Economics 103 – Statistics for Economists

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Lecture #19 – Hypothesis Testing II

Test for the mean of a normal population (variance known)

Relationship Between Confidence Intervals and Hypothesis Tests

P-values

A Simple Example

Suppose $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, \sigma^2 = 9)$ and we want to test

$$H_0\colon \mu = 2$$
$$H_1\colon \mu \neq 2$$

Step 1 – Specify Null Hypothesis H_0 and alternative Hypothesis H_1 ✓

Step 2 – Choose Test Statistic T_n

If $\bar{X}$ is far from 2 then $\mu = 2$ is implausible. Why?

Suppose $X_1, \ldots, X_{100} \sim$ iid $\text{N}(2, \sigma^2 = 9)$. What is the sampling distribution of $\bar{X}$?

(a) $\text{N}(0, 1)$

(b) $t(99)$

(c) $N(2, 0.3)$

(d) $N(2, 1)$

(e) $N(2, 0.09)$

If $\bar{X}_n$ is far from 2, then $\mu = 2$ is implausible

Since $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, 9)$, if $\mu = 2$ then $\bar{X} \sim N(2, 0.09)$

$$\begin{aligned} P(a \leq \bar{X} \leq b) &= P\left(\frac{a-2}{3/10} \leq \frac{\bar{X}-2}{3/10} \leq \frac{b-2}{3/10}\right) \\ &= P\left(\frac{a-2}{0.3} \leq Z \leq \frac{b-2}{0.3}\right) \end{aligned}$$

where $Z \sim N(0,1)$ so we see that if $H_0\colon \mu = 2$ is true then

$$\begin{aligned} P(1.7 \leq \bar{X} \leq 2.3) &= P(-1 \leq Z \leq 1) \approx 0.68 \\ P(1.4 \leq \bar{X} \leq 2.6) &= P(-2 \leq Z \leq 2) \approx 0.95 \\ P(1.1 \leq \bar{X} \leq 2.9) &= P(-3 \leq Z \leq 3) > 0.99 \end{aligned}$$

Step 2 – Choose Test Statistic T_n

- Reject $H_0: \mu = 2$ if the sample mean is far from 2.
- $\Rightarrow T_n$ should depend on the distance from $\bar{X}$ to 2, i.e. $|\bar{X} - 2|$.
- We can make our subsequent calculations much easier if we choose a scale for T_n that is convenient under H_0...

$$\mu = 2 \Rightarrow \quad \bar{X} - 2 \quad \sim \quad N(0, 0.09)$$

$$\frac{\bar{X} - 2}{0.3} \quad \sim \quad N(0, 1)$$

So we will set $T_n = \left|\frac{\bar{X} - 2}{0.3}\right|$

A Simple Example: $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, \sigma^2 = 9)$

Step 1 – $H_0\colon \mu = 2,\ H_1\colon \mu \neq 2$ ✓

Step 2 – $T_n = \left|\dfrac{\bar{X} - 2}{0.3}\right|$ ✓

Step 3 – If $\mu = 2$ then $\left(\dfrac{\bar{X} - 2}{0.3}\right) \sim N(0,1)$ ✓

Step 4 – Choose Critical Value c

(i) Specify significance level α.

(ii) Choose c so that $P(T_n > c) = \alpha$ under $H_0\colon \mu = 2$.

Choose c so that $P(T_n > c) = \alpha$ under H_0

$$T_n = \left|\frac{\bar{X}-2}{0.3}\right| \text{ and } \mu = 2 \implies \frac{\bar{X}-2}{0.3} \sim N(0,1)$$

$$\begin{aligned}
P\left(\left|\frac{\bar{X}-2}{0.3}\right| > c\right) &= \alpha \\
1 - P\left(\left|\frac{\bar{X}-2}{0.3}\right| \leq c\right) &= \alpha \\
P\left(\left|\frac{\bar{X}-2}{0.3}\right| \leq c\right) &= 1-\alpha \\
P\left(-c \leq \frac{\bar{X}-2}{0.3} \leq c\right) &= 1-\alpha
\end{aligned}$$

Hence: $c = \texttt{qnorm}(1-\alpha/2)$ which should look familiar!

A Simple Example: $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, \sigma^2 = 9)$

Step 1 – $H_0\colon \mu = 2,\; H_1\colon \mu \neq 2$ ✓

Step 2 – $T_n = \left|\dfrac{\bar{X} - 2}{0.3}\right|$ ✓

Step 3 – If $\mu = 2$ then $\left(\dfrac{\bar{X} - 2}{0.3}\right) \sim N(0,1)$ ✓

Step 4 – $c = \texttt{qnorm}(1 - \alpha/2)$ ✓

Step 5 – Look at the data: if $T_n > c$, reject H_0

- Suppose I choose $\alpha = 0.05$. Then $c \approx 2$.
- I observe a sample of 100 observations. Suppose $\bar{x} = 1.34$

$$T_n = \left|\frac{\bar{x} - 2}{0.3}\right| = \left|\frac{1.34 - 2}{0.3}\right| = 2.2$$

- Since $T_n > c$, I reject $H_0\colon \mu = 2$.

Reporting the Results of a Test

Our Example: $X_1, \ldots, X_{100} \sim \text{iid } N(\mu, 9)$

- $H_0\colon \mu = 2$ vs. $H_1\colon \mu \neq 2$
- $T_n = |(\bar{X}_n - 2)/0.3|$
- $\alpha = 0.05 \implies c \approx 2$

Suppose $\bar{x} = 1.34$

Then $T_n = 2.2$. Since this is greater than c for $\alpha = 0.05$, we reject $H_0\colon \mu = 2$ at the 5% significance level.

Suppose instead that $\bar{x} = 1.82$

Then $T_n = 0.6$. Since this is less than c for $\alpha = 0.05$, we fail to reject $H_0\colon \mu = 2$ at the 5% signifcance level.

General Version of Preceding Example

$X_1, \ldots, X_n \sim$ iid $\text{N}(\mu, \sigma^2)$ with σ^2 known and we want to test:

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu \neq \mu_0$$

where μ_0 is some specified value for the population mean.

