Part IIa

Tangent & Cotangent Bundles

In these notes, we develop quickly the necessary geometry required for formulating relativity. What we need are topological spaces that locally look like Euclidean spaces. The goal is description of analogues of straight lines and parallel transport in manifolds. The boring parts have been moved to the appendices.

1 | Differentiable Manifolds

Our starting point is a topological space $\mathcal{M}$ that's Hausdorff and admits a countable base for the topology. This condition is to make sure there are no pathological spaces we should be worried about. The Hausdorff condition makes the points distinguishable by the topology itself. The countable basis allows us to do analysis. On this topological space we want a differentiable structure, i.e., the structure that allows us to do calculus.

The differentiable structure allows us to define differentiable functions. We expect differentiable functions to have some form of local property, similar to continuous functions. Sheafs axiomatize this 'local property'. Given a topological space X, a sheaf is a way of describing a class of objects on X that have a local nature. To motivate the definition, consider the set of continuous functions on the space X. Denote by CU the set of real-valued continuous functions on U. Then every function, $f \in CU$ has the following local properties,

If $V \subset U$ then f restricted to V is a continuous map, $f|_V : V \to \mathbb{R}$. The map, $f \mapsto f|_V$ is a function $CU \to CV$. If $W \subset V \subset U$ are nested open sets then the restriction is transitive.

$$(f|_V)|_W = f|_W.$$

This can be summarised by saying the assignment $U \mapsto CU$ is a functor,

$$C : \mathcal{O}(X)^{\text{op}} \to \textbf{Sets},$$

where $\mathcal{O}(X)$ are open sets of X and the morphisms $V \to U$ are inclusions $V \subset U$. $\mathcal{O}(X)^{\text{op}}$ is the dual category of $\mathcal{O}(X)$ with same objects and the arrows reversed. To each such inclusion morphism in $\mathcal{O}(X)^{\text{op}}$ we get restriction morphism in **Sets**, $\{U \supset V\} \mapsto \{CU \to CV\}$ given by $f \mapsto f|_V$.

This captures the property of 'local' objects. The mathematical objects that have this property are called pre-sheaves. A pre-sheaf is a functor

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \textbf{Sets},$$

where morphisms in $\mathcal{O}(X)$ are inclusion maps and **Sets** has a class of morphisms called restriction maps $|_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that, $|_{VW} \circ |_{UV} = |_{UW}$.

We now need some way to extend structures defined 'locally' to bigger sets. We need a way to patch up this local structure. This can be achieved by axiomatizing the following 'collation' property of continuous functions, et $U = \bigcup_{i \in I} U_i$ be an open covering. If $f_i \in CU_i$ such that $f_i x = f_j x$ for every $x \in U_i \cap U_j$ then it means that there exists a continuous function $f \in CU$ such that $f_i = f|_{U_i}$. The maps $f_i \in CU_i$ and $f_j \in CU_j$ represent the restriction of same map f if,

$$f|_{U_i \cap U_j} = f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

So, what we have is an I-indexed family of functions $(f_i)_{i \in I} \in \prod_i CU_i$, and two maps

$$p(\textstyle\prod_i f_i) = \prod_{i,j} f_i|_{U_i \cap U_j}, \qquad q(\prod_i f_i) = \prod_{j,i} f_i|_{U_i \cap U_j}.$$

Note that the order of i and j is important here and thats what distinguishes the two maps. The above property of existence of the function f implies that $f|_{U_j}|_{U_i \cap U_j} = f|_{U_i}|_{U_i \cap U_j}$ which means that there is a map e from CU to $\prod_i CU_i$ such that $pe = pq$. $CU \to \prod_i CU_i$

$$CU \overset{e}{\dashrightarrow} \prod_i CU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j).$$

This is called the collation property. Sheaves are a special kind of pre-sheaves that have this collation property. This allows us to take stuff from local to global. The map e is called the equalizer of p and q.

Definition 1.1. A sheaf of sets $\mathcal{F}$ on a topological space X is a functor,

$$\mathcal{F} : \mathcal{O}(X)^{\text{op}} \to \textbf{Sets},$$

such that each open covering $U = \bigcup_{i \in I} U_i$ of an open set U of X yields an equalizer diagram.

$$\mathcal{F}U \overset{e}{\dashrightarrow} \prod_i \mathcal{F}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

We start with what we expect from the 'differentiable' functions. The differentiable functions are continuous functions and hence satisfy the locality requirements and should form a sheaf. The sheaf of 'differentiable functions' is our starting point.

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\text{op}} \to \textbf{Sets}.$$

Since each differentiable function is expected to be a continuous function as well we have, $\mathcal{A}^{\mathcal{M}}(U) \subseteq C^{\mathcal{M}}(U)$, where $C^{\mathcal{M}}$ is the sheaf of continuous functions, $C^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\text{op}} \to \textbf{Sets}$, on $\mathcal{M}$ i.e., $\mathcal{A}^{\mathcal{M}}$ is a subsheaf of $C^{\mathcal{M}}$.

