Generalized Bayesian Learning

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2018–2019

[https://bguedj.github.io]

8h de cours (4 $\times$ 2h, 6 février & 5 mars 2019)

Evaluation : projet à rendre. Deadline : **3 avril 2019 à 23h59**.

Slides : [https://bguedj.github.io/teaching/bguedj.pdf]

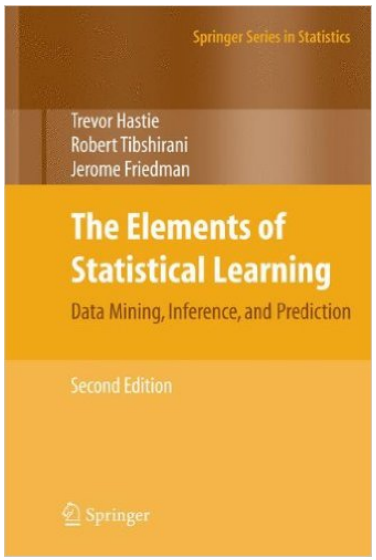
Projet : [https://bguedj.github.io/teaching/projet.pdf]

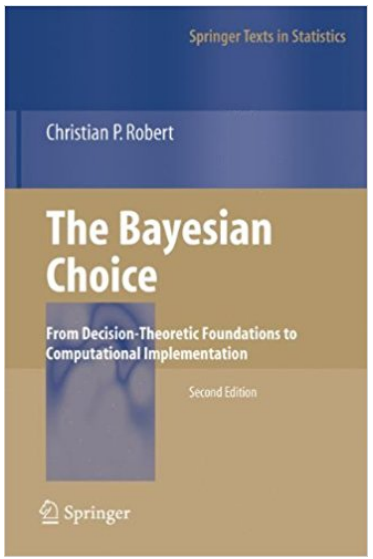
References

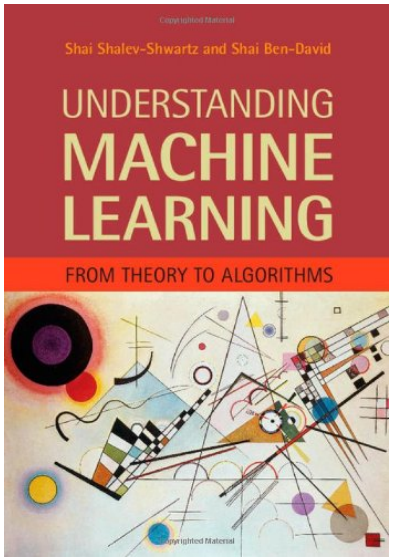
T. Hastie, R. Tibshirani and J. Friedman. *The Elements of Statistical Learning: Data Mining, Inference and Prediction*, Springer, 2009. [Link]

C. P. Robert. *The Bayesian Choice*, Springer, 2007. [Link]

S. Shalev-Shwartz and S. Ben-David. *Understanding Machine Learning: From Theory to Algorithms*, Cambridge University Press, 2014. [Link]





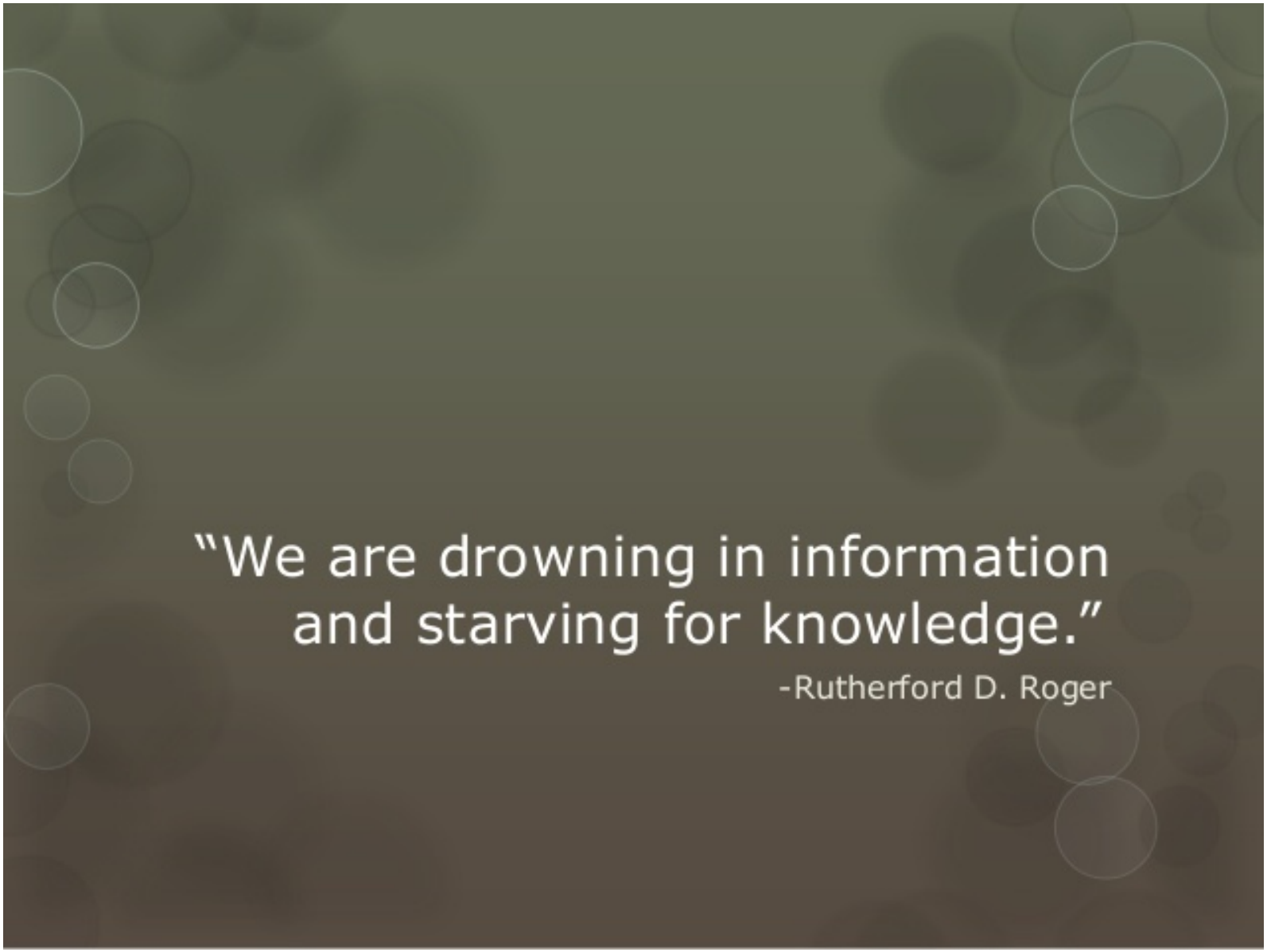


Outline

1. Introduction to statistical and machine learning
2. The Bayesian framework
3. Generalized Bayesian learning
4. Bayesian learning in practice: implementation

