Part II

Representable Functors

The philosophy of Yoneda lemma and representable functors is to utilize the nice properties of the target category of a functors. Category of sets can be used to study locally small categories. So, instead of studying the categories themselves we study the functor category to sets which have nicer properties.

1 | Category of Functors

A set is a collection of 'elements'. A category $\mathcal{A}$ is more sophisticated, it possesses 'objects' similar to how sets posses elements, but for each pair of objects, X and Y in $\mathcal{A}$, there is a set of relations between X and Y, called morphisms, denoted by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. The Yoneda Lemma allows us to define an object by its relations to other objects.

A functor $\mathcal{F}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of a mapping of objects of $\mathcal{A}$ to objects of $\mathcal{B}$, $X \mapsto \mathcal{F}X$ together with a map of the set of homomorphisms,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y).$$

the image of $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ denoted by $\mathcal{F}(f)$. That takes identity to identity and respects composition[1] i.e.,

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

They are called covariant functors. A contravariant functor is a functor from the opposite category, and hence should satisfy,

$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Whenever we say functor, we assume it to be covariant functor. A contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ can be thought of as a covariant functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$. A functor $\mathcal{F}$ is faithful if the map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}Y)$ is injective for all X, Y. It's full if the map is surjective. If it's a bijection the functor is called fully faithful.

A natural transformation κ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, denoted by,

$$\kappa : \mathcal{F} \Rightarrow \mathcal{G},$$

is a collection of mappings κ_X for every $X \in \mathcal{A}$, such that for all $f : X \to Y$, the diagram,

$$\begin{tikzcd} X \arrow[d, "f"] & \mathcal{F}X \arrow[r, "\kappa_X"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}X \arrow[d, "\mathcal{G}(f)"] \\ Y & \mathcal{F}Y \arrow[r, "\kappa_Y"'] & \mathcal{G}Y \end{tikzcd} \qquad \text{(natural transformation)}$$

[1]The composition $f \circ g$ assumes they are composable.

commutes, i.e., it respects the new objects and morphisms and satisfies the composition law,

$$(\kappa \circ \varphi)_X = \kappa_X \circ \varphi_X$$

The collection of all natural transformation between two functors $\mathcal{F}$ and $\mathcal{G}$ is denoted by,

$$\mathrm{Hom}_{\mathcal{B}^{\mathcal{A}}}(\mathcal{F}, \mathcal{G}).$$

We say two functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphic or naturally equivalent if the natural transformation between them is a natural isomorphism, denoted as, $\mathcal{F} \cong \mathcal{G}$. The collections of all functors from $\mathcal{A}$ to $\mathcal{B}$ together with the natural transformations as the morphisms between functors is a category, denoted by $\mathcal{B}^{\mathcal{A}}$. The nice thing about functor category $\mathcal{B}^{\mathcal{A}}$ is that if $\mathcal{B}$ has some nice properties then $\mathcal{B}^{\mathcal{A}}$ inherits these useful properties.

Subobjects generalize the notion of subsets to categories. These should correspond to objects within the same category, that 'are inside' a given object. Let A be aset, we can say that some set S is 'inside' A if there exists an injective map from S to A, or more categorically speaking S is 'inside' A if there exists a monomorphism $i : S \to A$. But we also want to treat objects isomorphic to S to be a subobject.

A subobject is hence an equivalence class of monomorphisms $i : S \to A$ where the equivalence relation is defined by saying $i \sim j$ if there exists an isomorphism f such that $f \circ i = j$.

1.1 | Equivalence of Categories

Equivalence of two categories can be thought of as giving two complementary description of same mathematical object. We can compare two categories $\mathcal{A}$ and $\mathcal{B}$ via the functors between them. The starting point is the functor category $\mathcal{B}^{\mathcal{A}}$.

A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if there is a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that

$$\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}, \quad \text{and} \quad \mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}},$$

where the identity functor $\mathbb{1}_{\mathcal{A}}$ sends objects of $\mathcal{A}$ to the same objects, and morphisms to the same morphisms. $\mathcal{G}$ is called quasi-inverse functor. In such a case, $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent.

