A gentle introduction to Bayesian statistics and modeling - Part I

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Probability according to Cicero

"That is probable which for the most part usually comes to pass, or which is a part of the ordinary beliefs of mankind, or which contains in itself some resemblance to these qualities, whether such resemblance be true or false."

– Cicero, ca. 80 BCE

The Frequentist vs. The Bayesian

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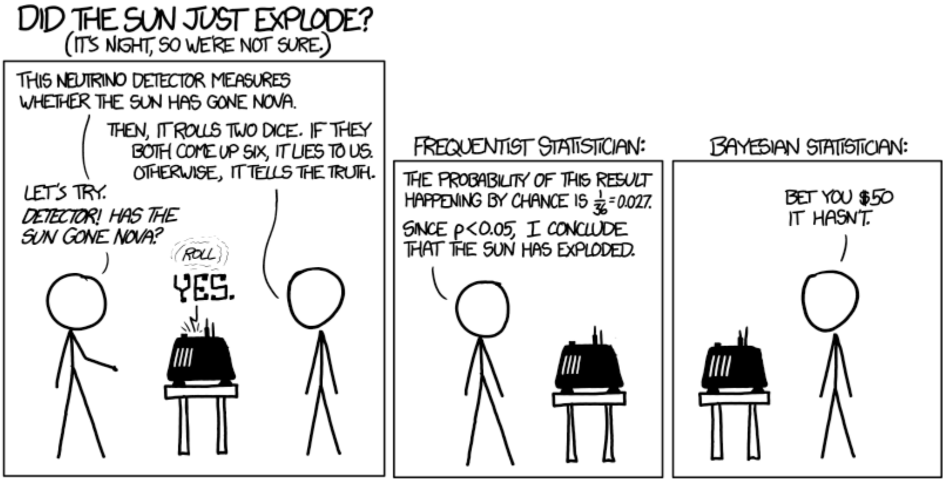

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Probabilities and uncertainties can only be expressed in terms of *repeated sampling* from some target population.

Null hypothesis significance testing

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4. Reject H_0 if $p < \alpha$ where α is an arbitrary significance threshold.

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- It says **absolutely nothing** about the alternative hypothesis, H_A.

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- When is the "true" linear relationship between two values ever *exactly* zero?
- When is the "true" difference between means every *exactly* zero?
- What is the point of falsifying something that we know *a priori* must be false?

Problems with NHST

3. Inference based on a test statistic $\tilde{z}$ is inference based on **hypothetical data** that was never observed.

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Student's t-test assumptions

- The sample mean is normally distributed.
- The sample variance is χ^2 distributed (or the samples are normally distributed).
- The sample mean and sample variance are independent.
- The variances of both populations are equal (relaxed for Welch's t-test).

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- Uncertainty should be quantified in terms of how well our hypotheses fit the data we have and not against hypothetical data that was never observed (see The Likelihood Principle).

Problems with NHST

These criticisms are not new...

- Berkson J. **Some difficulties of interpretation encountered in the application of the chi-square test.** Journal of the American Statistical Association. 1938.
- Rozeboom WW. **The fallacy of the null-hypothesis significance test.** Psychological bulletin. 1960.
- Berger JO, Sellke T. **Testing a point null hypothesis: The irreconcilability of p values and evidence.** Journal of the American statistical Association. 1987.
- Johnson DH. **The insignificance of statistical significance testing.** The journal of wildlife management. 1999.

Problems with NHST

. . . and have continued over time:

- Wasserstein RL, Schirm AL, Lazar NA. **Moving to a world beyond "p< 0.05"**. 2019.
- Amrhein V, Greenland S, McShane B. **Scientists rise up against statistical significance.** Nature. 2019 Mar;567(7748):305-7.
- Gelman A, Stern H. **The difference between "significant" and "not significant" is not itself statistically significant.** The American Statistician. 2006 Nov 1;60(4):328-31.
- Vasishth S, Mertzen D, Jäger LA, Gelman A. **The statistical significance filter leads to overoptimistic expectations of replicability.** Journal of Memory and Language. 2018 Dec 1;103:151-75.

