

PART IA

LOCALLY COMPACT GROUPS

1 | THE MODULAR FUNCTION

By a locally compact group we mean a topological group that is locally compact and Hausdorff. Let \mathcal{G} be a locally compact group. Let $C_c(\mathcal{G})$ denote the space of compactly supported continuous functions on \mathcal{G} . Let $C_c^+(\mathcal{G})$ denote the compactly supported continuous functions that are always non-negative. Note that $C_c(\mathcal{G})$ is the linear span of $C_c^+(\mathcal{G})$.

A left Haar measure on \mathcal{G} is a non-zero Radon measure λ , such that for all $x \in \mathcal{G}$,

$$\lambda(xE) = \lambda(E).$$

where E is a Borel subset of \mathcal{G} and $x \in \mathcal{G}$. Radon measures are measures that respect the topology of the underlying space. In case of locally compact spaces the Riesz representation theorem tells us that Radon measures are in bijection with linear functionals on the space of compactly supported continuous functions, given by $f \mapsto \int f d\lambda$, so the invariance translated to this setting means,

$$\int L_x f d\lambda = \int f d\lambda.$$

where $(L_x f)(y) = f(x^{-1}y)$. Haar measures are very useful, and they allow much of the analysis that was possible with Lebesgue measures on \mathbb{R}^n .

THEOREM 1.1. [2] *Every locally compact group has a left Haar measure unique up to scaling.*

If \mathcal{G} is non-commutative, the left Haar measures allows us to quantify the ‘amount of non-commutativity’. For each $x \in \mathcal{G}$, define, $\lambda_x(E) = \lambda(Ex)$. This is again a left Haar measure, and by uniqueness of Haar measures, there exists a number $\Delta(x) > 0$ such that,

$$\lambda_x = \Delta(x)\lambda.$$

The number $\Delta(x)$ is independent of the choice of λ by the uniqueness Haar measure. So, $x \mapsto \Delta(x)$ is a well defined function called the modular function. For any $x, y \in \mathcal{G}$, and $E \subset \mathcal{G}$, we have, $\lambda_{xy}(E) = \lambda(Exy) = \Delta(y)\Delta(x)\lambda(E)$, so,

$$xy \mapsto \Delta(x)\Delta(y),$$

or, Δ is a group homomorphism from the group \mathcal{G} to the multiplicative group of positive numbers denoted by \mathbb{R}_\times . If χ_E is the characteristic function on E , then we have, $\chi_E(xy) = \chi_{Ey^{-1}}(x)$. So,

$$\int \chi_E(xy) d\lambda(x) = \lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E) = \Delta(y^{-1}) \int \chi_E(x) d\lambda(x).$$

Since a general function $f \in L^1(\mathcal{G})$ can be approximated by simple functions, we have,

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda.$$

or equivalently, this can be abbreviated as,

$$d\lambda(xy) = \Delta(y) d\lambda(x). \quad (\text{right translation})$$

Since the map $y \mapsto R_y f$ is uniformly continuous, the map $y \mapsto \int R_y f d\lambda$ is a continuous map. So, Δ is continuous. So, the homomorphism Δ also respects the topological structure.

To each left Haar measure, we can associate a right Haar measure, $\rho(E) = \lambda(E^{-1})$. The modular function relates the two measures. If $f \in \mathcal{C}_c(\mathcal{G})$, then we have,

$$\int R_y f(x) \Delta(x^{-1}) d\lambda(x) = \Delta(y) \int f(xy) \Delta(xy^{-1}) d\lambda(x) = \int f(x) \Delta(x^{-1}) d\lambda(x).$$

In the last step we used [right translation](#). Thus the functional $f \mapsto \int f(x) \Delta(x^{-1}) d\lambda(x)$ is right invariant, and hence its associated Radon measure is a right Haar measure. Since ρ is a right Haar measure it must vary from the Radon measure associated to this functional by a constant, say κ . On symmetric neighborhoods, the left and right Haar measures are equal. If the constant κ is not 1, then by continuity of Δ , we can choose a small enough symmetric neighborhood of $1 \in \mathcal{G}$ such that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|\kappa - 1|$ on that neighborhood.

$$|\kappa - 1| \lambda(U) = |\kappa \rho(U) - \lambda(U)| = \left| \int_U [\Delta(x^{-1}) - 1] d\lambda(x) \right| \leq \frac{1}{2} |\kappa - 1| \lambda(U).$$

which can only happen if $\kappa = 1$. This gives us,

$$d\rho(x) = \Delta(x^{-1}) d\lambda(x)$$

This can be restated in a more convenient way as,

$$d\lambda(x^{-1}) = \Delta(x^{-1}) d\lambda(x). \quad (\text{inversion})$$

If $\Delta \equiv 1$, in such a case the left Haar measures do not measure any non-commutativity, such a locally compact group \mathcal{G} is called unimodular. Clearly, abelian groups are unimodular. For discrete groups, every element of the group has same measure, and hence whether we left translate or right translate, it would still have the same measure. So, discrete groups are also unimodular. Since Δ is a continuous homomorphism, it takes compact groups to compact subgroups of \mathbb{R}_\times , which can only be $\{1\}$. So, compact groups are unimodular, and if $K \subset \mathcal{G}$ is a compact subgroup, then $\Delta|_K \equiv 1$.

Let $[\mathcal{G}, \mathcal{G}]$ be the smallest closed subgroup of \mathcal{G} containing all elements of the form, $[x, y] = xyx^{-1}y^{-1}$, called the commutator subgroup of \mathcal{G} . It is a normal subgroups since, $z[x, y]z^{-1} = zxyx^{-1}y^{-1}z^{-1} = [zxz^{-1}, zyz^{-1}]$. Since Δ is a group homomorphism, we have,

$$\Delta([x, y]) = [\Delta(x), \Delta(y)]$$

since \mathbb{R}_\times is abelian, we have $[\Delta(x), \Delta(y)] = 1$. So, by the isomorphism theorem of groups, the group homomorphism $\Delta : \mathcal{G} \rightarrow \mathbb{R}_\times$ must factor through $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$. This implies that if $\mathcal{G}/[\mathcal{G}, \mathcal{G}]$ is compact, then \mathcal{G} is unimodular. This is intuitively obvious, the modular function was built to measure the non-commutativity and this just means that, $[\mathcal{G}, \mathcal{G}]$ does not contain any information about the non-commutativity contained in the group \mathcal{G} . Intuitively, the important thing to note about the modular function is that the left Haar measures can only measure non-commutativity of ‘large’ groups in some sense.

2 | CONVOLUTION & INVOLUTION OF FUNCTIONS

Given any two Radon measures μ, ν on \mathcal{G} , we can define a functional,

$$\mu * \nu : f \mapsto \iint f(xy) d\mu(x) d\nu(y).$$

for $f \in C_c(\mathcal{G})$. Clearly this is a linear map, and satisfies, $|(\mu * \nu)(f)| \leq \|f\|_{\sup} \|\mu\| \|\nu\|$, so it is a bounded linear functional, and hence defines a measure. This new measure is called the convolution of μ and ν , we may write,

$$\int f d(\mu * \nu) = \iint f(xy) d\mu(x) d\nu(y).$$

The order of the variables is important, the order of integration is not. If σ is some other measure, Fubini’s rule guarantees that convolution is an associative operation,

$$\begin{aligned} \int f d[\mu * (\nu * \sigma)] &= \iint f(xy) d\mu(x) d(\nu * \sigma)(y) \\ &= \iiint f(xyz) d\mu(x) d\nu(y) d\sigma(z) \\ &= \iint f(yz) d(\mu * \nu)(y) d\sigma(z) \\ &= \int f d[(\mu * \nu) * \sigma]. \end{aligned}$$

If \mathcal{G} is abelian, $f(xy) = f(yx)$ and by definition, the convolution will be commutative. Conversely, consider the delta measures, $\int f d(\delta_x * \delta_y) = \int \int f(uv) d\delta_x(u) d\delta_y(v) = f(xy) = \int f d\delta_{xy}$. So, if convolutions are commutative, we would have $\delta_{xy} = \delta_x * \delta_y = \delta_y * \delta_x = \delta_{yx}$. Or, equivalently $xy = yx$. So, \mathcal{G} is abelian if and only if convolutions commute.

