Illusions of criticality in high-dimensional autoregressive models

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Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

Contents

1 The problem

`sec:theprob`

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1}$$

`eq:model`

where $x(t)$ is an N-element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising the quantity

$$\frac{1}{2}\sum_{i,\mu}\left(x_{i\mu+1} - \sum_{j} A_{ij}x_{j\mu}\right)^2 = \frac{1}{2}\operatorname{Tr}\left(xU - Ax\right)\left(xU - Ax\right)^{\mathrm{T}}, \tag{2}$$

eq:minL

where U is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make U orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1}\delta_{\nu P}. \tag{3}$$

eq:Udef

The estimate of A is then

$$A = \left(xUx^{\mathrm{T}}\right)\left(xx^{\mathrm{T}}\right)^{-1}. \tag{4}$$

eq:Aest

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $x(t)$ is white noise. Then the true optimal A would be 0. However, with finite P the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega) = \left\langle \rho_A(\omega)\right\rangle_x, \qquad \rho_A = \sum_{i=1}^{N}\delta(\omega - \lambda_i), \tag{5}$$

eq:eigdist

where λ_i are the eigenvalues of A in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1. This quantity is most relevant in the limit of large N and P. We will keep the quantity $\alpha = P/N$ fixed in this limit.

Following [1], this can be computed from a potential:

$$\rho_A(\omega) = -\nabla^2\Phi_A(\omega), \qquad \Phi_A(\omega) = -\frac{1}{4\pi N}\ln\det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]. \tag{6}$$

eq:potential

We define a partition function

$$\Phi_A(\omega) = \frac{1}{4\pi N}\ln Z_A(\omega), \qquad Z_A(\omega) = \det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]^{-1}. \tag{7}$$

eq:partfn

The problem is now to compute $\left\langle \ln Z_A(\omega)\right\rangle_x$.

2 The solution

ec:solution

In §3 we will present a simplified derivation and in §4 we will fill in the gaps and justify the assumptions used in §4. The result will be:

$$rs = \frac{\alpha^2\left[-(\alpha-1)\left(1+|\omega|^2\right) \pm \sqrt{(\alpha-1)^2\left(1+|\omega|^2\right)^2 + (2\alpha-1)\left(1-|\omega|^2\right)^2}\right]}{(\alpha-1)\left(1-|\omega|^2\right)^2}, \tag{8}$$

eq:rssol

$$\zeta_- = \frac{1+|\omega|^2}{2\overline{\omega}} - \frac{\alpha^2}{2(\alpha-1)\overline{\omega}rs}, \tag{9}$$

eq:zmsol

$$\Phi(\omega) = \frac{1}{4\pi}\max_{\pm}\left[(1-\alpha)\ln(rs) - \alpha\ln\left(\left|\frac{\omega}{\zeta_-}\right|\right) - \mathrm{i}\alpha\arg(-\omega)\right], \tag{10}$$

eq:phisol

subject to the restrictions

$$rs \le \frac{\alpha^2(2\alpha^3-2\alpha+1)}{2(\alpha+1)(\alpha-1)^3\left(1+|\omega|^2\right)}. \tag{11}$$

eq:rsreq

$$|\zeta_-| \le 1, \tag{12}$$

eq:zmreq

The imaginary part of Φ is harmonic, and will therefore not contribute to the density (6). This means that everything has a rotation symmetry.

Can we have a crossover between the two choices in (8) as ω varies? This would require a point where the two solutions are equal:

$$\begin{aligned}(\alpha-1)^2\left(1+|\omega|^2\right)^2 + (2\alpha-1)\left(1-|\omega|^2\right)^2 &= 0,\\ \implies |\omega|^2 &= \frac{-(\alpha^2-4\alpha+2) \pm (\alpha-1)\sqrt{1-2\alpha}}{\alpha^2}.\end{aligned} \tag{13}$$

eq:rsdegener

This is only possible at $\alpha = 1$.

Let's look at two interesting limits.

First, $\alpha \to 1$:

$$rs \to \pm\frac{1}{(\alpha-1)\left(1-|\omega|^2\right)}, \tag{14}$$

eq:rsato1

which always satisfies (11),

$$\implies \zeta_- \to \frac{\left(1+|\omega|^2\right) \mp \left(1-|\omega|^2\right)}{2\overline{\omega}} = \quad \omega \quad \text{or} \quad \frac{1}{\overline{\omega}}, \tag{15}$$

eq:zmato1

with the first choice satisfying (12) for $|\omega| < 1$ and the other for $|\omega| > 1$,

$$\implies \Phi \to \begin{cases} 0 & \text{for } |\omega| < 1,\\ -\frac{\ln|\omega|}{2\pi} & \text{for } |\omega| > 1.\end{cases} \tag{16}$$

eq:phiato1

This is harmonic everywhere except $|\omega| = 1$. Applying Gauss' law to a circular loop of radius greater than 1, centred at the origin, tells us that the total charge enclosed is 1. Therefore:

$$\rho(\omega) \to \frac{\delta(|\omega| - 1)}{2\pi} \quad \text{as} \quad \alpha \to 1. \tag{17}$$

