Part I

Holomorphic Functions

Main aim of this part is to show holomorphic functions are analytic. Topics for these notes include, holomorphic functions, analytic functions, integration along paths, Cauchy's theorem, Cauchy integral formula, analytic continuation, Liouville theorem, fundamental theorem of algebra, the maximum modulus principle, Morera's theorem, Weierstrass & Montel's theorem.

1 | Holomorphic Functions

We are interested in differentiating functions defined on some open set $\Omega \subset \mathbb{C}$[1]. The topology on $\mathbb{C}$ comes from the standard euclidean norm on $\mathbb{C} = \mathbb{R}^2$. A function f defined on Ω is complex differentiable at a point $z \in \Omega$ if it's differentiable and the differential is complex linear i.e., there exists a complex linear function $df(z)$ such that,

$$\frac{f(z+h) - f(z)}{h} \xrightarrow[h \to 0]{} df(z)(h), \tag{1D}$$

The function f can be approximated infinitesimally by a complex linear function. If f is complex differentiable on Ω it's said to be holomorphic on Ω and the derivative is defined as,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

If f is holomorphic on all of $\mathbb{C}$ it's called entire.

If the function f is thought of as a map from Ω as an open subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, then f can be written as, $f = u + iv$. The condition of complex differentiability of f at $z \in \Omega$ can then be split into two requirements, real differentiable and complex linearity of the differential. This means that the differential is a real linear map,

$$df(z) : \mathbb{R}^2 \to \mathbb{R}^2.$$

The matrix form of $df(z)$ is called the Jacobian and in terms of partial derivatives in the standard basis, it is given by,

$$J_f(z) = (\partial_j f_i(z))_{i,j}$$

A real linear map T is complex linear if $T(i) = iT(1)$. This condition puts the required constraint. So the Jacobian matrix $J_f(z)$ will be of the form,

$$J_f(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{bmatrix} \partial_x u(z) & \partial_y u(z) \\ \partial_x v(z) & \partial_y v(z) \end{bmatrix}. \tag{2D}$$

[1] Note that whenever we talk about Ω, we assume it to be open subset of $\mathbb{C}$, and whenever we say $h \in \mathbb{C}$ is small we mean $\|h\|$ is small.

This is called the Cauchy-Riemann equation.[2]

The set of all holomorphic functions on Ω is denoted by $\mathcal{H}(\Omega)$. If f, and g are two complex differentiable functions on Ω and $\lambda \in \mathbb{C}$ then it follows directly from definition that, $f+g$, $f \cdot g$, and $\lambda \cdot f$ are also complex differentiable. So, $\mathcal{H}(\Omega)$ is an algebra over $\mathbb{C}$.

If f is a complex differentiable function on Ω and g is a complex differentiable function on V then the composition map $g \circ f$ is a complex differentiable function on Ω and,

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z).$$

which is the chain rule.

Clearly, all polynomials are holomorphic functions. We can generalise this to all power series within their disk of convergence. A formal power series is a series $\sum_{n \geq 1} a_n z^n$, denoted by $\mathbb{C}[z]$. Abel's theorem says that every power series has a radius of convergence, i.e., there exists $R > 0$ such that for any z in the ball of radius R the power series converges. The radius of convergence is given by,

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

A function $f : \Omega \to \mathbb{C}$ is said to be analytic if f is represented by a convergent power series expansion on a neighborhood around every point Ω.

The set of all analytic functions on Ω is denoted by $\mathcal{A}(\Omega)$. Clearly this set is an algebra. The main goal of this part is to prove that holomorphic functions and analytic functions are the same.

