Part IV

Abelian Categories

Abelian categories are the categories where we can do homological algebra. Since the goal is to quickly reach interesting parts, instead of writing down a minimal set of axioms and deducing the remaining properties, I will just state the basic properties directly. For the standard treatment, see [1],[3].

1 | Constructions with Limits

Abelian categories is a category that has all the properties needed to do homological algebra. The categorification of the standard properties of algebraic objects makes it applicable to category of sheaves.

1.1 | Important Constructions

1.2 | Limits & Colimits in Locally Small Categories

For **Sets**, the limit of a system

$$\mathcal{F} : \mathcal{I} \to \mathbf{Sets}$$

can be defined as,

$$\varprojlim \mathcal{F} := \mathrm{Cone}(1, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{I}}}(1, \mathcal{F})$$

here the initial/terminal object is the set with one element. For a locally small category $\mathcal{A}$ and a system $\mathcal{F} : \mathcal{I} \to \mathcal{A}$, limits can be defined representably using hom-sets. Consider the composite functor,

$$\begin{aligned} \widehat{\mathcal{F}}(X) := h^X \circ \mathcal{F} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}} &\to \mathbf{Sets} \\ I &\mapsto \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}I, X). \end{aligned}$$

$\widehat{\mathcal{F}}(X)$ is an inductive system in **Sets**.

Consider two functors, $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ and $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$. For any object $X \in \mathcal{A}$ we can construct the composite functor,

$$\mathcal{I} \xrightarrow{\mathcal{F}} \mathcal{A} \xrightarrow{h_X} \mathbf{Sets},$$

giving rise to the functors, $\widehat{\mathcal{F}}_X := \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(-), X)$, and $\widehat{\mathcal{G}}_X := \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}(-))$. This is a projective systems in the category of sets, $\widehat{\mathcal{F}}(X) : \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$, and the projective limit

$\varprojlim \widehat{\mathcal{F}}(X)$ exists since we are now in the category of sets. The limit of this inductive system $\mathcal{F}$, denoted by $\varinjlim \mathcal{F}$, can be defined as the representative of the functor,

$$X \mapsto \varprojlim \widehat{\mathcal{F}}(X).$$

Similarly, we can construct a composite functor for a projective system,

$$\begin{aligned} \check{\mathcal{G}}(X) \coloneqq h_X \circ \mathcal{G} : \mathcal{I}^{\mathrm{op}} &\to \mathcal{A} \to \mathbf{Sets} \\ I &\mapsto \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}I). \end{aligned}$$

This is a projective systems in the category of sets, $\check{\mathcal{G}}(X) : \mathcal{I}^{\mathrm{op}} \to \mathbf{Sets}$, the projective limit is the representative of the functor,

$$X \mapsto \varprojlim \check{\mathcal{G}}(X).$$

So, we have,

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim \mathcal{F}, X) \cong \varprojlim(\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, X)),$$

$$\mathrm{Hom}_{\mathcal{A}}(X, \varprojlim \mathcal{G}) \cong \varprojlim(\mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G})).$$

Note here that we should ideally write the limit as $\varprojlim_{\mathcal{I}}$ and $\varinjlim_{\mathcal{I}}$. But we have skipped this to avoid the notational nightmare.

1.2.1 | Products & Coproducts

Let $\mathcal{A}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{A}$ indexed by a set I, then we can consider the contravariant functor,

$$\mathcal{G} : Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $\mathcal{G}(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, P)$. This is called the product, denoted by, $\prod_{i\in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd} & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "f_j"] & \\ X_i & \prod_{i\in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $\mathcal{F}(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i\in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd} X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\ & Y & \end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

1.2.2 | Kernel & Cokernel

For sets, the kernel of two maps s, t is defined as the set $\ker(s,t) = \{x \in S \mid s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{A}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\ker(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"] \arrow[drr, bend left, "j"] & & \\ \ker(f,g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \end{tikzcd}$$

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object Z is called a zero object if for any object A, there exists a unique morphism $Z \to A$ and a unique morphism $A \to Z$. It's unique upto isomorphism and denoted by 0. Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \to \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{A}}(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(Z, X)\big).$$

This can be visualized by the diagram,

$$\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f,g) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd}
Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

For a category $\mathcal{I}$, we have two natural maps, $\mathrm{Cod} : \coprod \mathrm{Hom}_{\mathcal{I}}(I, J) \to \mathcal{I}$ that sends $f : I \to J$ to $I \in \mathcal{I}$ and the map $\mathrm{Dom} : \coprod \mathrm{Hom}_{\mathcal{I}}(I, J) \to \mathcal{I}$ that sends $f : I \to J$ to J.

