Part I

Holomorphic Functions

Main aim of this part is to show holomorphic functions are analytic. Topics for these notes include, holomorphic functions, analytic functions, integration along paths, Cauchy's theorem, Cauchy integral formula, Liouville theorem, fundamental theorem of algebra, the maximum modulus principle, and Morera's theorem.

1 | Holomorphic Functions

We are interested in differentiating functions defined on some open set $\Omega \subset \mathbb{C}$. A function f defined on Ω is complex differentiable at a point $z \in \Omega$ if it's differentiable and the differential is complex linear i.e., there exists a complex linear function $df(z)$ such that,

$$\frac{f(z+h) - f(z)}{h} \xrightarrow[h \to 0]{} df(z)(h), \tag{1D}$$

The function f can be approximated infinitesimally by a complex linear function. If f is complex differentiable on Ω it's said to be holomorphic on Ω and the derivative is defined as,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

If f is holomorphic on all of $\mathbb{C}$ it's called entire.

If the function f is thought of as a map from Ω as an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, then f can be written as, $f = u + iv$. The condition of complex differentiability of f at $z \in \Omega$ can then be split into two requirements, real differentiable and complex linearity of the differential. This means that the differential is a real linear map,

$$df(z) : \mathbb{R}^2 \to \mathbb{R}^2.$$

The matrix form of $df(z)$ is called the Jacobian and in terms of partial derivatives in the standard basis, it is given by,

$$J_f(z) = (\partial_j f_i(z))_{i,j}$$

A real linear map T is complex linear if $T(i) = iT(1)$. This condition puts the required constraint. So the Jacobian matrix $J_f(z)$ will be of the form,

$$J_f(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{bmatrix} \partial_x u(z) & \partial_y u(z) \\ \partial_x v(z) & \partial_y v(z) \end{bmatrix}. \tag{2D}$$

This is called the Cauchy-Riemann equation. The set of all holomorphic functions on Ω is denoted by $\mathcal{H}(\Omega)$. If f, and g are two complex differentiable functions on Ω and $\lambda \in \mathbb{C}$ then it follows directly from definition that, $f+g$, $f \cdot g$, and $\lambda \cdot f$ are also complex differentiable. So, $\mathcal{H}(\Omega)$ is an algebra.

If f is a complex differentiable function on Ω and g is a complex differentiable function on V then the composition map $g \circ f$ is a complex differentiable function on Ω and,

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

which is the chain rule.

Clearly, all polynomials are holomorphic functions. We can generalise this to all power series within their disk of convergence. A formal power series is a series $\sum_{n\geq 1} a_n z^n$, denoted by $\mathbb{C}[z]$. Abel's theorem says that every power series has a radius of convergence, i.e., there exists $R > 0$ such that for any z in the ball of radius R the power series converges. The radius of convergence

$$R = \lim_{n\to\infty} \left|\tfrac{a_n}{a_{n+1}}\right|.$$

A function $f : \Omega \to \mathbb{C}$ is said to be analytic if f is represented by a convergent power series expansion on a neighborhood around every point Ω.

The set of all analytic functions on Ω is denoted by $\mathcal{A}(\Omega)$. Clearly this set is an algebra. The main goal of this part is to prove that holomorphic functions and analytic functions are the same.

Theorem 1.1.

$$\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega).$$

Proof

Let $f \in \mathcal{A}(\Omega)$, without loss of generality assume the power series expansion of f is

$$f(z) = \sum_{n\geq 0} a_n z^n$$

for all $|z| < R$. Write, $f(z+h) = f(z) + g(z)h + r(z,h)$, where $g(z) = \sum_{n\geq 1} n a_n z^{n-1}$. By the radius of convergence formula, the radius of convergence of $g(z)$ is the same as $f(z)$. We have,

$$r(z,h) = \textstyle\sum_{n\geq 0} a_n B_n(z,h),$$

where $B_n(z,h) = (z+h)^n - z^n - nz^{n-1}h$ and $B_0 = B_1 = 0$. Let $|z| + |h| < r_1 < r_2 < R$, we have, $|a_n| \leq M/r_2^n$ for some $M < \infty$. So we have,

$$\begin{aligned} |B_n(z,h)| \leq |h|^2 \textstyle\sum_{k=0}^{n-2} \binom{n}{k+2} |h|^k |z|^{n-2-k} &\leq |h|^2 \textstyle\sum_{k=0}^{n-2} n^2 \binom{n-2}{k} |h|^k |z|^{n-2-k} \\ &= |h|^2 n^2 (|z| + |h|)^{n-2}. \end{aligned}$$