The rising of AI

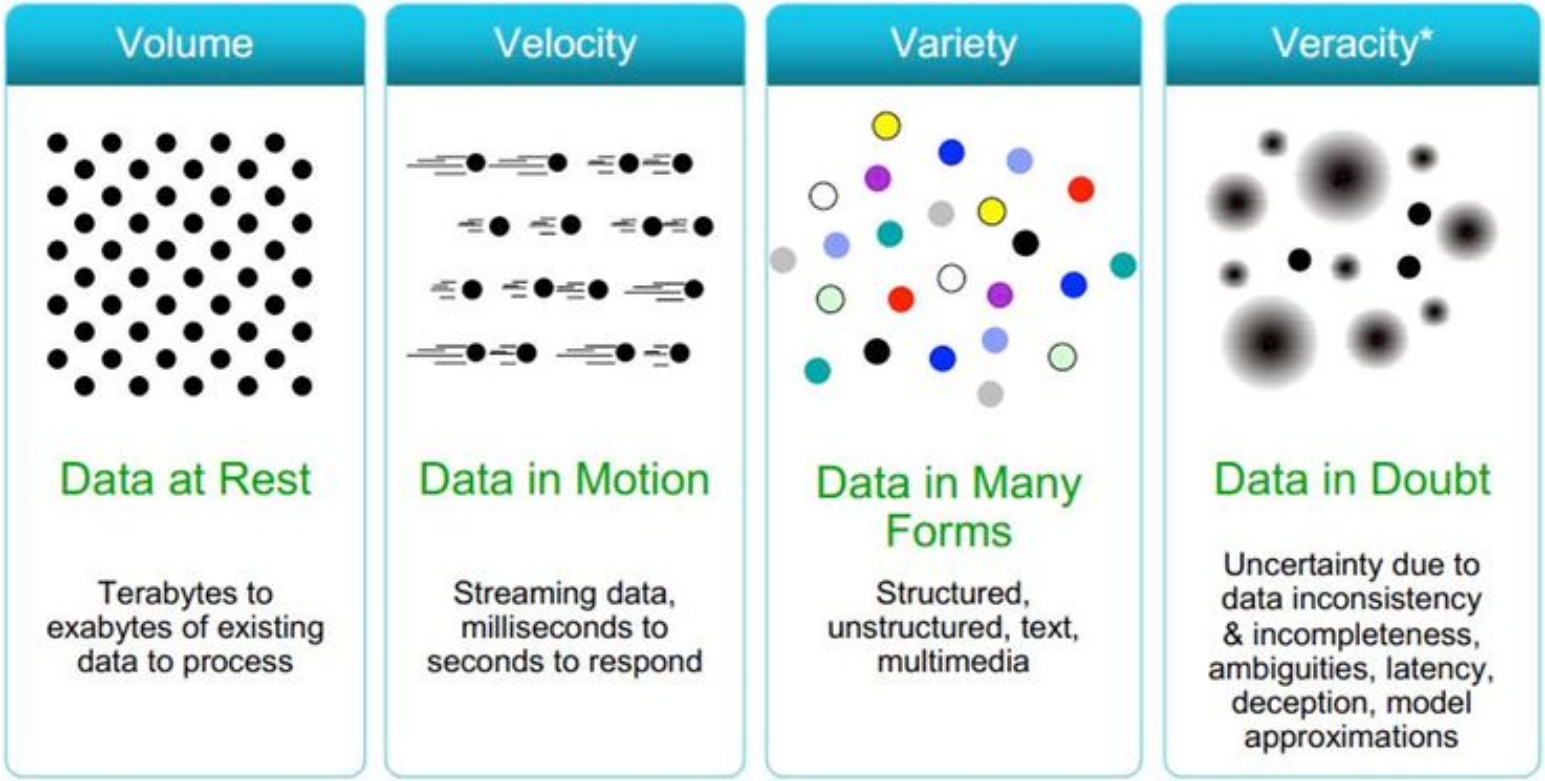
Introduction


"We are drowning in information
and starving for knowledge."
-Rutherford D. Roger

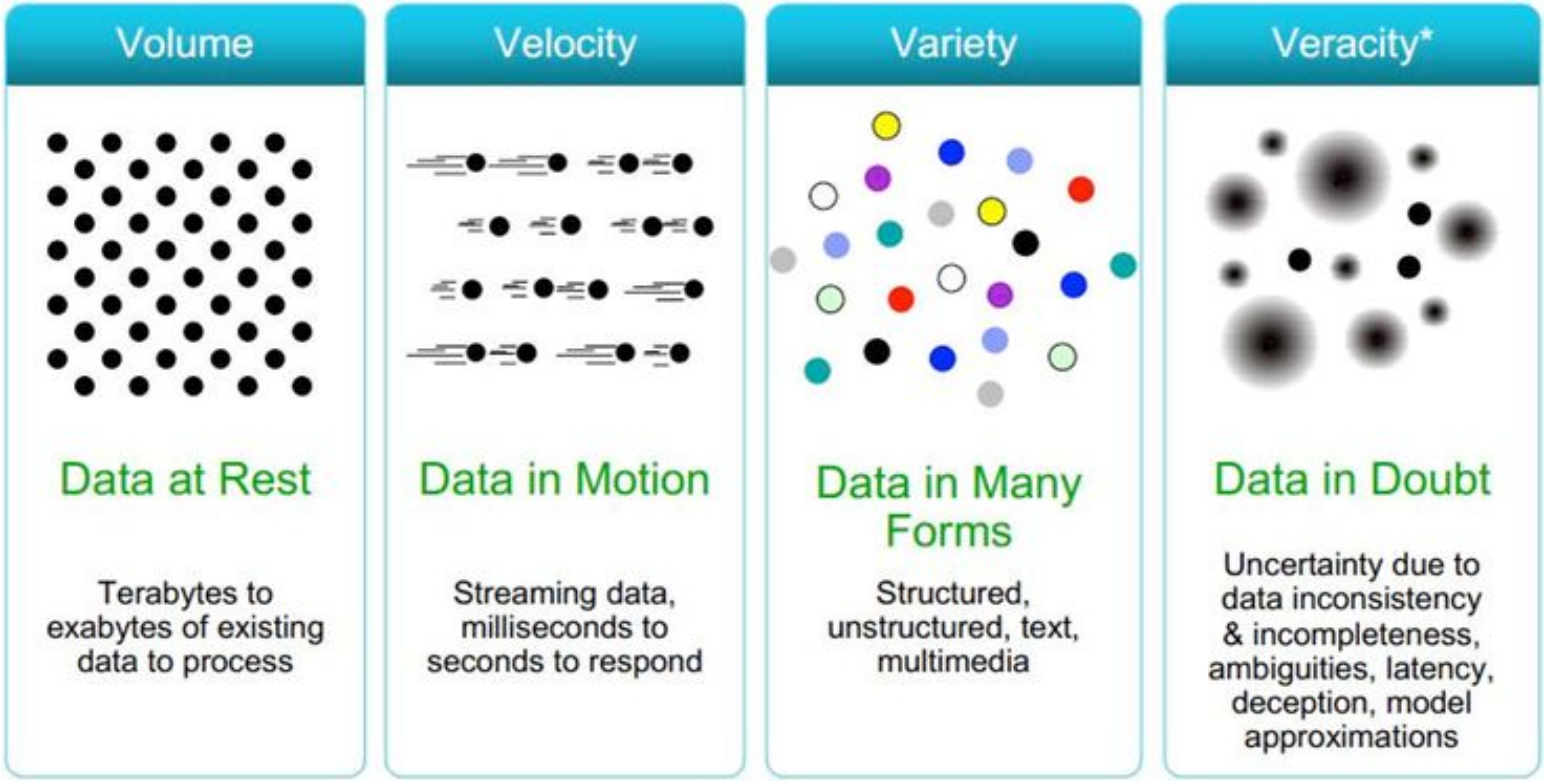
It is vital to remember that information - in the sense of raw data - is not knowledge, that knowledge is not wisdom, and that wisdom is not foresight. But information is the first essential step to all of these.

Arthur C Clarke

Big Data 4 V's


Volume
Data at Rest
Terabytes to exabytes of existing data to process
Velocity
Data in Motion
Streaming data, milliseconds to seconds to respond
Variety
Data in Many Forms
Structured, unstructured, text, multimedia
Veracity*
Data in Doubt
Uncertainty due to data inconsistency & incompleteness, ambiguities, latency, deception, model approximations

Big Data 4 V's



$\longrightarrow$ Value ($)

Data Scientists: voted 'most sexiest job' of the 21st century. Demand is expected to exceed supply by 50 to 60% (McKinsey, 2015)

Volume / Velocity

i	3	6	9	12	15	18	21	24	
10^i	kilo	mega	giga	tera	peta	exa	zeta	yotta	bytes

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- Facebook: 10 terabytes/day

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- Human brain may store about 2.5 petabytes of binary data

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- ...

