Part III

Quick Review of Measure Theory

These notes should serve as a quick review of measure theory which will be useful for the next part. We will skip the proofs involving limsups and liminfs and other set theoretic proofs. For important theorems proofs are sketched. The topics in this document will include measure spaces, Lebesgue and Lebesgue-Stieltjes measure, measurable maps, pushforwards, and integration.

1 | Measure Spaces

In this section we gather together the structures needed to measure 'sizes' of sets. Aim of this section is to identify the minimum structure needed to define such a measure. We want the measure of sets to be a positive number. i.e., $\mu(A) \in [0, \infty]$. The empty set should have zero measure, $\mu(\emptyset) = 0$. For disjoint sets, the total measure should be sum of measures of the individual sets.

$$\mu(\coprod_{i \in F} A_i) = \textstyle\sum_{i \in F} \mu(A_i),$$

where F is a finite set. If this holds for countable F then μ is called σ-additive.

One way to measure sets is to use geometric notions such as length of an interval to measure the sets. This as we will see in this section is not possible for all sets as there are certain pathological sets. Let $I_{\mathbb{R}}$ denote the set of all intervals of the form $(a, b] \subset \mathbb{R}$ where $a \leq b$. On this set, we can define the measure of the interval to be the length of the interval $\lambda : I_{\mathbb{R}} \to [0, \infty]$ defined by,

$$\lambda((a, b]) = b - a$$

It can be showed that the length of interval is σ-additive. Similarly we have the volume of cubes for higher dimensions. However the problem is that the length function cannot be defined for all subsets of $\mathbb{R}$ while also respecting translation invariance, i.e., $\mu(U + t) = \mu(U)$ for $t \in \mathbb{R}$. This was shown by Vitali, he constructed a set which if the measure satisfied the above rules would have both infinite measure and be a subset of a finite measure set.

Theorem 1.1. (Vitali's theorem) $\lambda : I_{\mathbb{R}} \to [0, \infty]$ *cannot be extended to* $\mathcal{P}(\mathbb{R})$.

So we can't extend the length function to all subsets so that it remains a measure. The aim is then to define a notion of measure coming from the length function to a large subset $\mathcal{B}(\mathbb{R})$ of $\mathcal{P}(\mathbb{R})$.

1.1 | σ-Algebras and Measures

σ-algebras act as the family of subsets of a set X that we can measure. So, what do we want to do? Our aim is to be able to define a measure on lots of subsets. So the definition of a σ-algebra is such that it has the nice behavior and leave out pathologies.

Let $\Sigma(X) \subset \mathcal{P}(X)$ be a collection of subsets of X that can be 'measured'. We expect the size of 'nothing' to be zero, so it should first be that the emptyset is measurable. Similarly the whole set X should be measurable, could have infinite measure but it must be measurable.

$$\emptyset \in \Sigma(X), \;\; X \in \Sigma(X). \tag{1S}$$

If A is measurable then we expect the size of A^c to be the size of X minus the size of A. So firstly, the set A^c is measurable.

$$A \in \Sigma(X), \;\text{ then } A^c \in \Sigma(X). \tag{2S}$$

If two sets $A \in \Sigma(X)$ and $B \in \Sigma(X)$ are measurable then we expect $A \cup B$ to be measurable and if they are disjoint the total size should be sum of the individual sizes,

$$A \cup B \in \Sigma(X) \tag{3S}$$

We can extend this to countable union as well while still having some nice behavior. Since our aim is to maximize the stuff we can measure we allow countable union also to be measurable.

$$\{A_i\}_{i\in\mathbb{N}} \subset \Sigma(X), \;\text{ then } \bigcup_{i\in\mathbb{N}} A_i \in \Sigma(X). \tag{4S}$$

