Illusions of criticality in high-dimensional autoregressive models

Subhaneil Lahiri and Surya Ganguli

September 13, 2012

Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

Contents

1 The problem

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1}$$

where $x(t)$ is an N-element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising the quantity

$$\frac{1}{2} \sum_{i,\mu} \left(x_{i\mu+1} - \sum_{j} A_{ij} x_{j\mu} \right)^2 = \frac{1}{2} \operatorname{Tr} (xU - Ax)(xU - Ax)^{\mathrm{T}}, \tag{2}$$

1

where U is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make U orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu1}\delta_{\nu P}. \tag{3}$$ eq:Udef

The estimate of A is then

$$A = \left(xUx^{\mathrm{T}}\right)\left(xx^{\mathrm{T}}\right)^{-1}. \tag{4}$$ eq:Aest

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $x(t)$ is white noise. Then the true optimal A would be 0. However, with finite P the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega) = \langle \rho_A(\omega) \rangle_x, \qquad \rho_A = \sum_{i=1}^{N} \delta(\omega - \lambda_i), \tag{5}$$ eq:eigdist

where λ_i are the eigenvalues of A in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1. This quantity is most relevant in the limit of large N and P. We will keep the quantity $\alpha = P/N$ fixed in this limit.

Following [1], this can be computed from a potential:

$$\rho_A(\omega) = -\nabla^2 \Phi_A(\omega), \qquad \Phi_A(\omega) = -\frac{1}{4\pi N} \ln \det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]. \tag{6}$$ eq:potential

We define a partition function

$$\Phi_A(\omega) = \frac{1}{4\pi N} \ln Z_A(\omega), \qquad Z_A(\omega) = \det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]^{-1}. \tag{7}$$ eq:partfn

The problem is now to compute $\langle \ln Z_A(\omega) \rangle_x$.

2 Simplified derivation

In this section, we will present a simplified version of the derivation. We will make the following simplifying assumption.

At some point, we will treat x as annealed, rather than quenched, disorder:

$$\langle \ln(\cdots) \rangle_x = \ln \langle \cdots \rangle_x. \tag{8}$$ eq:annealed

We will justify this assumption in §3 using the replica trick. We will see that, with a replica symmetric ansatz, the saddle point has zero off-diagonal replica overlaps. This means that there is no coupling between the replicas, which produces identical results to the annealed calculation.

We start with the representation of the determinant in (43). However, the matrix in (7) is not positive-definite when $\overline{\omega}$ is equal to one of the eigenvalues. We can fix this by adding $\epsilon^2 I$ and letting $\epsilon \to 0$ at the end.

$$Z_A(\omega) = \int \prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \exp\left(-z^\dagger(\overline{\omega} - A^\mathrm{T})(\omega - A)z - \epsilon^2 z^\dagger z\right). \qquad (9) \quad \boxed{\text{eq:partfnint}}$$

Looking at the expression (4) for A, we make the change of variables $z = (xx^\mathrm{T})w/P$.

$$Z_A(\omega) = \det(xx^\mathrm{T})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \mathrm{e}^{-F/P^2}$$
$$F = w^\dagger x(\overline{\omega} - U^\dagger)x^\mathrm{T}x(\omega - U)x^\mathrm{T}w + \epsilon^2 w^\dagger xx^\mathrm{T}xx^\mathrm{T}w. \qquad (10) \quad \boxed{\text{eq:partfnint}}$$

We now take advantage of (45) by introducing two standard complex Gaussian random vectors ($C = I$ in (44)), u and v:

$$Z_A(\omega) = \det(xx^\mathrm{T})^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \left\langle \mathrm{e}^{\mathrm{i}F'/P}\right\rangle_{u,v},$$
$$F' = w^\dagger x(\overline{\omega} - U^\dagger)x^\mathrm{T}u + u^\dagger x(\omega - U)x^\mathrm{T}w + \epsilon w^\dagger xx^\mathrm{T}v + \epsilon v^\dagger xx^\mathrm{T}w. \qquad (11) \quad \boxed{\text{eq:partfnint}}$$

