# Problemset 3

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## 1 Inventory Control with Forecasts

## 2 Inventory Pooling

We want to compare the optimal expected cost between a decentralized and a pooled inventory system with $n$ locations. Let the demand at each location $i$ be independently, normally distributed with mean $\mu$ and variance $\sigma^2$, i.e. $D_1, ...D_N \overset{iid}{\sim} N(\mu, \sigma^2)$. Then $\sum_{i=1}^{N} D_i \overset{d}{=} \sqrt{n}D_1 + \mu(n - \sqrt{n})$.

Let $h$ and $b$ denote the inventory holding cost and backorder cost, respectively. We already know that if $G(Q^*) = \mathbb{E}_D[h(Q^* - D)_+ + b(D - Q^*)_+]$ denotes the optimal expected cost for each location (with optimal order quantity $Q^* = \inf\left\{Q \geq 0 : \mathbb{P}\left(D_1 \leq Q\right) \geq \frac{b}{b+h}\right\}$), then the optimal expected cost in the decentralized inventory system is

$$G_D(Q^*) = \sum_{i=1}^{N} G(Q^*) = nG(Q^*)$$

To find the optimal expected cost in the pooled inventory system $G_P(Q_P^*)$, we first need to determine the optimal order quantity $Q_P^*$ under this system.

$$
\begin{aligned}
Q_P^* &= \inf\left\{Q \geq 0 : \mathbb{P}\left(\sum_{i=1}^{N} D_i \leq Q\right) \geq \frac{b}{b+h}\right\} \\
&= \inf\left\{Q \geq 0 : \mathbb{P}\left(\sqrt{n}D_1 + \mu(n-\sqrt{n}) \leq Q\right) \geq \frac{b}{b+h}\right\} \\
&= \inf\left\{\sqrt{n}X + \mu(n-\sqrt{n}) \geq 0 : \mathbb{P}\left(D_1 \leq X\right) \geq \frac{b}{b+h}\right\} \\
&= \sqrt{n}\inf\left\{X \geq 0 : \mathbb{P}\left(D_1 \leq X\right) \geq \frac{b}{b+h}\right\} + \mu(n-\sqrt{n}) \\
&= \sqrt{n}Q^* + \mu(n-\sqrt{n})
\end{aligned}
$$

With this we can now determine optimal expected cost in the pooled inventory system $G_P(Q_P^*)$:

$$
\begin{aligned}
G_P(Q_P^*) &= \mathbb{E}_{\{D\}}\left[h(Q_P^* - \sum_{i=1}^{N} D_i)_+ + b(\sum_{i=1}^{N} D_i - Q_P^*)_+\right] \\
&= \mathbb{E}_{\{D\}}\big[h((\sqrt{n}Q^* + \mu(n-\sqrt{n})) - (\sqrt{n}D_1 + \mu(n-\sqrt{n})))_+ \\
&+ b((\sqrt{n}D_1 + \mu(n-\sqrt{n})) - (\sqrt{n}Q^* + \mu(n-\sqrt{n})))_+\big] \\
&= \mathbb{E}_{\{D\}}\big[h(\sqrt{n}Q^* - \sqrt{n}D_1)_+ + b(\sqrt{n}D_1 - \sqrt{n}Q^*)_+\big] \\
&= \sqrt{n}\cdot\mathbb{E}_{\{D\}}\big[h(Q^* - D_1)_+ + b(D_1 - Q^*)_+\big] \\
&= \sqrt{n}G(Q^*)
\end{aligned}
$$

It follows that

$$
\frac{G_D(Q^*)}{G_P(Q_P^*)} = \frac{nG(Q^*)}{\sqrt{n}G(Q^*)} = \frac{n}{\sqrt{n}} = \sqrt{n}
$$

This result means that the expected optimal cost of the decentralized system is $\sqrt{n}$-times as large as the expected optimal cost of the pooled inventory system under the given assumptions.

## 3 An Investment Problem

We want to find the optimal investment strategy in a situation where the investor can make $N$ sequential investment decisions, each resulting in one of two possible outcomes: Either (1) the invested money doubles (with probability $1/2 < p < 1$, or (2) the invested money is lost (with probability $1-p$).