Let $\mathcal{A}$ be a category with products and kernels. Now suppose $\mathcal{F} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ is a projective system, for each $f : i \to j$, we get two morphisms in $\mathcal{A}$, we get two parallel maps, $\widehat{f} = f \circ \pi$ and $\widehat{g} = \mathbb{1}_{\mathcal{F}(i)} \circ \pi_{\mathcal{F}(i)}$, where π is the projection from the product $\mathcal{F}(i) \prod \mathcal{F}(j)$ to $\mathcal{F}(i)$,

$$\mathcal{F}(i) \prod \mathcal{F}(j) \overset{\widehat{f}}{\underset{\widehat{g}}{\rightrightarrows}} \mathcal{F}(i).$$

Using this we obtain two morphisms,

$$\textstyle\prod_{i \in \mathcal{I}} \mathcal{F}(i) \overset{\widehat{f}}{\underset{\widehat{g}}{\rightrightarrows}} \displaystyle\prod_{f \in \coprod \mathrm{Hom}_{\mathcal{I}}(I,J), I,J \in \mathcal{I}} \mathcal{F}(\mathrm{Cod}(f)).$$

It can then be showed that the colimit,

$$\varprojlim_{\mathcal{I}} \mathcal{F} = \ker(\widehat{f}, \widehat{g}).$$

and similarly for limit

$$\varinjlim_{\mathcal{I}} \mathcal{G} = \mathrm{coker}(\widehat{f}, \widehat{g}).$$

To prove this, one has to prove this for the category of sets, and then the proof for general case follows as it's defined representably in terms of limit for sets, see [3] f. So, if $\mathcal{A}$ possesses kernels and products it possesses colimits, and similarly if it possesses cokernels and coproducts, it possesses limits.

1.2.3 | Pullback or Fibered Product

Let $\mathcal{I}$ be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \rightarrow Z$ then for functors $\mathcal{F} : \mathcal{I} \rightarrow \mathcal{A}$, pullback $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$\mathcal{F}Y \xrightarrow{f} \mathcal{F}X \xleftarrow{g} \mathcal{F}Z$$

in a category $\mathcal{A}$ is the commutative square with vertex $\mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z$ such that any other commutative square factors through it, i.e.,

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left, "f_P"] \arrow[ddr, bend right, "g_P"'] & & \\
& \mathcal{F}Y \times_{\mathcal{F}X} \mathcal{F}Z \arrow[r, dashed] \arrow[d, dashed] & \mathcal{F}Z \arrow[d, "g"] \\
& \mathcal{F}Y \arrow[r, "f"'] & \mathcal{F}X
\end{tikzcd}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $\mathcal{G} : \mathcal{I}^{\mathrm{op}} \rightarrow \mathcal{A}$ as above,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{F}Z \arrow[d, dashed] \arrow[ddr, bend left, "f_P"] & \\
\mathcal{F}Y \arrow[r, dashed] \arrow[drr, bend right, "g_P"'] & \mathcal{F}Y \coprod_{\mathcal{F}X} \mathcal{F}Z \arrow[dr, dashed, "\exists !"] & \\
& & C
\end{tikzcd}$$

1.2.4 | Exponentiation

The categorical notions of product and coproduct correpsond to the arithematic operations such as multiplication and addition. We can similarly talk about exponentiation. In the category of sets, **Sets**, for $X, Z \in \mathcal{A}$, Z^X is the function set consisting of all functions $h : X \rightarrow Z$. Here we have the bijection,

$$\mathrm{Hom}(Y \times X, Z) \rightarrow \mathrm{Hom}(Y, Z^X).$$

for a function, $f : Y \times X \rightarrow Z$, this map sends each $y \in Y$ to the function $f(y, -) \in Z^X$. Conversely given a function $f' : Y \rightarrow Z^X$, we can define a map $f(y, x) = f'(y)(x)$. So,

$$\mathrm{Hom}(Y \times X, Z) \cong \mathrm{Hom}(Y, Z^X)$$

or equivalently, $(-)^X$ is the right adjoint of $(-) \times X$. By setting $Y = 1$, we obtain,

$$Z^X \cong \mathrm{Hom}(1, Z^X) \cong \mathrm{Hom}(X, Z).$$

Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$ be two functors, they are called an adjoint pair if

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X), Y) = \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G}(Y))$$

for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. $\mathcal{F}$ is a left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is a right adjoint to $\mathcal{F}$. This is denoted by, $\mathcal{F} \dashv \mathcal{G}$. Adjoints are unique upto isomorphism and is the representative of the functor,

$$X \mapsto \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X), Y)$$

This gives us, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}(X), \mathcal{G}(Y)) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{F} \circ \mathcal{G}(X), Y)$, and, $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X), \mathcal{F}(Y)) \cong \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{G} \circ \mathcal{F}(Y))$.