Theorem 1.1.

$$\mathcal{A}(\Omega) \subset \mathcal{H}(\Omega).$$

Proof

Let $f \in \mathcal{A}(\Omega)$, without loss of generality assume the power series expansion of f is

$$f(z) = \textstyle\sum_{n \geq 0} a_n z^n$$

for all $|z| < R$. Write, $f(z+h) = f(z) + g(z)h + r(z,h)$, where $g(z) = \sum_{n \geq 1} n a_n z^{n-1}$. By the radius of convergence formula, the radius of convergence of $g(z)$ is the same as $f(z)$. We have,

$$r(z,h) = \textstyle\sum_{n \geq 0} a_n B_n(z,h),$$

where $B_n(z,h) = (z+h)^n - z^n - nz^{n-1}h$ and $B_0 = B_1 = 0$. Let $|z| + |h| < r_1 < r_2 < R$, we have, $|a_n| \leq M/r_2^n$ for some $M < \infty$. So we have,

$$\begin{aligned} |B_n(z,h)| \leq |h|^2 \textstyle\sum_{k=0}^{n-2} \binom{n}{k+2} |h|^k |z|^{n-2-k} &\leq |h|^2 \textstyle\sum_{k=0}^{n-2} n^2 \binom{n-2}{k} |h|^k |z|^{n-2-k} \\ &= |h|^2 n^2 (|z| + |h|)^{n-2}. \end{aligned}$$

[2] Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can express $z \in \mathbb{C}$ as a vector $(x,y) \in \mathbb{R}^2$, with $z = x+iy$ and $dz = dx + idy$, $d\bar{z} = dx - idy$. For $f \in C^\infty(U)$, as a function on $\mathbb{R}^2$ we have, $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$. By writing $dx = \frac{dz + d\bar{z}}{2}$ and $dy = \frac{dz - d\bar{z}}{2i}$, we get,

$$df = \underbrace{\tfrac{1}{2}\left(\tfrac{\partial f}{\partial x} - i \tfrac{\partial f}{\partial y}\right)}_{:= \frac{\partial f}{\partial z}} dz + \underbrace{\tfrac{1}{2}\left(\tfrac{\partial f}{\partial x} + i \tfrac{\partial f}{\partial y}\right)}_{:= \frac{\partial f}{\partial \bar{z}}} d\bar{z}$$

So, $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$. f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ or $df = \frac{\partial f}{\partial z} dz$.

Hence,

$$|r(z,h)| \leq \sum_{n\geq 2} \frac{M}{r_2^n} n^2(|z|+|h|)^{n-2} \leq \frac{|h|^2 M}{r_2^2} \sum_{n\geq 0}(n+2)^2 \left(\frac{r_1}{r_2}\right)^n.$$

or, for some constant C, we have,

$$|r(z,h)| \leq C|h|^2.$$

or that $g(z) = f'(z)$ for $|z| < R$ because for $h \to 0$ we have $f(z+h) - f(z) \approx g(z)h$. So f is holomorphic. $\square$

To prove the other inclusion, i.e., $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$ we need more tools and complex integration. Let $\eta : [a,b] \to \mathbb{C}$ be a smooth curve. Let f be a continuous function defined atleast on the compact image $\eta([a,b])$. The path integral of f along η is defined by,

$$\int_\eta f(z)dz = \int_{[a,b]} f(\eta(t))\eta'(t)dt$$

since $(f \circ \eta) \cdot \eta'$ is continuous on $[a,b]$ the integral is well-defined. For a piecewise C^1-path the integral along $\eta = \eta_1 + \cdots + \eta_n$ is defined by, $\int_\eta f(z)dz = \sum_{i=1}^n \int_{\eta_i} f(z)dz$.

