Part II

Abelian Categories

1 | Yoneda Embedding

A set is a collection of ‘elements’. A category $\mathcal{C}$ is more sophisticated, it possesses ‘objects’ similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{C}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects.

A functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of a mapping of objects of $\mathcal{C}$ to objects of $\mathcal{D}$, $X \mapsto FX$ together with a map of the homomorphisms,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY).$$

the image of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ denoted by $F(f)$. That takes identity to identity and respects composition i.e.,

$$F(f \circ g) = F(f) \circ F(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$F(f \circ g) = F(g) \circ F(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ can be thought of as a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. A functor F is faithful if the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is injective for all X, Y. It’s full if the map is surjective. If it’s a bijection the functor is called fully faithful

1.1 | Yoneda Lemma

We want to study the objects in terms of the maps to or from the object. This information is contained in the functor $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)$. Each $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ tells us all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(X, \cdot)$. These are called hom functors.

We will focus on the hom functors here.

$$\begin{aligned} h_X : \ & \mathcal{C} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X). \end{aligned}$$

which maps a morphism $f : Y \to Z$ to a morphism $h_X(f) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ given by, $g \mapsto f \circ g$, corresponding to the composition,

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

We will denote this by,

$$\begin{aligned} f\circ = \mathrm{Hom}_{\mathcal{C}}(X, f) :\ \mathrm{Hom}_{\mathcal{C}}(X, Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ g &\mapsto f \circ g. \end{aligned}$$

similarly we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$s are all sets. Such categories are called locally small categories.

A natural transformation between two functors F and G from category $\mathcal{C}$ to $\mathcal{D}$ is a mapping κ such that for every $X \in \mathcal{C}$, there exists a map κ_X such that for all $f : X \to Y$, the diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & FX \arrow[r, "\kappa_X"] \arrow[d, "F(f)"'] & GX \arrow[d, "G(f)"] \\ Y & FY \arrow[r, "\kappa_Y"'] & GY \end{tikzcd} \tag{natural transformation}$$

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors F and G is denoted by,

$$\mathrm{Nat}(F, G).$$

We say two functors F and G are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $F \cong G$. The collections of all functors from $\mathcal{C}$ to $\mathcal{D}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{D}^{\mathcal{C}}$.

A functor $F : \mathcal{C} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{C}$,

$$F \cong h_X$$

in such a case, F is said to be represented by the object X. Where $\cong$ stands for natural isomorphism.

Theorem 1.1. (Yoneda Lemma) *For a functor $F : \mathcal{C} \to \mathbf{Sets}$ and any $A \in \mathcal{C}$, there is a natural bijection,*

$$\mathrm{Nat}(h_A, F) \cong F(A)$$

such that $\kappa \in \mathrm{Nat}(h_A, F) \leftrightarrow \kappa_A(\mathbb{1}_A) \in F(A)$.

Proof

In the natural transformation diagram, replace F by h_A, and G by F. For $A = X$, $\kappa_A : h_A A \to FA$. Now, $h_A A = \mathrm{Hom}_{\mathcal{C}}(A, A)$, which contains $\mathbb{1}_A$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Nat}(h_A, F) &\to FA \\ \kappa &\mapsto \kappa_A(\mathbb{1}_A). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $B \in \mathcal{C}$. For any $f : A \to B$, we have,

$$\begin{tikzcd} A \arrow[d, "f"] & h_A A \arrow[r, "\kappa_A"] \arrow[d, "h_A(f)"'] & FA \arrow[d, "F(f)"] & \mathbb{1}_A \arrow[r, mapsto, "\kappa_A"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ B & h_A B \arrow[r, "\kappa_B"'] & FB & f \arrow[r, mapsto, "\kappa_B"'] & \kappa_B(f) \end{tikzcd}$$

Hence $\kappa_B(f) = F(f)(\mu(\kappa))$, or the action of κ_B is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_B(f) = \varphi_B(f)$ for all $B \in \mathcal{C}$, so it's injective.

