section. The collection of the sections of the bundle $\mathcal{EC}$ is a sheaf over $\mathcal{M}$.

$$\Gamma\mathcal{EC} : \mathcal{O}(\mathcal{M})^{\text{op}} \to {}_{\mathcal{A}^{\mathcal{M}}}\text{Mod},$$

that assigns to each $U \in \mathcal{O}(\mathcal{M})$ the set $\mathcal{EC}U = \coprod_{x \in U} \mathcal{C}_x$. For each open subset $U \subset \mathcal{M}$ there is a function,

$$\begin{aligned} \eta_U : \mathcal{C}U &\to \Gamma\mathcal{EC}U, \\ f &\mapsto \hat{f}. \end{aligned}$$

This natural transformation of functors maps pre-sheaves to sheaves called sheafification of the pre-sheaf $\mathcal{C}$.

$${}^{\text{Sh}} : \text{PSh} \to \text{Sh}\,.$$

We will denote the sheafification of the pre-sheaf $\mathcal{C}$, $\Gamma\mathcal{EC}$ by $\mathcal{C}^{\text{Sh}}$. For the cotangent pre-sheaf we will call the sheafification as the cotangent sheaf, denoted by $\mathcal{T}^{\vee}$. This is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. It's also locally free, with the local isomorphism,

$$\{f_i\}_{i=1}^n \mapsto \sum_i f_i dx_i$$

of modules $(\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$ and $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ for a chart (φ, U). Smooth sections of the cotangent bundle, or elements of the cotangent sheaf are called differential forms.

1.2.3 | Locally Free Sheaves and Vector Bundles

A sheaf of $\mathcal{A}^{\mathcal{M}}$-modules $\mathcal{D}$ is said to be locally free of rank n if for every $x \in \mathcal{M}$ has a neighborhood U such that,

$$\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}.$$

Locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules in general give rise to vector bundles. Conversely to each vector bundle we can associate the sheaf of differentiable sections of π which is a locally free sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. There is a natural bijection between the sheaves of locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules and vector bundles. Every $\mathcal{A}^{\mathcal{M}}$-linear sheaf homomorphism gives a homomorphism of vector bundles. It's an equivalence of categories.

For a locally free sheaf $\mathcal{D}$, at each point $x \in \mathcal{M}$, we have a neighborhood U such that $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$. This composed with the evaluation gives us,

$$\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x = \mathbb{R}^n.$$

We can bundle the stalks $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x$ together,

$$\mathcal{VD} = \coprod_{x \in \mathcal{M}} \mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x,$$

together with the natural projection $\mathcal{D}_x/\mathcal{I}_x^{\mathcal{M}}\mathcal{D}_x \mapsto x$. Since $\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}})^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$ and using this identification, the topology and a differentiable structure can be provided to the bundle $\mathcal{VD}$. $\mathcal{VD}$ is the vector bundle associated with the locally free sheaf $\mathcal{D}$. This will become important in the future.

1.2.3.1 | Tensor Product, Exterior Product

We understand linear maps quite well, what we want to do is study multilinear maps using linear algebra. The idea of tensor products is to study multilinear maps as linear maps. If $\mathcal{E}$ and $\mathcal{F}$ are two locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules corresponding to vector bundles, then we can form the presheaf of $\mathcal{A}^{\mathcal{M}}U$-module, $\mathcal{E}U \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}U$. Whose stalk at each point x is given by $\mathcal{E}_x \otimes_{\mathcal{A}_x^{\mathcal{M}}} \mathcal{F}_x$.

$$\begin{aligned}\mathcal{E} \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{F} : \mathcal{O}(\mathcal{M})^{\text{op}} &\to \textbf{Sets}\\ U &\mapsto \mathcal{E}U \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}U\end{aligned}$$

Upon evaluation, i.e., quotienting by $\mathcal{I}_x$, this gives the tensor product of vector spaces $E_x \otimes_{\mathbb{R}} F_x$ at each x. Suppose $\mathcal{E}$ and $\mathcal{F}$ be locally free, then around each $x \in \mathcal{M}$ there exist neighborhoods U and V such that $\mathcal{E}(U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus k}$ and $\mathcal{F}(V) \cong (\mathcal{A}^{\mathcal{M}}V)^{\oplus l}$. In particular, on the intersection $U \cap V$,

$$\mathcal{E}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus k}, \;\; \mathcal{F}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus l}$$

For the sake of simplicity we will assume $U = V$. We have,

$$\mathcal{E}U \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{F}U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus kl}.$$

We are interested in tensor products when $\mathcal{E}$ and $\mathcal{F}$ are tangent and cotangent sheaves. Suppose $\mathcal{E}$ is a locally free sheaf of $\mathcal{A}^{\mathcal{N}}$-modules, and

$$\varkappa : \mathcal{M} \to \mathcal{N}$$

is a differentiable map, then we can consider the inverse image $\varkappa^{-1}(\mathcal{A}^{\mathcal{N}})$ can be defined

1.2.3.2 | Tensor Algebra, Exterior Algebra

We understand linear maps quite well, what we want to do is study multilinear maps using linear algebra. The idea of tensor products is to study multilinear maps as linear maps. Suppose we have a collection of $\mathcal{A}$-modules $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$, and a multilinear map,

$$\beta : \textstyle\prod_i \mathcal{V}_i \to \mathcal{W},$$

where $\mathcal{W}$ is an $\mathcal{A}$-module. What we want to do is study all such multilinear maps from the collection $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ to $\mathcal{W}$ as linear maps from the 'tensor product' $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ to $\mathcal{W}$ as a linear map. The algebraic tensor product of $\{\mathcal{V}_i\}_{i\in\mathcal{I}}$ is an $\mathcal{A}$-module $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$ together with a multilinear map,

$$t : \textstyle\prod_{i\in\mathcal{I}}\mathcal{V}_i \to \displaystyle\bigotimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i,$$

such that every other multilinear map from $\prod_{i\in\mathcal{I}}\mathcal{V}_i$ to $\mathcal{W}$ uniquely factors through $\otimes_{\mathcal{A}}^{i\in\mathcal{I}}\mathcal{V}_i$. This is the universal property of tensor products. This can be expressed in a commutative diagram by,

$$\begin{tikzcd} \prod_{i\in\mathcal{I}} \mathcal{V}_i \arrow[r, "t"] \arrow[dr, "\beta"'] & \bigotimes_{\mathcal{A}}^{i\in\mathcal{I}} \mathcal{V}_i \arrow[d, "\exists!\ \tilde{\beta}"] \\ & \mathcal{W} \end{tikzcd}$$

For the construction of the tensor product check wikipedia.

If $\mathcal{E}$ and $\mathcal{F}$ are two locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules corresponding to vector bundles, then we can form the presheaf of $\mathcal{A}^{\mathcal{M}}U$-module, $\mathcal{E}(U) \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}(U)$. Whose stalk at each point x is given by $\mathcal{E}_x \otimes_{\mathcal{A}^{\mathcal{M}}_x} \mathcal{F}_x$.

$$\begin{aligned} \mathcal{E} \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{F} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathcal{A}^{\mathcal{M}}}\mathrm{Mod} \\ U &\mapsto \mathcal{E}(U) \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}(U) \end{aligned}$$

Suppose $\mathcal{E}$ and $\mathcal{F}$ be locally free, then around each $x \in \mathcal{M}$ there exist neighborhoods U and V such that $\mathcal{E}(U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus k}$ and $\mathcal{F}(V) \cong (\mathcal{A}^{\mathcal{M}}V)^{\oplus l}$. In particular, on the intersection $U \cap V$,

$$\mathcal{E}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus k}, \quad \mathcal{F}(U \cap V) \cong (\mathcal{A}^{\mathcal{M}}(U \cap V))^{\oplus l}$$

For the sake of simplicity we will assume $U = V$. We have,

$$\mathcal{E}U \otimes_{\mathcal{A}^{\mathcal{M}}U} \mathcal{F}U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus kl}.$$

The tensor products of interest to us will be the tensor products of tangent sheaf $\mathcal{T}$ of $\mathcal{A}^{\mathcal{M}}$-modules and cotangent sheaf $\mathcal{T}^{\vee}$ of $\mathcal{A}^{\mathcal{M}}$-modules. We will denote $\mathcal{T}^{(k,l)}$ the sheaf consisting of tensor product of k tangent and l cotangent sheaves.

$$\mathcal{T}^{(k,l)} = \mathcal{T}^{\otimes_{\mathcal{A}^{\mathcal{M}}} k} \otimes_{\mathcal{A}^{\mathcal{M}}} (\mathcal{T}^{\vee})^{\otimes_{\mathcal{A}^{\mathcal{M}}} l}.$$

The sections of such tensor products are called tangent fields, (k,l)-type tensor field in particular. After evaluation at each stalk this will correspond to k times tensor product of tangent space, and l time tensor product of cotangent space.