Hence,

$$|r(z,h)| \leq \textstyle\sum_{n\geq 2} \frac{M}{r_2^n} n^2 (|z|+|h|)^{n-2} \leq \frac{|h|^2 M}{r_2^2} \sum_{n\geq 0} (n+2)^2 \left(\frac{r_1}{r_2}\right)^n.$$

or, for some constant C, we have,

$$|r(z,h)| \leq C|h|^2.$$

or that $g(z) = f'(z)$ for $|z| < R$ because for $h \to 0$ we have $f(z+h) - f(z) \approx g(z)h$. So f is holomorphic. $\square$

To prove the other inclusion, i.e., $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$ we need more tools and complex integration. Let $\eta : [a, b] \to \mathbb{C}$ be a smooth curve. Let f be a continuous function defined atleast on the compact image $\eta([a, b])$. The path integral of f along η is defined by,

$$\int_\eta f(z)dz = \int_{[a,b]} f(\eta(t))\eta'(t)dt$$

since $(f \circ \eta) \cdot \eta'$ is continuous on $[a, b]$ the integral is well-defined. For a piecewise C^1-path the integral along $\eta = \eta_1 + \cdots + \eta_n$ is defined by, $\int_\eta f(z)dz = \sum_{i=1}^n \int_{\eta_i} f(z)dz$.

Suppose we have a reparametrization of the interval $[a, b]$, given by the C^1-map $\varphi : [a', b'] \to [a, b]$ then we have,

$$\int_{\eta\circ\varphi} f(z)dz = \int_{[a',b']} f(\eta(\varphi(t)))\eta'(\varphi(t))\varphi'(t)dt = \int_{[a,b]} f(\eta(s))\eta'(s)ds = \int_\eta f(z)dz.$$

where $s = \varphi(t)$, $ds = \varphi'(t)dt$. Hence the path integral is invariant under reparametrization. Length of a path $\eta : [a, b] \to \mathbb{C}$ is defined by,

$$L(\eta) = \int_{[a,b]} |\eta'(t)|dt.$$

Below we list some immediate properties of the path integral,

If η is a path then, f, g in the domain containing η and $a, b \in \mathbb{C}$,

$$\int_\eta (af + bg)dz = a\int_\eta fdz + b\int_\eta gdz.$$

If the path η_1 starts at the end point of η_2 then,

$$\int_{\eta_1+\eta_2} fdz = \int_{\eta_1} fdz + \int_{\eta_2} fdz.$$

The reverse path can be written as $\widetilde{\eta}(s) = \eta(a + b - s)$, then by changing the variable, we have,

$$\int_{\widehat{\eta}} fdz = \int_\eta f(\eta(a + b - s))\eta'(a + b - s)(-1)ds = -\int_\eta fdz.$$

If $\eta([a, b]) \subseteq \Omega$ and $g \in \mathcal{H}(\Omega)$ with continuous derivative g', we have,

$$\int_{g\circ\eta} fdz = \int_\eta f(g(z))g'(z)dz.$$

for all $f \in \mathcal{H}(\Omega)$, $|\int_\eta fdz| \leq \int_{[a,b]} |f(\eta(t))||\eta'(t)|dt \leq \sup_{z\in(\eta([a,b]))} |f(z)| \int_{[a,b]} |\eta'(t)|dt = \|f\|_\eta L(\eta)$. and this can be extended to piecewise C^1 paths. So we have,

$$\left|\int_\eta fdz\right| \leq L(\eta)\|f\|_\eta.$$

where $\|f\|_\eta = \sup_{z\in\eta([a,b])} |f(z)|$. Using this we can show that if f_n converges to f uniformly in $\eta([a, b])$ then, $\lim_n \int_\eta f_n dz = \int_\eta fdz$.