Suppose we have a reparametrization of the interval $[a,b]$, given by the C^1-map $\varphi : [a',b'] \to [a,b]$ then we have,

$$\int_{\eta\circ\varphi} f(z)dz = \int_{[a',b']} f(\eta(\varphi(t)))\eta'(\varphi(t))\varphi'(t)dt = \int_{[a,b]} f(\eta(s))\eta'(s)ds = \int_\eta f(z)dz.$$

where $s = \varphi(t)$, $ds = \varphi'(t)dt$. Hence the path integral is invariant under reparametrization. Length of a path $\eta : [a,b] \to \mathbb{C}$ is defined by,

$$L(\eta) = \int_{[a,b]} |\eta'(t)|dt.$$

Below we list some immediate properties of the path integral,

If η is a path then, f, g in the domain containing η and $a, b \in \mathbb{C}$,

$$\int_\eta (af + bg)dz = a\int_\eta f dz + b\int_\eta g dz.$$

If the path η_1 starts at the end point of η_2 then,

$$\int_{\eta_1+\eta_2} f dz = \int_{\eta_1} f dz + \int_{\eta_2} f dz.$$

The reverse path can be written as $\widetilde{\eta}(s) = \eta(a+b-s)$, then by changing the variable, we have,

$$\int_{\widetilde{\eta}} f dz = \int_\eta f(\eta(a+b-s))\eta'(a+b-s)(-1)ds = -\int_\eta f dz.$$

If $\eta([a,b]) \subseteq \Omega$ and $g \in \mathcal{H}(\Omega)$ with continuous derivative g', we have,

$$\int_{g\circ\eta} f dz = \int_\eta f(g(z))g'(z)dz.$$

For all $f \in \mathcal{H}(\Omega)$, $|\int_\eta f dz| \le \int_{[a,b]} |f(\eta(t))||\eta'(t)|dt \le \sup_{z\in(\eta([a,b]))} |f(z)| \int_{[a,b]} |\eta'(t)|dt = \|f\|_\eta L(\eta)$. and this can be extended to piecewise C^1 paths. So we have,

$$\left| \int_\eta f dz \right| \le L(\eta)\|f\|_\eta.$$

where $\|f\|_\eta = \sup_{z\in\eta([a,b])} |f(z)|$. Using this we can show that if f_n converges to f uniformly in $\eta([a,b])$ then, $\lim_n \int_\eta f_n dz = \int_\eta f dz$.

Let $f \in \mathcal{H}(\Omega)$ with continuous derivative, for a smooth curve η, we have, $\frac{df(\eta(t))}{dt} = f'(\eta(t))\eta'(t)$ So, we have,

$$\int_\eta f' dz = \int_\eta f'(\eta(t))\eta'(t)dt = \int_{[a,b]} \frac{df(\eta(t))}{dt} dt = f(\eta(t))\big|_a^b = f(\eta(b)) - f(\eta(a)).$$

If f has an anti-derivative F, i.e., $F' = f$ then,

$$\int_\eta f(z)dz = F(\eta(b)) - F(\eta(a)).$$

So, if f has an anti-derivative in Ω then the path integral is independent of the path, and only depends on the end points. Let η_r be the circle $\{|z| = r\}$ for some positive number r, with the counter clockwise orientation, i.e., $\eta_r(t) = re^{it}$. Then,

$$\int_{\eta_r} z^n dz = \int_{[0,2\pi]} r^n e^{int} rie^{it} dt$$

For $n \ne -1$ the function $f(z) = z^n$ has the primitive $F(z) = \frac{z^{n+1}}{(n+1)}$, and $F(\eta_r(2\pi)) - F(\eta_r(0)) = 0$. So whenever $\int_{\eta_r} z^n dz = 0$ whenever $n \ne -1$. For $n = -1$ the integral becomes,

$$\int_{[0,2\pi]} r^{-1} e^{-it} rie^{it} dt = \int_{[0,2\pi]} i dt = 2\pi i$$

Theorem 1.2. (Cauchy-Goursat theorem) *$f \in \mathcal{H}(\Omega)$, if η is a loop in Ω that can be deformed to a point in Ω from within Ω, then,*

$$\int_\eta f(z)dz = 0$$

Proof

Let U be such that $\partial U = \eta$. By setting $u = f dz$, we have, $du = df \wedge dz$. By Stokes theorem, $\int_{\partial U} f dz = \int_U du$, which gives us,

$$\int_{\partial U} f dz = \int_U \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \right) \wedge dz$$

which will be zero for holomorphic functions, by Cauchy-Riemann equations. □

The Cauchy-Goursat theorem also tells us that for homotopic loops, and $f \in \mathcal{H}(\Omega)$, $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$, when the homotopy lies entirely inside Ω. This is because we can triangulate the homotopy, and $\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \sum_j \int_{\eta_j} f(z)dz = 0$, where the sum is over a finite collection of small loops.

