Complicate Steady States Analysis

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Example 1

Transition matrix

Let us consider the Markov chain with the following transition matrix

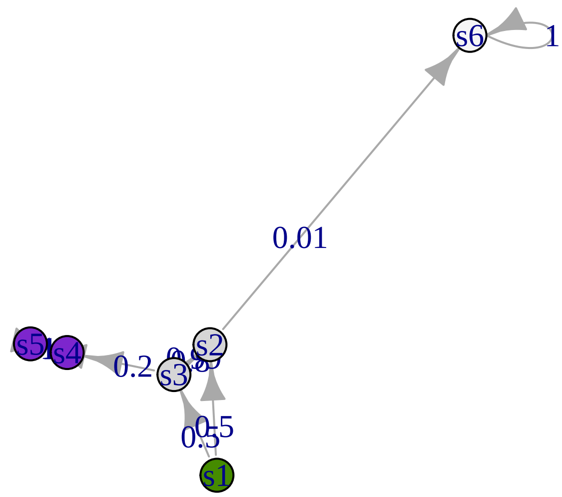
$$M = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 & p \\ 0 & 1-q & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $1 > p > 0$ close to zero, e.g. $p = 1\%$ and $1 > q > 0$, e.g. $q = 0.2$.

R implementation

From the diagram or M, it is easy that there are absorbing states (6) and recurrent states (4,5).

```
library(markovchain)
M<-matrix(c(0,0.5,0.50,0.0,0,0.00,0,0.0,0.99,0.0,0,0.01,0,0.8,0.00,
                0.2,0,0.00,0,0.0,0.00,0.0,1,0.00,
0,0.0,0.00,1.0,0,0.00,0,0.0,0.00,0.0,0,1.00), nrow=6, byrow=TRUE)
colnames(M) <- rownames(M) <- paste0("s",1:6)
chain1 <- as(M,"markovchain")
plot(chain1)
```



Invariant measure

We need to solve the following system in order to get the invariant measure

$$\mu M = \mu \Leftrightarrow \begin{cases} 0 = \mu_1 \\ 0.5\mu_1 + (1-q)\mu_3 = \mu_2 \\ 0.5\mu_1 + (1-p)\mu_2 = \mu_3 \\ q\mu_3 + \mu_5 = \mu_4 \\ \mu_4 = \mu_5 \\ p\mu_2 + \mu_6 = \mu_6 \end{cases} \Leftrightarrow \begin{cases} \mu_1 = 0 \\ (1-q)\mu_3 = \mu_2 \\ (1-p)\mu_2 = \mu_3 \\ q\mu_3 + \mu_4 = \mu_4 \\ \mu_5 = \mu_4 \\ p\mu_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = 0 \\ \mu_5 = \mu_4 \end{cases}$$

Using the probability condition $\sum_i \mu_i = 1$, we get $2\mu_4 + \mu_6 = 1$. So there exists an infinite number of invariant measures of the form (as long as $p > 0$)

$$\mu = (0, 0, 0, \nu, \nu, 1 - 2\nu), \nu \in [0, 1/2].$$

Note that there are independent of the value of p and q. Indeed, we have

```
mu <- c(0, 0, 0, 1/4, 1/4, 1/2)
mu %*% M
```

```
##      s1 s2 s3   s4   s5  s6
## [1,]  0  0  0 0.25 0.25 0.5
```

```
mu <- c(0, 0, 0, 0, 0, 1)
mu %*% M
```

```
##      s1 s2 s3 s4 s5 s6
## [1,]  0  0  0  0  0  1
```

```r
mu <- c(0, 0, 0, 1/2, 1/2, 0)
mu %*% M
```

```
##      s1 s2 s3  s4  s5 s6
## [1,]  0  0  0 0.5 0.5  0
```

```r
steadyStates(chain1)
```

```
##      s1 s2 s3  s4  s5 s6
## [1,]  0  0  0 0.0 0.0  1
## [2,]  0  0  0 0.5 0.5  0
```

Classification of states

Let $T^{x\to x}$ is the number of periods to go back to state x knowing that the chain starts in x.

- A state x is recurrent if $P(T^{x\to x} < +\infty) = 1$ (equivalently $P(T^{x\to x} = +\infty) = 0$).
  - A state x is null recurrent if in addition $E(T^{x\to x}) = +\infty$.
  - A state x is positive recurrent if in addition $E(T^{x\to x}) < +\infty$.
  - A state x is absorbing if in addition $P(T^{x\to x} = 1) = 1$.
- A state x is transient if $P(T^{x\to x} < +\infty) < 1$ (equivalently $P(T^{x\to x} = +\infty) > 0$).

