

Economics 103 – Statistics for Economists

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Lecture #9 – Discrete RVs III

Variance and Standard Deviation of a Random Variable

Binomial Random Variable

Variance and Standard Deviation of a RV

The Defs are the same for continuous RVs, but the method of calculating will differ.

Variance (Var)

$$\sigma^2 = \text{Var}(X) = E[(X - \mu)^2] = E[(X - E[X])^2]$$

Standard Deviation (SD)

$$\sigma = \sqrt{\sigma^2} = \text{SD}(X)$$

Key Point

Variance and std. dev. are *expectations of functions of a RV*

It follows that:

1. Variance and SD are constants
2. To derive facts about them you can use the facts you know about expected value

How To Calculate Variance for Discrete RV?

Remember: it's just a function of X !

$$\text{Recall that } \mu = E[X] = \sum_{\text{all } x} xp(x)$$

$$\text{Var}(X) = E[(X - \mu)^2] = \sum_{\text{all } x} (x - \mu)^2 p(x)$$

Shortcut Formula For Variance

This is *not* the definition, it's a shortcut for doing calculations:

$$\text{Var}(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

We'll prove this in an upcoming lecture.

Example: The Shortcut Formula



Let $X \sim \text{Bernoulli}(1/2)$. Calculate $\text{Var}(X)$.

$$E[X] = 0 \times 1/2 + 1 \times 1/2 = 1/2$$

$$E[X^2] = 0^2 \times 1/2 + 1^2 \times 1/2 = 1/2$$

$$E[X^2] - (E[X])^2 = 1/2 - (1/2)^2 = 1/4$$

Variance of Bernoulli RV – via the Shortcut Formula

Step 1 – $E[X]$

$$\mu = E[X] = \sum_{x \in \{0,1\}} p(x) \cdot x = (1-p) \cdot 0 + p \cdot 1 = p$$

Step 2 – $E[X^2]$

$$E[X^2] = \sum_{x \in \{0,1\}} x^2 p(x) = 0^2(1-p) + 1^2 p = p$$

Step 3 – Combine with Shortcut Formula

$$\sigma^2 = \text{Var}[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Variance of a Linear Transformation

$$\begin{aligned}\text{Var}(a + bX) &= E \left[\{(a + bX) - E(a + bX)\}^2 \right] \\&= E \left[\{(a + bX) - (a + bE[X])\}^2 \right] \\&= E \left[(bX - bE[X])^2 \right] \\&= E[b^2(X - E[X])^2] \\&= b^2 E[(X - E[X])^2] \\&= b^2 \text{Var}(X) = b^2 \sigma^2\end{aligned}$$

The key point here is that variance is defined in terms of expectation and expectation is linear.

Variance and SD are *NOT* Linear

$$\text{Var}(a + bX) = b^2\sigma^2$$

$$\text{SD}(a + bX) = |b|\sigma$$

These should look familiar from the related results for sample variance and std. dev. that you worked out on an earlier problem set.

Binomial Random Variable

Let X = the sum of n independent Bernoulli trials, each with probability of success p . Then we say that: $X \sim \text{Binomial}(n, p)$

Parameters

p = probability of “success,” n = # of trials

Support

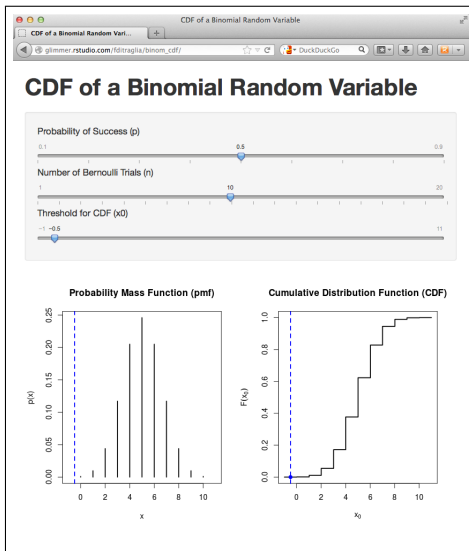
$\{0, 1, 2, \dots, n\}$

Probability Mass Function (pmf)

$$p(x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

http://fditraglia.shinyapps.io/binom_cdf/

Try playing around with all three sliders. If you set the second to 1 you get a Bernoulli.