For surjectivity we have to show that for all sets $u \in FA$, there exists a natural transformation φ such that $\varphi_A(\mathbb{1}_A) = u$. For $u \in FA$, and $f : A \to B$, construct the map,

$$\begin{aligned} \varphi : h_A &\to F \\ f &\mapsto F(f)(u). \end{aligned}$$

this satisfies the requirement that $\varphi_A(\mathbb{1}_A) = u$, because clearly, $\mathbb{1}_A \mapsto F(\mathbb{1}_A)(u) = \mathbb{1}_u(u) = u$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} B \arrow[d, "g"] & h_A B \arrow[r, "\varphi_B"] \arrow[d, "h_A(g)"'] & FB \arrow[d, "F(g)"] \\ C & h_A C \arrow[r, "\varphi_C"'] & FC \end{tikzcd}$$

commutes for all $B, C \in \mathcal{C}$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. For $f : A \to B$, by definition of φ,

$$F(g) \circ (\varphi_B(f)) = F(g) \circ F(f)(u)$$

which by functoriality of F is

$$= F(g \circ f)(u)$$

On the other hand, by definition of the hom functor, we have,

$$\varphi_C \circ (h_A(g)(f)) = \varphi_C(h_A(g \circ f))$$

which again by definition of φ is

$$= F(g \circ f)(u).$$

Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Nat}(h_A, F) \to FA$ is a bijection. $\square$

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof is same for the contravariant case, and the contravariant hom functor $h^X :\ \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ such that $Y \mapsto \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$. An immediate result of the Yoneda lemma is the Yoneda embedding functor,

$$Y_* :\ \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}}$$

which sends an object $A \in \mathcal{C}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(\cdot, A)$. Similarly we can define the contravariant Yoneda embedding,

$$Y^* :\ \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$$

which sends an object $Y \in \mathcal{C}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(\cdot, Y)$. These functors are fully faithful.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors, they are called an adjoint pair if

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) = \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. F is a left adjoint to G and G is a right adjoint to F. Adjoints are unique upto isomorphism and is the representative of the functor,

$$X \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

The isomorphism gives us,

$$\mathrm{Hom}_{\mathcal{C}}(G(X), G(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(F \circ G(X), Y)$$

and similarly,

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X, G \circ F(Y))$$

2 | Abelian Categories

The suitable category for doing homological algebra is called an abelian category denoted by **Ab**. An abelian category is a category that has kernels, cokernels, quotients, direct sums, direct products, etc. i.e., if $\alpha : A \to B$ is a morphism in the category then $\ker \alpha$ is also an object in the category and similarly for image and quotient, direct sums and direct products. We want to define an appropriate category for sheaves where we can do homological algebra. Since all of these can be expressed as a representable functor or in terms of a universal property, it's possible to define them categorically.

2.1 | Additive Categories

2.1.1 | Product & Coproduct

Let $\mathcal{C}$ be a category and consider a family $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$ indexed by a set I, then we can consider the contravariant functors,

$$G : Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$$

The product on the right side is the normal product in sets. Assuming the functor is representable, i.e., there exists an object P such that, $G(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, P)$. This is called the product, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_{i \in I} X_i) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd} & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "f_j"] & \\ X_i & \prod_{i \in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$$

This is a covariant functor, assuming it's representable, i.e., there exists an object C such that, $F(Y) = \mathrm{Hom}_{\mathcal{C}}(C, Y)$. The representative C is denoted by $\coprod_{i \in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} X_i, Y) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i \in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\
& Y &
\end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} X_i, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, \prod_{i \in I} X_i)$$

2.1.2 | Kernel & Cokernel

For sets, the kernel of two maps s, t is defined as the set $\ker(s, t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{C}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"] \arrow[drr, bend left, "j"] & & \\
\ker(f, g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object Z is called a zero object if for any object A, there exists a unique morphism $Z \to A$ and a unique morphism $A \to Z$. It's unique upto isomorphism and denoted by 0. Between any two objects $A, B \in \mathcal{C}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{C} \to \mathcal{C}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\left(\mathrm{Hom}_{\mathcal{C}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(Z, X)\right).$$

This can be visualized by the diagram,

$$\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f, g) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd}
Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