Convolutions define a product structure on the Radon measures on \mathcal{G} . The estimate, $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ turns the collection of all Radon measures $\mathcal{M}(\mathcal{G})$ on \mathcal{G} into a Banach algebra, called the measure algebra on \mathcal{G} . The delta measure at identity, δ_1 determines the identity on this algebra.

Consider the operation on $\mathcal{M}(\mathcal{G})$ defined by,

$$\mu^*(E) = \overline{\mu(E^{-1})}.$$

If $\mu, \nu \in \mathcal{M}(\mathcal{G})$ then we have,

$$\begin{aligned} \int f d(\mu * \nu)^* &= \int f(x^{-1}) d\overline{(\mu * \nu)} = \iint f((xy)^{-1}) d\bar{\mu}(x) d\bar{\nu}(y) \\ &= \iint f(y^{-1}x^{-1}) d\bar{\mu}(x) d\bar{\nu}(y) = \iint f(yx) d\mu^*(x) d\nu^*(y) = \int f d(\nu^* * \mu^*). \end{aligned}$$

So, $(\mu * \nu)^* = \nu^* * \mu^*$. The operation $\mu \mapsto \mu^*$ is an involution operation.

Given a left Haar measure λ , each function $f \in L^1(\mathcal{G})$ can be identified with the measure $f(x)d\lambda(x) \in \mathcal{M}(\mathcal{G})$. The convolution operation on $\mathcal{M}(\mathcal{G})$ gives a convolution operation on $L^1(\mathcal{G})$ given by,

$$f * g(x) = \int f(y)g(y^{-1}x)d\lambda(y).$$

With some manipulation and using (right translation), this is the same as,

$$f * g(x) = \int f(xy^{-1})g(y)\Delta(y^{-1})d\lambda(y). \quad (\text{convolution})$$

The involution on $\mathcal{M}(\mathcal{G})$, restricted to $L^1(\mathcal{G})$ is defined by the relation,

$$f^*(x)d\lambda(x) = \overline{f(x^{-1})}d\lambda(x^{-1}).$$

Which after some manipulations, and using (inversion), gives,

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \quad (\text{involution})$$

$L^1(\mathcal{G})$ with the convolution product and involution is a Banach algebra, called the L^1 group algebra of \mathcal{G} . These structures also can be defined for $L^p(\mathcal{G})$, for $p \in [1, \infty)$. $L^p(\mathcal{G})$ consists of functions such that $|f|^p$ is integrable. In such a case we set,

$$\|f\|_p = \left(\int_{x \in \mathcal{G}} |f|^p d\lambda(x) \right)^{1/p}$$

3 | REPRESENTATIONS & GROUP ALGEBRA

The action of groups of interest in quantum theory are as unitary operators, so, the objects of interest to us are unitary representations of locally compact groups. The physically important topology on operator algebras is the weak operator topology. The weak topology and strong topology coincide on unitary operators, so either topology works fine for us. ¹

A unitary representation (\mathcal{H}_π, π) of a locally compact group \mathcal{G} is a strongly continuous group homomorphism

$$\pi : \mathcal{G} \rightarrow U(\mathcal{H}_\pi),$$

where \mathcal{H}_π is some Hilbert space, and $U(\mathcal{H}_\pi)$ is the group of unitary operators on \mathcal{H}_π . The dimension of the Hilbert space \mathcal{H}_π is called the degree of the representation π . When there is no confusion, the representation will just be denoted by π . If π is an injection of \mathcal{G} into $U(\mathcal{H}_\pi)$ it is called a faithful representation.

Let (\mathcal{H}_π, π) be a representation of \mathcal{G} , a subspace \mathcal{M} of \mathcal{H}_π is said to be invariant under π if $\pi(x)\mathcal{M} \subseteq \mathcal{M}$ for all $x \in \mathcal{G}$. If \mathcal{M} is closed and $P_\mathcal{M}$ is the orthogonal projection onto the closed subspace \mathcal{M} the invariance implies that,

$$P_\mathcal{M}\pi(x) = \pi(x)P_\mathcal{M},$$

for all $x \in \mathcal{G}$. We can define a new representation $(\mathcal{M}, \pi^\mathcal{M})$ by defining,

$$\pi^\mathcal{M}(x) = P_\mathcal{M}\pi(x)P_\mathcal{M},$$

¹Suppose the net of unitary operators $\{T_\alpha\}$ converge to T strongly, then for any $|\varphi\rangle \in \mathcal{H}$, $\|(T_\alpha - T)\varphi\|^2 = \|T_\alpha\varphi\|^2 - 2\text{Re}\langle T_\alpha\varphi|T\varphi\rangle + \|T\varphi\|^2 = 2\|\varphi\|^2 - 2\text{Re}\langle T_\alpha\varphi|T\varphi\rangle$ So, the $\{T_\alpha\}$ converges to T in the strong topology only if the last term converges to $2\|T\varphi\|^2$.

It is called a subrepresentation of (\mathcal{H}_π, π) . This procedure of going to subrepresentation gives a decomposition of π . If \mathcal{M} is invariant under π then so is \mathcal{M}^\perp . This gives us a second subrepresentation. Now the original Hilbert space \mathcal{H} can be written as a direct sum, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and each operator $\pi(x)$ then decomposes as a direct sum $\pi(x) = \pi^\mathcal{M}(x) \oplus \pi^{\mathcal{M}^\perp}(x)$. So the representation can be written as $(\mathcal{H}_\pi, \pi) = (\mathcal{M}, \pi^\mathcal{M}) \oplus (\mathcal{M}^\perp, \pi^{\mathcal{M}^\perp})$.

Given a family of representations $(\mathcal{H}_\alpha, \pi_\alpha)_{\alpha \in I}$ of \mathcal{G} the direct sum of the representations $\{\pi_\alpha\}$ is defined as follows,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha,$$

consisting of vectors of the form $\varphi = \{\varphi_\alpha\}_{\alpha \in I}$ such that $\lim_F [\sum_{\alpha \in F} \|\varphi_\alpha\|^2] < \infty$ where F is a finite subset of I . The purpose of this definition is so that the norm is definable nicely. This Hilbert space together with the representation map,

$$\pi = \bigoplus_{\alpha \in I} \pi_\alpha,$$

is called direct sum of representations $\{(\mathcal{H}_\alpha, \pi_\alpha)\}_{\alpha \in I}$, denoted by, $\sum_{\alpha \in I}^\oplus \{(\mathcal{H}_\alpha, \pi_\alpha)\}$. A representation is trivial if $\pi(x) = 1$ for every $x \in \mathcal{G}$. These are uninteresting representations. A representation can however have a trivial part.