Where does the Binomial pmf come from?

Question

Suppose we flip a fair coin 3 times. What is the probability that we get exactly 2 heads?

Answer

Three basic outcomes make up this event: $\{HHT, HTH, THH\}$, each has probability $1/8 = 1/2 \times 1/2 \times 1/2$. Basic outcomes are mutually exclusive, so sum to get $3/8 = 0.375$

Where does the Binomial pmf come from?

Question

Suppose we flip an *unfair* coin 3 times, where the probability of heads is $1/3$. What is the probability that we get exactly 2 heads?

Answer

No longer true that *all* basic outcomes are equally likely, but those with exactly two heads *still are*

$$P(HHT) = (1/3)^2(1 - 1/3) = 2/27$$

$$P(THH) = 2/27$$

$$P(HTH) = 2/27$$

Summing gives $2/9 \approx 0.22$

Where does the Binomial pmf come from?

Starting to see a pattern?

Suppose we flip an unfair coin 4 times, where the probability of heads is $1/3$. What is the probability that we get exactly 2 heads?

HHTT TTHH

HTHT THTH

HTTH THTT

Six equally likely, mutually exclusive
basic outcomes make up this event:

$$\binom{4}{2} (1/3)^2 (2/3)^2$$

R Commands for Binomial(n, p) RV

Probability Mass Function

`dbinom(x, size, prob)`, where `size` is n and `prob` is p

Cumulative Distribution Function

`pbinom(q, size, prob)`, where `q` is x_0 , `size` is n and `prob` is p

Make Random Draws

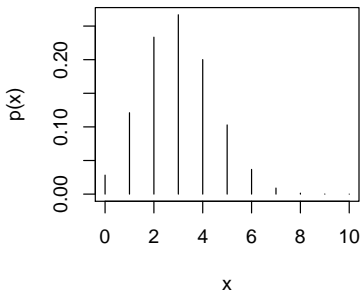
`rbinom(n, size, prob)`, where `n` is the number of draws, `size` is n and `prob` is p


```

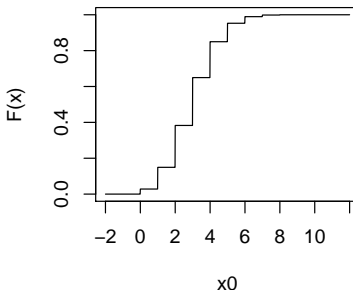
x <- 0:10
px <- dbinom(x, size = 10, prob = 0.3)
x0 <- seq(from = -2, to = 12, by = 0.01)
Fx <- pbinom(x0, size = 10, prob = 0.3)
par(mfrow = c(1, 2))
plot(x, px, type = 'h', ylab = 'p(x)', main = 'Binom(10, 0.3) pmf')
plot(x0, Fx, type = 'l', ylab = 'F(x)', main = 'Binom(10, 0.3) CDF')