$\mathcal{A} \subset \mathcal{P}(X)$ is an algebra if it satisfies conditions $(1S)$, $(2S)$, and $(3S)$. If an algebra $\Sigma(X) \subset \mathcal{P}(X)$ also satisfies the condition $(4S)$ it is called a σ-algebra. Clearly $\mathcal{P}(X)$ is a σ-algebra on X and $I_{\mathbb{R}}$ is not a σ-algebra. Immediate consequence of the definition is that all standard operations of sets such as union, intersection, difference, symmetric difference are also in $\Sigma(X)$.

$$A, B \in \Sigma(X), \;\;\; A \cup B, \; A \cap B, \; A \backslash B, \; A \Delta B \in \Sigma(X)$$

More importantly lim sup and lim inf also are measurable. Suppose $\Sigma_i(X) \subset \mathcal{P}(X)$ be σ-algebras, then it can be easily checked that $\Sigma(X) = \cap_i \Sigma_i(X)$ is also a σ-algebra. For any collection of subsets $S \subset \mathcal{P}(X)$, the σ-algebra generated by S, denoted by $\sigma(S)$ is the intersection of all σ-algebras containing S.

$$\sigma(S) = \bigcap_{S \subset \Sigma(X)} \Sigma(X).$$

It is hence the smallest σ-algebra containing S. Given a collection of subsets S, we can construct $\sigma(S)$ as follows, include all the elements of S, add all complements of elements of S, add X and $\emptyset$, add all countable unions and their complements. Since any σ-algebra containing S will have these sets it's the desired intersection. If $\Sigma(X)$ is a σ-algebra of X and $Y \subset X$ then $Y \cap \Sigma(X)$ is a σ-algebra of Y.

If (X, Γ) is a topological space where Γ is the collection of all open sets. The σ-algebra generated by the collection Γ is the smallest σ-algebra generated by Γ is called the Borel σ-algebra. Denoted by $\mathcal{B}(\Gamma)$, elements of $\mathcal{B}(\Gamma)$ are called Borel sets.

Theorem 1.2. *Let $f : Y \to X$ be a continuous map, then, $f^{-1}(\mathcal{B}(\Gamma)) = \mathcal{B}(f^{-1}(\Gamma))$.*

The proof of the theorem involves some basic set theory and we will not discuss it here. $\mathcal{B}(I_{\mathbb{R}})$ will be the σ-algebra where we will define the Lebesgue measure. Let $(\mathbb{R}, \Gamma_{\mathbb{R}})$ be the standard topology on $\mathbb{R}$. If $I_k^x = (x - 1/k, x]$ we get that $\{x\} = \cap_{k \in \mathbb{N}} I_i^x$. So $\{x\} \in \mathcal{B}(I_{\mathbb{R}})$. Hence we have,

$$\mathcal{B}(I_{\mathbb{R}}) = \mathcal{B}(\Gamma_{\mathbb{R}}).$$

Though $I_{\mathbb{R}}$ is not a σ-algebra, it has some nice properties. Open intervals, closed intervals, half closed intervals, also generate $\mathcal{B}(\mathbb{R})$. We are interested in generalising the length function on intervals.

The empty set can be taken to have zero length or the empty set belongs to $I = I_{\mathbb{R}}$.

$$\emptyset \in I \tag{1R}$$

Intersection of two intervals is again an interval, so we have,

$$A, B \in I_{\mathbb{R}} \implies A \cap B \in I \tag{2R}$$

An interval minus an interval is the disjoint union of two intervals. So in general we want

$$A \backslash B = \coprod_{i \in \{1, \ldots, n\}} C_i, \;\; C_i \in I. \tag{3R}$$

A subset $I \subset \mathcal{P}(X)$ is said to be a semi-ring if it satisfies the above three conditions. We can define 'pre-measures' on these semi-rings. If $\mathcal{I} \subseteq \mathcal{P}(X)$ is a semi-ring, and $A, B_1, \ldots B_m \in \mathcal{I}$, then by simple induction it can be showed that there exist $C_1, \ldots C_n \in \mathcal{I}$ disjoint, such that

$$A \backslash (\cup_{j=1}^m B_j) = \coprod_{i=1}^n C_i.$$

If $\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{H} \subseteq \mathcal{P}(Y)$ are two semi-rings, then it can be showed that $\mathcal{I} \times \mathcal{H}$ is a semi-ring. Proof is some set theory manipulations. We can now define pre-measure on semirings that captures the intuition we have about the length function on $I_{\mathbb{R}}$.