- $|\bar{X}_n - \mu_0|$ tells how far sample mean is from μ_0.
- Reject $H_0\colon \mu = \mu_0$ if sample mean is far from μ_0.
- Under $H_0\colon \mu = \mu_0$, $\dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$.
- Test statistic $T_n = \left|\dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}\right|$
- Reject $H_0\colon \mu = \mu_0$ if $T_n > \texttt{qnorm}(1 - \alpha/2)$

Suppose $X_1, \ldots, X_{64} \sim$ iid $\text{N}(\mu, \sigma^2 = 25)$ and we want to test $H_0\colon \mu = 0$ vs. $H_1\colon \mu \neq 0$ with $\alpha = 0.32$. If we observe $\bar{x} = 0.5$ what is our decision?

(a) Reject H_0

(b) Fail to Reject H_0

(c) Not enough information to determine.

$T_n = \left|\frac{0.5-0}{5/8}\right| = 0.5 \times 8/5 = 0.8$, $\texttt{qnorm}(1 - 0.32/2) \approx 1$

$\boxed{\text{Fail to reject } H_0}$

What is this test telling us to do?

Return to the example where $H_0\colon \mu = 2$ vs. $H_1\colon \mu \neq 2$ and $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, 9)$ with $\alpha = 0.05$:

$$\begin{aligned}
&\text{Reject } H_0 \quad \text{if} \quad \left|\frac{\bar{X}_n - 2}{0.3}\right| > 2 \\
&\text{Reject } H_0 \quad \text{if} \quad |\bar{X}_n - 2| > 0.6 \\
&\text{Reject } H_0 \quad \text{if} \quad (\bar{X}_n < 1.4) \text{ or } (\bar{X}_n > 2.6)
\end{aligned}$$

Reject $H_0\colon \mu = 2$ if $\bar{X}_n$ is far from 2. How far? Depends on choice of α along with sample size and population variance.

This looks suspiciously similar to a confidence interval...

$$\boxed{X_1, \ldots, X_n \sim \text{iid } \mathrm{N}(\mu, \sigma^2) \text{ where } \sigma^2 \text{ is known}}$$

$$\boxed{T_n = \left| \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \right|, \; c = \texttt{qnorm}(1 - \alpha/2), \; \text{Reject } H_0\colon \mu = \mu_0 \text{ if } T_n > c}$$

Another way of saying this is don't reject H_0 if:

$$\begin{aligned}
(T_n \leq c) \quad &\iff \quad \left(\left| \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \right| \leq c \right) \iff \left(-c \leq \frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} \leq c \right) \\
&\iff \quad \left(\bar{X}_n - c \times \frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{X}_n + c \times \frac{\sigma}{\sqrt{n}} \right)
\end{aligned}$$

In other words, don't reject $H_0\colon \mu = \mu_0$ at significance level α if μ_0 lies inside the $100 \times (1 - \alpha)\%$ confidence interval for μ.

CIs and Hypothesis Tests are Intimately Related

Our Simple Example

$X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, \sigma^2 = 9)$ and observe $\bar{x} = 1.34$

Test $H_0\colon \mu = 2$ vs. $H_1\colon \mu \neq 2$ with $\alpha = 0.05$

$T_n = 2.2$, $c = \texttt{qnorm}(1 - 0.05/2) \approx 2$. Since $T_n > c$ we reject.

95% Confidence Interval for μ

$1.34 \pm 2 \times 3/10$ i.e. 1.34 ± 0.6 or equivalently $(0.74, 1.94)$

Another way to carry out the test...

Since 2 lies outside the 95% confidence interval for μ, if our significance level is $\alpha = 0.05$ we reject $H_0\colon \mu = 2$.

$X_1, \ldots X_{100} \sim$ iid $\text{N}(\mu_X, 9)$ and $Y_1, \ldots, Y_{100} \sim$ iid $\text{N}(\mu_Y, 9)$

Two researchers: $H_0\colon \mu = 2$ vs. $H_1\colon \mu \neq 2$ with $\alpha = 0.05$

Researcher 1

- $\bar{x} = 1.34$
- $T_n = 2.2 > 2$
- Reject $H_0\colon \mu_X = 2$

Researcher 2

- $\bar{y} = 11.3$
- $T_n = 31 > 2$
- Reject $H_0\colon \mu_Y = 2$

Both researchers would report "reject H_0 at the 5% level" but Researcher 2 found much stronger evidence against H_0...

What if we had chosen a different significance level α?

$$\boxed{T_n = 2.2, \quad c = \texttt{qnorm}(1-\alpha/2), \quad \text{Reject } H_0\colon \mu = 2 \text{ if } T_n > c}$$

$\alpha = 0.32 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.32/2) \approx 0.99$ Reject

$\alpha = 0.10 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.10/2) \approx 1.64$ Reject

$\alpha = 0.05 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.05/2) \approx 1.96$ Reject

$\alpha = 0.04 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.04/2) \approx 2.05$ Reject

$\alpha = 0.03 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.03/2) \approx 2.17$ Reject

$\alpha = 0.02 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.02/2) \approx 2.33$ Fail to Reject

$\alpha = 0.01 \quad \Rightarrow \quad c = \texttt{qnorm}(1 - 0.01/2) \approx 2.58$ Fail to Reject

Result of Test Depends on Choice of α!

$\alpha = 0.32$	$\Rightarrow$	Reject
$\alpha = 0.10$	$\Rightarrow$	Reject
$\alpha = 0.05$	$\Rightarrow$	Reject
$\alpha = 0.04$	$\Rightarrow$	Reject
$\alpha = 0.03$	$\Rightarrow$	Reject
$\alpha = 0.02$	$\Rightarrow$	Fail to Reject
$\alpha = 0.01$	$\Rightarrow$	Fail to Reject

- If you reject H_0 at a given choice of α, you would also have rejected at any larger choice of α.
- If you fail to reject H_0 at a given choice of α, you would also have failed to reject at any smaller choice of α.

Question

If α is large enough we will reject; if α is small enough, we won't. Where is the dividing line between reject and fail to reject?

P-Value: Dividing Line Between Reject and Fail to Reject

$$\boxed{T_n = 2.2, \quad c = \texttt{qnorm}(1 - \alpha/2), \quad \text{Reject } H_0\colon \mu = 2 \text{ if } T_n > c}$$

Question

Given that we observed a test statistic of 2.2, what choice of α would put us just at the cusp of rejecting H_0?

Answer

Whichever α makes $c = 2.2$! At this α we just barely fail to reject.

Calculating the P-value

Definition of a P-value

Significance level α such that the critical value c exactly equals the observed value of the test statistic. Equivalently: α that lies exactly on boundary between Reject and Fail to Reject.