```

Binom(10, 0.3) pmf



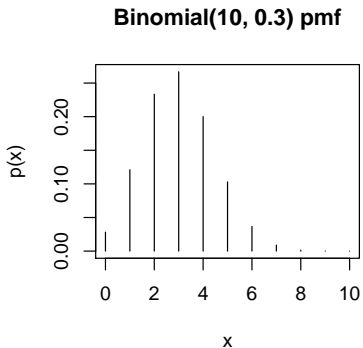
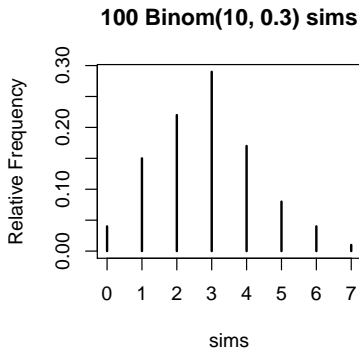
Binom(10, 0.3) CDF



```

set.seed(5545)
sims <- rbinom(100, size = 10, prob = 0.3)
par(mfrow = c(1, 2))
rel_freq <- prop.table(table(sims))
plot(rel_freq, main = '100 Binom(10, 0.3) sims',
     ylab = 'Relative Frequency')
plot(x, px, type = 'h', ylab = 'p(x)', main = 'Binomial(10, 0.3) pmf')

```



Lecture #10 – Discrete RVs IV

Joint vs. Marginal Probability Mass Functions

Conditional Probability Mass Function & Independence

Expectation of a Function of Two Discrete RVs, Covariance

Linearity of Expectation Reprise, Properties of Binomial RV

Multiple RVs *at once* - Definition of Joint PMF

Let X and Y be discrete random variables. The joint probability mass function $p_{XY}(x, y)$ gives the probability of each pair of realizations (x, y) in the support:

$$p_{XY}(x, y) = P(X = x \cap Y = y)$$

Example: Joint PMF in Tabular Form

		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

Plot of Joint PMF



What is $p_{XY}(1, 2)$?



		Y		
		1	2	3
X	0	1/8	0	0
	1	0	1/4	1/8
	2	0	1/4	1/8
	3	1/8	0	0

$$p_{XY}(1, 2) = P(X = 1 \cap Y = 2) = 1/4$$

$$p_{XY}(2, 1) = P(X = 2 \cap Y = 1) = 0$$

Properties of Joint PMF

1. $0 \leq p_{XY}(x, y) \leq 1$ for any pair (x, y)
2. The sum of $p_{XY}(x, y)$ over all pairs (x, y) in the support is 1:

$$\sum_x \sum_y p(x, y) = 1$$

Joint versus Marginal PMFs

Joint PMF

$$p_{XY}(x, y) = P(X = x \cap Y = y)$$

Marginal PMFs

$$p_X(x) = P(X = x)$$

$$p_Y(y) = P(Y = y)$$

You can't calculate a joint pmf from marginals alone but you *can* calculate marginals from the joint!

Marginals from Joint

$$p_X(x) = \sum_{\text{all } y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{\text{all } x} p_{XY}(x, y)$$

Why?

$$\begin{aligned} p_Y(y) &= P(Y = y) = P\left(\bigcup_{\text{all } x} \{X = x \cap Y = y\}\right) \\ &= \sum_{\text{all } x} P(X = x \cap Y = y) = \sum_{\text{all } x} p_{XY}(x, y) \end{aligned}$$

To get the marginals sum “into the margins” of the table.

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
					1

$$p_X(0) = 1/8 + 0 + 0 = 1/8$$

$$p_X(1) = 0 + 1/4 + 1/8 = 3/8$$

$$p_X(2) = 0 + 1/4 + 1/8 = 3/8$$

$$p_X(3) = 1/8 + 0 + 0 = 1/8$$

What is $p_Y(2)$?



		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	1

$$p_Y(1) = 1/8 + 0 + 0 + 1/8 = 1/4$$

$$p_Y(2) = 0 + 1/4 + 1/4 + 0 = 1/2$$

$$p_Y(3) = 0 + 1/8 + 1/8 + 0 = 1/4$$

Definition of Conditional PMF

How does the distribution of y change with x ?

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{P(Y = y \cap X = x)}{P(X = x)} = \frac{p_{XY}(x, y)}{p_X(x)}$$

Conditional PMF of Y given $X = 2$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8

$$p_{Y|X}(1|2) = \frac{p_{XY}(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$p_{Y|X}(2|2) = \frac{p_{XY}(2,2)}{p_X(2)} = \frac{1/4}{3/8} = 2/3$$

$$p_{Y|X}(3|2) = \frac{p_{XY}(2,3)}{p_X(2)} = \frac{1/8}{3/8} = 1/3$$

What is $p_{X|Y}(1|2)$?



