Barbara Andre HW 3, October 17, 2014 STAT 560

Q1

$$S_1 = \begin{bmatrix} 2 & .5 \\ .5 & 2 \end{bmatrix} \qquad \det(S_1) = (4 - .25) = 3.75$$

$$S_2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \qquad \det(S_2) = (25 - 16) = 9$$

$$\begin{aligned}
b_{0_1} &= \frac{1}{2}\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3.75} & -\frac{.5}{3.75} \\ -\frac{.5}{3.75} & \frac{2}{3.75} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\log(3.75) = 0.66 \\
b_1 &= -\begin{bmatrix} \frac{2}{3.75} & -\frac{.5}{3.75} \\ -\frac{.5}{3.75} & \frac{2}{3.75} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
c_1 &= \frac{1}{2}\begin{bmatrix} \frac{2}{3.75} & -\frac{.5}{3.75} \\ -\frac{.5}{3.75} & \frac{2}{3.75} \end{bmatrix} = \begin{bmatrix} \frac{1}{3.75} & -\frac{0.25}{3.75} \\ -\frac{0.25}{3.75} & \frac{1}{3.75} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
b_{0_2} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\log(9) = \frac{1}{2} \times \frac{2}{9} + 1.10 = 1.21 \\
b_2 &= -\begin{bmatrix} \frac{5}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} \frac{1}{9} \\ \frac{1}{9} \end{bmatrix} \\
c_2 &= \frac{1}{2}\begin{bmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{5}{18} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{5}{18} \end{bmatrix}
\end{aligned}$$

Bayes classifier: identify as class 1 if:

$$0.66 + \mathbf{x}\begin{bmatrix} \frac{1}{3.75} & -\frac{0.25}{3.75} \\ -\frac{0.25}{3.75} & \frac{1}{3.75} \end{bmatrix}\mathbf{x}^T \le 1.21 - \begin{bmatrix} \frac{1}{9} & \frac{1}{9} \end{bmatrix}\mathbf{x}^T + \mathbf{x}\begin{bmatrix} \frac{5}{18} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{5}{18} \end{bmatrix}\mathbf{x}^T$$

see graphic on attached

Q2 Bayes classifier: identify as class 1 if:

$$\begin{aligned}
\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)^T \mathbf{I}\left(\mathbf{x} - \frac{1}{2}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\right) &\ge \log\left(\frac{\pi_1}{\pi_2}\right) \\
\begin{bmatrix} 1 & 1 \end{bmatrix}\left(\mathbf{x} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) &\ge 0
\end{aligned}$$

Thus:

$$\begin{aligned}
\mathbf{v} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\mathbf{x}_0 &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}
\end{aligned}$$

Q3 Banknotes

Classifier	LDA		QDA		Nearest Centroid	
Fold	Train	Test	Train	Test	Train	Test
cv1	0	0.006	0	0.011	0	0
cv2	0	0.072	0	0.172	0	0.039
cv3	0	0.006	0	0.006	0	0.022
cv4	0	0.011	0	0.050	0	0.006
cv5	0	0.017	0	0.211	0	0.006
cv6	0	0.039	0	0.067	0	0.022
cv7	0	0.006	0	0.011	0	0.006
cv8	0	0.011	0	0.100	0	0.006
cv9	0	0.033	0	0.044	0	0.006
cv10	0	0.000	0	0.111	0	0
Mean misclassification rate	0	0.0161	0	0.078	0	0.011

Q4

$$\begin{aligned}
\Sigma_{11} &= \Sigma_{22} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad \Sigma_{11}^{-1} = (1-\rho)^2 \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \\
\Sigma_{12} &= \Sigma_{21} = \begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix} \\
B &= A = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\
&= (1-\rho^2)\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix}(1-\rho^2)\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\begin{bmatrix} \rho & \rho \\ \rho & \rho \end{bmatrix} \\
&= \begin{bmatrix} \frac{\rho^2}{1-\rho^2} & \frac{\rho^2}{1-\rho^2} \\ \frac{\rho^2}{1-\rho^2} & \frac{\rho^2}{1-\rho^2} \end{bmatrix}
\end{aligned}$$

$$P \overset{set}{=} \frac{\rho^2}{1-\rho^2}$$

To find the eigenvalues and eigenvectors, solve

$$\begin{aligned}
\begin{bmatrix} P-\lambda & P \\ P & P-\lambda \end{bmatrix} &= \mathbf{0} \\
(P-\lambda)^2 - P^2 &= 0 \\
P^2 - 2P\lambda + \lambda^2 - P^2 &= 0 \\
\lambda^2 - 2P\lambda &= 0 \\
\lambda^2 &= 2P\lambda \\
\lambda &= 2P = \frac{\rho^2}{1-\rho^2}
\end{aligned}$$

Find (x, y) such that

$$\begin{aligned}
\begin{bmatrix} P & P \\ P & P \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= 2P\begin{bmatrix} x \\ y \end{bmatrix} \\
x &= y \\
(x,y) &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \mathbf{a} = \mathbf{b}
\end{aligned}$$

The canonical variates are $\hat{U} = \mathbf{a}^T\mathbf{X}$ and $\hat{V} = \mathbf{b}^T\mathbf{Y}$.

$$\begin{aligned}
\hat{U} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\mathbf{X} = \frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2 \\
\hat{V} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\mathbf{Y} = \frac{1}{\sqrt{2}}Y_1 + \frac{1}{\sqrt{2}}Y_2
\end{aligned}$$

Q5

- Canonical Covariates pair and correlation

$$\begin{aligned}
U_1 &= X_1 - 0.003X_2 \\
V_1 &= -.52Y_1 - 0.85Y_2 \\
\rho_1^2 &= 0.107
\end{aligned}$$

$$\begin{aligned}
U_2 &= -.52X_1 + 0.85X_2 \\
V_2 &= = .92X_1 + 0.38X_2 \\
\rho_2^2 &= 0.029
\end{aligned}$$

- Hypothesis test
  Calculated statistic $= -(48 - 1 - \frac{1}{2}(2+2+1))\log\prod_{i=1}^{2}(1-\hat{\rho}^2) = 6.34$
  $\chi^2_{2*2}(.05) = 9.49$ The calculated statistic is less than the critical value.
  Do not reject $H_0 : \Sigma_{22} = 0$ at $\alpha = 0.05$.