Part IV

Integration & Exterior Derivatives

To develop calculus, we need the ability to integrate on manifolds. As it turns out integration on manifolds is very closely related to differential forms.

1 | Integration on Manifolds

So, to start with we need a measure on the manifold. Since we expect the measure to respect the topology of the manifold, it should be a Borel measure. By Riesz duality theorem, measure on a locally compact Hausdorff space can be identified with a linear functional,

$$\mu : \mathcal{C}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{C}_c^{\mathcal{M}}$ with $f \geq 0$. Where $\mathcal{C}_c^{\mathcal{M}}$ are all compactly supported continuous functions. The positivity condition makes the functional continuous.

Differentiable functions with compact support form a self-adjoint subalgebra of this algebra of compactly supported continuous functions, and separate points of the space $\mathcal{M}$. Hence by Stone-Weierstrass theorem, the algebra $\mathcal{A}_c^{\mathcal{M}}$ of compactly supported differentiable functions is dense in $\mathcal{C}_c^{\mathcal{M}}$. Since by Hahn-Banach theorem, linear functionals on subalgebras uniquely extend to the whole algebra, we can study functionals,

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

such that $\mu(f) \geq 0$ for all $f \in \mathcal{A}_c^{\mathcal{M}}$ with $f \geq 0$.

So, by a Borel measure on a differentiable manifold $\mathcal{M}$ we mean a linear form μ on the vector space $\mathcal{A}_c^{\mathcal{M}}$ of differentiable functions with a compact support on $\mathcal{M}$ which satisfies certain continuity requirement i.e., for a sequence of compactly supported differentiable functions, $\{f_i\}$, with support contained in the compact set K, if $\sup\{|f_i|\} \xrightarrow[i\to\infty]{} 0$ then $\mu(f_i) \to 0$.

$$\mu : \mathcal{A}_c^{\mathcal{M}} \to \mathbb{R}$$

The scalar $\mu(f)$ is denoted by $\int f d\mu$. On the space of compactly supported differentiable functions, we can define the sup norm making it into a Banach space. We can then start doing functional analysis. The measures under our consideration will be continuous linear functionals on $\mathcal{A}_c^{\mathcal{M}}$.

1.1 | Differentiable Measures

Let $\varkappa : \mathcal{M} \to \mathcal{N}$ be a differentiable map. For any function $g \in \mathcal{A}_c^{\mathcal{N}}$, the composition, $g \circ \varkappa \in \mathcal{A}_c^{\mathcal{M}}$. The composition has compact support because the manifold is Hausdorff, and hence the inverse image of compact set is compact. The image measure can then defined by,

$$(\varkappa^*(\mu))(g) = \mu(g \circ \varkappa).$$

The continuity and linearity follow from continuity of differentiable functions.

Now with this composition, we can start defining the Lie derivative. Let $\varkappa_t$ be the flow of a vector field X, The Lie derivative of a measure μ with respect to the vector field X is the functional,

$$f \mapsto \lim_{t\to 0} \frac{((\varkappa_t^{-1})^*(\mu))(f)-\mu(f)}{t} = \mu\Big(\underbrace{\lim_{t\to 0} \frac{f\circ\varkappa_t^{-1}-f}{t}}_{-L_X(f)=-X(f)} \Big).$$

We needed the continuity of the measure to take the limit inside. So we have,

$$L_X(\mu)(f) = -\mu(X(f)). \qquad \text{(Lie derivative)}$$

This will be our notion of differentiation of measures. Multiple differentiations will be defined as multiple iterations of the Lie derivative of the measure. We say a Borel measure μ is indefinitely differentiable if the k times differentiations is a Borel measure for all k.

If μ is a measure and $h \in \mathcal{A}^{\mathcal{M}}$, then the map,

$$f \mapsto \mu(hf)$$

is a linear form on $\mathcal{A}_c^{\mathcal{M}}$ and satisfies the continuity requirement i.e., if $\sup\{|hf_n|\}$ tends to zero then so does $\mu(hf_n)$. We will denote this measure by $h \cdot \mu$. Together with this notion of multiplication, $\mathcal{B}^{\mathcal{M}}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

Using $(L_X\mu)(f) = -\mu(Xf)$, the Lie derivative of μ with respect to hX is,

$$(L_{hX}\mu)(f) = -\mu(hX(f))$$

by using $X(hf) = fX(h) + hX(f)$, this is $= -\mu(X(hf) - fX(h)) = -\mu(X(hf)) + X(h)\mu(f)$.

$$(L_{hX}\mu)(f) = \underbrace{(L_X\mu)(hf)}_{h\cdot L_X(\mu)(f)} + X(h)\mu(f)$$

Since $h \cdot \mu(f) = \mu(hf)$, we have, $-\mu(hX(f)) = h - \cdot\mu(X(f)) = h \cdot L_X(\mu)$. So,

$$L_X(h \cdot \mu) = L_{hX}\mu = h \cdot L_X(\mu) + L_X(h) \cdot \mu. \qquad \text{(Leibniz rule)}$$

Similarly by expanding we get that,

$$L_{[X,Y]}(\mu) = L_X L_Y(\mu) - L_Y L_X(\mu). \qquad \text{(Lie bracket)}$$

So, the behavior of differentiable measures under Lie derivative is similar to that of differential forms. We are interested in measures that are locally translation invariant. Let V be a vector space, and μ be a measure. It's said to be translation invariant if $\varkappa_t^*\mu = \mu$ for $\varkappa_t = tv$ or equivalently, $L_{\partial_v}\mu = 0$ for all v.