Let $\mathcal{F}_n$ be the sheaf of differentiable functions on the Euclidean space $\mathbb{R}^n$. The 'locally looks like Euclidean space' means that the sheaf $\mathcal{A}^{\mathcal{M}}$ locally looks like differentiable functions over a Euclidean space.

Definition 1.2. A differentiable manifold is a Hausdorff, second countable topological space $\mathcal{M}$ together with a sheaf

$$\mathcal{A}^{\mathcal{M}} : \mathcal{O}(\mathcal{M})^{\text{op}} \to \textbf{Sets},$$

of subalgebras of $C^{\mathcal{M}}$ such that for any $x \in \mathcal{M}$ there is an open neighborhood $x \in U$ with a homeomorphism $U \cong_{\varphi} V \subseteq \mathbb{R}^n$, such that

$$(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{F}_n(V),$$

where $(\varphi_* \mathcal{A}^{\mathcal{M}})(U) = \mathcal{A}^{\mathcal{M}}(\varphi^{-1}(V))$. This is easier to see in the diagram,

$$V \xrightarrow{\varphi^{-1}} U \xrightarrow{\mathcal{A}^{\mathcal{M}}} \mathcal{A}^{\mathcal{M}}(U).$$

The pair $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ is called a differentiable manifold. The homeomorphisms φs are called coordinate charts and $\mathcal{A}^{\mathcal{M}}$ is called the differentiable structure. The homeomorphisms transfer the smooth on euclidean space to the manifold. The sections of $\mathcal{A}^{\mathcal{M}}$ over an open set $U \subset \mathcal{M}$ are called differentiable functions on U, and we can do calculus on them.

Clearly the Euclidean space is a differentiable manifold. For an open set $U \subset \mathcal{M}$ the pair $(U, \mathcal{A}^{\mathcal{M}}|_U)$ is a differentiable manifold. If $\cup_i U_i$ is an open cover of $\mathcal{M}$ then $(U_i, \mathcal{A}^{\mathcal{M}}|_{U_i})$ are open manifolds. Let $U_i \cong_{\varphi_i} V_i$ and U_i and U_j intersect, let $\varphi_i(U_i \cap U_j) = V_{ij}$ and $\varphi_j(U_i \cap U_j) = V_{ji}$ then,

$$V_{ij} \cong_{\varphi_j \circ \varphi_i^{-1}} V_{ji}.$$

So, a collection of differentiable manifolds $(U_i, \mathcal{A}_i)$ can be glued together to form a differentiable manifold if the homeomorphisms $\varphi_j \circ \varphi_i^{-1}$ map the restriction $\mathcal{A}_i|_{U_i \cap U_j}$ to $\mathcal{A}_j|_{U_i \cap U_j}$ i.e., differentiable maps are mapped to differentiable maps. This means that $\varphi_j \circ \varphi_i^{-1}$ is differentiable for every i, j.

$\mathcal{M}$ may be obtained by taking all the open sets U_i and pasting $U_{ij} \subset U_i$ to $U_{ji} \subset U_j$ together by the transition functions.

$$\coprod_{i,j} U_i \cap U_j \overset{p}{\underset{q}{\rightrightarrows}} \coprod_i U_i \xrightarrow{c} \mathcal{M}.$$

The map c sends all the points $x \in U_i$ to the same point $x \in \mathcal{M}$. c is the coequalizer of p and q in the category **Top** of topological spaces. This is parallel to the definition of sheaf.

A continuous map f of a differentiable manifold $\mathcal{M}$ into a differentiable manifold $\mathcal{N}$,

$$f : \mathcal{M} \to \mathcal{N},$$

is said to be differentiable if it locally maps differentiable functions to differentiable functions, i.e., for all $x \in \mathcal{M}$ if g is a differentiable function in a neighborhood U of $f(x)$ then $g \circ f$ is differentiable function on $f^{-1}(U)$. If $g \in \mathcal{A}^{\mathcal{N}}(U)$ then $g \circ f \in \mathcal{A}^{\mathcal{M}}(f^{-1}(U))$. Hence to each differentiable maps there is a homomorphism of the sheaf $\mathcal{A}_{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$ given by the map,

$$g \mapsto g \circ f.$$

This is the map

$$C^{\mathcal{N}} \to f_*(C^{\mathcal{M}}),$$

of sheaves on $\mathcal{N}$ which sends the subsheaf $\mathcal{A}^{\mathcal{N}}$ into $f_*(\mathcal{A}^{\mathcal{M}})$. Differentiable manifolds together with morphisms like is the category of smooth manifolds. f_* is called the structure homomorphism associated to f. A differentiable map $f : \mathcal{M} \to \mathcal{N}$ of differentiable manifolds is called a diffeomorphism if there is a differentiable inverse.

2 | Tangent and Cotangent Bundles

What we want to do is give a linear approximation of a manifold at each point. In order to do this, we use curves passing through the point, and linearize them, and then study them.

