Reconstruction of
The Lorentz Group

Bharath Ron

Quantum theory requires the notion of von Neumann algebras, and normal states on them. Relativity on the other hand uses semi-Riemannian geometric notions to formulate its ideas. We however expect quantum theory to be a universal framework capable of modeling and describing all observed phenomena. This makes it important to formulate the geometric ideas of relativity from within the operator algebraic framework of quantum theory.

We need a way to represent the geometric ideas of relativity on the the operator algebraic framework of quantum theory. Hence it becomes important to find a correspondence between geometric objects, and operator algebraic concepts. In this note, we attempt to recover the infinitesimal geometric data,(about the Lorentz groups of the Minkowski space) assuming that the vacuum state on the quasi-local algebra is invariant under such transformations. This turns the study of geometric data into the study operator algebraic data. In this sense, part of the geometric data (about transformation group that) can also be described from within the operator algebraic setting using states. This is similar to how in algebraic geometry one studies geometric data about spaces by studying commutative algebras. The main tool we use is the Tomita-Takesaki modular theory for von Neumann algebras.

1 | Tomita-Takesaki Modular Theory

Von Neumann algebras have algebraic and topological data. To be able to study a von Neumann algebra by its states we must understand how the state carries both the algebraic and topological data. Faithful states 'see' every element of the von Neumann algebra. Normal states respect the topological data of the von Neumann algebra. So we start with describing faithful normal states.

1.1 | Finite von Neumann Algebras

Since we expect the values of every single measurement instrument to be modeled using real numbers we expect the collection of effects corresponding to single measuring instrument to inherit these properties of real numbers. In this case the effects correspond to whether a measurement value lies in an interval. Since rational numbers are dense in real numbers, every interval has a rational number. Hence we expect any collection of mutually disjoint intervals to be of at most countable cardinality.

A von Neumann algebra is said to be σ-finite, if every collection of mutually orthogonal projections has at most a countable cardinality. For the reason discussed above, we will assume all von Neumann algebras of interest to us to be σ-finite.

Suppose $\mathcal{A}$ is a σ-finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$. Suppose $\{\kappa_i\}_{i\in\mathcal{I}} \subseteq \mathcal{H}$ be a maximal family of non-zero vectors such that $P_i \perp P_j$ whenever i and j are different. Here P_i is the projection to the subspace spanned by $\mathcal{A}'\kappa_i$. Since $\mathcal{A}$ is a von Neumann algebra, by von Neumann's density theorem we have $\mathcal{A}'' = \mathcal{A}$, and hence it follows that the projection P_i lies in $\mathcal{A}$. By maximality of $\{\kappa_i\}_{i\in\mathcal{I}}$ we have

$$\sum_{i\in\mathcal{I}} P_i = 1.$$

Hence $\{\kappa_i\}_{i\in\mathcal{I}}$ is cyclic for $\mathcal{A}'$, and equivalently separating for $\mathcal{A}$.[1] This is a mutually orthogonal collection of projection operators, and since $\mathcal{A}$ is σ-finite it follows that $\{P_i\}$ is countable, and hence the collection $\{\kappa_i\}_{\mathcal{I}}$ is countable. We will hence assume $\mathcal{I} \equiv \mathbb{N}$.

Normalise the collection of vectors $\{\kappa_i\}$ such that $\sum_{\mathbb{N}} \|\kappa_i\|^2 = 1$. Using this collection we can construct a faithful normal state by

$$\omega(A) = \sum_{i\in\mathbb{N}} \big\langle \kappa_i | A\kappa_i \big\rangle.$$

Hence we have proved the following theorem,

Theorem 1.1. (Existence) *If $\mathcal{A}$ is σ-finite then $\mathcal{A}$ has faithful normal states.*

σ-finite von Neumann algebras are convenient to study, because they have faithful normal states and one can use the Hilbert space structure to study them. Since this assumption is physically justified as well we will assume all von Neumann algebras appearing in the future to be σ-finite.

Let $\mathcal{A}$ be a von Neumann algebra together with a faithful normal state ω. Since ω is faithful, the Gelfand ideal is trivial. The state ω defines an inner product on the von Neumann algebra $\mathcal{A}$ given by,

$$\big\langle [A] | [B] \big\rangle_\omega := \omega(A^\dagger B).$$

where $[A]$ denotes the equivalence class determined by A, in this case each element of the von Neumann algebra determines an equivalence class uniquely. The Gelfand vector $[1]$ will be denoted by Ω. If $\mathcal{H}_\omega$ is the completion with respect to the inner product, then the left multiplication operator uniquely determines a representation π_ω of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\omega$, that is,

$$\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega).$$

By normality of the state ω, $\pi_\omega(\mathcal{A})$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\omega)$ and hence

$$\pi_\omega(\mathcal{A})'' = \pi_\omega(\mathcal{A}).$$

[1]A collection of vectors $\mathcal{K} \subset \mathcal{H}$ is separating for $\mathcal{A}$ if for every non-zero $A \in \mathcal{A}$ there exists some $\kappa \in \mathcal{K}$ such that $A\kappa$ is nonzero. $\mathcal{K} \subset \mathcal{H}$ is cyclic for $\mathcal{A}$ if $\mathcal{A}\mathcal{K}$ is dense in $\mathcal{H}$. It can be shown that $\mathcal{K}$ is separating for $\mathcal{A}$ if and only if it is cyclic for the commutant $\mathcal{A}'$.

Since ω is faithful $\omega(A^\dagger A) = \langle [A]|[A]\rangle_\omega$ is non-zero whenever A is non-zero, and hence it follows that $[A]$ is non-zero for every $A \in \mathcal{A}$. The Gelfand vector Ω is separating for $\pi_\omega(\mathcal{A})$ hence is cyclic for the commutant $\pi_\omega(\mathcal{A})'$.

By construction $\pi_\omega(\mathcal{A})\Omega$ is dense in $\mathcal{H}_\omega$, or Ω is cyclic for $\pi_\omega(\mathcal{A})$, or separating for the commutant $\pi_\omega(\mathcal{A})'$. Ω is cyclic and separating for both $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})'$. Hence we have proved the following lemma,

Lemma 1.2. *The vector Ω is cyclic and separating for both $\pi_\omega(\mathcal{A})$ and $\pi_\omega(\mathcal{A})'$.*

The data about the von Neumann algebra $\mathcal{A}$, as 'seen' by the state ω is contained in the dense subset $\pi_\omega(\mathcal{A})\Omega$, and similarly, the data about the commutant $\mathcal{A}'$ is contained in $\pi_\omega(\mathcal{A})'\Omega$. Modular theory studies the relation between a von Neumann algebra and its commutant by relating $\pi_\omega(\mathcal{A})\Omega$ and $\pi_\omega(\mathcal{A})'\Omega$.

1.2 | The Modular Group

We will assume that $\mathcal{A}$ is a von Neumann subalgebra on $\mathcal{H}_\omega$ with a cyclic and separating vector Ω, and denote $A\Omega$ by $[A]$. The approach of modular theory is to study the relation between $\mathcal{A}$ and $\mathcal{A}'$ by studying the relation between the dense subsets $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ using the inner product. The idea is to use the inner product $\langle\cdot|\cdot\rangle_\omega$ on $\mathcal{H}_\omega$ and exploit the fact that for every closed linear operator T on $\mathcal{H}_\omega$ we must have,

$$\big\langle \kappa|T\nu\big\rangle_\omega = \big\langle T^\dagger\kappa|\nu\big\rangle_\omega.$$

for every ν in the domain of T and κ in the domain of $T^\dagger$. Or similarly, for a closed anti-linear operator S we must have

$$\big\langle \kappa|S\nu\big\rangle_\omega = \big\langle \nu|S^\dagger\kappa\big\rangle_\omega.$$

If one of the arguments is from $\mathcal{A}\Omega$ and the other is from $\mathcal{A}'\Omega$, and if there exists an operator of this type, we would have related the two dense subsets. The goal is to try to find such a map S on $\mathcal{H}_\omega$. Observe that for any $A \in \mathcal{A}$ and $B \in \mathcal{A}'$ we have

$$\big\langle [B^\dagger]|[A]\big\rangle_\omega = \omega(BA) = \omega(AB) = \big\langle [A^\dagger]|[B]\big\rangle_\omega.$$

So, we should expect the map T or S to be related to the involutions $[A] \mapsto [A^\dagger]$ for $A \in \mathcal{A}$ and $[B] \mapsto [B^\dagger]$ for $B \in \mathcal{A}'$. Hence the starting point is to study the two maps,

$$\underline{S}_\mathcal{A} : [A] \mapsto [A^\dagger], \quad \underline{F}_\mathcal{A} : [B] \mapsto [B^\dagger],$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{A}'$, with domains $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ respectively.

Algebraic Properties

By construction and the assumption that ω is faithful and normal, both these sets are dense in $\mathcal{H}_\omega$. Hence $\underline{S}_\mathcal{A}$ and $\underline{F}_\mathcal{A}$ are densely defined anti-linear operators on $\mathcal{H}_\omega$. Let $\underline{S}^\dagger_\mathcal{A}$ and $\underline{F}^\dagger_\mathcal{A}$ be the adjoints of $\underline{S}_\mathcal{A}$ and $\underline{F}_\mathcal{A}$ respectively. For any $B \in \mathcal{A}'$ and for any $A \in \mathcal{A}$ we have $\langle \underline{F}_\mathcal{A}[B]|[A]\rangle_\omega = \langle [B^\dagger]|[A]\rangle_\omega = \langle [A^\dagger]|[B]\rangle_\omega = \langle \underline{S}_\mathcal{A}[A]|[B]\rangle_\omega$. By the anti-linearity of $\underline{S}_\mathcal{A}$ we have $\big\langle \underline{S}_\mathcal{A}[A]|[B]\big\rangle_\omega = \big\langle \underline{S}^\dagger_\mathcal{A}[B]|[A]\big\rangle_\omega$. Hence we get

$$\big\langle \underline{F}_\mathcal{A}[B]|[A]\big\rangle_\omega = \big\langle \underline{S}^\dagger_\mathcal{A}[B]|[A]\big\rangle_\omega.$$

So, whenever $[B]$ is in the domain of $\underline{F}_{\mathcal{A}}$ it is also in the domain of $\underline{S}^{\dagger}_{\mathcal{A}}$. Hence we have

$$\underline{F}_{\mathcal{A}} \subseteq \underline{S}^{\dagger}_{\mathcal{A}}.$$

Similarly we have $\underline{S}_{\mathcal{A}} \subseteq \underline{F}^{\dagger}_{\mathcal{A}}$. In particular they are densely defined. An operator is closeable if and only if it has a densely defined adjoint. Hence $\underline{S}_{\mathcal{A}}$ and $\underline{F}_{\mathcal{A}}$ are closable with closures $S_{\mathcal{A}}$ and $F_{\mathcal{A}}$ respectively. Since adjoints of densely defined operators are closed we have,

$$\underline{F}_{\mathcal{A}} \subseteq F_{\mathcal{A}} \subseteq \underline{S}^{\dagger}_{\mathcal{A}},$$

and similarly we have $\underline{S}_{\mathcal{A}} \subseteq S_{\mathcal{A}} \subseteq \underline{F}^{\dagger}_{\mathcal{A}}$.