1.1.1 | The Sheaf of Differentiable Measures

We now start looking at the collection of all indefinitely differentiable measures. Let $U \subseteq V$, then we have the natural inclusion of compactly supported functions functions $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$, by setting the functions to be equal to zero outside U.

Let $\mathcal{B}^{\mathcal{M}}U$ be the set of all differentiable measures on U, we have used $\mathcal{B}$ here for Borel. The inclusion $U \subset V$ gives rise to a restriction map of differentiable measures $\mu \mapsto \mu|_V$. The action of $\mu|_U$ on $\mathcal{A}_c^{\mathcal{M}}U$, is given by the action of μ on $\mathcal{A}_c^{\mathcal{M}}U \subseteq \mathcal{A}_c^{\mathcal{M}}V$. So,

$$\mathcal{B}^{\mathcal{M}} : U \mapsto \mathcal{B}^{\mathcal{M}}U$$

is a presheaf. Now, to patch these measures up, we need the notion of partition of unity. This is an important tool. What we intend to do is restrict the domain of functions to some regions so we can forget about the behaviour of the function outside some region.

Let $\{U_i\}_{i\in I}$ be a locally finite open cover of a differentiable manifold $\mathcal{M}$. The locally finite open covers exist because manifolds are locally compact. A partition of unity with respect to the cover $\{U_i\}_{i\in I}$ is a family of smooth functions $\{\varphi_i\}_{i\in I}$ with values in $[0,1]$ such that

$$\textstyle\sum_{i\in I}\varphi_i = 1$$

with support of φ_i contained in U_i. Once we have such a partition of unity, we can study the function $(\sum_{i\in I}\varphi_i)f$ instead of the function f. To show the existence, let $\{V_i\}_{i\in I}$ be an open cover with $\overline{V_i} \subset U_i$, we can construct functions ψ_i that have support in U_i.[1] Since the cover is locally finite the sum makes sense.

$$\varphi_i = \psi_i / (\textstyle\sum_{i\in I}\psi_i)$$

then acts as a partition of unity. This allows us to study the functions using the charts.

Let $\{U_i\}$ be a locally finite familly of open sets and μ_i be Borel measure on them. Suppose $\mu_i|_{U_i\cap U_j} = \mu_i|_{U_i\cap U_j}$ for all i,j, then we can define a measure μ on $U = \cup\{U_i\}$ by multiplying any function $f \in \mathcal{A}_c^{\mathcal{M}}U$ with a partition of unity associated with $\{U_i\}$, and then define,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f).$$

Since $\{U_i\}$ is locally finite, the sum is welldefined. Now, suppose $f \in \mathcal{A}_c^{\mathcal{M}}U_i$, then $\mu(f) = \sum_i \mu_i(\varphi_i f)$, since the support of f is contained in U_i, we have for every $U_i \cap U_j$, $\mu_i|_{U_i\cap U_j} = \mu_j|_{U_i\cap U_j}$ and hence we have,

$$\mu(f) = \textstyle\sum_i \mu_i(\varphi_i f) = \sum_i \mu_j(\varphi_i f)$$

By linearity of measures this is

$$= \mu_j((\textstyle\sum_i \varphi_i)f) = \mu_j(f).$$

Hence the collation property holds, i.e., there exists an equilizer map e such that,

$$\mathcal{B}^{\mathcal{M}}U \overset{e}{\dashrightarrow} \prod_i \mathcal{B}^{\mathcal{M}}U_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(U_i \cap U_j).$$

Since the partition of unity is a differentiable map, the linear map μ is also continuous, and hence is a Borel measure. The differentiable measures on a differentiable manifold $\mathcal{M}$ is a sheaf of $\mathcal{A}^{\mathcal{M}}$-modules.

[1] Let $S_1 \subset S_2$ be concentric spheres in $\mathbb{R}^n$ centered at 0 then we need to show that there exist differentiable function which is zero outside S_2 and non zero everywhere inside S_1. If we take S_2 to be the unit ball, the function, $\Phi(x) = \exp\left(\frac{1}{\sum_i x_i^2 - 1}\right)$ for x in the unit ball and zero outside works.