Around each point $x \in \mathcal{M}$, consider all the smooth functions $f \in \mathcal{A}^{\mathcal{M}}(U)$, i.e., $f : U \to \mathbb{R}$ defined in some open neighborhood U of $x \in \mathcal{M}$. For each smooth path $h : \mathbb{R} \to U$ which passes through x with $h(0) = x$ we can define a smooth map,

$$f \circ h : \mathbb{R} \to \mathbb{R},$$

which has a first derivative at 0. This gives us a pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0}.$$

To remove redundant information, we define the equivalences $f \equiv f'$ at x if $\langle f, h \rangle_x = \langle f', h \rangle_x$ for all h and $h \equiv h'$ at x if $\langle f, h \rangle_x = \langle f, h' \rangle_x$ for all f.

Under addition and scalar multiplication of functions, this set of all equivalence classes of functions f forms a real vector space, denoted T^x. The elements of T^x are called cotangent vectors at x. Each function f in the neighborhood of x determines a vector $[f]$.

The sheaf of differentiable functions has more algebraic structure. It's a sheaf of algebras over $\mathbb{R}$ or the sheaf of module over the ring $\mathbb{R}$. The category of modules over some ring possess direct limits. Inclusion is a preorder on the collection of open sets given by,

$$V \geq U \text{ if } V \subset U.$$

Let $\mathcal{D}$ be a directed collection of open sets. For the pre-sheaf $\mathcal{A}^{\mathcal{M}} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Sets}$, we get a directed system in **Sets** given by $\{\mathcal{A}^{\mathcal{M}} U\}_{U \in \mathcal{D}}$. We will focus on this particular directed system.

The stalk $\mathcal{A}^{\mathcal{M}}_x$ of a pre-sheaf $\mathcal{A}^{\mathcal{M}}$ at x is the direct limit of the directed system $\{\mathcal{A}^{\mathcal{M}} U_i\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is a directed set of open neighborhoods of x.

$$\mathcal{A}^{\mathcal{M}}_x = \varinjlim_{x \in U} \mathcal{A}^{\mathcal{M}} U.$$

Stalks are functors,

$$\begin{aligned} \mathrm{Stalk}_x : \mathrm{PSh}(X) &\to \mathbf{Sets} \\ \mathcal{A}^{\mathcal{M}} &\mapsto \mathcal{A}^{\mathcal{M}}_x. \end{aligned}$$

The elements of $\mathcal{A}^{\mathcal{M}}_x$ are called germs at x. If a germ is a direct limit of some element $f \in \mathcal{A}^{\mathcal{M}} U$ then we denote it by $\mathrm{germ}_x f$. Note that $\mathcal{A}^{\mathcal{M}}_x$ is an algebra, and in particular a ring. $\mathrm{germ}_x : \mathcal{A}^{\mathcal{M}} U \to \mathcal{A}^{\mathcal{M}}_x$, is a homomorphism of the respective category for each U.

An ideal $\mathcal{I}_x \subset \mathcal{A}^{\mathcal{M}}_x$ is a subalgebra such that if $\mathrm{germ}_x f \in \mathcal{I}_x$ then $\mathrm{germ}_x f \, \mathrm{germ}_x g \in \mathcal{I}_x$ for all $\mathrm{germ}_x g \in \mathcal{A}^{\mathcal{M}}_x$. A proper ideal cannot contain the identity because that would mean the whole algebra is contained in the ideal. For each $\mathrm{germ}_x f \in \mathcal{A}^{\mathcal{M}}_x$, we have the evaluation map, $\mathrm{germ}_x f \to f(x)$, gives us an algebra homomorphism,

$$ß : \mathcal{A}^{\mathcal{M}}_x \to \mathbb{R}.$$

The kernel of this evaluation map $\mathcal{I}_x = \ker(ß)$ is an ideal of $\mathcal{A}^{\mathcal{M}}_x$, consisting of all germs that vanish at x, i.e., $f(x) = 0$ and hence $f(x)g(x) = 0$ for all $g \in \mathcal{A}^{\mathcal{M}}$. Hence,

$$\mathcal{A}^{\mathcal{M}}_x / \mathcal{I}_x \cong \mathbb{R}.$$

Evaluation can hence be thought of as taking quotient with the maximal ideal $\mathcal{I}_x$. This is also the only maximal ideal, because all other functions have local inverse, because if a function f is non-zero in a small neighborhood it has an inverse, defined by, $\mathrm{germ}_x(1/f)$, and hence this would mean the constant function belongs to the ideal which means it's not proper ideal. So, no other proper ideal can exist.

Going back to the pairing,

$$\langle f, h\rangle_x = \left.\frac{d(f\circ h(t))}{dt}\right|_{t=0}, \tag{pairing}$$

The set of equivalence classes of paths h are called tangent vectors at x denoted by $T_x\mathcal{M}$. Each smooth path through x has a tangent vector denoted by τ_h. Using the pairing, the tangent vector τ_h determines a linear map, $D_{\tau_h} : T^x \to \mathbb{R}$ given by the action,

$$D_{\tau_h}([f]) = \langle f, h\rangle_x.$$

We would like to understand what T^x is. The set $T_x\mathcal{M}$ of all tangent vectors at x is isomorphic to the set of all linear maps $T^x \to \mathbb{R}$. That's to say, $T_x\mathcal{M}$ is the dual space of T^x, and hence is itself a vector space. We will hence denote T^x by $T_x^\vee\mathcal{M}$ or $\mathrm{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R})$.

