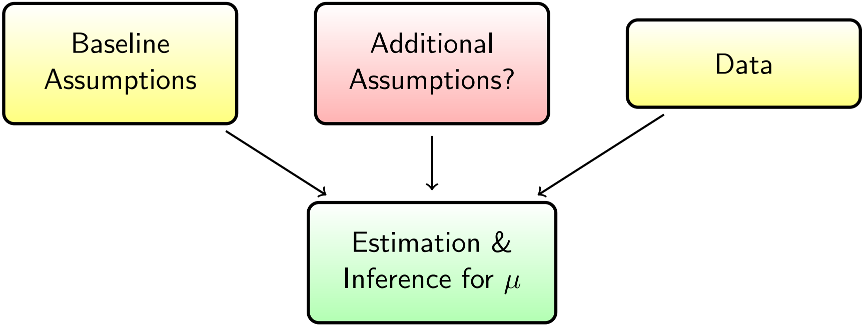
A Generalized Focused Information Criterion for GMM Model and Moment Selection

Francis J. DiTraglia

University of Pennsylvania

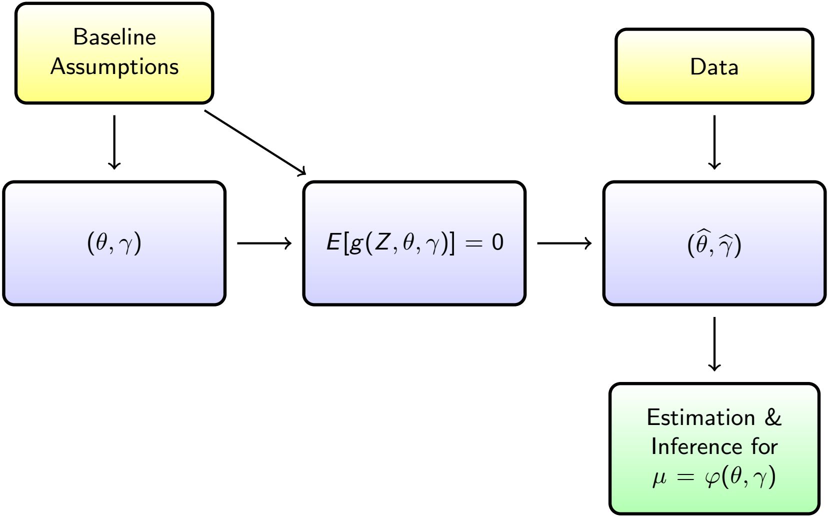
October 26, 2013

Generalized Focused Information Criterion (GFIC)

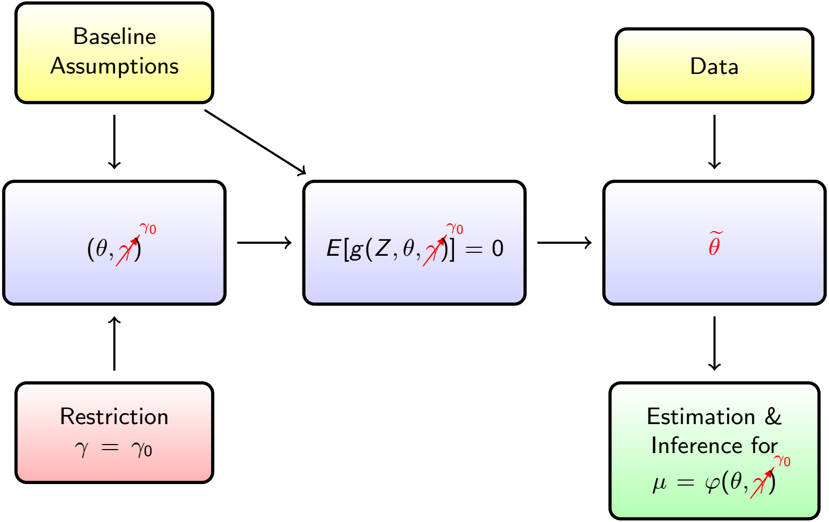


1. Choose False Assumptions on Purpose
2. Focused Choice of Assumptions
3. Local mis-specification
4. Averaging, Inference post-selection