Our Example

The observed value of the test statistic is 2.2 and the critical value is $\texttt{qnorm}(1-\alpha/2)$, so we need to solve:

$$
\begin{aligned}
2.2 &= \texttt{qnorm}(1-\alpha/2) \\
\texttt{pnorm}(2.2) &= \texttt{pnorm}\left(\texttt{qnorm}\left(1-\alpha/2\right)\right) \\
\texttt{pnorm}(2.2) &= 1-\alpha/2 \\
\alpha &= 2 \times [1-\texttt{pnorm}(2.2)] \approx 0.028
\end{aligned}
$$

How to use a p-value?

Alternative to Steps 4–5

Rather than choosing α, computing critical value c and reporting "Reject" or "Fail to Reject" at $100 \times \alpha\%$ level, just report p-value.

Example From Previous Slide

P-value for our test of $H_0\colon \mu = 2$ against $H_1\colon \mu \neq 2$ was ≈ 0.028

Using P-value to Test H_0

Using the p-value we can test H_0 for any α without doing any new calculations! For p-value $< \alpha$ reject; for p-value $\geq \alpha$ fail to reject.

Strength of Evidence Against H_0

P-value measures strength of evidence against the null. Smaller p-value $=$ stronger evidence against H_0. P-value does not measure size of effect.

Lecture #20 – Hypothesis Testing III

One-Sided Tests

Two-Sample Test For Difference of Means

Matched Pairs Test for Difference of Means

One-sided Test: Different Decision Rule

Same Example as Last Time

$X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, 9)$ and $H_0\colon \mu = 2$.

Three possible alternatives:

Two-sided	One-sided $(<)$	One-sided $(>)$
$H_1\colon \mu \neq 2$	$H_1\colon \mu < 2$	$H_1\colon \mu > 2$

Three corresponding decision rules:

- Two-sided: reject $\mu = 2$ whenever $|\bar{X}_n - 2|$ is too large.
- One-sided $(<)$: only reject $\mu = 2$ if $\bar{X}_n$ is far below 2.
- One-sided $(>)$: only reject $\mu = 2$ if $\bar{X}_n$ is far above 2.

One-sided ($>$) Example: $X_1, \ldots, X_{100} \sim$ iid $N(\mu, 9)$

Null and Alternative

Test $H_0\colon \mu = 2$ against $H_0\colon \mu > 2$ with $\alpha = 0.05$.

Test Statistic

Drop absolute value for one-sided test: $T_n = \dfrac{\bar{X}_n - 2}{0.3}$

Decision Rule

Reject $H_0\colon \mu = 2$ if test statistic is large and positive: $T_n > c$

Critical Value

Choose c so that $P(\text{type I error}) = P(T_n > c|\mu = 2) = 0.05$

Under H_0, $T_n \sim N(0,1)$

If $Z \sim N(0,1)$ what value of c ensures $P(Z > c) = 0.05$?

One-sided (<) Example: $X_1, \ldots, X_{100} \sim$ iid $\text{N}(\mu, 9)$

Null and Alternative

Test $H_0\colon \mu = 2$ against $H_1\colon \mu < 2$ with $\alpha = 0.05$.

Test Statistic

Drop absolute value for one-sided test: $T_n = \dfrac{\bar{X}_n - 2}{0.3}$

Decision Rule

Reject $H_0\colon \mu = 2$ if test statistic is large and negative: $T_n < c$

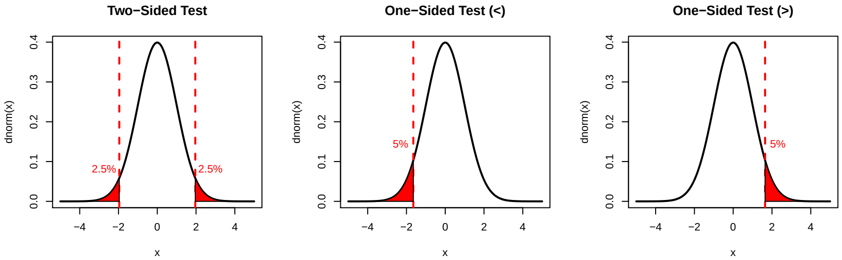
Critical Value

Choose c so that $P(\text{type I error}) = P(T_n < c|\mu = 2) = 0.05$

Under H_0, $T_n \sim N(0,1)$

If $Z \sim N(0,1)$ what value of c ensures $P(Z < c) = 0.05$?

Critical Values – Two-sided vs. One-sided Tests: $\alpha = 0.05$



Two-Sided

Splits $\alpha = 0.05$ between two tails: $c = \texttt{qnorm}(1 - 0.05/2) \approx 1.96$

One-Sided

One tail: $c = \texttt{qnorm}(0.05) \approx -1.64$ for $(<)$; $\texttt{qnorm}(0.95) \approx 1.64$ for $(>)$

Example: $X_1, \ldots, X_{100} \sim \text{iid N}(\mu, 9), \alpha = 0.05$

$$\boxed{\text{Suppose } \bar{x} = 1.5 \implies (\bar{x} - 2)/0.3 \approx -1.67}$$

Two-sided

$H_1: \mu \neq 2$

Reject if $|T_n| > 1.96$

$T_n = 1.67$

Fail to reject

One-sided ($<$)

$H_1: \mu < 2$

Reject if $T_n < -1.64$

$T_n = -1.67$

Reject

One-sided ($>$)

$H_1: \mu > 2$

Reject if $T_n > 1.64$

$T_n = -1.67$

Fail to reject

- If One-sided ($<$) rejects, then one-sided ($>$) doesn't and vice-versa.
- Two-sided and one-sided sometimes agree but sometimes disagree.
- One-sided test is "less stringent."