Definition 1.1. Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a semi-ring. A pre-measure on $\mathcal{I}$ is a map, $\mu : \mathcal{I} \to [0, \infty]$ with $\mu(\emptyset) = 0$ and μ is said to be σ-additive or a pre-measure if

$$\mu(\coprod_{i \geq 1} A_i) = \textstyle\sum_{i \geq 1} \mu(A_i).$$

μ is finitely additive or a content on $\mathcal{I}$, if, $\mu(\coprod_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. we say μ is a measure if μ is σ-additive and $\mathcal{I}$ is a σ-algebra.

A measure space is a triple $(X, \mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-algebra of the set X and μ is a measure on $\mathcal{A}$. $(X, \mathcal{A})$ is called a measurable space. The length function on $I_{\mathbb{R}}$ is a premeasure. It's called the Lebesgue-Borel pre-measure. Our aim is to extend the length pre-measure on $I_{\mathbb{R}}$ to $\mathcal{B}(\mathbb{R})$. Let $I \subseteq \mathcal{P}(X)$ be a semi-ring and μ is a pre-measure.

Suppose $A, B \in I$ with $B \subseteq A$, then $A = B \coprod (A \backslash B)$, but $A \backslash B = \coprod_{i=1}^n C_i$. So, we have,

$$\mu(A) = \mu(B) + \sum_{i=1}^n \mu(C_i) \geq \mu(B). \tag{monotone}$$

If $A \cup B \in I$ then we have, $A = (A \cap B) \cup (A \backslash B) = (A \cap B) \cup \coprod_{i=1}^{n} C_i$ and similarly $B = (A \cap B) \cup \coprod_{j=1}^{m} D_j$. So we have, $A \cup B = (A \cap B) \cup \coprod_{i=1}^{n} C_i \cup \coprod_{j=1}^{m} D_j$. So we get,

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B). \tag{parallelogram}$$

Let $\{A_i\}_{i \in \mathbb{N}} \subset I$ with $\cup_{i \in \mathbb{N}} A_i \in I$ then we can write $\cup_{i \in \mathbb{N}} A_i$ as a disjoint union by taking $A_i' = A_i \backslash (A_1 \cup \cdots \cup A_{i-1})$. Now each of these is a union of disjoint sets $A_i' = \coprod_{k=1}^{n} C_{k,i}$. So we have,

$$\mu(\cup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mu(A_i). \tag{σ-subadditivity}$$

Let $A_n \in I$ and $A_n \nearrow A$ then by taking $A_i' = A_i \backslash (A_1 \cup \cdots \cup A_{i-1}) = \coprod_{k=1}^{m_i} C_{k,i}$. Then A can be written as a countable union,

$$A = \bigcup_{i \in \mathbb{N}} A_i' = \bigcup_{i \in \mathbb{N}} \coprod_{k=1}^{m_i} C_{k,i}$$

Let $A_N = \bigcup_{n=1}^{N} \cup_{k=1}^{m_i} C_{k,i}$. This is a disjoint union and hence we have,

$$\mu(A) = \sum_{n \geq 1} \sum_{k=1}^{m_i} \mu(C_{k,i}) = \lim_{N \to \infty} \sum_{n=1}^{N} \sum_{k=1}^{m_i} \mu(C_{k,i}) = \lim_{N \to \infty} \mu(A_N)$$

$$\lim_{n \to \infty} \mu(A_n) = \mu(A) \tag{cont. from below}$$

Similarly continuity from above is proved with some similar argument.