		Y			
		1	2	3	
X	0	1/8	0	0	
	1	0	1/4	1/8	
	2	0	1/4	1/8	
	3	1/8	0	0	
		1/4	1/2	1/4	

$$p_{X|Y}(1|2) = \frac{p_{XY}(1,2)}{p_Y(2)} = \frac{1/4}{1/2} = 1/2$$

Similarly:

$$p_{X|Y}(0|2) = 0, \quad p_{X|Y}(2|2) = 1/2, \quad p_{X|Y}(3|2) = 0$$

Independent RVs: Joint Equals Product of Marginals

Definition

Two discrete RVs are **independent** if and only if

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

for all pairs (x, y) in the support.

Equivalent Definition

$$p_{Y|X}(y|x) = p_Y(y) \text{ and } p_{X|Y}(x|y) = p_X(x)$$

for all pairs (x, y) in the support.

Are X and Y Independent?



(A = YES, B = NO)

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$p_{XY}(2, 1) = 0$$

$$p_X(2) \times p_Y(1) = (3/8) \times (1/4) \neq 0$$

Therefore X and Y are *not* independent.

Expectation of Function of Two Discrete RVs

$$E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{XY}(x, y)$$

Some Extremely Important Examples

Same For Continuous Random Variables

Let $\mu_X = E[X]$, $\mu_Y = E[Y]$

Covariance

$$\sigma_{XY} = \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

Correlation

$$\rho_{XY} = \text{Corr}(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Shortcut Formula for Covariance

Much easier for calculating:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

I'll mention this again in a few slides...

Calculating $\text{Cov}(X, Y)$

		Y			
		1	2	3	
X	0	1/8	0	0	1/8
	1	0	1/4	1/8	3/8
	2	0	1/4	1/8	3/8
	3	1/8	0	0	1/8
		1/4	1/2	1/4	

$$E[X] = 3/8 + 2 \times 3/8 + 3 \times 1/8 = 3/2$$

$$E[Y] = 1/4 + 2 \times 1/2 + 3 \times 1/4 = 2$$

$$\begin{aligned} E[XY] &= 1/4 \times (2 + 4) + 1/8 \times (3 + 6 + 3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= 3 - 3/2 \times 2 = 0 \end{aligned}$$

$$\text{Corr}(X, Y) = \text{Cov}(X, Y) / [SD(X)SD(Y)] = 0$$

Hence, zero covariance (correlation) does *not* imply independence!

Zero Covariance versus Independence

While zero covariance (correlation) *does not* imply independence, independence *does* imply zero covariance (correlation).

You will prove this in an extension problem. . .

Linearity of Expectation, Again

Holds for Continuous RVs as well, but different proof.

In general $E[g(X, Y)] \neq g(E[X], E[Y])$. But if g is linear, then:

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

where X, Y are random variables and a, b, c are constants.

You will prove this as a review problem...

Application: Proof of Shortcut Formula for Variance

By the Linearity of Expectation,

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2\end{aligned}$$

Expected Value of Sum = Sum of Expected Values

Repeatedly applying the linearity of expectation,

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

regardless of how the RVs X_1, \dots, X_n are related to each other. In particular it **doesn't matter if they're dependent or independent.**

Independent and Identically Distributed (iid) RVs

Example

$$X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$$

Independent

Realization of one of the RVs gives no information about the others.

Identically Distributed

Each X_i is the same kind of RV, with the same values for any parameters. (Hence same pmf, cdf, mean, variance, etc.)

Recall: Binomial(n, p) Random Variable

Definition

Sum of n independent Bernoulli RVs, each with probability of “success,” i.e. 1, equal to p

Using Our New Notation

Let $X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$, $Y = X_1 + X_2 + \dots + X_n$.

