Illusions of criticality in high-dimensional autoregressive models

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Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

Contents

1 The problem

sec:theprob

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1}$$ eq:model

where $x(t)$ is an N-element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising

$$L = \frac{1}{2} \sum_{i,\mu} \left(x_{i\mu+1} - \sum_j A_{ij} x_{j\mu} \right)^2 = \frac{1}{2} \operatorname{Tr} \left(xU - Ax \right) \left(xU - Ax \right)^{\mathrm{T}}, \tag{2}$$ eq:minL

where U is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make U orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1}\delta_{\nu P}. \tag{3}$$

eq:Udef

The estimate of A is then

$$A = \left(xUx^{\mathrm{T}}\right)\left(xx^{\mathrm{T}}\right)^{-1}. \tag{4}$$

eq:Aest

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $x(t)$ is white noise. Then the true optimal A would be 0. However, with finite P the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega) = \left\langle \rho_A(\omega) \right\rangle_x, \qquad \rho_A = \sum_{i=1}^{N} \delta(\omega - \lambda_i), \tag{5}$$

eq:eigdist

where λ_i are the eigenvalues of A in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1.

Following [1], this can be computed from a potential:

$$\rho_A(\omega) = -\nabla^2\Phi_A(\omega), \qquad \Phi_A(\omega) = -\frac{1}{4\pi N}\ln\det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]. \tag{6}$$

eq:potential

We define a partition function

$$\Phi_A(\omega) = \frac{1}{4\pi N}\ln Z_A(\omega), \qquad Z_A(\omega) = \det\left[(\overline{\omega} - A^{\mathrm{T}})(\omega - A)\right]^{-1}. \tag{7}$$

eq:partfn

The problem is now to compute $\left\langle \ln Z_A(\omega) \right\rangle_z$.

Appendices

A Complex Gaussian integrals

sec:compgauss

First, Let's get all of the factors of 2 straight. Note that if we write $z = x + \mathrm{i}y$, then $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x\mathrm{d}y$. Let H be a positive-definite, $N \times N$ Hermitian matrix. Consider an integral of the form

$$\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i\right) \exp\left(-z^{\dagger}Hz\right).$$

We can diagonalise H with a unitary change of variables:

$$\begin{aligned}
\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right) &= \prod_i \int \mathrm{d}z_i \mathrm{d}\bar{z}_i \, \exp\left(-\lambda_i \left|z_i\right|^2\right) \\
&= \prod_i \int \mathrm{d}x_i \mathrm{d}y_i \, 2 \exp\left(-\lambda_i \left(x_i^2 + y_i^2\right)\right) \\
&= \prod_i \frac{2\pi}{\lambda_i} \\
&= \frac{(2\pi)^N}{\det H}.
\end{aligned} \tag{8}$$

eq:compgausint

The proper normalisation for a gaussian distribution is

$$P(z, z^\dagger) \mathrm{d}z \mathrm{d}z^\dagger = \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger C^{-1} z\right)}{\det C}. \tag{9}$$

eq:compgaussnorm

By completing the square, we can see that

$$\left\langle \exp\left(\zeta^\dagger z + z^\dagger \zeta\right) \right\rangle = \exp\left(\zeta^\dagger C \zeta\right). \tag{10}$$

eq:compgausslin

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\left\langle z z^\dagger \right\rangle = C. \tag{11}$$

eq:compgauscov

Now consider an integral of the form

$$\begin{aligned}
\left\langle \exp\left(-z^\dagger A z\right) \right\rangle &= \int \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger (C^{-1} + A) z\right)}{\det C} \\
&= \left(\det C \det\left(C^{-1} + A\right) \right)^{-1} \\
&= \det\left(I + CA\right)^{-1}.
\end{aligned} \tag{12}$$