THEOREM 1.3. (CAUCHY INTEGRAL FORMULA) *Let $B_r(z_0) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_\eta \frac{f(w)}{w-z} dw,$$

where $\eta(t) = z_0 + re^{it}$ and for all $z \in B_r(z_0)$.

PROOF

Assume $z_0 = 0$. For any $z \in B_r(0)$, from $B_r(0)$ remove a small ball centered around z, i.e,. let $U_\epsilon = B_r(0) \backslash B_\epsilon(z)$.

$$w \mapsto \tfrac{f(w)}{(w-z)}$$

is a holomorphic function on U_ϵ and Cauchy-Goursat theorem is applicable. The boundary circles of U_ϵ are homotopic relative to $\Omega \backslash \{z\}$.

$$0 = \tfrac{1}{2\pi i} \int_{\partial U_\epsilon} \tfrac{f(w)}{z-w} dw = \tfrac{1}{2\pi i} \int_{\partial B_r} \tfrac{f(w)}{z-w} dw - \tfrac{1}{2\pi i} \underbrace{\int_{\partial B_\epsilon(z)} \left(\tfrac{f(w)-f(z)}{z-w} \right) dw}_{\leq ML(\partial B_\epsilon(z)) = 2\pi M \epsilon} - \tfrac{f(z)}{2\pi i} \underbrace{\int_{\partial B_\epsilon(z)} \tfrac{1}{z-w} dw}_{\int_\gamma z^{-1} dz = 2\pi i}$$

The second term can be made arbitrarily small. Hence we have,

$$f(z) = \frac{1}{2\pi i} \int_\eta \frac{f(w)}{w-z} dw.$$

□

Note that in general case, $\frac{f(w)}{w-z} dw$ gets replaced by $\frac{f(w)}{(z-w)} dw + \frac{\partial f(w)}{d\bar{z}} \frac{dz \wedge d\bar{z}}{(z-w)}$ which for the case of holomorphic functions is the same.

THEOREM 1.4.

$$\mathcal{A}(\Omega) = \mathcal{H}(\Omega).$$

PROOF

For a small rectangle around z,

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(w)}{w-z} dw.$$

Whenever $|z - z_0| < |w - z_0|$, we have, $\frac{1}{w-z} = \frac{1}{(w-z_0)} \left(1 - \frac{z-z_0}{w-z_0}\right)^{-1} = \frac{1}{(w-z_0)} \sum_{n \geq 0} \left(\frac{z-z_0}{w-z_0}\right)^n$. So we have,

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} f(w) \sum_{n \geq 0} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} dw.$$

So,

$$f(z) = \textstyle\sum_{n \geq 0} a_n (z - z_0)^n$$

where,

$$a_n = \frac{1}{2\pi i} \int_{\partial R} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

So, $\mathcal{H}(\Omega) \subset \mathcal{A}(\Omega)$. □

An immediate consequence of the above is that every holomorphic function is infinitely differentiable. Every holomorphic function has a Taylor series expansion,

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

with $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw$, for some small circle γ around z_0. If $|f(z)| \leq M$ on Ω, by taking modulus on both sides we have,

$$|f^{(n)}(z)| \leq \frac{Mn!}{\mathrm{dist}(z,\partial\Omega)^n}$$

An immediate corollary of this is the Liouville's theorem, where $\Omega = B_R(0)$ where $R \to \infty$ and hence $\mathrm{dist}(z,\partial\Omega) \to \infty$ and we have, $f^{(k)} \equiv 0$ for all k

Corollary 1.5. (Liouville's Theorem) *Bounded entire functions are constant.* □

A corollary of Liouville's theorem is the fundamental theorem of algebra which says that every polynomial of positive degree, $P \in \mathbb{C}[z]$ has a complex zero i.e., $\exists z \in \mathbb{C}$ such that $P(z) = 0$ and has exactly as many zeros as the degree.