A vector φ in a Hilbert space \mathcal{H}_π is called cyclic for \mathcal{A} if $\{\pi(x)|\varphi\rangle\}_{x \in \mathcal{G}}$ is dense in \mathcal{H}_π . A cyclic representation of \mathcal{G} is a triple $(\mathcal{H}, \pi, |\varphi\rangle)$ where (\mathcal{H}_π, π) is a representation of \mathcal{G} and φ is a cyclic for $\pi(\mathcal{G})$.

Let (\mathcal{H}, π) be a nondegenerate representation of \mathcal{A} . Take a maximal family of nonzero vectors $\{|\Omega_\alpha\rangle\}_{\alpha \in I}$ in \mathcal{H} such that,

$$\langle \pi(A)\Omega_\alpha | \pi(B)\Omega_\beta \rangle = 0,$$

for all $A, B \in \mathcal{A}$ and $\alpha \neq \beta$. Define, $\mathcal{H}_\alpha = \overline{\{\pi(A)|\Omega_\alpha\rangle\}_{A \in \mathcal{A}}}$. This is an invariant subspace of \mathcal{H} . Define $\pi_\alpha = P_{\mathcal{H}_\alpha} \pi P_{\mathcal{H}_\alpha}$ where $P_{\mathcal{H}_\alpha}$ is projection onto \mathcal{H}_α . Then by construction the \mathcal{H}_α are mutually orthogonal and hence the representation (\mathcal{H}, π) is of the form,

$$\mathcal{H} = \bigoplus_{\alpha \in I} \{(\mathcal{H}_\alpha, \pi_\alpha)\}$$

By Zorn's lemma, every non-degenerate representation can be written as a direct sum of a family of cyclic subrepresentations. If it has no closed invariant subspaces the representation (\mathcal{H}_π, π) of \mathcal{G} is called irreducible.

Given two representations π_i and π_j of \mathcal{G} , an intertwining operator for π_i and π_j is a bounded linear operator,

$$T : \mathcal{H}_{\pi_i} \rightarrow \mathcal{H}_{\pi_j}$$

such that for every $x \in \mathcal{G}$, $T\pi_i(x) = \pi_j(x)T$. The collection of all intertwiners between π_i and π_j is denoted by $\mathcal{C}(\pi_i, \pi_j)$. If $\mathcal{C}(\pi_i, \pi_j)$ contains a unitary operator, the two representations are called unitarily equivalent.

The collection $\mathcal{C}(\pi)$ of bounded operators on \mathcal{H}_π that commute with $\pi(x)$ for all $x \in \mathcal{G}$ is called the commutant of π is an algebra. Since product operation is weakly continuous, $\mathcal{C}(\pi)$ is weakly closed. Since $T^*\pi(x) = [\pi(x^{-1})T]^* = [T\pi(x^{-1})]^* = \pi(x)T^*$, it is also closed under taking adjoints. Thus $\mathcal{C}(\pi)$ is a von Neumann algebra.

If π is reducible, $\mathcal{C}(\pi)$ contains non-trivial projections. Conversely, if $\mathcal{C}(\pi)$ contains bounded operators that are not multiples of identity, say T , then we can split it into the

canonical sum of two self-adjoint operators. Since these are linear combinations of the operators T and T^* which belong to $\mathcal{C}(\pi)$, they also belong to $\mathcal{C}(\pi)$. If an operator commutes with the self-adjoint operator, it also commutes with its spectral projections. So, $\pi(x)$ must also commute with the spectral projections of some self-adjoint operator. So, the subspace corresponding to the projection operator is an invariant subspace.

Let π_i and π_j be two unitary representations of \mathcal{G} . Then for every intertwining operator

$$T : \mathcal{H}_{\pi_i} \rightarrow \mathcal{H}_{\pi_j}$$

for π_i and π_j , the adjoint of the operator $T^* : \mathcal{H}_{\pi_j} \rightarrow \mathcal{H}_{\pi_i}$ is an intertwining operator for π_j and π_i because,

$$T^* \pi_j(x) = [\pi_j(x^{-1})T]^* = [T\pi_i(x^{-1})]^* = \pi_i(x)T^*.$$

So, $T^*T \in \mathcal{C}(\pi_i)$ and $TT^* \in \mathcal{C}(\pi_j)$. If π_i and π_j are two irreducible unitary representations of \mathcal{G} , then both of these should be a multiple of identity, that is, $T^*T = \kappa I$ or equivalently, $\kappa^{-1/2}T$ must be a unitary operator or T is zero. So, two irreducible representations π_i and π_j are equivalent if $\mathcal{C}(\pi_i, \pi_j)$ is one dimensional, or they are unitarily inequivalent. These two facts about irreducibility of representations is called Schur is lemma, which we will now state below.

THEOREM 3.1. (SCHUR'S LEMMA) *If π_i and π_j are irreducible representations of \mathcal{G} then either they are equivalent representations, or they have no intertwiners.*

If \mathcal{G} is abelian, then for every representation π of \mathcal{G} , the operators $\pi(x)$ commutes with all operators in $\pi(\mathcal{G})$, so $\pi(x) \in \mathcal{C}(\pi)$ for every $x \in \mathcal{G}$. If π is irreducible, then we should have $\pi(x) = \kappa_x I$, so every irreducible representation of an locally compact abelian group is one dimensional. Although this looks very surprising, it is because space of operators on a vector space contain a lot more information than groups. This is an important corollary of Schur is lemma.

COROLLARY 3.2. *If \mathcal{G} is abelian, and π is irreducible, then $\mathcal{H}_\pi = \mathbb{C}$.*

The unitary representations of the group \mathcal{G} are closely related to the $*$ -representations of the algebra of integrable functions $L^1(\mathcal{G})$. Once we have fixed a left Haar measure λ , let $f \in L^1(\mathcal{G})$, then for any unitary representation π of the group \mathcal{G} , we can define an operator $\pi(f)$ on the Hilbert space \mathcal{H}_π by,

$$\langle \pi(f)\varphi | \varkappa \rangle = \int f(x) \langle \pi(x)\varphi | \varkappa \rangle d\lambda(x).$$

Or, more compactly,

$$\pi(f) = \int f(x) \pi(x) d\lambda(x),$$

in the sense of strong operator topology. This satisfies,

$$|\langle \pi(f)\varphi | \varkappa \rangle| \leq \|f\|_1 \|\varphi\| \|\varkappa\|.$$

So, $\pi(f)$ is a bounded linear operator on \mathcal{H}_π with the norm $\|\pi(f)\| \leq \|f\|_1$. For any two $f, g \in L^1(\mathcal{G})$,

$$\begin{aligned}\pi(f * g) &= \iint f(y)g(y^{-1}x)\pi(x)d\lambda(y)d\lambda(x) = \iint f(y)g(x)\pi(yx)d\lambda(x)d\lambda(y) \\ &= \iint f(y)g(x)\pi(y)\pi(x)d\lambda(x)d\lambda(y) = \pi(f)\pi(g).\end{aligned}$$

So, we obtain an algebra homomorphism. Moreover,

$$\pi(f^*) = \int \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)d\lambda(x) = \int \overline{f(x)}\pi(x^{-1})d\lambda(x) = \int [f(x)\pi(x)]^*d\lambda(x) = \pi(f)^*,$$

hence, π is a $*$ -homomorphism. The left translation by x of the function f is given by,

$$\begin{aligned}\pi(L_x f) &= \int f(x^{-1}y)\pi(y)d\lambda(y) = \int f(y)\pi(xy)d\lambda(y) \\ &= \int f(y)\pi(x)\pi(y)d\lambda(y) = \pi(x)\pi(f).\end{aligned}$$