The derivative of a product, in the pairing, the map $D = D_{\tau_h}$ satisfies the following product rule,

$$D(fg) = f(x)D(g)(x) + g(x)D(f)(x). \tag{Leibniz}$$

for all $f, g \in \mathcal{A}^{\mathcal{M}}$. This is called the Leibniz property, and all the maps D with the Leibniz property are called derivations. Conversely, every derivation there is a corresponding curve h such that $D_{\tau_h} = D$.[1] The linear maps,

$$D : \mathcal{A}_x^{\mathcal{M}} \to \mathbb{R},$$

with the above Leibniz property at x are called derivations, denoted by $\mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$.

Our goal is to describe the collection of equivalence classes of functions f. The equivalence relation is $f \equiv f'$ iff $\langle f, h\rangle_x = \langle f', h\rangle_x$, for all h and these uniquely determine the map D_{τ_h}. By plugging in the constant 1, it can be checked that D annihilates constant functions.

$$D(\lambda) = 0 \quad \forall \lambda \in \mathbb{R}.$$

Hence all functions that differ by constant belong to the same equivalence class. Hence, for every $\mathrm{germ}_x\, f \in \mathcal{A}_x^{\mathcal{M}}$, we can consider the functions $\mathrm{germ}_x(f - f(x))$. These functions vanish at x, the action of D on the ideal $\mathcal{I}_x$ is sufficient to describe the map D. So, we have an surjection of the ideal $\mathcal{I}_x$ to the set of equivalence classes.

$$\mathcal{I}_x \twoheadrightarrow T_x^*\mathcal{M}.$$

Now we have to remove all the redundant information from $\mathcal{I}_x$. The kernel of the map is the ideal $\mathcal{I}_x^2 = \{\sum_{i,j} g_i f_j \;:\; f_i, g_j \in \mathcal{I}_x\}$. Hence, we can quotient it out of $\mathcal{I}_x$ and we have,

$$T_x^*\mathcal{M} \cong \mathcal{I}_x/\mathcal{I}_x^2,$$

or that the equivalence class for the function f only contains the first order information. The tangent space then is,

$$T_x\mathcal{M} \cong \mathrm{Hom}_{\mathbb{R}}(\mathcal{I}_x/\mathcal{I}_x^2, \mathbb{R}) = \mathcal{D}(\mathcal{A}_x^{\mathcal{M}}).$$

[1] The idea is to express it in local charts, and this should be of the form $\sum_i h_i \partial/\partial x_i$, and using this define a curve $h : t \mapsto \varphi^{-1}(t(c_i x_i))$. This works.

Now, for any derivation, $D \in \mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$, and $h \in \mathcal{A}^{\mathcal{M}}$,

$$h(x)D(fg) = h(x)f(x)D(g) + h(x)g(x)D(f).$$

If $D_1, D_2 \in \mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$, then their sum $D_1 + D_2 \in \mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$. $hD \in \mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$. So, $\mathcal{D}(\mathcal{A}_x^{\mathcal{M}})$ is an $\mathcal{A}_x^{\mathcal{M}}$-module. Using these derivations we can define the tangent sheaf and cotangent sheaf.

Define $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ to be derivations, i.e., maps such that for $f, g \in \mathcal{A}^{\mathcal{M}}U$,

$$D(fg) = fD(g) + gD(f).$$

That is to say, for all $f, g \in \mathcal{A}^{\mathcal{M}}U$,

$$D(fg) = fD(g) + gD(f).$$

Such operators $D : \mathcal{A}^{\mathcal{M}} \to \mathcal{A}^{\mathcal{M}}$ are called first order linear homogeneous differential operators. If $D_1, D_2 \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$, then we can define a new operator $[D_1, D_2]$ defined by,

$$[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f)).$$

For $f, g \in \mathcal{A}^{\mathcal{M}}U$, we have,

$$\begin{aligned}
[D_1, D_2](fg) &= D_1(D_2(fg)) - D_2(D_1(fg)) = D_1(fD_2(g) + gD_2(f)) - D_2(fD_1(g) + gD_1(f)) \\
&= fD_1(D_2(g)) + D_1(f)(D_2(g)) + D_1(g)D_2(f) + gD_1(D_2(f)) \\
&\qquad - fD_2(D_1(g)) + D_2(f)(D_1(g)) + D_2(g)D_1(f) + gD_2(D_1(f)) \\
&= f[D_1, D_2](g) + g[D_1, D_2](f).
\end{aligned}$$

So, $[D_1, D_2] \in \mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$. So, $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ is an $\mathbb{R}$-algebra. It's however not a $\mathcal{A}^{\mathcal{M}}U$-algebra because it's not $\mathcal{A}^{\mathcal{M}}U$-bilinear. Because for $g \in \mathcal{A}^{\mathcal{M}}U$,

$$\begin{aligned}
[D_1, gD_2](f) &= D_1(gD_2(f)) - gD_2(D_1(f)) \\
&= D_1(g)D_2(f) + gD_1(D_2(f)) - gD_2(D_1(f)) \\
&= (D_1(g)D_2 + g[D_1, D_2])(f).
\end{aligned}$$

So, it's not bilinear, and hence can't be an $\mathcal{A}^{\mathcal{M}}U$-algebra. Denote this sheaf by $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$.