2 | Solved Problems

1. Let μ^* be an outer measure on X, and $\mathcal{A} := \mathcal{A}(\mu^*)$ the Caratheodory σ-algebra of μ^*-measurable sets.

a) Show that a function $f : X \to \mathbb{R}$ is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable if and only if for all $Q \subseteq X$ and all $a \in \mathbb{R}$, we have

$$\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f > a)$$

$$\mathcal{A}(\mu^*) = \{A \mid \mu^*(A \cap Q) + \mu^*(A^c \cap Q) = \mu^*(Q)\ \forall Q \subset X\}.$$

[<=] f is $\mathcal{A}(\mu^*)$-$\mathcal{B}(\mathbb{R})$-measurable as soon as $f^{-1}(\mathcal{E}) \subseteq \mathcal{A}(\mu^*)$ because,

$$f^{-1}(\mathcal{B}(\mathbb{R})) = f^{-1}\sigma(\mathcal{E}) = \sigma(f^{-1}(\mathcal{E})) \subseteq \sigma(\mathcal{A}(\mu^*)) = \mathcal{A}(\mu^*).$$

Let $(a, b] \in \mathcal{E}$ = half open intervals of $\mathbb{R}$. Need to show, $f^{-1}((a, b]) \in \mathcal{A}(\mu^*)$, i.e., $\mu^*(Q) = \mu^*(Q \cap f^{-1}((a, b])) + \mu^*(Q \cap f^{-1}((a, b])^c)$ for all $Q \subseteq X$.

Let $A = f^{-1}((b, \infty))$, $B = f^{-1}((-\infty, a))$, and $C = f^{-1}((a, b])$. Assuming $\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f > a)$ we have,

$$\mu^*(Q) \geq \mu^*(Q \cap (B \cup C)) + \mu^*(Q \cap (B \cup C)^c) = \mu^*(Q \cap (B \cup C)) + \mu^*(Q \cap A).$$

(as A is the complement of the union of B and C)

$$\mu^*(Q \cap (B \cup C)) + \mu^*(Q \cap A) \geq \mu^*(Q \cap (A \cup B \cup C)) \qquad \text{(outer measure)}$$

So we have,

$$\mu^*(Q) = \mu^*(Q \cap f^{-1}((-\infty, b])) + \mu^*(Q \cap f^{-1}(b, \infty)).$$

that is, $f^{-1}((-\infty, b]) \in \mathcal{A}(\mu^*)$ for all b. So $f^{-1}((-\infty, b])^c \in \mathcal{A}(\mu^*)$ for all b. Hence we have,

$$f^{-1}((-\infty, b]) \cap f^{-1}((-\infty, a])^c = f^{-1}((a, b]) \in \mathcal{A}(\mu^*).$$

[=>] Let $O = (-\infty, a]$ and $f^{-1}(O) \in \mathcal{A}(\mu^*)$ and hence

$$\mu^*(Q) = \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f > a)$$

b) Show that a function $f : X \to \mathbb{R}$ is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable if and only if for all $Q \subseteq X$ and all $a < b \in \mathbb{R}$, we have

$$\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f \geq b)$$

[<=] If $\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f \geq b)$ for all $a < b$ we have, $\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f \geq a + \epsilon)$ for every $\epsilon > 0$. and hence,

$$\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f > a).$$

So it'll be measurable by previous part.

[=>] $(a, b) \in \mathcal{B}(\mathbb{R})$ and $f^{-1}((a, b)) \in \mathcal{A}(\mu^*)$. Let $A = f^{-1}([b, \infty))$, $B = f^{-1}((-\infty, a])$, and $C = f^{-1}((a, b))$.

$$\mu^*(Q) = \mu^*(Q \cap C) + \mu^*(Q \cap C^c) \geq \mu^*(Q \cap (A \cup B)).$$

Since μ^* is a measure it's restriction to $A \cup B$ is also a measure. Since A and B are disjoint we have,

$$\mu^*(Q \cap (A \cup B)) = \mu^*(Q \cap A) + \mu^*(Q \cup B).$$

or,

$$\mu^*(Q) \geq \mu^*(Q \cap f \leq a) + \mu^*(Q \cap f \geq b)$$

References