Then $Y \sim \text{Binomial}(n, p)$.

Expected Value of Binomial RV

Use the fact that a Binomial(n, p) RV is defined as the sum of n iid Bernoulli(p) Random Variables and the Linearity of Expectation:

$$\begin{aligned} E[Y] &= E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n] \\ &= p + p + \dots + p \\ &= np \end{aligned}$$

Variance of a Sum \neq Sum of Variances!

$$\begin{aligned} \text{Var}(aX + bY) &= E \left[\{(aX + bY) - E[aX + bY]\}^2 \right] \\ &\vdots \\ &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y) \end{aligned}$$

You'll fill in the missing steps as an extension problem...

Since $\sigma_{XY} = \rho\sigma_X\sigma_Y$, this is sometimes written as:

$$\text{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$$

$$\text{Independence} \Rightarrow \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

X and Y independent $\Rightarrow \text{Cov}(X, Y) = 0$. Hence:

$$\begin{aligned}\text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ &= \text{Var}(X) + \text{Var}(Y)\end{aligned}$$

Also true for three or more RVs

If X_1, X_2, \dots, X_n are independent, then

$$\text{Var}(X_1 + X_2 + \dots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$$

Crucial Distinction

Expected Value

Always true that

$$E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

Variance

Not true in general that

$$\text{Var}[X_1 + X_2 + \dots + X_n] = \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n]$$

except in the special case where X_1, \dots, X_n are independent (or at least uncorrelated).

Variance of Binomial Random Variable

Definition from Sequence of Bernoulli Trials

If $X_1, X_2, \dots, X_n \sim \text{iid Bernoulli}(p)$ then

$$Y = X_1 + X_2 + \dots + X_n \sim \text{Binomial}(n, p)$$

Using Independence

$$\begin{aligned}\text{Var}[Y] &= \text{Var}[X_1 + X_2 + \dots + X_n] \\ &= \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n] \\ &= p(1 - p) + p(1 - p) + \dots + p(1 - p) \\ &= np(1 - p)\end{aligned}$$

Lecture #11 – Continuous RVs I

Introduction: Probability as Area

Probability Density Function (PDF)

Relating the PDF to the CDF

Calculating the Probability of an Interval

Calculating Expected Value for Continuous RVs

Continuous RVs – What Changes?

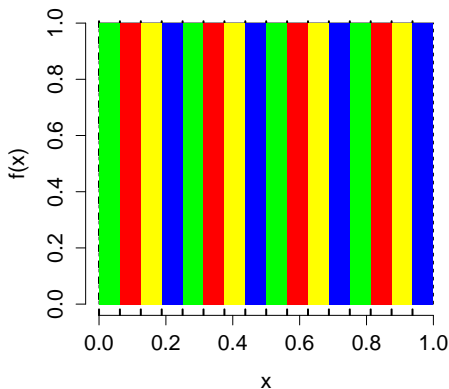
1. Probability Density Functions replace Probability Mass Functions
2. Integrals Replace Sums

Everything Else is Essentially Unchanged!

What is the probability of “Yellow?”



From Twister to Density – Probability as *Area*



For continuous RVs, probability is defined as *area under a curve*.

Zero area means zero probability!

Probability Density Function (PDF)

For a continuous random variable X ,

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

where $f(x)$ is the *probability density function* for X .

Extremely Important

For any realization x , $P(X = x) = 0$ since $\int_a^a f(x)dx = 0$. In other words, zero area means zero probability!

For a Continuous RV, Zero Probability \neq Impossible

It is *crucial* to specify the support set of a continuous RV:

- ▶ Any x outside the support set of X is *impossible*.
- ▶ Any x in the support set of X is a *possible outcome* even though $P(X = x) = 0$ for all x .

There is no way around this slightly awkward situation: it is a consequence of defining probability as the *area under a curve*.

