Part IV

Runge's Theorem & Homology

1 | Runge's Theorem

Using the Taylor expansion, every analytic function on a disc can be approximated uniformly by polynomials in z on any smaller disc. Every entire function can be approximated by polynomials uniformly on every compact set. Runge's theorem generalizes this.

Let $\Omega \subseteq \mathbb{C}$ and let $K \subset \Omega$ be a compact subset. For any continuous function $f \in C(K)$, we define the norm of the function f on K by,

$$|f|_K = \sup_{z \in K} |f(z)|.$$

Using this we can define a topology on $\mathcal{H}(\Omega)$ by taking as neighborhoods the sets,

$$B_{\epsilon,K}(f) = \{g \in \mathcal{H}(\Omega) \mid |f - g|_K < \epsilon\}$$

With this topology, a sequence $\{f_n\}$ converges in $\mathcal{H}(\Omega)$ if and only if $\{f_n\}$ converges uniformly on any compact set in Ω. The topology is called compact open topology.

Theorem 1.1. **(Runge Approximation theorem)** *Let $K \subset \Omega$ be a compact subset. Then the following conditions on Ω and K are equivalent.*

- *Every function which is analytic in a neighborhood of K can be approximated uniformly on K by functions in $\mathcal{H}(\Omega)$.*

References

[1] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000