Over most of the integration domain, we expect the real inner products $(w^\dagger w, u^\dagger u, \dots)$ will be $\mathcal{O}(N)$, whereas the complex inner products $(w^\dagger u, w^\mathrm{T}w, \dots)$ will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ, σ and τ, which are zero mean Gaussian random vectors:

$$\rho = x^\mathrm{T}w, \quad \langle \bar{\rho}_\mu \rho_\nu \rangle_x = Nr\delta_{\mu\nu}, \quad r = \frac{w^\dagger w}{N},$$
$$\sigma = x^\mathrm{T}u, \quad \langle \bar{\sigma}_\mu \sigma_\nu \rangle_x = Ns\delta_{\mu\nu}, \quad s = \frac{u^\dagger u}{N}, \qquad (12) \quad \boxed{\text{eq:rstdef}}$$
$$\tau = x^\mathrm{T}v, \quad \langle \bar{\tau}_\mu \tau_\nu \rangle_x = Nt\delta_{\mu\nu}, \quad t = \frac{v^\dagger v}{N},$$

with all other covariances negligible in the large N limit. We can now write

$$\langle \ln Z_A(\omega) \rangle_x = 2 \langle \ln \det(xx^\mathrm{T}) \rangle_x$$
$$+ \left\langle \ln \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi}\right] \mathrm{e}^{-N(s+t)-\xi^\dagger A\xi} \right\rangle_x,$$
$$\text{where} \quad \xi = \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix},$$
$$\qquad (13) \quad \boxed{\text{eq:potwuv}}$$
$$A = -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix},$$
$$\langle \xi\xi^\dagger \rangle_x = C = N \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}.$$

3

As we only care about the part of Φ that depends on ω, we can ignore the first term. We will simplify the second term using (8), (47) and the relation

$$\int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \right] f(r) = \frac{N^N}{\Gamma(N)} \int \mathrm{d}r \, r^{N-1} f(r), \tag{14}$$ `eq:spherical`

along with similar ones for u and v, to get

$$\begin{aligned}
\Phi(\omega) &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} (rst)^N \mathrm{e}^{-N(s+t)} \left\langle \mathrm{e}^{-\xi^\dagger A \xi} \right\rangle_x \\
&= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \frac{\exp[N(\ln(rst) - s - t)]}{\det(I + CA)}.
\end{aligned} \tag{15}$$ `eq:phiintsim`

As U is unitary, all the blocks in these matrices commute. Therefore, we can evaluate the determinant with some help from [2]. Also noting that the eigenvalues of U are $\exp(2\pi \mathrm{i}k/P)$, with $k \in \mathbb{Z}_P$:

$$\begin{aligned}
\ln\det(I + CA) &= \ln\det\left[\frac{1}{\alpha^3} \begin{pmatrix} \alpha & -\mathrm{i}r(\overline{\omega} - U^\dagger) & -\mathrm{i}\epsilon r \\ -\mathrm{i}s(\omega - U) & \alpha & 0 \\ -\mathrm{i}\epsilon t & 0 & \alpha \end{pmatrix} \right] \\
&= \ln\det\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - U^\dagger)(\omega - U)}{\alpha^2} \right] \\
&= \sum_{k=0}^{P-1} \ln\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-2\pi \mathrm{i}k/P})(\omega - \mathrm{e}^{2\pi \mathrm{i}k/P})}{\alpha^2} \right] \\
&= \frac{P}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \, \ln\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-\mathrm{i}\phi})(\omega - \mathrm{e}^{\mathrm{i}\phi})}{\alpha^2} \right] \\
&= \frac{P}{2\pi \mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln\left[\frac{G(\zeta)}{\alpha^2 \zeta} \right],
\end{aligned} \tag{16}$$ `eq:detsimple`

$(C^{-1} + A)$ IS NON-NORMAL! where the function $G(\zeta)$ is defined in §C, in particular (55). The denominator of the logarithm contributes a factor that is independent of ω, which can be safely ignored.

If we factorise $G(\zeta)$, we get some contour integrals of the form discussed in §B, the result of which appears in (54). From (56), we know that only one of the zeros of $G(\zeta)$ will ie inside the contour. We find that

$$\begin{aligned}
\ln\det(I + CA) &= \text{const.} + P[\ln(-rs\overline{\omega}) + \ln\zeta_+ + \ln\zeta_- - \ln(\min|\zeta_\pm|)] \\
&= \text{const.} + P\ln\left(-\frac{rs\omega}{\min|\zeta_\pm|} \right).
\end{aligned} \tag{17}$$ `eq:detsimple`

Now, if we use the saddle-point approximation of the integrals over r, s and t in (15), which becomes exact in the limit of large N and P, we find

$$\Phi(\omega) = \frac{1}{4\pi} \max_{r,s,t} \left[\ln(rst) - s - t - \alpha \ln\left(-\frac{rs\omega}{\min|\zeta_\pm|} \right) \right]. \tag{18}$$ `eq:phisaddle`