Testing $H_0\colon \mu = \mu_0$ when $X_1, \ldots, X_n \sim$ iid $\text{N}(\mu, \sigma^2)$

Two-Sided

Reject H_0 whenever $\left|\dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}\right| > \texttt{qnorm}(1-\alpha/2)$

One-Sided ($<$)

Reject H_0 whenever $\dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} < \texttt{qnorm}(\alpha)$

One-Sided ($>$)

Reject H_0 whenever $\dfrac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > \texttt{qnorm}(1-\alpha)$

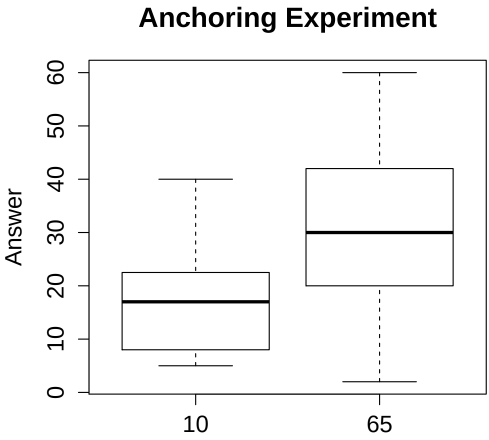
One-sided P-value

- Only makes sense to calculate one-sided p-value when sign of test stat. agrees with alternative:
  - Preceding example: $T_n = -1.67$
  - Calculate p-value for test vs. $H_1: \mu < 2$ but not $H_1: \mu > 2$
- Just as in two-sided test, p-value equals value of α for which c exactly equals the observed test statistic:
  - $c = \texttt{qnorm}(\alpha)$ for $(<)$
  - $c = \texttt{qnorm}(1-\alpha)$ for $(>)$
  - Example: $-1.67 = \texttt{qnorm}(\alpha) \iff \alpha = 0.047$
- Use and report one-sided p-value in same way as two-sided p-value

Comparing One-sided and Two-sided Tests

- Two-sided test is the default.
- Don't use one-sided unless you have a good reason!
- Relationship between CI and test only holds for two-sided.
- Why and when should we consider a one-sided test?
  - Suppose we know *a priori* that $\mu < 2$ is crazy/uninteresting
  - Test of $H_0: \mu = 2$ against $H_1: \mu > 2$ with significance level α has lower type II error rate than test against $H_1: \mu \neq 2$.
- If you use a one-sided test you must choose $(>)$ or $(<)$ *before looking at the data*. Otherwise the results are invalid.

The Anchoring Experiment



The Anchoring Experiment

Shown a "random" number and then asked what proportion of UN member states are located in Africa.

"Hi" Group – Shown 65 ($n_{Hi} = 46$)

Sample Mean: 30.7, Sample Variance: 253

"Lo" Group – Shown 10 ($n_{Lo} = 43$)

Sample Mean: 17.1, Sample Variance: 86

$\boxed{\text{Proceed via the CLT...}}$

In words, what is our null hypothesis?

(a) There is a *positive* anchoring effect: seeing a higher random number makes people report a higher answer.

(b) There is a *negative* anchoring effect: seeing a lower random number makes people report a lower answer.

(c) There *is* an anchoring effect: it could be positive or negative.

(d) There is *no* anchoring effect: people aren't influenced by seeing a random number before answering.

In symbols, what is our null hypothesis?

(a) $\mu_{Lo} < \mu_{Hi}$

(b) $\mu_{Lo} = \mu_{Hi}$

(c) $\mu_{Lo} > \mu_{Hi}$

(d) $\mu_{Lo} \neq \mu_{Hi}$

$\mu_{Lo} = \mu_{Hi}$ is *equivalent to* $\mu_{Hi} - \mu_{Lo} = 0$!

Anchoring Experiment

Under the null, what should we expect to be true about the values taken on by $\bar{X}_{Lo}$ and $\bar{X}_{Hi}$?

(a) They should be similar in value.

(b) $\bar{X}_{Lo}$ should be the smaller of the two.

(c) $\bar{X}_{Hi}$ should be the smaller of the two.

(d) They should be different. We don't know which will be larger.

What is our Test Statistic?

Sampling Distribution

$$\frac{(\bar{X}_{Hi} - \bar{X}_{Lo}) - (\mu_{Hi} - \mu_{Lo})}{\sqrt{\frac{S_{Hi}^2}{n_{Hi}} + \frac{S_{Lo}^2}{n_{Lo}}}} \approx N(0,1)$$

Test Statistic: Impose the Null

Under $H_0\colon \mu_{Lo} = \mu_{Hi}$

$$T_n = \frac{\bar{X}_{Hi} - \bar{X}_{Lo}}{\sqrt{\frac{S_{Hi}^2}{n_{Hi}} + \frac{S_{Lo}^2}{n_{Lo}}}} \approx N(0,1)$$

What is our Test Statistic?

$\bar{X}_{Hi} = 30.7$, $s^2_{Hi} = 253$, $n_{Hi} = 46$

$\bar{X}_{Lo} = 17.1$, $s^2_{Lo} = 86$, $n_{Lo} = 43$

Under $H_0\colon \mu_{Lo} = \mu_{Hi}$

$$T_n = \frac{\bar{X}_{Hi} - \bar{X}_{Lo}}{\sqrt{\frac{S^2_{Hi}}{n_{Hi}} + \frac{S^2_{Lo}}{n_{Lo}}}} \approx N(0,1)$$

Plugging in Our Data

$$T_n = \frac{\bar{X}_{Hi} - \bar{X}_{Lo}}{\sqrt{\frac{S^2_{Hi}}{n_{Hi}} + \frac{S^2_{Lo}}{n_{Lo}}}} \approx 5$$

Anchoring Experiment Example

Approximately what critical value should we use to test $H_0\colon \mu_{Lo} = \mu_{Hi}$ against the two-sided alternative at the 5% significance level?

α	0.10	0.05	0.01
`qnorm`$(1-\alpha)$	1.28	1.64	2.33
`qnorm`$(1-\alpha/2)$	1.64	1.96	2.58

... Approximately 2

Anchoring Experiment Example

Which of these commands would give us the p-value of our test of $H_0\colon \mu_{Lo} = \mu_{Hi}$ against $H_1\colon \mu_{Lo} < \mu_{Hi}$ at significance level α?

(a) `qnorm(`$1-\alpha$`)`

(b) `qnorm(`$1-\alpha/2$`)`

(c) `1 - pnorm(5)`

(d) `2 * (1 - pnorm(5))`

P-values for $H_0\colon \mu_{Lo} = \mu_{Hi}$

We plug in the value of the test statistic that we observed: 5

Against $H_1\colon \mu_{Lo} < \mu_{Hi}$

`1 - pnorm(5)` < 0.0000

Against $H_1\colon \mu_{Lo} \neq \mu_{Hi}$

`2 * (1 - pnorm(5))` < 0.0000

If the null is true (the two population means are equal) it would be extremely unlikely to observe a test statistic as large as this!

What should we conclude?

Which Exam is Harder?