We can immediately describe some of the algebraic properties of $S_{\mathcal{A}}$. Let $S^{\dagger}_{\mathcal{A}}$ be the adjoint of $S_{\mathcal{A}}$. Since $\underline{S}_{\mathcal{A}} = \underline{S}^{-1}_{\mathcal{A}}$, $S_{\mathcal{A}}$ is injective and we have

$$S^2_{\mathcal{A}} = 1$$

as an unbounded operator. Similarly we have $(S^{\dagger}_{\mathcal{A}})^2 = 1$. Both $S_{\mathcal{A}}$ and $S^{\dagger}_{\mathcal{A}}$ are involutions on $\mathcal{H}_\omega$ with dense ranges. They possess unique polar decompositions. Let the polar decomposition of $S_{\mathcal{A}}$ be given by

$$S_{\mathcal{A}} = J_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}}$$

where $\Delta_{\mathcal{A}} = S^{\dagger}_{\mathcal{A}}S_{\mathcal{A}}$ and $J_{\mathcal{A}}$ is an anti-unitary operator on $\mathcal{H}_\omega$, that is $J_{\mathcal{A}}J^{\dagger}_{\mathcal{A}} = J^{\dagger}_{\mathcal{A}}J_{\mathcal{A}} = 1$.

Since $S_{\mathcal{A}}$ is an involution it follows that $\Delta_{\mathcal{A}}$ is non-singular, positive and hence self-adjoint. Using the fact that $S^2_{\mathcal{A}} = 1$ and $(S^{\dagger}_{\mathcal{A}})^2 = 1$ we have $S^{\dagger}_{\mathcal{A}}S_{\mathcal{A}}S_{\mathcal{A}}S^{\dagger}_{\mathcal{A}} = 1$. Hence we have, $S_{\mathcal{A}}S^{\dagger}_{\mathcal{A}} = \Delta^{-1}_{\mathcal{A}}$, and we get,

$$J_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}}J_{\mathcal{A}} = \Delta^{-\frac{1}{2}}_{\mathcal{A}}.$$

Now we also have, $\Delta^{-1/2}_{\mathcal{A}} = J^2_{\mathcal{A}}J^{\dagger}_{\mathcal{A}}\Delta^{1/2}_{\mathcal{A}}J_{\mathcal{A}}$. Since $J^2_{\mathcal{A}}$ is a unitary operator and $J^{\dagger}_{\mathcal{A}}\Delta^{1/2}_{\mathcal{A}}J_{\mathcal{A}}$ is a positive self-adjoint operator. By the uniqueness of polar decomposition it follows that $J^2_{\mathcal{A}} = 1$ hence we have

$$J_{\mathcal{A}} = J^{\dagger}_{\mathcal{A}}.$$

Taking the adjoint of the polar decomposition we get,

$$S^{\dagger}_{\mathcal{A}} = \big(J_{\mathcal{A}}\Delta^{\frac{1}{2}}_{\mathcal{A}}\big)^{\dagger} = \Delta^{\frac{1}{2}}_{\mathcal{A}}J_{\mathcal{A}} = J_{\mathcal{A}}\Delta^{-\frac{1}{2}}_{\mathcal{A}}.$$

To study the properties of $\Delta_{\mathcal{A}}$, it is convenient to consider its spectral decomposition. Let $E_{\mathcal{A}} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}_\omega)$ be the spectral measure of $\Delta_{\mathcal{A}}$, that is,

$$\big\langle \eta \,|\Delta_{\mathcal{A}}\nu\big\rangle_\omega = \int_{\mathbb{R}^+} \lambda d\big\langle \eta \,|E_{\mathcal{A}}(\lambda)\nu\big\rangle_\omega.$$

for every ν in the domain of $\Delta_{\mathcal{A}}$. Note that it is suffucient to consider the integral over positive numbers because $\Delta_{\mathcal{A}}$ is positive, and hence support of the spectral measure $E_{\mathcal{A}}$ lies in the positive reals. The spectral measure for $\Delta^{-1}_{\mathcal{A}}$ corresponds to $J_{\mathcal{A}}E_{\mathcal{A}}J_{\mathcal{A}}$, and hence for any bounded Borel function f on $\mathbb{C}$, we have,

$$\begin{aligned}\big\langle f(\Delta^{-1}_{\mathcal{A}})\nu|\nu\big\rangle_\omega &= \int_{\mathbb{R}} f(\lambda)d\big\langle J_{\mathcal{A}}E_{\mathcal{A}}(\lambda)J_{\mathcal{A}}\nu \,|\nu\big\rangle_\omega \\ &= \int_{\mathbb{R}} f(\lambda)d\big\langle J_{\mathcal{A}}\nu \,|E_{\mathcal{A}}(\lambda)J_{\mathcal{A}}\nu\big\rangle_\omega \\ &= \big\langle f(\Delta_{\mathcal{A}})J_{\mathcal{A}}\nu \,|J_{\mathcal{A}}\nu\big\rangle_\omega = \big\langle J_{\mathcal{A}}\nu \,|\tilde{f}(\Delta_{\mathcal{A}})J_{\mathcal{A}}\nu\big\rangle_\omega.\end{aligned}$$

where $\tilde{f}(x) = \overline{f(x)}$. By the anti-unitarity of $J_{\mathcal{A}}$ we have,

$$\left\langle f(\Delta_{\mathcal{A}}^{-1})\nu \mid \nu \right\rangle_{\omega} = \left\langle J_{\mathcal{A}}\tilde{f}(\Delta_{\mathcal{A}})J_{\mathcal{A}}\nu \mid \nu \right\rangle_{\omega}.$$

Hence it follows that, $f(\Delta_{\mathcal{A}}^{-1}) = J_{\mathcal{A}}\tilde{f}(\Delta_{\mathcal{A}})J_{\mathcal{A}}$. By taking $f(x) = x^{-it}$ we obtain,

$$J_{\mathcal{A}}\Delta_{\mathcal{A}}^{it} = \Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}.$$

$J_{\mathcal{A}}$ is called the modular conjugation and $\Delta_{\mathcal{A}}$ is called the modular operator of ω.

Topological Properties

We now prove that it is indeed possible to understand the properties of the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$ as seen by the state ω, by studying the dense subset $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$. The difficulty lies in understanding how to go back from the dense subsets $\mathcal{A}\Omega$ to $\mathcal{A}$ and recover the von Neumann algebraic structures. The elements of $\mathcal{A}$ and $\mathcal{A}'$ act as bounded operators on the Hilbert space $\mathcal{H}_{\omega}$. We know that

$$S_{\mathcal{A}}[A] = A^{\dagger}\Omega$$

Hence we can think of the image of $\mathcal{A}\Omega$ under $S_{\mathcal{A}}$ as image of Ω under the action of the von Neumann algebra $\mathcal{A}$, and similarly for the commutant $\mathcal{A}'$. Hence the operators $S_{\mathcal{A}}$ allows us to between $\mathcal{A}\Omega$ and $\mathcal{A}$.

On the other hand, for $A \in \mathcal{A}$ and $B \in \mathcal{A}'$, and suppose additionally if we have $[A] \in \mathcal{A}'\Omega$ and $[B] \in \mathcal{A}\Omega$, then we have $B[A] = [BA] = [AB] = A[B]$. The goal of the discussion below is to prove that such vectors are dense in $\mathcal{H}_{\omega}$ and can be described in terms of domains of $S_{\mathcal{A}}$ and $F_{\mathcal{A}}$ respectively. Such vector as continuous actions of either $\mathcal{A}$ or $\mathcal{A}'$

$$\begin{array}{ccc} [B] & & [A] \\ \downarrow & & \downarrow \\ A[B] & = & B[A] \end{array}$$

Suppose ν be in the domain of $F_{\mathcal{A}}$, then consider the map

$$B_{\nu} : [A] \mapsto A\nu$$

If B_{ν} is continuous, then $AB_{\nu}[C] = AC\nu = B_{\nu}[AC] = B_{\nu}A[C]$ for all $C \in \mathcal{A}$. Hence we have $AB_{\nu}\nu = B_{\nu}A\nu$. Since ν is in the domain of $F_{\mathcal{A}}$, it also follows that

$$\langle [C]|B_{\nu}^{\dagger}[A]\rangle_{\omega} = \langle B_{\nu}[C]|[A]|\rangle_{\omega} = \langle C\nu|[A]\rangle_{\omega} = \langle \nu|[C^{\dagger}A]\rangle_{\omega}.$$

Since ν is already in the domain of $F_{\mathcal{A}}$ and using the fact that $\underline{F}_{\mathcal{A}} \subset \underline{S}_{\mathcal{A}}^{\dagger}$ we have $\langle \nu|[C^{\dagger}A]\rangle_{\omega} = \left\langle \nu|\underline{S}_{\mathcal{A}}[A^{\dagger}C]\right\rangle_{\omega} = \langle [A^{\dagger}C]|F_{\mathcal{A}}\nu\rangle_{\omega} = \langle [C]|A(F_{\mathcal{A}}\nu)\rangle_{\omega}$. Hence we showed that

$$A(F_{\mathcal{A}}\nu) = B_{\nu}^{\dagger}[A].$$

Hence $B_{\nu} \in \mathcal{A}'$. The domain of $S_{\mathcal{A}}^{\dagger}$ consists of all the vectors ν in $\mathcal{H}_{\omega}$ such that

$$[A] \mapsto \left\langle \nu|S_{\mathcal{A}}[A]\right\rangle_{\omega}$$

is continuous. Since $\langle \nu|S_{\mathcal{A}}[A]\rangle_{\omega} = \left\langle A\nu|\Omega\right\rangle_{\omega}$ it reduces to the case when $[A] \mapsto A\nu$ is continuous. By the discussion above, we know that there exists an operator B_{ν} such that $B_{\nu}A = AB_{\nu}$ for all ν as above.

Since $\mathcal{A}'\Omega$ is dense in $\mathcal{H}_\omega$, ν can be approximated by vectors in the domain of $\underline{F}_\mathcal{A}$, hence ν is in the domain of its closure $F_\mathcal{A}$, and B_ν is affiliated to $\mathcal{A}'$. Hence we have $S_\mathcal{A}^\dagger \subset F_\mathcal{A}$. Combinging with $S_\mathcal{A}^2 = 1$, we have proved the following lemma:

Lemma 1.3. (Takesaki)

$$S_\mathcal{A}^\dagger \equiv F_\mathcal{A} = J_\mathcal{A}\Delta_\mathcal{A}^{-1/2}.$$

This is not surprising since we anticipated that $\langle[B^\dagger]|[A]\rangle_\omega = \langle[A^\dagger]|[B]\rangle_\omega$ for all $A \in \mathcal{A}$ and $B \in \mathcal{A}'$. The support of the spectral measure $E_\mathcal{A}$ of $\Delta_\mathcal{A}$ belongs in $\mathbb{R}^+$. Since spectral measures resolve the identity, it follows that the projections $E_\mathcal{A}((0,t))$ converge strongly to the identity on $\mathcal{H}_\omega$, as t tends to infinity. If we denote the domain of $E_\mathcal{A}((0,t))$ by $\mathcal{D}(E_\mathcal{A}(t))$, we have,

$$\overline{\bigcup_{t\in\mathbb{R}^+} \mathcal{D}\big(E_\mathcal{A}(t)\big)} = \mathcal{H}_\omega.$$

By Takesaki's lemma above, the domain of $F_\mathcal{A}$ is the same as that of $\Delta^{-1/2}$, which is again the same as that of $\Delta_\mathcal{A}$ by positivity of the spectral measure. It follows that $\cup_t\mathcal{D}(E_\mathcal{A}(t)) \subseteq \mathcal{D}(S_\mathcal{A}) \cap \mathcal{D}(F_\mathcal{A})$. It follows that $\mathcal{D}(S_\mathcal{A}) \cap \mathcal{D}(F_\mathcal{A})$ is dense in $\mathcal{H}_\omega$. Hence we have proved the following;

Lemma 1.4.

$$\overline{\mathcal{D}\big(\Delta_\mathcal{A}^{\frac{1}{2}}\big) \cap \mathcal{D}\big(\Delta_\mathcal{A}^{-\frac{1}{2}}\big)} = \mathcal{H}_\omega.$$

This dense subset acts as a convenient subset to study behavior of operators, since it contains the properties both of the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$.