On the example above, using the transition matrix, we have

$$\begin{array}{l}
\forall k > 0, P(T^{s1\to s1} = k) = 0 \Rightarrow P(T^{s1\to s1} = +\infty) = 1,\\
P(T^{s4\to s4} = 2) = 1 \Rightarrow P(T^{s4\to s4} = +\infty) = 0,\\
P(T^{s5\to s5} = 2) = 1 \Rightarrow P(T^{s5\to s5} = +\infty) = 0,\\
P(T^{s6\to s6} = 1) = 1 \Rightarrow P(T^{s6\to s6} = +\infty) = 0.
\end{array}$$

So the states $s4, s5, s6$ are recurrent (since $E(T^{s4\to s4}) = 2 = E(T^{s5\to s5})$ and $E(T^{s6\to s6}) = 1$, they are positive recurrent), whereas the state $s1$ is transient. Furthermore, for the last two states, we have

$$\begin{array}{l}
P(T^{s2\to s2} = 2k) = 0.8^k(1-p)^k, p > 0 \Rightarrow P(T^{s2\to s2} = +\infty) = 0,\\
P(T^{s3\to s3} = 2k) = (1-p)^k 0.8^k, p > 0 \Rightarrow P(T^{s3\to s3} = +\infty) = 0.
\end{array}$$

So the states $s2, s3$ are recurrent (since $E(T^{s2\to s2}) = E(T^{s3\to s3}) = 1/(1-0.8(1-p)) < +\infty$, they are positive recurrent).

Analysis of the eigenvalues and eigenvectors

Let us compute the characteristic polynom of the matrix M

$$
\begin{aligned}
\chi_M(x) &= \begin{vmatrix} -x & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & -x & 1-p & 0 & 0 & p \\ 0 & 1-q & -x & q & 0 & 0 \\ 0 & 0 & 0 & -x & 1 & 0 \\ 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-x \end{vmatrix} = \begin{vmatrix} -x & 1/2 & 1/2 & 0 & 0 \\ 0 & -x & 1-p & 0 & 0 \\ 0 & 1-q & -x & q & 0 \\ 0 & 0 & 0 & -x & 1 \\ 0 & 0 & 0 & 1 & -x \end{vmatrix} (1-x) \\
&= -(1-x) \begin{vmatrix} -x & 1/2 & 1/2 & 0 \\ 0 & -x & 1-p & 0 \\ 0 & 1-q & -x & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} - x(1-x) \begin{vmatrix} -x & 1/2 & 1/2 & 0 \\ 0 & -x & 1-p & 0 \\ 0 & 1-q & -x & q \\ 0 & 0 & 0 & -x \end{vmatrix} \\
&= -(1-x) \begin{vmatrix} -x & 1/2 & 1/2 \\ 0 & -x & 1-p \\ 0 & 1-q & -x \end{vmatrix} + x^2(1-x) \begin{vmatrix} -x & 1/2 & 1/2 \\ 0 & -x & 1-p \\ 0 & 1-q & -x \end{vmatrix} \\
&= +x(1-x) \begin{vmatrix} -x & 1-p \\ 1-q & -x \end{vmatrix} - x^3(1-x) \begin{vmatrix} -x & 1-p \\ 1-q & -x \end{vmatrix} \\
&= [x(1-x) - x^3(1-x)](x^2 - (1-p)(1-q)) = x(1-x)^2(1+x)(x-(1-p)(1-q))(x+(1-p)(1-q))
\end{aligned}
$$

Therefore the eigenvalues are $\pm 1, 0, \pm(1-p)(1-q)$ with 1 of multiplicity 2. Hence the power of M is

$$
M^n = P \begin{vmatrix} 1^n & 0 & 0 & 0 & 0 & 0 \\ 0 & 1^n & 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-p)^n(1-q)^n & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-p)^n(1-q)^n \end{vmatrix} P^{-1}
$$

with an appropriate matrix P changing base in $\mathbb{R}^6$.

Numerical consideration

The classification of states is given by dedicated functions

```r
communicatingClasses(chain1)
```

```
## [[1]]
## [1] "s1"
##
## [[2]]
## [1] "s2" "s3"
##
## [[3]]
## [1] "s4" "s5"
##
## [[4]]
## [1] "s6"
```

```r
recurrentClasses(chain1)
```

```
## [[1]]
## [1] "s4" "s5"
```

```
##
## [[2]]
## [1] "s6"
```

```r
absorbingStates(chain1)
```

```
## [1] "s6"
```

```r
transientStates(chain1)
```

```
## [1] "s1" "s2" "s3"
```

