Markov Chains and Applications

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1. Introduction to Markov Chains

Markov chains are a mathematical principle that allows you to analyze the flow in networks between different elements based on probability. For example, your meal for the day can be depicted by a Markov Chain, where if you are starting at pizza, there is a .2 probability you have hotdog, .5 you have pizza, and .3 you have a burger. This creates loops of probabilities to forecast and show future stages. When connecting this to linear algebra, matrix can be used as well as a key equation to solving the value of future stages. That equation are as follows:

$$Px_i = x_{i+1}\ to\ find\ the\ next\ state\ where\ if\ both\ x\ equal\ then\ stable$$

Through these mathematical principles we are able to make multiple staged probability forecasts looking into the future to better understand and predict. Using principles of matrix multiplication we are able to find the next x matric to have probabilities of different stages until those x values hit an equilibrium. Sometimes we will have to use the transpose of P, which means P's values flipped over a diagonal so that each column adds up to 1.

Some concepts mentioned below are as follows: Matrix multiplication is the concept of taking a table and multiplying it by another using a formula of multiplying across and down to get the resultant. The higher power function is when you use the above $Px=x_{stable}$ function however raising P to a factor of 100 or more, this helps predict a large number of steps to find equilibrium where the result wont change. Note that all of the math mentioned above was done using MatLab to solve the values to be expected and use the concepts mentioned above to record vectors and perform matrix multiplication.

2. Use in Predicting Locations

One example of this principle can be seen in where a worker might be between different sites during different stages. Suppose that the worker started at site 2 and we wanted to forecast where

$$\begin{bmatrix} 0.10 & 0.20 & 0.30 & 0.15 & 0.25 \\ 0.05 & 0.35 & 0.10 & 0.40 & 0.10 \\ 0 & 0 & 0.35 & 0.55 & 0.10 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 \\ 0.33 & 0.32 & 0 & 0 & 0.35 \end{bmatrix}$$

they might be using a given probability seen below with chance of transition between sites, given the y values are starting locations and x are destination. Using this probability matrix we are able to use $(0,1,0,0,0)^T$ to multiply it by to forecast out where the person will be after 3 rounds to see what the probability they will be at site 5. Using the equation from section 1 we can forecast a probability of .1162. Now let’s say we wanted to see if 100 people started with 20 at each site that after 4 steps where people have gone to. Using the same principle but starting with $x=(20,20,20,20,20)^T$, we see that after 4 rounds 14, 22, 21,30, and 12 people at site 1-5 respectively. Finally we bring in the concept of having a steady state vector of P. This means what after a period of steps, the values of the resulting probabilities or locations do not change. Using the concept of P to a high-power multiplied by an input matrix, we see that a stable state is found saying that once a large number of steps are run, the resulting percentage for each state will be (.1408, .2195, .2154, .3032, .1211) respectively. This means that someone being at site 1 no matter the start after a lot of switches is 14.08% and the rest correlate with the remaining 4 sites.

3. Modeling Influenza using Markov Chains

This idea of Markov Chains can be applied to various scenarios, such as modeling the spread of

$$\begin{bmatrix} .84 & .16 \\ .40 & .60 \end{bmatrix}$$

influenza. To model the student body of Malady College we have a campus of 5000 students with infection rates seen below, where .16 is non infected becoming infected and .40 being recovering from. When using this to model, we can predict after how many people will have the flu on a set day while also finding equilibrium. For example, if 100 people had it initially, using the equation with starting vector [4900; 100] from above we see that we expect that 3,839 people in the study body won't have the flu while 1,161 will have the flu. We find using the high-power equation that we need roughly 3,571 non infected for equilibrium. By using the Px=x equation we find that this will happen after roughly 10 days. While it gets close in that 3000 range for a few days prior, it does not truly hit the equilibrium point until that 10th day where we get the expected values. Using the higher power function we can find the desired stable vector and then recursively solve until we find the stable vector. We can also use this to look at retrospective data and predict from that. For example, on the third day let's say we 1400 students infected. On day 0 or the origin date we can predict that we will have 1,093 infected people. To solve this we multiple the inverse of the transition matrix mentioned above by the vector X starting at [3600; 1400] to reach state 3, the resulting x from the recursive function gives values. The reason the numbers are so close is because we are almost at equilibrium

2. Conclusion

Using these principles of linear algebra, we are able to predict the flow of either people or flu or many other circumstances that can be depicted by a Markov Chain by applying principles of matrix multiplication.

3. Bibliography

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