Student	Exam 1	Exam 2	Difference
1	57.1	60.7	3.6
⋮	⋮	⋮	⋮
71	78.6	82.9	4.3
Sample Mean:	79.6	81.4	1.8
Sample Var.	117	151	124
Sample Corr.	0.54		

$\boxed{\text{Again, we'll use the CLT.}}$

One-Sample Hypothesis Test Using Differences

$\boxed{\text{Let } D_i = X_i - Y_i \text{ be (Midterm 2 Score - Midterm 1 Score) for student } i}$

Null Hypothesis

$H_0\colon \mu_1 = \mu_2$, i.e. both exams were of the same difficulty

Two-Sided Alternative

$H_1\colon \mu_1 \neq \mu_2$, i.e. one exam was harder than the other

One-Sided Alternative

$H_1\colon \mu_2 > \mu_1$, i.e. the second exam was easier

Decision Rules

$\boxed{\text{Let } D_i = X_i - Y_i \text{ be (Midterm 2 Score - Midterm 1 Score) for student } i}$

Test Statistic

$$\frac{\bar{D}_n}{\widehat{SE}(\bar{D}_n)} = \frac{1.8}{\sqrt{124/71}} \approx 1.36$$

Two-Sided Alternative

Reject $H_0\colon \mu_1 = \mu_2$ in favor of $H_1\colon \mu_1 \neq \mu_2$ if $|\bar{D}_n|$ is sufficiently large.

One-Sided Alternative

Reject $H_0\colon \mu_1 = \mu_2$ in favor of $H_1\colon \mu_2 > \mu_1$ if $\bar{D}_n$ is sufficiently large.

Reject against *Two-sided* Alternative with $\alpha = 0.1$?

$$\boxed{\frac{\bar{D}_n}{\widehat{SE}(\bar{D}_n)} = \frac{1.8}{\sqrt{124/71}} \approx 1.36}$$

α	0.10	0.05	0.01
`qnorm`$(1-\alpha)$	1.28	1.64	2.33
`qnorm`$(1-\alpha/2)$	1.64	1.96	2.58

(a) Reject

(b) Fail to Reject

(c) Not Sure

Reject against *One-sided* Alternative with $\alpha = 0.1$?

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(a) Reject

(b) Fail to Reject

(c) Not Sure

P-Values for the Test of $H_0: \mu_1 = \mu_2$

$$\boxed{\frac{\bar{D}_n}{\widehat{SE}(\bar{D}_n)} = \frac{1.8}{\sqrt{124/71}} \approx 1.36}$$

One-Sided $H_1: \mu_2 > \mu_1$

`1 - pnorm(1.36)` $=$ `pnorm(-1.36)` ≈ 0.09

Two-Sided $H_1: \mu_1 \neq \mu_2$

`2 * (1 - pnorm(1.36))` $=$ `2 * pnorm(-1.36)` ≈ 0.18

Lecture #21 – Testing/CI Roundup

One-sample Test for Proportion

Test for Difference of Proportions

Statistical vs. Practical Significance

Data-Dredging

Tests for Proportions

Basic Idea

The population *can't be* normal (it's Bernoulli) so we use the CLT to get approximate sampling distributions (c.f. Lecture 18).

There's a small twist!

Bernoulli has a *single* unknown parameter (p) so $SE(\widehat{p})$ is *known* under H_0: don't have to estimate it. (C.f. Review Question #194)

Tests for Proportions: One-Sample Example

From Pew Polling Data

54% of a random sample of 771 registered voters correctly identified 2012 presidential candidate Mitt Romney as Pro-Life.

Sampling Model

$X_1, \ldots, X_n \sim \text{iid Bernoulli}(p)$

Sample Statistic

Sample Proportion: $\widehat{p} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$

Suppose I wanted to test $H_0: p = 0.5$

Tests for Proportions: One Sample Example

Under $H_0\colon p = 0.5$ what is the standard error of $\widehat{p}$?

(a) 1

(b) $\sqrt{\widehat{p}(1-\widehat{p})/n}$

(c) $\sigma/\sqrt{n}$

(d) $1/(2\sqrt{n})$

(e) $p(1-p)$

$p = 0.5 \implies \sqrt{0.5(1-0.5)/n} = 1/(2\sqrt{n})$

Under the null we know the SE! Don't have to estimate it!

One-Sample Test for a Population Proportion

Sampling Model

$X_1, \ldots, X_n \sim$ iid Bernoulli(p)

Null Hypothesis

$H_0\colon\, p =$ Known Constant p_0

Test Statistic

$$T_n = \frac{\widehat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \approx N(0,1) \text{ under } H_0 \text{ provided } n \text{ is large}$$

One-Sample Example H_0: $p = 0.5$

54% of a random sample of 771 registered voters knew Mitt Romney is Pro-Life.

$$
\begin{aligned}
T_n &= \frac{\widehat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = 2\sqrt{771}(0.54 - 0.5) \\
&= 0.08 \times \sqrt{771} \approx 2.2
\end{aligned}
$$

One-Sided p-value

`1 - pnorm(2.2)` ≈ 0.014

Two-Sided p-value

`2 * (1 - pnorm(2.2))` ≈ 0.028

Tests for Proportions: Two-Sample Example

From Pew Polling Data

53% of a random sample of 238 Democrats correctly identified Mitt Romney as Pro-Life versus 61% of 239 Republicans.

Sampling Model

Republicans: $X_1, \ldots, X_n \sim$ iid Bernoulli(p) independent of

Democrats: $Y_1, \ldots, Y_m \sim$ iid Bernoulli(q)

Sample Statistics

Sample Proportions: $\widehat{p} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i, \quad \widehat{q} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} Y_i$

$\boxed{\text{Suppose I wanted to test } H_0 \colon p = q}$

A More Efficient Estimator of the SE Under H_0

Don't Forget!

Standard Error (SE) means "std. dev. of sampling distribution" so you should know how to prove that that:

$$SE(\widehat{p} - \widehat{q}) = \sqrt{\frac{p(1-p)}{n} + \frac{q(1-q)}{m}}$$

Under H_0: $p = q$

Don't know values of p and q: only that they are equal.

Pooled SE Estimator is More Efficient Under H_0

Unpooled SE

$$\widehat{SE} = \sqrt{\frac{\widehat{p}(1-\widehat{p})}{n} + \frac{\widehat{q}(1-\widehat{q})}{m}}$$

Pooled SE

$$\widehat{SE}_{Pooled} = \sqrt{\widehat{\pi}(1-\widehat{\pi})\left(\frac{1}{n}+\frac{1}{m}\right)} \quad \text{where} \quad \widehat{\pi} = \frac{n\widehat{p}+m\widehat{q}}{n+m}$$

Why Pool?