4

One can show that (see §3, in particular (40) and (41)) the maximum has

$$r \sim \mathcal{O}(\epsilon^{-1}), \qquad s \sim \mathcal{O}(\epsilon), \qquad t \sim \mathcal{O}(1), \qquad rs \sim \mathcal{O}(1). \tag{19}$$

If we take $\epsilon \to 0$, we find that Φ depends on r, s and t in the combinations rs and t:

$$\Phi(\omega) = \frac{1}{4\pi} \max_{rs,t} \left[(1-\alpha)\ln(rs) + \ln t - t - \alpha \ln\left(-\frac{\omega}{\min|\zeta_\pm|} \right) \right]$$
$$\frac{\partial \Phi}{\partial(rs)} = \frac{1-\alpha}{rs} + \alpha\,\mathfrak{Re}\left(\frac{\alpha^2}{rs\,G'(\min|\zeta_\pm|)} \right), \tag{20}$$
$$\frac{\partial \Phi}{\partial t} = \frac{1}{t} - 1.$$

Setting these derivatives to zero requires

$$|\zeta_-| < 1, \qquad [\alpha^2 + rs\left(1 + |\omega|^2\right)]^2 > 4(rs)^2 |\omega|^2, \tag{21}$$

in which case

$$t = 1,$$
$$rs = \frac{\alpha^2 \left[-(\alpha-1)\left(1+|\omega|^2\right) \pm \sqrt{(\alpha-1)^2\left(1+|\omega|^2\right)^2 - (2\alpha-1)\left(1-|\omega|^2\right)^2} \right]}{(\alpha-1)\left(1-|\omega|^2\right)^2},$$
$$\zeta_- = \frac{1+|\omega|^2}{2\overline{\omega}} - \frac{\alpha^2}{2(\alpha-1)\overline{\omega}rs},$$
$$\Phi(\omega) = \frac{1}{4\pi}\left[(1-\alpha)\ln(rs) - \alpha\ln\left(-\frac{\omega}{|\zeta_-|} \right) \right] \tag{22}$$

3 Full, replica-tastic derivation

The starting point for this version of the derivation will be (11) and (13):

$$\Phi(\omega) = \text{const.} + \frac{1}{4\pi N}\left\langle \ln \widetilde{Z} \right\rangle_x,$$
$$\widetilde{Z} = \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi} \right] \mathrm{e}^{-F''}, \tag{23}$$
$$F'' = u^\dagger u + v^\dagger v - \frac{\mathrm{i}}{P}\left[w^\dagger x(\overline{\omega} - U^\dagger)x^{\mathrm{T}}u + \epsilon w^\dagger x x^{\mathrm{T}}v + \text{c.c.} \right].$$

We will use the replica trick., i.e. we rewrite the logarithm as

$$\ln \widetilde{Z} = \left. \frac{\partial(\widetilde{Z}^n)}{\partial n} \right|_{n=0}. \tag{24}$$

For integer n, we can compute $\widetilde{Z}^n$ by creating n replicas of the system, We then let $n \to 0$ after averaging over x. We index these replicas with $a, b = 1, \ldots, n$:

$$\Phi(\omega) = \frac{1}{4\pi N} \frac{\partial}{\partial n} \left\langle \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \mathrm{e}^{-F'''} \right\rangle_x \Bigg|_{n=0},$$

$$F''' = \sum_a u_a^\dagger u_a + v_a^\dagger v_a - \frac{\mathrm{i}}{P} \left[w_a^\dagger x(\overline{\omega} - U^\dagger)x^\mathrm{T} u_a + \epsilon w_a^\dagger x x^\mathrm{T} v_a + \text{c.c.} \right].$$

(25) `eq:phirep`

Over most of the integration domain, we expect the Hermitian overlaps $(w_a^\dagger w_b, u_a^\dagger u_b, \ldots)$ will be $\mathcal{O}(N)$, whereas the non-Hermitian overlaps $(w_a^\dagger u_b, w_a^\mathrm{T} w_b, \ldots)$ will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ_a, σ_a and τ_a, which are zero mean Gaussian random vectors:

$$\rho_a = x^\mathrm{T} w_a, \quad \langle \bar{\rho}_{\mu a}\rho_{\nu b}\rangle_x = N\delta_{\mu\nu}R_{ab}, \quad R_{ab} = \frac{w_a^\dagger w_b}{N},$$

$$\sigma_a = x^\mathrm{T} u_a, \quad \langle \bar{\sigma}_{\mu a}\sigma_{\nu b}\rangle_x = N\delta_{\mu\nu}S_{ab}, \quad S_{ab} = \frac{u_a^\dagger u_b}{N},$$

$$\tau_a = x^\mathrm{T} v_a, \quad \langle \bar{\tau}_{\mu a}\tau_{\nu b}\rangle_x = N\delta_{\mu\nu}T_{ab}, \quad T_{ab} = \frac{v_a^\dagger v_b}{N},$$