If $P(z)$ has no complex zeros, the function $f(z) = 1/P(z)$ is an entire function. If $P(z) = \sum_{i=0}^n a_i z^i$ with $a_n \neq 0$ we have,

$$|P(z)| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i||z|^i \geq \frac{1}{2}|a_n|R^n$$

for all $|z| = R$ with large enough R. Hence $f(z) = 1/P(z)$ is bounded and hence must be constant by Liouville's theorem. Hence there must be $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$. Every polynomial can be factored,

$$P(z) = a_n \prod_{i=1}^n (z-\alpha_i).$$

Another consequence of this is the principle of analytic continuation. Suppose $f \in \mathcal{H}(\Omega)$, Ω be a connected open set. If $f|_U \equiv 0$ for $U \subset \Omega$. Let $E_n = \{z \mid f^{(n)}(z) = 0\}$, and let

$$E = \bigcap_{n\geq 0} E_n$$

Since f is a continuous function E_n is a closed set, and hence E is a closed set. Since $f \in \mathcal{H}(\Omega)$ it can be locally written as a power series, $f(z) = \sum_{n\geq 0} a_n z^n$, So, if $f|_U \equiv 0$ we have $a_n = 0$ in a neighborhood, and hence E must be open. So E is both open and closed, and since it's non empty, as $U \subset E$ it must be Ω, and hence $f \equiv 0$ on Ω.

Theorem 1.6. (Maximum Modulus Principle) *$f \in \mathcal{H}(\Omega)$, $f \in C(\overline{\Omega})$, Ω bounded, then,*

$$|f(z)| \leq \max_{w\in\partial\Omega} |f(w)|.$$

Sketch of Proof

The first step is to prove the same for small ϵ neighborhoods. Suppose there exists a maximum modulus and attained at z_0, let U be an ϵ neighborhood of z_0. Let $\gamma_\rho = \rho e^{it}$ be the circle around z_0 of radius $\rho < \epsilon$. By Cauchy integral formula,

$$f(z_0) = \tfrac{1}{2\pi i}\int_{\gamma_\rho} \tfrac{f(z)}{z-z_0}dz = \tfrac{1}{2\pi i}\int_{[0,2\pi]} \tfrac{f(z_0+\rho e^{it})}{\rho e^{it}} \cdot i\rho e^{it}dt = \tfrac{1}{2\pi i}\int_{[0,2\pi]} f(z_0+\rho e^{it})dt.$$

Taking modulus we get,

$$|f(z_0)| \le \frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0+\rho e^{it})|dt.$$

Since the maximum modulus is attained for z_0, we have, $|f(z_0)| \le |f(z)|$ for all $z \in U$. Hence we have,

$$\frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0+\rho e^{it})|dt \le \frac{1}{2\pi i}\int_{[0,2\pi]} |f(z_0)|dt = |f(z_0)|.$$

So we have,

$$0 = \frac{1}{2\pi i}\int_{[0,2\pi]} \underbrace{(|f(z_0)| - |f(z_0+\rho e^{it})|)}_{\ge 0} dt$$

Hence we have $|f(z_0)| = |f(z)|$ for all $z \in \gamma_\rho$. Since ρ was arbitrary the equality holds for all $z \in U$. Now for the general case, let $z_0 \in D$ for which $|f(z_0)| \ge |f(z)|$ for all $z \in D$. For any $w \in D$ consider a path joining z_0 and w, the ϵ neighborhoods cover the line, and by compactness of the path, only finitely many such ϵ-neighborhoods are required. For each of these ϵ-neighborhoods, the function f is constant, and hence we will get that $|f(w)| = |f(z_0)|$ □