Variety / Veracity

categorical
Marital status
maried

integer
Children
3

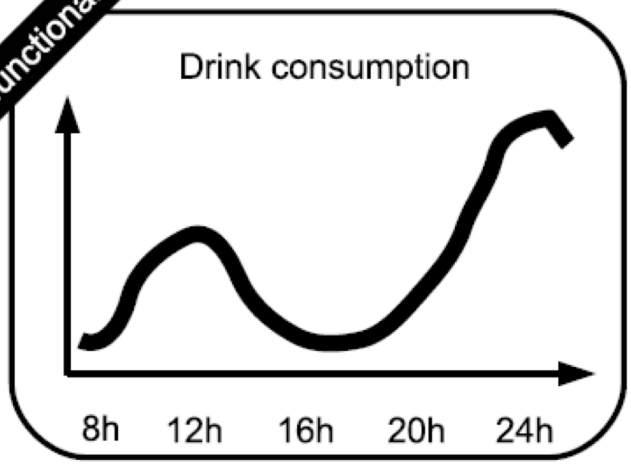
missing
Size (m)
?

rank
Drink preference
beer > soda > water

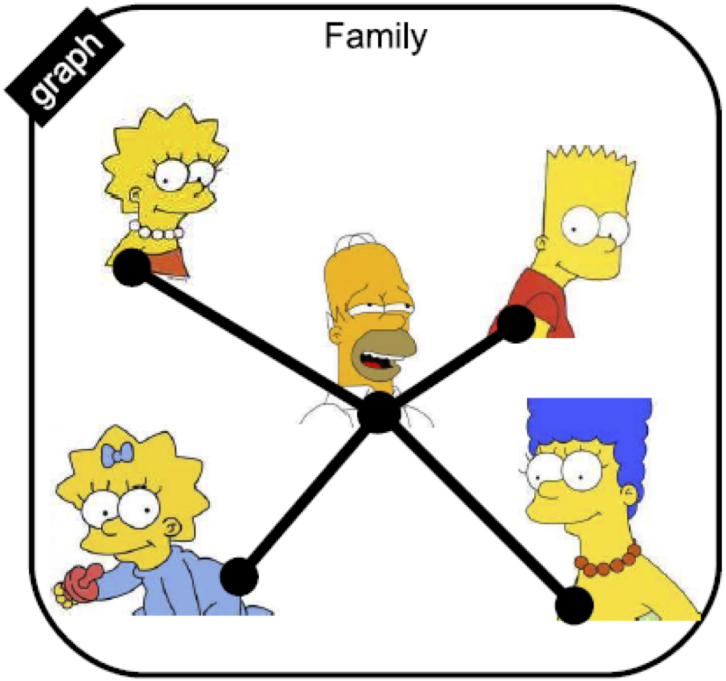
ordinal
Intelligence
low

continuous
Weight (kg)
119.5

functional

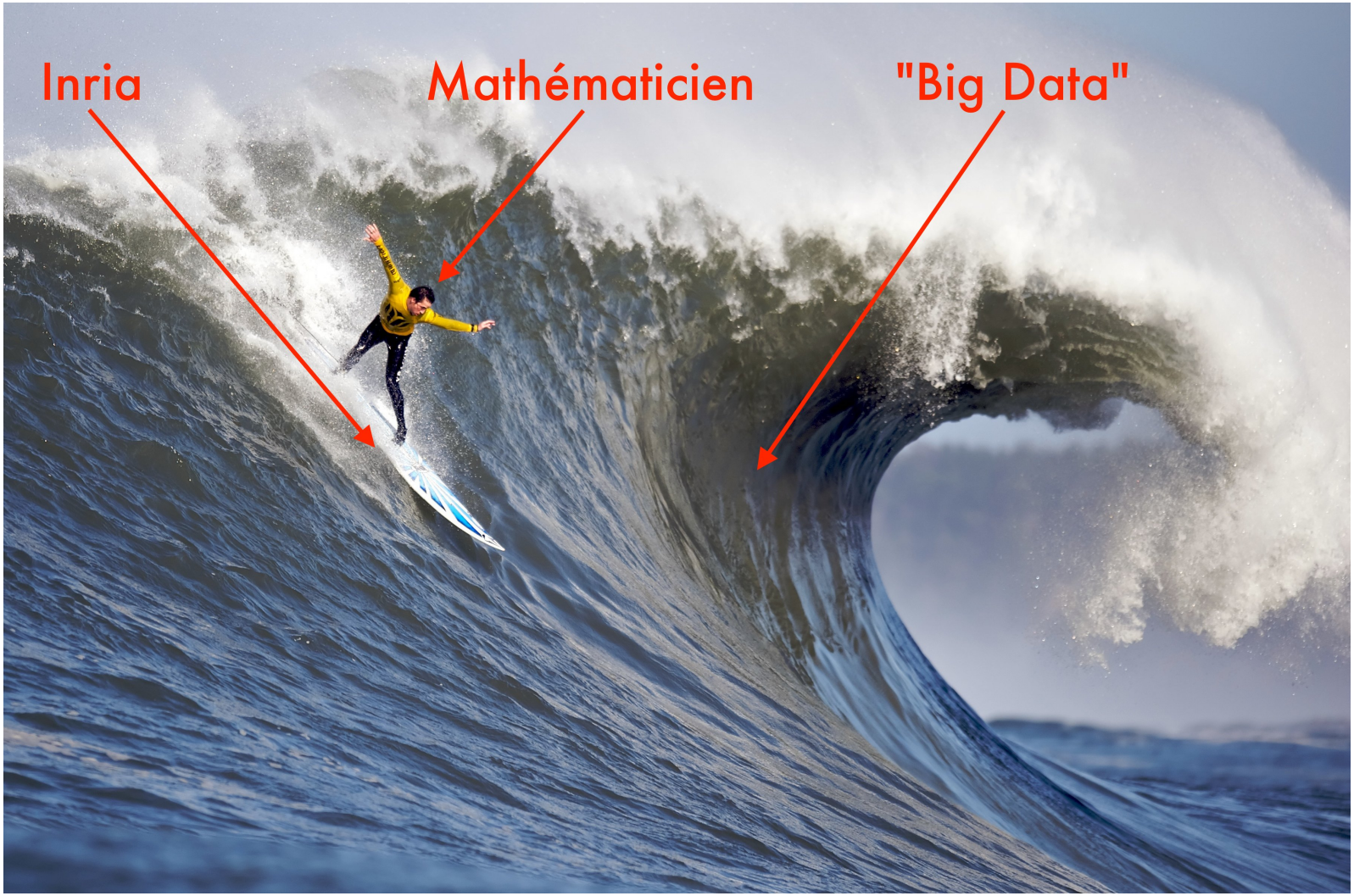


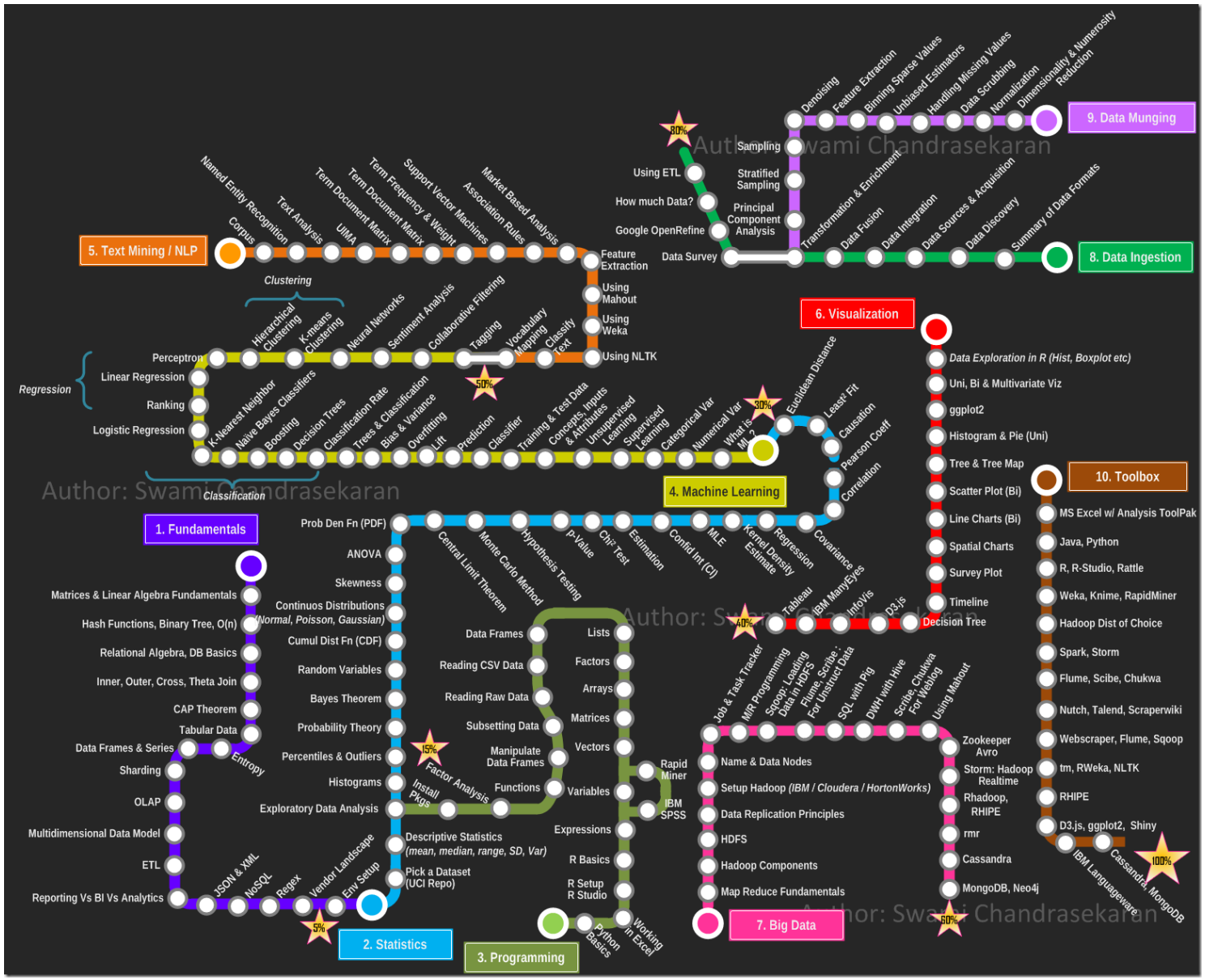
graph

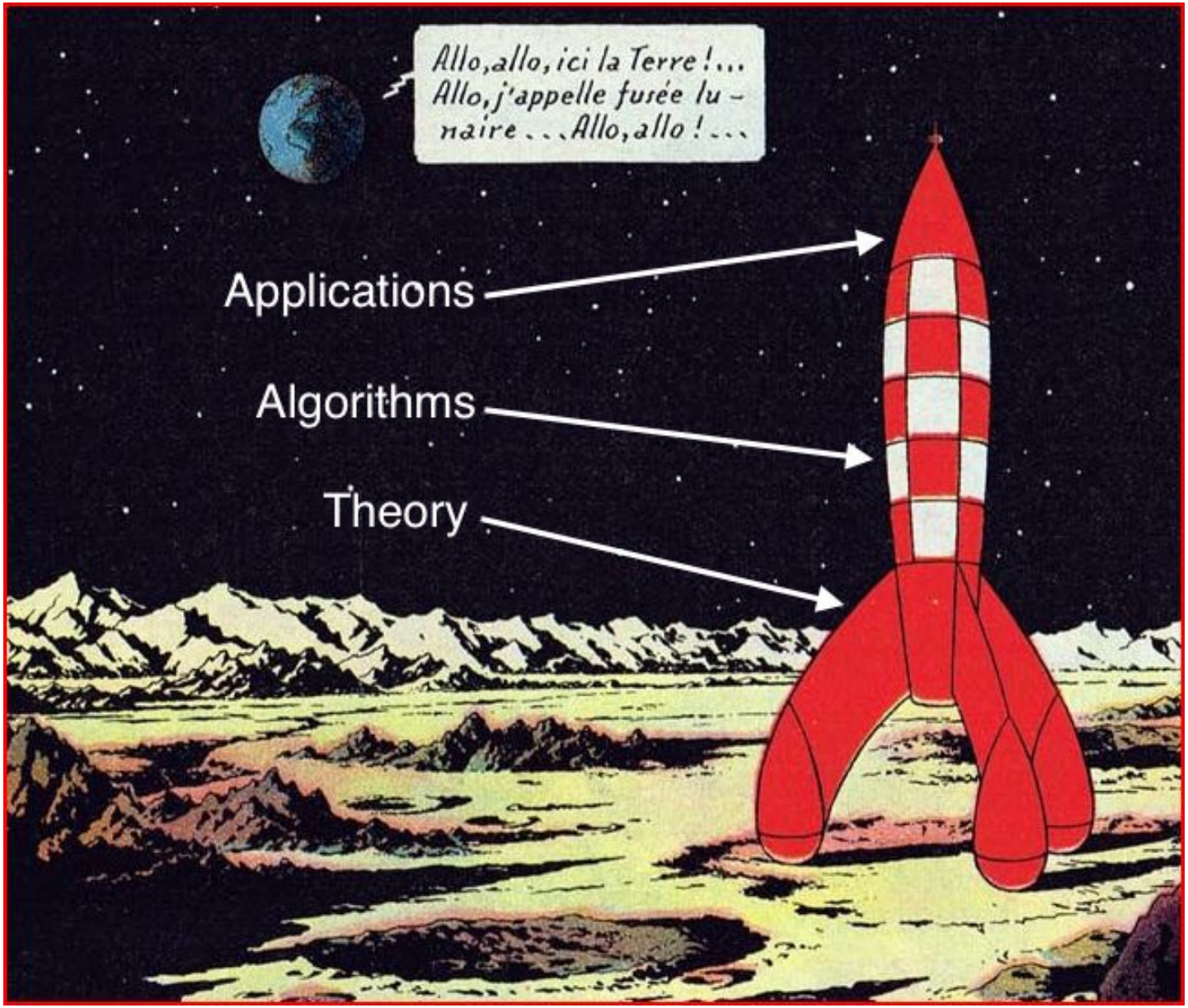


And so on...

My job (allegory)




1. Fundamentals
Matrices & Linear Algebra Fundamentals
Hash Functions, Binary Tree, O(n)
Relational Algebra, DB Basics
Inner, Outer, Cross, Theta Join
CAP Theorem
Tabular Data
Data Frames & Series
Entropy
Sharding
OLAP
Multidimensional Data Model
ETL
Reporting Vs BI Vs Analytics
JSON & XML
NoSQL
Regex
Vendor Landscape
Env Setup
5%
2. Statistics
Prob Den Fn (PDF)
ANOVA
Skewness
Continuos Distributions (Normal, Poisson, Gaussian)
Cumul Dist Fn (CDF)
Random Variables
Bayes Theorem
Probability Theory
Percentiles & Outliers
Histograms
Exploratory Data Analysis
Descriptive Statistics (mean, median, range, SD, Var)
Pick a Dataset (UCI Repo)
Central Limit Theorem
Monte Carlo Method
Hypothesis Testing
p-Value
Chi² Test
Estimation
Confid Int (CI)
MLE
Kernel Density Estimate
Regression
Covariance
15%
Factor Analysis
Install Pkgs
3. Programming
Data Frames
Reading CSV Data
Reading Raw Data
Subsetting Data
Manipulate Data Frames
Functions
Lists
Factors
Arrays
Matrices
Vectors
Variables
Expressions
R Basics
R Setup R Studio
Python Basics
Working in Excel
Rapid Miner
IBM SPSS
4. Machine Learning
Regression
Perceptron
Linear Regression
Ranking
Logistic Regression
Classification
K-Nearest Neighbor
Naive Bayes Classifiers
Boosting
Decision Trees
Classification Rate
Trees & Classification
Bias & Variance
Overfitting
Lift
Prediction
Classifier
Training & Test Data
Concepts, Inputs & Attributes
Unsupervised Learning
Supervised Learning
