14D006 Stochastic Models and Optimization

Problemset 3

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1 Inventory Control with Forecasts

2 Inventory Pooling

We want to compare the optimal expected cost between a decentralized and a pooled inventory system with n locations. Let the demand at each location i be independently, normally distributed with mean μ and variance σ^2, i.e. $D_1, ...D_N \overset{iid}{\sim} N(\mu, \sigma^2)$. Then $\sum_{i=1}^{N} D_i \overset{d}{=} \sqrt{n}D_1 + \mu(n - \sqrt{n})$.

Let h and b denote the inventory holding cost and backorder cost, respectively. We already know that if $G(Q^*) = \mathbb{E}_D[h(Q^* - D)_+ + b(D - Q^*)_+]$ denotes the optimal expected cost for each location (with optimal order quantity $Q^* = \inf\left\{Q \geq 0 : \mathbb{P}\left(D_1 \leq Q\right) \geq \frac{b}{b+h}\right\}$), then the optimal expected cost in the decentralized inventory system is

$$G_D(Q^*) = \sum_{i=1}^{N} G(Q^*) = nG(Q^*)$$

To find the optimal expected cost in the pooled inventory system $G_P(Q_P^*)$, we first need to determine the optimal order quantity Q_P^* under this system.

$$
\begin{aligned}
Q_P^* &= \inf\left\{Q \geq 0 : \mathbb{P}\left(\sum_{i=1}^{N} D_i \leq Q\right) \geq \frac{b}{b+h}\right\} \\
&= \inf\left\{Q \geq 0 : \mathbb{P}\left(\sqrt{n}D_1 + \mu(n-\sqrt{n}) \leq Q\right) \geq \frac{b}{b+h}\right\} \\
&= \inf\left\{\sqrt{n}X + \mu(n-\sqrt{n}) \geq 0 : \mathbb{P}\left(D_1 \leq X\right) \geq \frac{b}{b+h}\right\} \\
&= \sqrt{n}\inf\left\{X \geq 0 : \mathbb{P}\left(D_1 \leq X\right) \geq \frac{b}{b+h}\right\} + \mu(n-\sqrt{n}) \\
&= \sqrt{n}Q^* + \mu(n-\sqrt{n})
\end{aligned}
$$

With this we can now determine optimal expected cost in the pooled inventory system $G_P(Q_P^*)$:

$$
\begin{aligned}
G_P(Q_P^*) &= \mathbb{E}_{\{D\}}\left[h(Q_P^* - \sum_{i=1}^{N} D_i)_+ + b(\sum_{i=1}^{N} D_i - Q_P^*)_+\right] \\
&= \mathbb{E}_{\{D\}}\big[h((\sqrt{n}Q^* + \mu(n-\sqrt{n})) - (\sqrt{n}D_1 + \mu(n-\sqrt{n})))_+ \\
&+ b((\sqrt{n}D_1 + \mu(n-\sqrt{n})) - (\sqrt{n}Q^* + \mu(n-\sqrt{n})))_+\big] \\
&= \mathbb{E}_{\{D\}}\big[h(\sqrt{n}Q^* - \sqrt{n}D_1)_+ + b(\sqrt{n}D_1 - \sqrt{n}Q^*)_+\big] \\
&= \sqrt{n} \cdot \mathbb{E}_{\{D\}}\big[h(Q^* - D_1)_+ + b(D_1 - Q^*)_+\big] \\
&= \sqrt{n}G(Q^*)
\end{aligned}
$$

It follows that

$$
\frac{G_D(Q^*)}{G_P(Q_P^*)} = \frac{nG(Q^*)}{\sqrt{n}G(Q^*)} = \frac{n}{\sqrt{n}} = \sqrt{n}
$$

This result means that the expected optimal cost of the decentralized system is $\sqrt{n}$-times as large as the expected optimal cost of the pooled inventory system under the given assumptions.

3 An Investment Problem

4 Asset Selling with Maintenance Cost

5 Scheduling Problem