- Under H_0, $p = q$. Call their common value "π"
- More accurate to estimate *1 parameter* (π) with a *big* sample ($n+m$) vs. *2 parameters* (p, q) with smaller samples (n, m).

Two-Sample Test for Proportions

Sampling Model

$X_1, \ldots, X_n \sim$ iid Bernoulli(p) indep. of $Y_1, \ldots, Y_m \sim$ iid Bernoulli(q)

Sample Statistics

Sample Proportions: $\widehat{p} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i, \quad \widehat{q} = \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} Y_i$

Null Hypothesis

$H_0\colon p = q \quad \Leftarrow \boxed{\text{ i.e. } p - q = 0}$

Pooled Estimator of SE under H_0

$$\widehat{\pi} = \frac{n\widehat{p} + m\widehat{q}}{n+m}, \quad \widehat{SE}_{Pooled} = \sqrt{\widehat{\pi}(1-\widehat{\pi})\left(1/n + 1/m\right)}$$

Test Statistic

$T_n = \dfrac{\widehat{p} - \widehat{q}}{\widehat{SE}_{Pooled}} \approx N(0,1)$ under H_0 provided n and m are large

Two-Sample Example H_0: $p = q$

53% of 238 Democrats knew Romney is Pro-Life vs. 61% of 239 Republicans

$$\widehat{\pi} = \frac{n\widehat{p} + m\widehat{q}}{n+m} = \frac{239 \times 0.61 + 238 \times 0.53}{239 + 238} \approx 0.57$$

$$\begin{aligned} \widehat{SE}_{Pooled} &= \sqrt{\widehat{\pi}(1-\widehat{\pi})\left(1/n + 1/m\right)} = \sqrt{0.57 \times 0.43(1/239 + 1/238)} \\ &\approx 0.045 \end{aligned}$$

$$T_n = \frac{\widehat{p} - \widehat{q}}{\widehat{SE}_{Pooled}} = \frac{0.61 - 0.53}{0.045} \approx 1.78$$

One-Sided P-Value

`1 - pnorm(1.78)`≈ 0.04

Two-Sided P-Value

`2 * (1 - pnorm(1.78))`≈ 0.08

Terminology: Statistical Significance

Definition

If we reject H_0 in a test with significance level α, then we say that the results are "statistically significant at the α% level.

Example: Anchoring Experiment

In a two-sided test, we found a difference betwen the "Hi" and "Lo" groups that was statistically significant at the 5% level.

Example: Previous Slide

In a two-sided test, we found a difference between the share of Republicans and Democrats who knew that Romney is pro-life that was statistically significant at the 10% level.

Statistical Significance $\neq$ Practical Importance

Problem

People confuse "significance" in the statistical sense with the everyday English word meaning "important."

Statistically Significant Does Not Mean Important

- A difference can be unimportant but statistically significant.
- A difference can be important but statistically insignificant.

A p-value measures the *strength of evidence against* H_0; it does *not* measure the size of an effect!

Statistically Significant but Not Practically Important

I flipped a coin 10 million times (in R) and got 4990615 heads.

Test of $H_0\colon p = 0.5$ against $H_1\colon p \neq 0.5$

$$T = \frac{\widehat{p} - 0.5}{\sqrt{0.5(1-0.5)/n}} \approx -5.9 \implies \text{p-value} \approx 0.000000003$$

Approximate 95% Confidence Interval

$$\widehat{p} \pm \texttt{qnorm}(1 - 0.05/2)\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \implies (0.4988, 0.4994)$$

$\boxed{\text{Actual } p \text{ was } 0.499}$

Practically Important But Not Statistically Significant

Vickers: "What is a P-value Anyway?" (p. 62)

Just before I started writing this book, a study was published reporting about a 10% lower rate of breast cancer in women who were advised to eat less fat. If this indeed the true difference, low fat diets could reduce the incidence of breast cancer by tens of thousands of women each year – astonishing health benefit for something as simple and inexpensive as cutting down on fatty foods. The p-value for the difference in cancer rates was 0.07 and here is the key point: this was widely misinterpreted as indicating that low fat diets don't work. For example, the New York Times editorial page trumpeted that "low fat diets flub a test" and claimed that the study provided "strong evidence that the war against all fats was mostly in vain." However failure to prove that a treatment is effective is not the same as proving it ineffective.

Data-Dredging and the Replication Crisis

Reading Assignment

On Piazza: "The Economist - Trouble in the Lab."

Basic Idea

- Journals usually publish only "statistically significant" results.
- You test a large number of null hypotheses with $\alpha = 0.05$.
- Suppose all of these nulls are actually *TRUE*.
- You'll reject 5% of the time: each rejection is a Type I error.
- Cheating in academia: carry out lots of ridiculous hypothesis tests and only report the "statistically significant" results.

Green Jelly Beans Cause Acne!

xkcd #882

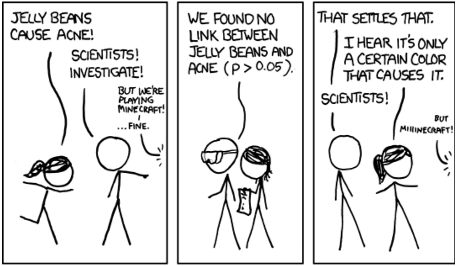


Figure: Reading this comic strip is part of your homework!