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Just like p-values, CIs represent **pre-observational probabilities** over the sampling procedure itself.

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- The basic theory of random sampling and relative frequency are deceptively intuitive. . .
- . . . but using the corresponding procedures to answer common research questions is actually not!

When to still be frequentist

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- Nonparametrics (e.g. bootstrapping) for large, high-dimensional datasets

Recommendations

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- **Do not** equate "statistical significance" with *scientific significance*.
- **Do not** selectively show results based on p-values or confidence intervals.
- **Do not** highlight certain results as "statistically significant" based on their p-values or confidence intervals.
- **Avoid** NHST in favor of either the more rigorous Neyman-Pearson theory or a Bayesian approach.

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- **Do** be as transparent as possible when reporting statistical results.
- **Do** prefer problem-specific error metrics and goodness of fit measures like R^2 to p-values.
- **Do** use resampling or out-of-sample testing to assess robustness of your results.

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In general, it's good to be Bayesian by default :)

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...or it can be an arbitrarily large set of parameters for a complex model.

Statistical inference via Bayes rule

Once we have a summary of $P(\theta|\mathcal{D})$, we can ask questions like:

What is the probability that $\mu > 0$?

What is the probability that $\mu_1 > \mu_2$ for groups 1 and 2?

What is the probability that the slope of a linear regression $\beta > 0$?

Maximum a posteriori (MAP) estimation

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Notice that maximum likelihood estimation (MLE) is just a special case of MAP.

Estimating the posterior distribution

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Unfortunately, that's pretty hard!

$$P(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta)P(\theta)}{\int_\theta P(\mathcal{D}|\theta)P(\theta)d\theta}$$

The integral on the bottom is typically intractable for non-trivial models.

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Modern state-of-the-art samplers are typically variants **Hamiltonian Monte Carlo** (HMC) which require gradients.

Hamiltonian Monte Carlo

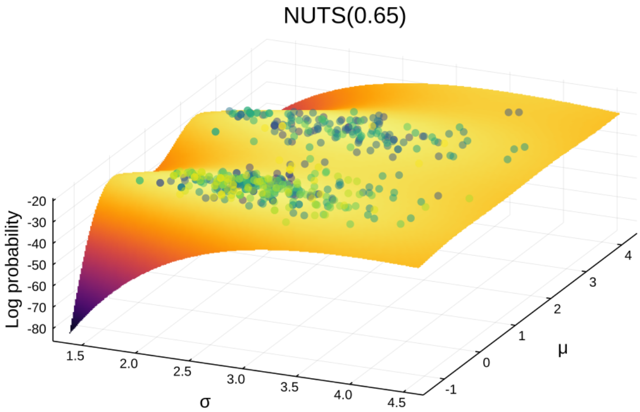

Figure 1: NUTS sampler visualization (source: Turing.jl documentation)