eq:rhoato1

Now, $\alpha \to \infty$:

$$rs = -\frac{2\alpha^2\left(1 + |\omega|^2\right)}{\left(1 - |\omega|^2\right)^2} + \mathcal{O}(\alpha) \quad \text{or}$$

$$\frac{\alpha}{1 + |\omega|^2}\left[1 + \frac{1 + 4|\omega|^2 + |\omega|^4}{\alpha\left(1 + |\omega|^2\right)^2} + \frac{1 + 6|\omega|^2 + 18|\omega|^4 + 6|\omega|^6 + |\omega|^8}{\alpha^2\left(1 + |\omega|^2\right)^4} + \mathcal{O}(\alpha^{-3})\right], \tag{18}$$

eq:rsatoinft

both of which satisfy (11),

$$\Longrightarrow \zeta_- = \frac{1 + |\omega|^2}{2\overline{\omega}} \quad \text{or} \quad \frac{\omega}{\alpha\left(1 + |\omega|^2\right)}\left[1 + \frac{2|\omega|^2}{\alpha\left(1 + |\omega|^2\right)^2} + \mathcal{O}(\alpha^{-2})\right], \tag{19}$$

eq:zmatoinft

only the second of which satisfies (12)

$$\Longrightarrow \Phi = -\frac{\ln\left(1 + |\omega|^2\right)}{4\pi} + \mathcal{O}(\alpha^{-1}), \tag{20}$$

eq:phiatoinf

in which we have dropped (infinite) constants. This results in

$$\rho(\omega) \to \frac{1}{\pi\left(1 + |\omega|^2\right)^2} \quad \text{as} \quad \alpha \to \infty. \tag{21}$$

eq:rhoatoinf

This should be a delta function at zero.

3 Simplified derivation

simplederiv

In this section, we will present a simplified version of the derivation. We will make the following simplifying assumption.

At some point, we will treat x as annealed, rather than quenched, disorder:

$$\langle \ln(\cdots) \rangle_x = \ln \langle \cdots \rangle_x . \tag{22}$$

eq:annealed

We will justify this assumption in §4 using the replica trick. We will see that, with a replica symmetric ansatz, the saddle point has zero off-diagonal replica overlaps. This means that there is no coupling between the replicas, which produces identical results to the annealed calculation.

We start with the representation of the determinant in (58). However, the matrix in (7) is not positive-definite when $\overline{\omega}$ is equal to one of the eigenvalues. We can fix this by adding $\epsilon^2 I$ and letting $\epsilon \to 0$ at the end.

$$Z_A(\omega) = \int \prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \exp\left(-z^\dagger(\overline{\omega} - A^\mathrm{T})(\omega - A)z - \epsilon^2 z^\dagger z\right). \tag{23}$$ eq:partfnint

Looking at the expression (4) for A, we make the change of variables $z = (xx^\mathrm{T})w/P$.

$$\begin{aligned} Z_A(\omega) &= \det(xx^\mathrm{T})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \mathrm{e}^{-F/P^2} \\ F &= w^\dagger x(\overline{\omega} - U^\dagger)x^\mathrm{T} x(\omega - U)x^\mathrm{T} w + \epsilon^2 w^\dagger xx^\mathrm{T} xx^\mathrm{T} w. \end{aligned} \tag{24}$$ eq:partfnint

We now take advantage of (60) by introducing two standard complex Gaussian random vectors ($C = I$ in (59)), u and v:

$$\begin{aligned} Z_A(\omega) &= \det(xx^\mathrm{T})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \left\langle \mathrm{e}^{\mathrm{i}F'/P} \right\rangle_{u,v}, \\ F' &= w^\dagger x(\overline{\omega} - U^\dagger)x^\mathrm{T} u + u^\dagger x(\omega - U)x^\mathrm{T} w + \epsilon w^\dagger xx^\mathrm{T} v + \epsilon v^\dagger xx^\mathrm{T} w. \end{aligned} \tag{25}$$ eq:partfnint

Over most of the integration domain, we expect the real inner products ($w^\dagger w$,$u^\dagger u$,...) will be $\mathcal{O}(N)$, whereas the complex inner products ($w^\dagger u$,$w^\mathrm{T} w$,...) will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ, σ and τ, which are zero mean Gaussian random vectors:

$$\begin{aligned} \rho &= x^\mathrm{T} w, & \langle \overline{\rho}_\mu \rho_\nu \rangle_x &= N r \delta_{\mu\nu}, & r &= \frac{w^\dagger w}{N}, \\ \sigma &= x^\mathrm{T} u, & \langle \overline{\sigma}_\mu \sigma_\nu \rangle_x &= N s \delta_{\mu\nu}, & s &= \frac{u^\dagger u}{N}, \\ \tau &= x^\mathrm{T} v, & \langle \overline{\tau}_\mu \tau_\nu \rangle_x &= N t \delta_{\mu\nu}, & t &= \frac{v^\dagger v}{N}, \end{aligned} \tag{26}$$ eq:rstdef

with all other covariances negligible in the large N limit. We can now write

$$\begin{aligned} \langle \ln Z_A(\omega) \rangle_x =& 2 \left\langle \ln\det(xx^\mathrm{T}) \right\rangle_x \\ &+ \left\langle \ln \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi} \right] \mathrm{e}^{-N(s+t) - \xi^\dagger A \xi} \right\rangle_x, \\ \text{where} \quad \xi =& \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\ A =& -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}, \\ \left\langle \xi\xi^\dagger \right\rangle_x =& C = N \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}. \end{aligned} \tag{27}$$ eq:potwuv

