BGSE MSc in Data Science - Stochastic Models and Optimization
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Problem Set 3

Problem 1 (Inventory Control with Forecasts - [B05] Exercise 4.3)

Consider an inventory control problem with no fixed ordering cost. At the beginning of each period k the inventory manager, in addition to knowing the current inventory level x_k, receives an accurate forecast that the demand w_k will be selected in accordance with one out of two probability distributions P_l, P_s (large demand, small demand). The a priori probability of a large demand forecast, q, is also known to the manager.

(a) Obtain the optimal ordering policy for the standard Newsvendor problem (i.e., single period and accounting for inventory holding and backorder costs, but *not* for variable ordering cost). Is it an open-loop policy like in the no-forecast case, or a closed-loop one?

(b) Obtain the optimal inventory replenishment policy for the standard multi-period problem (accounting for variable ordering, inventory holding, and backorder costs). *Hint:* Consider an augmented state for the system (x_k, y_k), with x_k governed by the standard inventory dynamics while $y_{k+1} = \xi_k$, where ξ_k takes the value "large demand" with probability q, and the value "small demand" otherwise.

Problem 2 (Inventory Pooling)

Consider the single period multi-location Newsvendor model: n different locations face independent and Normally distributed demands with mean μ and variance σ^2. The goal is to cover these demands with the minimum expected cost. The inventory holding cost and backorder cost parameters, h and b respectively, are the same in every location.

One approach is to cover each demand from an individual inventory repository at each location; we call this the *decentralized system*. Clearly, this is equivalent to having n standard (single period, single location) Newsvendor problems. Thus, the optimal order quantity at each location is Q^* and the optimal expected cost of the whole system equal to $nG(Q^*)$.

An alternative strategy is to satisfy all demands from a central inventory repository; we call this the *pooled system*. Let Q_p^* be the optimal order quantity in the pooled system and $G_p(Q_p^*)$ the optimal expected cost.

Show that $nG(Q^*)/G_p(Q_p^*) = \sqrt{n}$. How do you interpret this result?

Hint: First show that

$$Q_p^* = \sqrt{n}Q^* + \mu\big(n - \sqrt{n}\big),$$

and then use it to prove the desired result. Take advantage of the fact that if $D_1, D_2, \ldots, D_n$ are independent random variables, distributed identically to a Normal distribution with mean μ and finite variance, then

$$\sum_{i=1}^{n} D_i \stackrel{d}{=} \sqrt{n} D_1 + \mu\big(n - \sqrt{n}\big).$$

Problem 3 (An Investment Problem - [B05] Exercise 4.13)

An investor has the opportunity to make (up to) N sequential investments: at time k he may invest any amount $u_k \geq 0$ that does not exceed his current wealth x_k (defined to be his initial wealth, x_0, plus his gain or minus his loss thus far). He wins his investment back and as much more with probability p, where $1/2 < p < 1$, and he loses his investment with probability $(1-p)$. Find the optimal investment strategy.

Hint: Consider the logarithm of the investor's wealth after the N^{th} investment as the objective function, and prove that $J_k(x_k) = A_k + \ln(x_k)$, where A_k is independent of x_k.

Problem 4 (Asset Selling with Maintenance Cost - [B05] Exercise 4.16)

Suppose that a person wants to sell a house and an offer comes at the beginning of each day. We assume that successive offers are independent and an offer is w_j with probability p_j, $j = 1, \ldots, n$, where w_j are given nonnegative scalars. Any offer not immediately accepted is not lost but may be accepted at any later date. Also, a maintenance cost c is incurred for each day that the house remains unsold. The objective is to maximize the seller's profit when there is a deadline to sell the house within N days. Characterize the optimal policy.

Hint: Determine the one-step stopping set and use the fact that

$$h(x) = \sum_{j=1}^{n} p_j \max(x, w_j) - x$$

is a nonincreasing function.

Problem 5 (Scheduling Problem - [B05] Exercise 4.28)

Consider a quiz contest where a person is given a list of N questions and can answer these questions in any order she chooses. Question i will be answered correctly with probability p_i, and the contestant will then receive a reward $R_i > 0$; if the question is not answered correctly then the quiz terminates and the contestant is allowed to keep her previous earnings minus a penalty $F_i \geq 0$. The problem is to choose the ordering of questions so as to maximize expected rewards.

(a) Use an interchange argument to show that it is optimal to answer the questions in order of decreasing $\big(p_i R_i - (1-p_i)F_i\big)/(1-p_i)$.

(b) Solve a variant of the problem where there is a no-cost option to stop answering questions.

Hint: A simpler version of the quiz problem is solved in Lecture 6 of Dimitri Bertsekas' slides.