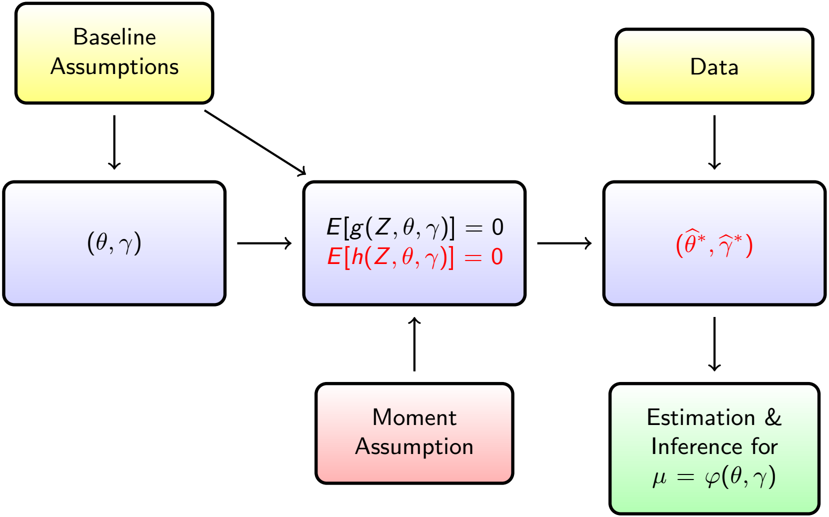
GMM Framework



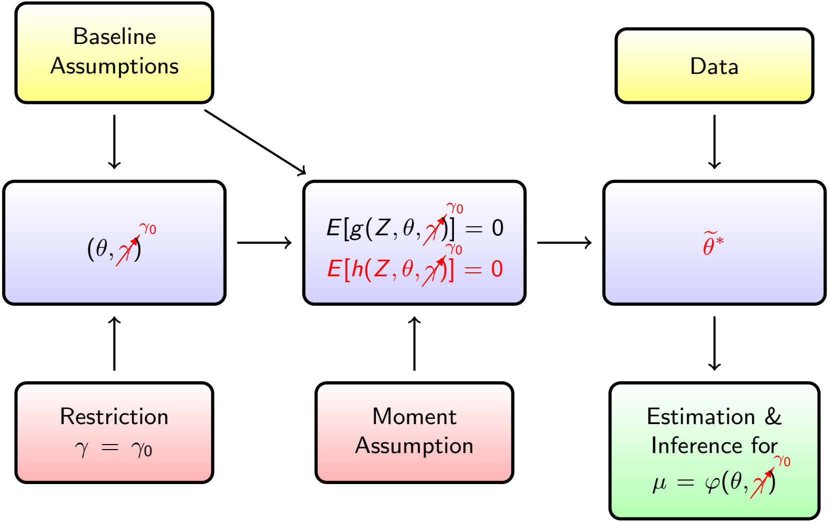
Additional Assumption – Parameter Restriction