Let $x_k$ denote the wealth at time $k$ (initial wealth $x_0$), let $u_k \in U_k(x_k) = [0, x_k]$ be the investment (decision) at time $k$ and let

$$w_k(u_k) = \begin{cases} 2u_k & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

be the outcome of the investment at time $k$.

First, note that the state transition has the following form:

$$x_{k+1} = f(x_k, u_k, w_k) = x_k - u_k + w_k = \begin{cases} x_k + u_k & \text{with probability } p \\ x_k - u_k & \text{with probability } (1-p) \end{cases}$$

The DP algorithm has the form (note that we can consider the logarithm of the investor's wealth after the $N^{th}$ investment as the objective function without changing the optimal strategy):

$$J_N(x_N) = \ln(x_N)$$

$$\begin{aligned} J_k(x_k) &= \max_{u_k \in U_k(x_k)} \mathbb{E}_{w_k}\left[J_{k+1}(x_{k+1})\right] \\ &= \max_{u_k \in U_k(x_k)} \left[pJ_{k+1}(x_k + u_k) + (1-p)J_{k+1}(x_k - u_k)\right] \end{aligned}$$

We want to find the strategy that yields the maximum expected wealth at time $N$, given initial wealth $x_0$: $J_0(x_0)$.

First, consider the optimal strategy for a one-period investment horizon, i.e. $J_0(x_0) = J_{N-1}(x_{N-1}) = \max_{u_k \in U_k(x_k)}\left[p\ln(x_k + u_k) + (1-p)\ln(x_k - u_k)\right]$. We can find the optimal investment strategy $u^*_{N-1}$ using first-order conditions:

$$\begin{aligned} \frac{\partial}{\partial u_{N-1}} J_{N-1}(x_{N-1}) &= 0 \\ \frac{\partial}{\partial u_{N-1}}\left[p\ln(x_{N-1} + u_{N-1}) + (1-p)\ln(x_{N-1} - u_{N-1})\right] &= 0 \\ \frac{p}{x_{N-1} + u^*_{N-1}} - \frac{1-p}{x_{N-1} - u^*_{N-1}} &= 0 \\ p(x_{N-1} - u^*_{N-1}) &= (1-p)(x_{N-1} + u^*_{N-1}) \\ u^*_{N-1} &= (2p-1)x_{N-1} \end{aligned}$$

It follows that $J_{N-1}(x_{N-1})$ has the form:

$$\begin{aligned}
J_{N-1}(x_{N-1}) &= \max_{u_{N-1}} \mathbb{E}_{w_{N-1}}\left[\ln J_N(x_N)\right] \\
&= \max_{u_{N-1}} \left[p\ln(x_{N-1}+u_{N-1}) + (1-p)\ln(x_{N-1}-u_{N-1})\right] \\
&= p\ln(x_{N-1}+(2p-1)x_{N-1}) + (1-p)\ln(x_{N-1}-(2p-1)x_{N-1}) \\
&= p\ln(2p\cdot x_{N-1}) + (1-p)\ln((2-2p)\cdot x_{N-1}) \\
&= p\ln(2p) + \ln(x_{N-1}) + (1-p)\ln(2-2p) + \ln(x_{N-1}) \\
&= p\ln(2p) - p\ln(2-2p) + \ln(2-2p) + \ln(x_{N-1}) \\
&= A_{N-1} + \ln(x_{N-1})
\end{aligned}$$

where $A_{N-1} = p\ln(2p) - p\ln(2-2p) + \ln(2-2p)$ is independent of $x_{N-1}$. We see that $J_{N-1}(x_{N-1})$ contains the term $\ln(x_{N-1})$ and so the optimal strategy (known as "Kelly strategy") generalizes to all $k = 0, 1, .., (N-1)$ in the multi-period problem:

$$u_k^* = (2p-1)x_k$$

and therefore $J_k(x_k) = A_k + \ln(x_k)$ for some $A_K$ independent of $x_k$.

## 4 Asset Selling with Maintenance Cost

## 5 Scheduling Problem