The tensor algebra of $\mathcal{T}$ is defined as the direct sum,

$$T^{\bullet}_{\mathcal{A}^{\mathcal{M}}}\mathcal{T} = \bigoplus_{i\geq 0} \mathcal{T}^i,$$

together with the multiplication defined by tensor product and extending linearly.

We usually encounter multilinear maps with additional properties. These will be the usual types of multilinear functionals we encounter while doing calculus. These are bilinear forms that are alternating, i.e., when the entries repeat the form should be zero. An example of such a multilinear map is the oriented area.

A k-linear map $\alpha : \mathcal{V} \times \cdots \mathcal{V} \to \mathcal{W}$ is called alternating, if the value is zero whenever two entries are the same. An exterior power of degree l is the universal vector space $\bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ together with an alternating multilinear map $i : \mathcal{V} \times \cdots \times \mathcal{V} \to \bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V}$ such that for all alternating multilinear maps α, there exists a unique linear map $\tilde{\alpha}$ such that the following diagram commutes,

$$\begin{tikzcd} \prod^l \mathcal{V} \arrow[r, "i"] \arrow[dr, "\alpha"'] & \bigwedge^l_{\mathcal{A}^{\mathcal{M}}} \mathcal{V} \arrow[d, "\exists!\ \tilde{\alpha}"] \\ & \mathcal{W} \end{tikzcd}$$

An exterior algebra of $\mathcal{A}^{\mathcal{M}}$-algebra $\mathcal{T}$, is the direct sum of all exterior powers, denoted by $\bigwedge^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T}$. This is also the quotient of the tensor algebra by the two-sided ideal $\mathcal{K}$ generated by all elements of the form $\tau \otimes \tau$ for all $\tau \in \mathcal{T}$. This quotient is called the exterior algebra,

$$\bigwedge\nolimits^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T} = T^{\bullet}_{\mathcal{A}^{\mathcal{M}}} \mathcal{T}/\mathcal{K}.$$

This quotient puts all the elements in the tensor algebra that have same two entries into the equivalence class of zero. By doing this we are removing all the elements that can't be distinguished by alternating multilinear maps. We will stop writing the subscript $\mathcal{A}^{\mathcal{M}}$ when there is no confusion. The image of the tensor product $\mathcal{T}^k$ in $\bigwedge \mathcal{T}$ is denoted by $\bigwedge^k \mathcal{T}$. The image of $\tau_1 \otimes \cdots \otimes \tau_n$ is denoted by $\tau_1 \wedge \cdots \wedge \tau_n$. This is a graded algebra. By expanding $(\tau_1 + \tau_2) \otimes (\tau_1 + \tau_2)$ we see that,

$$\tau_1 \wedge \tau_2 = -\tau_2 \wedge \tau_1,$$

in $\bigwedge^2 \mathcal{T}$. Similarly, in $\bigwedge^l \mathcal{T}$,

$$\tau_1 \wedge \cdots \wedge \tau_l = (-1)^{\mathrm{sgn}(\sigma)} \tau_{\sigma(1)} \wedge \cdots \wedge \tau_{\sigma(l)},$$

where σ is a permutation of $\{1, \dots, l\}$. The exterior algebra is a skew-commutative algebra.

A differential k-form or k-form is an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields. Or equivalently, sections of the sheafification of the kth exterior power of cotangent pre-sheaf. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$\omega \wedge \kappa(X_1, \dots, X_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(X_{\sigma(1),\dots,X_{\sigma(k)}}) \kappa(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}),$$

where σ is a permutation of the set $\{1, \dots k + l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ is the sign of the permutation. This is a graded algebra.

1.2.4 | Differential of a map

If the tangent space at x is interpreted as the linear approximation of the manifold $\mathcal{M}$ at x, then the differential of a map is interpreted as the linear approximation of the map $\varkappa$.

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map of manifold $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ into $(\mathcal{N}, \mathcal{A}^{\mathcal{N}})$. Let $x \in \mathcal{M}$, the tangent space is defined to be the collection of the equivalence classes of such curves passing through x. Using the map $\varkappa$, we can push forward the curve h to $\mathcal{N}$, given by the composition, $\varkappa \circ h : \mathbb{R} \to \varkappa(U)$.

$$\mathbb{R} \xrightarrow{h} \mathcal{M} \xrightarrow{\varkappa} \mathcal{N}$$

By a differential of the function $\varkappa$, we mean a map $D\varkappa(x)$ that takes the equivalence class τ_h to the equivalence class $\tau_{\varkappa \circ h}$. The new pairing that arises from the map is,

$$\langle g, \varkappa \circ h \rangle_{\varkappa(x)} = \left. \frac{d(g \circ \varkappa \circ h(t))}{dt} \right|_{t=0},$$

for all $g \in \mathcal{A}^{\mathcal{N}}(\varkappa(U))$.

$$\begin{aligned} D\varkappa(x) :\ & T_x\mathcal{M} \to T_{\varkappa(x)}\mathcal{N} \\ & \tau_h \mapsto \tau_{\varkappa \circ h} \end{aligned}$$

It gives a vector bundle homomorphism of $T\mathcal{M}$ into $\varkappa^*(T\mathcal{N})$, usually denoted by $D\varkappa$. In terms of local coordinates this will be the Jacobian of the map. For compositions, we have,

$$(\varkappa \circ \varphi)^* = \varkappa^* \circ \varphi^*.$$

[3] Let $[f] \in \mathcal{I}_x^{\mathcal{N}}/(\mathcal{I}_x^{\mathcal{N}})^2$, with the representative $f \in \mathcal{A}^{\mathcal{N}}U$, Then we have the pull back, given by the composition,

$$\mathcal{M} \xrightarrow{\varkappa} \mathcal{N} \xrightarrow{f} \mathbb{R}$$

Now, $f \circ \varkappa \in \mathcal{A}^{\mathcal{M}}(\varkappa^{-1}U)$. The new pairing that arises from the map is,

$$\langle f \circ \varkappa, h \rangle_x = \left. \frac{d(f \circ \varkappa \circ h(t))}{dt} \right|_{t=0},$$

for all curves $h : \mathbb{R} \to \mathcal{M}$ with $h(0) = x$. This gives us the pullback map,

$$\begin{aligned} T^\vee_{\varkappa(x)}\mathcal{N} &\to T^\vee_x\mathcal{M} \\ [f] &\mapsto [f \circ \varkappa]. \end{aligned}$$

It gives a vector bundle homomorphism of $\varkappa_*(T^\vee\mathcal{N})$ into $T^\vee\mathcal{M}$, and usually denoted by $D\varkappa^\dagger$. In terms of local coordinates this will be the adjoint of the Jacobian. For compositions, we have,

$$(\varkappa \circ \varphi)_* = \varphi_* \circ \varkappa_*.$$

A map $\varkappa : \mathcal{M} \to \mathcal{N}$ corresponds to a corresponding linear map on the tensor product bundle, described by its action on the tangent vectors and the cotangent vectors as above. So, in terms of local coordinates it will be a tensor product of the Jacobians and adjoints of the Jacobians. We will denote this map by $\varkappa^*$.

1.3 | Lie Derivative

A smooth function $\varkappa : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if $D\varkappa(x)$ is invertible for all $x \in \mathcal{M}$. This would mean that there exists an smooth inverse $\varkappa^{-1}$. The set of all diffeomorphisms of a manifold, i.e., diffeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ is a group. We will call such maps diffeomorphism 'of' $\mathcal{M}$. A one parameter group of diffeomorphisms of $\mathcal{M}$ is a collection of diffeomorphisms,

$$\varkappa : t \mapsto \varkappa_t$$

where each $\varkappa_t$ is a diffeomorphism of $\mathcal{M}$ such that, $\varkappa_0 = \mathbb{1}_{\mathcal{M}}$,

$$\varkappa_t \circ \varkappa_s = \varkappa_{t+s} \quad \forall t, s \in \mathbb{R},$$

and $\varkappa_t$ is a smooth as a map from $\mathcal{M} \times \mathbb{R}$ to $\mathcal{M}$. Where $\mathcal{M} \times \mathbb{R}$ has the differentiable structure of a product manifold. The one parameter group $t \mapsto \varkappa_t$ determines at each $x \in \mathcal{M}$ smooth curves,

$$t \mapsto \varkappa_t(x).$$

At each point x, this gives a tangent vector, the equivalence class of curves $[\varkappa_t(x)]$. Hence we obtain at each point $x \in \mathcal{M}$ a vector in the tangent space at x.

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[3] In terms of local coordinates, we can express the map $\varkappa$ as $\widehat{\varkappa} = \psi \circ \varkappa \circ \varphi^{-1}$, and

$$d(\widehat{\varkappa} \circ \varphi \circ c)/dt\big|_{t=0} = \underbrace{D\widehat{\varkappa}(\varphi(x))}_{J_{ij}} \circ d(\varphi \circ c)/dt\big|_{t=0}.$$

$J_{ij} = \frac{\partial \widehat{\varkappa}_j}{\partial x_i}$. is the Jacobian of the map.