As we only care about the part of Φ that depends on ω, we can ignore the first term. We will simplify the second term using (22), (62) and the relation

$$\int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \right] f(r) = \frac{N^N}{\Gamma(N)} \int \mathrm{d}r \, r^{N-1} f(r), \tag{28}$$ eq:spherical

along with similar ones for u and v, to get

$$\begin{aligned} \Phi(\omega) &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} (rst)^N \mathrm{e}^{-N(s+t)} \left\langle \mathrm{e}^{-\xi^\dagger A \xi} \right\rangle_x \\ &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \frac{\exp[N(\ln(rst) - s - t)]}{\det(I + CA)}. \end{aligned} \tag{29}$$ eq:phiintsim

As U is unitary, all the blocks in these matrices commute. Therefore, we can evaluate the determinant with some help from [2]. Also noting that the eigenvalues of U are $\exp(2\pi \mathrm{i}k/P)$, with $k \in \mathbb{Z}_P$:

$$\begin{aligned} \ln\det(I + CA) &= \ln\det \left[\frac{1}{\alpha^3} \begin{pmatrix} \alpha & -\mathrm{i}r(\overline{\omega} - U^\dagger) & -\mathrm{i}\epsilon r \\ -\mathrm{i}s(\omega - U) & \alpha & 0 \\ -\mathrm{i}\epsilon t & 0 & \alpha \end{pmatrix} \right] \\ &= \ln\det \left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - U^\dagger)(\omega - U)}{\alpha^2} \right] \\ &= \sum_{k=0}^{P-1} \ln \left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega - \mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2} \right] \\ &= \frac{P}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \, \ln \left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-\mathrm{i}\phi})(\omega - \mathrm{e}^{\mathrm{i}\phi})}{\alpha^2} \right] \\ &= \frac{P}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln \left[\frac{G(\zeta)}{\alpha^2 \zeta} \right], \end{aligned} \tag{30}$$ eq:detsimple

$(C^{-1} + A)$ IS NON-NORMAL! where the function $G(\zeta)$ is defined in §C, in particular (70). The denominator of the logarithm contributes a factor that is independent of ω, which can be safely ignored.

If we factorise $G(\zeta)$, we get some contour integrals of the form discussed in §B, the result of which appears in (69). From (71), we know that only one of the zeros of $G(\zeta)$ will ie inside the contour. We find that

$$\begin{aligned} \ln\det(I + CA) &= \text{const.} + P[\ln(-rs\overline{\omega}) + \ln\zeta_+ + \ln\zeta_- - \ln(\min|\zeta_\pm|)] \\ &= \text{const.} + P \ln \left(-\frac{rs\omega}{\min|\zeta_\pm|} \right). \end{aligned} \tag{31}$$ eq:detsimple

Now, if we use the saddle-point approximation of the integrals over r, s and t in (29), which becomes exact in the limit of large N and P, we find

$$\Phi(\omega) = \frac{1}{4\pi} \max_{r,s,t} \left[\ln(rst) - s - t - \alpha \ln \left(-\frac{rs\omega}{\min|\zeta_\pm|} \right) \right]. \tag{32}$$ eq:phisaddle

One can show that (see §4, in particular (55) and (56)) the maximum has

$$r \sim \mathcal{O}(\epsilon^{-1}), \qquad s \sim \mathcal{O}(\epsilon), \qquad t \sim \mathcal{O}(1), \qquad rs \sim \mathcal{O}(1). \tag{33}$$

eq:saddle0e

If we take $\epsilon \to 0$, we find that Φ depends on r, s and t in the combinations rs and t:

$$\begin{aligned}
\Phi(\omega) &= \frac{1}{4\pi} \max_{rs,t} \left[(1-\alpha)\ln(rs) + \ln t - t - \alpha \ln\left(-\frac{\omega}{\min|\zeta_\pm|} \right) \right] \\
\frac{\partial \Phi}{\partial (rs)} &= \frac{1-\alpha}{rs} + \alpha\, \mathfrak{Re} \left(\frac{\alpha^2}{rsG'(\min|\zeta_\pm|)} \right), \\
\frac{\partial \Phi}{\partial t} &= \frac{1}{t} - 1.
\end{aligned} \tag{34}$$

eq:phimaxsim

Setting these derivatives to zero requires

$$|\zeta_-| \leq 1, \qquad [\alpha^2 + rs\left(1 + |\omega|^2\right)]^2 \geq 4(rs)^2 \, |\omega|^2 , \tag{35}$$