1.3 | Additive Categories

A category $\mathcal{A}$ is said to be linear if for each $X, Y \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are **abelian groups**, and the **composition is bilinear**. This allows us to add and subtract morphisms in the same hom-sets. The axiom also tells us that each hom-set is nonempty, i.e., every element in the category is connected to every other element of the category. The bilinearity just says that composition of morphisms respect the structure of hom-sets.

Since we would like to talk about kernels and cokernels, and we have zero morphisms in the hom-sets, we need to have the zero object in them. A linear category is called additive if it has a zero object and the functors,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i) \tag{product}$$

and,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y) \tag{coproduct}$$

are representable for a family $\{X_i\}_{i \in I}$ of objects in $\mathcal{A}$ indexed by a set I.

By definition of the **zero object** we also have, $\mathrm{Hom}_{\mathcal{A}}(0, X) = \mathrm{Hom}_{\mathcal{A}}(X, 0) = 0$. The zero object is needed for the description of kernels and cokernels. The representative of the contravariant functor, product, is called the **product** of the family $\{X_i\}_{i \in I}$, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
& Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"'] \arrow[dr, "f_j"] & \\
X_i & \prod_{i \in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom-set. If all $X_i = X$ then this is denoted by X^I.

The representative of the covariant functor coproduct, is called the **coproduct** of the family $\{X_i\}_{i \in I}$, denoted by $\coprod_{i \in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i\in I} X_i$, visualized by the diagram,

$$\begin{tikzcd} X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\ & Y & \end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

Assuming $\mathcal{A}$ is additive, every morphism $f : X \to Y$ in $\mathcal{A}$, the compositions, $Z \to X \to Y$, and $X \to Y \to Z$ give us two morphisms of abelian groups, $\mathrm{Hom}_{\mathcal{A}}(Z, f) : \mathrm{Hom}_{\mathcal{A}}(Z, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, Y)$. and, $\mathrm{Hom}_{\mathcal{A}}(f, Z) : \mathrm{Hom}_{\mathcal{A}}(X, Z) \to \mathrm{Hom}_{\mathcal{A}}(Y, Z)$.

If the functor,

$$Z \mapsto \ker(\mathrm{Hom}_{\mathcal{A}}(Z, f)), \tag{kernel}$$

is representable it's representative is called the **kernel** of f, denoted by $\ker f$.

$$\mathrm{Hom}_{\mathcal{A}}(Z, \ker f) \cong \ker(\mathrm{Hom}_{\mathcal{A}}(Z, f))$$

In other words, the kernel of a morphism $f : X \to Y$ is such that every morphism $g : Z \to X$ with $f \circ g = 0_{Z,Y,}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd} Z \arrow[d, dashed] \arrow[dr, "g"] & & \\ \ker(f) \arrow[r, "\alpha"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Similarly, if the functor,

$$Z \mapsto \ker(\mathrm{Hom}_{\mathcal{A}}(f, Z)), \tag{cokernel}$$

is representable, it's representative is called **cokernel** of f, denoted by coker f.

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}\, f, Z) \cong \ker(\mathrm{Hom}_{\mathcal{A}}(f, Z))$$

In other words, the cokernel of a morphism $f : X \to Y$ is such that every morphism $g : Y \to Z$ with $g \circ f = 0_{X,Z}$ uniquely factors through $\mathrm{coker}(f)$, visualized by the diagram,

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[r, "\beta"] \arrow[dr, "g"'] & \mathrm{coker}(f) \arrow[d, dashed] \\ & & Z \end{tikzcd}$$

Note that the equality $\ker f \cong 0$ and resp. coker $f \cong 0$ is equivalent to saying that is injective or monomorphism and resp. a surjective or epimorphism.