(26) `eq:reprstdef`

with all other covariances negligible in the large N limit.

$$\Phi(\omega) = \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \left\langle \mathrm{e}^{-N\,\mathrm{Tr}(S+T)-\xi^\dagger A\xi} \right\rangle_x \Bigg|_{n=0},$$

$$= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia}\mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia}\mathrm{d}\bar{v}_{ia}}{2\pi} \right] \mathrm{e}^{-NE(R,S,T)} \Bigg|_{n=0},$$

(27) `eq:phirepxi`

where

$$E(R,S,T) = \mathrm{Tr}(S+T) + \frac{1}{N} \ln \det(I + CA),$$

$$\xi = \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix},$$

$$A = -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \otimes I,$$

$$\langle \xi\xi^\dagger \rangle_x = C = NI \otimes \begin{pmatrix} R & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix}.$$

(28) `eq:overlapen`

It will be helpful to separate the integrals over w, u and v into an integral over values of w, u and v with the same overlap and an integral over values of the overlap. This can be done by inserting factors like the following into the integral:

$$\int \mathrm{d}R_{ab} N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = 1.$$

(29) `eq:overlapde`

We define

$$S(R) = \frac{1}{N} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi} \right] N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = \ln \det R + \text{const.} \qquad (30) \quad \boxed{\texttt{eq:overlapen}}$$

with the final expression valid in the large N limit. Then (27) reduces to

$$\Phi(\omega) = \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \mathrm{d}R_{ab}\mathrm{d}S_{ab}\mathrm{d}T_{ab}\, e^{-N(E(R,S,T)-S(R)-S(S)-S(T))}\bigg|_{n=0}. \qquad (31) \quad \boxed{\texttt{eq:phioverla}}$$

We will perform this integral in the large N limit with the saddle point method.

We make the following, replica-symmetric ansätze for the saddle-point:

$$R_{ab} = r_0\delta_{ab} + r_1, \qquad S_{ab} = s_0\delta_{ab} + s_1, \qquad T_{ab} = t_0\delta_{ab} + t_1. \qquad (32) \quad \boxed{\texttt{eq:RSansatze}}$$

A matrix of this form has $(n-1)$ eigenvalues equal to r_0 and one eigenvalue equal to $(r_0 + nr_1)$. Therefore,

$$S(R) = \ln \det R = (n-1)\ln r_0 + \ln(r_0 + nr_1) = n\ln r_0 + \frac{nr_1}{r_0} + \mathcal{O}(n^2). \qquad (33) \quad \boxed{\texttt{eq:RSentropy}}$$

The replica symmetric form of $R, S <, T$ and the unitarity of U means that all the blocks in (28) commute. Then, according to [2],

$$\ln \det(I + CA) = \ln \det \left[\frac{1}{\alpha^3} \begin{pmatrix} \alpha & -\mathrm{i}(\overline{\omega} - U^\dagger) \otimes R & -\mathrm{i}\epsilon I \otimes R \\ -\mathrm{i}(\omega - U) \otimes S & \alpha & 0 \\ -\mathrm{i}\epsilon I \otimes T & 0 & \alpha \end{pmatrix} \right] \qquad (34) \quad \boxed{\texttt{eq:detrep}}$$

$$= \ln \det \left[\frac{\alpha^2 + \epsilon^2 I \otimes RT + (\overline{\omega} - U^\dagger)(\omega - U) \otimes RS}{\alpha^2} \right].$$

Note that

$$(RS)_{ab} = r_0s_0\delta_{ab} + r_0s_1 + r_1s_0 + \mathcal{O}(n), \qquad (35) \quad \boxed{\texttt{eq:overlappr}}$$

and similar for RT, so these will have a similar eigenvalue structure to R, with the same eigenvectors. Also, the eigenvalues of U are $\exp(2\pi\mathrm{i}k/P)$, with $k \in \mathbb{Z}_P$. Therefore:

$$\begin{aligned}
\ln \det(I + CA) &= \sum_{k=0}^{P-1} n \left\{ \ln\left[\frac{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega} - e^{-2\pi\mathrm{i}k/P})(\omega - e^{2\pi\mathrm{i}k/P})}{\alpha^2} \right] \right. \\
&\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0) + (r_0 s_1 + r_1 s_0)(\overline{\omega} - e^{-2\pi\mathrm{i}k/P})(\omega - e^{2\pi\mathrm{i}k/P})}{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega} - e^{-2\pi\mathrm{i}k/P})(\omega - e^{2\pi\mathrm{i}k/P})} \right\} \\
&\quad + \mathcal{O}(n^2) \\
&= \frac{nP}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \left\{ \ln\left[\frac{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega} - e^{-\mathrm{i}\phi})(\omega - e^{\mathrm{i}\phi})}{\alpha^2} \right] \right. \qquad (36) \quad \boxed{\texttt{eq:detrepint}} \\
&\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0) + (r_0 s_1 + r_1 s_0)(\overline{\omega} - e^{-\mathrm{i}\phi})(\omega - e^{\mathrm{i}\phi})}{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0(\overline{\omega} - e^{-\mathrm{i}\phi})(\omega - e^{\mathrm{i}\phi})} \right\} \\
&= \frac{nP}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \left\{ \ln\left[\frac{G(\zeta)}{\alpha^2\zeta} \right] \right. \\
&\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta + (r_0 s_1 + r_1 s_0)(\overline{\omega}\zeta - 1)(\omega - \zeta)}{G(\zeta)} \right\},
\end{aligned}$$

where the function $G(\zeta)$ is defined in §C, in particular (55). The first integral was computed in (17). The second can be computed with the residue theorem. The integrand has poles at $\zeta \in \{0, \zeta_+, \zeta_-\}$, with only one of the last two lying inside the contour (57).

$$
\begin{aligned}
\frac{1}{2\pi i} \int \frac{d\zeta}{\zeta} & \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta + (r_0 s_1 + r_1 s_0)(\overline{\omega}\zeta - 1)(\omega - \zeta)}{G(\zeta)} \\
&= \frac{(r_0 s_1 + r_1 s_0)(-\omega)}{G(0)} + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta_\pm + (r_0 s_1 + r_1 s_0)(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{\zeta_\pm G'(\zeta_\pm)} \\
&= \frac{r_0 s_1 + r_1 s_0}{r_0 s_0}\left(1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_\pm)}\right) + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)}{G'(\zeta_\pm)}.
\end{aligned}
\tag{37}
$$
`eq:repcontou`

Combining all of this gives:

$$
\begin{aligned}
\Phi(\omega) = \frac{1}{4\pi} \max_{r_{0,1}, s_{0,1}, t_{0,1}} & \left\{ \ln(r_0 s_0 t_0) + \frac{r_1}{r_0} + \frac{s_1}{s_0} + \frac{t_1}{t_0} - (s_0 + s_1 + t_0 + t_1) \right. \\
& \left. - \alpha \ln\left(-\frac{r_0 s_0 \omega}{|\zeta_\pm|}\right) - \frac{\alpha(r_0 s_1 + r_1 s_0)}{r_0 s_0}\left(1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_\pm)}\right) \right. \\
& \left. - \frac{\alpha \epsilon^2(r_0 t_1 + r_1 t_0)}{G'(\zeta_\pm)} \right\}.
\end{aligned}
\tag{38}
$$
`eq:phimaxrep`

The maximum can be found by setting to zero the following derivatives

$$
\begin{aligned}
\frac{\partial(4\pi\Phi)}{\partial r_1} &= \frac{1}{r_0} - \frac{\alpha}{r_0} + \frac{\alpha^3}{r_0 G'(\zeta_\pm)}, \\
\frac{\partial(4\pi\Phi)}{\partial s_1} &= \frac{1}{s_0} - 1 - \frac{\alpha}{s_0} + \frac{\alpha(\alpha^2 + \epsilon r_0 t_0)}{s_0 G'(\zeta_\pm)}, \\
\frac{\partial(4\pi\Phi)}{\partial t_1} &= \frac{1}{t_0} - 1 - \frac{\alpha \epsilon^2 r_0}{G'(\zeta_\pm)},
\end{aligned}
\tag{39}
$$
`eq:phidiff1`

Solving these equations gives

$$
s_0 = \frac{(\alpha - 1)\epsilon^2 r_0}{\alpha^2}, \qquad t_0 = 1 - s_0, \qquad G'(\zeta_\pm) = \frac{\alpha^3}{\alpha - 1}.
\tag{40}
$$
`eq:repsaddle`