1.2 | Differential Forms

Differential forms are closely related to 'oriented volume'. The multilinear maps of interest to us are alternating multilinear maps, alternating multilinear maps also carry with them information about orientation. Orientation makes the order of vectors important. To study such multilinear maps we can restrict ourselves to the study of exterior powers instead of studying the much larger tensor product. So the starting point is the cotangent pre-sheaf.

$$\mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends each open set U to $\mathcal{I}^{\mathcal{M}}U/(\mathcal{I}^{\mathcal{M}}U)^2$ which we will denote by $\mathcal{C}U$. We can consider the exterior algebra of this cotangent pre-sheaf.

$$\bigwedge^k \mathcal{C} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

which sends U to $\wedge^k \mathcal{C}U$. The elements of the stalks of this pre-sheaf will be called k-forms. We can now bundle these stalks and consider the sheaf of sections of this bundle.

$$(\bigwedge^k \mathcal{C})^{\mathrm{Sh}} : \mathcal{O}(\mathcal{M})^{\mathrm{op}} \to \mathbf{Sets},$$

This is a sheaf of vector spaces over $\mathbb{R}$ usually denoted by Ω^k. We can similarly consider the exterior algebra, which consists of the direct sum of all the exterior powers.

$$\Omega^\bullet = \bigoplus_{i=0}^{\infty} \Omega^k$$

Note that $\Omega^k = 0$ for $k > n$ where n is the dimension of the manifold and $\Omega^0 = \mathcal{A}^{\mathcal{M}}$. A differential k-form is a section of sheafification of kth exterior power of cotangent pre-sheaf. Equivalently it's an alternating $\mathcal{A}^{\mathcal{M}}$-multilinear form of degree k on the space of vector fields.

1.2.1 | Exterior Product

Although we haven't yet described the relation between differential forms and volumes, it's useful and use it to motivate other definitions involving differential forms. We want to be able to multiply two lengths and find out area. This is the idea of exterior product. Given two differential forms which intuitively measure some sort of length, we want to define an 'oriented area'. Let ω and κ be two differential forms. These give us a map,

$$\tau \mapsto (\omega(\tau), \kappa(\tau))$$

for each tangent vector τ. Now, we can define $(\omega \wedge \kappa)(\tau_1, \tau_2)$ to be the area of the parallelogram with sides $(\omega(\tau_1), \kappa(\tau_1))$ and $(\omega_1(\tau_2), \omega_2(\tau_2))$. Now, the area of the parallelogram is given by,

$$(\omega \wedge \kappa)(\tau_1, \tau_2) = \begin{vmatrix} \omega(\tau_1) & \kappa(\tau_1) \\ \omega(\tau_2) & \kappa(\tau_2) \end{vmatrix}.$$

This is called the exterior product of the differential forms ω and κ. We can generalize this to more general volumes. Let ω be a differential k-form and κ be differential l-form. The exterior product of two differential forms ω and κ is defined to be the differential form,

$$(\omega \wedge \kappa)(\tau_1, \ldots, \tau_{k+l}) = \sum_{\sigma} \epsilon_\sigma \omega(\tau_{\sigma(1)}, \ldots, \tau_{\sigma(k)}) \kappa(\tau_{\sigma(k+1)}, \ldots, \tau_{\sigma(k+l)}),$$

where σ is a partition[2] of the set $\{1, \dots k+l\}$ and $\epsilon_\sigma = (-1)^{\mathrm{sgn}(\sigma)}$ where $\mathrm{sgn}(\sigma)$ is the sign of the partition. Note that $\mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$. This is a bilinear map. The exterior algebra with the above product is a $\mathbb{Z}$-graded algebra. We can now list the basic properties of the exterior product.

Consider,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = \sum_\sigma \epsilon_\sigma \kappa(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)})\omega(\tau_{\sigma(k+1)}, \dots, \tau_{\sigma(k+l)}),$$

We can act this by a permutation, γ, such that, $(\gamma(1), \gamma(2), \dots, \gamma(k+l)) = (k+1, \dots, k+l, 1, \dots, k)$. The sign of this permutation is, $\mathrm{sgn}(\gamma) = (-1)^{kl}$. So we have,

$$(\kappa \wedge \omega)(\tau_1, \dots, \tau_{k+l}) = (-1)^{kl}(\omega \wedge \kappa)(\tau_1, \dots, \tau_{k+l}).$$

Some basic combinatorics argument shows us that,

$$(\omega \wedge \kappa) \wedge \xi = \omega \wedge (\kappa \wedge \xi)$$

At each point x, every cotangent vector can be written in terms of local coordinates φ as, $[f] = \sum_{i=1}^n \left[\frac{\partial f}{\partial x_i}(x)\right] dx_i$. where dx_i is the equivalence class corresponding to the function $\varphi_i(x) - x_i$. Since we expect differential forms to be smooth sections of the cotangent sheaf, every differential form can be written as,

$$\omega = \textstyle\sum_{i=1}^n a_i dx_i.$$

where $a_i \in \mathcal{A}^{\mathcal{M}}$. Similarly, differential k-forms can be written as,

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1, \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

where each $a_{i_1, \dots i_k} \in \mathcal{A}^{\mathcal{M}}$.