Categorical Var
Numerical Var
What is ML?
50%
Clustering
Hierarchical Clustering
K-means Clustering
Neural Networks
Sentiment Analysis
Collaborative Filtering
Tagging
Vocabulary Mapping
Classify Text
Using NLTK
Using Mahout
Using Weka
Named Entity Recognition
Text Analysis
Corpus
UIMA
Term Document Matrix
Term Document Matrix
Term Frequency & Weight
Support Vector Machines
Association Rules
Market Based Analysis
Feature Extraction
5. Text Mining / NLP
30%
Euclidean Distance
Least² Fit
Causation
Pearson Coeff
Correlation
6. Visualization
Data Exploration in R (Hist, Boxplot etc)
Uni, Bi & Multivariate Viz
ggplot2
Histogram & Pie (Uni)
Tree & Tree Map
Scatter Plot (Bi)
Line Charts (Bi)
Spatial Charts
Survey Plot
Timeline
Decision Tree
40%
Tableau
IBM ManyEyes
InfoVis
D3.js
80%
Sampling
Stratified Sampling
Principal Component Analysis
Transformation & Enrichment
Using ETL
How much Data?
Google OpenRefine
Data Survey
9. Data Munging
Denoising
Feature Extraction
Binning Sparse Values
Unbiased Estimators
Handling Missing Values
Data Scrubbing
Normalization
Dimensionality & Numerosity Reduction
8. Data Ingestion
Data Fusion
Data Integration
Data Sources & Acquisition
Data Discovery
Summary of Data Formats
7. Big Data
Job & Task Tracker
MR Programming
Sqoop: Loading Data in HDFS
Flume, Scribe : For Unstruct Data
SQL with Pig
DWH with Hive
Scribe, Chukwa For Weblog
Using Mahout
Name & Data Nodes
Setup Hadoop (IBM / Cloudera / HortonWorks)
Data Replication Principles
HDFS
Hadoop Components
Map Reduce Fundamentals
Zookeeper Avro
Storm: Hadoop Realtime
Rhadoop, RHIPE
rmr
Cassandra
MongoDB, Neo4j
60%
10. Toolbox
MS Excel w/ Analysis ToolPak
Java, Python
R, R-Studio, Rattle
Weka, Knime, RapidMiner
Hadoop Dist of Choice
Spark, Storm
Flume, Scibe, Chukwa
Nutch, Talend, Scraperwiki
Webscraper, Flume, Sqoop
tm, RWeka, NLTK
RHIPE
D3.js, ggplot2, Shiny
IBM Languageware
Cassandra, MongoDB
100%


