Illusions of criticality in high-dimensional autoregressive models

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Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

Contents

1 The problem

sec:theprob

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1}$$ eq:model

where $x(t)$ is an N-element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising the quantity

$$\frac{1}{2} \sum_{i,\mu} \left(x_{i\mu+1} - \sum_j A_{ij} x_{j\mu} \right)^2 = \frac{1}{2} \operatorname{Tr} (xU - Ax) (xU - Ax)^{\mathrm{T}}, \qquad (2) \quad \boxed{\texttt{eq:minL}}$$

where U is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make U orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1} \delta_{\nu P}. \qquad (3) \quad \boxed{\texttt{eq:Udef}}$$

The estimate of A is then

$$A = \left(xUx^{\mathrm{T}} \right) \left(xx^{\mathrm{T}} \right)^{-1}. \qquad (4) \quad \boxed{\texttt{eq:Aest}}$$

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $x(t)$ is white noise. Then the true optimal A would be 0. However, with finite P the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega) = \langle \rho_A(\omega) \rangle_x, \qquad \rho_A = \sum_{i=1}^{N} \delta(\omega - \lambda_i), \qquad (5) \quad \boxed{\texttt{eq:eigdist}}$$

where λ_i are the eigenvalues of A in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1.

Following [1], this can be computed from a potential:

$$\rho_A(\omega) = -\nabla^2 \Phi_A(\omega), \qquad \Phi_A(\omega) = -\frac{1}{4\pi N} \ln \det \left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A) \right]. \qquad (6) \quad \boxed{\texttt{eq:potential}}$$

We define a partition function

$$\Phi_A(\omega) = \frac{1}{4\pi N} \ln Z_A(\omega), \qquad Z_A(\omega) = \det \left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A) \right]^{-1}. \qquad (7) \quad \boxed{\texttt{eq:partfn}}$$

The problem is now to compute $\langle \ln Z_A(\omega) \rangle_x$.

2 Simplified derivation

sec:simplederiv

In this section, we will present a simplified version of the derivation. We will make two simplifying assumptions.

First, we will treat x as annealed, rather than quenched, disorder:

$$\langle \ln Z_A(\omega) \rangle_x = \ln \langle Z_A(\omega) \rangle_x . \tag{8}$$

eq:annealed

We will justify this assumption in §3 using the replica trick. We will see that, with a replica symmetric ansatz, the saddle point has zero off-diagonal replica overlaps. This means that there is no coupling between the replicas, which produces identical results to the annealed calculation.

Second, we will assume factorisation of an expectation value:

$$\left\langle \det(xx^{\mathrm{T}})^2 \cdots \right\rangle_x = \left\langle \det(xx^{\mathrm{T}})^2 \right\rangle_x \langle \cdots \rangle_x . \tag{9}$$

eq:factorass

This will also be justified, in the large N limit, in §3.

We start with the representation of the determinant in (13). However, the matrix in (7) is not positive-definite when $\overline{\omega}$ is equal to one of the eigenvalues. We can fix this by adding $\epsilon^2 I$ and letting $\epsilon \to 0$ at the end.

$$Z_A(\omega) = \int \prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \exp\left(-z^\dagger (\overline{\omega} - A^{\mathrm{T}})(\omega - A) z - \epsilon^2 z^\dagger z \right) . \tag{10}$$

eq:partfnintz

Looking at the expression (4) for A, we make the change of variables $z = (xx^{\mathrm{T}})w/P$.

$$Z_A(\omega) = \det(xx^{\mathrm{T}})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \exp\left(-\frac{1}{P^2} w^\dagger x(\overline{\omega} - U)x^{\mathrm{T}} x(\omega - U)x^{\mathrm{T}} w - \frac{\epsilon^2}{P^2} w^\dagger xx^{\mathrm{T}} xx^{\mathrm{T}} w \right) . \tag{11}$$

eq:partfnintw

We now take advantage of (15) by introducing two standard complex Gaussian random vectors ($C = I$ in (14)), u and v:

$$Z_A(\omega) = \det(xx^{\mathrm{T}})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \left\langle \mathrm{e}^{\mathrm{i}S} \right\rangle_{u,v} ,$$
$$S = \frac{w^\dagger x(\overline{\omega} - U)x^{\mathrm{T}} u + u^\dagger x(\omega - U)x^{\mathrm{T}} w + \epsilon w^\dagger xx^{\mathrm{T}} v + \epsilon v^\dagger xx^{\mathrm{T}} w}{P} . \tag{12}$$

eq:partfnintwuv

3 Full, replica-tastic derivation

sec:replicader

Appendices

A Complex Gaussian integrals

sec:compgauss

First, Let's get all of the factors of 2 straight. Note that if we write $z = x + \mathrm{i}y$, then $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x\mathrm{d}y$. Let H be a positive-definite, $N \times N$ Hermitian matrix. Consider an integral of the form

$$\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right).$$

We can diagonalise H with a unitary change of variables:

$$\begin{aligned}
\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right) &= \prod_i \int \mathrm{d}z_i \mathrm{d}\bar{z}_i \, \exp\left(-\lambda_i \left|z_i\right|^2\right) \\
&= \prod_i \int \mathrm{d}x_i \mathrm{d}y_i \, 2 \exp\left(-\lambda_i \left(x_i^2 + y_i^2\right)\right) \\
&= \prod_i \frac{2\pi}{\lambda_i} \\
&= \frac{(2\pi)^N}{\det H}.
\end{aligned} \tag{13}$$

eq:compgausint

The proper normalisation for a Gaussian distribution is

$$P(z, z^\dagger)\mathrm{d}z\mathrm{d}z^\dagger = \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger C^{-1} z\right)}{\det C}. \tag{14}$$

eq:compgaussnorm

By completing the square, we can see that

$$\left\langle \exp\left(\zeta^\dagger z + z^\dagger \zeta\right) \right\rangle = \exp\left(\zeta^\dagger C \zeta\right). \tag{15}$$

eq:compgausslin

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\left\langle z z^\dagger \right\rangle = C. \tag{16}$$

eq:compgauscov

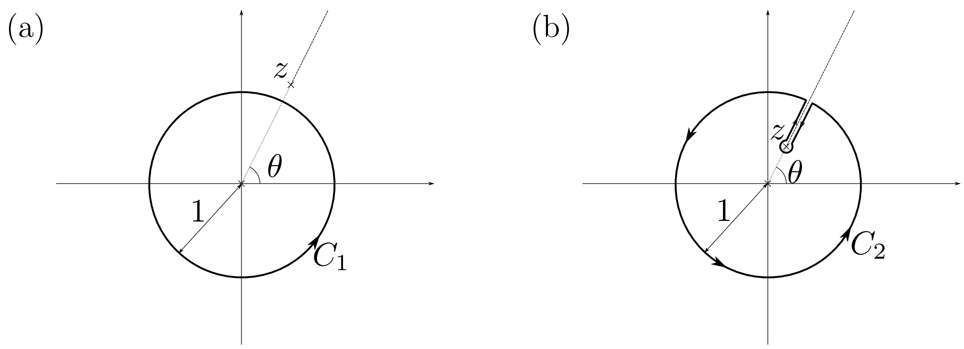

Figure 1: Contours used to evaluate (18), (a) when $|z| > 1$, (b) when $|z| < 1$. `fig:contours`

Now consider an integral of the form

$$\begin{aligned}\left\langle \exp\left(-z^\dagger A z\right)\right\rangle &= \int \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi}\right) \frac{\exp\left(-z^\dagger (C^{-1}+A) z\right)}{\det C} \\ &= \left(\det C \det\left(C^{-1}+A\right)\right)^{-1} \\ &= \det\left(I + CA\right)^{-1}.\end{aligned} \tag{17}$$

`eq:compgaussquad`

B Contour integrals for determinants

`sec:contourints`

In evaluating determinants, we will come across contour integrals of the form

$$I(z) = \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z-\zeta), \tag{18}$$

`eq:contourint`

where the contour is the unit circle in a counter-clockwise direction. The contour might not be closed because of the branch cut. We choose the branch of the logarithm so that

$$\arg\left(\frac{\zeta - z}{z}\right) \in [0, 2\pi], \tag{19}$$

`eq:branch`

and we define $\theta = \arg z$. The branch cut is shown in fig.1.

If $|z| > 1$, we can use the contour C_1 in fig.1(a). Using the residue theorem:

$$I(z) = \frac{1}{2\pi\mathrm{i}} \int_{C_1} \frac{\mathrm{d}\zeta}{\zeta} \ln(z-\zeta) = \ln z. \tag{20}$$

`eq:intout`

If $|z|>1$, we can use the contour C_2 in fig.1(b):

$$C_2: \quad \begin{array}{ll} \zeta = \mathrm{e}^{\mathrm{i}\phi}, & \phi \in [\theta+\delta, \theta+2\pi-\delta], \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta+2\pi-\delta)} + xz\left(1-\mathrm{e}^{\mathrm{i}(2\pi-\delta)}\right), & x \in [0,1], \\ \zeta = z - x\mathrm{e}^{\mathrm{i}(\theta+2\pi-\delta)}, & x \in [|z|-1, -\epsilon], \\ \zeta = z + \epsilon\mathrm{e}^{-\mathrm{i}\phi}, & \phi \in [-\theta-2\pi+\delta, -\theta-\delta], \\ \zeta = z + x\mathrm{e}^{\mathrm{i}(\theta+\delta)}, & x \in [\epsilon, 1-|z|\,.] \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta+\delta)} + (1-x)z\left(1-\mathrm{e}^{\mathrm{i}\delta}\right), & x \in [0,1], \end{array} \tag{21}$$

eq:contout

Using the residue theorem:

$$\frac{1}{2\pi\mathrm{i}}\int_{C_2}\frac{\mathrm{d}\zeta}{\zeta}\ln(z-\zeta) = \ln z. \tag{22}$$

eq:intin

If we let $\delta, \epsilon \to 0$, the second, fourth and sixth parts of the contour integral vanish, and the first part gives $I(z)$ in (18). We're left with

$$\begin{aligned} \ln z = & I(z) - \frac{1}{2\pi\mathrm{i}}\int_{|z|-1}^{0}\frac{\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}x}{z-x\mathrm{e}^{\mathrm{i}\theta}}\ln\left(x\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) + \frac{1}{2\pi\mathrm{i}}\int_{0}^{1-|z|}\frac{\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}x}{z+x\mathrm{e}^{\mathrm{i}\theta}}\ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\ = & I(z) - \frac{1}{2\pi\mathrm{i}}\int_{0}^{1-|z|}\frac{\mathrm{d}x}{|z|+x}\ln\left(-x\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) + \frac{1}{2\pi\mathrm{i}}\int_{0}^{1-|z|}\frac{\mathrm{d}x}{|z|+x}\ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\ = & I(z) - \int_{0}^{1-|z|}\frac{\mathrm{d}x}{|z|+x} \\ = & I(z) + \ln|z|\,. \end{aligned} \tag{23}$$

eq:intinlim

Therefore:

$$I(z) = \frac{1}{2\pi\mathrm{i}}\int\frac{\mathrm{d}\zeta}{\zeta}\ln(z-\zeta) = \ln z - [\ln|z|]_- = \mathrm{i}\arg z + [\ln|z|]_+, \tag{24}$$

eq:countourintresu

where $[x]_\pm = x\theta(\pm x)$ and $\theta(x)$ is the Heaviside step function.

C The quadratic function $\Gamma(\zeta)$

sec:Gamma

In evaluating determinants in §2 and §3, we come across the function

$$\Gamma(\zeta) = (\alpha^2+\epsilon^2 rt)\zeta + rs(\overline{\omega}\zeta-1)(\omega-\zeta) = -rs\overline{\omega}(\zeta-\zeta_+)(\zeta-\zeta_-). \tag{25}$$

eq:Gammadef

We will collect some useful features of $\zeta_\pm$ here.

First, by comparing the two forms of $\Gamma(\zeta)$, we see that:

$$\zeta_+\zeta_- = \frac{\omega}{\overline{\omega}}, \tag{26}$$ eq:zpzm

$$\zeta_+ + \zeta_- = \frac{\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2}{rs\overline{\omega}}, \tag{27}$$ eq:zppzm

$$\Gamma'(\zeta_\pm) = \mp rs\overline{\omega}(\zeta_+ - \zeta_-), \tag{28}$$ eq:Gprime

and (26) tells us that $|\zeta_+|\,|\zeta_-| = 1$. Solving the equation $\Gamma(\zeta_\pm) = 0$ gives

$$\zeta_\pm = \frac{\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2 \pm \sqrt{[\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2] - 4(rs)^2\,|\omega|^2}}{2rs\overline{\omega}}, \tag{29}$$ eq:zetapm

$$\zeta_+ - \zeta_- = \frac{\sqrt{\left[\alpha^2 + \epsilon^2 rt + rs(1+|\omega|^2)\right] - 4(rs)^2\,|\omega|^2}}{rs\overline{\omega}}, \tag{30}$$ eq:zpmzm

$$\Gamma'(\zeta_\pm) = \mp\sqrt{[\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2] - 4(rs)^2\,|\omega|^2}. \tag{31}$$ eq:Gprimesol

Differentiating the equation $\Gamma(\zeta_\pm) = 0$ gives

$$\frac{\partial\zeta_\pm}{\partial r} = -\frac{\epsilon^2 t\zeta_\pm + s(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{\Gamma'(\zeta_\pm)} = \frac{\alpha^2\zeta_\pm}{r\Gamma'(\zeta_\pm)}, \tag{32}$$ eq:dzpmdr

$$\frac{\partial\zeta_\pm}{\partial s} = -\frac{r(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{\Gamma'(\zeta_\pm)} = \frac{(\alpha^2 + \epsilon^2 rt)\zeta_\pm}{s\Gamma'(\zeta_\pm)}, \tag{33}$$ eq:dzpmds

$$\frac{\partial\zeta_\pm}{\partial r} = -\frac{\epsilon^2 r\zeta_\pm}{\Gamma'(\zeta_\pm)}. \tag{34}$$ eq:dzpmdt

It will also de helpful to note that

$$\frac{\mathrm{d}\,|\zeta_\pm|}{|\zeta_\pm|} = \mathfrak{Re}\left(\frac{\mathrm{d}\zeta_\pm}{\zeta_\pm}\right). \tag{35}$$ eqdabsz

References

ers1988asymmetric [1] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, "Spectrum of Large Random Asymmetric Matrices," *Phys. Rev. Lett.* **60** (May, 1988) 1895–1898.