That is,

$$\pi(L_x f) = \pi(x)\pi(f)$$

Similarly, for the right translation,

$$\begin{aligned}\Delta(x^{-1})\pi(R_{x^{-1}} f) &= \Delta(x^{-1}) \int f(yx^{-1})\pi(y)d\lambda(y) = \int f(y)\pi(xy)d\lambda(y) \\ &= \int f(y)\pi(y)\pi(x)d\lambda(y) = \pi(f)\pi(x).\end{aligned}$$

That is,

$$\Delta(x^{-1})\pi(R_{x^{-1}} f) = \pi(f)\pi(x).$$

This will be reversed if we had chosen a right Haar measure to define integration. So, the map $f \mapsto \pi(f)$ is a $*$ -homomorphism from $L^1(\mathcal{G})$ to $\mathcal{B}(\mathcal{H}_\pi)$. Since π is strongly continuous, we can find a neighborhood V of 1, such that for all $x \in V$,

$$\|\pi(x)\varphi - \varphi\| < \|\varphi\|.$$

Now consider the function, $f = |V|^{-1}\chi_V$, we get,

$$\|\pi(f)\varphi - \varphi\| = |V|^{-1} \left\| \int_V [\pi(x)\varphi - \varphi]d\lambda(x) \right\| < \|\varphi\|$$

So, $\|\pi(x)\varphi\| \neq 0$, or $\pi(x)\varphi \neq 0$. Hence,

$$\pi : f \mapsto \pi(f)$$

is a non-degenerate $*$ -representation of $L^1(\mathcal{G})$ on \mathcal{H}_π . So, a representation of the locally compact group \mathcal{G} gives rise to a representation of integrable functions on the same Hilbert space. Conversely, given a non-degenerate $*$ -representation of $L^1(\mathcal{G})$ on a Hilbert space \mathcal{H}_π gives rise to a unique unitary representation of \mathcal{G} . The idea is to use the functions f approaching the delta function at x , to get an operator for $\pi(x)$. See §3.2, [2].

THEOREM 3.3. *The von Neumann algebras generated by $\pi(\mathcal{G})$ and $\pi(L^1(\mathcal{G}))$ are identical.*

IDEA OF PROOF

The idea is to approximate functions in $L^1(\mathcal{G})$ by compactly supported functions, and then approximate the representatives of these compactly supported functions using the representatives of the group elements.

If $g \in \mathcal{C}_c(\mathcal{G})$, we can divide up the support of g into a finite partition, say $E = \{E_i\}$. Now we can approximate the function g , using the value g at some point $x_i \in E_i$, by,

$$\Sigma_E = \sum_i g(x_i) \pi(x_i) |E_i|$$

Given any $\epsilon > 0$, by compactness of the support of g , and continuity of the map $x \mapsto g(x)\pi(x)\varphi$, for each φ , there exists a partition $E = \{E_i\}$ of the support of g such that, $\|g(x)\pi(x)\varphi - g(y)\pi(y)\varphi\| < \epsilon$, whenever $x, y \in E_j$ for some j . Thus,

$$\|\Sigma_E \varphi - \pi(g)\varphi\| < \epsilon |\cup_i E_i|.$$

Thus every neighborhood of $\pi(g)$ with respect to the strong topology, contains sums Σ_E .

Now every function $f \in L^1(\mathcal{G})$ is the L^1 limit of functions in $\mathcal{C}_c(\mathcal{G})$. So, by continuity of π , $\pi(f)$ is norm limit of operators in $\pi(\mathcal{C}_c(\mathcal{G}))$. So, the von Neumann algebra generated by $\pi(L^1(\mathcal{G}))$ is contained in the von Neumann algebra generated by $\pi(\mathcal{G})$. Conversely, each $\pi(x)$ is the strong limit of $\pi(L_x \phi_U)$, as $U \rightarrow \{1\}$. Where ϕ_U is an ‘approximate identity’. See [2] §2.5 for approximate identities, and §3.2 for a complete proof. \square

3.1 | FUNCTIONS OF POSITIVE TYPE

In the C^* algebra case representations and states are closely related to each other, we need to find analogues of states for group algebras. A function of positive type is analogous to states on operator algebras. We can do similar constructions with group algebras that we did with C^* -algebras.

Let $L^\infty(\mathcal{G})$ denote the collection of all bounded measurable functions on \mathcal{G} . With the sup-norm, $\|f\|_\infty := \sup_{x \in \mathcal{G}} |f(x)|$, $L^\infty(\mathcal{G})$ is a Banach space.

Every function $\omega \in L^\infty$ defines a linear functional on $L^1(\mathcal{G})$ defined by,

$$f \mapsto \hat{\omega}(f) := \int_{x \in \mathcal{G}} f(x) \omega(x) d\lambda(x)$$

This is bounded because,

$$|\hat{\omega}(f)| = \left| \int_{x \in \mathcal{G}} f(x) \omega(x) d\lambda(x) \right| \leq \|\omega\|_\infty \left| \int_{x \in \mathcal{G}} f(x) d\lambda(x) \right| = \|\omega\|_\infty \|f\|_1.$$

So, $\hat{\omega}$ is a bounded linear functional on the group algebra $L^1(\mathcal{G})$. A function $\omega \in L^\infty(\mathcal{G})$ is said to be a function of positive type if for any function $f \in L^1(\mathcal{G})$,

$$\hat{\omega}(f^* * f) = \int (f^* * f) \omega \geq 0.$$

Expanding the integral, by [convolution](#) and [involution](#), we have,

$$\begin{aligned} \int (f^* * f) \omega &= \iint \Delta(y^{-1}) \overline{f(y^{-1})} f(y^{-1}x) \omega(x) d\lambda(y) d\lambda(x) \\ &= \iint \overline{f(y)} f(yx) \omega(x) d\lambda(x) d\lambda(y) = \iint f(x) \overline{f(y)} \omega(y^{-1}x) d\lambda(x) d\lambda(y). \end{aligned}$$

We can restate that a function $\omega \in L^\infty(\mathcal{G})$ is of positive type if,

$$\omega(f^* * f) = \iint f(x) \overline{f(y)} \omega(y^{-1}x) d\lambda(x) d\lambda(y) \geq 0 \quad (\text{positive type})$$

Let ω be a function of positive type, consider the function complex conjugate, $\bar{\omega}$,

$$\int (f^* * f) \bar{\omega} = \iint \overline{f(y)} f(yx) \overline{\omega(x)} d\lambda(y) d\lambda(x) = \int \overline{[(\bar{f})^* * \bar{f}]} \omega \geq 0$$

So, if ω is a function of positive type, then so is $\bar{\omega}$. The set of all continuous functions of positive type on \mathcal{G} is denoted by $\mathcal{S}(\mathcal{G})$.