eq:saddlereq

in which case

$$\begin{aligned}
t &= 1, \\
rs &= \frac{\alpha^2 \left[-(\alpha-1)\left(1+|\omega|^2\right) \pm \sqrt{(\alpha-1)^2\left(1+|\omega|^2\right)^2 + (2\alpha-1)\left(1-|\omega|^2\right)^2} \right]}{(\alpha-1)\left(1-|\omega|^2\right)^2}, \\
\zeta_- &= \frac{1+|\omega|^2}{2\overline{\omega}} - \frac{\alpha^2}{2(\alpha-1)\overline{\omega} rs}, \\
\Phi(\omega) &= \frac{1}{4\pi} \max_{\pm} \left[(1-\alpha)\ln(rs) - \alpha\ln\left(-\frac{\omega}{|\zeta_-|}\right) \right].
\end{aligned} \tag{36}$$

eq:saddlesol

With the aid of this expression for rs, the second condition (35) reduces to

$$rs \leq \frac{\alpha^2(2\alpha^3 - 2\alpha + 1)}{2(\alpha+1)(\alpha-1)^3\left(1+|\omega|^2\right)}. \tag{37}$$

eq:rsreqsimp

4 Full, replica-tastic derivation

replicader

The starting point for this version of the derivation will be (25) and (27):

$$\begin{aligned}
\Phi(\omega) &= \text{const.} + \frac{1}{4\pi N} \left\langle \ln \widetilde{Z} \right\rangle_x , \\
\widetilde{Z} &= \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi} \right] \mathrm{e}^{-F''}, \\
F'' &= u^\dagger u + v^\dagger v - \frac{\mathrm{i}}{P} \left[w^\dagger x(\overline{\omega} - U^\dagger) x^{\mathrm{T}} u + \epsilon w^\dagger x x^{\mathrm{T}} v + \text{c.c.} \right].
\end{aligned} \tag{38}$$

eq:phiint

We will use the replica trick., i.e. we rewrite the logarithm as

$$\ln \widetilde{Z} = \left. \frac{\partial(\widetilde{Z}^n)}{\partial n} \right|_{n=0} . \tag{39}$$

eq:replicatr

For integer n, we can compute $\widetilde{Z}^n$ by creating n replicas of the system, We then let $n \to 0$ after averaging over x. We index these replicas with $a, b = 1, \ldots, n$:

$$\begin{aligned}
\Phi(\omega) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \left. \left\langle \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \mathrm{e}^{-F'''} \right\rangle_x \right|_{n=0} , \\
F''' &= \sum_a u_a^\dagger u_a + v_a^\dagger v_a - \frac{\mathrm{i}}{P} \left[w_a^\dagger x(\overline{\omega} - U^\dagger) x^{\mathrm{T}} u_a + \epsilon w_a^\dagger x x^{\mathrm{T}} v_a + \text{c.c.} \right] .
\end{aligned} \tag{40}$$

eq:phirep

Over most of the integration domain, we expect the Hermitian overlaps ($w_a^\dagger w_b, u_a^\dagger u_b, \ldots$) will be $\mathcal{O}(N)$, whereas the non-Hermitian overlaps ($w_a^\dagger u_b, w_a^{\mathrm{T}} w_b, \ldots$) will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ_a, σ_a and τ_a, which are zero mean Gaussian random vectors:

$$\begin{aligned}
\rho_a &= x^{\mathrm{T}} w_a, & \langle \bar{\rho}_{\mu a} \rho_{\nu b} \rangle_x &= N \delta_{\mu\nu} R_{ab}, & R_{ab} &= \frac{w_a^\dagger w_b}{N}, \\
\sigma_a &= x^{\mathrm{T}} u_a, & \langle \bar{\sigma}_{\mu a} \sigma_{\nu b} \rangle_x &= N \delta_{\mu\nu} S_{ab}, & S_{ab} &= \frac{u_a^\dagger u_b}{N}, \\
\tau_a &= x^{\mathrm{T}} v_a, & \langle \bar{\tau}_{\mu a} \tau_{\nu b} \rangle_x &= N \delta_{\mu\nu} T_{ab}, & T_{ab} &= \frac{v_a^\dagger v_b}{N},
\end{aligned} \tag{41}$$

eq:reprstdef

with all other covariances negligible in the large N limit.

$$\begin{aligned}
\Phi(\omega) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \left. \left\langle \mathrm{e}^{-N \operatorname{Tr}(S+T) - \xi^\dagger A \xi} \right\rangle_x \right|_{n=0} , \\
&= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \left. \mathrm{e}^{-N E(R,S,T)} \right|_{n=0} ,
\end{aligned} \tag{42}$$

eq:phirepxi

where

$$\begin{aligned}
E(R, S, T) &= \operatorname{Tr}(S + T) + \frac{1}{N} \ln\det(I + CA), \\
\xi &= \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\
A &= -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \otimes I, \\
\left\langle \xi \xi^\dagger \right\rangle_x &= C = N I \otimes \begin{pmatrix} R & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix} .
\end{aligned} \tag{43}$$

eq:overlapen

It will be helpful to separate the integrals over w, u and v into an integral over values of w, u and v with the same overlap and an integral over values of the overlap. This can be done by inserting factors like the following into the integral:

$$\int \mathrm{d}R_{ab} N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = 1. \tag{44}$$