1.2.1 | The Modular Relation

We now relate the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$ in terms of the modular relations. We will then solve this relation in terms of the modular operator and the modular conjugation. The goal is to use the Hilbert space $\mathcal{H}_\omega$ to relate the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$, and then remove any terms involving the Hilbert space directly. We will follow [**?**], since they treat both $\mathcal{A}$ and $\mathcal{A}'$ on an equal footing.

For any $A, B, C \in \mathcal{A}$, we have $BCA^\dagger = (AC^\dagger B^\dagger)^\dagger$. Since we can view the algebra $\mathcal{A}$ in terms of the dense subset $\mathcal{A}\Omega$, viewing C as the vector $[C]$ in $\mathcal{A}\Omega$, we have,

$$BS_\mathcal{A}AS_\mathcal{A}[C] = S_\mathcal{A}AS_\mathcal{A}B[C].$$

So, when viewed from the subset $\mathcal{A}\Omega$, the operator $S_\mathcal{A}AS_\mathcal{A}$ acts as if it is in the commutant of $\mathcal{A}$. We use this heuristic relation as motivation to construct an element of $\mathcal{A}'$ for each element in the von Neumann algebra $\mathcal{A}$.

Sakai-Radon-Nikodym Theorem

We prove a useful technical lemma, due to Sakai, which will provide us with a link between the von Neumann algebra $\mathcal{A}$ and its commutant $\mathcal{A}'$ on the Hilbert space $\mathcal{H}_\omega$. If ω is a faithful normal state on $\mathcal{A}$, it contains non-trivial data about every operator in $\mathcal{A}$.

Now, suppose $\varphi \ll \omega$. This by definition means that for every operator $C \in \mathcal{A}$,

$$(\omega - \varphi)\big(C^\dagger C\big) \geq 0.$$

Heuristically this means that the state ω contains more data about $\mathcal{A}$ than the data contained in φ about $\mathcal{A}$ or that φ has 'less' data about the von Neumann algebra $\mathcal{A}$ than the data contained in ω. Quantifying 'how much?' usually give rise to a Radon-Nikodym type relation between φ and ω. We will describe below such a Radon-Nikodym type relation due to Sakai.

Let φ be a state on $\mathcal{A}$. Then, $\widehat{\varphi}(C,B) = \varphi(C^\dagger B)$ defines a sesquilinear form. Since φ a positive linear functional we have, $\widehat{\varphi}\big(\mu C - B, \mu C - B\big) \geq 0$. Upon expanding and taking

$$\mu = \widehat{\varphi}(C,B)/\widehat{\varphi}(C,C)$$

we obtain a Cauchy-Schwarz like inequality,

$$\big|\varphi(C^\dagger B)\big| \leq \big|\varphi(C^\dagger C)\big|^{\frac{1}{2}}\big|\varphi(B^\dagger B)\big|^{\frac{1}{2}}.$$

If $\varphi \ll \omega$, we have, $\varphi(C^\dagger C) \leq \omega(C^\dagger C)$. Hence we must have

$$\big|\varphi(C^\dagger B)\big| \leq \big|\omega(C^\dagger C)\big|^{\frac{1}{2}}\big|\omega(B^\dagger B)\big|^{\frac{1}{2}}.$$

We now use the fact that for a von Neumann algebra $\mathcal{A}$, the collection of all normal states $\mathcal{A}_*$ is the predual of $\mathcal{A}$, and hence every continous linear functional on $\mathcal{A}_*$ can be thought of as an element of $\mathcal{A}$. The idea is to exploit this fact about von Neumann algebras, along with the Hahn-Banach separation theorem to show that φ can be constructed using ω and some operator A. Heuristically, A quantifies the extra data contained in ω.

Theorem 1.5. (Sakai) *Let φ be such that*

$$\big|\varphi(C^\dagger B)\big| \leq \big|\omega(C^\dagger C)\big|^{\frac{1}{2}}\big|\omega(B^\dagger B)\big|^{\frac{1}{2}}$$

then for any fixed $\lambda \in \mathbb{R}_+$, there exists some $A \in \mathcal{A}$, with $\|A\| \leq 1$ such that

$$\varphi(X) = \omega\big(\lambda AX + \lambda^{-1}XA\big).$$

Proof

Fix $\lambda \in \mathbb{R}_+$. Construct a parametrized collection of linear functionals on $\mathcal{A}$ given by,

$$\omega_{\lambda,A} : X \mapsto \omega(\lambda AX + \lambda^{-1}XA).$$

for every $A \in \mathcal{A}$. The idea is to show that if φ is not of the type $\omega_{\lambda,A}$ then φ can be separated by a continuous linear functional using the Hahn-Banach separation theorem. Since continuous linear functionals on von Neumann algebras are more constrained, we end up arriving at a contradiction.

The role of λ here is to absorb some of the wildness and to ensure A is well behaved. Since $\omega_{\lambda,A}$ is constructed out of ω, it clearly has 'less data' about the von Neumann algebra $\mathcal{A}$ than ω itself. If A was the identity, then $\omega_{\lambda,I}$ is only a scaling of ω, or would have the same data once normalised.

By normality of the state ω it follows that whenever A is near to C in $\mathcal{A}$, $\omega_{\lambda,A}(X)$ must be near to $\omega_{\lambda,C}(X)$ as complex numbers. By linearity of ω we also have, $\omega_{\lambda,(A+C)}(X) = \omega_{\lambda,A}(X) + \omega_{\lambda,C}(X)$. So the mapping

$$A \mapsto \omega_{\lambda,A}$$

defines a continuous linear map from $\mathcal{A}$ to $\mathcal{A}_*$. Hence the set $\omega_{\lambda,1} \equiv \{\omega_{\lambda,A} \mid \|A\| \leq 1\}$ is a convex subset of $\mathcal{A}_*$, since it is the image of a closed set it is weakly compact in $\mathcal{A}_*$. We now prove that φ belongs to this set, which would prove the lemma.

Suppose there exists no A such that $\varphi = \omega_{\lambda,A}$. Then by the Hahn-Banach separation theorem there exists a continuous real linear functional κ on $\mathcal{A}_*$ which separates the compact convex subset ω_λ in $\mathcal{A}_*$ from the point φ in $\mathcal{A}_*$. That is, the interval, $\kappa(\omega_{\lambda,A})$ for $\|A\| \leq 1$ and $\kappa(\varphi)$ are disjoint. Assume without loss of generality that $\kappa(\varphi) > \kappa(\omega_{\lambda,A})$ for every $\|A\| \leq 1$.

Since $\mathcal{A}$ is a von Neumann algebra, $\mathcal{A}_*$ is its predual, and hence κ corresponds to an element H in $\mathcal{A}$. Let $H \in \mathcal{A}$ correspond to the functional κ then we have,

$$\mathrm{Re}(\varphi(H)) > \big|\omega_{\lambda,1}(H)\big| \equiv \sup\Big\{\mathrm{Re}(\omega_{\lambda,A}(H)) \mid \omega_{\lambda,A} \in \omega_{\lambda,1}\Big\}.$$

Consider the polar decomposition, $H = U|H| = |H^\dagger|U$ where U is a unitary operator and both U and $|H|$ are in $\mathcal{A}$. We can now use U and $|H|$ to arrive at a contradiction. We have,

$$\omega_{\lambda,\frac{U}{2}^\dagger}(H) = \tfrac{1}{2}\lambda\omega\big(U^\dagger H\big) + \tfrac{1}{2}\lambda^{-1}\omega\big(HU^\dagger\big) = \tfrac{1}{2}\lambda\omega\big(|H|\big) + \tfrac{1}{2}\lambda^{-1}\omega\big(|H^\dagger|\big).$$

By unitarity of U we have $\|U\|^2 = \|U^\dagger U\| = 1$, and hence $\omega_{\lambda,\frac{U}{2}^\dagger} \in \omega_{\lambda,1}$. Hence we have

$$\mathrm{Re}\Big(\omega_{\lambda,\frac{U}{2}^\dagger}(H)\Big) \leq \big|\omega_\lambda(H)\big| < \mathrm{Re}(\varphi(H))$$

Combining this with the AM-GM inequality we have,

$$\begin{aligned}\mathrm{Re}(\varphi(H)) > |\omega_{\lambda,1}(H)| &\geq \tfrac{1}{2}\big(\lambda\omega\big(|H|\big) + \lambda^{-1}\omega\big(|H^\dagger|\big)\big)\\ &\geq \omega\big(|H|\big)^{\frac{1}{2}}\omega\big(|H^\dagger|\big)^{\frac{1}{2}}.\end{aligned}$$

Since we assumed that $|\varphi(C^\dagger B)| \leq |\omega(C^\dagger C)|^{1/2}|\omega(B^\dagger B)|^{1/2}$, for every $C, B \in \mathcal{A}$, we have,

$$\begin{aligned}\big|\mathrm{Re}(\varphi(H))\big| \leq |\varphi(H)| &= \big|\varphi(U|H|^{\frac{1}{2}}|H|^{\frac{1}{2}})\big|\\ &\leq \big|\omega(|H|)\big|^{\frac{1}{2}}\big|\omega(U|H|U)\big|^{\frac{1}{2}} = \big|\omega(|H|)\big|^{\frac{1}{2}}\big|\omega(|H^\dagger|)\big|^{\frac{1}{2}}.\end{aligned}$$

Which implies that $\omega(|H|)^{1/2}\omega(|H^\dagger|)^{1/2} > \omega(|H|)^{1/2}\omega(|H^\dagger|)^{1/2}$ which is absurd and we arrive at a contradiction. Hence there must exist some A such that $\varphi = \omega_{\lambda,A}$. □

Relation between $\mathcal{A}$ and $\mathcal{A}'$

We now use the Sakai-Radon-Nikodym theorem to relate the von Neumann algebra $\mathcal{A}$ with its commutant $\mathcal{A}'$. By construction, the Hilbert space $\mathcal{H}_\omega$ describes all the data about the von Neumann algebra $\mathcal{A}$ contained in the state ω. We can now ask how much of this data remains after the action of $\mathcal{A}'$. Instead of the cyclic and separating vector Ω, we consider the action of $\mathcal{A}$ on elements of $\mathcal{A}'\Omega$ and what we can say about this action using the state ω. So, we are interested in data of the form, $\omega(XB)$ for $X \in \mathcal{A}$ and $B \in \mathcal{A}'$.