Theorem 1.7. (Open Mapping Theorem) *Let Ω be connected open set. $f \in \mathcal{H}(\Omega)$. If f is not a constant map, then f is an open map from Ω to $\mathbb{C}$ i.e., $f(U)$ is open for all U open.*

Sketch of Proof

Without loss of generality assume that $f(x) = 0$.[3] Let $B_\epsilon(x)$ be a neighborhood of x contained in Ω such that $f(z) \ne 0$ for $z \in \overline{B_\epsilon(x)}$. This happens because in a neighborhood U we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(z-x)^n$$

and $a_0 = 0$, and hence we can write it as $(z-x)^k \sum_{n=k}^{\infty} a_n(z-x)^n$, so in a small enough neighborhood V, we will have, $\{f(z) = 0 \mid z \in V\} = \{x\}$.

Let $\delta = \inf\{|f(z)| \;:\; |z-x| \le \epsilon\}$ i.e., the smallest value taken by f at the boundary of $B_\epsilon(x)$. Now we claim that if $w \notin f(B_\epsilon)$ then it belongs to a closed set. Hence showing that $f(B_\epsilon(x))$ is open in $\mathbb{C}$. To show this, consider the function $\varphi(z) = 1/(f(z)-w) \in \mathcal{H}(\Omega)$. By Maximum modulus principle, it takes a maximum value at the boundary and hence,

$$\tfrac{1}{|w|} = |\varphi(x)| \le \sup_{|z-x|=\epsilon} |\varphi(x)| = \tfrac{1}{\inf_{|z-x|=\epsilon}\{|f(z)-w|\}}.$$

But now we have, $|f(z)-w| \ge |f(z)| - |w| \ge \delta - |w|$. Which gives us $|w| \ge \frac{1}{2}\delta$. □

[3] Otherwise consider $f - f(x)$.

Morera's theorem provides a sort of converse to Cauchy's theorem. We now have the necessary tools for it's proof.

Theorem 1.8. (Morera's Theorem) *$f \in C(\Omega)$, if for every rectangle, $R \subset \Omega$,*

$$\int_{\partial R} f dz = 0,$$

then $f \in \mathcal{H}(\Omega)$.

Sketch of Proof

The idea is to find a local holomorphic primitive F of f, i.e., $F' = f$, and then by infinite differentiability of holomorphic functions we would have proved that f is itself holomorphic.

We will prove for the case when Ω is a disc of radius r around the origin. Let $\gamma(t) = tz$ be the straight line joining 0 and z. Set,

$$F(z) := \int_0^z f(w)dw = z \int_{[0,1]} f(tz)dt$$

for all $|z| < r$. For small h, it follows that

$$F(z+h) - F(z) = \int_z^{z+h} f(w)dw$$

here we used the assumption about integral being zero for rectangles in the region Ω, and did Riemann integral type trick for the line joining z and $z+h$. The line segment is parametrized as $z + th$ with $t \in [0,1]$. We obtain,

$$\tfrac{F(z+h)-F(z)}{h} = \int_{[0,1]} f(z+th)dt \xrightarrow[h \to 0]{} f(z)$$

Hence $F \in \mathcal{H}(\Omega)$ and $F' = f$, and hence $f \in \mathcal{H}(\Omega)$. $\square$

Theorem 1.9. (Branch of Logarithm) *Let Ω be simply connected, Let $f \in \mathcal{H}(\Omega)$ such that $f \neq 0$ on Ω. Then there exists $g \in \mathcal{H}(\Omega)$ with $e^{g(z)} = f(z)$. g is unique upto constant addition by $2\pi n$.*