`eq:overlapde`

We define

$$S(R) = \frac{1}{N} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia} \mathrm{d}\overline{w}_{ia}}{2\pi} \right] N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = \ln \det R + \text{const.} \tag{45}$$

`eq:overlapen`

with the final expression valid in the large N limit. Then (42) reduces to

$$\Phi(\omega) = \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \mathrm{d}R_{ab} \mathrm{d}S_{ab} \mathrm{d}T_{ab}\, \mathrm{e}^{-N(E(R,S,T) - S(R) - S(S) - S(T))} \Big|_{n=0} . \tag{46}$$

`eq:phioverla`

We will perform this integral in the large N limit with the saddle point method.

We make the following, replica-symmetric ansätze for the saddle-point:

$$R_{ab} = r_0 \delta_{ab} + r_1, \qquad S_{ab} = s_0 \delta_{ab} + s_1, \qquad T_{ab} = t_0 \delta_{ab} + t_1. \tag{47}$$

`eq:RSansatze`

A matrix of this form has $(n-1)$ eigenvalues equal to r_0 and one eigenvalue equal to $(r_0 + nr_1)$. Therefore,

$$S(R) = \ln \det R = (n-1) \ln r_0 + \ln(r_0 + nr_1) = n \ln r_0 + \frac{nr_1}{r_0} + \mathcal{O}(n^2). \tag{48}$$

`eq:RSentropy`

The replica symmetric form of $R, S <, T$ and the unitarity of U means that all the blocks in (43) commute. Then, according to [2],

$$\begin{aligned} \ln \det(I + CA) &= \ln \det \left[\frac{1}{\alpha^3} \begin{pmatrix} \alpha & -\mathrm{i}(\overline{\omega} - U^\dagger) \otimes R & -\mathrm{i}\epsilon I \otimes R \\ -\mathrm{i}(\omega - U) \otimes S & \alpha & 0 \\ -\mathrm{i}\epsilon I \otimes T & 0 & \alpha \end{pmatrix} \right] \\ &= \ln \det \left[\frac{\alpha^2 + \epsilon^2 I \otimes RT + (\overline{\omega} - U^\dagger)(\omega - U) \otimes RS}{\alpha^2} \right]. \end{aligned} \tag{49}$$

`eq:detrep`

Note that

$$(RS)_{ab} = r_0 s_0 \delta_{ab} + r_0 s_1 + r_1 s_0 + \mathcal{O}(n), \tag{50}$$

`eq:overlappr`

and similar for RT, so these will have a similar eigenvalue structure to R, with the same eigenvectors. Also, the eigenvalues of U are $\exp(2\pi \mathrm{i}k/P)$, with $k \in \mathbb{Z}_P$. Therefore:

$$
\begin{aligned}
\ln\det(I+CA) = \sum_{k=0}^{P-1} n \Biggl\{ \ln\left[\frac{\alpha^2+\epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2}\right] \\
+\frac{\epsilon^2(r_0t_1+r_1t_0)+(r_0s_1+r_1s_0)(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2+\epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}\Biggr\} \\
+\mathcal{O}(n^2) \\
= \frac{nP}{2\pi}\int_0^{2\pi}\mathrm{d}\phi \Biggl\{ \ln\left[\frac{\alpha^2+\epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}{\alpha^2}\right] \\
+\frac{\epsilon^2(r_0t_1+r_1t_0)+(r_0s_1+r_1s_0)(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}{\alpha^2+\epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}\Biggr\} \\
= \frac{nP}{2\pi\mathrm{i}}\int\frac{\mathrm{d}\zeta}{\zeta}\Biggl\{\ln\left[\frac{G(\zeta)}{\alpha^2\zeta}\right] \\
+\frac{\epsilon^2(r_0t_1+r_1t_0)\zeta+(r_0s_1+r_1s_0)(\overline{\omega}\zeta-1)(\omega-\zeta)}{G(\zeta)}\Biggr\},
\end{aligned}
\tag{51}
$$

eq:detrepint

where the function $G(\zeta)$ is defined in §C, in particular (70). The first integral was computed in (31). The second can be computed with the residue theorem. The integrand has poles at $\zeta \in \{0, \zeta_+, \zeta_-\}$, with only one of the last two lying inside the contour (72).