1.3.1 | Lie Derivative of Functions

Each one parameter group of diffeomorphisms determines a vector field. The converse also holds locally. Given a vector field X on a differentiable manifold $\mathcal{M}$ there exists a one-parameter group of diffeomorphisms $\varkappa$ such that $X_\varkappa = X$. This is due to the existence and uniqueness of solutions to ODEs.

For any $f \in \mathcal{A}^{\mathcal{M}}$ we have the smooth composition, $f \circ \varkappa_t : \mathcal{M} \to \mathbb{R}$. For fixed $x \in \mathcal{M}$, and varying t, this also corresponds to the smooth map,

$$\begin{aligned} f(\varkappa_{(\cdot)}(x)) :\ & \mathbb{R} \to \mathbb{R} \\ & t \mapsto f(\varkappa_t(x)). \end{aligned}$$

So, we can differentiate this function.

$$(X_\varkappa f)(x) = \lim_{t\to 0} \frac{f(\varkappa_t(x)) - f(x)}{t} = \left.\frac{d(f \circ h(t))}{dt}\right|_{t=0} = \langle f, h \rangle_x,$$

where $h = \varkappa_t(x)$. This can also be thought as the function,

$$\begin{aligned} X_\varkappa(\cdot) : \mathcal{A}^{\mathcal{M}} & \to \mathcal{A}^{\mathcal{M}}. \\ f & \mapsto (X_\varkappa f). \end{aligned}$$

It is called the differentiation of the function f with respect to $\varkappa$ at x. $\varkappa_t$ moves x to $\varkappa_t(x)$, so what the above limit is doing is measuring the infinitesimal change to the function f 'along' $\varkappa_t$. It's also called the Lie derivative of the function f along the vector field X, denoted by,

$$L_{X_\varkappa}(f) = X_\varkappa(f).$$

For $f, g \in \mathcal{A}^{\mathcal{M}}$, by directly plugging into the definition, we find that,

$$X_\varkappa(f+g)(x) = X_\varkappa(f)(x) + X_\varkappa(g)(x),$$

and for the product,

$$X_\varkappa(fg)(x) = f(x)(X_\varkappa g)(x) + g(x)(X_\varkappa f)(x).$$

So, $X_\varkappa$ is linear map, and

$$X_\varkappa(fg) = f(X_\varkappa g) + g(X_\varkappa f).$$

Hence, $X_\varkappa$ is a homogeneous first order operator. This abstract definition, while less intuitive can be extended to tensors product case later.

The flow of a vector field X is the one parameter group of diffeomorphisms $\varkappa$ such that $X_\varkappa = X$. A flow is said to be a global flow if it's defined for all $\mathbb{R}$ and at every point $x \in \mathcal{M}$. If a vector field gives rise to a global flow, it's called complete. For compact manifolds they do exist as we can always find local solutions and patch them up for finite cover. Let X be a vector field and $\varkappa_t$ be its corresponding flow, then the orbit of a point $x \in \mathcal{M}$, $t \mapsto \varkappa_t(x)$ is called the integral curve for X.

The integral curve is the constant map if and only if the vector field is zero. Such points are called singularities. $\varkappa_t$ fixes a point x if and only if x is a singularity of X. When x is not a singularity, $X(x) \neq 0$, and hence by continuity, the integral curve has injective differential nearby. Hence the integral curve is an immersed one dimensional manifold.

1.3.1.1 | Frobenius Theorem

1.3.2 | Lie Derivative of Tensor Fields

Let X be a vector field with the corresponding flow $\varkappa$. These are diffeomorphisms, $\varkappa_t : \mathcal{M} \to \mathcal{M}$. Let $\mathcal{T}^{(k,l)}$ be a tensor sheaf, consisting of tensor product of k tangent and l cotangent sheaves. A tensor fields are sections of the tensor sheaf. The diffeomorphism gives an isomorphism of the sheaf. If $\mathcal{T}_x^{(k,l)}$ is the stalk of $\mathcal{T}^{(k,l)}$ at x, then we have the induced isomorphism,

$$\varkappa_t^* : \mathcal{T}_x^{(k,l)}/\mathcal{I}_x^{\mathcal{M}} \to \mathcal{T}_{\varkappa_t(x)}^{(k,l)}/\mathcal{I}_{\varkappa_t(x)}^{\mathcal{M}}.$$

So, using the flow of the vector field we can push forward the tensor field R. Since the association $t \mapsto \varkappa_t^*$ is also smooth the map[4], we have,

$$\varkappa_{(\cdot)}^*(R(x)) : \; t \mapsto \varkappa_t^*(R(\varkappa_t(x))).$$

from $\mathbb{R}$ to the vector bundle $\mathcal{V}\mathcal{T}^{(k,l)}$ associated with the locally free sheaf $\mathcal{T}^{(k,l)}$. In particular we can talk about taking the limit,

$$L_X(R)(x) = \lim_{t\to 0} \frac{\varkappa_t^*(R(\varkappa_t(x))) - R(x)}{t}, \qquad \text{(Lie derivative)}$$

called the Lie derivative of the tensor field R with respect to the vector field X at $x \in \mathcal{M}$.

To study how the Lie derivative of the tensor product, exterior product and symmetric product of tensor fields, we can study Lie derivative of multilinear maps. Let β be an $\mathcal{A}^{\mathcal{M}}$-bilinear sheaf homomorphism,

$$\beta : \mathcal{V} \times \mathcal{V}' \to \mathcal{W},$$

where $\mathcal{V}, \mathcal{V}'$ and $\mathcal{W}$ are tensor sheaves. The action of $\varkappa_t$ is given by,

$$\varkappa_t^*(\beta(R, R')) = \beta(\varkappa_t^* R, \varkappa_t^* R').$$

Plugging this in the Lie derivative, we get,

$$\begin{aligned} L_X(\beta(R,R'))(x) &= \lim_{t\to 0} \tfrac{\varkappa_t^*\beta(R,R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} = \lim_{t\to 0} \tfrac{\beta(\varkappa_t^* R, \varkappa_t^* R')(\varkappa_t(x)) - \beta(R,R')(x)}{t} \\ &= \beta\big(\lim_{t\to 0} \tfrac{\varkappa_t^* R(\varkappa_t(x)) - R(x)}{t}, \lim_{t\to 0} \varkappa_t^* R'(\varkappa_t(x))\big) + \beta\big(R, \lim_{t\to 0} \tfrac{\varkappa_t^* R'(\varkappa_t(x)) - R'(x)}{t}\big) \\ &= \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \end{aligned}$$

Here, we added and subtracted a term, and then took the limit inside. To take the limit inside, we would need the bilinear form to be continuous.

The Lie derivative of $\beta(R, R')$ with respect to X is,

$$L_X(\beta(R,R'))(x) = \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \qquad \text{(product rule)}$$

Here we will have to add and subtract $\beta(\varkappa_t^* R(\varkappa_t(x)), R'(x))$ and use $\mathcal{A}^{\mathcal{M}}$-bilinearity of β to get the Lie derivative inside, this is similar to how product rule is proved. The result is also valid for just $\mathbb{R}$-bilinearity. So, by taking the bilinear map β to be the tensor product, $(R, R') \mapsto R \otimes R'$, we have,

$$L_X(R \otimes R') = L_X R \otimes R' + R \otimes L_X R'.$$

[4]we have to carefully look at a bunch of maps, and this will turn out to be smooth

Similarly, for the differential forms, which are sections of exterior powers of cotangent sheaf,

$$L_X(\omega \wedge \omega') = L_X\omega \wedge \omega' + \omega \wedge L_X\omega'.$$

We can define a pairing of a 1-form and a vector field, $\beta(\omega, Y) = \omega(Y)$. This yields,

$$L_X(\omega(Y)) = \beta(L_X\omega, Y) + \beta(\omega, L_X Y)(x) = (L_X(\omega))(Y) + \omega(L_X(Y)).$$

Let Y be a vector field, it's a derivation,

$$Y : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}.$$

The map $(Y, f) \mapsto Y(f)$ is a $\mathbb{R}$-bilinear map. So, the product rule is applicable. Note that $Y(f) \in \mathcal{A}^{\mathcal{M}}$, and the Lie derivative of functions is given by,

$$L_X(g) = X(g)$$

Hence, by plugging in $g = Y(f)$, we have,

$$\underbrace{L_X(Y(f))}_{X(Y(f))} = (L_X(Y))(f) + Y(\underbrace{L_X(f)}_{X(f)})$$

So, we have, $(L_X(Y))(f) = X(Y(f)) - Y(X(f)) = [X, Y](f)$. This is the Lie bracket, and it should be interpreted as the Lie derivative of Y with respect to the vector field X. Now, we can consider the bilinear map, $(Y, Z) \mapsto [Y, Z]$. By applying product rule, we get,

$$L_X([Y, Z]) = [[X, Y], Z]] + [Y, [X, Z]].$$

But by previous calculation of Lie derivative of vector fields, we have, $L_X([Y, Z]) = [X, [Y, Z]]$. This yields us the so called Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{Jacobi identity}$$