Definition 2.1. A Lie algebras over a commutative ring $\mathcal{A}$ is an $\mathcal{A}$-module together with a bilinear operation, $(D_1, D_2) \mapsto [D_1, D_2]$, such that, $[D, D] = 0$ and,

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0 \qquad \text{(Jacobi's identity)}$$

A homomorphism of Lie algebras is an algebra homomorphism $f : \mathcal{V} \to \mathcal{W}$ such that,

$$f([X, Y]) = [f(X), f(Y)].$$

The sheaf $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ is a sheaf of Lie algebras over $\mathbb{R}$. The restriction map is a Lie algebra homomorphism. This sheaf of Lie algebras is called the Tangent sheaf. The Tangent sheaf,

$$\begin{aligned}
\mathcal{D}(\mathcal{A}^{\mathcal{M}}) :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to \mathbf{Sets} \\
U &\mapsto \mathcal{D}(\mathcal{A}^{\mathcal{M}}U).
\end{aligned}$$

is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules and sheaf of $\mathbb{R}$-Lie algebras.

$\{f_i\}_{i=1}^n \mapsto \sum_i f_i \frac{\partial}{\partial x_i}$ is an isomorphism of modules $(\mathcal{A}^{\mathcal{M}}(U))^{\oplus n}$ and $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U)$ for a chart (φ, U), see appendix for details. Such sheaves of modules are called locally free modules. The tangent sheaf consists of the sections of the tangent bundle where the tangent bundle $T\mathcal{M}$ is the disjoint union,

$$T\mathcal{M} = \coprod_{x \in \mathcal{M}} T_x\mathcal{M}.$$

These sections correspond to vector fields. The stalks of this sheaf consist of germs of vector fields.

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) = \varinjlim_{x \in U} \mathcal{D}(\mathcal{A}^{\mathcal{M}}U).$$

So, $\mathrm{germ}_x D : \mathcal{A}^{\mathcal{M}}_x \to \mathcal{A}^{\mathcal{M}}_x$. The evaluation of the germs at the point x should give us vectors of the tangent space and they do. We evaluate $\mathrm{germ}_x D(f) \mapsto (Df)(x)$ at x. So the composition of the derivation with this evaluation map corresponds to a derivation at x.

The evaluation map gave us isomorphism, $\mathcal{A}^{\mathcal{M}}_x/\mathcal{I}_x \cong \mathbb{R}$. For locally free sheaves, for every $x \in \mathcal{M}$, there exists a neighborhood U such that $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) = (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$. So, we have,

$$\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}}) \cong \mathbb{R}^n.$$

In this sense $\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$ is the space of valuation of the sections at x. This is the same as the tangent space at x.

$$T_x\mathcal{M} = \mathcal{D}_x(\mathcal{A}^{\mathcal{M}})/\mathcal{I}_x\mathcal{D}_x(\mathcal{A}^{\mathcal{M}})$$

We have a projection from the tangent bundle to the base space,

$$\pi : T\mathcal{M} \to \mathcal{M}$$

which sends $T_x\mathcal{M} \mapsto x$. Since $\mathcal{D}(\mathcal{A}^{\mathcal{M}}U) \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$, $\pi^{-1}(U)$ can be identified with $U \times \mathbb{R}^n$. So, the set $\pi^{-1}(U)$ can be given a differentiable structure of a product. These can be patched up to get a differentiable structure on $T\mathcal{M}$. This topology is Hausdorff and has a countable basis because locally it's a product of Hausdorff spaces with countable basis.[2] Smooth sections of this bundle are called vector fields.

Similarly we can consider the cotangent sheaf,

$$\begin{aligned} \mathcal{I}/\mathcal{I}^2 :\ & \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets} \\ & U \mapsto \mathcal{I}(U)/\mathcal{I}(U)^2 \end{aligned}$$

where $\mathcal{I}(U)$ is the maximal ideal of $\mathcal{A}^{\mathcal{M}}U$. This is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. It's also locally free, with the local isomorphism $\{f_i\}_{i=1}^n \mapsto \sum_i f_i dx_i$ of modules $(\mathcal{A}^{\mathcal{M}}(U))^{\oplus n}$ and $\mathcal{I}(U)/\mathcal{I}(U)^2$ for a chart (φ, U).