$$
\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\int\frac{\mathrm{d}\zeta}{\zeta}\,\frac{\epsilon^2(r_0t_1+r_1t_0)\zeta+(r_0s_1+r_1s_0)(\overline{\omega}\zeta-1)(\omega-\zeta)}{G(\zeta)} \\
=\frac{(r_0s_1+r_1s_0)(-\omega)}{G(0)}+\frac{\epsilon^2(r_0t_1+r_1t_0)\zeta_\pm+(r_0s_1+r_1s_0)(\overline{\omega}\zeta_\pm-1)(\omega-\zeta_\pm)}{\zeta_\pm G'(\zeta_\pm)} \\
=\frac{r_0s_1+r_1s_0}{r_0s_0}\left(1-\frac{\alpha^2+\epsilon^2r_0t_0}{G'(\zeta_\pm)}\right)+\frac{\epsilon^2(r_0t_1+r_1t_0)}{G'(\zeta_\pm)}.
\end{aligned}
\tag{52}
$$

eq:repcontou

Combining all of this gives:

$$
\begin{aligned}
\Phi(\omega) = \frac{1}{4\pi}\max_{r_{0,1},s_{0,1},t_{0,1}}\Biggl\{\ln(r_0s_0t_0)+\frac{r_1}{r_0}+\frac{s_1}{s_0}+\frac{t_1}{t_0}-(s_0+s_1+t_0+t_1) \\
-\alpha\ln\left(-\frac{r_0s_0\omega}{|\zeta_\pm|}\right)-\frac{\alpha(r_0s_1+r_1s_0)}{r_0s_0}\left(1-\frac{\alpha^2+\epsilon^2r_0t_0}{G'(\zeta_\pm)}\right) \\
-\frac{\alpha\epsilon^2(r_0t_1+r_1t_0)}{G'(\zeta_\pm)}\Biggr\}.
\end{aligned}
\tag{53}
$$

eq:phimaxrep

To find the maximum, we must set to zero the following derivatives

$$\begin{aligned}
\frac{\partial(4\pi\Phi)}{\partial r_1} &= \frac{1}{r_0} - \frac{\alpha}{r_0} + \frac{\alpha^3}{r_0 G'(\zeta_\pm)}, \\
\frac{\partial(4\pi\Phi)}{\partial s_1} &= \frac{1}{s_0} - 1 - \frac{\alpha}{s_0} + \frac{\alpha(\alpha^2 + \epsilon r_0 t_0)}{s_0 G'(\zeta_\pm)}, \\
\frac{\partial(4\pi\Phi)}{\partial t_1} &= \frac{1}{t_0} - 1 - \frac{\alpha \epsilon^2 r_0}{G'(\zeta_\pm)}.
\end{aligned} \tag{54}$$

eq:phidiff1

Solving these equations gives

$$s_0 = \frac{(\alpha-1)\epsilon^2 r_0}{\alpha^2}, \qquad t_0 = 1 - s_0, \qquad G'(\zeta_\pm) = \frac{\alpha^3}{\alpha - 1}. \tag{55}$$

eq:repsaddle

Expression (74) for $G'(\zeta_\pm)$ tells us that it must be ζ_- that lies inside the unit circle. Solving for r_0 is a mess, but we can see that no solutions exist for $r_0 < \mathcal{O}(\epsilon^{-1})$ or $r_0 > \mathcal{O}(\epsilon^{-1})$ as $\epsilon \to 0$, as the last equation would reduce to

$$\begin{aligned}
r_0 \sim \mathcal{O}(\epsilon^{-1+\delta}) \implies && \alpha^2 &= \frac{\alpha^3}{\alpha-1}, \\
r_0 \sim \mathcal{O}(\epsilon^{-1-\delta}) \implies && \frac{(\alpha-1)\left(1-|\omega|^2\right)^2 \epsilon^2 r_0^2}{\alpha^2} &= \frac{\alpha^3}{\alpha-1}.
\end{aligned} \tag{56}$$

eq:r0e

Combined with (55), this justifies (33).

We must also set to zero the following derivatives

$$\begin{aligned}
\frac{\partial(4\pi\Phi)}{\partial r_0} &= \frac{\partial(4\pi\Phi)}{\partial r_1} - \mathrm{i}\alpha\,\mathfrak{Im}\left[\frac{\alpha^2}{r_0 G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\
\frac{\partial(4\pi\Phi)}{\partial s_0} &= \frac{\partial(4\pi\Phi)}{\partial s_1} - \mathrm{i}\alpha\,\mathfrak{Im}\left[\frac{\alpha^2 \epsilon r_0 t_0}{s_0 G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\
\frac{\partial(4\pi\Phi)}{\partial t_0} &= \frac{\partial(4\pi\Phi)}{\partial t_1} + \mathrm{i}\alpha\,\mathfrak{Im}\left[\frac{\epsilon^2 r_0}{G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1.
\end{aligned} \tag{57}$$

eq:phidiff0

Looking at (55) and (36), we see that r_0, s_0, t_0 and $G'(\zeta_\pm)$ are all real. Therefore imaginary parts above will vanish, and the saddle-point will have $r_1 = s_1 = t_1 = 0$. This justifies the annealed assumption in (22).