Suppose $\mathcal{A}$ has all kernels and cokernels, then, $\alpha : \ker f \to X$ has a cokernel, and it's called **coimage** of f, i.e.,

$$\mathrm{coim}(f) = \mathrm{coker}(\alpha)$$

The **image** of f is defined as the kernel of the morphisms, $\beta : Y \to \operatorname{coker} f$, i.e.,

$$\operatorname{Im}(f) = \ker(\beta)$$

This can be summarised in a diagram as,

$$\begin{tikzcd}
\ker(f) \arrow[r, "\alpha"] & X \arrow[r] \arrow[d, dashed] \arrow[dr, "f"] & \operatorname{coker}(\alpha) \arrow[d, dashed] & \\
& \ker(\beta) \arrow[r] & Y \arrow[r, "\beta"'] & \operatorname{coker}(f)
\end{tikzcd}$$

So, by universal property of kernel, there is a canonical morphism;

$$\operatorname{coim}(f) \to \operatorname{Im}(f)$$

An additive category $\mathcal{A}$ is called abelian if every morphism $f : X \to Y$ has kernel and cokernel, and the canonical morphism,

$$\operatorname{coim}(f) \cong \operatorname{Im}(f)$$

is an isomorphism. In such a case, every morphism $f : X \to Y$ splits as,

$$0 \to \ker f \to X \to \operatorname{Im}\ f \to Y \to \operatorname{coker} f \to 0.$$

A sequence of morphisms,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called an exact sequence if $g \circ f = 0$ and $\operatorname{Im}(f) \cong \ker g$.

2 | Abelian Categories

Abelian categories is a category that has all the properties needed to do homological algebra. The categorification of the standard properties of algebraic objects makes it applicable to category of sheaves.

A category $\mathcal{A}$ is said to be linear if for each $X, Y \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are abelian groups, and the composition is bilinear. This allows us to add and subtract morphisms in the same hom-sets. The axiom also tells us that each hom-set is nonempty, i.e., every element in the category is connected to every other element of the category. The bilinearity just says that composition of morphisms respect the structure of hom-sets.

Since we would like to talk about kernels and cokernels, and we have zero morphisms in the hom-sets, we need to have the zero object in them. A linear category is called additive if it has a zero object and the functors,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i) \tag{product}$$

and,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y) \tag{coproduct}$$

are representable for a family $\{X_i\}_{i \in I}$ of objects in $\mathcal{A}$ indexed by a set I.

The representative of product is called the product of the family $\{X_i\}_{i \in I}$, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

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$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $F(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i \in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i \in I} X_i$, visualized by the diagram,

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 & Y &
\end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i\in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i\in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

By definition of the zero object we also have, $\mathrm{Hom}_{\mathcal{A}}(0, X) = \mathrm{Hom}_{\mathcal{A}}(X, 0) = 0$. The zero object is needed for the description of kernels and cokernels.

(Product & Coproduct) $\mathcal{A}$ contains products and coproducts.

(Zero) $\mathcal{A}$ has a zero object, i.e., an object that's both initial and terminal.

By definition of the zero object we also have, $\mathrm{Hom}_{\mathcal{A}}(0, X) = \mathrm{Hom}_{\mathcal{A}}(X, 0) = 0$. The zero object is needed for the description of kernels and cokernels. A category $\mathcal{A}$ which satisfies abelianness, has products and coproducts, and has a zero object is called an additive category.

(Kernel & Cokernel) Every morphism in $\mathcal{A}$ has a kernel and cokernel.

Finally we need the ability to take quotients, or that we want the first isomorphism theorem to hold. This can be said categorically by requiring that the image and coimage are isomorphic.

(Isomorphism)

$$\mathrm{coim}(f) \cong \mathrm{Im}(f)$$

A category $\mathcal{A}$ that satisfies the five axioms is called an abelian category. In such a case, for every morphism $f : X \to Y$ there exists a sequence

$$\ker f \xrightarrow{k} X \xrightarrow{i} \mathrm{Im}\ f \xrightarrow{j} Y \xrightarrow{c} \mathrm{coker}\, f.$$

such that $j \circ i = f$. Such a sequence is called a canonical decomposition. This is equivalent to the isomorphism theorem, $\mathrm{Im}\ f \cong X/\ker f$.

A sequence of morphisms in an abelian category $\mathcal{A}$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called exact if $\mathrm{Im}\ f = \ker g$. Everything that goes into Y via f gets killed by g.

2.1 | Contains Products & Coproducts

Let $\mathcal{A}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{A}$ indexed by a set I, then we can consider the contravariant functor,

$$G : Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

The product on the right side is the standard product in the category of sets. Assuming the functor is representable, i.e., there exists an object P such that, $G(Y) = \mathrm{Hom}_{\mathcal{A}}(Y, P)$. This is called the product, denoted by, $\prod_{i \in I} X_i$. So by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(Y, X_i)$$

This isomorphism can be translated into the universal property definition as follows, given an object Y and a family of morphisms $f_i : Y \to X_i$ this family factorizes uniquely through $\prod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
 & Y \arrow[dl, "f_i"'] \arrow[d, dashed, "\exists! h"'] \arrow[dr, "f_j"] & \\
X_i & \prod_{i \in I} X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j
\end{tikzcd}$$

The order of I is unimportant as composition with a permutation of I also belongs to the same hom set. If all $X_i = X$ then this is denoted by X^I.