Properties of PDFs

1. $f(x) \geq 0$ for all x in the support of X and zero otherwise.
2. $\int_{-\infty}^{\infty} f(x) dx = 1$

Warning: $f(x)$ is not a probability

Can have $f(x) > 1$ for some x as long as $\int_{-\infty}^{\infty} f(x) dx = 1$.

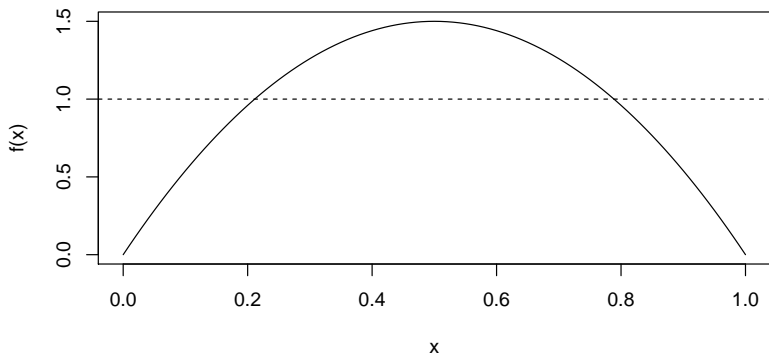
Relating the CDF to the PDF

$$F(x_0) \equiv P(X \leq x_0) = \int_{-\infty}^{x_0} f(x) dx$$

Example: Suppose X has Support Set $[0, 1]$

Let $f(x) = 6x(1 - x)$ for $x \in [0, 1]$ and zero otherwise.

```
curve(6 * x * (1 - x), from = 0, to = 1, ylab = 'f(x)')  
abline(h = 1, lty = 2)
```



Example: Suppose X has Support Set $[0, 1]$

Let $f(x) = 6x(1 - x)$ for $x \in [0, 1]$ and zero otherwise.

Is f a valid PDF?

1. Is $f(x) \geq 0$ for $x \in [0, 1]$ and zero otherwise?
2. Does the total area under f equal one?

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) dx &= \int_0^1 6x(1 - x) dx = 6 \int_0^1 (x - x^2) dx \\ &= 6 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = 1\end{aligned}$$

So yes, f is a valid PDF ✓

Integrating a Function in R

```
pdf <- function(x) {  
  6 * x * (1 - x)  
}  
  
integrate(pdf, lower = 0, upper = 1)  
  
## 1 with absolute error < 1.1e-14
```

You can use this to check your work!

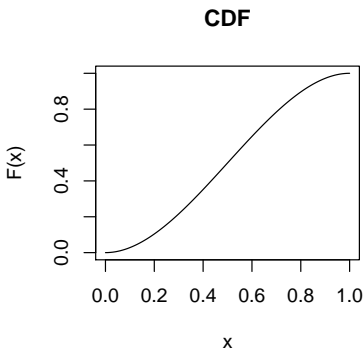
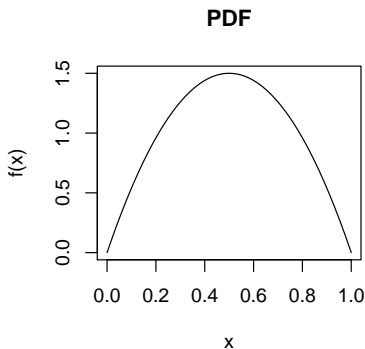
Example: $f(x) = 6x(1 - x)$ for $x \in [0, 1]$, zero otherwise.