Given a representation π of \mathcal{G} , every vector $\varphi \in \mathcal{H}_\pi$ defines a function,

$$\Phi(x) = \langle \pi(x)\varphi | \varphi \rangle.$$

By the continuity requirements of the representation, this is a continuous function on \mathcal{G} . $\Phi(y^{-1}x) = \langle \pi(y^{-1}x)\varphi | \varphi \rangle = \langle \pi(y^{-1})\pi(x)\varphi | \varphi \rangle = \langle \pi(x)\varphi | \pi(y)\varphi \rangle$. So for every $f \in L^1(\mathcal{G})$,

$$\begin{aligned} \hat{\Phi}(f^* * f) &= \iint f(x) \overline{f(y)} \Phi(y^{-1}x) d\lambda(x) d\lambda(y) \\ &= \iint \langle f(x)\pi(x)\varphi | f(y)\pi(y)\varphi \rangle d\lambda(x) d\lambda(y) = \|\pi(f)\varphi\|^2 \geq 0 \end{aligned}$$

So, $\Phi \in \mathcal{S}(\mathcal{G})$. Consider the space of all square integrable functions on \mathcal{G} , consisting of functions g such that $\int |g|^2 d\lambda < \infty$. This together with the inner product,

$$\langle h, g \rangle = \int_{x \in \mathcal{G}} \overline{h(x)} g(x) d\lambda(x).$$

$L^2(\mathcal{G})$ is a Hilbert space. On this space, we have a representation of the group \mathcal{G} , given by,

$$[\pi_L(x)g](y) = L_x g(y) = g(x^{-1}y).$$

The left invariance of the left Haar measure ensures that this is a unitary representation. It is called the left regular representation of \mathcal{G} on $L^2(\mathcal{G})$. The right regular representation is similarly defined. Let $g \in L^2(\mathcal{G})$, let $\tilde{g}(x) = \overline{g(x^{-1})}$, then we have,

$$\langle \pi_L(x)g | g \rangle = \int g(x^{-1}y) \overline{g(y)} d\lambda(y) = \overline{g * \tilde{g}(x)}.$$

Since every function of the type $\langle \pi(x)\varphi | \varphi \rangle$ arising from representations are functions of positive type, the above defined function is a function of positive type. Since the conjugate of functions of positive type are also functions of positive type, we have,

$$g * \tilde{g} \in \mathcal{S}(\mathcal{G}).$$

In particular, $\mathcal{S}(\mathcal{G})$ is non-empty. So, for each $g \in L^2(\mathcal{G})$, $g * \tilde{g}$ defines a function of positive type. So, functions of positive type do indeed exist.

3.1.1 | CONSTRUCTION OF REPRESENTATIONS

Every unitary representations give rise to functions of positive type. The converse is also true, that is every function of positive type arises from a unitary representation. The key ingredient to proving this is to use functions of positive type to construct Hilbert spaces, much like GNS construction in the case of C^* -algebras.

The goal is to define an inner product using the function of positive type on $L^1(\mathcal{G})$ and consider the completion, similar to how it is done in GNS construction. Given a function ω of positive type, define a hermitian form on $L^1(\mathcal{G})$ by,

$$\langle f, g \rangle_\omega = \int (g^* * f)\omega = \iint f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y).$$

By Fubini is theorem this satisfies,

$$|\langle f, g \rangle_\omega| \leq \|\omega\|_\infty \|f\|_1 \|g\|_1. \quad (\text{bounded})$$

This is positive semi-definite since ω is of **positive type**. So, \langle, \rangle_ω defines a positive definite Hermitian form on $L^1(\mathcal{G})$. So, this satisfies the Schwarz inequality,

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle_\omega \langle g, g \rangle_\omega.$$

So, we can consider the ideal $\mathcal{J}_\omega = \{f \mid \langle f, f \rangle_\omega = 0\}$. By the above inequality, $f \in \mathcal{J}_\omega$ if and only if $\langle f, g \rangle_\omega = 0$ for all $g \in L^1(\mathcal{G})$. So, \langle, \rangle_ω defines an inner product on the quotient space $L^1(\mathcal{G})/\mathcal{J}_\omega$. If the image of $f \in L^1(\mathcal{G})$ in \mathcal{H}_ω by $[f]$, the inner product is given by,

$$\langle [f], [g] \rangle = \langle f, g \rangle_\omega$$

Denote the completion of this inner product space by \mathcal{H}_ω . We have a natural action of the group \mathcal{G} on the Hilbert space \mathcal{H}_ω given by the left translation.

$$\pi_\omega(x)[f] = [L_x f].$$

This action is a unitary operator because,

$$\begin{aligned} \langle L_x f, L_x g \rangle_\omega &= \iint f(x^{-1}y)\overline{g(x^{-1}z)}\omega(z^{-1}y)d\lambda(y)d\lambda(z) \\ &= \iint f(y)\overline{g(z)}\omega((xz)^{-1}(xy))d\lambda(y)d\lambda(z) = \langle f, g \rangle_\omega. \end{aligned}$$

In the last step, we used the left invariance of the Haar measure λ . This also tells us that the action keeps \mathcal{J}_ω invariant, and hence L_x gives rise to a unitary operator on \mathcal{H}_ω . We also have an action of $L^1(\mathcal{G})$ on \mathcal{H}_ω given by,

$$\pi_\omega(g)[f] = [g * f].$$

THEOREM 3.4.

$$\exists \Omega \in \mathcal{H}_\omega \ni \pi_\omega(f)|\Omega\rangle = [f], \omega(x) = \langle \pi_\omega(x)\Omega, \Omega \rangle.$$

almost everywhere.

PROOF

We want to think of ω as a limit of functions $L^1(\mathcal{G})$ functions. The idea is that if $\psi \in L^1(\mathcal{G})$ and $\omega \in L^\infty(\mathcal{G})$ then $\psi * \omega \in L^1(\mathcal{G})^2$ and consider the product of ω with an approximate identity to obtain the required sequence. $\hat{f}_i(f) \rightarrow \hat{\omega}(f)$ because $\psi_{U_i} * \omega \rightarrow \omega$ in appropriate topologies. Now all we are left with is some routine checks, which can be found in §3.3 [2]. \square

The above theorem says that the information about the action of ω is determined by its infinitesimal behavior around the identity and $\Omega = [\{\psi_{U_i}\}]$. So, every function of positive type agrees with continuous functions locally almost everywhere. So, there is a correspondence between cyclic representations and continuous functions of positive type.

Given a function of positive type $\omega \in \mathcal{S}(\mathcal{G})$, there exists some representation π_ω such that

$$|\omega(x)| = |\langle \pi_\omega(x)\Omega, \Omega \rangle| \leq \|\Omega\|^2 = \omega(1)$$

Similarly,

$$\omega(x^{-1}) = \langle \pi_\omega(x^{-1})\Omega, \Omega \rangle = \langle \Omega, \pi_\omega(x)\Omega \rangle = \overline{\omega(x)}$$

Suppose π_i and π_j are two cyclic representations of \mathcal{G} with cyclic vectors Ω_i and Ω_j then it follows that $T\pi_i(x)\Omega_i = \pi_j(x)\Omega_j$ extends to a linear isometry on the spans, and by continuity to a unitary map between the Hilbert spaces \mathcal{H}_{π_i} and \mathcal{H}_{π_j} . So, $\pi_i(x)T = T\pi_j(x)$ for all $x \in \mathcal{G}$. So, any two cyclic representations are unitarily equivalent. So, every cyclic representation is unitarily equivalent to the one we constructed above.