Definition 2.2. A sheaf of $\mathcal{A}^{\mathcal{M}}$-modules $\mathcal{D}$ is said to be locally free of rank n if for every $x \in \mathcal{M}$ has a neighborhood U such that,

$$\mathcal{D}|_U \cong (\mathcal{A}^{\mathcal{M}}U)^{\oplus n}$$

[2] Note that in the case of Etale space, the properties of individual elements of the sheaf are used to get a topology, in the case of tangent bundle we used the properties of the sheaf itself to get a topology. We first quotiented the stalks with the maximal ideal of the ring of functions and then bundled them, and didn't care about the properties of the individual elements for the topology.

A differentiable map $\pi : E \to \mathcal{M}$ such that for any $x \in \mathcal{M}$ there exists an open neighborhood U such that

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

is a diffeomorphism and each fiber $\pi^{-1}(x)$ has the structure of a vector space for every $x \in \mathcal{M}$ is called a vector bundle. Φ is called a local trivialization of the bundle E. A homomorphism of a vector bundle E into F is a map which makes the diagram,

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & \mathcal{M} & \end{tikzcd}$$

commute, and the induced maps on the fibers are all linear.

Locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules in general give rise to vector bundles. Conversely to each vector bundle we can associate the sheaf of differentiable sections of π which is a locally free sheaf of $\mathcal{A}^{\mathcal{M}}$-modules. There is a natural bijection between the sheaves of locally free sheaves of $\mathcal{A}^{\mathcal{M}}$-modules and vector bundles. Every $\mathcal{A}^{\mathcal{M}}$-linear sheaf homomorphism gives a homomorphism of vector bundles. It's an equivalence of categories.

2.1 | Differential of a map

If the tangent space at x is interpreted as the linear approximation of the manifold $\mathcal{M}$ at x, then the differential of a map is interpreted as the linear approximation of the map $\varkappa$.

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map of manifold $(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ into $(\mathcal{N}, \mathcal{A}^{\mathcal{N}})$. Let $x \in \mathcal{M}$, using the pairing,

$$\langle f, h \rangle_x = \left. \frac{d(f \circ h(t))}{dt} \right|_{t=0},$$

where $h : \mathbb{R} \to U$ is a curve with $h(0) = x$, and for all $f \in \mathcal{A}^{\mathcal{M}}U$, the tangent space is defined to be the collection of the equivalence classes of such curves. Using the map $\varkappa$, we can push forward the curve h to $\mathcal{N}$, given by the composition, $\varkappa \circ h : \mathbb{R} \to \varkappa(U)$.

$$\mathbb{R} \xrightarrow{h} \mathcal{M} \xrightarrow{\varkappa} \mathcal{N}$$

By a differential of the function $\varkappa$, we mean a map $D\varkappa(x)$ that takes the equivalence class τ_h to the equivalence class $\tau_{\varkappa \circ h}$. The new pairing that arises from the map is,

$$\langle g, \varkappa \circ h \rangle_{\varkappa(x)} = \left. \frac{d(g \circ \varkappa \circ h(t))}{dt} \right|_{t=0},$$

for all $g \in \mathcal{A}^{\mathcal{N}}(\varkappa(U))$.

$$D\varkappa(x) : \; T_x\mathcal{M} \to T_{\varkappa(x)}\mathcal{N}$$
$$\tau_h \mapsto \tau_{\varkappa \circ h}$$

It gives a vector bundle homomorphism of $T\mathcal{M}$ into $\varkappa^*(T\mathcal{N})$, usually denoted by $D\varkappa$. If the local coordinates for $\mathcal{M}$ and $\mathcal{N}$ are (x_i) and (y_j) respectively. The differential of $\varkappa$ is the map,

$$\tfrac{\partial}{\partial x_i} \mapsto \textstyle\sum_j \left[\tfrac{\partial}{\partial x_i}(\widehat{\varkappa}_j)\right] \tfrac{\partial}{\partial y_j}.$$

where $\widehat{\varkappa} = y \circ \varkappa \circ x^{-1}$. $J_{ij} = \frac{\partial}{\partial x_i}(\widehat{\varkappa}_j)$ is called the Jacobian of the map $\varkappa$. Inverse function theorem, implicit function theorem, etc. are applicable here.

2.2 | Vector Fields and Frobenius Theorem

A smooth function $\varkappa : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if $D\varkappa(x)$ is invertible for all $x \in \mathcal{M}$. This would mean that there exists an smooth inverse $\varkappa^{-1}$. The set of all diffeomorphisms of a manifold, i.e., $\varkappa$ is a map from $\mathcal{M}$ to $\mathcal{M}$ is a group. We will call such maps diffeomorphism 'of' $\mathcal{M}$. A one parameter group of diffeomorphisms of $\mathcal{M}$ is a collection of diffeomorphisms,

$$\varkappa : t \mapsto \varkappa_t$$

where each $\varkappa_t$ is a diffeomorphism of $\mathcal{M}$ such that, $\varkappa_0 = \mathbb{I}_{\mathcal{M}}$ and