Using the tangency pairing,

$$\langle f, h \rangle_x = \left.\frac{d(f \circ h(t))}{dt}\right|_{t=0},$$

we can pair a differential form and a vector field pointwise which measures the length of X using the differential form ω at each point. This is called a contraction or interior product of ω with X. Denoted by

$$\iota_X \omega \coloneqq \langle \omega, X \rangle$$

This can also be extended to differential k-forms.

$$\iota_X \omega(X_1, \dots, X_{k-1}) \coloneqq \omega(X, X_1, \dots, X_{k-1}).$$

We can now start listing down the algebraic properties of the contraction. For exterior product of differential forms is given by,

$$\iota_X(\omega \wedge \kappa) = (\iota_X \omega) \wedge \kappa + (-1)^p \omega \wedge (\iota_X \kappa).$$

From the anti-symmetry of differential forms, we have, $\omega(X, Y, \dots) = -\omega(Y, X, \dots)$. So,

$$\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$$

The contraction provides a map,

$$\iota_X : \Omega^k \to \Omega^{k-1}.$$

For a differential k-form, and vector fields $X_1 \dots X_k$, we denote the evaluation by,

$$\langle \omega, (X_1 \dots X_k) \rangle = \omega(X_1 \dots X_k).$$

[2] a partition is a permutation such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

1.2.2 | Exterior Differentiation

Differential forms define at each point, the notion of length, area, volume, etc. Now we want to do calculus with them i.e., differentiate and integrate stuff. A k-form provides some sort of k-volume on tangent spaces. They are more complicated and Lie derivative doesn't describe how they change correctly. We want to define a notion of differentiation that captures all the ways in which it changes.

For a function $f \in \mathcal{A}^{\mathcal{M}}$, the differential is the flow of the 0-volume.

$$f : \mathcal{M} \to \mathbb{R}$$

gives us the map $df(x)$ of equivalence classes of curves, $\tau_h \mapsto \tau_{f \circ h}$. This is a map from $T_x\mathcal{M}$ to $\mathbb{R}$. Hence $df(x)$ is an element in the stalk of the cotangent pre-sheaf $\mathcal{C}$. Covectors can be thought of as assigning to each tangent vector its 'length'. Since this depends smoothly on the point x, it's a differential form i.e., $df \in \Omega^1$. Note here that this is the reason why the equivalence classes of functions $\varphi_i(x) - x_i$ were written as dx_i.

The definition of exterior derivative is not very intuitive. We try to motivate the definition of 'exterior derivative' of differential forms below. Although this motivation is not sufficient to characterize the definition, it can help understand what's happening.

For a differential form ω, and vector fields X and Y, we have the pairings $\langle \omega, X \rangle$ and $\langle \omega, y \rangle$. Each of the pairings are differentiable functions on $\mathcal{M}$ i.e.,

$$\langle \omega, X \rangle, \langle \omega, Y \rangle \in \mathcal{A}^{\mathcal{M}}.$$

We are interested in understanding how the function $\langle \omega, X \rangle$ changes along another vector field Y, and $\langle \omega, Y \rangle$ changes along X. The change of $\langle \omega, X \rangle$ along Y is given by the new pairing, $L_Y(\langle \omega, X \rangle) = \langle d(\langle \omega, X \rangle), Y \rangle$, so the difference,

$$L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle)$$

is a differential 2-form. We want to take into account all the changes that are happening, so we should also take into account how X changes with respect to Y, as measured by the differential form ω which is $\omega(L_X(Y))$. So, we define the exterior derivative as,

$$(d\omega)(X, Y) = L_X(\langle \omega, Y \rangle) - L_Y(\langle \omega, X \rangle) - \omega(L_X(Y)).$$

So, we have,

$$(d\omega)(X, Y) = X(\langle \omega, Y \rangle) - Y(\langle \omega, X \rangle) - \omega([X, Y]).$$

The negative signs are used so that this is a differential form. It's not yet clear why we have to take $\omega(L_X(Y))$ and not $\omega(L_Y(X))$, but we will not try to motivate that here. For a differential k-form, the exterior derivative is defined as,

$$\begin{aligned} (d\omega)(X_1 \dots X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1 \dots \widehat{X_i} \dots X_{k+1})) \\ &+ \sum_{1 \le j < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_{k+1}). \end{aligned} \tag{exterior derivative}$$

The explicit form of the exterior derivative in terms of local coordinates, is given by $d\omega = \sum \partial a_i / \partial x_i dx_i \wedge dx_I$, where $\omega = \sum a_i dx_I$. So, the exterior derivative is a map,

$$d : \Omega^k \to \Omega^{k+1}$$

We can now start listing all the properties of the exterior derivative. From the definition, it follows that the exterior derivative is linear,

$$d(\lambda\omega + \mu\kappa) = \lambda d(\omega) + \mu d(\kappa).$$

For a vector field X, we have,

$$\begin{aligned}(d\iota_X + \iota_X d)(\omega)(X_1 \dots X_k) = &\sum(-1)^{i+1} X_i(\iota_X \omega(X_1 \dots \widehat{X_i} \dots X_k)) \\ &+ \sum(-1)^{i+j}(\iota_X\omega)([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) \\ &+ (d\omega)(X, X_1 \dots \widehat{X_i} \dots X_j \dots X_k)\end{aligned}$$

which on expanding gives,

$$= X(\omega(X_1 \dots X_k)) + \sum(-1)^i \omega([X_i, X_j], X_1 \dots \widehat{X_i} \dots X_j \dots X_k) = (L_X(\omega))(X_1 \dots X_k)$$

This is called Cartan formula, and can be written compactly as,

$$d\iota_X + \iota_X d = L_X \qquad \text{(Cartan formula)}$$

Let ω be a differential k-form and κ be a differential l-form, it can be checked that,

$$d(\omega \wedge \kappa) = (d\omega) \wedge \kappa + (-1)^k \omega \wedge (d\kappa)$$

Such maps are called derivations of odd type. It can be showed that[3],

$$d \circ d = 0.$$

The last property is very important and allows us to study the topological properties of the manifolds by studying the differential forms. The proofs of all these are very simple in local coordinates.