Allo, allo, ici la Terre !...
Allo, j'appelle fusée lu-
naire ...Allo, allo !...
Applications
Algorithms
Theory

A foretaste of Learning Theory

{Statistical,Machine} Learning: building automatic procedures to infer general rules from examples.

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In the (~~rather~~ not so?) long term: mimic the inductive functioning of the humain brain to develop an artificial intelligence.

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In the Big Data Era, very dynamic field at the crossroads of Computer Science, Optimization and Statistics.

Probabilistic framework: n-sample $\mathcal{D}_n = (\mathbf{X}_i, \mathbf{Y}_i)_{i=1}^n$ of i.i.d. replications of some random variable

$$(\mathbf{X}, \mathbf{Y}) \in \mathcal{X} \times \mathcal{Y}, \quad \dim(\mathcal{X}) = d.$$

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We want to infer the link between the explanatory variable $\mathbf{X}$ and the response variable $\mathbf{Y}$, *i.e.*, use $\mathcal{D}_n$ to build up $\hat{\phi}$ such that $\hat{\phi}(\mathbf{X})$ is a "good" approximation of $\mathbf{Y}$.

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- Classification: $\mathcal{Y}$ is discrete.
- Regression: $\mathcal{Y}$ is a continuum.

- Online learning: observations are revealed over time.
- Batch learning: all observations are revealed at once.

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Big Data Era: easy/cheap to collect massive amounts of data, hence typically $\mathbf{X}_1, \ldots, \mathbf{X}_n, \ldots \in \mathbb{R}^{n \times d}$ where d and n may be (extremely) large.

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- Tall data: $n \gg d$

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- Good ol' statistics: n and d small
- Tall data: $n \gg d$
- Fat data: $d \gg n$
- Big/massive data: n and d huge

- Supervised learning: all of the $\mathbf{Y}_i$s are observed.

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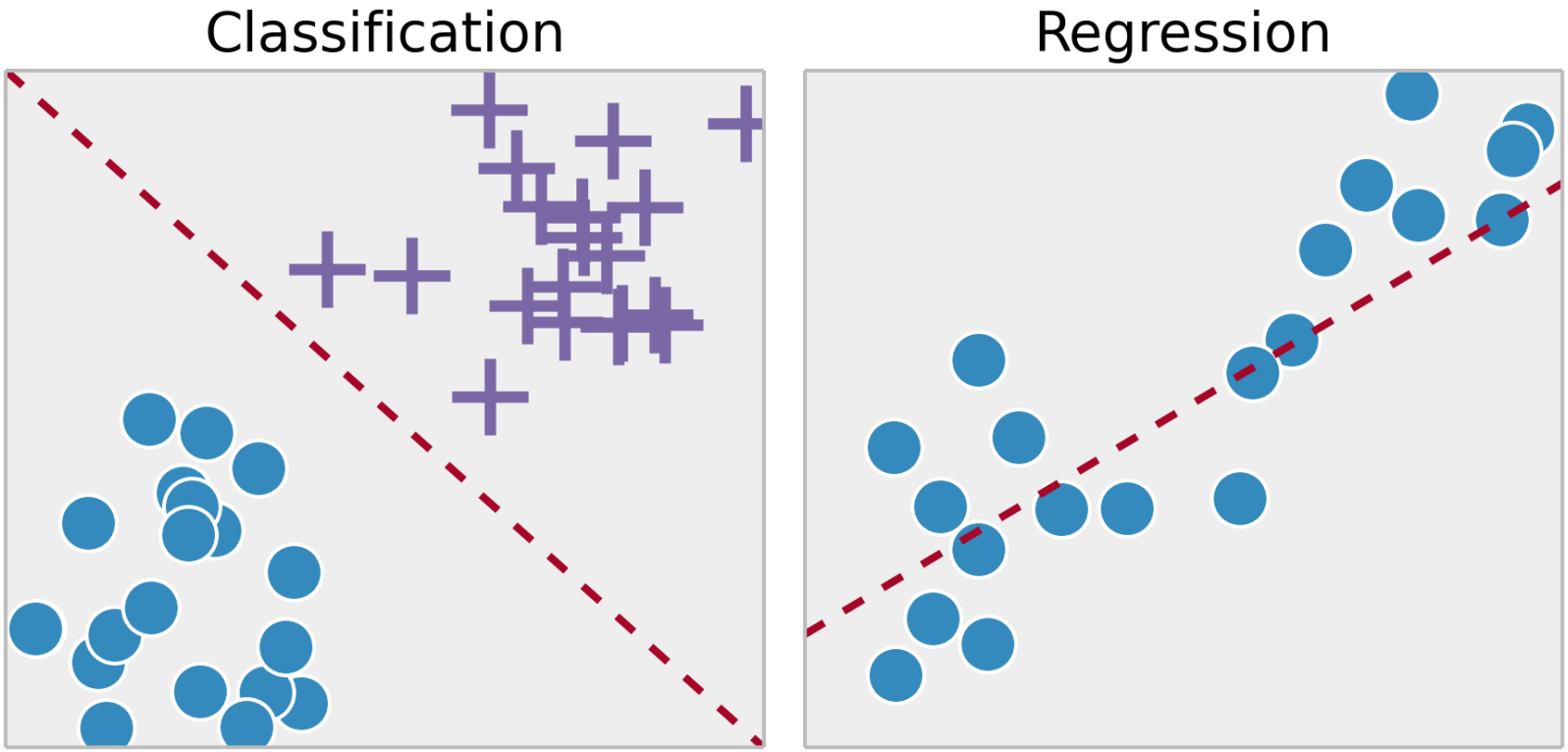

- Semi-supervised learning: some of the $\mathbf{Y}_i$s are observed (labeling is expensive or difficult).

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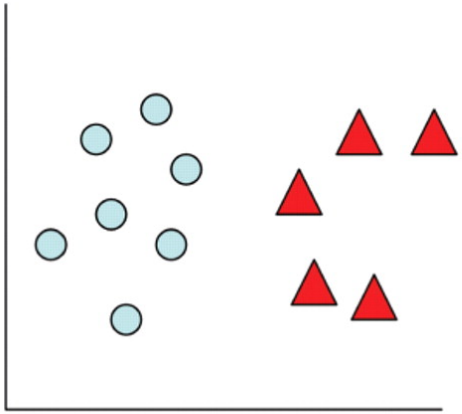
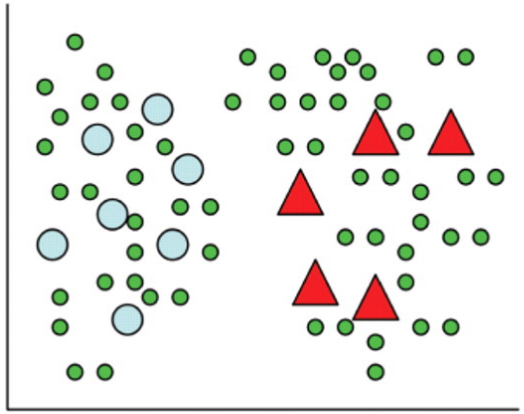
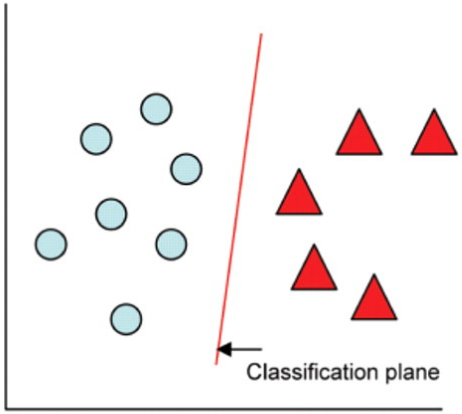