Expression (59) for $G'(\zeta_\pm)$ tells us that it must be ζ_- that lies inside the unit circle. Solving for r_0 is a mess, but we can see that no solutions exist for $r_0 < \mathcal{O}(\epsilon^{-1})$ or $r_0 > \mathcal{O}(\epsilon^{-1})$ as $\epsilon \to 0$, as the last equation would reduce to

$$
\begin{aligned}
r_0 \sim \mathcal{O}(\epsilon^{-1+\delta}) &\implies & \alpha^2 &= \frac{\alpha^3}{\alpha - 1}, \\
r_0 \sim \mathcal{O}(\epsilon^{-1-\delta}) &\implies & \frac{(\alpha - 1)\left(1 - |\omega|^2\right)^2 \epsilon^2 r_0^2}{\alpha^2} &= \frac{\alpha^3}{\alpha - 1}.
\end{aligned}
\tag{41}
$$
`eq:r0e`

Combined with (40), this justifies (19).

We must also set to zero the following derivatives

$$
\frac{\partial(4\pi\Phi)}{\partial r_0} = \frac{\partial(4\pi\Phi)}{\partial r_1} - i\alpha\,\mathfrak{Im}\left[\frac{\alpha^2}{r_0 G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1,
$$

$$
\frac{\partial(4\pi\Phi)}{\partial s_0} = \frac{\partial(4\pi\Phi)}{\partial s_1} - i\alpha\,\mathfrak{Im}\left[\frac{\alpha^2\epsilon r_0 t_0}{s_0 G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \qquad (42) \quad \boxed{\texttt{eq:phidiff0}}
$$

$$
\frac{\partial(4\pi\Phi)}{\partial t_0} = \frac{\partial(4\pi\Phi)}{\partial t_1} + i\alpha\,\mathfrak{Im}\left[\frac{\epsilon^2 r_0}{G'(\zeta_\pm)}\right] + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1.
$$

Looking at (40) and (22), we see that r_0, s_0, t_0 and $G'(\zeta_\pm)$ are all real. Therefore imaginary parts above will vanish, and the saddle-point will have $r_1 = s_1 = t_1 = 0$. This justifies the annealed assumption in (8).

Appendices

A   Complex Gaussian integrals

$\boxed{\texttt{:compgauss}}$

First, Let's get all of the factors of 2 straight. Note that if we write $z = x + iy$, then $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x\mathrm{d}y$. Let H be a positive-definite, $N \times N$ Hermitian matrix (or just a normal matrix whose eigenvalues have positive real parts). Consider an integral of the form

$$
\int \left(\prod_i \mathrm{d}z_i\mathrm{d}\bar{z}_i\right) \exp\left(-z^\dagger H z\right).
$$

We can diagonalise H with a unitary change of variables:

$$
\begin{aligned}
\int \left(\prod_i \mathrm{d}z_i\mathrm{d}\bar{z}_i\right) \exp\left(-z^\dagger H z\right) &= \prod_i \int \mathrm{d}z_i\mathrm{d}\bar{z}_i \,\exp\left(-\lambda_i |z_i|^2\right) \\
&= \prod_i \int \mathrm{d}x_i\mathrm{d}y_i\, 2\exp\left(-\lambda_i\left(x_i^2 + y_i^2\right)\right) \\
&= \prod_i \frac{2\pi}{\lambda_i} \\
&= \frac{(2\pi)^N}{\det H}.
\end{aligned}
\qquad (43) \quad \boxed{\texttt{eq:compgausi}}
$$

The proper normalisation for a Gaussian distribution is

$$
P(z, z^\dagger)\mathrm{d}z\mathrm{d}z^\dagger = \left(\prod_i \frac{\mathrm{d}z_i\mathrm{d}\bar{z}_i}{2\pi}\right) \frac{\exp\left(-z^\dagger C^{-1} z\right)}{\det C}. \qquad (44) \quad \boxed{\texttt{eq:compgauss}}
$$

By completing the square, we can see that

$$
\left\langle \exp\left(\zeta^\dagger z + z^\dagger \zeta\right) \right\rangle = \exp\left(\zeta^\dagger C \zeta\right). \qquad (45) \quad \boxed{\texttt{eq:compgauss}}
$$

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\left\langle zz^\dagger \right\rangle = C. \tag{46}$$

Now consider an integral of the form

$$\begin{aligned}
\left\langle \exp\left(-z^\dagger A z\right)\right\rangle &= \int \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi}\right) \frac{\exp\left(-z^\dagger(C^{-1}+A)z\right)}{\det C} \\
&= \left(\det C \det\left(C^{-1}+A\right)\right)^{-1} \\
&= \det\left(I+CA\right)^{-1}.
\end{aligned} \tag{47}$$

ONLY WORKS IF $(C^{-1}+A)$ IS POS DEF, OR AT LEAST NORMAL!