Additional Assumption – Moment Conditions

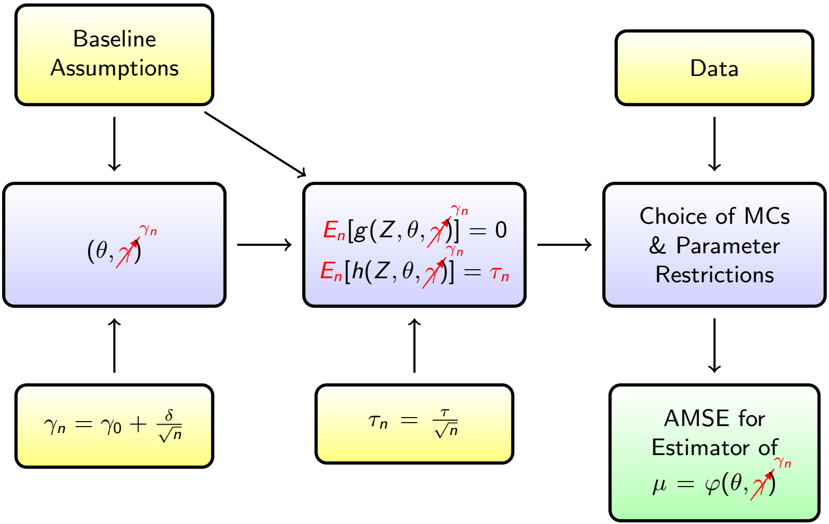


Both at Once



Baseline Assumptions
Data
$(\theta, \cancelto{\gamma_0}{\gamma})$
$E[g(Z, \theta, \cancelto{\gamma_0}{\gamma})] = 0$
$E[h(Z, \theta, \cancelto{\gamma_0}{\gamma})] = 0$
$\widetilde{\theta}^*$
Restriction
$\gamma = \gamma_0$
Moment Assumption
Estimation & Inference for
$\mu = \varphi(\theta, \cancelto{\gamma_0}{\gamma})$

GFIC Asymptotics – Local Mis-Specification



Overview of GFIC Derivation

Asymptotic Normality of GMM Estimator

- Biased unless γ estimated, no MCs from h used
- Smaller variance if γ set to γ_0, MCs from h used

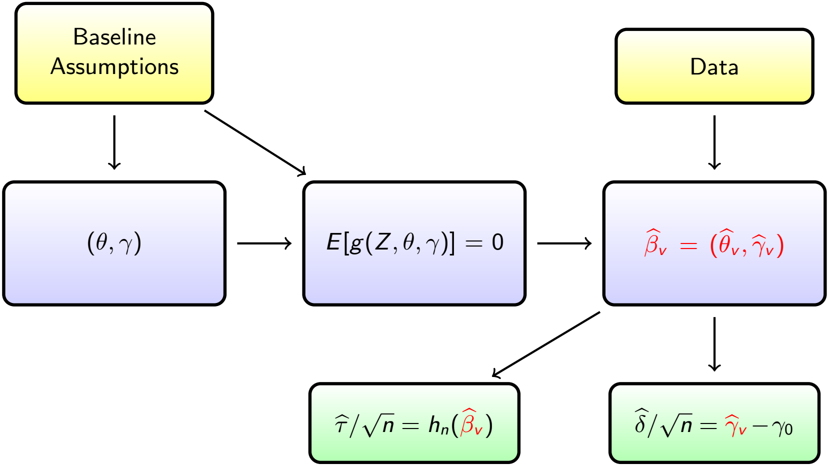
Asymptotic Normality of Target Parameter $\widehat{\mu}(b, c)$

- Inherits bias-variance tradeoff from $\widehat{\beta}(b, c)$
- $\text{AMSE}\left(\widehat{\mu}(b, c)\right)$ depends on $B = \begin{bmatrix} \tau\tau' & \tau\delta' \\ \delta\tau' & \delta\delta' \end{bmatrix}$

GFIC = Asymptotically Unbiased Estimator of $\widehat{\text{AMSE}}\left(\widehat{\mu}(b, c)\right)$

- Asymptotically unbiased estimator $\widehat{B}$ of B
- Select (b, c) to minimize GFIC

Estimating δ, τ



Estimating δ, τ – Details

Limit Distribution of Bias Parameter Estimators

$$\begin{bmatrix} \widehat{\delta} \\ \widehat{\tau} \end{bmatrix} = \sqrt{n} \begin{bmatrix} (\widehat{\gamma}_v - \gamma_0) \\ h_n(\widehat{\beta}_v) \end{bmatrix} \to_d \begin{bmatrix} \delta \\ \tau \end{bmatrix} + \Psi \; N(0, \Omega)$$

- Both Ψ and Ω can be estimated consistently!

Asymptotically Unbiased Estimator of B

$$\begin{aligned} B &= \begin{bmatrix} \tau\tau' & \tau\delta' \\ \delta\tau' & \delta\delta' \end{bmatrix} \\ \widehat{B} &= \begin{bmatrix} \widehat{\tau}\widehat{\tau}' & \widehat{\tau}\widehat{\delta}' \\ \widehat{\delta}\widehat{\tau}' & \widehat{\delta}\widehat{\delta}' \end{bmatrix} - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}' \end{aligned}$$

Generalized Focused Information Criterion

Purpose

Simultaneous Model and Moment Selection for GMM Estimation

Key Features

- Local mis-specification framework
- Estimator of AMSE of user-specified target parameter
- Focused Selection
- Select "wrong" specification on purpose