To study such data consider the linear functionals of the form

$$\omega_B(X) = \big\langle [X] | [B] \big\rangle_\omega,$$

for $B \in \mathcal{A}'$ and note that $\omega(X) = \omega_I(X)$. Since $B^\dagger B \leq \|B\|^2$, for every $B \in \mathcal{A}'$ and $C \in \mathcal{A}$ we have $B^\dagger C^\dagger CB = C^\dagger B^\dagger BC \leq C^\dagger \|B\|^2 C$ and by the positivity of the state ω we have

$\omega(B^\dagger C^\dagger CB) \leq \|B\|^2\omega(C^\dagger C)$. Hence we obtain,

$$\begin{aligned}\omega\big(C^\dagger BC\big) &\leq \big|\omega((B^\dagger C)^\dagger B^\dagger C)\big|^{\frac{1}{2}}\big|\omega(C^\dagger C)\big|^{\frac{1}{2}}\\ &= \big|\omega(C^\dagger BB^\dagger C)\big|^{\frac{1}{2}}\big|\omega(C^\dagger C)\big|^{\frac{1}{2}}\\ &\leq \big|\omega(C^\dagger\|B\|^2 C)\big|^{\frac{1}{2}}\big|\omega(C^\dagger C)\big|^{\frac{1}{2}} = \|B\|\,\omega(C^\dagger C).\end{aligned}$$

Hence we have, $\omega_B(C^\dagger C) \leq \|B\|\omega(C^\dagger C)$ or equivalently $\omega_B \ll \omega$. The assumptions of the theorem of Sakai above are satisfied. by ω_B. For every $\lambda \in \mathbb{R}_+$ there exists some A in $\mathcal{A}$ such that, $\omega_B(X) = \omega_{\lambda,A}(X)$

$$\omega_B(X) = \omega(\lambda XA + \lambda^{-1}AX).$$

We now use this application of Sakai's theorem to obtain a relation between $\mathcal{A}$ and $\mathcal{A}'$.

Theorem 1.6. (The Modular Relation) *Let* $\lambda \in \mathbb{R}_+$. $\forall B \in \mathcal{A}'$, $\exists A \in \mathcal{A}$ *with*

$$\lambda\big\langle\Delta_{\mathcal{A}}^{\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{-\frac{1}{2}}\eta\big\rangle_\omega + \lambda^{-1}\big\langle\Delta_{\mathcal{A}}^{-\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{\frac{1}{2}}\eta\big\rangle_\omega = \big\langle\nu|J_{\mathcal{A}}BJ_{\mathcal{A}}\eta\big\rangle_\omega,$$

$\forall \eta,\nu \in \mathcal{D}\big(\Delta_{\mathcal{A}}^{-1/2}\big) \cap \mathcal{D}\big(\Delta_{\mathcal{A}}^{1/2}\big)$.

Proof

Using the fact that $\omega(A) = \langle\Omega|[A]\rangle_\omega$ we can express the above relation in terms of the inner product on $\mathcal{H}_\omega$, $\langle[B]|[X]\rangle_\omega = \lambda\langle[AX]|\Omega\rangle_\omega + \lambda^{-1}\langle[XA]|\Omega\rangle_\omega$. Hence we obtain a relation between the elements of $\mathcal{A}$ and $\mathcal{A}'$,

$$\big\langle[X]|[B]\big\rangle_\omega = \lambda\big\langle[X]|[A^\dagger]\big\rangle_\omega + \lambda^{-1}\big\langle[X^\dagger]|[A]\big\rangle_\omega,$$

upon rearranging we get, $\langle[X^\dagger]|[A]\rangle_\omega = \lambda\langle[X]|[B] - \lambda[A^\dagger]\rangle_\omega$. Since $A, X \in \mathcal{A}$ we observe that $[A],[X]$ are in the domain of $S_{\mathcal{A}}$. Since the right hand side is well-defined we conclude that $[A] \in \mathcal{D}(S_{\mathcal{A}}^\dagger)$.

Hence we can express the vector $[B]$ in the set $\mathcal{A}'\Omega$ in terms of a vector $[A]$ in the set $\mathcal{A}\Omega$ as $[B] = (\lambda S_{\mathcal{A}} + \lambda^{-1}F_{\mathcal{A}})[A]$ or equivalently,

$$[B] = \Big[\underbrace{\lambda J_{\mathcal{A}}\Delta_{\mathcal{A}}^{\frac{1}{2}} + \lambda^{-1}J_{\mathcal{A}}\Delta_{\mathcal{A}}^{-\frac{1}{2}}}_{G_\lambda}\Big][A].$$

Let $\nu \in \mathcal{D}(\Delta_{\mathcal{A}}^{1/2}) \cap \mathcal{D}(\Delta_{\mathcal{A}}^{-1/2})$. Set

$$G_1\nu = \eta.$$

If $\eta \in \mathcal{D}(F_{\mathcal{A}})$ then so does $F_{\mathcal{A}}\eta$. Since every vector η in the domain of $F_{\mathcal{A}}$ corresponds to an operator B_η affiliated with $\mathcal{A}'$ such that $B_\eta\Omega = \eta$. Hence there exists a sequence $\{B_i\} \subset \mathcal{A}'$ with $[B_i] \to \eta$. For $\lambda = 1$ we obtain for each $B_i \in \mathcal{A}'$ there is an element $A_i \in \mathcal{A}$ such that

$$G_1[A_i] = [B_i].$$

Since the spectrum of $(\Delta_{\mathcal{A}}^{1/2} + \Delta_{\mathcal{A}}^{-1/2})$ does not contain 0 it follows that G_1 has a bounded inverse. By letting $\nu_i = G_1^{-1}[B_i]$ we have, $[A_i] = G_1^{-1}[B_i] = \nu_i \to \nu$.

So we can view any $\nu, \eta \in \mathcal{D}(F_{\mathcal{A}})$ similar to vectors in $\mathcal{A}\Omega$ and let $A_\nu, A_\eta \in \mathcal{A}$ be such vectors, we have

$$\begin{aligned}\langle \eta | S_{\mathcal{A}} A S_{\mathcal{A}} \nu \rangle &= \langle [A_\eta] | S_{\mathcal{A}} A S_{\mathcal{A}} [A_\nu] \rangle \\ &= \langle [A_\eta] | [(A A_\nu^\dagger)^\dagger] \rangle = \langle S_{\mathcal{A}}[A] | S_{\mathcal{A}}^\dagger [A_\nu^\dagger A_\eta] \rangle\end{aligned}$$

With a similar manipulation we obtain

$$\langle \eta | F_{\mathcal{A}} A F_{\mathcal{A}} \nu \rangle = \langle [A_\nu^\dagger A_\eta] | [A^\dagger] \rangle.$$

By taking $[X] = [A_\nu^\dagger A_\eta]$ and rearranging we obtain

$$\lambda \langle \nu | S_{\mathcal{A}} A S_{\mathcal{A}} \eta \rangle_\omega + \lambda^{-1} \langle \nu | F_{\mathcal{A}} A F_{\mathcal{A}} \eta \rangle_\omega = \langle \nu | B \eta \rangle_\omega,$$

We now take $J_{\mathcal{A}}\nu$ and $J_{\mathcal{A}}\eta$ instead of ν and η. Subsituting $S_{\mathcal{A}} = J_{\mathcal{A}} \Delta_{\mathcal{A}}^{1/2}$ and $F_{\mathcal{A}} = J_{\mathcal{A}} \Delta_{\mathcal{A}}^{-1/2}$, and using the fact, $J_{\mathcal{A}}^2 = 1$, it follows that,

$$\begin{aligned}\langle J_{\mathcal{A}}\nu | S_{\mathcal{A}} A S_{\mathcal{A}} J_{\mathcal{A}} \eta \rangle_\omega &= \langle J_{\mathcal{A}}\nu | J_{\mathcal{A}} \Delta_{\mathcal{A}}^{\frac{1}{2}} A J_{\mathcal{A}} \Delta_{\mathcal{A}}^{\frac{1}{2}} J_{\mathcal{A}} \eta \rangle_\omega \\ &= \langle \Delta_{\mathcal{A}}^{\frac{1}{2}} J_{\mathcal{A}}^2 \nu | A \Delta_{\mathcal{A}}^{-\frac{1}{2}} J_{\mathcal{A}}^2 \eta \rangle_\omega = \langle \Delta_{\mathcal{A}}^{\frac{1}{2}} \nu | A \Delta_{\mathcal{A}}^{-\frac{1}{2}} \eta \rangle_\omega.\end{aligned}$$

Similarly we have,

$$\langle J_{\mathcal{A}}\nu | F_{\mathcal{A}} A F_{\mathcal{A}} J_{\mathcal{A}} \eta \rangle_\omega = \langle \Delta_{\mathcal{A}}^{-\frac{1}{2}} \nu | A \Delta_{\mathcal{A}}^{\frac{1}{2}} \eta \rangle_\omega.$$

Hence we have,

$$\lambda \langle \Delta_{\mathcal{A}}^{\frac{1}{2}} \nu | A \Delta_{\mathcal{A}}^{-\frac{1}{2}} \eta \rangle_\omega + \lambda^{-1} \langle \Delta_{\mathcal{A}}^{-\frac{1}{2}} \nu | A \Delta_{\mathcal{A}}^{\frac{1}{2}} \eta \rangle_\omega = \langle \nu | J_{\mathcal{A}} B J_{\mathcal{A}} \eta \rangle_\omega.$$

whenever ν and η are in the domain of $\Delta_{\mathcal{A}}^{1/2}$ and $\Delta_{\mathcal{A}}^{-1/2}$. □

1.2.2 | Tomita's Theorem

The modular relation gives rise to a relation between $\mathcal{A}$ and $\mathcal{A}'$ for each $\lambda \in \mathbb{R}_+$. We now remove the dependence on λ. The idea is to think of the relation as the Fourier transform of a complex, rapidly decreasing function. The parameter λ can then be viewed in terms of characters. We need to develop certain tools to be able to do this.