This can also be written as,

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

For $f, g \in \mathcal{A}^{\mathcal{M}}U$, by expanding it can be verified that,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f).$$

So, $[X, Y] \in \mathcal{T}U$. It can also be checked that $[,]$ is $\mathbb{R}$-bilinear. So, $\mathcal{T}U$ is an $\mathbb{R}$-algebra for all U. It's however not a $\mathcal{A}^{\mathcal{M}}U$-algebra because it's not $\mathcal{A}^{\mathcal{M}}U$-bilinear. Because for $g \in \mathcal{A}^{\mathcal{M}}U$,

$$\begin{aligned}
[X, gY](f) &= X(gY(f)) - gY(X(f)) \\
&= X(g)Y(f) + gX(Y(f)) - gY(X(f)) \\
&= (X(g)Y + g[X, Y])(f).
\end{aligned}$$

So, it's not $\mathcal{A}^{\mathcal{M}}$-bilinear, and hence can't be an $\mathcal{A}^{\mathcal{M}}U$-algebra. The tangent sheaf is an $\mathbb{R}$-algebra. If X and Y are two commuting vector fields, i.e., $[X, Y] = 0$ then the flows of the corresponding vector fields commute in the group of diffeomorphisms of $\mathcal{M}$.

A Lie algebras over a commutative ring $\mathcal{A}$ is an $\mathcal{A}$-module together with a bilinear operation, $(X, Y) \mapsto [X, Y]$, such that, $[X, X] = 0$ and satisfies the Jacobi identity. A homomorphism of Lie algebras is an algebra homomorphism $f : \mathcal{V} \to \mathcal{W}$ such that,

$$f([X, Y]) = [f(X), f(Y)].$$

The sheaf $\mathcal{T}U$ is a sheaf of Lie algebras over $\mathbb{R}$. The restriction map is a Lie algebra homomorphism.

Consider the $\mathbb{R}$-bilinear map, $(\omega, Y) \mapsto \omega(Y) \in \mathcal{A}^{\mathcal{M}}$, applying the product rule we have,

$$L_X(\omega(Y)) = \beta(L_X\omega, Y) + \beta(\omega, L_X Y)(x) = (L_X(\omega))(Y) + \omega(L_X(Y)).$$

So, the Lie derivative of a 1-form ω is given by,

$$(L_X(\omega))(Y) = X(\omega(Y)) - \omega([X, Y]).$$

Similarly, the Lie derivative of a k-form is given by,

$$(L_X(\alpha))(X_1, \ldots, X_k) = X(\alpha(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).$$

It follows that,

$$L_{[X,Y]}\omega = L_X L_Y \omega - L_Y L_X \omega.$$

and,

$$L_X(h\omega) = L_X(h) \cdot \omega + h \cdot L_X(\omega).$$

The Lie derivative allows us to differentiate a tensor field with respect to a vector field. What we want is a notion of differentiation of a tensor field with respect to a tangent vector at a point. This we cannot do with Lie derivative, the behavior of the vector field in a neighborhood was important as we took the limit. Or more algebraicly speaking,

$$(L_{fX}\omega)(Y) = (fX)(\omega(Y)) - \omega([fX, Y]) = f(L_X\omega)(Y) + (Y(f))\omega(X)$$

or, L is not $\mathcal{A}^{\mathcal{M}}$-linear. Changing the vector field X at x with a function $f \in \mathcal{A}^{\mathcal{M}}$, also depends on the behavior of the function f in the neighborhood and not just its value at x. In order to differentiate tensor fields with respect to a tangent vector we need the notion of connection, which we will discuss later.

1.4 | Exterior Derivative; de Rham Complex

Some functions such as oriented area, volume, etc. carry a lot of geometric information about the manifold. The goal of this section is to study all such maps. So, the multilinear maps of interest to us are alternating multilinear maps. To study such multilinear maps we can restrict ourselves to the study of exterior powers instead of studying the much larger tensor product. A differential k-form or k-form is an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields. Or equivalently, sections of kth exterior power of cotangent sheaf. Instead of studying the geometric information contained in one such multilinear map, we consider all of them.

1.5 | Algebra of Differential Operators

2 | $\mathcal{D}$-Modules, Jets & Connections

3 | Integration & Exterior Derivative

To develop calculus, we need the ability to integrate on manifolds. As it turns out integration on manifolds is very closely related to differential forms.

3.1 | Integration on Manifolds

So, to start with we need a measure on the manifold. Since we expect the measure to respect the topology of the manifold, it should be a Borel measure. By Riesz duality theorem, measure on a locally compact Hausdorff space can be identified with a linear functional,

$$\mu : \mathcal{C}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{C}_c^{\mathcal{M}}$ with $f \geq 0$. Where $\mathcal{C}_c^{\mathcal{M}}$ are all compactly supported continuous functions. The positivity condition makes the functional continuous.

Differentiable functions with compact support form a self-adjoint subalgebra of this algebra of compactly supported continuous functions, and separate points of the space $\mathcal{M}$. Hence by Stone-Weierstrass theorem, the algebra $\mathcal{A}_c^{\mathcal{M}}$ of compactly supported differentiable functions is dense in $\mathcal{C}_c^{\mathcal{M}}$. Since by Hahn-Banach theorem, linear functionals on subalgebras uniquely extend to the whole algebra, we can study functionals,

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{A}_c^{\mathcal{M}}$ with $f \geq 0$.

Firstly, we try to integrate a differential form along a smooth path $h : I \to \mathcal{M}$. What we want to do is, measure the 'length' of the equivalence class corresponding to the curve h at each point, and add it up.

So, by a Borel measure on a differentiable manifold $\mathcal{M}$ we mean a linear form μ on the vector space $\mathcal{A}_c^{\mathcal{M}}$ of differentiable functions with a compact support on $\mathcal{M}$ which satisfies certain continuity requirement i.e., for a sequence of compactly supported differentiable functions, $\{f_i\}$, with support contained in the compact set K, if $\sup\{|f_i|\} \xrightarrow[i\to\infty]{} 0$ then $\mu(f_i) \to 0$.

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

The scalar $\mu(f)$ is denoted by $\int f d\mu$. On the space of compactly supported differentiable functions, we can define the sup norm making it into a Banach space. We can then start doing functional analysis. The measures under our consideration will be continuous linear functionals on $\mathcal{A}_c^{\mathcal{M}}$.

3.1.1 | Differentiable Measures

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map. For any function $g \in \mathcal{A}_c^{\mathcal{N}}$, the composition, $g \circ \varkappa \in \mathcal{A}_c^{\mathcal{M}}$. The composition has compact support because the manifold is Hausdorff, and hence the inverse image of compact set is compact. The image measure can then defined by,

$$(\varkappa^*(\mu))(g) = \mu(g \circ \varkappa).$$

The continuity and linearity follow from continuity of differentiable functions.

Now with this composition, we can start defining the Lie derivative. Let $\varkappa_t$ be the flow of a vector field X, The Lie derivative of a measure μ with respect to the vector field X is the functional,

$$f \mapsto \lim_{t\to 0} \tfrac{((\varkappa_t^{-1})^*(\mu))(f)-\mu(f)}{t} = \mu\Big(\underbrace{\lim_{t\to 0} \tfrac{f\circ\varkappa_t^{-1}-f}{t}}_{-L_X(f)=-X(f)} \Big).$$

We needed the continuity of the measure to take the limit inside. So we have,

$$L_X(\mu)(f) = -\mu(X(f)). \qquad \text{(Lie derivative)}$$

This will be our notion of differentiation of measures. Multiple differentiations will be defined as multiple iterations of the Lie derivative of the measure. We say a Borel measure μ is indefinitely differentiable if the k times differentiations is a Borel measure for all k.

If μ is a measure and $h \in \mathcal{A}^{\mathcal{M}}$, then the map,

$$f \mapsto \mu(hf)$$

is a linear form on $\mathcal{A}_c^{\mathcal{M}}$ and satisfies the continuity requirement i.e., if $\sup\{|hf_n|\}$ tends to zero then so does $\mu(hf_n)$. We will denote this measure by $h \cdot \mu$. Together with this notion of multiplication, $\mathcal{B}^{\mathcal{M}}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

Using $(L_X\mu)(f) = -\mu(Xf)$, the Lie derivative of μ with respect to hX is,

$$(L_{hX}\mu)(f) = -\mu(hX(f))$$

by using $X(hf) = fX(h)+hX(f)$, this is $= -\mu(X(hf) - fX(h)) = -\mu(X(hf)) + X(h)\mu(f)$.

$$(L_{hX}\mu)(f) = \underbrace{(L_X\mu)(hf)}_{h\cdot L_X(\mu)(f)} + X(h)\mu(f)$$

Since $h \cdot \mu(f) = \mu(hf)$, we have, $-\mu(hX(f)) = h - \cdot\mu(X(f)) = h \cdot L_X(\mu)$. So,

$$L_X(h \cdot \mu) = L_{hX}\mu = h \cdot L_X(\mu) + L_X(h) \cdot \mu. \qquad \text{(Leibniz rule)}$$

Similarly by expanding we get that,

$$L_{[X,Y]}(\mu) = L_X L_Y(\mu) - L_Y L_X(\mu). \qquad \text{(Lie bracket)}$$

So, the behavior of differentiable measures under Lie derivative is similar to that of differential forms. We are interested in measures that are locally translation invariant. Let V be a vector space, and μ be a measure. It's said to be translation invariant if $\varkappa_t^*\mu = \mu$ for $\varkappa_t = tv$ or equivalently, $L_{\partial_v}\mu = 0$ for all v. and hence the set of all invariant measures in $\mathcal{B}^{\mathcal{M}}$ is one dimensional.