Lemma 1.1. *$\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories iff $\mathcal{F}$ is fully faithful and for every object $Y \in \mathcal{B}$ there exists an object X such that $\mathcal{F}X$ is isomorphic to Y.*

Proof

$\Rightarrow$ Suppose $\mathcal{F}$ is an equivalence of categories then there exists a functor $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}\mathcal{G} \cong \mathbb{1}_{\mathcal{B}}$ and $\mathcal{G}\mathcal{F} \cong \mathbb{1}_{\mathcal{A}}$. So, by this, there exists for each $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, isomorphisms,

$$\varphi_X : \mathcal{G}\mathcal{F}X \to X, \quad \varkappa_Y : \mathcal{F}\mathcal{G}Y \to Y.$$

So, each object $Y \in \mathcal{B}$ is isomorphic to the object $\mathcal{F}X$ where $X = \mathcal{G}Y$. To show that this is fully faithful, we have to show it gives homset isomorphism. Let $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$ then we have the following diagram which commutes,

$$\begin{array}{ccc} \mathcal{G}\mathcal{F}X & \xrightarrow{\varphi_X} & X \\ {\scriptstyle \mathcal{G}\mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{G}\mathcal{F}X' & \xrightarrow{\varphi_{X'}} & X' \end{array}$$

note here that φ_X is invertible. Hence we can construct f as,

$$f = \varphi_{X'} \circ \mathcal{GF}(f) \circ \varphi_X^{-1}$$

So, each f can be constructed from $\mathcal{F}(f)$. Given any map $g \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}X, \mathcal{F}X')$, set,

$$f = \varphi_{X'} \circ \mathcal{G}(g) \circ \varphi_X^{-1} \in \mathrm{Hom}_{\mathcal{A}}(X, X').$$

So we have $\mathcal{G}(g) = \mathcal{GF}(f)$ and this gives us a hom set isomorphism or that $\mathcal{F}$ is fully faithful.

$\Leftarrow$ Assuming to each $Y \in \mathcal{B}$ there corresponds $X_Y \in \mathcal{A}$ such that there exists an isomorphism $\varkappa_Y : \mathcal{F}X_Y \to Y$. We have to construct a quasi-inverse functor using these isomorphisms. Set $\mathcal{G}Y = X_Y$, and for each morphism $g \in \mathrm{Hom}_{\mathcal{B}}(Y, Y')$, set,

$$\mathcal{G}(g) = \varkappa_{Y'}^{-1} \circ g \circ \varkappa_Y$$

then we have $\mathcal{G}(g) \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{FG}Y, \mathcal{FG}Y')$ which is same as $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}Y, \mathcal{G}Y')$ because we had assumed $\mathcal{F}$ is fully faithful, i.e., the hom sets are isomorphic. It's easyto check that $\mathcal{G}$ is a functor and is a quasi-inverse to $\mathcal{F}$. $\square$

Quotient categories can be defined when we have an equivalence relation on the collection of morphisms. The objects remain the same, and the hom-sets get quotiented.

2 | Representable Functors

Many of the definitions and properties of algebraic objects can be expressed in categorical language. Representable functor define new properties using functors we understand well. Definitions are simpler to study and they inherit many interesting properties from nicely behaved categories such as the category of sets.

Each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ tells us about all the relations the object X has with other object Y. The thing we should be studying is the functor $h_X = \mathrm{Hom}_{\mathcal{A}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{A}}(-, X)$. These are called hom functors.

$$\begin{aligned} h^X :\ & \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets} \\ & Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X). \end{aligned}$$

which maps each morphism $f : Y \to Z$ to a morphism of hom sets given by the composition,

$$Y \xrightarrow{f} Z \xrightarrow{g} X$$

We will denote this by,

$$\begin{aligned} h^X(f) :\ & h^X(Y) \to h^X(Z) \\ & g \mapsto g \circ f. \end{aligned}$$

Similarly, we can define the contravariant hom functor. Note that we are assuming here that $\mathrm{Hom}_{\mathcal{A}}(Y, X)$s are all sets. Such categories are called locally small categories.