Let us compute power of M. Note that even powers of M yield to a matrix different than odd powers since $P(T^{s4\to s4} = 2) = P(T^{s5\to s5} = 2) = 1$.

```r
library(expm)
```

```
## Loading required package: Matrix
```

```
##
## Attaching package: 'expm'
```

```
## The following object is masked from 'package:Matrix':
##
##     expm
```

```r
M %^% 1000
```

```
##    s1           s2           s3        s4        s5         s6
## s1  0 1.163927e-51 1.440359e-51 0.4807692 0.4759615 0.04326923
## s2  0 2.304575e-51 0.000000e+00 0.9519231 0.0000000 0.04807692
## s3  0 0.000000e+00 2.304575e-51 0.0000000 0.9615385 0.03846154
## s4  0 0.000000e+00 0.000000e+00 1.0000000 0.0000000 0.00000000
## s5  0 0.000000e+00 0.000000e+00 0.0000000 1.0000000 0.00000000
## s6  0 0.000000e+00 0.000000e+00 0.0000000 0.0000000 1.00000000
```

```r
M %^% 1001
```

```
##    s1           s2           s3        s4        s5         s6
## s1  0 1.152287e-51 1.152287e-51 0.4759615 0.4807692 0.04326923
## s2  0 0.000000e+00 2.281529e-51 0.0000000 0.9519231 0.04807692
## s3  0 1.843660e-51 0.000000e+00 0.9615385 0.0000000 0.03846154
## s4  0 0.000000e+00 0.000000e+00 0.0000000 1.0000000 0.00000000
## s5  0 0.000000e+00 0.000000e+00 1.0000000 0.0000000 0.00000000
## s6  0 0.000000e+00 0.000000e+00 0.0000000 0.0000000 1.00000000
```

Invariant measures are the eigenvectors associated to the eigen value 1 for the transposed matrix M^T such that $\mu_i \geq 0$. If in addition $\sum_i \mu_i = 1$, then it is a probability measure.

Let us compute the eigenvalue. We don't retrieve the theoretical values $\pm 1, 0, \pm(1-p)(1-q)$.

```r
c(1, -1, 0, .99*.8, -.99*.8)
```

```
## [1]  1.000 -1.000  0.000  0.792 -0.792
```

```r
eigen(t(M))
```

```
## eigen() decomposition
## $values
## [1]  1.0000000  1.0000000 -1.0000000  0.8899438 -0.8899438  0.0000000
##
## $vectors
```

```
##                [,1] [,2]          [,3]        [,4]         [,5]
## [1,]  0.000000e+00    0  0.000000e+00  0.00000000  0.000000000
## [2,]  1.161938e-15    0  5.551115e-17  0.48246436 -0.482927050
## [3,]  1.442406e-15    0 -1.665335e-16  0.53670772  0.537222429
## [4,]  7.063412e-01    0 -7.071068e-01 -0.45926896  0.459709403
## [5,]  7.063412e-01    0  7.071068e-01 -0.51606512 -0.516560027
## [6,] -4.652096e-02    1 -2.168404e-19 -0.04383801  0.002555246
##                [,6]
## [1,]  7.758348e-01
## [2,] -3.918357e-01
## [3,] -4.848967e-01
## [4,]  2.153374e-17
## [5,]  9.697934e-02
## [6,]  3.918357e-03
```

Keeping those equalling 1, we obtain

```
eigen(t(M))$vectors[,eigen(t(M))$values >= 1]
```

```
##                [,1] [,2]
## [1,]  0.000000e+00    0
## [2,]  1.161938e-15    0
## [3,]  1.442406e-15    0
## [4,]  7.063412e-01    0
## [5,]  7.063412e-01    0
## [6,] -4.652096e-02    1
```

So numerically, only one (probability) measure is obtained $(0, 0, 0, 0, 0, 1)$. That's rather logical.

One can get the "boundary" measures by looking to the subset of recurrent states, extracting positive eigenvectors associated to the eigenvalue 1 and normalize ti. Indeed

```
Msub <- M[rownames(M) %in% unlist(recurrentClasses(chain1)), colnames(M) %in% unlist(recurrentClasses(c
eigen(Msub)$vectors[,eigen(Msub)$values >= 1]
```

```
##      [,1]      [,2]
## [1,]    0 0.7071068
## [2,]    0 0.7071068
## [3,]    1 0.0000000
```

```
t(eigen(Msub)$vectors[,eigen(Msub)$values >= 1]) / colSums(eigen(Msub)$vectors[,eigen(Msub)$values >= 1
```

```
##      [,1] [,2] [,3]
## [1,]  0.0  0.0    1
## [2,]  0.5  0.5    0
```

The `steadyStates` function in (Spedicato 2017) implements the abovementioned algorithm when negative values are found in the eigenvectors of the full matrix corresponding to unitary eigenvalues.

References

Spedicato, Giorgio Alfredo. 2017. "Discrete Time Markov Chains with R." *The R Journal*. https://journal.r-project.org/archive/2017/RJ-2017-036/index.html.