What is the CDF of X ?

$$\begin{aligned} F(x_0) &\equiv P(X \leq x_0) = \int_{-\infty}^{x_0} f(x) \, dx = \int_0^{x_0} 6x(1 - x) \, dx \\ &= 6 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^{x_0} = 3x_0^2 - 2x_0^3 \end{aligned}$$

$$F(x_0) = \begin{cases} 0, & x_0 < 0 \\ 3x_0^2 - 2x_0^3, & 0 \leq x_0 \leq 1 \\ 1, & x_0 > 1 \end{cases}$$

```
par(mfrow = c(1,2))  
curve(6 * x * (1 - x), from = 0, to = 1, ylab = 'f(x)', main = 'PDF')  
curve(3 * x^2 - 2 * x^3, from = 0, to = 1, ylab = 'F(x)', main = 'CDF')
```



```
par(mfrow = c(1,1))
```

Relationship between PDF and CDF

Integrate PDF to get CDF

$$F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(x) dx$$

Differentiate CDF to get PDF

$$f(x) = \frac{d}{dx} F(x)$$

This is just the First Fundamental Theorem of Calculus.

Example: $f(x) = 6x(1 - x)$ for $x \in [0, 1]$, zero otherwise.

Differentiate CDF to get PDF

$$\begin{aligned} f(x) &= \frac{d}{dx} F(x) = \frac{d}{dx} (3x^2 - 2x^3) \\ &= 6x - 6x^2 \\ &= 6x(1 - x) \end{aligned}$$

Key Idea: Probability of an Interval for a Continuous RV

$$P(a \leq X \leq b) = \int_a^b f(x) dx = F(b) - F(a)$$

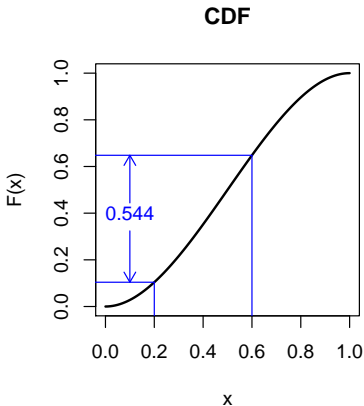
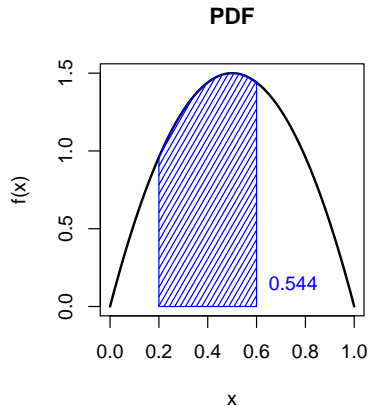
This is just the Second Fundamental Theorem of Calculus.

Example: $f(x) = 6x(1 - x)$ for $x \in [0, 1]$, zero otherwise.

Two equivalent ways of calculating $P(0.2 \leq X \leq 0.6)$

```
cdf <- function(x0) {  
  3 * x0^2 - 2 * x0^3  
}  
cdf(0.6) - cdf(0.2)  
  
## [1] 0.544  
  
integrate(pdf, lower = 0.2, upper = 0.6)  
  
## 0.544 with absolute error < 6e-15
```


Example: $f(x) = 6x(1 - x)$ for $x \in [0, 1]$, zero otherwise.



$$P(0.2 \leq X \leq 0.6) = 0.544$$

Expected Value for Continuous RVs

$$E[X] = \int_{-\infty}^{\infty} xf(x) dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) dx$$

Integrals Replace Sums!

What about all those rules for expected value?

- ▶ The only difference between expectation for continuous versus discrete is how we do the *calculation*.
- ▶ Sum for discrete; integral for continuous.
- ▶ All *properties* of expected value **continue to hold!**
- ▶ Includes linearity, shortcut for variance, etc.

Variance of Continuous RV

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

where

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x) dx$$

Shortcut formula still holds for continuous RVs!

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

Example: $f(x) = 6x(1 - x)$ for $x \in [0, 1]$, zero otherwise.

$$E[X] = \int_{-\infty}^{\infty} xf(x) dx = \int_0^1 x \cdot 6x(1 - x) = 6 \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 x^2 \cdot 6x(1 - x) = 6 \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{3}{10}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{3}{10} - \left(\frac{1}{2} \right)^2 = 1/20$$

Complete the algebra at home and check using `integrate` in R.