A contravariant functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ is called representable if for some $X \in \mathcal{A}$,

$$\mathcal{F} \cong h^X \tag{representable}$$

in such a case, $\mathcal{F}$ is said to be represented by the object X. We are especially interested in contravariant functors because they correspond to pre-sheaves. For covariant functors, $\mathcal{G} : \mathcal{A} \to \mathbf{Sets}$, this will be $\mathcal{G} \cong h_X$. Where $\cong$ stands for natural isomorphism.

2.1 | Yoneda Embedding

Yoneda embedding and representable functors allow us to use the nice properties (ability to take limits) of the category of sets to study more complex categories that are not so nice. We want to study the objects in terms of the maps to or from the object. This information is contained in the functors $\mathrm{Hom}_{\mathcal{A}}(-, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, -)$. Yoneda lemma establishes a connection between objects $X \in \mathcal{A}$ and the functor $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$.

Theorem 2.1. (Yoneda Lemma) *For a functor $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$ and any $X \in \mathcal{A}$, there is a natural bijection,*

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \cong \mathcal{F}X \tag{strong Yoneda}$$

such that $\kappa \in \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \leftrightarrow \kappa_X(\mathbb{1}_X) \in \mathcal{F}X$.

Proof

In the natural transformation diagram, replace $\mathcal{F}$ by h^X, and $\mathcal{G}$ by $\mathcal{F}$. $\kappa_X : h^X X \to \mathcal{F}X$. Now, $h^X X = \mathrm{Hom}_{\mathcal{A}}(X, X)$, which contains $\mathbb{1}_X$. Using this we construct a map,

$$\begin{aligned} \mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) &\to \mathcal{F}X \\ \kappa &\mapsto \kappa_X(\mathbb{1}_X). \end{aligned}$$

We have to now check that this is a bijection. We show this by showing κ is determined by $\mu(\kappa)$ for all $Y \in \mathcal{A}$. For any $f : Y \to X$, we have,

$$\begin{tikzcd} X & h^X X \arrow[r, "\kappa_X"] \arrow[d, "h^X(f)"'] & \mathcal{F}X \arrow[d, "\mathcal{F}(f)"] & \mathbb{1}_X \arrow[r, mapsto, "\kappa_X"] \arrow[d, mapsto] & \mu(\kappa) \arrow[d, mapsto] \\ Y \arrow[u, "f"] & h^X Y \arrow[r, "\kappa_Y"'] & \mathcal{F}Y & f \arrow[r, mapsto, "\kappa_Y"'] & \kappa_Y(f) \end{tikzcd}$$

Hence $\kappa_Y(f) = \mathcal{F}(f)(\mu(\kappa))$, or the action of κ_Y is determined by $\mu(\kappa)$. So, if $\mu(\kappa) = \mu(\varphi)$ then $\kappa_Y(f) = \varphi_Y(f)$ for all $Y \in \mathcal{A}$, so it's injective.

For surjectivity we have to show that for all sets $x \in \mathcal{F}X$, there exists a natural transformation φ such that $\varphi_X(\mathbb{1}_X) = x$. For $x \in \mathcal{F}X$, and $f : Y \to X$, construct the map,

$$\begin{aligned} \varphi : h^X &\to \mathcal{F} \\ f &\mapsto \mathcal{F}(f)(x). \end{aligned}$$

this satisfies the requirement that $\varphi_X(\mathbb{1}_X) = x$, because clearly, $\mathbb{1}_X \mapsto \mathcal{F}(\mathbb{1}_X)(x) = \mathbb{1}_x(x) = x$. We must make sure it's indeed a natural transformation, i.e., check if the naturality diagram,

$$\begin{tikzcd} Y & h^X Y \arrow[r, "\varphi_Y"] \arrow[d, "h^X(g)"'] & \mathcal{F}Y \arrow[d, "\mathcal{F}(g)"] \\ Z \arrow[u, "g"] & h^X Z \arrow[r, "\varphi_Z"'] & \mathcal{F}Z \end{tikzcd}$$