Similarly we can consider the functor,

$$Y \mapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y)$$

This is a covariant functor. Assuming it's representable there exists an object C such that, $F(Y) = \mathrm{Hom}_{\mathcal{A}}(C, Y)$. The representative C is denoted by $\coprod_{i \in I} X_i$ and by definition we have,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

This isomorphism can be translated as follows, given an object Y and a family of morphisms $f_i : X_i \to Y$ this family factorizes uniquely through $\coprod_{i \in I} X_i$, visualized by the diagram,

$$\begin{tikzcd}
X_j \arrow[r, "\epsilon_j"] \arrow[dr, "f_j"'] & \coprod_{i \in I} X_i \arrow[d, dashed, "h"'] & X_i \arrow[l, "\epsilon_i"'] \arrow[dl, "f_i"] \\
 & Y &
\end{tikzcd}$$

In algebra, for modules, etc. the coproduct is denoted by $\oplus$, and is called direct sum. It follows directly from definition that,

$$\mathrm{Hom}_{\mathcal{A}}(\coprod_{i \in I} X_i, Y) = \mathrm{Hom}_{\mathcal{A}}(Y, \prod_{i \in I} X_i)$$

When the indexing category is discrete, i.e., the only morphisms are the identity morphisms, the limit and colimit in such a case corresponds to products and coproducts.

2.2 | Contains Zero Object, Kernel, Cokernel

To be able to describe kernel and cokernel we have to first have a zero object, i.e,. an object that's both initial and terminal. An object 0 is called a zero object if for any object X, there exists a unique morphism $0 \to X$ and a unique morphism $X \to 0$. It's unique upto isomorphism and denoted by 0. Between any two objects $X, Y \in \mathcal{A}$, there exists a unique morphism $0_{X,Y}$ given by the composition,

$$X \to 0 \to Y$$

The kernel of a map $f : X \to Y$ is a morphism $\iota : \ker(f) \to X$ such that $f \circ \iota = 0_{\ker(f),Y}$ and any other morphism $i : K \to X$ with $f \circ i = 0_{K,Y}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
K \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : Y \to \operatorname{coker}(f)$ with $\iota \circ f = 0_{X,\operatorname{coker}(f)}$, and for any morphism $k : Y \to C$ with $k \circ f = 0_{X,C}$ will factor uniquely through $\operatorname{coker}(f)$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\iota"] \arrow[dr, bend right, "k"'] & \operatorname{coker}(f) \arrow[d, dashed, "e"] \\
 & & C
\end{tikzcd}$$

If a category $\mathcal{A}$ possesses products and kernels then it has colimits, and similarly if it has coproducts and cokernels then it possesses limits. Converse also holds since products and kernels can be thought of as limits. For details on this see [3].

2.2.1 | Contains Images, & Coimages

Image of a map $f : X \to Y$ is a factorisation,

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[dr, bend right, "f"'] & \operatorname{Im}(f) \arrow[d, tail, "i"] \\
 & Y
\end{tikzcd}$$

where i is monic(injective), and e is epic(surjective), and $f = i \circ e$. Whenever a category has kernels and cokernels we have the following diagram,

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, tail, bend left, "f"] & & \\
\ker(p) \arrow[r] & Y \arrow[r, "p"'] & \operatorname{coker}(f)
\end{tikzcd}$$

So, image exists whenever the category has kernels and cokernels and is given by,

$$\operatorname{Im}(f) = \ker(\operatorname{coker}(f))$$

With every map $f : X \to Y$, we have the following exact factorization,

$$\ker(f) \xrightarrow{k} X \xrightarrow{i} \operatorname{coim}(f) \cong \operatorname{Im}(f) \xrightarrow{j} Y \xrightarrow{c} \operatorname{coker}(f).$$

such that $j \circ i = f$. This is called a canonical decomposition.