Let f be a function bounded on the strip $S \equiv \{-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$. Then we can define a meromorphic function by,

$$g(z) = \pi f(z)(\sin(\pi z))^{-1}.$$

This is a meromorphic function with a pole at $z = 0$. By the power series expansion of sin, we have $\sin(\pi z) = \pi z - \pi^3 z^3/3! + \cdots$, it follows that it is a simple pole. Hence, the residue is given by,

$$\mathrm{res}_g(0) = \lim_{z \to 0} \pi f(z) \big(\pi - \tfrac{\pi^3 z^2}{3!} + \cdots\big)^{-1} = f(0).$$

By the residue theorem, for any loop $\gamma \subseteq \mathcal{H}(S \backslash E)$, we have

$$\sum_{x \in E} \mathrm{res}_g(x) n(\gamma, x) = (2\pi i)^{-1} \int_\gamma g(z) dz,$$

where $n(\gamma, x)$ is the winding number for the loop γ around the point x, which corresponds to the number of times the loop 'winds' around the point. As z tends to infinity, since the function f is assumed to be bounded, the meromorphic function g rapidly tends to zero for $|\chi| < \pi$. Since there is only one pole for g, and at 0, for any loop γ not passing through 0 we have,

$$f(0)n(\gamma, 0) = (2\pi i)^{-1}\Big[\int g(\gamma(t))\gamma'(t)dt\Big].$$

Now, choose the loop to be the one along the infinite rectangle, then we can split the curve into four parts, two of which are along the lines, $z = \pm\frac{1}{2}$, given by, $\gamma_{\pm}(t) = \pm it \pm \frac{1}{2}$, the other two curves are at infinity, and since g rapidly dies at infinity, they will not have any contribution. Hence we have,

$$f(0) = (2\pi i)^{-1}\int_{\mathbb{R}}\Big[ig\big(it + \tfrac{1}{2}\big) - ig\big(it - \tfrac{1}{2}\big)\Big]dt$$

Substituting $g(z) = \pi f(z)(\sin(\pi z))^{-1}$ we obtain,

$$f(0) = \tfrac{1}{2}\int_{\mathbb{R}}\bigg[\Big[\frac{f(it + \frac{1}{2})}{\sin\big(i\pi t + \frac{\pi}{2}\big)}\Big] - \Big[\frac{f(it - \frac{1}{2})}{\sin\big(i\pi t - \frac{\pi}{2}\big)}\Big]\bigg]dt.$$

Using the fact that $\sin(i\pi t + \pi/2) = \cos(i\pi t) = \cosh(\pi t)$, and $\sin(i\pi t - \pi/2) = -\cos(i\pi t) = -\cosh(\pi t)$, and by substituting $\cosh(\pi t) = (e^{\pi t} + e^{-\pi t})/2$, we obtain,

$$f(0) = \int_{\mathbb{R}}\Big[\frac{f\big(it + \frac{1}{2}\big) + f\big(it - \frac{1}{2}\big)}{e^{\pi t} + e^{-\pi t}}\Big]dt.$$

We now use this formula to obtain an operator equation using the modular relation;

Lemma 1.7. (Operator Equation) *If $A \in \mathcal{A}, B \in \mathcal{A}'$ are such that*

$$\lambda\big\langle\Delta_{\mathcal{A}}^{\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{-\frac{1}{2}}\eta\big\rangle_\omega + \lambda^{-1}\big\langle\Delta_{\mathcal{A}}^{-\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{\frac{1}{2}}\eta\big\rangle_\omega = \big\langle\nu|J_{\mathcal{A}}BJ_{\mathcal{A}}\eta\big\rangle_\omega$$

for any $\eta, \nu \in \mathcal{D}(\Delta_{\mathcal{A}}^{-1/2}) \cap \mathcal{D}(\Delta_{\mathcal{A}}^{1/2})$, then,

$$A = \int_{\mathbb{R}}\lambda^{2it}\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t} + e^{-\pi t}}\Big]dt.$$

Proof

For every $\nu, \eta \in \mathcal{D}(\Delta_{\mathcal{A}}^{-1/2}) \cap \mathcal{D}(\Delta_{\mathcal{A}}^{1/2})$, consider the complex valued function

$$f_{\nu,\eta}(z) = \lambda^{2z}\Big\langle\Delta_{\mathcal{A}}^{\overline{z}}\nu \mid A\Delta_{\mathcal{A}}^{-z}\eta\Big\rangle_\omega.$$

By strong continuity of $\Delta_{\mathcal{A}}^{z}$, normality of the action of $\mathcal{A}$ on $\mathcal{H}_\omega$, and by skew-linearity with respect to the first argument in the inner product, $f_{\nu,\eta}$ is an analytic function, and we have,

$$\begin{aligned} f_{\nu,\eta}\big(it + \tfrac{1}{2}\big) &= \lambda^{2(it+\frac{1}{2})}\Big\langle\Delta_{\mathcal{A}}^{\overline{it+\frac{1}{2}}}\nu \mid A\Delta_{\mathcal{A}}^{-it-\frac{1}{2}}\eta\Big\rangle_\omega \\ &= \lambda^{2(it+\frac{1}{2})}\Big\langle\Delta_{\mathcal{A}}^{\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\nu \mid A\Delta_{\mathcal{A}}^{-\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\eta\Big\rangle_\omega. \end{aligned}$$

Similarly we have,

$$\begin{aligned} f_{\nu,\eta}\big(it - \tfrac{1}{2}\big) &= \lambda^{2(it-\frac{1}{2})}\Big\langle\Delta_{\mathcal{A}}^{\overline{it-\frac{1}{2}}}\nu \mid A\Delta_{\mathcal{A}}^{-it+\frac{1}{2}}\eta\Big\rangle_\omega \\ &= \lambda^{2(it-\frac{1}{2})}\Big\langle\Delta_{\mathcal{A}}^{-\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\nu \mid A\Delta_{\mathcal{A}}^{\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\eta\Big\rangle_\omega. \end{aligned}$$

$f_{\nu,\eta}$ is also bounded on the strip $S = \{-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$, so $g(z) = \pi f_{\nu,\mu}(z)(\sin(\pi z))^{-1}$ decreases rapidly as z tends to infinity. Hence by the calculation prior this lemma, we have,

$$f_{\nu,\eta}(0) = \int_{\mathbb{R}} \Big[\frac{f_{\nu,\eta}(it+\frac{1}{2}) + f_{\nu,\eta}(it-\frac{1}{2})}{e^{\pi t}+e^{-\pi t}}\Big] dt.$$

Since $f_{\nu,\eta}(0) = \langle \nu|A\eta\rangle_\omega$, we have,

$$\langle \nu|A\eta\rangle_\omega = \int_{\mathbb{R}} \lambda^{2(it+\frac{1}{2})}\Big[\frac{\langle \Delta_{\mathcal{A}}^{\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\nu|A\Delta_{\mathcal{A}}^{-\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\eta\rangle_\omega}{e^{\pi t}+e^{-\pi t}}\Big]dt + \int_{\mathbb{R}} \lambda^{2(it-\frac{1}{2})}\Big[\frac{\langle \Delta_{\mathcal{A}}^{-\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\nu|A\Delta_{\mathcal{A}}^{\frac{1}{2}}\Delta_{\mathcal{A}}^{-it}\eta\rangle_\omega}{e^{\pi t}+e^{-\pi t}}\Big]dt.$$

Note that $\Delta_{\mathcal{A}}^{-it}\nu, \Delta_{\mathcal{A}}^{-it}\eta \in \mathcal{D}(\Delta_{\mathcal{A}}^{-1/2}) \cap \mathcal{D}(\Delta_{\mathcal{A}}^{1/2})$. Hence using the modular relation;

$$\lambda\langle \Delta_{\mathcal{A}}^{\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{-\frac{1}{2}}\eta\rangle_\omega + \lambda^{-1}\langle \Delta_{\mathcal{A}}^{-\frac{1}{2}}\nu|A\Delta_{\mathcal{A}}^{\frac{1}{2}}\eta\rangle_\omega = \langle \nu|J_{\mathcal{A}}BJ_{\mathcal{A}}\eta\rangle_\omega,$$

we get,

$$\begin{aligned}\langle \nu|A\eta\rangle_\omega &= \int_{\mathbb{R}} \lambda^{2it}\Big\langle \Delta_{\mathcal{A}}^{-it}\nu \,\Big|\, \Big[\frac{J_{\mathcal{A}}BJ_{\mathcal{A}}}{e^{\pi t}+e^{-\pi t}}\Big]\Delta_{\mathcal{A}}^{-it}\eta\Big\rangle_\omega dt \\ &= \Big\langle \nu \,\Big|\, \Big[\int_{\mathbb{R}} \lambda^{2it}\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]dt\Big]\eta\Big\rangle_\omega.\end{aligned}$$

Since the choice of ν, η was arbitrary it follows that,

$$A = \int_{\mathbb{R}} \lambda^{2it}\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]dt$$

on the dense subset $\mathcal{D}(\Delta_{\mathcal{A}}^{-1/2}) \cap \mathcal{D}(\Delta_{\mathcal{A}}^{1/2})$, and hence it holds on $\mathcal{H}_\omega$. □

With this lemma, we got rid of ν and η and showed that A and B can be related directly by an operator equation. We now have all the ingredients to prove Tomita's theorem, which establishes a relation between von Neumann algebras and their commutants in terms of the modular operator and the modular conjugation.

For every $B \in \mathcal{A}'$,

$$\int_{\mathbb{R}} \lambda^{2it}\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]dt \in \mathcal{A}.$$

Consider the function,

$$f_{\nu,\eta}(t) = \Big\langle \eta \,\Big|\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]\nu\Big\rangle_\omega.$$

The damping factor $1/(e^{\pi t}+e^{-\pi t})$ ensures that this function decreases faster than any polynomial in t, and allows us to do Fourier analysis. We now use the positivity of λ. Since $\lambda \in \mathbb{R}_+$ there exists χ such that $\lambda = e^{\chi/2}$. The Fourier transform of $f_{\nu,\eta}(t)$ is

$$\begin{aligned}\mathcal{F}f_{\nu,\eta}(\chi) &= \int_{\mathbb{R}} \lambda^{2it}\Big\langle \eta \,\Big|\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]\nu\Big\rangle_\omega dt \\ &= \Big\langle \eta \,\Big|\Big[\int_{\mathbb{R}} \lambda^{2it}\Big[\frac{\Delta_{\mathcal{A}}^{it}J_{\mathcal{A}}BJ_{\mathcal{A}}\Delta_{\mathcal{A}}^{-it}}{e^{\pi t}+e^{-\pi t}}\Big]dt\Big]\nu\Big\rangle_\omega.\end{aligned}$$

We now prove Tomita's theorem via injectivity of Fourier transforms.

Theorem 1.8. (Tomita) *If $J_\mathcal{A}$ and $\Delta_\mathcal{A}$ are as before, then*

$$J_\mathcal{A}\mathcal{A}J_\mathcal{A} = \mathcal{A}',\ \ \Delta_\mathcal{A}^{it}\mathcal{A}\Delta_\mathcal{A}^{-it} = \mathcal{A},\ \ \forall t \in \mathbb{R}.$$

Proof

Similar to $f_{\nu,\eta}$, consider the function,

$$g_{\nu,\eta}(t) = \Big\langle \eta \Big| \, U\Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] U^\dagger \nu \Big\rangle_\omega,$$

where U is a unitary operator in $\mathcal{A}'$. The Fourier transform of $g_{\nu,\eta}$ is given by,

$$\begin{aligned}\mathcal{F}g_{\nu,\eta}(\chi) &= \int_\mathbb{R} \lambda^{2it} \Big\langle \eta \Big| \, U\Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] U^\dagger \nu \Big\rangle_\omega dt \\ &= \Big\langle U^\dagger \eta \Big| \Big[\int_\mathbb{R} \lambda^{2it} \Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt \Big] U^\dagger \nu \Big\rangle_\omega .\end{aligned}$$

Since $\langle U\nu | U\eta\rangle_\omega = \langle \nu|\eta\rangle_\omega$ it follows that,

$$\mathcal{F}g_{\nu,\eta}(\chi) = \Big\langle \eta \Big| \Big[\int_\mathbb{R} \lambda^{2it} \Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}}\Big] dt \Big] \nu \Big\rangle_\omega = \mathcal{F}f_{\nu,\eta}(\chi).$$