eq:compgaussquad

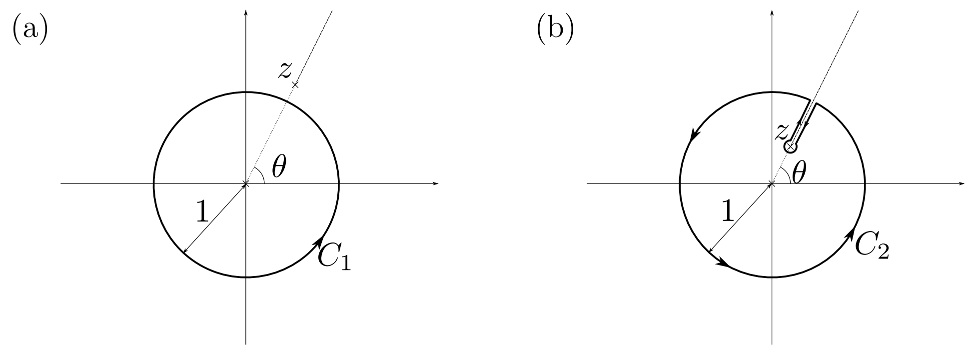
B Contour integrals for determinants

sec:contourints

In evaluating determinants, we will come across contour integrals of the form

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta), \tag{13}$$

eq:contourint



figcontout figcontin

Figure 1: Contours used to evaluate (13), (a) when $|z| > 1$, (b) when $|z| < 1$. fig:contours

where the contour is the unit circle in a counter-clockwise direction. The contour might not be closed because of the branch cut. We choose the branch of the logarithm so that

$$\arg\left(\frac{\zeta - z}{z}\right) \in [0, 2\pi], \tag{14}$$ eq:branch

and we define $\theta = \arg z$. The branch cut is shown in fig.1.

If $|z| > 1$, we can use the contour C_1 in fig.1(a). Using the residue theorem:

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int_{C_1} \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{15}$$ eq:intout

If $|z| > 1$, we can use the contour C_2 in fig.1(b):

$$C_2 : \begin{cases} \zeta = \mathrm{e}^{\mathrm{i}\phi}, & \phi \in [\theta + \delta, \theta + 2\pi - \delta], \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta + 2\pi - \delta)} + xz\left(1 - \mathrm{e}^{\mathrm{i}(2\pi - \delta)}\right), & x \in [0, 1], \\ \zeta = z - x\mathrm{e}^{\mathrm{i}(\theta + 2\pi - \delta)}, & x \in [|z| - 1, -\epsilon], \\ \zeta = z + \epsilon \mathrm{e}^{-\mathrm{i}\phi}, & \phi \in [-\theta - 2\pi + \delta, -\theta - \delta], \\ \zeta = z + x\mathrm{e}^{\mathrm{i}(\theta + \delta)}, & x \in [\epsilon, 1 - |z|\,.] \\ \zeta = \mathrm{e}^{\mathrm{i}(\theta + \delta)} + (1 - x)z\left(1 - \mathrm{e}^{\mathrm{i}\delta}\right), & x \in [0, 1], \end{cases} \tag{16}$$ eq:contout

Using the residue theorem:

$$\frac{1}{2\pi \mathrm{i}} \int_{C_2} \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z. \tag{17}$$ eq:intin

If we let $\delta, \epsilon \to 0$, the second, fourth and sixth parts of the contour integral vanish, and the first part gives $I(z)$ in (13). We're left with

$$
\begin{aligned}
\ln z &= I(z) - \frac{1}{2\pi \mathrm{i}} \int_{|z|-1}^{0} \frac{\mathrm{e}^{\mathrm{i}\theta} \mathrm{d}x}{z - x\mathrm{e}^{\mathrm{i}\theta}} \ln\left(x\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{e}^{\mathrm{i}\theta} \mathrm{d}x}{z + x\mathrm{e}^{\mathrm{i}\theta}} \ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\
&= I(z) - \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \ln\left(-x\mathrm{e}^{\mathrm{i}(\theta+2\pi)}\right) + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \ln\left(-x\mathrm{e}^{\mathrm{i}\theta}\right) \\
&= I(z) - \int_{0}^{1-|z|} \frac{\mathrm{d}x}{|z| + x} \\
&= I(z) + \ln|z| \,.
\end{aligned} \tag{18}
$$

eq:intinlim

Therefore:

$$
I(z) = \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}\zeta}{\zeta} \ln(z - \zeta) = \ln z - [\ln|z|]_{-}, \tag{19}
$$

eq:countourintresu

where $[x]_{\pm} = x\theta(\pm x)$ and $\theta(x)$ is the Heaviside step function.

C The quadratic function $\Gamma(\zeta)$

sec:Gamma

References

ers1988asymmetric

[1] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, "Spectrum of Large Random Asymmetric Matrices," *Phys. Rev. Lett.* **60** (May, 1988) 1895–1898.