Appendices

A Complex Gaussian integrals

::compgauss

First, Let's get all of the factors of 2 straight. Note that if we write $z = x + \mathrm{i}y$, then $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x\mathrm{d}y$. Let H be a positive-definite, $N \times N$ Hermitian matrix (or just a normal matrix whose eigenvalues have positive real parts). Consider an integral of the form

$$\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right).$$

We can diagonalise H with a unitary change of variables:

$$\begin{aligned} \int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right) &= \prod_i \int \mathrm{d}z_i \mathrm{d}\bar{z}_i \, \exp\left(-\lambda_i \left|z_i\right|^2\right) \\ &= \prod_i \int \mathrm{d}x_i \mathrm{d}y_i \, 2 \exp\left(-\lambda_i \left(x_i^2 + y_i^2\right)\right) \\ &= \prod_i \frac{2\pi}{\lambda_i} \\ &= \frac{(2\pi)^N}{\det H}. \end{aligned} \tag{58}$$

`eq:compgausi`

The proper normalisation for a Gaussian distribution is

$$P(z, z^\dagger) \mathrm{d}z \mathrm{d}z^\dagger = \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger C^{-1} z\right)}{\det C}. \tag{59}$$

`eq:compgauss`

By completing the square, we can see that

$$\left\langle \exp\left(\zeta^\dagger z + z^\dagger \zeta\right) \right\rangle = \exp\left(\zeta^\dagger C \zeta\right). \tag{60}$$

`eq:compgauss`

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\left\langle z z^\dagger \right\rangle = C. \tag{61}$$

`eq:compgausc`

Now consider an integral of the form

$$\begin{aligned} \left\langle \exp\left(-z^\dagger A z\right) \right\rangle &= \int \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger (C^{-1} + A) z\right)}{\det C} \\ &= \left(\det C \det\left(C^{-1} + A\right)\right)^{-1} \\ &= \det\left(I + CA\right)^{-1}. \end{aligned} \tag{62}$$

`eq:compgauss`

ONLY WORKS IF $(C^{-1} + A)$ IS POS DEF, OR AT LEAST NORMAL!

Does it matter if the matrix is normal? Suppose we diagonalise with a non-unitary

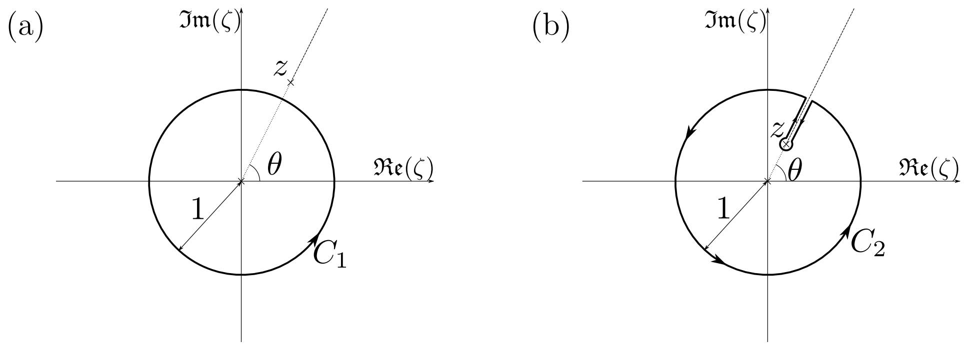

Figure 1: Contours used to evaluate (63), (a) when $|z| > 1$, (b) when $|z| < 1$. Branch cut indicated by dashed line. fig:contours

transformation. What Jacobian factor would we pick up?

$$
\begin{aligned}
\begin{matrix} z' = Sz, \\ \bar{z}' = S^{-1}\bar{z}, \end{matrix} \quad \implies \quad & \begin{matrix} x' = \dfrac{z' + \bar{z}'}{2} = \dfrac{S + S^{-1}}{2}x - \dfrac{S - S^{-1}}{2\mathrm{i}}y, \\ y' = \dfrac{z' - \bar{z}'}{2\mathrm{i}} = \dfrac{S - S^{-1}}{2\mathrm{i}}x - \dfrac{S + S^{-1}}{2}y, \end{matrix} \\
\implies \det J &= \det \begin{pmatrix} \frac{S+S^{-1}}{2} & -\frac{S-S^{-1}}{2\mathrm{i}} \\ \frac{S-S^{-1}}{2\mathrm{i}} & \frac{S+S^{-1}}{2} \end{pmatrix} \\
&= \det \left[\left(\frac{S + S^{-1}}{2} \right)^2 - \left(\frac{S - S^{-1}}{2\mathrm{i}} \right)^2 \right] \\
&= \det SS^{-1} = 1.
\end{aligned}
$$

The change in contours can be undone, as the integrand is analytic in x and y.

B Contour integrals for determinants

contourints

In evaluating determinants, we will come across contour integrals of the form

$$
I(z) = \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta), \tag{63}
$$

eq:contourin

where the contour is the unit circle in a counter-clockwise direction. The contour might not be closed because of the branch cut. We choose the branch of the logarithm so that

$$
\arg\left(\frac{\zeta - z}{z}\right) \in [0, 2\pi], \tag{64}
$$

eq:branch

and we define $\theta = \arg z$. The branch cut is shown in fig.1.