By injectivity of Fourier transformation it follows that $f_{\nu,\eta} = g_{\nu,\eta}$. Since the choice of ν and η was also arbitrary it follows that $U(\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}) = (\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it})U$. Since the choice of the unitary $U \in \mathcal{A}'$ was arbitrary we conclude that

$$\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it} \in \mathcal{A} \ \ \forall t \in \mathbb{R}.$$

By taking $t = 0$ we have, $J_\mathcal{A}\mathcal{A}'J_\mathcal{A} \subseteq \mathcal{A}$ and by symmetry we also have, $J_\mathcal{A}\mathcal{A}J_\mathcal{A} \subseteq \mathcal{A}'$. Hence we have,

$$J_\mathcal{A}\mathcal{A}J_\mathcal{A} = \mathcal{A}'.$$

Hence every $A \in \mathcal{A}$ is of the form, $A = J_\mathcal{A} B J_\mathcal{A}$ for some $B \in \mathcal{A}'$. We have, $\Delta_\mathcal{A}^{it} A \Delta_\mathcal{A}^{-it} = \Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it} \in \mathcal{A}$. Hence it follows that,

$$\Delta_\mathcal{A}^{it}\mathcal{A}\Delta_\mathcal{A}^{-it} \in \mathcal{A}$$

this completes the proof. □

Summary of Modular Theory

Since the Tomita-Takesaki modular theory is a tedious theory, we will give below a quick summary of key takeaways for the convenience of the reader. Let ω be a faithful normal state for a von Neumann algebra $\mathcal{A}$, and $\mathcal{H}_\omega$ be the corresponsing GNS construction with Gelfand vector Ω. Let $[A] \equiv A\Omega$ denotes the vector corresponding to the equivalence class determined by an element $A \in \mathcal{A}$. By construction of $\mathcal{H}_\omega$ we have,

$$\overline{\mathcal{A}\Omega} = \overline{\mathcal{A}'\Omega} = \mathcal{H}_\omega.$$

We can think about the von Neumann algebra $\mathcal{A}$ as points of the dense subset $\mathcal{A}\Omega$ as well as by its action on the Hilbert space $\mathcal{H}_\omega$, similarly for $\mathcal{A}'$. Since $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$ are dense in $\mathcal{H}_\omega$, we can make use of the structure of Hilbert spaces to relate $\mathcal{A}$ and $\mathcal{A}'$.

The maps $[A] \mapsto [A^\dagger]$ and $[B] \mapsto [B^\dagger]$ for $A \in \mathcal{A}$ and $B \in \mathcal{A}'$ are closable since both are densely defined with domains $\mathcal{A}\Omega$ and $\mathcal{A}'\Omega$. Let $S_\mathcal{A}$ and $F_\mathcal{A}$ be their respective closures. Immediately by their definition we have

$$S_\mathcal{A}^2 = F_\mathcal{A}^2 = 1$$

Let the polar decomposition of $S_\mathcal{A}$ is given by

$$S_\mathcal{A} = J_\mathcal{A} \Delta_\mathcal{A}^{\frac{1}{2}}.$$

By definition of polar decomposition, $J_\mathcal{A}$ is an anti-unitary operator, that is,

$$J_\mathcal{A} = J_\mathcal{A}^\dagger, \;\; J_\mathcal{A}^\dagger J_\mathcal{A} = J_\mathcal{A} J_\mathcal{A}^\dagger = 1$$

$$\Delta_\mathcal{A} = S_\mathcal{A}^\dagger S_\mathcal{A} > 0$$

For every Borel function f on reals we have,

$$f(\Delta_\mathcal{A}^{-1}) = J_\mathcal{A} \tilde{f}(\Delta_\mathcal{A}) J_\mathcal{A}$$

In particular,

$$J_\mathcal{A} \Delta_\mathcal{A}^{it} = \Delta_\mathcal{A}^{it} J_\mathcal{A}.$$

$$S_\mathcal{A}^\dagger = F_\mathcal{A} = J_\mathcal{A} \Delta_\mathcal{A}^{-\frac{1}{2}}$$

$$F_\mathcal{A}^\dagger = S_\mathcal{A}$$

Every element in the domain of $S_\mathcal{A}$ can be thought to be affiliated with $\mathcal{A}$, that is, for all $\nu \in \mathcal{D}(S_\mathcal{A})$, there exists a sequence $\{A_i\} \subset \mathcal{A}$ such that $A_i\Omega \to \nu$. Similarly for $F_\mathcal{A}$. More importantly,

$$\overline{\mathcal{D}\big(\Delta_\mathcal{A}^{\frac{1}{2}}\big) \cap \mathcal{D}\big(\Delta_\mathcal{A}^{-\frac{1}{2}}\big)} = \mathcal{H}_\omega$$

This acts as a convenient dense subset of $\mathcal{H}_\omega$ and allows us to study how an operator acts on $\mathcal{H}_\omega$ without actually worry about all of $\mathcal{H}_\omega$. It is now sufficient to study how $\mathcal{A}$ and $\mathcal{A}'$ act on this dense subset.

Making use of a Radon-Nikodym type theorem due to Sakai, we then related $\mathcal{A}$ and $\mathcal{A}'$ by studying how they act on the intersections of the domains $\mathcal{D}(S_\mathcal{A})$ and $\mathcal{D}(F_\mathcal{A})$. This relation is described by the following modular relation. For every $B \in \mathcal{A}'$ and $\lambda \in \mathbb{R}^+$, there exists $A \in \mathcal{A}$ such that

$$\lambda \big\langle \Delta_\mathcal{A}^{\frac{1}{2}} \nu | A \Delta_\mathcal{A}^{-\frac{1}{2}} \eta \big\rangle_\omega + \lambda^{-1} \big\langle \Delta_\mathcal{A}^{-\frac{1}{2}} \nu | A \Delta_\mathcal{A}^{\frac{1}{2}} \eta \big\rangle_\omega = \big\langle \nu | J_\mathcal{A} B J_\mathcal{A} \eta \big\rangle_\omega,$$

for all $\eta, \nu \in \mathcal{D}\big(\Delta_\mathcal{A}^{-1/2}\big) \cap \mathcal{D}\big(\Delta_\mathcal{A}^{1/2}\big)$.

Using the density of $\mathcal{D}(\Delta_\mathcal{A}^{1/2}) \cap \mathcal{D}(\Delta_\mathcal{A}^{-1/2})$ and some complex analytic tools we got rid of ν and η get an operator equation,

$$A = \int_\mathbb{R} \lambda^{2it} \Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}} \Big] dt.$$

By taking

$$f_{\nu,\eta}(t) = \Big\langle \eta \Big| \Big[\frac{\Delta_\mathcal{A}^{it} J_\mathcal{A} B J_\mathcal{A} \Delta_\mathcal{A}^{-it}}{e^{\pi t} + e^{-\pi t}} \Big] \nu \Big\rangle_\omega,$$

we observe that its Fourier transform is given by

$$\mathcal{F}f_{\nu,\eta} = \Big\langle \eta \Big| \Big[\int_{\mathbb{R}} \lambda^{2it} \Big[\frac{\Delta_{\mathcal{A}}^{it} J_{\mathcal{A}} B J_{\mathcal{A}} \Delta_{\mathcal{A}}^{-it}}{e^{\pi t} + e^{-\pi t}} \Big] dt \Big] \nu \Big\rangle_{\omega}.$$

Using the injectivity of Fourier transform; we obtained the main result of the Tomita-Takesaki modular theory, which is Tomita's theorem:

$$J_{\mathcal{A}} \mathcal{A} J_{\mathcal{A}} = \mathcal{A}', \;\; \Delta_{\mathcal{A}}^{it} \mathcal{A} \Delta_{\mathcal{A}}^{-it} = \mathcal{A}, \;\; \forall t \in \mathbb{R}.$$

The automorphism group,

$$\sigma_t^{\omega}(A) := \Delta_{\mathcal{A}}^{it} A \Delta_{\mathcal{A}}^{-it}$$

is called the modular automorphism group of $\mathcal{A}$ associated with the faithful normal state ω. The modular automorphism was constructed as a unitary group on the GNS Hilbert space $\mathcal{H}_{\omega}$, and it 'rotates' the Hilbert space keeping the Gelfand vector Ω fixed, and keeping the dense subset $\mathcal{A}\Omega$ within itself.

The modular group has been used as a tool for the analysis of operator algebras. We will now look into its use in the analysis of operator algebras in the quantum field theory setting. Before discussing the use of modular theory in the analysis of quantum field theories we discuss the so called KMS condition which shows that the modular group, a seemingly abstract mathematical object is naturally related to concrete physical phenomena.

2 | Half-Sided Modular Inclusions

Let $\mathcal{A}$ be a von Neumann algebra together with a faithful normal state ω. Then the von Neumann algebra can be assumed to be a concrete von Neumann algebra on the GNS Hilbert space $\mathcal{H}_\omega$ with a cyclic and separating vector Ω. In terms of the cyclic and separating vector Ω, the state ω is given by

$$\omega(A) = \langle \Omega | A\Omega \rangle_\omega.$$

If $\Delta_{\mathcal{A}}$ and $J_{\mathcal{A}}$ are the modular operator and modular conjugations respectively corresponding to the state ω, then by Tomita's theorem, $J_{\mathcal{A}}\mathcal{A}J_{\mathcal{A}} = \mathcal{A}'$, and $\Delta_{\mathcal{A}}^{it}\mathcal{A}\Delta_{\mathcal{A}}^{-it} = \mathcal{A}$, for all $t \in \mathbb{R}$.

We are interested in studying symmetries of quantum systems. The symmetries of the state correspond to strongly continuous unitary groups on $\mathcal{H}_\omega$, that keep ω invariant. If $U(s)$ is a strongly continuous one-parameter group unitaries, then by Stone's theorem there exists a self-adjoint operator P such that,

$$U : s \mapsto e^{isP} \;\; \forall s \in \mathbb{R}.$$

We are in particular interested in strongly continuous one-parameter groups generated by positive operators.

Let $\mathcal{A}$ be a von Neumann algebra, and Ω be a cyclic and separating vector. Let $U(s) = e^{isP}$ be a one-parameter group, where P is positive. If

$$U(s)\mathcal{A}U(s)^\dagger \subseteq \mathcal{A}, \;\; \forall s \in \mathbb{R}^+,$$

then the triple $(\mathcal{A}, U, \Omega)$ is called a Borchers triple. Note here that s being in $\mathbb{R}^+$ is important because if $U(s)\mathcal{A}U(s)^\dagger \subset \mathcal{A}$ for all $s \in \mathbb{R}$ together with $U(s)\Omega = \Omega$ it would imply that $U(s) = 1$ for all $s \in \mathbb{R}$.

2.1 | Borchers' Theorems

If $(\mathcal{A}, U, \Omega)$ is a causal Borchers triple, Borchers' theorem relates the one paramter groups of the Borchers triples and the modular automorphisms corresponding to the state ω. We will prove the theorem following [?].

Theorem 2.1. (Borchers) *Let $(\mathcal{A}, U, \Omega)$ be a Borchers triple. Then*

$$\Delta_{\mathcal{A}}^{it}U(s)\Delta_{\mathcal{A}}^{-it} = U(e^{-2\pi t}s),$$

$$J_{\mathcal{A}}U(s)J_{\mathcal{A}} = U(s)^\dagger$$

for all $s, t \in \mathbb{R}$.