$$\varkappa_t \circ \varkappa_s = \varkappa_{t+s} \quad \forall t, s \in \mathbb{R}.$$

and $\varkappa_t$ is a smooth as a map from $\mathcal{M} \times \mathbb{R}$ to $\mathcal{M}$. Where $\mathcal{M} \times \mathbb{R}$ has the differentiable structure of a product manifold. For fixed $x \in \mathcal{M}$, $\varkappa_t(x)$ is a smooth curve in $\mathcal{M}$. The one parameter group $t \mapsto \varkappa_t$ determines at each $x \in \mathcal{M}$ the equivalence class of curves $[\varkappa_t(x)]$. Hence we obtain at each point $x \in \mathcal{M}$ a vector in the tangent space at x. Each one parameter group of diffeomorphisms determines a vector field.

For any $f \in \mathcal{A}^{\mathcal{M}}$ we have the smooth composition, $f \circ \varkappa_t : \mathcal{M} \to \mathbb{R}$. For fixed $x \in \mathcal{M}$, and varying t, this also corresponds to the smooth map,

$$\begin{aligned} f(\varkappa_{(\cdot)}(x)) :\ & \mathbb{R} \to \mathbb{R} \\ & t \mapsto f(\varkappa_t(x)) \end{aligned}$$

So, we can differentiate this function.

$$(\Xi_\varkappa f)(x) = \lim_{t \to 0} \frac{f(\varkappa_t(x)) - f(x)}{t} = \left.\frac{d(f \circ h(t))}{dt}\right|_{t=0} = \langle f, h \rangle_x$$

where $h = \varkappa_t(x)$. This can also be thought as the function,

$$\begin{aligned} \Xi_\varkappa(\cdot) : \mathcal{A}^{\mathcal{M}} & \to \mathcal{A}^{\mathcal{M}}. \\ f & \mapsto (\Xi_\varkappa f) \end{aligned}$$

It is called the differentiation of the function f with respect to $\varkappa$ at x. $\varkappa_t$ moves x to $\varkappa_t(x)$, so what the above limit is doing is measuring the infinitesimal change to the function f as 'along' $\varkappa_t$.

For $f, g \in \mathcal{A}^{\mathcal{M}}$, by directly plugging into the definition, we find that,

$$\Xi_\varkappa(f + g)(x) = \Xi_\varkappa(f)(x) + \Xi_\varkappa(g)(x)$$

and for the product,

$$\begin{aligned} \Xi_\varkappa(fg)(x) =& \lim_{t \to 0} f(\varkappa_t(x))\big[\big[g(\varkappa_t(x)) - g(x)\big]/t\big] + \lim_{t \to 0} g(x)\big[\big[f(\varkappa_t(x)) - f(x)\big]/t\big] \\ =& f(x)(\Xi_\varkappa g)(x) + g(x)(\Xi_\varkappa f)(x) \end{aligned}$$

Here we used the fact that $\varkappa_0 = \mathbb{I}_{\mathcal{M}}$. So, $\Xi_\varkappa$ is linear map, and

$$\Xi_\varkappa(fg) = f(\Xi_\varkappa g) + g(\Xi_\varkappa f)$$

Hence, $\Xi_\varkappa$ is a homogeneous first order operator. The converse also holds locally. Given a vector field Ξ on a differentiable manifold $\mathcal{M}$ there exists a one-parameter group of diffeomorphisms $\varkappa$ such that $\Xi_\varkappa = \Xi$. This is due to the existence and uniqueness of solutions to ODEs.

Locally in the charts, the vector field takes the form $\sum_i a_i \partial/\partial x_i$. What we need to do is find functions, $\varkappa(t, x) : (-\epsilon, \epsilon) \times U \to \mathbb{R}^n$ such that $\frac{\partial}{\partial t}\varkappa_i(t, x) = a_i(x)$, with $\varkappa_i(x) = x_i$. The existence and uniqueness of such solution follows from the existence and uniqueness of solutions to ODEs.

Definition 2.3. The flow of a vector field Ξ is the one parameter group of diffeomorphisms $\varkappa$ such that $\Xi_\varkappa = \Xi$.

If a vector field gives rise to a global flow, it's called complete.

A | Appendices

Partition of Unity

This is an important tool. What we intend to do is restrict the domain of functions to some regions so we can forget about the behaviour of the function outside some region.

Let $\{U_i\}_{i\in I}$ be a locally finite open cover of a differentiable manifold $\mathcal{M}$. This happens because manifolds are locally compact. A partition of unity with respect to the cover $\{U_i\}_{i\in I}$ is a family of functions $\{\varphi_i\}_{i\in I}$ with values in $[0,1]$ such that

$$\textstyle\sum_{i\in I}\varphi_i = 1$$

with support of φ_i contained in U_i. Once we have such a partition of unity, we can study the function $(\sum_{i\in I}\varphi_i)f$ instead of the function f. To show the existence, let $\{V_i\}_{i\in I}$ be an open cover with $\overline{V_i}\subset U_i$, we can construct functions ψ_i that have support in U_i.[3] Since the cover is locally finite the sum makes sense.

$$\varphi_i = \psi_i/(\textstyle\sum_{i\in I}\psi_i)$$

then acts as a partition of unity. This allows us to study the functions using the charts.