If $|z| > 1$, we can use the contour C_1 in fig.1(a). Using the residue theorem:

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int_{C_1} \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{65}$$

eq:intout

If $|z| < 1$, we can use the contour C_2 in fig.1(b):

$$C_2 : \begin{array}{ll} \zeta = \mathrm{e}^{\mathrm{i}\phi}, & \phi \in [\theta + \delta, \theta + 2\pi - \delta], \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta + 2\pi - \delta)} + xz\left(1 - \mathrm{e}^{\mathrm{i}(2\pi - \delta)}\right), & x \in [0, 1], \\ \zeta = z - x\mathrm{e}^{\mathrm{i}(\theta + 2\pi - \delta)}, & x \in [|z| - 1, -\epsilon], \\ \zeta = z + \epsilon \mathrm{e}^{-\mathrm{i}\phi}, & \phi \in [-\theta - 2\pi + \delta, -\theta - \delta], \\ \zeta = z + x\mathrm{e}^{\mathrm{i}(\theta + \delta)}, & x \in [\epsilon, 1 - |z|\,.] \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta + \delta)} + (1 - x)z\left(1 - \mathrm{e}^{\mathrm{i}\delta}\right), & x \in [0, 1], \end{array} \tag{66}$$

eq:contout

Using the residue theorem:

$$\frac{1}{2\pi \mathrm{i}} \int_{C_2} \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{67}$$

eq:intin

If we let $\delta, \epsilon \to 0$, the second, fourth and sixth parts of the contour integral vanish, and the first part gives $I(z)$ in (63). We're left with

$$\begin{aligned} \ln z &= I(z) - \frac{1}{2\pi \mathrm{i}} \int_{|z|-1}^{0} \frac{\mathrm{e}^{\mathrm{i}\theta} \mathrm{d}x}{z - x\mathrm{e}^{\mathrm{i}\theta}} \ln\left(x\mathrm{e}^{\mathrm{i}(\theta + 2\pi)}\right) + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{e}^{\mathrm{i}\theta} \mathrm{d}x}{z + x\mathrm{e}^{\mathrm{i}\theta}} \ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\ &= I(z) - \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \ln\left(-x\mathrm{e}^{\mathrm{i}(\theta + 2\pi)}\right) + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\ &= I(z) - \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \\ &= I(z) + \ln|z|\,. \end{aligned} \tag{68}$$

eq:intinlim

Therefore:

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z - [\ln|z|]_- = \mathrm{i} \arg z + [\ln|z|]_+, \tag{69}$$

eq:countouri

where $[x]_\pm = x\theta(\pm x)$ and $\theta(x)$ is the Heaviside step function.

C The quadratic function $G(\zeta)$

sec:Gamma

In evaluating determinants in §3 and §4, we come across the function

$$G(\zeta) = (\alpha^2 + \epsilon^2 rt)\zeta + rs(\overline{\omega}\zeta - 1)(\omega - \zeta) = -rs\overline{\omega}(\zeta - \zeta_+)(\zeta - \zeta_-). \tag{70}$$

eq:Gammadef

We will collect some useful features of $\zeta_\pm$ here.

First, by comparing the two forms of $G(\zeta)$, we see that:

$$\zeta_+\zeta_- = \frac{\omega}{\overline{\omega}}, \tag{71}$$ $\boxed{\texttt{eq:zpzm}}$

$$\zeta_+ + \zeta_- = \frac{\alpha^2 + \epsilon^2 rt + rs\left(1 + |\omega|^2\right)}{rs\overline{\omega}}, \tag{72}$$ $\boxed{\texttt{eq:zppzm}}$

$$G'(\zeta_\pm) = \mp rs\overline{\omega}(\zeta_+ - \zeta_-), \tag{73}$$ $\boxed{\texttt{eq:Gprime}}$

and (71) tells us that $|\zeta_+|\,|\zeta_-| = 1$. Solving the equation $G(\zeta_\pm) = 0$ gives

$$G'(\zeta_\pm) = \mp\sqrt{\left[\alpha^2 + \epsilon^2 rt + rs\left(1 + |\omega|^2\right)\right]^2 - 4(rs)^2\,|\omega|^2}. \tag{74}$$ $\boxed{\texttt{eq:Gprimesol}}$

Differentiating the equation $G(\zeta_\pm) = 0$ gives

$$\begin{aligned}
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 t\zeta_\pm + s(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{\alpha^2\zeta_\pm}{rG'(\zeta_\pm)},\\
\frac{\partial\zeta_\pm}{\partial s} &= -\frac{r(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{(\alpha^2 + \epsilon^2 rt)\zeta_\pm}{sG'(\zeta_\pm)},\\
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 r\zeta_\pm}{G'(\zeta_\pm)}.
\end{aligned} \tag{75}$$ $\boxed{\texttt{eq:dzpmdrst}}$

It will also be helpful to note that

$$\frac{\mathrm{d}\,|\zeta_\pm|}{|\zeta_\pm|} = \mathfrak{Re}\left(\frac{\mathrm{d}\zeta_\pm}{\zeta_\pm}\right). \tag{76}$$ $\boxed{\texttt{eqdabsz}}$

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