Proof

Let $\nu \in \mathcal{A}\Omega$ and $\eta \in \mathcal{A}'\Omega$, consider the function

$$f^+_{\nu,\eta}(z) = \left\langle \left[\Delta_{\mathcal{A}}^{iz}U(e^{2\pi z}s)\Delta_{\mathcal{A}}^{-iz}\right]\nu \mid \eta \right\rangle_\omega, \quad s \in \mathbb{R}_+.$$

This is a bounded function whenever $\mathrm{Im}(e^{2\pi z}s)$ is positive,[2] which is the case on the sets

$$S(k, k+\tfrac{1}{2}) \equiv \{x+iy \mid y \in [k, k+\tfrac{1}{2}]\}$$

and hence $U(e^{2\pi z}s)$ is a bounded operator for all $z \in S(k, k+\frac{1}{2})$.

Since $U(s)\mathcal{A}U(s)^\dagger \subseteq \mathcal{A}$ for all $s \in \mathbb{R}_+$, it follows that for any $A \in \mathcal{A}$ and $B \in \mathcal{A}'$, $[U(s)^\dagger B U(s), A] = U(s)^\dagger [B, U(s)AU(s)^\dagger] U(s) = 0$. So we have $U(s)^\dagger \mathcal{A}' U(s) \subseteq \mathcal{A}'$, for all $s \in \mathbb{R}_+$. If we define

$$V(s) = J_{\mathcal{A}} U(s)^\dagger J_{\mathcal{A}}$$

By Tomita's theorem, $J_{\mathcal{A}}\mathcal{A}J_{\mathcal{A}} = \mathcal{A}'$, it follows that $V(s)\mathcal{A}V(s)^\dagger \subseteq \mathcal{A}$, for all $s \in \mathbb{R}_+$. So, $V(s)$ has similar properties as U. For similar reasons as above, the function,

$$f^-_{\nu,\eta}(z) = \left\langle \Delta^{iz}_{\mathcal{A}} V(e^{2\pi z}s) \Delta^{-iz}_{\mathcal{A}} \nu \mid \eta \right\rangle_\omega,$$

is bounded operator on the sets, $S(k-\frac{1}{2}, k) = \{x+iy \mid y \in [k-\frac{1}{2}, k]\}$ for all $s \in \mathbb{R}_+$.

We now show that we can define a well defined analytic function on the complex plain by taking these functions on small patches and then glue them up together. The idea is to make use of the Schwarz' reflection principle from complex analysis, which states that, if an analytic function is defined on the upper half-plane and has well-defined (non-singular) real values on the real axis, then it can be extended to the conjugate function on the lower half-plane.

To show that $f^+_{\nu,\eta}$ and $f^-_{\nu,\eta}$ glue up to give an analytic function we must verify that they agree on the common domains, that is, we have to verify that $f^+_{\nu,\eta}(t+i/2) = f^-_{\nu,\eta}(t)$. We have

$$\begin{aligned}
f^+_{\nu,\eta}(t+\tfrac{i}{2}) &= \left\langle \Delta^{it}_{\mathcal{A}} \Delta^{-\frac{1}{2}}_{\mathcal{A}} U(e^{2\pi t + i\pi}s) \Delta^{-it}_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} \nu \mid \eta \right\rangle_\omega \\
&= \left\langle \Delta^{it}_{\mathcal{A}} \Delta^{-\frac{1}{2}}_{\mathcal{A}} J^2_{\mathcal{A}} U(-e^{2\pi t}s) \Delta^{-it}_{\mathcal{A}} J^2_\omega \Delta^{\frac{1}{2}}_{\mathcal{A}} \nu \mid \eta \right\rangle_\omega \\
&= \left\langle \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} V(e^{2\pi t}s) \Delta^{-it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} \nu \mid \eta \right\rangle_\omega.
\end{aligned}$$

In the second step we used the fact that $J^2_\omega = I$, and in the third step we used $\Delta^{-1/2}_{\mathcal{A}} J_{\mathcal{A}} = J_{\mathcal{A}} \Delta^{1/2}_{\mathcal{A}}$, and $J_{\mathcal{A}} \Delta^{it}_{\mathcal{A}} = \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}}$, and substituted $J_{\mathcal{A}} U(s)^\dagger J_{\mathcal{A}} = V(s)$.

By Tomita's theorem, we have, $\Delta^{it}_{\mathcal{A}} A \Delta^{-it}_{\mathcal{A}} \in \mathcal{A}$ for every $A \in \mathcal{A}$. Hence we must have $V(e^{2\pi t}s) \Delta^{it}_{\mathcal{A}} A^\dagger \Delta^{-it}_{\mathcal{A}} V(e^{2\pi t}s)^\dagger \in \mathcal{A}$ and. Since Ω is invariant under the action of V and $\Delta_{\mathcal{A}}$, we have, $\Delta^{it}_{\mathcal{A}} V(e^{2\pi t}s)^\dagger \Omega = \Omega$. Plugging these into the above equation, we obtain,

$$\begin{aligned}
f^+_{\nu,\eta}(t+\tfrac{i}{2}) &= \left\langle \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} V(e^{2\pi t}s) \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} [A] \mid \eta \right\rangle_\omega \\
&= \left\langle \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} \big(V(e^{2\pi t}s) \Delta^{it}_{\mathcal{A}} A^\dagger \big) \Omega \mid \eta \right\rangle_\omega \\
&= \left\langle \Delta^{it}_{\mathcal{A}} J_{\mathcal{A}} \Delta^{\frac{1}{2}}_{\mathcal{A}} \big(V(e^{2\pi t}s) \Delta^{it}_{\mathcal{A}} A^\dagger \Delta^{it}_{\mathcal{A}} V(e^{2\pi t}s)^\dagger \big) \Omega \mid \eta \right\rangle_\omega \\
&= \left\langle \Delta^{it}_{\mathcal{A}} V(e^{2\pi t}s) \Delta^{-it}_{\mathcal{A}} A \Omega \mid \eta \right\rangle_\omega = f^-_{\nu,\eta}(t).
\end{aligned}$$

[2] The problematic part is $U(e^{2\pi z}s)$, by Stone's theorem, we can write $U(s) = e^{isP}$, and hence we have

$$\|U(e^{2\pi z}s)\| = \left\| e^{i\mathrm{Re}(e^{2\pi z}s)P} e^{-\mathrm{Im}(e^{2\pi z}s)P} \right\| \le \left\| e^{-\mathrm{Im}(e^{2\pi z}s)P} \right\|.$$

Here we used the fact that the first term in the product is a unitary operator whose norm must be 1. The second term is a bounded operator whenever $\mathrm{Im}(e^{2\pi z}s)$ is positive.

Similarly we also have,

$$f^-_{\nu,\eta}\big(t + \tfrac{i}{2}\big) = f^+_{\nu,\eta}(t).$$

Hence the function,

$$F_{\nu,\eta}(x+iy) = \begin{cases} f^+_{\nu,\eta}(x+iy) & \forall y \in \big[n, n+\frac{1}{2}\big] \\ f^-_{\nu,\eta}(x+iy) & \forall y \in \big[n-\frac{1}{2}, n\big] \end{cases}$$

is a bounded, analytic function on $\mathbb{C}$. By Liouville's theorem, $F_{\nu,\eta}$ must be a constant function. Hence we have, $F_{\nu,\eta}(0) = F_{\nu,\eta}(t)$. Since ν, η are arbitrary, and are dense inside $\mathcal{H}_\omega$, it follows that

$$\Delta^{it}_{\mathcal{A}} U(e^{2\pi t}s) \Delta^{-it}_{\mathcal{A}} = U(s)$$

or equivalently,

$$\Delta^{it}_{\mathcal{A}} U(s) \Delta^{-it}_{\mathcal{A}} = U(e^{-2\pi t}s)$$

Similarly, we must have $F_{\nu,\eta}(0) = F_{\nu,\eta}\big(\frac{i}{2}\big)$, which yields,

$$U(s) = J_{\mathcal{A}} U(-s) J_{\mathcal{A}}.$$

□

We now prove a convenient generalisation of the previous theorem. Let $\mathcal{B} \subset \mathcal{A}$ be a von Neumann subalgebra, ω is a faithful normal state for both $\mathcal{A}$ and $\mathcal{B}$, equivalently the Hilbert space $\mathcal{H}_\omega$ has a common cyclic and separating vector Ω, that is

$$\overline{\mathcal{B}\Omega} = \overline{\mathcal{A}\Omega} = \mathcal{H}_\omega.$$

We let $\Delta_{\mathcal{A}}$, $\Delta_{\mathcal{B}}$ and $J_{\mathcal{A}}$, $J_{\mathcal{B}}$ denote the modular operator and modular conjugation corresponding to $\mathcal{A}$ and $\mathcal{B}$ respectively. All unitary groups we will encounter will be assumed to be strongly continuous.

Theorem 2.2. (Borchers) *Let $U(s)$ with $U(s)\Omega = \Omega$ $\forall s \in \mathbb{R}$ be a unitary group with analytic extension on $S(0, \frac{1}{2})$, continuous on its boundary. If*

$$\begin{aligned} U(s)\mathcal{B}U(s)^\dagger &\subset \mathcal{A} & \forall s \in \mathbb{R} \\ U\big(s+\tfrac{i}{2}\big)\mathcal{B}'U\big(s+\tfrac{i}{2}\big)^\dagger &\subset \mathcal{A}' & \forall s \in \mathbb{R} \end{aligned}$$

then

$$\begin{aligned} \Delta^{it}_{\mathcal{A}} U(s) \Delta^{-it}_{\mathcal{B}} &= U(s-t) & \forall s,t \in \mathbb{R} \\ J_{\mathcal{A}} U(s) J_{\mathcal{B}} &= U\big(s+\tfrac{i}{2}\big) & \forall s \in \mathbb{R}. \end{aligned}$$

Proof

The idea of proof is the same as the proof of Borchers' theorem in the last section, with a lot more tedious manipulations. We will only sketch some of these manipulations and give the main ideas as to what we are trying to achieve with those manipulations. Let $\nu \in \mathcal{B}\Omega$ and $\eta \in \mathcal{A}'\Omega$. For a fixed $s \in \mathbb{R}$ define the two functions in the variable t by,

$$\begin{aligned} f^+_{\nu,\eta}(t) &= \Big\langle \big[\Delta^{it}_{\mathcal{A}} U(s+t) \Delta^{-it}_{\mathcal{B}}\big]\nu \mid \eta \Big\rangle_\omega, \\ f^-_{\nu,\eta}(t) &= \Big\langle \big[\Delta^{it}_{\mathcal{B}} U(s+t)^\dagger \Delta^{-it}_{\mathcal{A}}\big]\eta \mid \nu \Big\rangle_\omega. \end{aligned}$$

By assumption $\nu = [B] \in \mathcal{B}\Omega$ and $\eta = [A'^{\dagger}] \in \mathcal{A}'\Omega$ $B \in \mathcal{B}$ and $A' \in \mathcal{A}'$. So we have,

$$f^{+}_{\nu,\eta}(t) = \Big\langle \Big[\Delta^{it}_{\mathcal{A}} U(s+t) \Delta^{-it}_{\mathcal{B}} [B] \mid [A'^{\dagger}]\Big\rangle_{\omega} = \Big\langle A' \Big[\underbrace{\Delta^{it}_{\mathcal{A}} \Big[U(s+t)\big[\Delta^{-it}_{\mathcal{B}} B \Delta^{it}_{\mathcal{B}}\big] U(s+t)\Big] \Delta^{-it}_{\mathcal{A}}}_{\in\mathcal{A}} \Big] \Omega \mid \Omega \Big\rangle_{\omega}$$