3.1.2 | STRUCTURE OF $\mathcal{S}(\mathcal{G})$

The set $\mathcal{S}(\mathcal{G})$ of all continuous functions of positive type, then by linearity of integration, we have for every finite set $\{\omega_i\}_{i \in \mathcal{I}} \subset \mathcal{S}(\mathcal{G})$ and $\lambda_i \in [0, 1]$ with $\sum_i \lambda_i = 1$, we have,

$$\int (f^* * f)(\sum_i \lambda_i \omega_i) = \sum_{i \in \mathcal{I}} \lambda_i \int (f^* * f)\omega_i \geq 0.$$

for all $f \in L^1(\mathcal{G})$. So, the set $\mathcal{S}(\mathcal{G})$ is a convex set, and since scaling of functions of positive type by positive numbers also gives a function of positive type it is a convex cone. Some special subsets of $\mathcal{S}(\mathcal{G})$ can be singled out,

$$\mathcal{S}_1 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty = 1\}, \quad \mathcal{S}_0 = \{\omega \in \mathcal{S}(\mathcal{G}) \mid \omega(1) = \|\omega\|_\infty \leq 1\}.$$

Let $\mathcal{E}(\mathcal{S}_i)$ be the extreme points of \mathcal{S}_i .

THEOREM 3.5. *Let $\omega \in \mathcal{S}(\mathcal{G})$, then π_ω is irreducible iff $\omega \in \mathcal{E}(\mathcal{S}_1)$.*

PROOF

\Leftarrow Suppose π_ω is a reducible representation of \mathcal{G} , say, $\mathcal{H}_\omega = \mathcal{M} \oplus \mathcal{M}^\perp$ where \mathcal{M} is invariant under π_ω . Let Ω be a cyclic vector for the representation π_ω , then Ω cannot belong to either \mathcal{M} or \mathcal{M}^\perp , because otherwise $\pi_\omega(\mathcal{G})\Omega$ can only belong to either \mathcal{M} or \mathcal{M}^\perp . But, if Ω is a cyclic vector, then $\pi_\omega(\mathcal{G})\Omega$ must be dense in \mathcal{H}_ω .

So, $\Omega = \alpha\varphi + \beta\kappa$ where φ and κ are unit vectors in \mathcal{M} and \mathcal{M}^\perp respectively. Let $\omega_\varphi = \langle \pi_\omega\varphi|\varphi \rangle$ and $\omega_\kappa = \langle \pi_\omega\kappa|\kappa \rangle$. Then $\omega_\varphi, \omega_\kappa \in \mathcal{S}(\mathcal{G})$.

$$\begin{aligned} \omega_\varphi(x) - \omega_\kappa(x) &= \langle \pi_\omega\varphi|\varphi \rangle - \langle \pi_\omega\kappa|\kappa \rangle \\ &= \langle \pi_\omega(x)\Omega|\alpha^{-1}\varphi - \beta^{-1}\kappa \rangle. \end{aligned}$$

²this is where almost everywhere is used.

this is due to some manipulation using $\Omega = \alpha\varphi + \beta\chi$. Since Ω is cyclic, and $\alpha^{-1}\varphi - \beta^{-1}\chi$ is non-zero, there must exist some $x \in \mathcal{G}$ such that $\langle \pi_\omega(x)\Omega | \alpha^{-1}\varphi - \beta^{-1}\chi \rangle$ is non-zero. So, in particular for some $x \in \mathcal{G}$,

$$\omega_\varphi(x) - \omega_\chi(x) \neq 0$$

or that $\omega_\varphi \neq \omega_\chi$ as continuous functions on \mathcal{G} . Since $\omega_\varphi(1) = \omega_\chi(1) = 1$ they are linearly independent, because if not, we would have $\omega_\varphi = \delta\omega_\chi$, in which case we should have $\omega_\varphi(1) = \delta\omega_\chi(1)$.

$$\omega(x) = \langle \pi_\omega(x)\Omega | \Omega \rangle = \alpha^2 \langle \pi_\omega\varphi | \varphi \rangle + \beta^2 \langle \pi_\omega\chi | \chi \rangle = \alpha^2\omega_\varphi(x) + \beta^2\omega_\chi(x).$$

So, ω is not an extreme point in $\mathcal{S}(\mathcal{G})$.

\Rightarrow Suppose π_ω is irreducible, and suppose ω is not an extreme point of $\mathcal{S}(\mathcal{G})$ then there exist ω_φ and ω_χ in $\mathcal{S}(\mathcal{G})$ such that $\omega = \omega_\varphi + \omega_\chi$. For any $f, g \in L^1(\mathcal{G})$, we have,

$$\langle f, f \rangle_{\omega_\varphi} = \langle f, f \rangle_\omega - \langle f, f \rangle_{\omega_\chi} \leq \langle f, f \rangle_\omega.$$

So,

$$|\langle f, g \rangle_{\omega_\varphi}| \leq \langle f, f \rangle_{\omega_\varphi}^{1/2} \langle g, g \rangle_{\omega_\varphi}^{1/2} \leq \langle f, f \rangle_\omega^{1/2} \langle g, g \rangle_\omega^{1/2}.$$

This gives us a bounded Hermitian form on \mathcal{H}_ω , $(f, g) \rightarrow \langle f, g \rangle_{\omega_\varphi}$. So by Riesz lemma, there exists a bounded self-adjoint operator T on \mathcal{H}_ω such that,

$$\langle f, g \rangle_{\omega_\varphi} = \langle T[f], [g] \rangle_\omega.$$

By the left action of \mathcal{G} on $L^1(\mathcal{G})$, we have,

$$\begin{aligned} \langle T\pi_\omega(x)[f], [g] \rangle_\omega &= \langle T[L_x f], [g] \rangle_\omega = \langle L_x f, g \rangle_{\omega_\varphi} = \langle f, L_{x^{-1}}g \rangle_{\omega_\varphi} \\ &= \langle T[f], \pi(x^{-1})[g] \rangle_{\omega_\varphi} = \langle \pi_\omega(x)T[f], [g] \rangle_\omega. \end{aligned}$$

So, $T \in \mathcal{C}(\pi_\omega)$. By Schur is lemma, this means that $T = c\mathbb{I}$ or that $\langle f, g \rangle_{\omega_\varphi} = c\langle f, g \rangle_\omega$. Since,

$$\langle f, g \rangle_\omega = \int (g^* * f)\omega = \iint f(x)\overline{g(y)}\omega(y^{-1}x)d\lambda(x)d\lambda(y),$$

we must have, $\omega_\varphi = c\omega$. or, ω is an extreme point of $\mathcal{S}(\mathcal{G})$. \square

If ω_i is a net of functions of positive types that converge weakly to ω , then the weak* limit ω is also a function of positive type.³ So, \mathcal{S}_0 is a weak* closed unit ball in $L^\infty(\mathcal{G})$. Now, by Alaoglu is theorem, \mathcal{S}_0 is weak* compact and by Krein-Milman theorem it is the weak* closure of its extreme points.

The constant zero-function, which maps every element of the group to zero is a function of positive type by definition. Since $\|\omega\|_\infty = \omega(1)$, we have that $\|0\|_\infty = \omega(1) = 0$. This implies that the constant zero-function cannot be a mixture of other functions of positive type. Suppose ω be an extreme point in \mathcal{S}_1 , and suppose it's a mixture of two other functions of positive type, that is, $\omega = \mu_i\omega_i + \mu_j\omega_j$ where $\mu_i + \mu_j = 1$. Then $\omega(1) = \mu_i\omega_i(1) + \mu_j\omega_j(1)$, this can only happen if $\omega_i(1) = \omega_j(1) = 1$. Which means that if $\omega \in \mathcal{S}_1$, then it can only be a mixture of other ω_i 's in \mathcal{S}_1 . Let $\omega \in \mathcal{S}(\mathcal{G})$, then we can construct the state $\omega/\omega(1) \in \mathcal{S}_1$. So, we can always write ω as a mixture,

$$\omega = (1 - \omega(1))0 + (\omega(1))\omega/\omega(1)$$

³recall that weak* limit implies that $\hat{\omega}_i(f) \rightarrow \hat{\omega}(f)$ for all f .