Didn't Discuss Today

- Dynamic Panel Example
- Works well in simulations
- Provides framework for model and moment averaging
- Valid post-selection confidence intervals

Supplementary Material

Simple Dynamic Panel Example – Large N, Small T

True Data Generating Process

$$y_{it} = \gamma y_{it-1} + \theta x_{it} + \eta_i + v_{it}$$

- Dynamics unless $\gamma = 0$
- Correlated effects $\eta_i \Rightarrow$ first differences
- x_{it} predetermined but *not* strictly exogenous

Goal – Estimate θ with minimum MSE

- Model Selection Decision: set $\gamma = 0$?
- Moment Selection Decision: treat x_{it} as strictly exogenous?

Anderson & Hsiao–esque 2SLS Estimators (1982)

LW Moment Conditions:

$$\mathbb{E}\left[\begin{pmatrix} y_{i,t-2} \\ x_{i,t-1} \end{pmatrix}(\Delta y_{it} - \gamma \Delta y_{i,t-1} - \theta \Delta x_{it})\right] = 0, \text{ for } t = 3, \ldots, T$$

LS Adds the Moment Conditions:

$$\mathbb{E}\left[x_{it}\left(\Delta y_{it} - \gamma \Delta y_{i,t-1} - \theta \Delta x_{it}\right)\right] = 0, \text{ for } t = 3, \ldots, T$$

Only the LW conditions are correct

Anderson & Hsiao–esque 2SLS Estimators (1982)

W Moment Conditions:

$$\mathbb{E}\left[x_{i,t-1}\left(\Delta y_{it} - \theta \Delta x_{it}\right)\right] = 0, \text{ for } t = 2, \ldots, T$$

S Adds the Moment Conditions:

$$\mathbb{E}\left[x_{it}\left(\Delta y_{it} - \theta \Delta x_{it}\right)\right] = 0, \text{ for } t = 2, \ldots, T$$

None of these moment conditions are correct

Why Use an Incorrect Specification?

$$\Delta y_{it} = \gamma \Delta y_{it-1} + \theta \Delta x_{it} + \Delta v_{it}$$

Wrong Model

- γ small $\implies$ ignore dynamics
- Adds small bias
- Much lower variance: extra time period, fewer parameters

Invalid MCs

- $E[x_{it} v_{it-1}]$ small $\implies$ add x_{it} as instrument for period t
- Adds small bias
- Much lower variance: x_{it} is a strong instrument for Δx_{it}

Simulation Setup

Similar to Andrews & Lu (2001)

- $y_{i0} = 0$
- $y_{it} = \gamma y_{it-1} + 0.5x_{it} + \eta_i + v_{it} \qquad (t = 1, \ldots, T)$

$$\begin{bmatrix} x_i \\ \eta_i \\ v_i \end{bmatrix} \sim \text{iid } N\left(\begin{bmatrix} 0_T \\ 0 \\ 0_T \end{bmatrix}, \begin{bmatrix} I_T & 0.2\iota_T & \sigma_{xv}\Gamma \\ 0.2\iota_T' & 1 & 0_T' \\ \sigma_{xv}\Gamma' & 0_T & I_T \end{bmatrix} \right)$$