Does it matter if the matrix is normal? Suppose we diagonalise with a non-unitary transformation. What Jacobian factor would we pick up?

$$\begin{aligned}
\begin{matrix} z' = Sz, \\ \bar{z}' = S^{-1}\bar{z}, \end{matrix} \implies \quad & \begin{aligned} x' &= \frac{z'+\bar{z}'}{2} = \frac{S+S^{-1}}{2}x - \frac{S-S^{-1}}{2i}y, \\ y' &= \frac{z'-\bar{z}'}{2i} = \frac{S-S^{-1}}{2i}x - \frac{S+S^{-1}}{2}y, \end{aligned} \\
\implies \det J &= \det\begin{pmatrix} \frac{S+S^{-1}}{2} & -\frac{S-S^{-1}}{2i} \\ \frac{S-S^{-1}}{2i} & \frac{S+S^{-1}}{2} \end{pmatrix} \\
&= \det\left[\left(\frac{S+S^{-1}}{2}\right)^2 - \left(\frac{S-S^{-1}}{2i}\right)^2\right] \\
&= \det SS^{-1} = 1.
\end{aligned}$$

The change in contours can be undone, as the integrand is analytic in x and y.

B Contour integrals for determinants

In evaluating determinants, we will come across contour integrals of the form

$$I(z) = \frac{1}{2\pi i} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z-\zeta), \tag{48}$$

where the contour is the unit circle in a counter-clockwise direction. The contour might not be closed because of the branch cut. We choose the branch of the logarithm so that

$$\arg\left(\frac{\zeta-z}{z}\right) \in [0, 2\pi], \tag{49}$$

and we define $\theta = \arg z$. The branch cut is shown in fig.1.

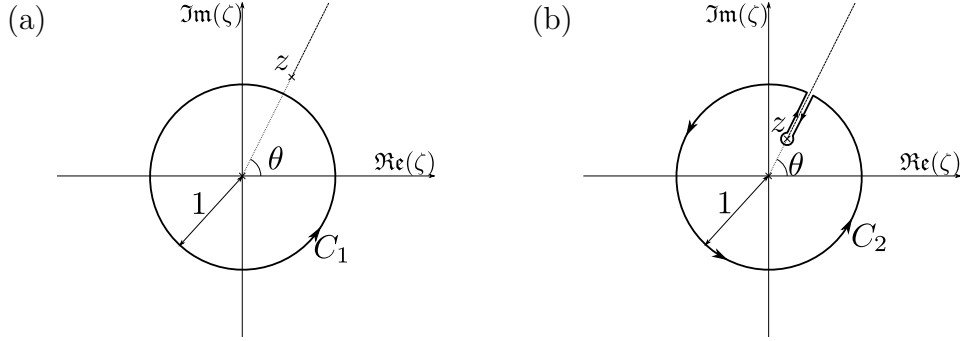Figure 1: Contours used to evaluate (48), (a) when $|z| > 1$, (b) when $|z| < 1$. Branch cut indicated by dashed line.

`fig:contours`

If $|z| > 1$, we can use the contour C_1 in fig.1(a). Using the residue theorem:

$$I(z) = \frac{1}{2\pi i} \int_{C_1} \frac{d\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{50}$$

`eq:intout`

If $|z| < 1$, we can use the contour C_2 in fig.1(b):

$$C_2: \quad \begin{aligned} \zeta &= e^{i\phi}, & \phi &\in [\theta + \delta, \theta + 2\pi - \delta], \\ \zeta &= e^{i(\theta + 2\pi - \delta)} + xz\left(1 - e^{i(2\pi - \delta)}\right), & x &\in [0, 1], \\ \zeta &= z - xe^{i(\theta + 2\pi - \delta)}, & x &\in [|z| - 1, -\epsilon], \\ \zeta &= z + \epsilon e^{-i\phi}, & \phi &\in [-\theta - 2\pi + \delta, -\theta - \delta], \\ \zeta &= z + xe^{i(\theta + \delta)}, & x &\in [\epsilon, 1 - |z|.] \\ \zeta &= e^{i(\theta + \delta)} + (1 - x)z\left(1 - e^{i\delta}\right), & x &\in [0, 1], \end{aligned} \tag{51}$$