Let $f \in \mathcal{H}(\Omega)$ with continuous derivative, for a smooth curve η, we have, $\frac{df(\eta(t))}{dt} = f'(\eta(t))\eta'(t)$ So, we have,

$$\int_\eta f' dz = \int_\eta f'(\eta(t))\eta'(t)dt = \int_{[a,b]} \frac{df(\eta(t))}{dt} dt = f(\eta(t))\big|_a^b = f(\eta(b)) - f(\eta(a)).$$

If f has an anti-derivative F, i.e., $F' = f$ then,

$$\int_\eta f(z)dz = F(\eta(b)) - F(\eta(a)).$$

So, if f has an anti-derivative in Ω then the path integral is independent of the path, and only depends on the end points. Let η_r be the circle $\{|z| = r\}$ for some positive number r, with the counter clockwise orientation, i.e., $\eta_r(t) = re^{it}$. Then,

$$\int_{\eta_r} z^n dz = \int_{[0,2\pi]} r^n e^{int} rie^{it} dt$$

For $n \neq -1$ the function $f(z) = z^n$ has the primitive $F(z) = \frac{z^{n+1}}{(n+1)}$, and $F(\eta_r(2\pi)) - F(\eta_r(0)) = 0$. So whenever $\int_{\eta_r} z^n dz = 0$ whenever $n \neq -1$. For $n = -1$ the integral becomes,

$$\int_{[0,2\pi]} r^{-1} e^{-it} rie^{it} dt = \int_{[0,2\pi]} idt = 2\pi i$$

Theorem 1.2. (Cauchy-Goursat theorem) *$f \in \mathcal{H}(\Omega)$, if η is a loop in Ω that can be deformed to a point in Ω from within Ω, then,*

$$\int_\eta f(z)dz = 0$$

Proof

The proof is a simple application of Green's theorem together with the Cauchy-Riemann equation. For rectangles $R \subset \mathbb{R}^2$, the Green's theorem says that,

$$\int_{\partial R} a(x,y)dx + b(x,y)dy = \iint_R \Big(-\frac{\partial}{\partial y} a(x,y) + \frac{\partial}{\partial x} b(x,y) \Big) dxdy$$

However this formula extends to all regions diffeomorphic to a disc and bound by finitely many smooth curves. Let $\eta = \partial U$ for some $U \subset \Omega$,

$$\begin{aligned} \int_\eta f(z)dz &= \int_\eta udx - vdy + i(udy + vdx) \\ &= \iint_U (-u_y + v_x)dxdy + i \iint_U (-v_y + u_x)dxdy \end{aligned}$$

By Cauchy-Riemann equation the right side is zero and hence the integral is zero. □

Theorem 1.3. (Cauchy Integral Formula) *Let $B_r(z_0) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_\eta \frac{f(\xi)}{\xi - z} d\xi,$$

where $\eta(t) = z_0 + re^{it}$ and for all $z \in B_r(z_0)$.

PROOF

Assume $z_0 = 0$. For any $z \in B_r(0)$, from $B_r(0)$ remove a small ball centered around z, i.e,. let $U_\epsilon = B_r(0) \backslash B_\epsilon(z)$.

$$\xi \mapsto \frac{f(\xi)}{(\xi - z)}$$

is a holomorphic function on U_ϵ and Cauchy-Goursat theorem is applicable. The boundary circles of U_ϵ are homotopic relative to $\Omega \backslash \{z\}$.

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_{\partial U_\epsilon} \frac{f(\xi)}{z - \xi} d\xi &= \frac{1}{2\pi i} \int_{\partial B_r} \frac{f(\xi)}{z - \xi} d\xi \\ &- \frac{1}{2\pi i} \underbrace{\int_{\partial B_\epsilon(z)} \frac{f(\xi) - f(z)}{z - \xi} d\xi}_{\leq ML(\partial B_\epsilon(z))} - \frac{f(z)}{2\pi i} \underbrace{\int_{\partial B_\epsilon(z)} \frac{1}{z - \xi} d\xi}_{\int_\gamma z^{-1} dz = 2\pi i} \end{aligned}$$

The second term can be made arbitrarily small. Hence we have,

$$f(z) = \frac{1}{2\pi i} \int_\eta \frac{f(\xi)}{\xi - z} d\xi.$$

□

THEOREM 1.4.

$$\mathcal{A}(\Omega) = \mathcal{H}(\Omega).$$

PROOF

For a small rectangle around z,

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(w)}{w - z} dw.$$

Whenever $|z - z_0| < |w - z_0|$, we have, $\frac{1}{w-z} = \frac{1}{(w-z_0)} \left(1 - \frac{z-z_0}{w-z_0}\right)^{-1} = \frac{1}{(w-z_0)} \sum_{n \geq 0} \left(\frac{z-z_0}{w-z_0}\right)^n$. So we have,