1.3 | Invariant Forms vs. Invariant Measures

To motivate and make this relation precise, we start translation invariant measures on Euclidean space. The translational invariance makes sure that we can determine the measure of any measurable set if we know its value for some model set, say, a cube, because cubes generate the Borel σ-algebra, and the translational invariance allows us to measure any scaled copy of the cube.

1.3.1 | Motivating Example, $\mathbb{R}^n$

The translation invariant measures are determined uniquely upto a constant multiplication, and for Euclidean space it corresponds to the Lebesgue measure, upto scalar multiplication. So, for the vector field $X = \partial_v$ with the flow given by translations by $\varkappa_t = tv$, the Lie derivative $L_{\partial_v}\nu$ of the Lebesgue measure ν along ∂_v is given by,

$$L_{\partial_v}\nu(f) = \nu\big(\lim_{t\to 0} \tfrac{f\circ\varkappa_t^{-1} - f}{t}\big) = \nu\big(\tfrac{f-f}{t}\big) = 0.$$

Here we used the translational invariance. So, it exists and equals 0. Since we have,

$$(L_{hX}\mu)(f) = (L_X\mu)(hf) + X(h)\mu(f)$$

[3] Proof of this using the coordinate expression is very simple and only involves using the fact that $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$. In fact all computations are easier done in local coordinates.

The Lie derivative along any vector field $\sum_i h_i \partial_i$ exists and equals,

$$(L_{(\sum_i h_i \partial_i)}\nu)(f) = \sum_i \partial_i h_i \nu(f).$$

Hence Lebesgue measures are differentiable measures. Denote the set of all Lebesgue measures on $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. Lebesgue measures only vary by a scalar multiple, and are determined hence by their action on cubes. Each Lebesgue measure ν determines a map,

$$\begin{aligned} \widehat{\nu}:\ &\textstyle\prod^n \mathbb{R}^n \to \mathbb{R}. \\ &(v_i)_{i=1}^n \mapsto \nu([v_i]_{i=1}^n), \end{aligned}$$

where $[v_i]_{i=1}^n$ is the n-dimensional cube in $\mathbb{R}^n$ determined by the vectors $\{v_i\}$. $\nu([v_i]_{i=1}^n)$ is non-zero only if $\{v_i\}$ forms a basis of $\mathbb{R}^n$. Because otherwise, $[v_i]_{i=1}^n$ is a measure zero set, they are $< n$ dimensional sheets. The translation invariant, additivity, and continuity guarantee that,

$$\nu([v_1, \ldots rv_i, \ldots v_n]) = r\nu([v_i]_{i=1}^n).$$

So the map, $(v_i) \mapsto \nu([v_i])$ is multilinear in v_i, and since if any two v_is are equal we should have the measure to be zero, it's an alternating multilinear map and must factor through $\wedge^n \mathbb{R}^n$.

$$\begin{tikzcd} \prod^n \mathbb{R}^n \arrow[r, "i"] \arrow[dr, "\widehat{\nu}"'] & \bigwedge^n \mathbb{R}^n \arrow[d, "\exists!\ e_\nu"] \\ & \mathbb{R} \end{tikzcd}$$

So we have, $\nu([v_i]_{i=1}^n) = e_\nu(\wedge_{i=1}^n v_i)$. So, to each Lebesgue measure on $\mathbb{R}^n$ we have an associated differential n-form. Equivalently, we get a map from the space of invariant measures into the one-dimensional space $(\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}$.

$$\nu \mapsto e_\nu$$

The wedge product however is order sensitive, and the measure is not. So we should have,

$$e_\nu(\wedge^n v_i) = \pm\nu([v_i]_{i=1}^n).$$

This can be interpreted in the following sense, consider $(\wedge^n \mathbb{R}^n)^\vee \backslash \{0\} \cong \mathbb{R}\backslash\{0\}$, which has two connected components. The assignment of the sign 1 (respectively -1) in the definition is based upton whether $v_1 \wedge \cdots \wedge v_n$ belongs to the chosen component (or not). The choice of a connected component is equivalent to chosing a basis for $(\wedge^n \mathbb{R}^n)^\vee$. Such a choice of basis in $(\wedge^n \mathbb{R}^n)^\vee$ is called a volume element of $\mathbb{R}^n$, and such a volume element fixes the sign of Lebsgue measures. The map,

$$E:\ \mathcal{L}(\mathbb{R}^n) \to (\wedge^n \mathbb{R}^n)^\vee \cong \mathbb{R}, \quad \nu \mapsto e_\nu.$$

is an isomorphism of the space of Lebesgue measures $\mathcal{L}(\mathbb{R}^n)$ and the space of differential n-forms $(\wedge^n \mathbb{R}^n)^\vee$, and this isomorphism depends on the choice of basis for $(\wedge^n \mathbb{R}^n)^\vee$, and the two isomorphisms differ by constant multiple (-1). The choice of basis is called the orientation of the vector space $\mathbb{R}^n$. A Euclidean space has two orientations. corresponding to the choice. In this case,

