Illusions of criticality in high-dimensional autoregressive models

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Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

Contents

1 The problem

sec:theprob

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1}$$ eq:model

where $x(t)$ is an N-element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising the quantity

$$\frac{1}{2}\sum_{i,\mu}\left(x_{i\mu+1}-\sum_j \hat{A}_{ij}x_{j\mu}\right)^2 = \frac{1}{2}\operatorname{Tr}\left(x\mathcal{S}-\hat{A}x\right)\left(x\mathcal{S}-\hat{A}x\right)^{\mathrm{T}}, \tag{2}$$ eq:minL

where $\mathcal{S}$ is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make $\mathcal{S}$ orthogonal:

$$\mathcal{S}_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1}\delta_{\nu P}. \tag{3}$$ eq:Udef

The estimate of A is then

$$\hat{A} = \left(x\mathcal{S}x^{\mathrm{T}}\right)\left(xx^{\mathrm{T}}\right)^{-1} = x\mathcal{S}x^*, \tag{4}$$ eq:Aest

where x^* is the pseudoinverse of x, i.e. $xx^* = I$.

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $A = 0$ and $x(t)$ is just white noise. For large enough P, we should get $\hat{A} = 0$. However, with finite P the estimate (4) will not be zero.

Moreover if $P = N$, generically x will be invertible, so $x^* = x^{-1}$. This would mean that (4) is a similarity transform of $\mathcal{S}$, and will therefore have the same eigenvalues: $\exp(2\pi \mathrm{i}k/P)$ for $k \in \mathbb{Z}_P$. We would be fooled into thinking the system was critical (eigenvalues of absolute value 1) when in reality there is only noise. This raises the question, how large must $\alpha = P/N$ be to avoid this problem?

We will look at the average eigenvalue distribution:

$$\rho(\omega,\overline{\omega}) = \left\langle \rho_{\hat{A}}(\omega,\overline{\omega})\right\rangle_x, \qquad \rho_{\hat{A}}(\omega,\overline{\omega}) = \sum_{i=1}^{N}\delta^{(2)}(\omega-\lambda_i), \tag{5}$$ eq:eigdist

where λ_i are the eigenvalues of $\hat{A}$ in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1. This quantity is most relevant in the limit of large N and P. We will keep α fixed in this limit.

Following [1], this can be computed from a potential:

$$\rho_{\hat{A}}(\omega,\overline{\omega}) = -\nabla^2\Phi_{\hat{A}}(\omega,\overline{\omega}), \qquad \Phi_{\hat{A}}(\omega,\overline{\omega}) = -\frac{1}{4\pi N}\ln\det\left[(\overline{\omega}-\hat{A}^{\mathrm{T}})(\omega-\hat{A})\right]. \tag{6}$$ eq:potential

We define a partition function

$$\Phi_{\hat{A}}(\omega,\overline{\omega}) = \frac{1}{4\pi N}\ln Z_{\hat{A}}(\omega,\overline{\omega}), \qquad Z_{\hat{A}}(\omega,\overline{\omega}) = \det\left[(\overline{\omega}-\hat{A}^{\mathrm{T}})(\omega-\hat{A})\right]^{-1}. \tag{7}$$ eq:partfn

The problem is now to compute $\left\langle \ln Z_{\hat{A}}(\omega,\overline{\omega})\right\rangle_x$.

2 The solution

ec:solution

In §3 we will present a simplified derivation and in §4 we will fill in the gaps and justify the assumptions used in §4. The result from (35), with some constant pieces dropped, will be:

$$\begin{aligned} q &= \frac{\sqrt{\alpha^2\left(1+|\omega|^2\right)^2 - 4(2\alpha-1)\,|\omega|^2} - (\alpha-1)\left(1+|\omega|^2\right)}{\left(1-|\omega|^2\right)^2}, \\ \Phi(\omega,\overline{\omega}) &= \frac{1}{4\pi}\left[(1-\alpha)\ln q + \alpha\ln\left(\frac{1+|\omega|^2-q^{-1}}{2\,|\omega|^2}\right)\right], \end{aligned} \tag{8}$$