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} f(w) \sum_{n \geq 0} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} dw.$$

So,

$$f(z) = \textstyle\sum_{n \geq 0} a_n (z - z_0)^n$$

where $a_n = \frac{1}{2\pi i} \int_{\partial R} \frac{f(w)}{(w-z_0)^{n+1}} dw$. So $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$. □

An immediate consequence of the above is that every holomorphic function is infinitely differentiable. Every holomorphic function has a Taylor series expansion,

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

with $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0} dw$, for some small circle γ around z_0. If $|f(z)| \leq M$ on Ω, by taking modulus on both sides we have,

$$|f^{(n)}(z)| \leq \frac{M n!}{\text{dist}(z, \partial\Omega)^n}$$

An immediate corollary of this is the Liouville's theorem, where $\Omega = B_R(0)$ where $R \to \infty$ and hence $\text{dist}(z, \partial\Omega) \to \infty$ and we have, $f^{(k)} \equiv 0$ for all k

Corollary 1.5. (Liouville's Theorem) *Bounded entire functions are constant.* □

A corollary of Liouville's theorem is the fundamental theorem of algebra which says that every polynomial of positive degree, $P \in \mathbb{C}[z]$ has a complex zero i.e., $\exists z \in \mathbb{C}$ such that $P(z) = 0$ and has exactly as many zeros as the degree.

if $P(z)$ has no complex zeros, the function $f(z) = 1/P(z)$ is an entire function. If $P(z) = \sum_{i=0}^{n} a_i z^i$ with $a_n \neq 0$ we have,

$$|P(z)| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i||z|^i \geq \frac{1}{2}|a_n|R^n$$

for all $|z| = R$ with large enough R. Hence $f(z) = 1/P(z)$ is bounded and hence must be constant by Liouville's theorem. Hence there must be $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$. Every polynomial can be factored,

$$P(z) = a_n \prod_{i=1}^{n} (z - \alpha_i).$$

Theorem 1.6. (Maximum Modulus Principle) $f \in \mathcal{H}(\Omega)$, $f \in C(\overline{\Omega})$, Ω *bounded, then,*

$$|f(z)| \leq \max_{\xi \in \partial\Omega} |f(\xi)|.$$

Sketch of Proof

The first step is to prove the same for small ϵ neighborhoods. Suppose there exists a maximum modulus and attained at z_0, let U be an ϵ neighborhood of z_0. Let $\gamma_\rho = \rho e^{it}$ be the circle around z_0 of radius $\rho < \epsilon$. By Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{[0,2\pi]} \frac{f(z_0 + \rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it} dt = \frac{1}{2\pi i} \int_{[0,2\pi]} f(z_0 + \rho e^{it}) dt.$$

Taking modulus we get,

$$|f(z_0)| \leq \frac{1}{2\pi i} \int_{[0,2\pi]} |f(z_0 + \rho e^{it})| dt.$$

Since the maximum modulus is attained for z_0, we have, $|f(z_0)| \leq |f(z)|$ for all $z \in U$. Hence we have,

$$\frac{1}{2\pi i} \int_{[0,2\pi]} |f(z_0 + \rho e^{it})| dt \leq \frac{1}{2\pi i} \int_{[0,2\pi]} |f(z_0)| dt = |f(z_0)|.$$

So we have,

$$0 = \frac{1}{2\pi i} \int_{[0,2\pi]} \underbrace{(|f(z_0)| - |f(z_0 + \rho e^{it})|)}_{\geq 0} dt$$

Hence we have $|f(z_0)| = |f(z)|$ for all $z \in \gamma_\rho$. Since ρ was arbitrary the equality holds for all $z \in U$. Now for the general case, let $z_0 \in D$ for which $|f(z_0)| \geq |f(z)|$ for all $z \in D$. For any $w \in D$ consider a path joining z_0 and w, the ϵ neighborhoods cover the line, and by compactness of the path, only finitely many such epsilon neighborhoods are required. For each of these epsilon neighborhoods, the function f is constant, and hence we will get that $|f(w)| = |f(z_0)|$ □

Morera's theorem provides a sort of converse to Cauchy's theorem. We now have the necessary tools for it's proof.

Theorem 1.7. (Morera's Theorem) *$f \in C(\Omega)$, $\mathcal{T}$ be the collection of all triangles in Ω*

References