And now a simulation example of Data Dredging using R. . .

```r
# Function to calculate the p-value for a two-sided
#test for difference of means
get_p_value <- function(x, y) {
  xbar <- mean(x)
  ybar <- mean(y)
  n <- length(x)
  m <- length(y)
  s_x <- sd(x)
  s_y <- sd(y)
  SE <- sqrt(s_x^2 / n + s_y^2 / m)
  test_stat <- abs(xbar - ybar) / SE
  return(2 * (1 - pnorm(test_stat)))
}
```

```r
# Test get_p_value using the anchoring  experiment
# example from our previous lecture
data_url <- 'http://ditraglia.com/econ103/old_survey.csv'
survey <- read.csv(data_url)
anchoring <- survey[, c('rand.num', 'africa.percent')]
rand_num <- na.omit(anchoring$rand.num)
africa_percent <- na.omit(anchoring$africa.percent)

x <- subset(africa_percent, rand_num == 65)
y <- subset(africa_percent, rand_num == 10)
get_p_value(x, y)

## [1] 6.682931e-07
```

```r
# Use *real* student test scores as the outcome
data_url <- 'http://ditraglia.com/econ103/midterms.csv'
midterms <- read.csv(data_url)
scores <- na.omit(midterms$Midterm1)
n_students <- length(scores)

# Generate fake "grouping variables" (0/1) indep. of scores
set.seed(987654321)

n_fake <- 500
# Empty matrix to store grouping variables:
fake <- matrix(NA, nrow = n_students, ncol = n_fake)

for(i in 1:n_fake) {
  fake[, i] <- rbinom(n_students, size = 1, prob = 0.5)
}
```

```r
# Use each grouping variable to split students into x and y
# and calculate p-value for test of difference of means

p_values <- rep(NA, n_fake) # empty vector to store results

for(i in 1:n_fake) {
  group_indicator <- fake[,i]
  x <- subset(scores, group_indicator == 1)
  y <- subset(scores, group_indicator == 0)
  p_values[i] <- get_p_value(x, y)
}
# How many of the tests were statistically significant?
sum(p_values < 0.05)

## [1] 20
```

```r
# Grouping variable with the lowest p-value
group_indicator <- fake[, which.min(p_values)]
x <- subset(scores, group_indicator == 1)
y <- subset(scores, group_indicator == 0)

# These results look convincing, but are spurious!
mean(x) - mean(y)

## [1] -7.974127

sqrt(var(x) / length(x) + var(y) / length(y))

## [1] 2.240852
```

Lecture #22 – Regression II

The Population Regression Model

Inference for Regression

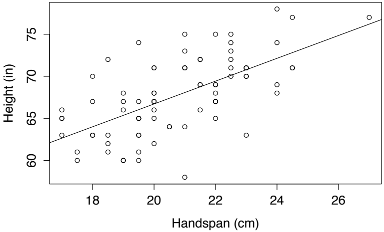
Inference for Regression: Height and Handspan

Residual Standard Deviation and R^2

Multiple Regression

Beyond Regression as a Data Summary

Based on a sample of Econ 103 students, we made the following graph of handspan against height, and fitted a linear regression:



The estimated slope was about 1.4 inches/cm and the estimated intercept was about 40 inches.

What if anything does this tell us about the relationship between height and handspan *in the population*?

The Population Regression Model

Question

If we want to predict Y using X in the *population* using a line, how should we choose the slope and intercept?

Optimization Problem

Choose β_0, β_1 to minimize $E[(Y - \beta_0 - \beta_1 X)^2]$

Solution

$$\beta_1 = \frac{Cov(X,Y)}{Var(X)}, \quad \beta_0 = E[Y] - \beta_1 E[X]$$

...you will derive this as an extension problem.

The Regression Error Term: ε

Definition

$\varepsilon \equiv Y - \beta_0 - \beta_1 X$ (Hence: $Y = \beta_0 + \beta_1 X + \varepsilon$)

Interpretation

ε is the part of Y that isn't predicted by X

Properties

- $E[\varepsilon] = 0$
- $Cov(X, \varepsilon) = 0$
- $Var(\varepsilon) = Var(Y) - Cov(X, Y)^2/Var(X)$

...using the expressions for β_0 and β_1 from the previous slide.

The Population Regression Coefficients: β_0, β_1

Recall

$$Y = \beta_0 + \beta_1 X + \varepsilon, \quad \beta_1 = \frac{Cov(X,Y)}{Var(X)}, \quad \beta_0 = E[Y] - \beta_1 E[X]$$

Interpretation

- β_0, β_1 are population parameters: *unknown constants*
- If $X = 0$, we predict $Y = \beta_0$.
- If two people differ by one unit in X, we predict that they will differ by β_1 units in Y.

The only problem is, we don't know β_0, β_1...

Estimating β_0, β_1

Random Sample

Observe $(Y_1, X_1), \ldots, (Y_n, X_n) \sim$ iid with $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$.

Estimators of β_0, β_1

$$\widehat{\beta}_1 = \frac{S_{XY}}{S_X^2} = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2}, \quad \widehat{\beta}_0 = \bar{Y}_n - \widehat{\beta}_1 \bar{X}_n$$

Under random sampling, the estimators $(\widehat{\beta}_0, \widehat{\beta}_1)$ have sampling distributions...

Sampling Uncertainty: Pretend the Class is our Population

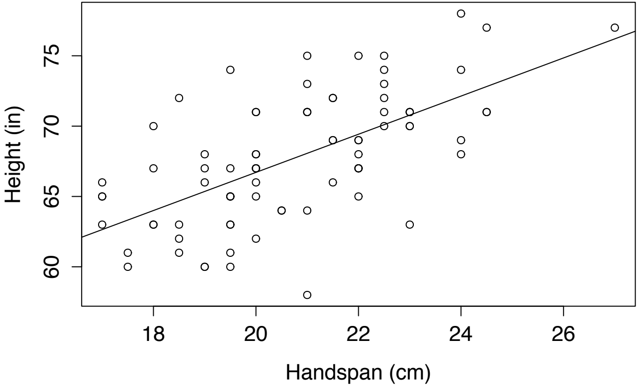
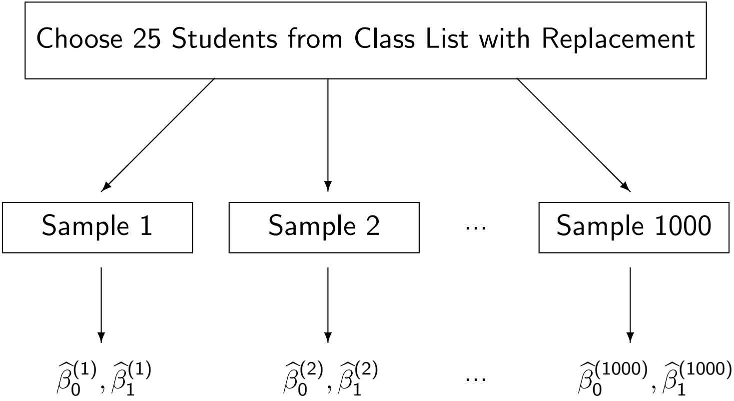