So, we have,

$$\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}.$$

THEOREM 3.6. *The convex hull of $\mathcal{E}(\mathcal{S}_1)$ is weak* dense in \mathcal{S}_1 .*

PROOF

We have to show that every function of positive type, ω of norm 1, is a weak* limit of mixtures of functions in $\mathcal{E}(\mathcal{S}_1)$. Note that by Alaoglu's theorem $\mathcal{S}(\mathcal{G})$ is a compact set in the weak* topology, in particular it's weak* closed. So, there exists a net of functions of positive type $\{\omega_i\} \subset \mathcal{S}(\mathcal{G})$ such that $\omega_i \rightarrow \omega$ weakly. We must ensure that if $\omega \in \mathcal{S}_1$ then these ω_i 's can themselves be described as mixtures $\omega_i = \sum_j \mu_j \psi_j$ such that each $\psi_j \in \mathcal{S}_1$.

Since $\mathcal{E}(\mathcal{S}_0) = \mathcal{E}(\mathcal{S}_1) \cup \{0\}$, we can write each ω_i as $\mu_1 \psi_1 + \cdots + \mu_{n_i} \psi_{n_i} + \mu_{n_i+1} 0$ where each $\psi_j \in \mathcal{E}(\mathcal{S}_1)$. We can now remove the 0 component of the mixture. We can consider the new net,

$$\omega'_i = \frac{1}{\omega_i(1)} \sum_{j=1}^{n_i} \mu_j \psi_j$$

Each of these $\omega'_i \in \mathcal{S}_1$, by construction, and hence ω'_i is in the convex hull of $\mathcal{E}(\mathcal{S}_1)$ and $\omega = \lim_i \omega'_i$. \square

TOPOLOGY OF COMPACT CONVERGENCE

We want to define a notion of nearness on the functions of positive type, that corresponds to how they act on the functions. That's to say, a function ω is nearer to \varkappa if for all f , the value of $\omega(f)$ is near to $\varkappa(f)$, this is the idea of uniform convergence. This is however a very strict requirement. We can relax this a bit more and require that ω and \varkappa are close to each other if in some neighborhood they are close to each other. This is the idea of the topology of compact convergence. The neighborhoods of a function ω consist of sets,

$$N(\omega, \epsilon, K) = \{\varkappa \mid |\varkappa(x) - \omega(x)| < \epsilon, \quad \forall x \in K\}.$$

This implies that if f has a compact support, then $\omega(f)$ and $\varkappa(f)$ will be near each other. Both uniform topology and the topology of compact convergence coincide on compact sets.

3.1.3 | GELFAND-RAIKOV THEOREM

THEOREM 3.7. (GELFAND-RAIKOV)

EXISTENCE OF HAAR MEASURES

Existence of Haar measures on locally compact groups is a very important tool for analysis of locally compact groups. We need to construct a functional on the space of compactly supported continuous functions that is translation invariant. All that we have to work with is the set of continuous functions on the locally compact group. Compactly supported functions are well behaved, so, we can try to exploit their nice properties.

Let f, g be two compactly supported positive functions. Note that the support of each of these functions can be covered by finitely many open sets. We can relate the two functions using this. We can dominate the function f by a finite linear combination of scaled translations of g . Note that this is possible because f attains a maximum, and g is a non-zero positive function. So, there exists some $x \in \mathcal{G}$ such that $g(x) > 0$, and we can find a number λ such that $\lambda g(x) > \|f\|_{\text{sup}}$. By continuity of translation we can find a neighborhood around x where this holds. We can now transfer via translation L_y all the problematic parts to this neighborhood. Since the support of f is compact, we can cover it with finitely many translations of this neighborhood, hence we can make sense of a sum. So, there exists a finite collection of real numbers λ_i and translations L_{x_i} such that,

$$f \leq \sum_i \lambda_i (L_{x_i}(g)).$$

Consider the set of all finite collections of numbers $\{\lambda_i\}$ such that $f \leq \sum_{i=1}^n \lambda_i (L_{x_i}(g))$. This is nonempty due to the above reason. Denote this set by $\mathcal{D}(f : g)$. The intuitive meaning here is that $\sum_i \lambda_i (L_{x_i}g)$ is an approximation of f in terms of translations of the function g . Using this we can define the ‘best’ approximation based on how little scaling we have to do with the translations of g . This defines some sort of index of the function f with respect to g .

Since each element of this set is a finite collection of real numbers, we can add them up. So, we have a ‘sum’ function on $\mathcal{D}(f : g)$,

$$\begin{aligned} \sum : \mathcal{D}(f : g) &\rightarrow \mathbb{R} \\ \{\lambda_i\} &\mapsto \sum_i \lambda_i. \end{aligned}$$

Using this function, we can define a new object,

$$(f : g) := \inf_{\{\lambda_i\} \in \mathcal{D}(f:g)} \left(\sum_i \lambda_i \right).$$

We will call this the ‘approximation index’ of the function f with respect to g . This approximation index allows us to reduce the problem of constructing a special linear functional on the space of compactly supported continuous functions to doing some stuff with some ‘reference function’. This will be made precise below. Note that this is where the ‘locally compact’ part has been important. The Riesz representation tells us that the construction of Radon measures is equivalent to constructing some special linear functionals on compactly supported continuous functions. Then the compactness of the supports of the functions ensure that the sets $\mathcal{D}(f : g)$ are non-empty for each compactly supported functions f and g .

Note that the translation L_y is a linear map on the space of continuous functions, so we have,

$$L_{y^{-1}} \left(\sum_i \lambda_i (L_{x_i}g) \right) = \sum_i \lambda_i (L_{y^{-1}x_i}g).$$

Since the approximation index is an infimum, the approximation index of any translation L_y of the function f with respect to g will be the same. So, we have,

$$(f : g) = (L_y f : g). \quad (\text{invariance})$$

Scaling the functions f by λ just scales the sets $\mathcal{D}(f : g)$, so we have,

$$(\lambda f : g) = \lambda(f : g). \quad (\text{scaling})$$

If $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(h : g)$, then the disjoint union, $\{\lambda_i\} \coprod \{\mu_j\} \in \mathcal{D}(f + h : g)$. So, we have,

$$(f + h : g) \leq (f : g) + (h : g). \quad (\text{subadditivity})$$

Note that if this was additive, we would be done, we would have constructed a linear functional that is invariant under left translation. It is however possible choose g such that the inequality gets closer to an equality. The key lies in how well we can approximate the given function f , in terms of linear combinations of translations of g . So, the problem is now choosing an appropriate g . We will need some further properties of the functionals $(f : g)$ to do this.