eq:phisol

The potential has a rotation symmetry. Therefore, we can express the eigenvalue density in terms of the radial density

$$\rho(|\omega|) = \int \rho(\omega,\overline{\omega})\,|\omega|\,\mathrm{d}\phi = 2\pi\,|\omega|\,\rho(\omega,\overline{\omega}) = -\frac{\partial}{\partial|\omega|}\left(|\omega|\,\frac{\partial(2\pi\Phi)}{\partial|\omega|}\right). \tag{9}$$

eq:radialrho

We find

$$\rho(|\omega|) = \frac{2(\alpha-1)\,|\omega|\,q}{\sqrt{\alpha^2\left(1+|\omega|^2\right)^2 - 4(2\alpha-1)\,|\omega|^2}}. \tag{10}$$

eq:radialrho

Let's look at two interesting limits.

First, $\alpha \to 1$:

$$\begin{aligned} q &\to \frac{1}{\left|1-|\omega|^2\right|}, \\ \implies 1+|\omega|^2-q^{-1} &\to \left(1+|\omega|^2\right) - \left|1-|\omega|^2\right| = \begin{cases} 2\,|\omega|^2 & \text{for } |\omega|<1, \\ 2 & \text{for } |\omega|>1. \end{cases} \\ \implies \Phi &\to \begin{cases} 0 & \text{for } |\omega|<1, \\ -\frac{\ln|\omega|}{2\pi} & \text{for } |\omega|>1. \end{cases} \end{aligned} \tag{11}$$

eq:rszmphiat

This is harmonic everywhere except $|\omega| = 1$. Applying Gauss' law to a circular loop of radius greater than 1, centred at the origin, tells us that the total charge enclosed is 1. Therefore:

$$\rho(|\omega|) \to \delta(|\omega|-1) \quad \text{as} \quad \alpha \to 1. \tag{12}$$

eq:rhoato1

Now, $\alpha \to \infty$:

$$q = \frac{1}{1+|\omega|^2}\left[1 + \frac{2\,|\omega|^2}{\alpha\left(1+|\omega|^2\right)^2} + \frac{8\,|\omega|^4}{\alpha^2\left(1+|\omega|^2\right)^4} + \mathcal{O}(\alpha^{-3})\right], \tag{13}$$

eq:rsatoinft

which leads to

$$1+|\omega|^2-q^{-1} = \frac{2\,|\omega|^2}{\alpha\left(1+|\omega|^2\right)}\left[1 + \frac{2\,|\omega|^2}{\alpha\left(1+|\omega|^2\right)^2} + \mathcal{O}(\alpha^{-2})\right]. \tag{14}$$

eq:zmatoinft

Dropping constants:

$$\Phi = -\frac{\ln\left(1+|\omega|^2\right)}{4\pi} + \mathcal{O}(\alpha^{-1}). \tag{15}$$

eq:phiatoinf

This results in

$$\rho(|\omega|) \to \frac{2\,|\omega|}{\left(1+|\omega|^2\right)^2} \quad \text{as} \quad \alpha \to \infty. \tag{16}$$

eq:rhoatoinf

This should be $\delta(|\omega|)$.

3 Simplified derivation

simplederiv

In this section, we will present a simplified version of the derivation. We will make the following simplifying assumption: at some point, we will treat x as annealed, rather than quenched, disorder:

$$\left\langle \ln(\cdots)\right\rangle_x = \ln\left\langle \cdots \right\rangle_x . \tag{17}$$

eq:annealed

We will justify this assumption in §4 using the replica trick. We will see that, with a replica symmetric ansatz, the saddle point has zero off-diagonal replica overlaps. This means that there is no coupling between the replicas so it gives identical results to the annealed calculation.