- $E[x_{it}v_{it-1}] = \sigma_{xv}$ but $E[x_{it}v_{is}] = 0, s \neq t-1$

Vary γ and σ_{xv} over a grid

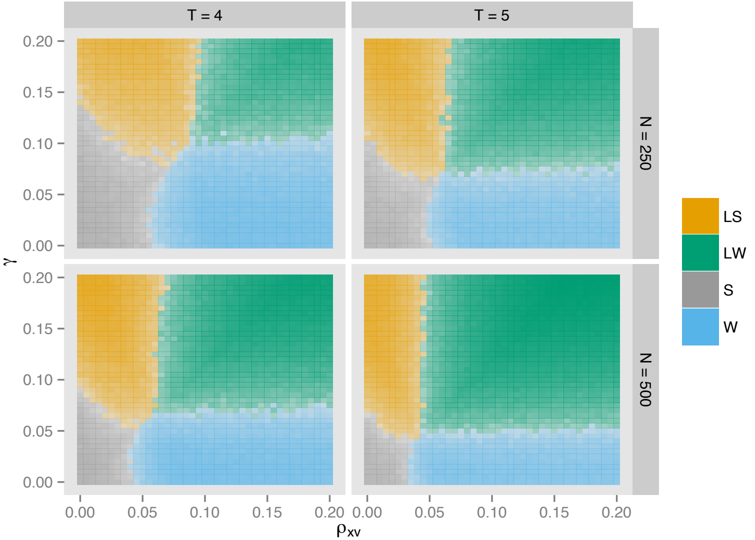


Figure: Minimum RMSE Specification at each combination of parameter values. Shading gives RMSE relative to second best specification.

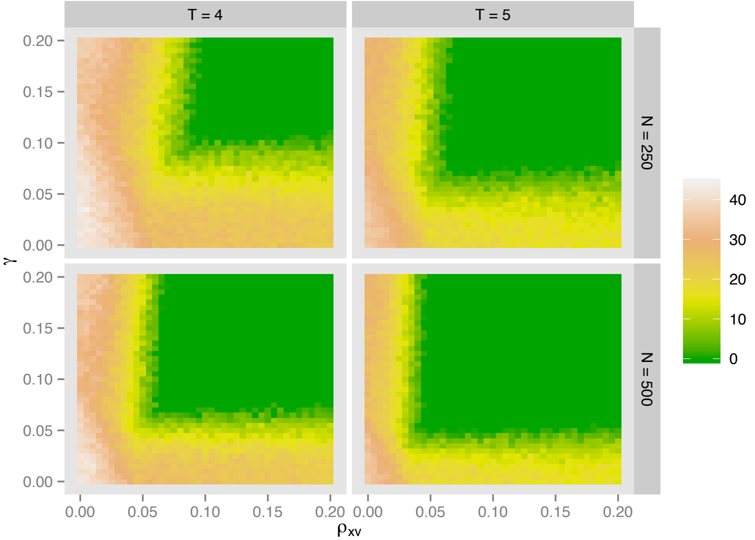


Figure: % RMSE Advantage of Best Specification (vs. LW)

Competing Procedure: Downward J-test

1. Use S unless J-test rejects.
2. If S rejected, use W unless J-test rejects.
3. If W rejected, use LS unless J-test rejects.
4. Only use LW if all others rejected.

Competing Procedures: Andrews & Lu (2001)

J-test Statistic Minus Penalty Term

$$\begin{array}{rl} \text{BIC-Type} & J - (|c| - |b|)\log n \\ \text{AIC-Type} & J - 2(|c| - |b|) \\ \text{HQ-Type} & J - 2.01(|c| - |b|)\log\log n \end{array}$$

- $|b| = \#$ (parameters estimated)
- $|c| = \#$(MCs used)
- Select specification with *lowest* value of criterion

	$N = 250$		$N = 500$	
	$T = 4$	$T = 5$	$T = 4$	$T = 5$
LW	19	10	13	7
LS	30	44	54	79
W	24	34	46	64
S	31	50	64	94
GFIC	17	13	15	10
J-test 10%	32	45	55	74
J-test 5%	31	47	57	79
GMM-BIC	32	48	62	87
GMM-HQ	32	46	57	77
GMM-AIC	31	39	47	57

Table: Average RMSE minus Pointwise Optimal (% points)

	$N = 250$		$N = 500$	
	$T = 4$	$T = 5$	$T = 4$	$T = 5$
LW	0	0	0	0
LS	42	81	94	154
W	49	88	105	158
S	48	92	107	171
GFIC	3	8	6	11
J-test 10%	43	78	91	140
J-test 5%	45	83	98	153
GMM-BIC	48	89	106	168
GMM-HQ	46	85	102	154
GMM-AIC	39	68	81	118

Table: Worst-case RMSE minus Minimax Optimal (% points)

Valid Post-Selection Inference

Post Selection Estimator

Randomly Weighted Average of candidate estimators (0-1 weights).

Standard CIs are Invalid

Nonstandard limit distribution since weights are *data dependent*

What about consistent selection?