2.1.3 | Inductive Objects & Projective Objects

Inductive and projective systems generalize directed systems. An inductive system in $\mathcal{C}$ indexed by a category $\mathcal{I}$ is an association to each object in $i \in \mathcal{I}$ an object in $X_i \in \mathcal{C}$ such that if there is an arrow $i \to j$ we must have an arrow $X_i \to X_j$. In other words, it's a functor,

$$I : \mathcal{I} \to \mathcal{C}.$$

Similarly a projective system in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor,

$$P : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}.$$

We would like to take limits of these systems. Since we work with locally small categories, our approach will be to define limits for the category of sets, then use this definition to define inductive and projective limits representably using hom sets of categories. So we will now define limits in the category of sets. Consider the category of sets, **Sets**. Consider a projective system in **Sets**,

$$D : \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$$

Let $\mathcal{C}$ be a locally small category, let $P : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ be a projective system. For any object $X \in \mathcal{C}$ we can construct the composite functor,

$$\mathcal{I} \xrightarrow{P} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,\_)} \mathbf{Sets}.$$

This is a projective system in the category of sets, $\mathcal{I} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,F\_)} \mathbf{Sets}$ and the limit exists.

The limit of this projective system can be defined as,

$$\lim P = \{\}$$

In case of sets, let $\mathcal{I}$ and $\mathcal{C}$ be sets, and consider a

Let $\mathcal{C}$ and $\mathcal{I}$ be two categories,

Let I be an indexing set, and $\mathcal{C}$ a category. A projective system in $\mathcal{C}$ indexed by I is a functor $\alpha : I^{\mathrm{op}} \to \mathcal{C}$

The category of modules has the property that the hom-sets are abelian groups. A linear category is a category $\mathcal{C}$ whose hom-sets, $\mathrm{Hom}_{\mathcal{C}}(A,B)$ are abelian groups for all $A, B \in \mathcal{C}$ and composition is bilinear, i.e., if $f, f' \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $g, g' \in \mathrm{Hom}_{\mathcal{C}}(B,C)$ we must have,

$$(g + g') \circ f = g \circ f + g' \circ f \text{ and } g \circ (f + f') = g \circ f + g' \circ f.$$

A linear category with zero object and direct sums is called an additive category. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is additive if each $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a group homomorphism.

$$\begin{tikzcd}
A \arrow[d, "e"'] \arrow[dr, tail, bend left, "f"] & & \\
\ker(p) \arrow[r, "I"'] & B \arrow[r, "p"'] & \text{coker(p)}
\end{tikzcd}$$

$$\begin{tikzcd}
\ker(f) \arrow[r, "I"] & A \arrow[r, "p"] \arrow[dr, two heads, bend right, "f"'] & \operatorname{coker}(i) \arrow[d, dashed, "e"] \\
& & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, two heads, "p"] \arrow[dr, bend right, "f"'] & \operatorname{im}(p) \arrow[d, tail, "I"] \\
& \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, dashed, "e"'] \arrow[dr, bend left, "f"] & & \\
\ker(i) \arrow[r, "i"'] & B \arrow[r, "p"'] & \operatorname{coker}(f)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "g"] & I \arrow[r, "m"] & B \arrow[r, "\varphi"] \arrow[d, "p"] & C \\
& ker \arrow[u, dashed, "v"] \arrow[ur, bend right, "I"'] & coker \arrow[ur, dashed, bend right, "\psi"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& ker \arrow[d, "k"] & & \\
A \arrow[ur, dashed, bend left, "w"] \arrow[r, "e"'] & ker \arrow[r, "I"] \arrow[d, "g"] & B \arrow[r, "p"] & coker \\
& T & &
\end{tikzcd}$$

$$\begin{tikzcd}
& ker \arrow[d, dashed, "\delta"] \arrow[dr, bend left, "I"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "e'"'] & I' \arrow[r, "i'"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
& ker \arrow[d, "I"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "f"'] & I' \arrow[r, "g"] \arrow[d, "p"'] & B \arrow[dl, dashed, bend left, "t"] \\
& coker &
\end{tikzcd}$$

A pullback for a diagram $X \xrightarrow{f} B \xleftarrow{g} Y$ in a category $\mathcal{C}$ is the commutative square with vertex P such that any other commutative square factors through P,

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_Q"] \arrow[ddr, bend right, "g_Q"'] & & \\
& P \arrow[r, dashed, "f_P"] \arrow[d, dashed, "g_P"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f'"'] & B
\end{tikzcd}$$

The vertex of the pullback P is called the fibered product, denoted by $X \times_B Y$. In case of **Sets** the pullback always exists and consists of all elements (x, y) such that $f(x) = g(y)$ in B. Now consider the functor category $\mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$, with X, Y, B being functors to **Sets**.

3 | Categories of Complexes

References