If $f \leq h$ then any upper approximation of h by translations of g is also an upper approximation of f by translations of g . Hence we have,

$$f \leq h \Rightarrow (f : g) \leq (h : g). \quad (\text{monotonicity})$$

Let f attain a maximum at x , that is $f(x) = \|f\|_{\sup}$ and g attain a maximum at y , that is, $g(y) = \|g\|_{\sup}$. Then the translation $L_{yx^{-1}}(f)$ attains a maximum at y , by $(f : g) = (L_{yx^{-1}}f : g)$, we obtain,

$$(f : g) \geq \|f\|_{\sup} / \|g\|_{\sup}. \quad (\text{boundedness})$$

Suppose $f \leq \sum_i \lambda_i (L_{x_i} g)$ and $g \leq \sum_j \mu_j (L_{y_j} h)$, then by linearity of translations, we have, $f \leq \sum_{i,j} \lambda_i \mu_j (L_{x_i y_j} h)$. That is to say, $\{\lambda_i \mu_j\} \in \mathcal{D}(f : h)$ whenever $\{\lambda_i\} \in \mathcal{D}(f : g)$ and $\{\mu_j\} \in \mathcal{D}(g : h)$. So, we have,

$$(f : h) \leq (f : g)(g : h). \quad (\text{product})$$

Now we choose a ‘reference function’ F , and define a normalized index,

$$I_g(f) := (f : g) / (F : g).$$

This gives us a functional $I_g : C_c^+(\mathcal{G}) \rightarrow \mathbb{R}$, that is left invariant, subadditive and monotone. By, $(F : g) \leq (F : f)(f : g)$ and $(f : g) \leq (f : F)(F : g)$ we have,

$$f_F \equiv (F : f)^{-1} \leq I_g(f) \leq (f : F) \equiv f^F \quad (\text{rescaling})$$

Now the problem is to choose an appropriate g such that (subadditivity) becomes an equality. In order to choose such a g , we have to understand what makes the approximations $(f : g)$ not ‘close’ to the function f . The function is determined by its value at each point. So, given a number, we can scale it appropriately at each point to get f . What we are trying to do is to approximate the function f , by scaling a bunch of functions on some open sets. So, smaller open sets means better approximation. So, to make the approximation closer to the actual value of f , we could use g that has smaller support.

Let f_i, f_j be two compactly supported continuous functions. Let g be a positive compactly supported continuous function such that $g \equiv 1$ on the support of the functions $(f_i + f_j)$. Let $\delta > 0$ be a number (will be specified a bit later). Then we can consider the function,

$$h = f_i + f_j + \delta \cdot g$$

Using this we can construct new functions,

$$h_i = f_i/h, \quad h_j = f_j/h.$$

This implies that $h_i \in \mathcal{C}_c^+(\mathcal{G})$, and $h_i \equiv 0$ whenever $f_i \equiv 0$. By continuity of h_i , for every $\delta > 0$ we can find a small enough neighborhood V of $1 \in \mathcal{G}$ such that

$$|h_i(x) - h_i(y)| \leq \delta$$

whenever $x, y \in V$. For any function φ with support in V , consider an approximate of h by finitely many translations of φ , $h \leq \sum_k \lambda_k L_{x_k} \varphi$, then we have,

$$f_i(x) = h(x)h_i(x) \leq \sum_j \lambda_k L_{x_k} \varphi(x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)h_i(x) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta)$$

whenever $x, x_k \in V$. Since $h_i + h_j \leq 1$, we get,

$$(f_i : \varphi) + (f_j : \varphi) \leq \sum_k \lambda_k \varphi(x_k^{-1}x)(h_i(x_k) + \delta) + \sum_k \lambda_k \varphi(x_k^{-1}x)(h_j(x_k) + \delta) \leq (1 + 2\delta) \sum_k \lambda_k.$$

Hence, by taking the infimum of all such $\sum_k \lambda_k$, we get,

$$\begin{aligned} I_\varphi(f_i) + I_\varphi(f_j) &\leq (1 + 2\delta)I_\varphi(h) \leq (1 + 2\delta)(I_\varphi(f_i) + I_\varphi(f_j) + \delta I_\varphi(g)) \\ &= (I_\varphi(f_i + f_j)) + \delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g). \end{aligned}$$

The last step is by [\(subadditivity\)](#) and [\(scaling\)](#). Then by [\(rescaling\)](#) we can choose δ small enough such that the last part is less than ϵ , $\delta(2I_\varphi(f_i + f_j) + I_\varphi(g)) + 2\delta^2 I_\varphi(g) < \epsilon$. Hence we can choose a function φ with a small support such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

The choice of δ determines the support of the function used for approximation. This can be chosen such that $I_\varphi(f_i) + I_\varphi(f_j)$ is as close to $I_\varphi(f_i + f_j)$ as we want. However this choice of δ depends on the functions f_i and f_j .

Each function $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ gives us a map $f \mapsto I_\varphi(f) \in \mathbb{R}$. The goal is to find a functional that is linear. For each φ the value of $I_\varphi(f)$ lies in the interval $X_f \equiv [f_F, f^F]$. So, we can embed all such functionals inside the compact Hausdorff space,

$$X = \prod_{f \in \mathcal{C}_c^+(\mathcal{G})} X_f.$$

This is compact Hausdorff space because it is a product of compact Hausdorff spaces. It consists of all real valued functions from the space of compactly supported functions whose value at f lies in the interval X_f .

Consider for each neighborhood V of $1 \in \mathcal{G}$, the compact subsets $K(V)$ of X , consisting of closures of the collection of all functionals of the form I_φ with support of φ inside V .

$$X = \bigcup_{1 \in V} K(V)$$

Clearly, $V \subseteq W$ implies $K(V) \subseteq K(W)$. So, these sets satisfy the ‘finite intersection property’ $\cap_{i=1}^n K(V_i) \supset K(\cap_{i=1}^n V_i)$. This is a strict subset because there always exist functions with

domain that is larger than $\cap_i V_i$. Since X is compact, the intersection $\cap_i K(V_i)$ is nonempty.⁴ So there exists some point I in X that lies in all of the $K(V)$'s.

Every neighborhood of I contains I_φ 's. So, for any collection of functions $\{f_i\}_{i=1}^n \subset \mathcal{C}_c^+(\mathcal{G})$ and $\epsilon > 0$, there exists some $\varphi \in \mathcal{C}_c^+(\mathcal{G})$ such that $|I(f_i) - I_\varphi(f_i)| < \epsilon$, by definition of neighborhoods in product topology.

This implies that I is left translation invariant $L_y I = I$, and commutes with scaling. Since by adjusting the support for all $\epsilon > 0$, we can find a function φ such that,

$$I_\varphi(f_i) + I_\varphi(f_j) \leq I_\varphi(f_i + f_j) + \epsilon.$$

This implies that, $I(f_i) + I(f_j) \leq I(f_i + f_j) + \epsilon$ for all ϵ , and since I belongs to every $K(V)$, ϵ can be arbitrarily small, hence,

$$I(f_i) + I(f_j) = I(f_i + f_j).$$

This has a unique extension to $\mathcal{C}_c(\mathcal{G})$, because any function can be written as $f = g - h$ with $g, h \in \mathcal{C}_c^+(\mathcal{G})$, and if $f = g' - h'$, then we have $g + h' = g' + h$ and hence, $I(g) + I(h') = I(g') + I(h)$, or, $I(f) = I(g) - I(h)$ is well defined. This linear functional corresponds to a left Haar measure.

UNIQUENESS

⁴This is an alternate characterization of compactness. The equivalence of the two definitions is basically rewriting the compactness definition of open sets and unions in terms of closed sets and intersections using De Morgan's laws.

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