commutes for all $Y, Z \in \mathcal{A}$, $g \in \mathrm{Hom}_{\mathcal{A}}(Z, Y)$. For $f : Y \to X$, by definition of φ,

$$\mathcal{F}(g) \circ (\varphi_Y(f)) = \mathcal{F}(g) \circ \mathcal{F}(f)(x)$$

which by functoriality of $\mathcal{F}$ is $= \mathcal{F}(f \circ g)(x)$. On the other hand, by definition of the hom functor, we have,

$$\varphi_Z \circ (h_X(g)(f)) = \varphi_Z(h_X(f \circ g))$$

which again by definition of φ is $= \mathcal{F}(f \circ g)(x)$. Hence the diagram commutes, and φ is a natural transformation. The map $\mu : \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, \mathcal{F}) \to \mathcal{F}X$ is a bijection. $\square$

So, the information about objects is contained in their associated hom functors, for locally small categories. The proof covariant version is exactly the same, just have to reverse the arrows on the category $\mathcal{A}$. The Yoneda lemma gives us an embedding of the category $\mathcal{A}$ inside the functor category $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, given by,

$$X \mapsto h^X.$$

This embedding is called the Yoneda embedding $h^{(-)} : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, which sends an object $X \in \mathcal{A}$ to the sets of morphisms $\mathrm{Hom}_{\mathcal{A}}(-, X)$. These functors are fully faithful by Yoneda lemma, because by replacing the functor $\mathcal{F}$ by h^Y we have,

$$\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h^X, h^Y) \cong h^Y(X) = \mathrm{Hom}_{\mathcal{A}}(X, Y). \tag{weak Yoneda}$$

Similarly for the covariant embedding, in which case this will be $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{A}}(Y, X)$.

Given a contravariant functor, $\mathcal{F} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Sets}$, the strong Yoneda tells us that we can think of the action of $\mathcal{F}$ on the element X as natural transformations to the hom functor h^X in the functor category. So, every functor $\mathcal{F}$ can extended and be thought of as a representable functor,

$$\begin{aligned} h^{\mathcal{F}} : (\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}})^{\mathrm{op}} &\to \mathbf{Sets} \\ \mathcal{G} &\mapsto \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}}(\mathcal{G}, \mathcal{F}) \end{aligned}$$

where elements $X \in \mathcal{A}$ are to be interpreted as the elements $h^X \in \mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$. The following frequently used corollary of Yoneda lemma allows us to compare objects of locally small category by their hom-functors, and hence using the properties of the category of sets.

Lemma 2.2. (Yoneda Principle)

$$h^X \cong h^Y \Rightarrow X \cong Y.$$

$\square$

Note that, strong Yoneda associates to each set in $\mathcal{F}X$ a natural transformation between h^X and $\mathcal{F}$. If the functor $\mathcal{F}$ is representable, i.e., there exists $Y \in \mathcal{A}$ such that there exists a natural isomorphism,

$$\mathcal{F} \xrightarrow[\alpha]{\cong} h^Y$$

Let $\mu(\alpha)$ be the corresponding element in $\mathcal{F}Y = \mathrm{Hom}_{\mathcal{A}}(Y, Y)$. The pair $(Y, \mu(\alpha))$ is called a universal object for $\mathcal{F}$. It's such that for any other object $Z \in \mathcal{A}$, and each $g \in \mathcal{F}X = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ there exists a unique morphism $f : X \to Y$ such that,

$$\mathcal{F}(f)(\mu(\alpha)) = g.$$

2.2 | Group Objects

The notion of representable functor allows us to extend various algebraic notions to categories. The algebraic notion important for us is that of groups. An object $G \in \mathcal{A}$ is called a group object if there is a functor

$$\widehat{G} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Grps}$$

such that G is the representative for the functor $\circ\, \widehat{G}$. This means that the hom-functor,

$$\begin{aligned} h^G : \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ X &\mapsto \mathrm{Hom}_{\mathcal{A}}(X, G). \end{aligned}$$

decomposes into

$$\mathcal{A}^{\mathrm{op}} \to \mathbf{Grps} \to \mathbf{Sets}$$

References

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