`eq:contout`

Using the residue theorem:

$$\frac{1}{2\pi i} \int_{C_2} \frac{d\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{52}$$

`eq:intin`

If we let $\delta, \epsilon \to 0$, the second, fourth and sixth parts of the contour integral vanish, and the first part gives $I(z)$ in (48). We're left with

$$\begin{aligned} \ln z &= I(z) - \frac{1}{2\pi i} \int_{|z|-1}^{0} \frac{e^{i\theta} dx}{z - xe^{i\theta}} \ln\left(xe^{i(\theta + 2\pi)}\right) + \frac{1}{2\pi i} \int_{0}^{1-|z|} \frac{e^{i\theta} dx}{z + xe^{i\theta}} \ln\left(-xe^{i\theta}\right) \\ &= I(z) - \frac{1}{2\pi i} \int_{0}^{1-|z|} \frac{dx}{|z| + x} \ln\left(-xe^{i(\theta + 2\pi)}\right) + \frac{1}{2\pi i} \int_{0}^{1-|z|} \frac{dx}{|z| + x} \ln\left(-xe^{i\theta}\right) \\ &= I(z) - \int_{0}^{1-|z|} \frac{dx}{|z| + x} \\ &= I(z) + \ln|z|. \end{aligned} \tag{53}$$

`eq:intinlim`

11

Therefore:

$$I(z) = \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z - [\ln|z|]_- = \mathrm{i}\arg z + [\ln|z|]_+, \qquad (54) \quad \boxed{\texttt{eq:countouri}}$$

where $[x]_\pm = x\theta(\pm x)$ and $\theta(x)$ is the Heaviside step function.

C The quadratic function $G(\zeta)$

In evaluating determinants in §2 and §3, we come across the function

$$G(\zeta) = (\alpha^2 + \epsilon^2 rt)\zeta + rs(\overline{\omega}\zeta - 1)(\omega - \zeta) = -rs\overline{\omega}(\zeta - \zeta_+)(\zeta - \zeta_-). \qquad (55) \quad \boxed{\texttt{eq:Gammadef}}$$

We will collect some useful features of $\zeta_\pm$ here.

First, by comparing the two forms of $G(\zeta)$, we see that:

$$\zeta_+\zeta_- = \frac{\omega}{\overline{\omega}}, \qquad (56) \quad \boxed{\texttt{eq:zpzm}}$$

$$\zeta_+ + \zeta_- = \frac{\alpha^2 + \epsilon^2 rt + rs\left(1 + |\omega|^2\right)}{rs\overline{\omega}}, \qquad (57) \quad \boxed{\texttt{eq:zppzm}}$$

$$G'(\zeta_\pm) = \mp rs\overline{\omega}(\zeta_+ - \zeta_-), \qquad (58) \quad \boxed{\texttt{eq:Gprime}}$$

and (56) tells us that $|\zeta_+|\,|\zeta_-| = 1$. Solving the equation $G(\zeta_\pm) = 0$ gives

$$G'(\zeta_\pm) = \mp\sqrt{\left[\alpha^2 + \epsilon^2 rt + rs\left(1 + |\omega|^2\right)\right]^2 - 4(rs)^2\,|\omega|^2}. \qquad (59) \quad \boxed{\texttt{eq:Gprimesol}}$$

Differentiating the equation $G(\zeta_\pm) = 0$ gives

$$\begin{aligned}
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 t\zeta_\pm + s(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{\alpha^2\zeta_\pm}{rG'(\zeta_\pm)}, \\
\frac{\partial\zeta_\pm}{\partial s} &= -\frac{r(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{(\alpha^2 + \epsilon^2 rt)\zeta_\pm}{sG'(\zeta_\pm)}, \\
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 r\zeta_\pm}{G'(\zeta_\pm)}.
\end{aligned} \qquad (60) \quad \boxed{\texttt{eq:dzpmdrst}}$$

It will also be helpful to note that

$$\frac{\mathrm{d}\,|\zeta_\pm|}{|\zeta_\pm|} = \mathfrak{Re}\left(\frac{\mathrm{d}\zeta_\pm}{\zeta_\pm}\right). \qquad (61) \quad \boxed{\texttt{eqdabsz}}$$

References

$\boxed{\texttt{Sasymmetric}}$ [1] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, "Spectrum of Large Random Asymmetric Matrices," *Phys. Rev. Lett.* **60** (May, 1988) 1895–1898.

$\boxed{\texttt{eterminants}}$ [2] J. Silvester, "Determinants of block matrices," *The Mathematical Gazette* **84** (2000) no. 501, 460–467.