$$\mathcal{K}^{\mathcal{M}} = \Omega^n \mathbb{R}^n = \mathcal{A}^{\mathbb{R}^n} \otimes_{\mathbb{R}} (\Lambda^n \mathbb{R}^n)^\vee$$

and similarly we can tensor the space of invariant measures with $\mathcal{A}^{\mathbb{R}^n}$ of differentiable functions on $\mathbb{R}^n$, this is a subsheaf of the differentiable measures, $\mathcal{B}^{\mathbb{R}^n}$ consisting of measures of the form $f \cdot \nu$ for $f \in \mathcal{A}^{\mathbb{R}^n}$ and $\nu \in \mathcal{L}(\mathbb{R}^n)$. So the isomorphism above yields an isomorphism of these sheaves. The elements of $\mathcal{K}^{\mathcal{M}}$ are called volume forms.

1.4 | Sheaf of Densities

Although invariant measures and differential n-forms are closely related in Euclidean space, it might not be the case in general differentiable manifold. The manifold might have twists which might make such an association that patches up nicely impossible. We want to study homomorphism from the sheaf of volume forms $\mathcal{K}^{\mathcal{M}}$ to the sheaf invariant measures in $\mathcal{B}^{\mathcal{M}}$.

1.4.1 | Orientation Sheaf

We now have two pre-sheaves, $\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}} \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}})$ of $\mathcal{A}^{\mathcal{M}}$-modules, for any $U \subset \mathcal{M}$, consider the new pre-sheaves, the restructions, $\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U \in \mathrm{PSh}(U, \mathcal{A}^{\mathcal{M}}|_U)$. We can now consider all the natural transformations between these pre-sheaves. This gives us an association,

$$U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathcal{A}^{\mathcal{M}}|_U)}(\mathcal{K}^{\mathcal{M}}|_U, \mathcal{B}^{\mathcal{M}}|_U).$$

Since the elements are natural transformations, the diagram,

$$\begin{tikzcd} U \arrow[d, "|_V"] & \mathcal{K}^{\mathcal{M}}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{K}^{\mathcal{M}}(|_V)"'] & \mathcal{B}^{\mathcal{M}}U \arrow[d, "\mathcal{B}^{\mathcal{M}}(|_V)"] \\ V & \mathcal{K}^{\mathcal{M}}V \arrow[r, "\kappa_V"'] & \mathcal{B}^{\mathcal{M}}V. \end{tikzcd}$$

commutes for each natural transformation κ for every $V \subset U$. Hence we have a restriction map for the natural transformations. Hence the association is a pre-sheaf itself. This is called the internal hom of $\mathcal{K}^{\mathcal{M}}$ and $\mathcal{B}^{\mathcal{M}}$, denoted by,

$$\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}) \in \mathrm{PSh}(\mathcal{M}, \mathcal{A}^{\mathcal{M}}).$$

Sometimes also written as $(\mathcal{B}^{\mathcal{M}})^{\mathcal{K}^{\mathcal{M}}}$. We will now show that the internal hom is also a sheaf, i.e., it satisfies the collation property,

$$\begin{tikzcd} \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U \arrow[r, dashed, "e"] & \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i,j} \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i \cap U_j). \end{tikzcd}$$

To show that $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf, we have to show the sequence is exact at $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U$ and at $\prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$. This means that we have to show that e is injective, and e is the co-equalizer for p and q.

Proposition 1.1. *If $\mathcal{B}^{\mathcal{M}}$ is a sheaf then so is $\mathcal{H}\mathrm{om}(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$.*

Proof

First, we have to show that e is injective. Let $\{U_i\}_{i\in I}$ be a cover of U. For every natural transformation $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})U$, and $U_i \subset U$, we have,

$$\begin{tikzcd} U \arrow[d, "|_{U_i}"] & \mathcal{K}^{\mathcal{M}}U \arrow[r, "\kappa_U"] \arrow[d, "\mathcal{K}^{\mathcal{M}}(|_{U_i})"'] & \mathcal{B}^{\mathcal{M}}U \arrow[d, "\mathcal{B}^{\mathcal{M}}(|_{U_i})"] \\ U_i & \mathcal{K}^{\mathcal{M}}U_i \arrow[r, "\kappa_{U_i}"'] & \mathcal{B}^{\mathcal{M}}U_i. \end{tikzcd}$$

Suppose $\kappa \in \ker(e)$, then $e(\kappa) = \prod_i \kappa|_{U_i} = 0$. So, for any $U_i \in \{U_i\}$, $\kappa|_{U_i} = 0$. This means every section of $f \in \mathcal{K}^{\mathcal{M}}U_i$ is mapped by κ to zero.

$$\kappa(f)|_{U_i} = 0.$$

For any $V \subset U$, we have on the intersection,

$$\kappa(f)|_{U_i \cap V} = 0.$$

Now, $\{V \cap U_i\}$ is a cover of V, and $\mathcal{B}^{\mathcal{M}}V \ni \kappa(f) = 0$. So, κ must be zero.