Proof

Consider the function $\frac{f'}{f} \in \mathcal{H}(\Omega)$. It's in $\mathcal{H}(\Omega)$ because f doesn't vanish on Ω. Since any two curves with same end points in a simply connected space are homotopic, we have a well defined integral,

$$g(z) := \textstyle\int_{z_0}^{z} \frac{f'(w)}{f(w)} dw.$$

that doesn't depend on the path from z_0 to z. Now,

$$\frac{g(z+h) - g(z)}{h} = \int_{[0,1]} \frac{f'(z+th)}{f(z+th)} dt \to \frac{f'(z)}{f(z)} \quad \text{as } h \to 0.$$

So, $g \in \mathcal{H}(\Omega)$ and $(fe^g)' = 0$ which means that $e^g = cf$ which is means $e^{g-k} = f$ for some constant k, unique upto addition by $2\pi in$. $\square$

Let f be a continuous function on Ω, and γ be a piecewise differentiable curve in Ω, we can define a function g on $\mathbb{C}\backslash\text{Im}\gamma$ by,

$$g(w) = \int_\gamma \frac{f(z)}{z-w} dw.$$

For fixed w, and for small $h \in \mathbb{C}$, we have,

$$\tfrac{(g(w+h)-g(w))}{h} = \int_\gamma f(z)\big(\tfrac{1}{z-w-h} - \tfrac{1}{z-w}\big) \cdot \tfrac{1}{h} dz$$

as $h \to 0$, we have, $\big(\frac{1}{z-w-h} - \frac{1}{z-w}\big) \cdot \frac{1}{h} \to \frac{1}{(z-w)^2}$ so we have,

$$g'(w) = \int_\gamma \frac{f(z)}{(z-w)^2} dz.$$

In particular $g \in \mathcal{H}(\Omega)$.

Theorem 1.10. (Weierstrass) *$\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$, $\{f_n\} \to f$ uniformly, then $f \in \mathcal{H}(\Omega)$ and $\{f_n'\} \to f'$ uniformly.*

Proof

Let $B_r(z_0)$ be a ball around $z_0 \in \Omega$ with radius r, such that $\overline{B_r(z_0)} \subset \Omega$, for $|w| < \rho < r$, and $\gamma_r = re^{2\pi i t}$, we have by Cauchy's integral formula,

$$f_n(w) = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f_n(z)}{z-w} dw.$$

which by assumption converges uniformly to the continuous function f,

$$f(w) = \lim_{n\to\infty} f_n(w) = \lim_{n\to\infty} \tfrac{1}{2\pi i} \int_\gamma \tfrac{f_n(z)}{z-w} dw = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f(z)}{z-w} dw.$$

This function must hence be holomorphic in $B_r(z_0)$, and since the choice of z_0 was arbitrary we have that $f \in \mathcal{H}(\Omega)$. Since, $f_n'(w) = \int_\gamma \frac{f_n(z)}{(z-w)^2} dz$ and $\frac{1}{|z-w|^2} \leq \frac{1}{(r-\rho)^2}$ for $|z| = r$, the limit $\lim_{n\to\infty} f_n'(w)$ exists uniformly for $|w| \leq \rho$ and hence,

$$f'(w) = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f(z)}{(z-w)^2} dz$$

for any compact set $K \subset \Omega$, we can cover K with balls of the above type, and from this we can pick out a finite cover, from this a maximum bound can be picked hence proving the uniform convergence of the above limit. $\square$

Theorem 1.11. (Montel) *Let $\mathcal{F} \subset \mathcal{H}(\Omega)$ such that for any compact set $K \subset \Omega$ there exists $M_K > 0$ such that for all $f \in \mathcal{F}$,*

$$|f(z)| \leq M_K.$$

Then for any sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}_{k\geq 1} \subset \{f_n\}_{n\geq 1}$ which converges uniformly on every compact subset of Ω. The limit is given by the Weierstrass theorem.

Proof

References

[1] W Schlag, A Course in Complex Analysis and Riemann Surfaces, AMS, 2010

[2] R Narasimhan, Complex Analysis in One Variable, Second Edition Springer, 2000