3.1.2 | The Sheaf of Differentiable Measures

We now start looking at the collection of all indefinitely differentiable measures. Let $U \subseteq V$, then we have the natural inclusion of compactly supported functions functions $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$, by setting the functions to be equal to zero outside U.

Let $\mathcal{B}^{\mathcal{M}}U$ be the set of all differentiable measures on U, we have used $\mathcal{B}$ here for Borel. The inclusion $U \subset V$ gives rise to a restriction map of differentiable measures $\mu \mapsto \mu|_V$. The action of $\mu|_U$ on $\mathcal{A}_c^{\mathcal{M}}U$, is given by the action of μ on $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$. So,

$$\mathcal{B}^{\mathcal{M}} : U \mapsto \mathcal{B}^{\mathcal{M}}U$$

is a presheaf. Now, to patch these measures up, we need the notion of partition of unity. This is an important tool. What we intend to do is restrict the domain of functions to some regions so we can forget about the behaviour of the function outside some region.

Let $\{U_i\}_{i\in I}$ be a locally finite open cover of a differentiable manifold $\mathcal{M}$. The locally finite open covers exist because manifolds are locally compact. A partition of unity with respect to the cover $\{U_i\}_{i\in I}$ is a family of smooth functions $\{\varphi_i\}_{i\in I}$ with values in $[0,1]$ such that

$$\textstyle\sum_{i\in I} \varphi_i = 1$$

with support of φ_i contained in U_i. Once we have such a partition of unity, we can study the function $(\sum_{i\in I} \varphi_i)f$ instead of the function f. To show the existence, let $\{V_i\}_{i\in I}$ be an open cover with $\overline{V_i} \subset U_i$, we can construct functions ψ_i that have support in U_i.[1] Since the cover is locally finite the sum makes sense.

$$\varphi_i = \psi_i / (\textstyle\sum_{i\in I} \psi_i)$$

then acts as a partition of unity. This allows us to study the functions using the charts.

Let $\{U_i\}$ be a locally finite familly of open sets and μ_i be Borel measure on them. Suppose $\mu_i|_{U_i\cap U_j} = \mu_i|_{U_i\cap U_j}$ for all i, j, then we can define a measure μ on $U = \cup\{U_i\}$ by multiplying any function $f \in \mathcal{A}_c^{\mathcal{M}}U$ with a partition of unity associated with $\{U_i\}$, and then define,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f).$$

Since $\{U_i\}$ is locally finite, the sum is welldefined. Now, suppose $f \in \mathcal{A}_c^{\mathcal{M}}U_i$, then $\mu(f) = \sum_i \mu_i(\varphi_i f)$, since the support of f is contained in U_i, we have for every $U_i \cap U_j$, $\mu_i|_{U_i\cap U_j} = \mu_j|_{U_i\cap U_j}$ and hence we have,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f) = \sum_i \mu_j(\varphi_i f)$$

By linearity of measures this is

$$= \mu_j((\textstyle\sum_i \varphi_i)f) = \mu_j(f).$$

Hence the collation property holds, i.e., there exists an equilizer map e such that,

$$\mathcal{B}^{\mathcal{M}}U \overset{e}{\dashrightarrow} \prod_i \mathcal{B}^{\mathcal{M}}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(U_i \cap U_j).$$

Since the partition of unity is a differentiable map, the linear map μ is also continuous, and hence is a Borel measure. The differentiable measures on a differentiable manifold $\mathcal{M}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

[1] Let $S_1 \subset S_2$ be concentric spheres in $\mathbb{R}^n$ centered at 0 then we need to show that there exist differentiable function which is zero outside S_2 and non zero everywhere inside S_1. If we take S_2 to be the unit ball, the function, $\Phi(x) = \exp\left(\frac{1}{\sum_i x_i^2 - 1}\right)$ for x in the unit ball and zero outside works.

3.2 | Differential Forms

Differential forms are closely related to 'oriented volume'. The multilinear maps of interest to us are alternating multilinear maps, alternating multilinear maps also carry with them information about orientation. Orientation makes the order of vectors important. To study such multilinear maps we can restrict ourselves to the study of exterior powers instead of studying the much larger tensor product. So the starting point is the cotangent pre-sheaf.

$$\mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends each open set U to $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ which we will denote by $\mathcal{C}U$. We can consider the exterior algebra of this cotangent pre-sheaf.

$$\bigwedge^k \mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends U to $\wedge^k \mathcal{C}U$. The elements of the stalks of this pre-sheaf will be called k-forms. We can now bundle these stalks and consider the sheaf of sections of this bundle.

$$(\bigwedge^k \mathcal{C})^{\mathrm{Sh}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

This is a sheaf of vector spaces over $\mathbb{R}$ usually denoted by Ω^k. We can similarly consider the exterior algebra, which consists of the direct sum of all the exterior powers.

$$\Omega^\bullet = \bigoplus_{i=0}^{\infty} \Omega^k$$

Note that $\Omega^k = 0$ for $k > n$ where n is the dimension of the manifold and $\Omega^0 = \mathcal{A}^{\mathcal{M}}$. A differential k-form is a section of sheafification of kth exterior power of cotangent pre-sheaf. Equivalently it's an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields.

3.2.1 | Exterior Product

Although we haven't yet described the relation between differential forms and volumes, it's useful and use it to motivate other definitions involving differential forms. We want to be able to multiply two lengths and find out area. This is the idea of exterior product. Given two differential forms which intuitively measure some sort of length, we want to define an 'oriented area'. Let ω and κ be two differential forms. These give us a map,

$$\tau \mapsto (\omega(\tau), \kappa(\tau))$$

for each tangent vector τ. Now, we can define $(\omega \wedge \kappa)(\tau_1, \tau_2)$ to be the area of the parallelogram with sides $(\omega(\tau_1), \kappa(\tau_1))$ and $(\omega_1(\tau_2), \omega_2(\tau_2))$. Now, the area of the parallelogram is given by,

$$(\omega \wedge \kappa)(\tau_1, \tau_2) = \begin{vmatrix} \omega(\tau_1) & \kappa(\tau_1) \\ \omega(\tau_2) & \kappa(\tau_2) \end{vmatrix}.$$

This is called the exterior product of the differential forms ω and κ. We can generalize this to more general volumes. Let ω be a differential k-form and κ be differential l-form. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$(\omega \wedge \kappa)(\tau_1, \ldots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(k)}) \kappa(\tau_{\sigma(k+1)}, \ldots, \tau_{\sigma(k+l)}),$$

where σ is a partition[2] of the set $\{1,\dots k+l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ where $\mathrm{sgn}(\sigma)$ is the sign of the partition. Note that $\mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$. This is a bilinear map. The exterior algebra with the above product is a $\mathbb{Z}$-graded algebra. We can now list the basic properties of the exterior product.

Consider,

$$(\kappa \wedge \omega)(\tau_1,\dots,\tau_{k+l}) = \sum_\sigma \epsilon_\sigma \kappa(\tau_{\sigma(1)},\dots,\tau_{\sigma(k)})\omega(\tau_{\sigma(k+1)},\dots,\tau_{\sigma(k+l)}),$$

We can act this by a permutation, γ, such that, $(\gamma(1),\gamma(2),\dots,\gamma(k+l)) = (k+1,\dots,k+l,1,\dots,k)$. The sign of this permutation is, $\mathrm{sgn}(\gamma) = (-1)^{kl}$. So we have,

$$(\kappa\wedge\omega)(\tau_1,\dots,\tau_{k+l}) = (-1)^{kl}(\omega\wedge\kappa)(\tau_1,\dots,\tau_{k+l}).$$

Some basic combinatorics argument shows us that,

$$(\omega\wedge\kappa)\wedge\xi = \omega\wedge(\kappa\wedge\xi)$$

At each point x, every cotangent vector can be written in terms of local coordinates φ as, $[f] = \sum_{i=1}^n \left[\frac{\partial f}{\partial x_i}(x)\right]dx_i$. where dx_i is the equivalence class corresponding to the function $\varphi_i(x) - x_i$. Since we expect differential forms to be smooth sections of the cotangent sheaf, every differential form can be written as,

$$\omega = \textstyle\sum_{i=1}^n a_i dx_i.$$

where $a_i \in \mathcal{A}^{\mathcal{M}}$. Similarly, differential k-forms can be written as,

$$\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\dots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

where each $a_{i_1,\dots i_k} \in \mathcal{A}^{\mathcal{M}}$.