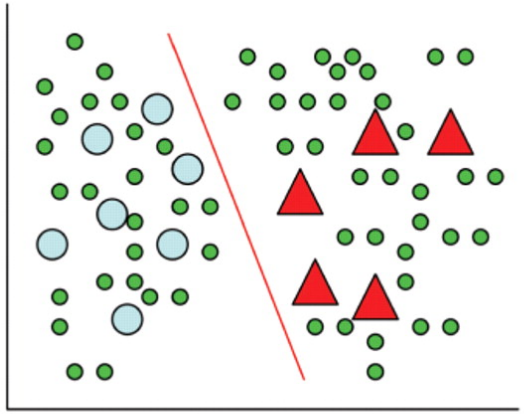
Labeled Data
(a)



Labeled and Unlabeled Data
(b)



Supervised Learning
(c)



Semi-Supervised Learning
(d)

- Unsupervised learning: none of the $\mathbf{Y}_i$s are observed (detect patterns).

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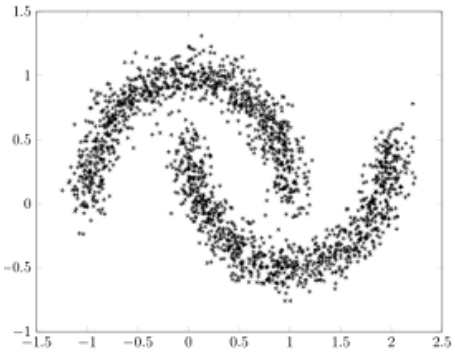

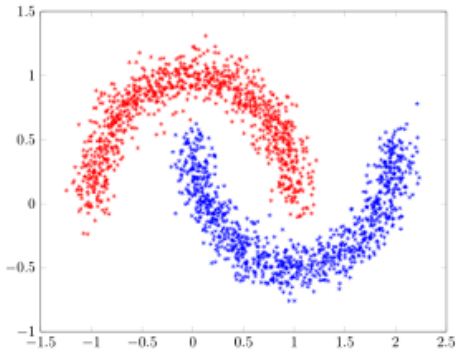

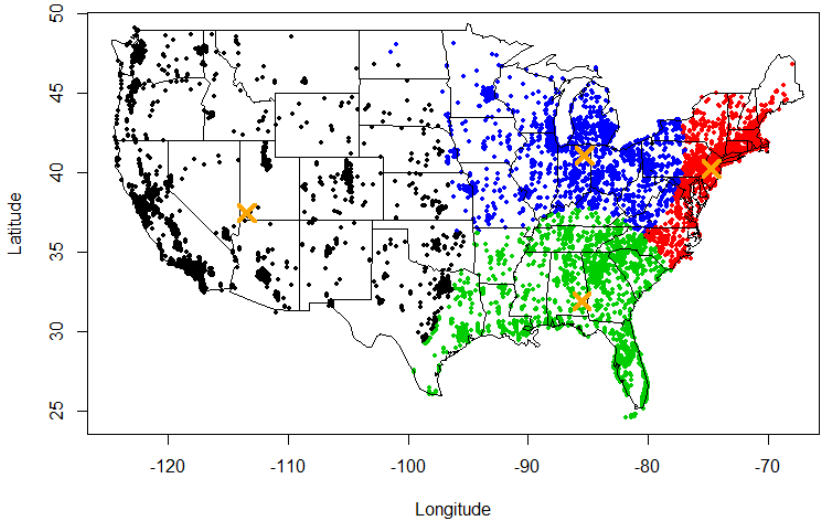

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- Reinforcement learning: feedback (possibly adversarial) from the environment (robotics, adversarial environments, training...).

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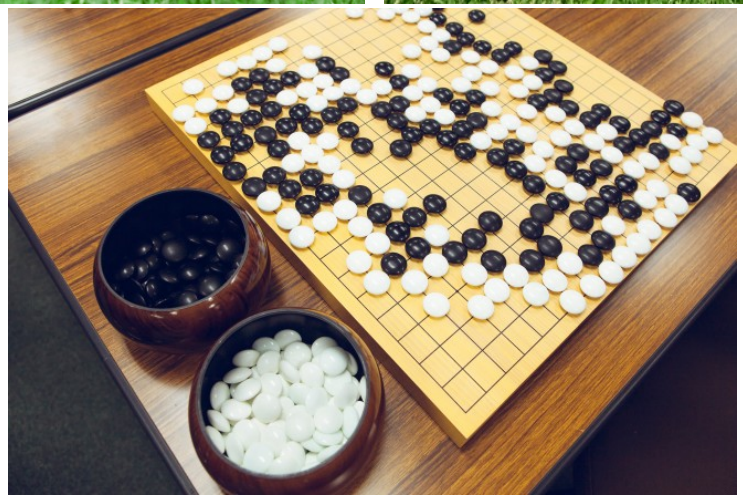

Loss function:

$$\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+.$$

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A predictor is any mapping

$$\hat{\phi} : (\mathcal{X} \times \mathcal{Y})^n \to \mathcal{Y}^{\mathcal{X}}.$$

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Risk

$$R(\hat{\phi}) = \mathbb{E}\left[\ell\left(\hat{\phi}(\mathbf{X}), Y\right)\right],$$

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Empirical risk

$$R_n(\hat{\phi}) = \frac{1}{n}\sum_{i=1}^{n}\left[\ell\left(\hat{\phi}(\mathbf{X}_i), Y_i\right)\right].$$

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- Absolute loss (regression): $\ell(a, b) = |a - b|$.
- 0-1 loss (classification): $\ell(a, b) = \mathbb{1}_{\{a \neq b\}}$.
- Hinge loss (classification): $\ell(a, b) = \max(0, 1 - ab)$.

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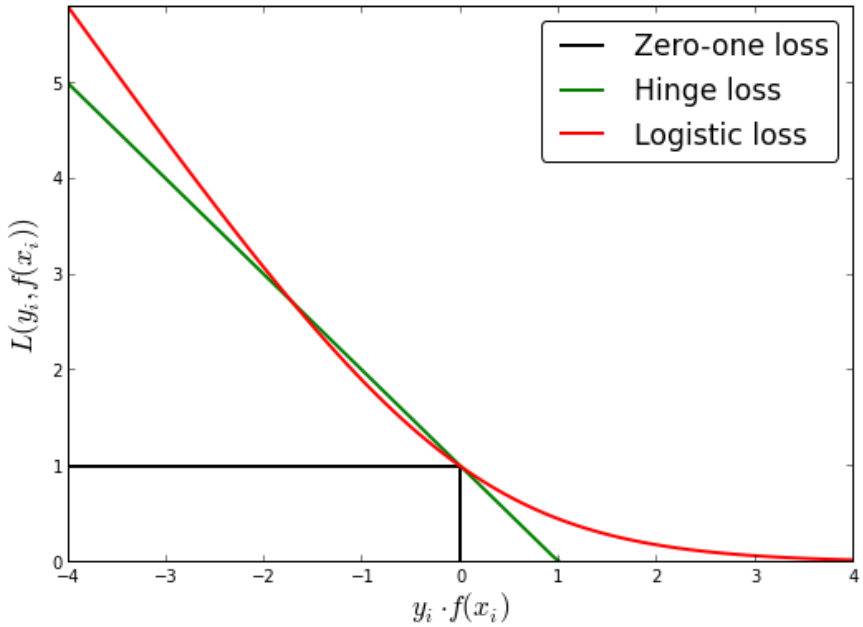