For sets, the equalizer of two maps s, t is defined as the set $\mathrm{Eq}(s,t) = \{x \in S | s(x) = t(x)\}$. Using this, for any two maps $f, g : Y \rightrightarrows Z$, we have set maps,

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

given by the action, $h \mapsto f \circ h$. Using these set maps we can define the functor,

$$Y \mapsto \mathrm{Eq}\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(X, Z)\big).$$

This is a covariant functor from the category $\mathcal{A}$ to **Sets**. Assuming this functor is representable, the representative denoted by $\mathrm{Eq}(f, g)$ is called the equalizer of f, g.

This isomorphism can be translated as follows, given an object X and morphisms $i : X \to Y$ and $j : X \to Z$ such that $i \circ f = j \circ g$, uniquely factors through $\ker(f, g)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!"] \arrow[dr, "i"'] \arrow[drr, bend left, "j"] & & \\
\mathrm{Eq}(f,g) \arrow[r, "e"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd} \tag{kernel}$$

Between any two objects $A, B \in \mathcal{A}$, there exists a unique morphism $0_{A,B}$ given by the composition,

$$A \to 0 \to B$$

In this case, the kernel of a map f is defined as the equalizer of the maps $f, 0 : \mathcal{A} \to \mathcal{A}$, $\ker(f) = \ker(f, 0)$. The kernel of a map $f : Y \to Z$ is a morphism $\iota : \ker(f) \to A$ such that $f \circ \iota = 0_{\ker(f),B}$ and any other morphism $i : X \to Y$ with $f \circ i = 0_{K,B}$ uniquely factors through $\ker(f)$, visualized by the diagram,

$$\begin{tikzcd}
X \arrow[d, dashed, "e"] \arrow[dr, bend left, "i"] & & \\
\ker(f) \arrow[r, "\iota"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Here we have not written the zero morphism from X to Z. Similarly we can define coequalizer and cokernel. Given two maps $f, g : Y \rightrightarrows Z$, we have set maps, $\mathrm{Hom}_{\mathcal{A}}(Y, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, X)$ given by the action, $h \mapsto h \circ f$. Coequalizer is the representative of the functor,

$$Y \mapsto \ker\big(\mathrm{Hom}_{\mathcal{A}}(Y, X) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(Z, X)\big).$$

This can be visualized by the diagram,

$$\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr, bend right, "l"'] & Z \arrow[r, "\iota"] \arrow[dr, "k"] & \mathrm{coker}(f,g) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

The cokernel of a morphism f is a morphism $\iota : X \to \mathrm{coker}(f)$ with $\iota \circ f = 0_{A,\mathrm{coker}(f)}$, and for any morphism $k : B \to L$ with $k \circ f = 0_{A,L}$ will factor uniquely through $\mathrm{coker}(f)$.

$$\begin{tikzcd}
Y \arrow[r, "f"] & Z \arrow[r, "\iota"] \arrow[dr, bend right, "k"] & \mathrm{coker}(f) \arrow[d, dashed, "e"] \\
& & X
\end{tikzcd}$$

For a category $\mathcal{I}$, we have two natural maps, Cod : $\coprod \mathrm{Hom}_{\mathcal{I}}(I,J) \to \mathcal{I}$ that sends $f : I \to J$ to $I \in \mathcal{I}$ and the map Dom : $\coprod \mathrm{Hom}_{\mathcal{I}}(I,J) \to \mathcal{I}$ that sends $f : I \to J$ to J.

Let $\mathcal{A}$ be a category with products and kernels. Now suppose $F : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ is a projective system, for each $f : i \to j$, we get two morphisms in $\mathcal{A}$, we get two parallel maps, $\widehat{f} = f \circ \pi$ and $\widehat{g} = \mathbb{1}_{F(i)} \circ \pi_{F(i)}$, where π is the projection from the product $F(i) \prod F(j)$ to $F(i)$,

$$F(i) \prod F(j) \overset{\widehat{f}}{\underset{\widehat{g}}{\rightrightarrows}} F(i).$$

Using this we obtain two morphisms,

$$\prod\nolimits_{i \in \mathcal{I}} F(i) \overset{\widehat{f}}{\underset{\widehat{g}}{\rightrightarrows}} \prod_{f \in \coprod \mathrm{Hom}_{\mathcal{I}}(I,J), I,J \in \mathcal{I}} F(\mathrm{Cod}(f)).$$

It can then be showed that the colimit,

$$\varprojlim\nolimits_{\mathcal{I}} F = \ker(\widehat{f}, \widehat{g}).$$

and similarly for limit

$$\varinjlim\nolimits_{\mathcal{I}} G = \mathrm{coker}(\widehat{f}, \widehat{g}).$$

To prove this, one has to prove this for the category of sets, and then the proof for general case follows as it's defined representably in terms of limit for sets, see [3] f. So, if $\mathcal{A}$ possesses kernels and products it possesses colimits, and similarly if it possesses cokernels and coproducts, it possesses limits.