Using the tangency pairing,

$$\langle f,h\rangle_x = \left.\frac{d(f\circ h(t))}{dt}\right|_{t=0},$$

we can pair a differential form and a vector field pointwise which measures the length of X using the differential form ω at each point. This is called a contraction or interior product of ω with X. Denoted by

$$\iota_X\omega \coloneqq \langle\omega, X\rangle$$

This can also be extended to differential k-forms.

$$\iota_X\omega(X_1,\dots,X_{k-1}) \coloneqq \omega(X,X_1,\dots,X_{k-1}).$$

We can now start listing down the algebraic properties of the contraction. For exterior product of differential forms is given by,

$$\iota_X(\omega\wedge\kappa) = (\iota_X\omega)\wedge\kappa + (-1)^p\omega\wedge(\iota_X\kappa)\,.$$

From the anti-symmetry of differential forms, we have, $\omega(X,Y,\dots) = -\omega(Y,X,\dots)$. So,

$$\iota_X\iota_Y\omega = -\iota_Y\iota_X\omega$$

The contraction provides a map,

$$\iota_X : \Omega^k \to \Omega^{k-1}.$$

For a differential k-form, and vector fields $X_1\dots X_k$, we denote the evaluation by,

$$\langle\omega,(X_1\dots X_k)\rangle = \omega(X_1\dots X_k).$$

[2]a partition is a permutation such that $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(k+l)$.

3.2.1.1 | Lie Derivative of Differential Forms

Recall that Lie derivative of a tensor field along a vector field X with the flow $\varkappa_t$ at $x \in \mathcal{M}$ is defined as the limit,

$$L_X(R)(x) = \lim_{t \to 0} \frac{\varkappa_t^*(R(\varkappa_t(x))) - R(x)}{t}, \tag{Lie derivative}$$

Let β be an $\mathcal{A}^{\mathcal{M}}$-bilinear sheaf homomorphism, $\beta : \mathcal{V} \times \mathcal{V}' \to \mathcal{W}$, where $\mathcal{V}, \mathcal{V}'$ and $\mathcal{W}$ are tensor sheaves. The action of $\varkappa_t$ is given by,

$$\varkappa_t^*(\beta(R, R')) = \beta(\varkappa_t^* R, \varkappa_t^* R').$$

Here, we added and subtracted a term, and then took the limit inside. To take the limit inside, we would need the bilinear form to have some appropriate notion of continuity.

The Lie derivative of $\beta(R, R')$ with respect to X is,

$$L_X(\beta(R, R'))(x) = \beta(L_X R, R')(x) + \beta(R, L_X R')(x). \tag{product rule}$$

The result is also valid for just $\mathbb{R}$-bilinearity. For the differential forms, the exterior product is a biliear map, and hence we have,

$$L_X(\omega \wedge \omega') = (L_X \omega) \wedge \omega' + \omega \wedge (L_X \omega').$$

For the $\mathbb{R}$-bilinear map, $(\omega, Y) \mapsto \omega(Y) \in \mathcal{A}^{\mathcal{M}}$, applying the product rule we have,

$$L_X(\omega(Y)) = \beta(L_X \omega, Y) + \beta(\omega, L_X Y)(x) = (L_X(\omega))(Y) + \omega(L_X(Y)).$$

So, the Lie derivative of a 1-form ω is given by,

$$(L_X(\omega))(Y) = X(\omega(Y)) - \omega([X, Y]).$$

Similarly, the Lie derivative of a k-form is given by,

$$(L_X(\alpha))(X_1, \dots, X_k) = X(\alpha(X_1, \dots, X_k)) - \sum_{i=1}^{k} \alpha(X_1, \dots, [X, X_i], \dots, X_k).$$

3.2.2 | Exterior Differentiation

Differential forms define at each point, the notion of length, area, volume, etc. Now we want to do calculus with them i.e., differentiate and integrate stuff. A k-form provides some sort of k-volume on tangent spaces. They are more complicated and Lie derivative doesn't describe how they change correctly. We want to define a notion of differentiation that captures all the ways in which it changes.

For a function $f \in \mathcal{A}^{\mathcal{M}}$, the differential is the flow of the 0-volume.

$$f : \mathcal{M} \to \mathbb{R}$$

gives us the map $df(x)$ of equivalence classes of curves, $\tau_h \mapsto \tau_{f \circ h}$. This is a map from $T_x\mathcal{M}$ to $\mathbb{R}$. Hence $df(x)$ is an element in the stalk of the cotangent pre-sheaf $\mathcal{C}$. Covectors can be thought of as assigning to each tangent vector its 'length'. Since this depends smoothly on the point x, it's a differential form i.e., $df \in \Omega^1$. Note here that this is the reason why the equivalence classes of functions $\varphi_i(x) - x_i$ were written as dx_i.

The definition of exterior derivative is not very intuitive. We try to motivate the definition of 'exterior derivative' of differential forms below. Although this motivation is not sufficient to characterize the definition, it can help understand what's happening.

For a differential form ω, and vector fields X and Y, we have the pairings $\langle \omega, X \rangle$ and $\langle \omega, y \rangle$. Each of the pairings are differentiable functions on $\mathcal{M}$ i.e.,

$$\langle \omega, X \rangle, \langle \omega, Y \rangle \in \mathcal{A}^{\mathcal{M}}.$$

We are interested in understanding how the function $\langle \omega, X \rangle$ changes along another vector field Y, and $\langle \omega, Y \rangle$ changes along X. The change of $\langle \omega, X \rangle$ along Y is given by the new pairing, $L_Y(\langle \omega, X \rangle) = \langle d(\langle \omega, X \rangle), Y \rangle$, so the difference,

$$L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle)$$

is a differential 2-form. We want to take into account all the changes that are happening, so we should also take into account how X changes with respect to Y, as measured by the differential form ω which is $\omega(L_X(Y))$. So, we define the exterior derivative as,

$$(d\omega)(X, Y) = L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle) - \omega(L_X(Y)).$$

So, we have,

$$(d\omega)(X, Y) = X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \omega([X, Y]).$$

The negative signs are used so that this is a differential form. It's not yet clear why we have to take $\omega(L_X(Y))$ and not $\omega(L_Y(X))$, but we will not try to motivate that here. For a differential k-form, the exterior derivative is defined as,

$$\begin{aligned}(d\omega)(X_1 \dots X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1 \dots \widehat{X_i} \dots X_{k+1})) \\ &+ \sum_{1 \le j < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_{k+1}).\end{aligned} \tag{exterior derivative}$$

The explicit form of the exterior derivative in terms of local coordinates, is given by $d\omega = \sum \partial a_i / \partial x_i dx_i \wedge dx_I$, where $\omega = \sum a_i dx_I$. So, the exterior derivative is a map,

$$d : \Omega^k \to \Omega^{k+1}$$

We can now start listing all the properties of the exterior derivative. From the definition, it follows that the exterior derivative is linear,

$$d(\lambda\omega + \mu\kappa) = \lambda d(\omega) + \mu d(\kappa).$$

For a vector field X, we have,

$$\begin{aligned}(d\iota_X + \iota_X d)(\omega)(X_1 \dots X_k) = &\sum (-1)^{i+1} X_i(\iota_X \omega(X_1 \dots \widehat{X_i} \dots X_k)) \\ &+ \sum (-1)^{i+j} (\iota_X \omega)([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) \\ &+ (d\omega)(X, X_1 \dots \widehat{X_i} \dots X_j \dots X_k)\end{aligned}$$

which on expanding gives,

$$= X(\omega(X_1 \dots X_k)) + \sum (-1)^i \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) = (L_X(\omega))(X_1 \dots X_k)$$

This is called Cartan formula, and can be written compactly as,

$$d\iota_X + \iota_X d = L_X \tag{Cartan formula}$$

Let ω be a differential k-form and κ be a differential l-form, it can be checked that,

$$d(\omega \wedge \kappa) = (d\omega) \wedge \kappa + (-1)^k \omega \wedge (d\kappa)$$

Such maps are called derivations of odd type. It can be showed that[3],

$$d \circ d = 0.$$

The last property is very important and allows us to study the topological properties of the manifolds by studying the differential forms. The proofs of all these are very simple in local coordinates.

