14D006 Stochastic Models and Optimization

Problemset 4

Daniel Bestard, Michael Cameron, Hans-Peter Höllwirth, Akhil Lohia

March 14, 2017

1 Linear-Quadratic Problem with Forecasts

First of all let's set up the problem in order to make the proof. The dynamics of the problem is linear function of the form:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

and the cost function is a quadratic function of the form:

$$\mathop{\mathbb{E}}_{w_k}\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\}$$

where

$$g_N(x_N) = x_N' Q_N x_N$$

$$g_k(x_k, u_k, w_k) = x_k' Q_k x_k + u_k' R_k u_k$$

The matrices A_k, B_k, Q_k and R_k are given and the last two are positive semidefinite symmetric and positive definite symmetric, respectively.

The DP-algorithm that solves the minimization problem is:

$$J_N(x_N) = x_N' Q_N x_N$$

$$J_k(x_k) = \min_{u_k} \mathop{\mathbb{E}}_{w_k|y_k}\left\{ x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(A_k x_k + B_k u_k + w_k) \right\}$$

By induction we get that:

$$
\begin{aligned}
J_{N-1}(x_{N-1}) = \underset{u_{N-1}}{min} \underset{w_{N-1}|y_{N-1}}{\mathbb{E}} \Big\{ & x'_{N-1}Q_{N-1}x_{N-1} + u'_{N-1}R_{N-1}u_{N-1} + \\
& + \big(A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1}\big)' Q_N \big(A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1}\big) \Big\} \\
& = x'_{N-1}Q_{N-1}x_{N-1} + x'_{N-1}A'_{N-1}Q_N A_{N-1}x_{N-1} + \\
& + \underset{u_{N-1}}{min} \Big\{ u'_{N-1}R_{N-1}u_{N-1} + u'_{N-1}B'_{N-1}Q_N B_{N-1}u_{N-1} + 2x'_{N-1}A'_{N-1}Q_N B_{N-1}u_{N-1} \Big\} + \\
& + \underset{w_{N-1}|y_{N-1}}{\mathbb{E}} \Big\{ w'_{N-1}Q_N w_{N-1} + 2x'_{N-1}A'_{N-1}Q_N w_{N-1} \Big\} + \\
& + \underset{u_{N-1}}{min} \underset{w_{N-1}|y_{N-1}}{\mathbb{E}} \Big\{ 2u'_{N-1}B'_{N-1}Q_N w_{N-1} \Big\}
\end{aligned}
$$

By differentiating the previous expression and setting it to 0, we obtain the following result:

$$
\big(R_{N-1} + B'_{N-1}Q_N B_{N-1}\big)u^*_{N-1} = -B'_{N-1}Q_N A_{N-1}x_{N-1} - B'_{N-1}Q_N \mathbb{E}[w_{N-1}|y_{N-1}]
$$

Given the definitions provided previously we can note that the matrix $R_{N-1}+B'_{N-1}Q_N B_{N-1}$ is positive definite, which means that we can invert it and obtain the following optimal value:

$$
\begin{aligned}
u^*_{N-1} &= -\big(R_{N-1} + B'_{N-1}Q_N B_{N-1}\big)^{-1}\big(B'_{N-1}Q_N A_{N-1}x_{N-1} + B'_{N-1}Q_N \mathbb{E}[w_{N-1}|y_{N-1}]\big) \\
&= -\big(R_{N-1} + B'_{N-1}Q_N B_{N-1}\big)^{-1} B'_{N-1}Q_N \big(A_{N-1}x_{N-1} + \mathbb{E}[w_{N-1}|y_{N-1}]\big)
\end{aligned}
$$

Note that the previous expression is already of the form of the expression to be proved. If we substitute back the optimal value u^*_{N-1} in $J_{N-1}(x_{N-1})$ and continue doing this we get the general expression provided in the exercise, which is:

$$
\mu^*_k(x_k, y_k) = -\big(R_k + B'_{N-1}K_{k+1}B_k\big)^{-1} B'_k K_{k+1}\big(A_k x_k + \mathbb{E}[w_k|y_k]\big) + \alpha_k
$$

where K_{k+1} is a function of A_k, B_k, Q_k and R_k.

2 Computational Assignment on Linear-Quadratic Control

3 Asset Selling with Offer Estimation

4 Inventory Control with Demand Estimation

5 Robust Dynamic Programming