Local Description

To get explicit description of the tangent space in terms of the local coordinates lets consider the definition of a differentiable manifold, 1.2. From the definition,

$$\mathcal{A}^{\mathcal{M}}(\varphi^{-1}(U)) = \mathcal{F}_n(V)$$

for the local chart, φ. Since $\mathcal{F}_n$ consist of smooth functions on $\mathbb{R}^n$, we can describe them in terms of their Taylor expansion. If we denote the local coordinates by x_i, we have,

$$f(y) = f(x) + \textstyle\sum_{i=1}^n \big[\frac{\partial f}{\partial y_i}(x)\big](\varphi_i(x)-x_i) + \sum_{i,j}\frac{1}{2!}\big[\frac{\partial^2 f}{\partial x_i\partial x_j}(\varphi(x))\big](\varphi_i(x)-x_i)(\varphi_j(x)-x_j)+\cdots$$

Since the elements should be in $\mathcal{I}_x$, we have, $f(x)=0$, and since we are quotienting out by $\mathcal{I}_x^2$, the higher order terms will go. The equivalence classes $[\varphi_i(x)-x_i]$ form a basis for the cotangent space. We denote them by dx_i. We can also do this more directly from the definition, in local charts, we obtain the familiar formula for derivative of a composition of functions,

$$\langle f,h\rangle_x = \textstyle\sum_{i=1}^n \big(\frac{\partial f}{\partial x_i}\big)\big|_x\big(\frac{d(\varphi_i\circ h)}{dt}\big)\big|_{t=0}$$

So, each cotangent vector $[f]$ at x has n real coordinates $(\partial f/\partial x_i)$ evaluated at x. Which gives n coordinate functions, $\{\partial/\partial x_i\}_{i=1}^n$ for every cotangent space. and n dual coordinates $\{dx_i\}_{i=1}^n$ for every tangent space. We can write,

$$[f] = \textstyle\sum_{i=1}^n\big(\frac{\partial f}{\partial x_i}\big)\big|_x dx_i, \quad \tau_h = \sum_{i=1}^n\big(\frac{d(x_i\circ h)}{dt}\big)\big|_{t=0}\big(\frac{\partial}{\partial x_i}\big)$$

A general vector in the cotangent space is hence of the form, $\sum_{i=1}^n f_i dx_i$. A general vector in the tangent space is of the form, $\sum_{i=1}^n h_i\partial/\partial x_i$.

Let $\varkappa:\mathcal{M}\to\mathcal{N}$ be a differentiable map of manifold $(\mathcal{M},\mathcal{A}^{\mathcal{M}})$ into $(\mathcal{N},\mathcal{A}^{\mathcal{N}})$. The map in terms of local coordinates, with charts, φ and ψ respectively, then we consider the map $\widehat{\varkappa}=\psi\circ\varkappa\circ\varphi^{-1}$. This is of the form,

$$(x_i)_{i\in I}\mapsto(\widehat{\varkappa}_j((x_i)_{i\in I}))_{j\in J}.$$

Let $x\in\mathcal{M}$, using the pairing,

$$\langle f,h\rangle_x = \frac{d(f\circ h(t))}{dt}\bigg|_{t=0},$$

[3] Let $S_1\subset S_2$ be concentric spheres in $\mathbb{R}^n$ centered at 0 then we need to show that there exist differentiable function which is zero outside S_2 and non zero everywhere inside S_1. If we take S_2 to be the unit ball, the function, $\Phi(x)=\exp\big(\frac{1}{\sum_i x_i^2-1}\big)$ for x in the unit ball and zero outside works.

This corresponds to the map,

$$\tau_h = \sum_{i=1}^{n} \left(\frac{d(x_i \circ h)}{dt}\right)\Big|_{t=0} \left(\frac{\partial}{\partial x_i}\right) \mapsto \tau_{\varkappa \circ h} = \sum_{i=1}^{n} \left(\frac{d(y_i \circ \varkappa \circ h)}{dt}\right)\Big|_{t=0} \left(\frac{\partial}{\partial y_i}\right)$$

Now, we can write it as,

$$\frac{d(\widehat{\varkappa} \circ \varphi \circ c)}{dt}\Big|_{t=0} = \underbrace{D\widehat{\varkappa}(\varphi(x))}_{J_{ij}} \circ \frac{d(\varphi \circ c)}{dt}\Big|_{t=0}.$$

J_{ij} is called the Jacobian of the map $\varkappa$ at x and it will be,

$$J_{ij} = \frac{\partial \widehat{\varkappa}_j}{\partial x_i}.$$

References

[1] S Mac Lane, L Moerdijk, Sheaves in Geometry and Logic: A First Introduction to Topos Theory, Springer, 1992

[2] S Ramanan, Global Calculus, Graduate Studies in Mathematics, AMS, 2004