3.2.3 | Graded Derivations & de Rham Complex

A complex of $\mathcal{A}$-modules is a sequence $\{\Omega^i\}$ of modules and homomorphisms $\partial_i : M^i \to M^{i+1}$ such that successive composites $\partial_i \circ \partial_{i-1} = 0$, i.e., $\mathrm{im}(\partial_{i-1}) \subseteq \ker(\partial_i)$.

$$\cdots \xrightarrow{\partial_{i-2}} M^{i-1} \xrightarrow{\partial_{i-1}} M^i \xrightarrow{\partial_i} M^{i+1} \xrightarrow{\partial_{i+1}} \cdots$$

$$\mathrm{Im}(\partial_i) \subseteq \ker(\partial_{i+1}).$$

The homomorphisms ∂_i are called differentials of the complex usually the subscript i is not written explicitly. The complex itself is written as $\Omega^\bullet$. We can study certain quotient modules.

$$H^i = \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i-1})}.$$

These are called cohomology groups. In our case we have the $\mathcal{A}^\mathcal{M}$-modules of differential k-forms, Ω^k and d is the exterior derivative on the differential forms. The complex formed for such a complex is called the de Rham complex. The de Rham complex is then given by,

$$\cdots \xrightarrow{d} \Omega^{i-1} \xrightarrow{d} \Omega^i \xrightarrow{d} \Omega^{i+1} \xrightarrow{d} \cdots$$

Also written as, $(\Omega^\bullet, d)$. It's a $\mathbb{Z}$-graded algebra with respect to the exterior product.

A graded derivation of degree r on $\Omega^\bullet$ is a family of maps $D : \Omega^\bullet \to \Omega^\bullet$, such that,

$$D : \Omega^k \mapsto \Omega^{k-r}.$$

and such that,

$$D(\lambda\omega + \mu\kappa) = \lambda D(\omega) + \mu D(\kappa).$$
$$D(\omega \wedge \kappa) = D(\omega) \wedge \kappa + (-1)^{kr} \omega \wedge D(\kappa),$$

and D is a morphism of sheaves,

$$\begin{array}{ccc} \Omega^k U & \xrightarrow{D} & \Omega^{k-r} U \\ \big\downarrow |_V & & \big\downarrow |_V \\ \Omega^k V & \xrightarrow{D} & \Omega^{k-r} V \end{array}$$

[3] Proof of this using the coordinate expression is very simple and only involves using the fact that $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$. In fact all computations are easier done in local coordinates.

Suppose D_1 and D_2 are two graded derivations of degree r_1 and r_2. Then, we can define a new derivation,

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1.$$

Some some algebraic computation it can be showed that these are derivations of $(r_1 + r_2)$-degree.

3.2.4 | Invariant Forms vs. Invariant Measures

To motivate and make this relation precise, we start translation invariant measures on Euclidean space. The translational invariance makes sure that we can determine the measure of any measurable set if we know its value for some model set, say, a cube, because cubes generate the Borel σ-algebra, and the translational invariance allows us to measure any scaled copy of the cube.

3.2.4.1 | Motivating Example, $\mathbb{R}^n$

The translation invariant measures are determined uniquely upto a constant multiplication, and for Euclidean space it corresponds to the Lebesgue measure, upto scalar multiplication. So, for the vector field $X = \partial_v$ with the flow given by translations by $\varkappa_t = tv$, the Lie derivative $L_{\partial_v}\nu$ of the Lebesgue measure ν along ∂_v is given by,

$$L_{\partial_v}\nu(f) = \nu\big(\lim_{t\to 0} \tfrac{f\circ\varkappa_t^{-1} - f}{t}\big) = \nu\big(\tfrac{f-f}{t}\big) = 0.$$

Here we used the translational invariance. So, it exists and equals 0. Since we have,

$$(L_{hX}\mu)(f) = (L_X\mu)(hf) + X(h)\mu(f)$$

The Lie derivative along any vector field $\sum_i h_i\partial_i$ exists and equals,

$$(L_{(\sum_i h_i\partial_i)}\nu)(f) = \textstyle\sum_i \partial_i h_i \nu(f).$$

Hence Lebesgue measures are differentiable measures. Denote the set of all Lebesgue measures on $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. Lebesgue measures only vary by a scalar multiple, and are determined hence by their action on cubes. Each Lebesgue measure ν determines a map,

$$\begin{aligned} \widehat{\nu}:\ & \textstyle\prod^n \mathbb{R}^n \to \mathbb{R}. \\ & (v_i)_{i=1}^n \mapsto \nu([v_i]_{i=1}^n), \end{aligned}$$

where $[v_i]_{i=1}^n$ is the n-dimensional cube in $\mathbb{R}^n$ determined by the vectors $\{v_i\}$. $\nu([v_i]_{i=1}^n)$ is non-zero only if $\{v_i\}$ forms a basis of $\mathbb{R}^n$. Because otherwise, $[v_i]_{i=1}^n$ is a measure zero set, they are $< n$ dimensional sheets. The translation invariant, additivity, and continuity guarantee that,

$$\nu([v_1, \dots rv_i, \dots v_n]) = r\nu([v_i]_{i=1}^n).$$

So the map, $(v_i) \mapsto \nu([v_i])$ is multilinear in v_i, and since if any two v_is are equal we should have the measure to be zero, it's an alternating multilinear map and must factor through $\wedge^n\mathbb{R}^n$.

$$\begin{tikzcd} \prod^n \mathbb{R}^n \arrow[r, "i"] \arrow[dr, "\widehat{\nu}"'] & \bigwedge^n \mathbb{R}^n \arrow[d, "\exists!\ e_\nu"] \\ & \mathbb{R} \end{tikzcd}$$

So we have, $\nu([v_i]_{i=1}^n) = e_\nu(\wedge_{i=1}^n v_i)$. So, to each Lebesgue measure on $\mathbb{R}^n$ we have an associated differential n-form. Equivalently, we get a map from the space of invariant measures into the one-dimensional space $(\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}$.

$$\nu \mapsto e_\nu$$

The wedge product however is order sensitive, and the measure is not. So we should have,

$$e_\nu(\wedge^n v_i) = \pm\nu([v_i]_{i=1}^n).$$

This can be interpreted in the following sense, consider $(\wedge^n \mathbb{R}^n)^\vee \backslash \{0\} \cong \mathbb{R}\backslash\{0\}$, which has two connected components. The assignment of the sign 1 (respectively -1) in the definition is based upton whether $v_1 \wedge \cdots \wedge v_n$ belongs to the chosen component (or not). The choice of a connected component is equivalent to chosing a basis for $(\wedge^n \mathbb{R}^n)^\vee$. Such a choice of basis in $(\wedge^n \mathbb{R}^n)^\vee$ is called a volume element of $\mathbb{R}^n$, and such a volume element fixes the sign of Lebesgue measures. The map,

$$E : \mathcal{L}(\mathbb{R}^n) \to (\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}, \quad \nu \mapsto e_\nu.$$

is an isomorphism of the space of Lebesgue measures $\mathcal{L}(\mathbb{R}^n)$ and the space of differential n-forms $(\wedge^n \mathbb{R}^n)^\vee$, and this isomorphism depends on the choice of basis for $(\wedge^n \mathbb{R}^n)^\vee$, and the two isomorphisms differ by constant multiple (-1). The choice of basis is called the orientation of the vector space $\mathbb{R}^n$. A Euclidean space has two orientations. corresponding to the choice. In this case,

$$\mathcal{K}^{\mathbb{R}^n} = \Omega^n \mathbb{R}^n = \mathcal{A}^{\mathbb{R}^n} \otimes_{\mathbb{R}} (\Lambda^n \mathbb{R}^n)^\vee$$

and similarly we can tensor the space of invariant measures with $\mathcal{A}^{\mathbb{R}^n}$ of differentiable functions on $\mathbb{R}^n$, this is a subsheaf of the differentiable measures, $\mathcal{B}^{\mathbb{R}^n}$ consisting of measures of the form $f \cdot \nu$ for $f \in \mathcal{A}^{\mathbb{R}^n}$ and $\nu \in \mathcal{L}(\mathbb{R}^n)$. So the isomorphism above yields an isomorphism of these sheaves. The elements of $\mathcal{K}^{\mathbb{R}^n}$ are called volume forms.

3.3 | Sheaf of Densities

Although invariant measures and differential n-forms are closely related in Euclidean space, it might not be the case in general differentiable manifold. The manifold might have twists which might make such an association that patches up nicely impossible. We want to study homomorphism from the sheaf of volume forms $\mathcal{K}^{\mathcal{M}}$ to the sheaf invariant measures in $\mathcal{B}^{\mathcal{M}}$.