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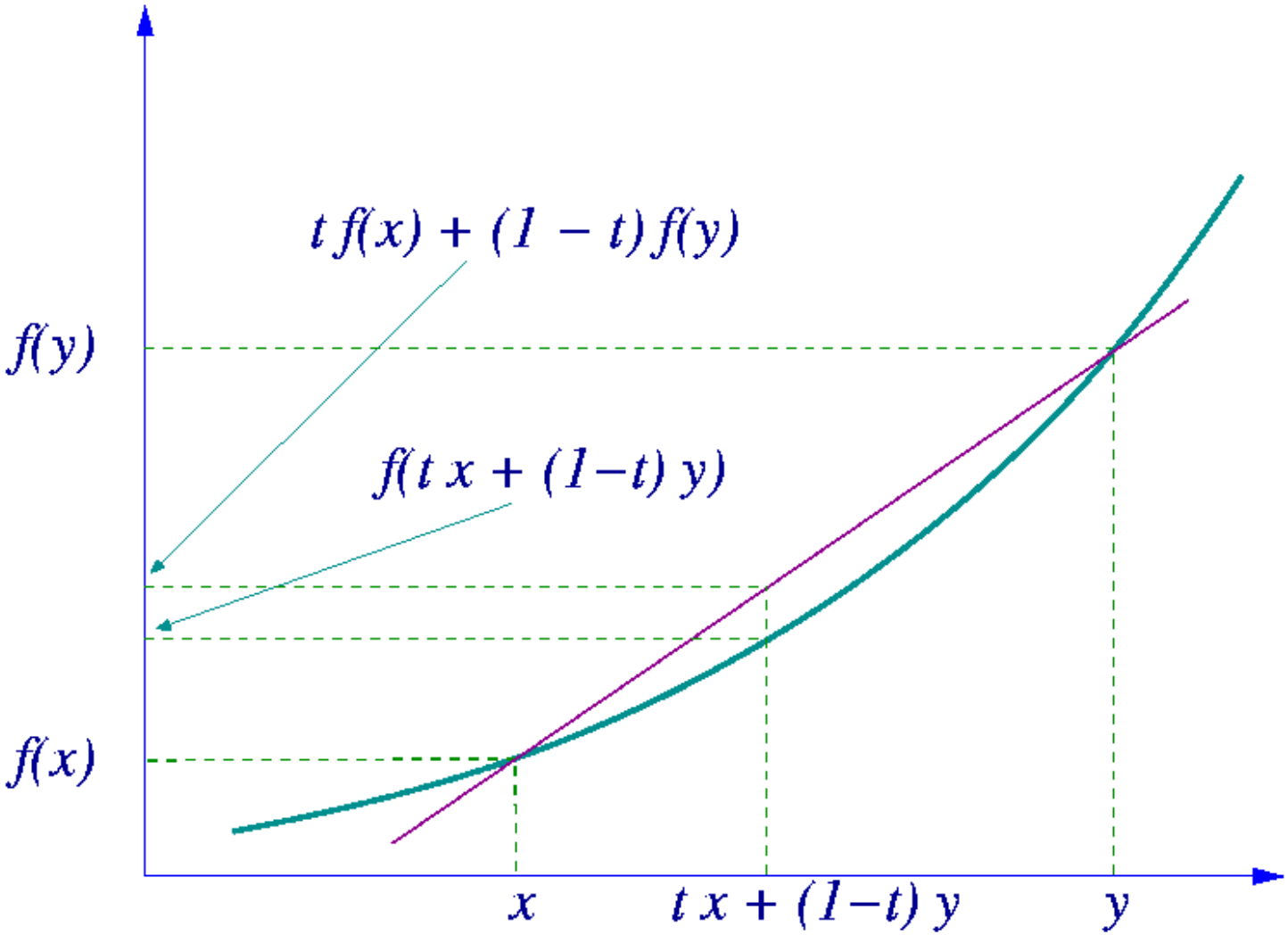

Statistical Learning vs. Machine Learning

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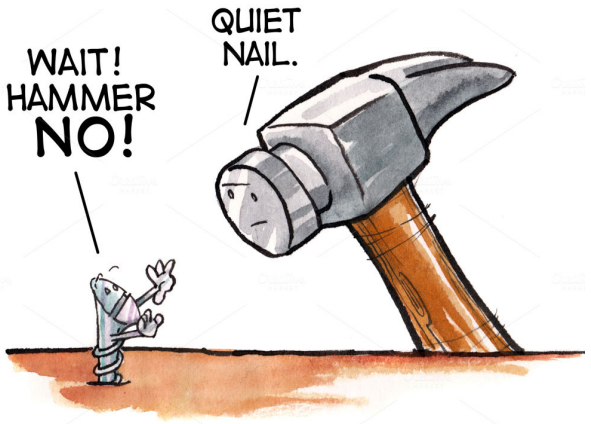
All models are wrong

but some are useful

George E.P. Box

If the only tool you have is a hammer, you tend to see every problem as a nail.

(Abraham Maslow)



A primer on probability distributions

All words are hyperlinks.

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- [Normal]

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- [Inverse Gaussian (a.k.a Wald)]

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The Bayesian paradigm

Introductory example

Consider observations $\mathbf{x} = (x_1, \ldots, x_n)$ generated from a probability distribution with density $f(\cdot|\theta)$.

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The associated likelihood is the inverted density:

$$\mathcal{L}(\theta|x) = f(x|\theta).$$

Example $f(\cdot|\theta) = \mathcal{N}(\theta, 1)$.

Bayes' Theorem

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Inversion of probabilities a.k.a actualisation principle.

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If A and B are events such that $\mathbb{P}(B) \neq 0$,

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A) + \mathbb{P}(B|A^c)\mathbb{P}(A^c)} \\ &= \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.\end{aligned}$$

(due to Thomas Bayes, published in 1764)

Who was Thomas Bayes?

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Reverend Thomas Bayes (ca. 1702–1761) Presbyterian minister in Kent from 1731. Election to the Royal Society based on a tract of 1736 where he defended the views and philosophy of Newton. Sole probability paper, "Essay Towards Solving a Problem in the Doctrine of Chances", published posthumously in 1763 and containing the seeds of Bayes' Theorem.

A new paradigm

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- Inference based on the distribution of θ conditional on $\mathbf{X}$ $\pi(\theta|\mathbf{x})$, called *posterior distribution*

$$\pi(\theta|\mathbf{x}) = \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int f(\mathbf{x}|\theta)\pi(\theta)\mathrm{d}\theta}.$$

A Bayesian model

. . . is made of a parametric (in this course) statistical model defined through its likelihood $f(\mathbf{x}|\theta)$ and a prior distribution on the parameter $\pi(\theta)$.