Now to show that e is the equaliser of p and q, i.e., given $(\kappa_i)_{i \in I} \in \prod_i \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U_i)$ which agrees on intersection, i.e.,

$$\kappa_i|_{U_i \cap U_j} = \kappa_j|_{U_i \cap U_j},$$

we have to show there exists a section, $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ such that $\kappa|_{U_i} = \kappa_i$. Now, we use the fact that $\mathcal{B}^{\mathcal{M}}$ is a sheaf to patch these natural transformations.

Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, we have for all $V \subset U$,

$$\mathcal{K}^{\mathcal{M}}V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

here the first map comes from the natural transformation, $\mathcal{K}^{\mathcal{M}}V \ni f \mapsto \kappa_i(f|_{V \prod U_i})$. Since $\mathcal{B}^{\mathcal{M}}$ is a sheaf, this must uniquely factor through $\mathcal{B}^{\mathcal{M}}V$, by definition of equaliser. Hence, we have,

$$\mathcal{K}^{\mathcal{M}}V \xrightarrow{\exists !} \mathcal{B}^{\mathcal{M}}V \xrightarrow{\kappa_V} \prod_i \mathcal{B}^{\mathcal{M}}(V \cap U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} \mathcal{B}^{\mathcal{M}}(V \cap (U_i \cap U_j)).$$

Let this unique map be κ_V, then clearly we have, $V \mapsto \kappa_V \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})(U)$ defines the patched up element that equalizes the diagram, and hence the internal hom is a sheaf whenever $\mathcal{B}^{\mathcal{M}}$ is a sheaf. $\square$

The natural transformations $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ are $\mathcal{A}^{\mathcal{M}}$-module homomorphisms. In particular, $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is a sheaf of $\mathbb{R}$-modules. However not all of these are important to us. We are interested in those which map invariant volume forms to invariant measures. Since the information about the measure being invariant has to do with Lie derivatives, we just have to preserve that structure, i.e, performing Lie derivation before the homomorphism should be the same as taking Lie derivation after the homomorphism.

Hence, the natural transformation of interest to us should preserve the Lie derivative. $\kappa \in \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$ is said to be flat if

$$\kappa L_X = L_X \kappa. \tag{flat}$$

The collection of all flat homomorphisms is denoted by $OR_{\mathcal{M}}$.

Flat homomorphisms take invariant forms to invariant measures. The set of all flat homomorphisms $\mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules and is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$. Flat homomorphisms do exist when $\mathcal{M} = \mathbb{R}^n$, as described in 1.3.1.

Proposition 1.2. *If $\mathcal{M}$ is connected, κ, φ flat, then there exists $\lambda \in \mathbb{R}$ such that $\kappa = \lambda\varphi$.*

Proof

It's enough to describe the action on invariant forms. Let ω be an invariant form, i.e., in local coordinates, $L_{\partial_v}\omega = 0$. Since κ, φ are flat homomorphisms, $\kappa(\omega)$ and $\varphi(\omega)$ are invariant measures, because $\kappa L_X = L_X\kappa$, $\varphi L_X = L_X\varphi$, and κ, φ are homomorphisms,

$$L_{\partial_v}(\kappa(\omega)) = L_{\partial_v}(\varphi(\omega)) = 0.$$

Locally, these invariant measures are Lebesgue measures and hence must vary by a constant multiple. On intersections, this constant is preserved. Since the manifold is connected, there can't be any abrupt change to this constant multiple. So,

$$\kappa(\omega) = \lambda(\varphi(\omega))$$

for some $\lambda \in \mathbb{R}$. $\square$

The pre-sheaf of flat homomorphisms,

$$\begin{aligned} OR_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathbb{R}}\mathrm{Mod} \\ U &\mapsto OR_{\mathcal{M}}(U), \end{aligned}$$

where $OR_{\mathcal{M}}(U)$ is the collection of all flat homomorphisms, $\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$ is a sheaf of $\mathbb{R}$-modules. $OR_{\mathcal{M}}$ is a subsheaf of $\mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$,

$$OR_{\mathcal{M}} \rightarrowtail \mathcal{H}om(\mathcal{K}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}})$$

Since in local coordinates each flat homomorphism varies from a standard flat homomorphism by a constant real multiple, it's a locally constant sheaf of $\mathbb{R}$-modules of rank 1.

The existence of global sections of $OR_{\mathcal{M}}$, i.e., global flat homomorphisms depends geometrically on whether the manifold has 'twists' or not. To intuitively motivate this twisting, consider for example, the case of a mobius strip. We can choose local coordinates around a point $x \in \mathcal{M}$, if there existed a flat homomorphism, then it should patch up nicely with local restrictions, however, when we go around the strip and return, the order of the basis we had chosen will be reversed. So, although we have local flat homomorphisms that agree on intersections, there doesn't exist a global patch up of them. The only possible flat homomorphism is the trivial homomorphism, which sends everything to zero. This twisting can hence be made precise in terms of the orientation sheaf.

Since flat homomorphisms vary by constant multiples, we can just consider equivalence classes. Choose in a local chart a choice of ordered basis, this determines locally, a standard flat homomorphism

$$\kappa : \mathcal{K}^{\mathcal{M}}|_U \to \mathcal{B}^{\mathcal{M}}|_U$$

Now, we can consider all flat homomorphisms that vary by an integral multiple of κ, i.e., flat homomorphisms of the type $\lambda \cdot \kappa$ with $\lambda \in \mathbb{Z}$. This collection is a locally constant sheaf, denoted by $OZ_{\mathcal{M}}$, and the sheaf is called the local system of 'twisted integers'.