3.3.1 | Orientation Sheaf

We now have two pre-sheaves, $\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}} \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ of $\mathcal{A}^{\mathcal{M}}$-modules, for any $U \subset \mathcal{M}$, consider the new pre-sheaves, the restructions, $\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U \in \mathrm{PSh}(U, \mathcal{A}^{\mathcal{M}}|_U)$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U, \mathcal{A}^{\mathcal{M}}|_U)}(\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{array}{ccccc} U & & \mathcal{K}^{\mathcal{M}}U & \xrightarrow{\kappa_U} & \mathcal{B}^{\mathcal{M}}U \\ \big\downarrow |_V & & \mathcal{K}^{\mathcal{M}}(|_V)\big\downarrow & & \big\downarrow \mathcal{B}^{\mathcal{M}}(|_V) \\ V & & \mathcal{K}^{\mathcal{M}}V & \xrightarrow[\kappa_V]{} & \mathcal{B}^{\mathcal{M}}V. \end{array}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom of $\mathcal{K}^{\mathcal{M}}$ and $\mathcal{B}^{\mathcal{M}}$, denoted by,

$$\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}) \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}}).$$

Sometimes also written as $(\mathcal{B}^{\mathcal{M}})^{\mathcal{K}^{\mathcal{M}}}$. We will now show that the internal hom is also a sheaf, i.e., it satisfies the collation property,

$$\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U \overset{e}{\dashrightarrow} \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i \cap U_j).$$

To show that $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U$ and at $\prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 3.3.1. *If $\mathcal{B}^{\mathcal{M}}$ is a sheaf then so is $\mathcal{H}\mathrm{om}(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i \in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{K}, \mathcal{B}^{\mathcal{M}})U$, and $U_i \subset U$, we have,

$$\begin{array}{ccc} U & \qquad & \mathcal{K}^{\mathcal{M}}U \xrightarrow{\kappa_U} \mathcal{B}^{\mathcal{M}}U \\ \big\downarrow{\scriptstyle |_{U_i}} & & {\scriptstyle \mathcal{K}^{\mathcal{M}}(|_{U_i})}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle \mathcal{B}^{\mathcal{M}}(|_{U_i})} \\ U_i & & \mathcal{K}^{\mathcal{M}}U_i \xrightarrow[\kappa_{U_i}]{} \mathcal{B}^{\mathcal{M}}U_i. \end{array}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{K}^{\mathcal{M}}U_i$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \cap V} = 0.$$

Now, $\{V \cap U_i\}$ is a cover of V, and $\mathcal{B}^{\mathcal{M}}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \cap U_j} = \kappa_j|_{U_i \cap U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{B}^{\mathcal{M}}$ is a sheaf to patch these natural transformations.

Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{K}^{\mathcal{M}}V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

here the first map comes from the natural transformation, $\mathcal{K}V \ni f \mapsto \kappa_i(f|_{V \coprod U_i})$. Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, this must uniquely factor through $\mathcal{B}^{\mathcal{M}}V$, by definition of equaliser. Hence, we have,

$$\mathcal{K}^{\mathcal{M}}V \xrightarrow{\exists!} \mathcal{B}^{\mathcal{M}}V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ defines the patched up element that equalizes the diagram, and hence the internal hom is a sheaf whenever $\mathcal{B}^{\mathcal{M}}$ is a sheaf. □

The natural transformations $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ are $\mathcal{A}^{\mathcal{M}}$-module homomorphisms. In particular, $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf of $\mathbb{R}$-modules. However not all of these are important to us. We are interested in those which map invariant volume forms to invariant measures. Since the information about the measure being invariant has to do with Lie derivatives, we just have to preserve that structure, i.e, performing Lie derivation before the homomorphism should be the same as taking Lie derivation after the homomorphism.

Hence, the natural transformation of interest to us should preserve the Lie derivative. $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is said to be flat if

$$\kappa L_X = L_X \kappa. \tag{flat}$$

The collection of all flat homomorphisms is denoted by $OR_{\mathcal{M}}$.

Flat homomorphisms take invariant forms to invariant measures. The set of all flat homomorphisms $\mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules and is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$. Flat homomorphisms do exist when $\mathcal{M} = \mathbb{R}^n$, as described in 3.2.4.1.

Proposition 3.3.2. *If $\mathcal{M}$ is connected, κ, φ flat, then there exists $\lambda \in \mathbb{R}$ such that $\kappa = \lambda\varphi$.*

Proof

It's enough to describe the action on invariant forms. Let ω be an invariant form, i.e., in local coordinates, $L_{\partial_v}\omega = 0$. Since κ, φ are flat homomorphisms, $\kappa(\omega)$ and $\varphi(\omega)$ are invariant measures, because $\kappa L_X = L_X\kappa$, $\varphi L_X = L_X\varphi$, and κ, φ are homomorphisms,

$$L_{\partial_v}(\kappa(\omega)) = L_{\partial_v}(\varphi(\omega)) = 0.$$

Locally, these invariant measures are Lebesgue measures and hence must vary by a constant multiple. On intersections, this constant is preserved. Since the manifold is connected, there can't be any abrupt change to this constant multiple. So,

$$\kappa(\omega) = \lambda(\varphi(\omega))$$

for some $\lambda \in \mathbb{R}$. □

The pre-sheaf of flat homomorphisms,

$$\begin{aligned} OR_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathbb{R}}\mathrm{Mod} \\ U &\mapsto OR_{\mathcal{M}}(U), \end{aligned}$$

where $OR_{\mathcal{M}}(U)$ is the collection of all flat homomorphisms, $\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules. $OR_{\mathcal{M}}$ is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$,

$$OR_{\mathcal{M}} \rightarrowtail \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$$

Since in local coordinates each flat homomorphism varies from a standard flat homomorphism by a constant real multiple, it's a locally constant sheaf of $\mathbb{R}$-modules of rank 1.

The existence of global sections of $OR_{\mathcal{M}}$, i.e., global flat homomorphisms depends geometrically on whether the manifold has 'twists' or not. To intuitively motivate this twisting, consider for example, the case of a mobius strip. We can choose local coordinates around a point $x \in \mathcal{M}$, if there existed a flat homomorphism, then it should patch up nicely with local restrictions, however, when we go around the strip and return, the order of the basis we had chosen will be reversed. So, although we have local flat homomorphisms that agree on intersections, there doesn't exist a global patch up of them. The only possible flat homomorphism is the trivial homomorphism, which sends everything to zero. This twisting can hence be made precise in terms of the orientation sheaf.

Since flat homomorphisms vary by constant multiples, we can just consider equivalence classes. Choose in a local chart a choice of ordered basis, this determines locally, a standard flat homomorphism

$$\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$$

Now, we can consider all flat homomorphisms that vary by an integral multiple of κ, i.e., flat homomorphisms of the type $\lambda \cdot \kappa$ with $\lambda \in \mathbb{Z}$. This collection is a locally constant sheaf, denoted by $OZ_{\mathcal{M}}$, and the sheaf is called the local system of 'twisted integers'.

A connected manifold $\mathcal{M}$ is called oriented if this is the constant sheaf. Equivalently, we say $\mathcal{M}$ is oriented if the étale space of the sheaf of twisted integers, $OZ_{\mathcal{M}}$, is $\mathcal{M} \times \mathbb{Z}$. In such a case, there are two trivializations, and each of which is called an orientation on $\mathcal{M}$. Clearly,

$$OR_{\mathcal{M}} = OZ_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathbb{R}$$

The existence of a flat homomorphisms means that at each point in the manifold, we can associate an invariant measure, and these invariant measures patch up nicely. Assuming a flat homomorphism $\kappa : \mathcal{K}^{\mathcal{M}} \to \mathcal{B}^{\mathcal{M}}$ exists, the sheaf of densities is the image $\mathcal{S}_{\mathcal{M}} := \kappa(\mathcal{K}^{\mathcal{M}}) \subseteq \mathcal{B}^{\mathcal{M}}$.

$$\begin{aligned} \mathcal{S}_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathbb{R}}\mathrm{Mod} \\ U &\mapsto \kappa(\mathcal{K}^{\mathcal{M}}|_U), \end{aligned}$$

where $\kappa(\mathcal{K}^{\mathcal{M}}|_U) = \{\kappa(\omega) \mid \omega \in \mathcal{K}^{\mathcal{M}}|_U\}$. Since flat homomorphisms vary only by a constant multiple, no information is lost by choosing a flat homomorphism.

On $\mathbb{R}^n$, $\mathcal{S}_{\mathbb{R}^n}$ consists of all measures of the form, $f d\mu$ where μ is the Lebesgue measure on $\mathbb{R}^n$. So, in the case of a differential manifold, by local isomorphism of the sheaf of differentiable functions, in any coordinate system, these measures can be expressed as $f d\mu$.

3.3.2 | Pullback of Sheaf of Densities

Now that we have the sheaf of densities, we can study the behavior of the sheaf under diffeomorphisms. We are interested in how the elements of the sheaf change, and get a local formula for this change in terms of coordinates. So, it's good enough to restrict to diffeomorphisms from domains in $\mathbb{R}^n$.

Theorem 3.3.3. (Change of Variable Formula)

3.3.3 | Adjoint of Differential Operators

3.3.3.1 | Stokes Theorem

[1] S Ramanan, Global Calculus, Springer, 2000

[2] M Kashiwara, P Schapira, Sheaves on Manifolds, Springer-Verlag, 1994