Figure: Estimated Slope = 1.4, Estimated Intercept = 40

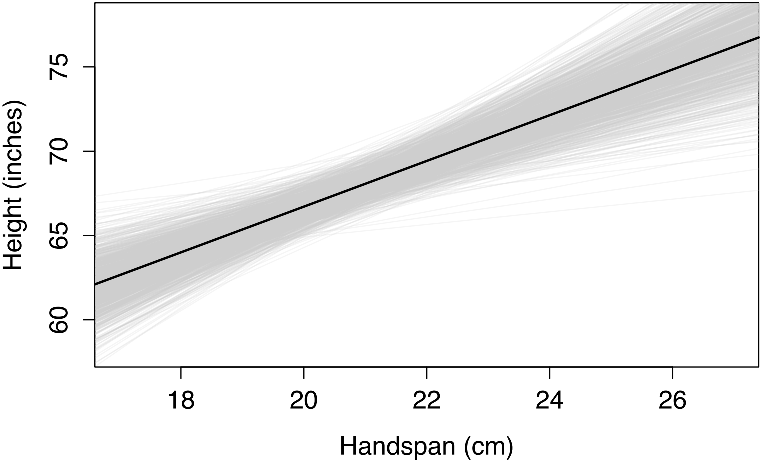
Sampling Distribution of Regression Coefficients $\widehat{\beta}_0$ and $\widehat{\beta}_1$



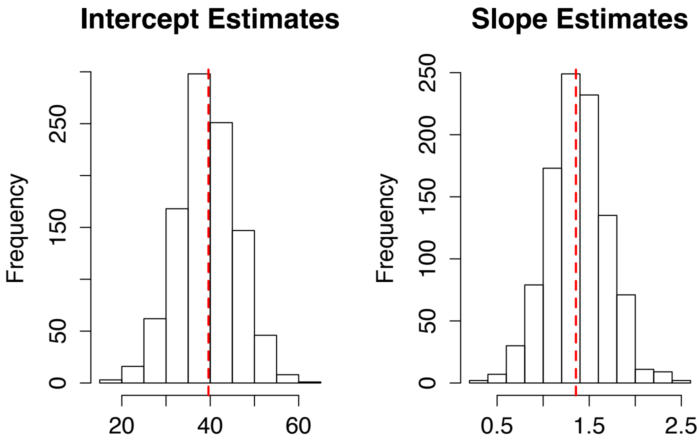
Repeat 1000 times $\rightarrow$ get 1000 different pairs of estimates

Sampling Distribution: long-run relative frequencies

1000 Replications, $n = 25$



Population: Intercept = 40, Slope = 1.4



Based on 1000 Replications, $n = 25$

Inference for Linear Regression

Central Limit Theorem

$$\frac{\widehat{\beta} - \beta}{\widehat{SE}(\widehat{\beta})} \approx N(0,1)$$

How to calculate $\widehat{SE}$?

R will do this for us, but we won't cover the details in Econ 103.

You'll have to wait for Econ 104!

Height $= \beta_0 + \epsilon$

```
lm(formula = height ~ 1, data = student.data)
            coef.est coef.se
(Intercept) 67.74    0.51
---
n = 80, k = 1

> mean(student.data$height)
[1] 67.7375

> sd(student.data$height)/sqrt(length(student.data$height))
[1] 0.5080814
```

Dummy Variable (aka Binary Variable)

A predictor variable that takes on only two values: 0 or 1. Used to represent two categories, e.g. Male/Female.

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46    0.56
sexMale      6.10    0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45

> mean(male$height) - mean(female$height)
[1] 6.09868

> sqrt(var(male$height)/length(male$height) +
  var(female$height)/length(female$height))
[1] 0.7463796
```

Height $= \beta_0 + \beta_1$ Male $+\epsilon$

What is the ME for an approximate 95% confidence interval for the difference of population means of height: (men - women)?

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46     0.56
sexMale      6.10     0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45
```

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

What is the ME for an approximate 95% CI for β_1?

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

What are `residual sd` and `R-squared`?

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Fitted Values and Residuals

Fitted Value $\widehat{y}_i$

Predicted y-value for person i given her x-variables using estimated regression coefficients: $\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots + \widehat{\beta}_k x_{ik}$

Residual $\widehat{\epsilon}_i$

Person i's *vertical deviation* from regression line: $\widehat{\epsilon}_i = y_i - \widehat{y}_i$.

The residuals are *stand-ins* for the unobserved errors ϵ_i.

Residual Standard Deviation: $\widehat{\sigma}$

- Idea: use residuals $\widehat{\epsilon}_i$ to estimate σ

$$\widehat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} \widehat{\epsilon}_i^2}{n-k}}$$

- Measures avg. distance of y_i from regression line.
  - E.g. if Y is points scored on a test and $\widehat{\sigma} = 16$, the regression predicts to an accuracy of about 16 points.
- Same units as Y (Exam practice: verify this)
- Denominator $(n-k) =$ (# Datapoints - # of X variables)

Proportion of Variance Explained: R^2

aka Coefficient of Determination

$$R^2 \approx 1 - \frac{\widehat{\sigma^2}}{s_y^2}$$

- R^2 = proportion of $Var(Y)$ "explained" by the regression.
  - Higher value $\implies$ greater proportion explained
- Unitless, between 0 and 1
- Generally harder to interpret than $\widehat{\sigma}$, but...
- For simple linear regression $R^2 = (r_{xy})^2$ and this where its name comes from!

Height $= \beta_0 + \beta_1$ Handspan $+\epsilon$

```
lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60     3.96
handspan     1.36     0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
> cor(student.data$height, student.data$handspan)^2
[1] 0.3954669
```

Which Gives Better Predictions: Sex (a) or Handspan (b)?

```
lm(formula = height ~ sex, data = student.data)
            coef.est coef.se
(Intercept) 64.46    0.56
sexMale      6.10    0.76
---
n = 80, k = 2
residual sd = 3.38, R-Squared = 0.45

lm(formula = height ~ handspan, data = student.data)
            coef.est coef.se
(Intercept) 39.60    3.96
handspan     1.36    0.19
---
n = 80, k = 2
residual sd = 3.56, R-Squared = 0.40
```

Simple vs. Multiple Regression

Terminology

Y is the "outcome" and X is the "predictor."

Simple Regression

One predictor variable: $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$

Multiple Regression

More than one predictor variable:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$$

Multiple Regression

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_k X_{ik} + \epsilon_i$$

Ceteris Paribus Interpretation

If two individuals differ by one unit in X_j but have the *same values for all other predictors*, we predict they will differ by β_j units in Y.

Estimating $\beta_0, \beta_1, \ldots, \beta_k$

The formulas require matrix algebra: R will do it for us.

Inference for Multiple Regression

$\dfrac{\widehat{\beta}_j - \beta_j}{\widehat{SE}(\widehat{\beta}_j)} \approx N(0,1)$ if n is large. R will calculate the SE for us.

Bring Your Laptop Next Time: We'll be Using R