Here we used the assumption that $U(s)\mathcal{B}U(s)^{\dagger} \subset \mathcal{A}$ for all $s \in \mathbb{R}$. This allows us to exchange elements of $\mathcal{A}$ and $\mathcal{A}'$ and it ensures that

$$f^{+}_{\nu,\eta}(t) = f^{-}_{\nu,\eta}(t).$$

Since $U(s)$ extends to an analytic function on the strip $S(0, \frac{1}{2})$, the function $f^{+}_{\nu,\eta}(t)$ posses bounded extensions into the strip $S(0, \frac{1}{2})$, and $f^{-}_{\nu,\eta}(t)$ into the strip $S(-\frac{1}{2}, 0)$.

$$\begin{aligned}
f^{+}_{\nu,\eta}(t + \tfrac{i}{2}) &= \Big\langle \Big[\Delta^{i(t+\frac{i}{2})}_{\mathcal{A}} U\big(s+t+\tfrac{i}{2}\big) \Delta^{-i(t+\frac{i}{2})}_{\mathcal{B}}\Big][B] \mid [A'^{\dagger}]\Big\rangle_{\omega} \\
&= \Big\langle \Big[\Delta^{-\frac{1}{2}}_{\mathcal{A}} \Delta^{it}_{\mathcal{A}} U\big(s+t+\tfrac{i}{2}\big) \Delta^{-it}_{\mathcal{B}} \Delta^{\frac{1}{2}}_{\mathcal{B}}\Big][B] \mid [A'^{\dagger}]\Big\rangle_{\omega} \\
&= \Big\langle \Big[J_{\mathcal{A}} \Delta^{it}_{\mathcal{A}} U\big(s+t+\tfrac{i}{2}\big) \Delta^{-it}_{\mathcal{B}} J_{\mathcal{B}}\Big][B^{\dagger}] \mid [A']\Big\rangle_{\omega} \\
&= \Big\langle \Big[A'^{\dagger} J_{\mathcal{A}} \Delta^{it}_{\mathcal{A}} U\big(s+t+\tfrac{i}{2}\big) \Delta^{-it}_{\mathcal{B}} J_{\mathcal{B}} B^{\dagger}\Big]\Omega \mid \Omega\Big\rangle_{\omega}.
\end{aligned}$$

Using $U(s+t+\frac{i}{2})^{\dagger}\Omega = \Omega$ and $\Delta_{\mathcal{A}}\Omega = \Delta_{\mathcal{B}}\Omega = \Omega$ we obtain that,

$$f^{+}_{\nu,\eta}(t+\tfrac{i}{2}) = \Big\langle \underbrace{\big[\,\Delta^{-it}_{\mathcal{A}} (\underbrace{J_{\mathcal{A}} A'^{\dagger} J_{\mathcal{A}}}_{\in\mathcal{A}}) \Delta^{it}_{\mathcal{A}}\,\big]}_{\in\mathcal{A}} \underbrace{\Big[\,U\big(s+t+\tfrac{i}{2}\big)\big[\Delta^{-it}_{\mathcal{B}} (\underbrace{J_{\mathcal{B}} B^{\dagger} J_{\mathcal{B}}}_{\in\mathcal{B}'}) \Delta^{it}_{\mathcal{B}}\big] U\big(s+t+\tfrac{i}{2}\big)\,\Big]}_{\in\mathcal{A}'} \Omega | \Omega \Big\rangle.$$

Here we used the assumption that $U(s+\frac{i}{2})\mathcal{B}'U(s+\frac{i}{2})^{\dagger} \subset \mathcal{A}'$. This allows us to exchange the two underbraced terms, which ensures that

$$f^{+}_{\nu,\eta}(t+\tfrac{i}{2}) = f^{-}_{\nu,\eta}(t-\tfrac{i}{2}).$$

Hence by Schwarz reflection principle we obtain an entire analytic function,

$$F_{\nu,\eta}(x+iy) = \begin{cases} f^{-}_{\nu,\eta}(x+iy) & \forall y \in \big[n-\frac{1}{2}, n\big] \\ f^{+}_{\nu,\eta}(x+iy) & \forall y \in \big[n, n+\frac{1}{2}\big] \end{cases}$$

Since the function is bounded by $\sup\{\|[B^{\dagger}]\|\|[A]\|, \|[A^{\dagger}]\|\|[B]\}$ by Liouville's theorem it must correspond to a constant function. Hence by evaluating $F_{\nu,\eta}$ using both $f^{+}_{\nu,\eta}$ and $f^{-}_{\nu,\eta}$ at $t \in \mathbb{R}$ and using the density of the choices ν and η it follows that

$$\Delta^{it}_{\mathcal{A}} U(s) \Delta^{-it}_{\mathcal{B}} = U(s-t), \quad \forall s,t \in \mathbb{R}.$$

Similarly, we must also have $f^{+}_{\nu,\eta}(\frac{i}{2}) = f^{-}_{\nu,\eta}\big(-\frac{i}{2}\big)$, and using $J_{\mathcal{A}}$ and $J_{\mathcal{B}}$ to flip the order in the inner product, we obtain

$$J_{\mathcal{A}} U(s) J_{\mathcal{B}} = U(s+\tfrac{i}{2}).$$

□

2.1.1 | Wiesbrock's Half-Sided Translations

We prove next a result due to Wiesbrock, [?], which constructs a strongly continuous one-parameter group of unitaries that satisfy the conditions of Borchers' theorem starting from a a given 'good' von Neumann subalgebra $\mathcal{B} \subset \mathcal{A}$.

The pair $(\mathcal{B} \subset \mathcal{A}, \Omega)$ is called a $\pm$half-sided modular inclusion ($\pm$HSMI) if

$$\Delta_{\mathcal{A}}^{-it} \mathcal{B} \Delta_{\mathcal{A}}^{it} \subset \mathcal{B}, \ \forall t \in \mathbb{R}_{\mp}.$$

Suppose $U(s)$ is a group of unitaries for $s \in \mathbb{R}$ satisfying the conditions of Borchers theorem, that is there exists $P \geq 0$ such that $U(s) = e^{isP}$ and $U(s)\Omega = \Omega$ such that

$$U(s)\mathcal{A}U(s)^\dagger \subset \mathcal{A}, \ \forall s \in \mathbb{R}^+.$$

Then by Borchers theorem we have

$$\Delta_{\mathcal{A}}^{it} U(s) \Delta_{\mathcal{A}}^{-it} = U(e^{-2\pi t} s), \ \forall s, t \in \mathbb{R}.$$

By Tomita's theorem we have $\Delta_{\mathcal{A}}^{it} \mathcal{A} \Delta_{\mathcal{A}}^{-it} = \mathcal{A}$. Combining the two we have

$$\begin{aligned}
\Delta_{\mathcal{A}}^{-it}\big(U(s)\mathcal{A}U(s)^\dagger\big)\Delta_{\mathcal{A}}^{it} &= (\Delta_{\mathcal{A}}^{-it} U(s) \Delta_{\mathcal{A}}^{it})\big(\Delta_{\mathcal{A}}^{-it} \mathcal{A} \Delta_{\mathcal{A}}^{it}\big)(\Delta_{\mathcal{A}}^{-it} U(s)^\dagger \Delta_{\mathcal{M}}^{it}) \\
&= U(e^{2\pi t} s)\big(\Delta_{\mathcal{A}}^{-it} \mathcal{A} \Delta_{\mathcal{A}}^{it}\big) U(-e^{2\pi t} s) \\
&= U(s)\big(U((e^{2\pi t} - 1)s)\mathcal{A}U((e^{2\pi t} - 1)s)^\dagger\big)U(s)^\dagger.
\end{aligned}$$

Since $(e^{2\pi t} - 1)s \geq 0 \ \forall s, t \in \mathbb{R}^+$ and by assumption $U(s)\mathcal{A}U(s)^\dagger \subset \mathcal{A}$, $\forall s \in \mathbb{R}^+$, we have

$$\Delta_{\mathcal{A}}^{it}\Big[U(s)\mathcal{A}U(s)^\dagger\Big]\Delta_{\mathcal{A}}^{-it} \subset \Big[U(s)\mathcal{A}U(s)^\dagger\Big], \ \forall s, t \in \mathbb{R}^+.$$

So, $(U(s)\mathcal{A}U(s)^\dagger \subset \mathcal{A}, \Omega)$ is a $-$half-sided modular inclusion for each $s \in \mathbb{R}^+$. So unitary groups satisfying conditions of Borchers theorem gives rise to half-sided modular inclusions. That is, whenever $(\mathcal{A}, U, \Omega)$ is a Borchers triple then $\forall s \in \mathbb{R}^+, (U(s)\mathcal{A}U(s)^\dagger \subset \mathcal{A}, \Omega)$ is a half-sided modular inclusion.

The following theorem due to Wiesbrock proves the converse is also true, and constructs a Borchers triple starting from a half-sided modular inclusion. This acts as a starting point to investigations attempting to recover the space-time symmetries from the vacuum state on the quasi-local algebra.

Theorem 2.3.

On the other hand we can also construct a strongly continuous one-parameter group of unitaries when we are given a $-$half-sided modular inclusion, $(\mathcal{B} \subset \mathcal{A}, \Omega)$ by taking

$$U(e^{2\pi t} - 1) = \Delta_{\mathcal{A}}^{-it} \Delta_{\mathcal{B}}^{it}, \ \forall t \in \mathbb{R}.$$

This guess is based on the above construction.

we have proved the following theorem,

Theorem 2.4. (Wiesbrock) *If $(\mathcal{B} \subset \mathcal{A}, \Omega)$ is a half-sided modular inclusion. Then,*

$$U(e^{2\pi t} - 1) = \Delta_{\mathcal{A}}^{-it} \Delta_{\mathcal{B}}^{it}, \ \ \forall t \in \mathbb{R}$$

satisfies the conditions of Borchers' theorem.

2.1.2 | Intermediate Algebras

The half-sided modular inclusions are peculiar subalgebras. One such peculiarity is that if $(\mathcal{B} \subset \mathcal{A}, \Omega)$ is a half-sided modular inclusion, there there exists no intermediate algebra $\mathcal{C}$, $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ that is a type I factor.

Theorem 2.5. *Let $(\mathcal{B} \subset \mathcal{A}, \Omega)$ be a $\pm$half-sided modular inclusion and let $\mathcal{C}$ be an intermediate von Neumann subalgebra ($\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$). Then $\Delta_{\mathcal{C}}^{it} \Delta_{\mathcal{A}}^{-it}$ converges strongly to an isometry $U = J_{\mathcal{C}} U J_{\mathcal{A}}$ as $t \to \mp\infty$.*

2.2 | Double Cones, Lightcones & Wedges

2.3 | The Bisognano-Wichmann Property

3 | Modular Nuclearity

3.1 | Collection of Dense Subsets

3.2 | Reconstruction of P

4 | Algebraic Characterisation of the Vacuum

4.1 | Reconstruction of Translations

4.1.1 | Buchholz-Summers Approach

4.1.2 | Geometric Modular Action