No *pointwise* effect on the limiting distribution, but the same is *not* true uniformly (Pötscher, 1991).

Post-Selection Inference via Model Average Estimators

Consider an estimator of the form

$$\widehat{\mu} = \sum_{(b,c)\in\mathcal{BC}} \widehat{\omega}(b,c)\widehat{\mu}(b,c)$$

where $\widehat{\omega}(b,c)$ is a set of data-dependent weights.

Some Notation

$$F = \begin{bmatrix} \nabla_{\gamma'} g\left(Z, \gamma_0, \theta_0\right) & \nabla_{\theta'} g\left(Z, \gamma_0, \theta_0\right) \\ \nabla_{\gamma'} h\left(Z, \gamma_0, \theta_0\right) & \nabla_{\theta'} h\left(Z, \gamma_0, \theta_0\right) \end{bmatrix}$$

$$F = \begin{bmatrix} F_\gamma & F_\theta \end{bmatrix} = \begin{bmatrix} G_\gamma & G_\theta \\ H_\gamma & H_\theta \end{bmatrix} = \begin{bmatrix} G \\ H \end{bmatrix}$$

$$\Omega = Var \begin{bmatrix} g(Z, \gamma_0, \theta_0) \\ h(Z, \gamma_0, \theta_0) \end{bmatrix} = \begin{bmatrix} \Omega_{gg} & \Omega_{gh} \\ \Omega_{hg} & \Omega_{hh} \end{bmatrix}$$

These expressions are evaluated *in the limit* where all MCs have expectation zero at (γ_0, θ_0).

Limit Distribution of GMM Estimators

$\sqrt{n}\left(\widehat{\beta}(b,c) - \beta_0^{(b)}\right)$ converges in distribution to

$$\boxed{-K(b,c)\Xi_c\left(\mathcal{N} + \left[\begin{array}{c} 0 \\ \tau \end{array}\right] - F_\gamma \delta\right)}$$

$$\begin{aligned} K(b,c) &= \left[F(b,c)' W_c F(b,c)\right]^{-1} F(b,c)' W_c \\ \Xi_c &= \text{Moment Selection Matrix} \\ \mathcal{N} &\sim \mathrm{N}(0,\Omega) \end{aligned}$$

Limit Distribution of Target Parameter Estimators

$\sqrt{n}\left(\widehat{\mu}(b,c) - \mu_n\right)$ converges in distribution to

$$\boxed{-\nabla_\beta \varphi_0' \Xi_b' K(b,c) \Xi_c \left(\mathcal{N} + \left[\begin{array}{c} 0 \\ \tau \end{array} \right] - F_\gamma \delta \right) - \nabla_\gamma \varphi_0' \delta}$$

$$\begin{array}{rcl}
\mu & = & \varphi(\theta, \gamma) \\
\varphi_0 & = & \varphi(\gamma_0, \theta_0) \\
\mu_n & = & \varphi(\theta_0, \gamma_0 + \delta/\sqrt{n}) \\
\Xi_b & = & \text{Model Selection Matrix} \\
\Xi_c & = & \text{Moment Selection Matrix} \\
\mathcal{N} & \sim & N(0, \Omega)
\end{array}$$

Limit Distribution of $(\widehat{\delta}, \widehat{\tau})$

$$\boxed{\begin{bmatrix} \widehat{\delta} \\ \widehat{\tau} \end{bmatrix} = \sqrt{n} \begin{bmatrix} (\widehat{\gamma}_v - \gamma_0) \\ h_n(\widehat{\beta}_v) \end{bmatrix} \to_d \begin{bmatrix} \delta \\ \tau \end{bmatrix} + \Psi \mathscr{N}}$$

$$\begin{aligned} \Psi &= \begin{bmatrix} -K_v^{\gamma} & \mathbf{0} \\ -HK_v & I \end{bmatrix} \\ \mathscr{N} &\sim N(0, \Omega) \end{aligned}$$

Key Point: Joint Convergence

$$\boxed{\sqrt{n}\left(\widehat{\mu}(b,c) - \mu_n\right) \text{ converge } \textit{jointly } \forall (b,c) \in \mathcal{BC} \textit{ along with } (\widehat{\delta}, \widehat{\tau})}$$

- Only source of randomness in the limit is $\mathscr{N}$
- Everything except δ and τ is consistently estimable.
- Just need to impose some conditions on the weights...