Consequences

- Semantic drift from unknown to random

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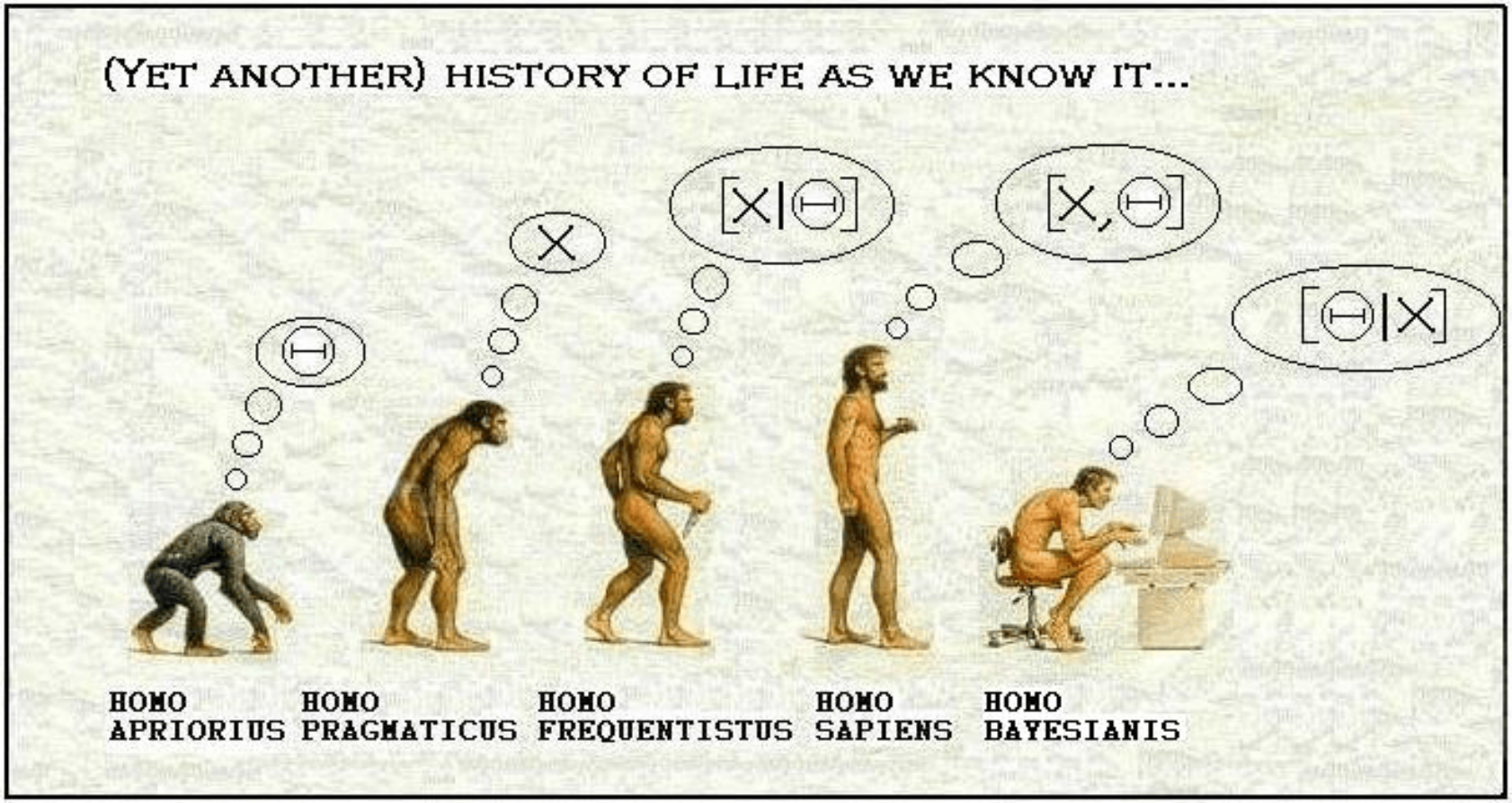
- Semantic drift from unknown to random
- Actualization of θ by extracting the information contained in the observation $\mathbf{x}$
- Allows incorporation of imperfect information in the decision process

The advantages of being a Bayesian



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- The *marginal distribution* of $\mathbf{x}$

$$m(\mathbf{x}) = \int \varphi(\theta, \mathbf{x}) \mathrm{d}\theta = \int f(\mathbf{x}|\theta)\pi(\theta)\mathrm{d}\theta.$$

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- The *predictive distribution* of y when $y \sim g(\cdot|\theta, \mathbf{x})$

$$g(y|\mathbf{x}) = \int g(y|\theta, \mathbf{x})\pi(\theta|\mathbf{x})\mathrm{d}\theta.$$

A comprehensive example normal-normal

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which means $\theta|\mathbf{x} \sim \mathcal{N}\left(\frac{10x+a}{11}, \frac{10}{11}\right)$.

A comprehensive example uniform-binomial

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Bayes' question: given X, what inference can we make on p?

Mathematical translation

Derive the posterior distribution of p given X, when $p \sim \mathcal{U}(0,1)$ and $X \sim \mathcal{B}(n,p)$.

Resolution 1/2

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and

$$\mathbb{P}(X = x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \mathrm{d}p,$$

Resolution 2/2

then

$$\mathbb{P}(a < p < b | X = x) = \frac{\int_a^b \binom{n}{x} p^x (1-p)^{n-x} \mathrm{d}p}{\int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \mathrm{d}p}$$
$$= \frac{\int_a^b \binom{n}{x} p^x (1-p)^{n-x} \mathrm{d}p}{\mathcal{B}(x+1, n-x+1)},$$

i.e., $p|x \sim \mathcal{B}(x+1, n-x+1)$.
(Beta distribution)

Pour se remettre dans le bain

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1. Quelle est la différence entre *statistical learning* et *machine learning* ?
2. Donner la définition d'un algorithme d'apprentissage.
3. Quels sont les quatre grands types d'apprentissage ?
4. Que définissent *fat data* et *tall data* ?
5. Comment compare-t-on les performances d'algorithmes d'apprentissage ?
6. Quelle notion est souvent cruciale au moment de choisir une bonne fonction de perte ? En donner la définition.
7. Donner quelques exemples de fonctions de perte.
8. Enoncer le théorème de Bayes.
9. Quel est le rôle de la distribution *a priori* ?
10. Donner la définition d'un modèle bayésien.
11. Quelles sont les quatre distributions importantes en bayésien ?

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- Coherent and complete inferential scope and unique motor of inference.
- Usually known up to a constant! $m(\mathbf{x})$ may be intractable.

Prior distributions

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A prior on θ may depend on additional parameters: those are called hyperparameters.

Conjugacy and explicit calculus of posteriors

Conjugate priors are a specific family of distributions with nice analytical properties.

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A family $\mathcal{F}$ of probability distributions is *conjugate* for a likelihood $f(x|\theta)$ if for every $\pi \in \mathcal{F}$, the posterior distribution $\pi(\theta|x)$ also belongs to $\mathcal{F}$.

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Only of interest when $\mathcal{F}$ is parameterized: switching from the prior to the posterior is reduced to an update of parameters.

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- Most importantly: tractability and simplicity

Exponential families

The family of distributions

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Main interest: allow for conjugate priors

$$\pi(\theta|\mu,\lambda) = K(\mu,\lambda)\exp(\theta\mu - \lambda\psi(\theta)), \quad \lambda > 0.$$

Classical exponential families and conjugate priors

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[Examples of exponential families]

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Classical exponential families and conjugate priors

[Examples of exponential families]

$f(x\|\theta)$	$\pi(\theta)$	$\pi(\theta\|x)$
Normal	Normal	Normal
$\mathcal{N}(\theta, \sigma^2)$	$\mathcal{N}(\mu, \tau^2)$	$\mathcal{N}(\rho(\sigma^2\mu + \tau^2 x), \rho\sigma^2\tau^2)$
		$\rho^{-1} = \sigma^2 + \tau^2$

Classical exponential families and conjugate priors

[Examples of exponential families]

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Normal $\mathcal{N}(\theta, \sigma^2)$	Normal $\mathcal{N}(\mu, \tau^2)$	Normal $\mathcal{N}(\rho(\sigma^2\mu + \tau^2 x), \rho\sigma^2\tau^2)$ $\rho^{-1} = \sigma^2 + \tau^2$
Poisson $\mathcal{P}(\theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	Gamma $\mathcal{G}(\alpha + x, \beta + 1)$

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Binomial $\mathcal{B}(n, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	Beta $\mathcal{B}e(\alpha + x, \beta + n - x)$

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Negative binomial $\mathrm{Neg}(m, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	Beta $\mathcal{B}e(\alpha + m, \beta + x)$

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Normal $\mathcal{N}(\mu, 1/\theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	Gamma $\mathcal{G}(\alpha + 1/2, \beta + (\mu - x)^2/2)$

Non-informative priors: Jeffreys priors

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Pros & Cons:

- Relates to information theory

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Jeffreys prior is defined as

$$\pi^\star(\theta) \propto \det(I(\theta))^{1/2}.$$

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- Parameterization invariant

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- Depends on data: incoherence with the likelihood principle

Example

If $x \sim \mathcal{B}(n, \theta)$, Jeffreys' prior is

$$\pi(\theta) \propto \mathcal{B}e(1/2, 1/2).$$

If $n \sim \mathrm{Neg}(x, \theta)$, Jeffreys' prior is

$$\pi(\theta) \propto \theta^{-1}(1-\theta)^{-1/2}$$

Non-informative priors: Laplace priors

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Continuous extension: $\pi(\theta) \propto 1$. This is no longer a probability distribution yet if $\int f(x|\theta)\mathrm{d}\theta < +\infty$, the posterior is well-defined as a probability distribution. Modeling is crucial. Weakness: lack of reparameterization invariance.

Pour se remettre dans le bain

- Expliquer l'intérêt de la conjugaison, et en donner la définition.
- Quel est l'intérêt d'utiliser une vraisemblance issue d'une famille exponentielle naturelle ?
- Donner des exemples de lois conjuguées.
- Donner deux exemples de méthodes de construction de priors non-informatifs. Quelles sont les limites de ces méthodes ?