Simulating a Beta(2, 2) Random Variable

Our example from above is a special case of the *Beta distribution*. The command `rbeta(n, 2, 2)` makes `n` draws for this RV. These simulations agree with our calculations from above:

```
set.seed(12345)
sims <- rbeta(10000, 2, 2)
mean(sims)

## [1] 0.5007002

var(sims)

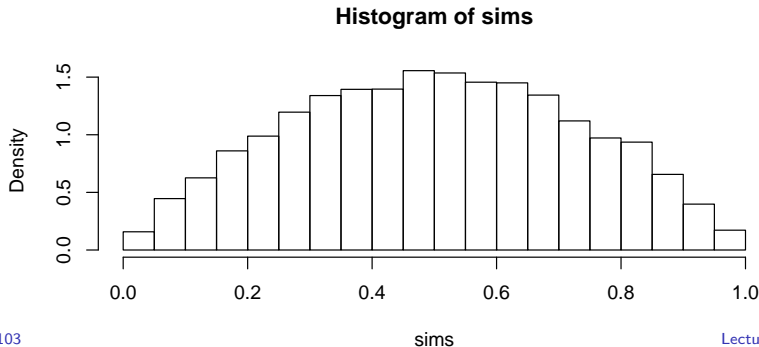
## [1] 0.05012776
```

Simulating a Beta(2, 2) Random Variable

```
mean(sims^2)

## [1] 0.3008234

hist(sims, freq = FALSE)
```



The Uniform Random Variable

Several of your review questions along with one of your extension questions will involve the so-called *Uniform Random Variable*:

Uniform(0,1) Random Variable

$f(x) = 1$ for $x \in [0, 1]$, zero otherwise.

Uniform(a,b) Random Variable

$f(x) = 1/(b - a)$ for $x \in [a, b]$, zero otherwise.

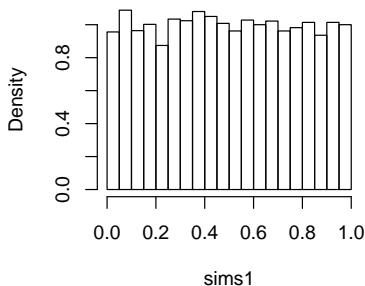
Simulating from a Uniform RV

`runif(n, a, b)` makes `n` draws from a $\text{Uniform}(a, b)$ RV.

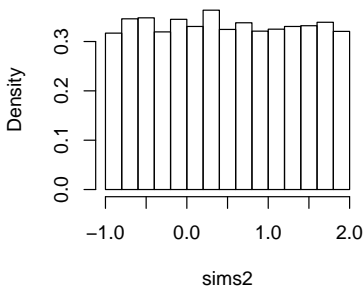
Simulating Uniform Random Variables

```
sims1 <- runif(10000, 0, 1)
sims2 <- runif(10000, -1, 2)
par(mfrow = c(1, 2))
hist(sims1, freq = FALSE)
hist(sims2, freq = FALSE)
```

Histogram of sims1



Histogram of sims2



We don't have time to cover these in Econ 103:

Joint Density

$$P(a \leq X \leq b \cap c \leq Y \leq d) = \int_c^d \int_a^b f(x, y) \, dx dy$$

Marginal Densities

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$$

Independence in Terms of Joint and Marginal Densities

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

Conditional Density

$$f_{Y|X} = f_{XY}(x, y)/f_X(x)$$

So where does that leave us?

What We've Accomplished

We've covered all the basic properties of RVs on this [Handout](#).

Where are we headed next?

Next up is the most important RV of all: the normal RV. After that it's time to do some statistics!

How should you be studying?

If you *master* the material on RVs (both continuous and discrete) and in particular the normal RV the rest of the semester will seem easy. If you don't, you're in for a rough time. . .