2.2.2 | Pullback or Fibered Product

Let $\mathcal{I}$ be the indexing category with three objects X, Y, Z and two morphisms, $Y \leftarrow X \to Z$ then for functors $F : \mathcal{I} \to \mathcal{A}$, pullback $FY \times_{FX} FZ$ is defined to be the limit of this functor. In terms of universal property, a pullback for a diagram

$$FY \xrightarrow{f} FX \xleftarrow{g} FZ$$

in a category $\mathcal{A}$ is the commutative square with vertex $FY \times_{FX} FZ$ such that any other commutative square factors through it, i.e.,

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "f_P"] \arrow[ddr, bend right, "g_P"'] & & \\
& FY \times_{FX} FZ \arrow[r, dashed] \arrow[d, dashed] & FZ \arrow[d, "g"] \\
& FY \arrow[r, "f"'] & FX
\end{tikzcd}$$

The limit is called the fibered product. The categories that have the fibered product are called fibered categories. In case of **Sets** the pullback always exist because limits exist and the pullback consists of all elements (x, y) such that $f(x) = g(y)$.

Similarly, a pushforward corresponds to the limit of the functor $G : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}$ as above,

$$\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "g"'] & FZ \arrow[d, dashed] \arrow[ddr, bend left, "f_P"] & \\
FY \arrow[r, dashed] \arrow[drr, bend right, "g_P"'] & FY \coprod_{FX} FZ \arrow[dr, dashed, "\exists!"] & \\
 & & C
\end{tikzcd}$$

2.3 | Hom-sets are Abelian Groups, Composition Bilinear

A linear category $\mathcal{A}$ is a category where for every $X, Y \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are abelian groups and composition is bilinear, i.e., if $f, f' \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $g, g' \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$ we must have,

$$(g + g') \circ f = g \circ f + g' \circ f$$

and,

$$g \circ (f + f') = g \circ f + g' \circ f.$$

With composition as product and $\mathbb{1}_X$ as unit, $\mathrm{Hom}_{\mathcal{A}}(X, X)$ is an associative ring. $0_{X,Y}$ is the zero element of the abelian group $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. Immediately we see that $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq \varnothing$ since it's a group and contains atleast $0_{X,Y}$. By definition of the zero object we also have,

$$\mathrm{Hom}_{\mathcal{A}}(0, X) = \mathrm{Hom}_{\mathcal{A}}(X, 0) = 0.$$

An abelian category is an additive category $\mathcal{A}$ in which every morphism has a kernel and cokernel and every injective morphism is the kernel of its cokernel, i.e., the

In the category of sets, the image of a map $f : X \to Y$ is the subset of Y whose elements are images of some element in X. So, in terms of maps, we can factor the map f through the image of f. So, for a category $\mathcal{A}$, we can define image, coimage, kernel, and cokernel via the following diagram,

$$\begin{tikzcd}
\ker(f) \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{coker}(f) \\
& \operatorname{coim}(f) \arrow[r, dashed] \arrow[ur, dashed] & \operatorname{Im}(f) \arrow[u] &
\end{tikzcd}$$

For every $Z \in \mathcal{A}$ the hom functor $\mathrm{Hom}_{\mathcal{A}}(Z, -)$ is a contravariant functor,

$$\mathrm{Hom}_{\mathcal{A}}(Z, -) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}.$$

which sends each $X \in \mathcal{A}$ to the abelian group $\mathrm{Hom}_{\mathcal{A}}(Z, X)$. Since we are assuming the category to be linear, these hom sets are abelian groups. To every map $f : X \to Y$, the functor associates the map of abelian groups,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(Z, f) : \mathrm{Hom}_{\mathcal{A}}(Z, X) &\to \mathrm{Hom}_{\mathcal{A}}(Z, Y) \\ g &\mapsto f \circ g. \end{aligned}$$

we will denote $\mathrm{Hom}_{\mathcal{A}}(Z, f)$ by f_* for the sake of brevity. Then, we have the following, sequence of homomorphisms of abelian groups,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Z, \ker(f)) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(Z, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(Z, Y)$$

corresponding to the sequence $\ker(f) \xrightarrow{i} X \xrightarrow{f} Y$, where 0 is the group with a single element, the identity, and it gets mapped to $0_{Z,\ker(f)}$. Since the hom functors preserve limits, and the fact that kernels and cokernels are limits, we have,

$$\mathrm{Im}(i_*) = \ker(f_*).$$

So,

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Z, \ker(f)) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{A}}(Z, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(Z, Y),$$

is an exact sequence of abelian groups.