We start with the representation of the determinant in (59). However, the matrix in (7) is not positive-definite when ω is equal to one of the eigenvalues. We can fix this by adding $\epsilon^2 I$ and letting $\epsilon \to 0$ at the end.

$$Z_{\hat{A}}(\omega,\overline{\omega}) = \int \prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \exp\left(-z^\dagger(\overline{\omega}-\hat{A}^{\mathrm{T}})(\omega-\hat{A})z - \epsilon^2 z^\dagger z\right). \tag{18}$$

eq:partfnint

Looking at the expression (4) for $\hat{A}$, we make the change of variables $z = (xx^{\mathrm{T}})w/P$.

$$\begin{gathered} Z_{\hat{A}}(\omega,\overline{\omega}) = \det\left(\frac{xx^{\mathrm{T}}}{P}\right)^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \mathrm{e}^{-F/P^2} \\ F = w^\dagger x(\overline{\omega}-\mathcal{S}^\dagger)x^{\mathrm{T}}x(\omega-\mathcal{S})x^{\mathrm{T}}w + \epsilon^2 w^\dagger xx^{\mathrm{T}}xx^{\mathrm{T}}w. \end{gathered} \tag{19}$$

eq:partfnint

We now take advantage of (61) by introducing two standard complex Gaussian random vectors ($C = I$ in (60)), u and v:

$$\begin{gathered} Z_{\hat{A}}(\omega,\overline{\omega}) = \det\left(\frac{xx^{\mathrm{T}}}{P}\right)^2 \int \prod_i \frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \left\langle \mathrm{e}^{\mathrm{i}F'/P}\right\rangle_{u,v}, \\ F' = w^\dagger x(\overline{\omega}-\mathcal{S}^\dagger)x^{\mathrm{T}}u + u^\dagger x(\omega-\mathcal{S})x^{\mathrm{T}}w + \epsilon w^\dagger xx^{\mathrm{T}}v + \epsilon v^\dagger xx^{\mathrm{T}}w. \end{gathered} \tag{20}$$

eq:partfnint

Over most of the integration domain, we expect the real inner products ($w^\dagger w$,$u^\dagger u$,...) will be $\mathcal{O}(N)$, whereas the complex inner products ($w^\dagger u$,$w^{\mathrm{T}} w$,...) will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ, σ and τ, which are zero mean Gaussian random vectors:

$$\begin{aligned} \rho &= x^{\mathrm{T}} w, & \left\langle \overline{\rho}_\mu \rho_\nu \right\rangle_x &= N r \delta_{\mu\nu}, & r &= \frac{w^\dagger w}{N}, \\ \sigma &= x^{\mathrm{T}} u, & \left\langle \overline{\sigma}_\mu \sigma_\nu \right\rangle_x &= N s \delta_{\mu\nu}, & s &= \frac{u^\dagger u}{N}, \\ \tau &= x^{\mathrm{T}} v, & \left\langle \overline{\tau}_\mu \tau_\nu \right\rangle_x &= N t \delta_{\mu\nu}, & t &= \frac{v^\dagger v}{N}, \end{aligned} \tag{21}$$

eq:rstdef

with all other covariances negligible in the large N limit. We can now write

$$\begin{aligned} \left\langle \ln Z_{\hat{A}}(\omega, \overline{\omega}) \right\rangle_x =& 2 \left\langle \ln \det \left(\frac{x x^{\mathrm{T}}}{P} \right) \right\rangle_x \\ &+ \left\langle \ln \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi} \right] \mathrm{e}^{-N(s+t) - \xi^\dagger A \xi} \right\rangle_x , \\ \text{where} \quad \xi =& \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\ A =& -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - \mathcal{S}^\dagger & \epsilon \\ \omega - \mathcal{S} & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}, \\ \left\langle \xi \xi^\dagger \right\rangle_x =& C = N \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}. \end{aligned} \tag{22}$$

eq:potwuv

As we only care about the part of Φ that depends on ω, we can ignore the first term. We will simplify the second term using the assumption (17), the identity (63) and the relation