A connected manifold $\mathcal{M}$ is called oriented if this is the constant sheaf. Equivalently, we say $\mathcal{M}$ is oriented if the étale space of the sheaf of twisted integers, $OZ_{\mathcal{M}}$, is $\mathcal{M} \times \mathbb{Z}$. In such a case, there are two trivializations, and each of which is called an orientation on $\mathcal{M}$. Clearly,

$$OR_{\mathcal{M}} = OZ_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathbb{R}$$

The existence of a flat homomorphisms means that at each point in the manifold, we can associate an invariant measure, and these invariant measures patch up nicely. Assuming a flat homomorphism $\kappa : \mathcal{K}^{\mathcal{M}} \to \mathcal{B}^{\mathcal{M}}$ exists, the sheaf of densities is the image $\mathcal{S}_{\mathcal{M}} \coloneqq \kappa(\mathcal{K}^{\mathcal{M}}) \subseteq \mathcal{B}^{\mathcal{M}}$.

$$\begin{aligned} \mathcal{S}_{\mathcal{M}} :\ \mathcal{O}(\mathcal{M})^{\mathrm{op}} &\to {}_{\mathbb{R}}\mathrm{Mod} \\ U &\mapsto \kappa(\mathcal{K}^{\mathcal{M}}|_U), \end{aligned}$$

where $\kappa(\mathcal{K}^{\mathcal{M}}|_U) = \{\kappa(\omega) \mid \omega \in \mathcal{K}^{\mathcal{M}}|_U\}$.

$$\mathcal{K}^{\mathcal{M}} \otimes_{\mathbb{Z}} OZ_{\mathcal{M}} \cong \mathcal{S}_{\mathcal{M}}$$

On $\mathbb{R}^n$, $\mathcal{S}_{\mathbb{R}^n}$ consists of all measures of the form, $f d\mu$ where μ is the Lebesgue measure on $\mathbb{R}^n$. So, in the case of a differential manifold, by local isomorphism of the sheaf of differentiable functions, in any coordinate system, these measures can be expressed as $f d\mu$. Of course these measures are not invariant.

1.4.2 | Pullback of Sheaf of Densities

Now that we have the sheaf of densities, we can study the behavior of the sheaf under diffeomorphisms. We are interested in how the elements of the sheaf change, and get a local formula for this change in terms of coordinates. So, it's good enough to restrict to diffeomorphisms between domains in $\mathbb{R}^n$.

Consider a diffeomorphism $\varphi : U \to V$ of open sets of $\mathbb{R}^n$. We say φ is compatible with the orientation if we choose an orientation, i.e., a choice of an ordered basis, which determines, a standard flat homomorphism of U and V, inherited from $\mathbb{R}^n$, then φ doesn't change the orientation.

Theorem 1.3. (Change of Variable Formula) *φ be a diffeomorphism from an open set $U \subseteq \mathbb{R}^n$ to an open set V, then for any $f \in \mathcal{A}^{\mathcal{M}}V$ with compact support,*

$$\int f(y) d\mu(y) = \int f(\varphi(x)) \big| \det(d\varphi) \big| d\mu(x).$$

Proof

Using the isomorphism

$$\mathcal{K}^{\mathcal{M}} \otimes_{\mathbb{Z}} OZ_{\mathcal{M}} \cong \mathcal{S}_{\mathcal{M}}$$

we can compute the induced map on $\mathcal{S}_{\mathcal{M}}$ by φ by its action on $\mathcal{K}^{\mathcal{M}}$ and $OZ_{\mathcal{M}}$.

1.5 | Adjoint of Differential Operators

The philosophy of differential geometry is to try to study a given topological space using a certain class of functions on them, the differentiable functions. We need to be able to manipulate differentiable functions to extract out the properties of the topological space, these manipulations are the differential operators.

Just like the rest of mathematics it's useful to first understand the linear case better. To understand the structure of linear differential operators, we should understand the standard linear algebraic operations on the linear differential operators. In this section, the focus will be on the adjoint of a linear differential operator.

Differential operators are local, in the sense that they form a sheaf. So we should expect the adjoint operation on the differential operator to be also a local operation. In order to describe the notion of adjoint of a linear differential operator completely locally, we should not rely on global properties such as inner products on the involved vector bundles nor metric/orientation on the base manifold. So, we start with the notion of vector valued densities on a differential manifold.

In order to define adjoints of linear operators we need a notion of inner product on the vector space. This inner product on the vector space of differential functions can be obtained from the notion of intergration on differentiable manifolds.

Let $\mathcal{E}$ be a locally free sheaf corresponding to a vector bundle. An $\mathcal{E}$ valued density is a section of the sheaf

$$\mathcal{E} \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{K}^{\mathcal{M}} \otimes_{\mathbb{R}} OR_{\mathcal{M}} \cong \mathcal{E} \otimes_{\mathcal{A}^{\mathcal{M}}} \mathcal{S}^{\mathcal{M}}$$

the manipulations we do

1.5.1 | Stokes Theorem

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