Lemma 2.1. *Given $f : X \to Y$,*

An exact category is a category with zero object, kernels and cokernels, and for every morphism $f : X \to Y$ the induced morphisms

$$\mathrm{coker}(f) \to \mathrm{Im}(f)$$

is always an isomorphism.

$$
\begin{tikzcd}
A \arrow[d, "e"'] \arrow[dr, tail, bend left, "f"] & & \\
\ker(p) \arrow[r, "I"'] & B \arrow[r, "p"'] & \text{coker(p)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\ker(f) \arrow[r, "I"] & A \arrow[r, "p"] \arrow[dr, two heads, bend right, "f"'] & \operatorname{coker}(i) \arrow[d, dashed, "e"] \\
 & & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, two heads, "p"] \arrow[dr, bend right, "f"'] & \operatorname{im}(p) \arrow[d, tail, "I"] \\
 & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, dashed, "e"'] \arrow[dr, bend left, "f"] & & \\
\ker(i) \arrow[r, "i"'] & B \arrow[r, "p"'] & \operatorname{coker}(f)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "g"] & I \arrow[r, "m"] & B \arrow[r, "\varphi"] \arrow[d, "p"] & C \\
 & ker \arrow[u, dashed, "v"] \arrow[ur, bend right, "I"'] & coker \arrow[ur, dashed, bend right, "\psi"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & ker \arrow[d, "k"] & & \\
A \arrow[ur, dashed, bend left, "w"] \arrow[r, "e"'] & ker \arrow[r, "I"] \arrow[d, "g"] & B \arrow[r, "p"] & coker \\
 & T & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & ker \arrow[d, dashed, "\delta"] \arrow[dr, bend left, "I"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "e'"'] & I' \arrow[r, "i'"'] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & ker \arrow[d, "I"] & \\
A \arrow[ur, bend left, "e"] \arrow[r, "f"'] & I' \arrow[r, "g"] \arrow[d, "p"'] & B \arrow[dl, dashed, bend left, "t"] \\
 & coker &
\end{tikzcd}
$$

Now consider the functor category $\mathbf{Sets}^{\mathcal{A}^{\mathrm{op}}}$, with X, Y, B being functors to **Sets**.

So, limits and colimits can be used to characterize categories that have the necessary properties that have the appropriate structure to do homological algebra.

Definition 2.1. A category $\mathcal{A}$ is called exact if there are zero objects, all morphisms in $\mathcal{A}$ have kernels and cokernels, for every morphism f, the induced morphism $\operatorname{coker}(f) \to \operatorname{Im}(f)$ is an isomorphism

So, in such categories we can start talking about exact sequences. Snake lemma holds for exact category, i.e., given a short exact sequence of complexes in $\mathcal{A}$ one can associate a long exact sequence consisting of kernels and images.

A linear category is a category $\mathcal{A}$ whose hom-sets, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are abelian groups for all $X, Y \in \mathcal{A}$ A linear category with zero object and direct sums and products is called an additive category. A functor $F : \mathcal{A} \to \mathcal{D}$ between additive categories is additive if each $F_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a group homomorphism. In such a case,

$$F(0) \cong 0, \quad F(X \oplus Y) = F(X) \oplus F(Y).$$

An abelian category is one where each morphism has kernel and cokernel. In particular, this means that given a map, $f : X \to Y$, splits uniquely as,

3 | Diagram Chasing

3.1 | Salamander Lemma

3.1.1 | Corollaries of Salamander Lemma

4 | Limits and Colimits

Lemma 4.1. (Five Lemma) *Consider two exact sequences $A^\bullet$ and $B^\bullet$,*

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r] \arrow[d, "u_i"] & A^{i+1} \arrow[r] \arrow[d, "u_{i+1}"] & A^{i+2} \arrow[r] \arrow[d, "u_{i+2}"] & A^{i+3} \arrow[r] \arrow[d, "u_{i+3}"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & B^{i+2} \arrow[r] & B^{i+3} \arrow[r] & \cdots
\end{tikzcd}$$

References

[1] S Gelfand, Y Manin, Methods of Homological Algebra Springer, 2003

[2] F Borceux, Handbook of Categorical Algebra 2: Categories and Structures Cambridge University Press, 1994

[3] M Kashiwara, P Schapira, Categories and Sheaves, Springer, 2000