Requirements for the Weights

Weights Sum to 1

$\sum_{(b,c)\in\mathcal{BC}} \widehat{\omega}(b,c) = 1$

Joint Convergence

$\widehat{\omega}(b,c) \to_d \psi(\mathscr{N}, \delta, \tau | b, c)$ jointly for all $(b,c) \in \mathcal{BC}$

Limit Function ψ

Depends *only* on $\mathscr{N}, \delta, \tau$, and consistently estimable quantities.

Assumptions cover GFIC, J-test, Andrews & Lu (2001), etc.

Limit Distribution of Averaging Estimator

Weights Sum to 1

$$\sqrt{n}(\widehat{\mu} - \mu_n) = \sum_{(b,c)\in\mathcal{BC}} \widehat{\omega}(b,c)\sqrt{n}\left(\widehat{\mu}\left(b,c\right) - \mu_n\right)$$

Joint Convergence in Distribution

$$\sqrt{n}\left(\widehat{\mu} - \mu_n\right) \to_d \Lambda(\tau,\delta)$$

$$\boxed{\Lambda(\tau,\delta) = -\nabla_\beta\varphi_0' \sum_{(b,c)\in\mathcal{BC}} \psi(\mathscr{N},\delta,\tau|b,c)\left\{\Xi_b' K(b,c)\Xi_c \mathscr{N} + M(b,c)\begin{bmatrix}\delta\\ \tau\end{bmatrix}\right\}}$$

Non-normal limit distribution that depends on (δ,τ)

“Bootstrapping the Limit Experiment”

Suppose δ and τ were known:

(i) For each $j = 1, 2, \ldots, J$, generate $\mathcal{N}_j \sim N(0, \widehat{\Omega})$

(ii) For each for $j = 1, 2, \ldots, J$ set

$$\Lambda_j(\tau, \delta) = -\nabla_\beta \widehat{\varphi}_0' \sum_{(b,c)\in\mathcal{BC}} \widehat{\psi}(\mathcal{N}_j, \delta, \tau | b, c) \left\{ \Xi_b' \widehat{K}(b,c) \Xi_c \mathcal{N}_j + \widehat{M}(b,c) \left[\begin{array}{c} \delta \\ \tau \end{array} \right] \right\}$$

(iii) Using $\{\Lambda_j(\delta, \tau)\}_{j=1}^J$, calculate $\widehat{a}(\delta, \tau)$, $\widehat{b}(\delta, \tau)$ such that

$$P\left\{ \widehat{a}(\delta, \tau) \leq \Lambda(\delta, \tau) \leq \widehat{b}(\delta, \tau) \right\} = 1 - \alpha$$

Accounting for Estimated (δ, τ)

Let $R(\alpha_1)$ be a $(1-\alpha_1)\times 100\%$ confidence region for (δ, τ).

1. For each $(\delta, \tau) \in R(\alpha_1)$ construct a confidence interval

$$\mathbb{P}\left\{\widehat{a}(\delta,\tau) \leq \Lambda(\delta,\tau) \leq \widehat{b}(\delta,\tau)\right\} = 1-\alpha_2$$

   using the simulation procedure from the previous slide.

2. Define

$$\begin{aligned} \widehat{a}_{min}(\widehat{\delta},\widehat{\tau}) &= \min_{(\delta,\tau)\in R(\alpha_1)} \widehat{a}(\delta,\tau) \\ \widehat{b}_{max}(\widehat{\delta},\widehat{\tau}) &= \max_{(\delta,\tau)\in R(\alpha_1)} \widehat{b}(\delta,\tau) \end{aligned}$$

3. The following CI has asymptotic coverage of *at least* $1-(\alpha_1+\alpha_2)$

$$\mathrm{CI}_{sim} = \left[\widehat{\mu} - \frac{\widehat{b}_{max}(\widehat{\delta},\widehat{\tau})}{\sqrt{n}}, \quad \widehat{\mu} - \frac{\widehat{a}_{min}(\widehat{\delta},\widehat{\tau})}{\sqrt{n}}\right]$$