Interpreting the output

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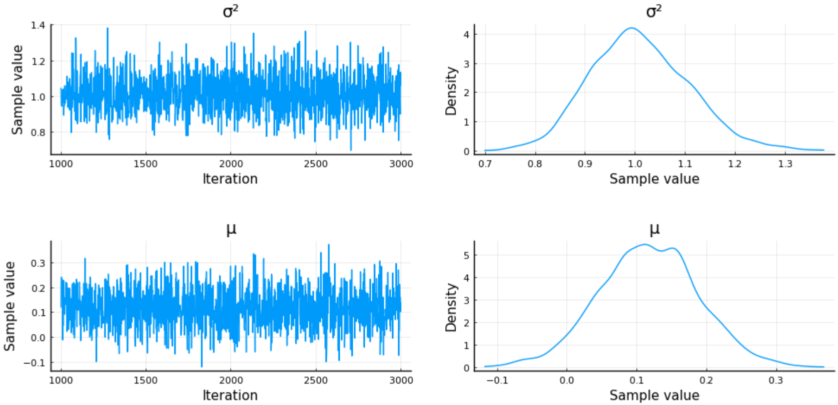
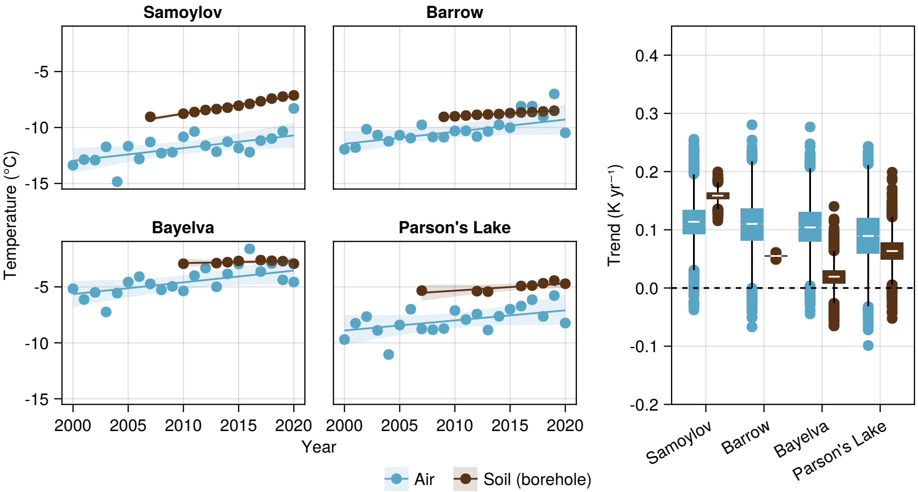


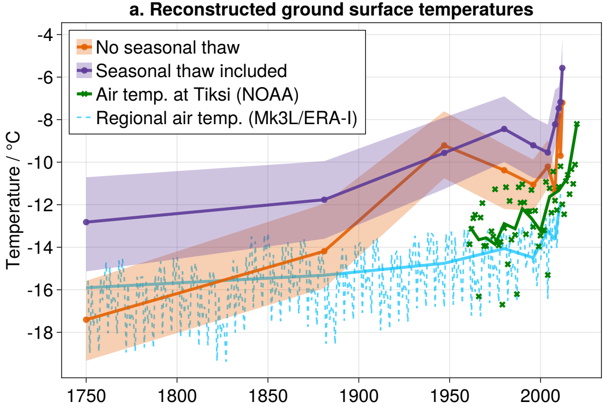
Overview of algorithms for Bayesian inference

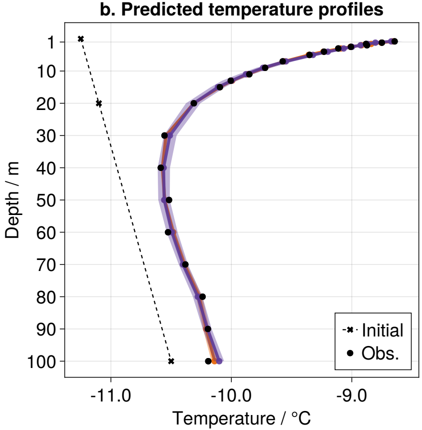
	Forward runs	Gradients required?	Parallelizable?	Asymptotic convergence
MCMC	10^4 to 10^6	No	No	Yes
HMC	10^3 to 10^5	Yes	No	Yes
VI	10^2 to 10^4	Yes	Batch	No
SMC	10^3 to 10^5	No	Yes	Yes
EKS	10^3 to 10^4	No	Yes	Gaussian
IS	10^1 to 10^3	No	Yes	No

Use case: Multi-site trend analysis



Use case: Surface temperature reconstruction





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- MCMC is computationally expensive, especially with large numbers of parameters.
- Practical analysis of the inference results can be time consuming.
- Software support is not as comprehensive as for “classical” frequentist analyses.

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- Probabilistic programming

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- Bayesian generalized linear models

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- Model diagnostics and criticism

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- Bayesian generalized linear models
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- Example application to observational data from the Arctic

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Follow along!

Part II of the workshop consists of a series of Jupyter notebooks.

You can directly open these notebooks on Google Colab if you do not want to bother setting up a new python environment!