14D006 Stochastic Models and Optimization

Problemset 4

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1 Linear-Quadratic Problem with Forecasts

First of all let's set up the problem in order to make the proof. The dynamics of the problem is linear function of the form:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

and the cost function is a quadratic function of the form:

$$\mathop{\mathbb{E}}_{w_k}\left\{ g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \right\}$$

where

$$g_N(x_N) = x_N' Q_N x_N$$

$$g_k(x_k, u_k, w_k) = x_k' Q_k x_k + u_k' R_k u_k$$

The matrices A_k, B_k, Q_k and R_k are given and the last two are positive semidefinite symmetric and positive definite symmetric, respectively.

The DP-algorithm that solves the minimization problem is:

$$J_N(x_N) = x_N' Q_N x_N$$

$$J_k(x_k) = \min_{u_k} \mathop{\mathbb{E}}_{w_k|y_k} \left\{ x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(A_k x_k + B_k u_k + w_k) \right\}$$

By induction we get that:

$$\begin{aligned}
J_{N-1}(x_{N-1}) = &\min_{u_{N-1} w_{N-1}|y_{N-1}} \mathbb{E} \Big\{ x'_{N-1} Q_{N-1} x_{N-1} + u'_{N-1} R_{N-1} u_{N-1} + \\
&+ \big(A_{N-1} x_{N-1} + B_{N-1} u_{N-1} + w_{N-1}\big)' Q_N \big(A_{N-1} x_{N-1} + B_{N-1} u_{N-1} + w_{N-1}\big) \Big\} \\
&= x'_{N-1} Q_{N-1} x_{N-1} + x'_{N-1} A'_{N-1} Q_N A_{N-1} x_{N-1} + \\
&+ \min_{u_{N-1}} \Big\{ u'_{N-1} R_{N-1} u_{N-1} + u'_{N-1} B'_{N-1} Q_N B_{N-1} u_{N-1} + 2 x'_{N-1} A'_{N-1} Q_N B_{N-1} u_{N-1} \Big\} + \\
&+ \mathop{\mathbb{E}}_{w_{N-1}|y_{N-1}} \Big\{ w'_{N-1} Q_N w_{N-1} + 2 x'_{N-1} A'_{N-1} Q_N w_{N-1} \Big\} + \\
&+ \min_{u_{N-1} w_{N-1}|y_{N-1}} \mathbb{E} \Big\{ 2 u'_{N-1} B'_{N-1} Q_N w_{N-1} \Big\}
\end{aligned}$$

By differentiating the previous expression and setting it to 0, we obtain the following result:

$$\big(R_{N-1} + B'_{N-1} Q_N B_{N-1}\big) u^*_{N-1} = -B'_{N-1} Q_N A_{N-1} x_{N-1} - B'_{N-1} Q_N \mathbb{E}[w_{N-1}|y_{N-1}]$$

Given the definitions provided previously we can note that the matrix $R_{N-1} + B'_{N-1} Q_N B_{N-1}$ is positive definite, which means that we can invert it and obtain the following optimal value:

$$\begin{aligned}
u^*_{N-1} &= -\big(R_{N-1} + B'_{N-1} Q_N B_{N-1}\big)^{-1} \big(B'_{N-1} Q_N A_{N-1} x_{N-1} + B'_{N-1} Q_N \mathbb{E}[w_{N-1}|y_{N-1}]\big) \\
&= -\big(R_{N-1} + B'_{N-1} Q_N B_{N-1}\big)^{-1} B'_{N-1} Q_N \big(A_{N-1} x_{N-1} + \mathbb{E}[w_{N-1}|y_{N-1}]\big)
\end{aligned}$$

Note that the previous expression is already of the form of the expression to be proved. If we substitute back the optimal value u^*_{N-1} in $J_{N-1}(x_{N-1})$ and continue doing this we get the general expression provided in the exercise, which is:

$$\mu^*_k(x_k, y_k) = -\big(R_k + B'_{N-1} K_{k+1} B_k\big)^{-1} B'_k K_{k+1} \big(A_k x_k + \mathbb{E}[w_k|y_k]\big) + \alpha_k$$

where K_{k+1} is a function of A_k, B_k, Q_k and R_k.

2 Computational Assignment on Linear-Quadratic Control

3 Asset Selling with Offer Estimation

4 Inventory Control with Demand Estimation

5 Robust Dynamic Programming

Consider a variation of the basic problem in which we do not have a probabilistic description of uncertainty. Instead, we want to find the closed-loop policy $\pi = \{\mu_0(.), .., \mu_{N-1}(.)\}$ with $\mu_k(x_k) \in U_k(x_k)$ that minimizes the maximum possible cost:

$$J_\pi(x_0) = \max_{w_k \in W_k(x_k, \mu_k(x_k))} \left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right]$$

5.1 DP formulation

Using the principle of optimality, the DP like recursion for this variation of the basic problem looks like:

$$\begin{aligned} J_N(x_N) &= g_N(x_N) \\ J_k(x_k) &= \min_{u_k \in U_k(x_k)} \max_{w_k \in W_k(x_k, \mu_k(x_k))} \left[g_k(x_k, \mu_k(x_k), w_k) + J_{k+1}(f_k(x_k, \mu_k(x_k), w_k)) \right] \end{aligned}$$

Note that, when we compare the DP formulation to the original basic problem, instead of minimizing the expected cost over w_k, here we minimize the maximum possible cost that can result from an action $u_k = \mu_k(x_k)$.

5.2 Reachability of a target tube

Now assume that at each stage k, the state x_k must belong to a given set X_k. A cost structure that fits the reachability problem within the general formulation looks as follows:

$$\begin{aligned} J_N(x_N) &= \begin{cases} 0 & \text{if } x_N \in X_N \\ 1 & \text{if } x_N \notin X_N \end{cases} \\ J_k(x_k) &= \min_{u_k \in U_k(x_k)} \max_{w_k \in W_k(x_k, \mu_k(x_k))} \left[J_{k+1}(f_k(x_k, \mu_k(x_k), w_k)) \right] \end{aligned}$$

The set $\bar{X}_k$, i.e. the set that we must reach at stage k in order to be able to maintain the state of the system in the desired tube henceforth, can be computed recursively as follows:

$$\bar{X}_N = X_N$$
$$\bar{X}_k = \{x_k : \exists \mu_k(x_k) \in U_k(x_k). \forall w_k \in W_k(x_k, \mu_k(x_k)). f_k(x_k, \mu_k(x_k), w_k) \in \bar{X}_{k+1}\}$$

For $x_k \in \bar{X}_k$, there must exist at least one action $u_k = \mu_k(x_k)$ in set $U_k(x_k)$ such that every possible outcome w_k in the outcome set $W_k(x_k, \mu_k(x_k))$ takes us to a state $x_{k+1} = f_k(x_k, \mu_k(x_k), w_k) \in \bar{X}_{k+1}$.