$$\int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \right] f(r) = \frac{N^N}{\Gamma(N)} \int \mathrm{d}r \, r^{N-1} f(r), \tag{23}$$

eq:spherical

along with similar ones for u and v, to get

$$\begin{aligned} \Phi(\omega, \overline{\omega}) &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} (rst)^N \mathrm{e}^{-N(s+t)} \left\langle \mathrm{e}^{-\xi^\dagger A \xi} \right\rangle_x \\ &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{\mathrm{d}r}{r} \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \frac{\exp[N(\ln(rst) - s - t)]}{\det(I + CA)}. \end{aligned} \tag{24}$$

eq:phiintsim

$(C^{-1} + A)$ IS NON-NORMAL!

As $\mathcal{S}$ is unitary, all the blocks in these matrices commute. Therefore, we can evaluate the determinant with some help from [2]. Also noting that the eigenvalues of $\mathcal{S}$ are

$\exp(2\pi \mathrm{i} k/P)$, with $k \in \mathbb{Z}_P$:

$$\begin{aligned}
\ln\det(I + CA) &= \ln\det\left[\frac{1}{\alpha}\begin{pmatrix} \alpha & -\mathrm{i}r(\overline{\omega} - \mathcal{S}^\dagger) & -\mathrm{i}\epsilon r \\ -\mathrm{i}s(\omega - \mathcal{S}) & \alpha & 0 \\ -\mathrm{i}\epsilon t & 0 & \alpha \end{pmatrix}\right] \\
&= \ln\det\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathcal{S}^\dagger)(\omega - \mathcal{S})}{\alpha^2}\right] \\
&= \sum_{k=0}^{P-1} \ln\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega - \mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2}\right] \qquad (25) \\
&= \frac{P}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \, \ln\left[\frac{\alpha^2 + \epsilon^2 rt + rs(\overline{\omega} - \mathrm{e}^{-\mathrm{i}\phi})(\omega - \mathrm{e}^{\mathrm{i}\phi})}{\alpha^2}\right] \\
&= \frac{P}{2\pi\mathrm{i}}\int \frac{\mathrm{d}\zeta}{\zeta} \, \ln\left[\frac{G(\zeta)}{\alpha^2\zeta}\right],
\end{aligned}$$

eq:detsimple

where the function $G(\zeta)$ is defined in §C, in particular (71). From (72), we know that only one of the zeros of $G(\zeta)$ will lie inside the contour. We define:

$$\{\zeta_>, \zeta_<\} = \{\zeta_+, \zeta_-\}, \qquad |\zeta_>| \geq 1, \qquad |\zeta_<| \leq 1. \tag{26}$$

eq:zetainout

If we factorise $G(\zeta)$, this contour integral is of the form discussed in §B, the result of which appears in (70). We find that

$$\ln\det(I + CA) = P\ln\left(\frac{rs\overline{\omega}\zeta_>}{\alpha^2}\right) = P\ln\left(\frac{rs\omega}{\alpha^2\zeta_<}\right). \tag{27}$$

eq:detsimple

Now, if we use the saddle-point approximation of the integrals over r, s and t in (24), which becomes exact in the limit of large N and P, we find

$$\Phi(\omega, \overline{\omega}) = \frac{1}{4\pi}\max_{r,s,t}\left[\ln(rst) - s - t - \alpha\ln\left(\frac{rs\omega}{\alpha^2\zeta_<}\right)\right]. \tag{28}$$

eq:phisaddle

One can show that (see §4, in particular (54) and (55)) the maximum has

$$r \sim \mathcal{O}(\epsilon^{-1}), \qquad s \sim \mathcal{O}(\epsilon), \qquad t \sim \mathcal{O}(1), \qquad rs \sim \mathcal{O}(1). \tag{29}$$

eq:saddleOe

If we take $\epsilon \to 0$, we find that Φ depends on r, s and t in the combinations rs and t:

$$\begin{aligned}
\Phi(\omega, \overline{\omega}) &= \frac{1}{4\pi}\max_{rs,t}\left[(1-\alpha)\ln(rs) + \ln t - t - \alpha\ln\left(\frac{\omega}{\alpha^2\zeta_<}\right)\right] \\
\frac{\partial\Phi}{\partial(rs)} &= \frac{1-\alpha}{rs} + \frac{\alpha^3}{rsG'(\zeta_<)}, \qquad (30) \\
\frac{\partial\Phi}{\partial t} &= \frac{1}{t} - 1.
\end{aligned}$$

eq:phimaxsim

Setting these derivatives to zero gives

$$t = 1, \qquad G'(\zeta_\pm) = \frac{\alpha^3}{\alpha - 1}, \tag{31}$$

eq:saddlecon

which can be solved for rs provided that

$$|\zeta_-| \leq 1, \tag{32}$$

eq:saddlereq

in which case

$$rs = \frac{\alpha^2 \left[-(\alpha-1)\left(1+|\omega|^2\right) \pm \sqrt{(\alpha-1)^2\left(1+|\omega|^2\right)^2 + (2\alpha-1)\left(1-|\omega|^2\right)^2} \right]}{(\alpha-1)\left(1-|\omega|^2\right)^2}, \tag{33}$$

eq:saddlesol

Using (73) and (74)

$$\zeta_- = \frac{1+|\omega|^2}{2\overline{\omega}} - \frac{\alpha^2}{2(\alpha-1)\overline{\omega} rs}, \tag{34}$$

eq:saddlesol

and (32) requires that we pick the positive root for rs. Furthermore, with this choice, (32) is satisfied everywhere. Finally, dropping some constant pieces,

$$\Phi(\omega, \overline{\omega}) = \frac{1}{4\pi} \left[(1-\alpha)\ln(rs) - \alpha \ln\left(\frac{\omega}{\zeta_<}\right) \right]. \tag{35}$$

eq:saddlesol

This will simplify if expressed in terms of $q = (\alpha - 1)rs/\alpha^2$.

4 Full, replica-tastic derivation

replicader

The starting point for this version of the derivation will be (20) and (22):

$$\begin{aligned}
\Phi(\omega, \overline{\omega}) &= \text{const.} + \frac{1}{4\pi N}\left\langle \ln \widetilde{Z} \right\rangle_x, \\
\widetilde{Z} &= \int \prod_i \left[\frac{\mathrm{d}w_i \mathrm{d}\overline{w}_i}{2\pi} \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i}{2\pi} \frac{\mathrm{d}v_i \mathrm{d}\bar{v}_i}{2\pi} \right] \mathrm{e}^{-F''}, \\
F'' &= u^\dagger u + v^\dagger v - \frac{\mathrm{i}}{P}\left[w^\dagger x(\overline{\omega} - \mathcal{S}^\dagger)x^{\mathrm{T}} u + \epsilon w^\dagger x x^{\mathrm{T}} v + (\text{c.c.}) \right].
\end{aligned} \tag{36}$$

eq:phiint

We will use the replica trick, i.e. we rewrite the logarithm as

$$\ln \widetilde{Z} = \left. \frac{\partial(\widetilde{Z}^n)}{\partial n} \right|_{n=0}. \tag{37}$$

eq:replicatr

For integer n, we can compute $\widetilde{Z}^n$ by creating n replicas of the system. We then let $n \to 0$ after averaging over x. We index these replicas with $a, b = 1, \ldots, n$:

$$
\begin{aligned}
\Phi(\omega, \overline{\omega}) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \left\langle \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia} \mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia} \mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia} \mathrm{d}\bar{v}_{ia}}{2\pi} \right] \mathrm{e}^{-F'''} \right\rangle_x \Bigg|_{n=0}, \\
F''' &= \sum_a u_a^\dagger u_a + v_a^\dagger v_a - \frac{\mathrm{i}}{P} \left[w_a^\dagger x (\overline{\omega} - \mathcal{S}^\dagger) x^{\mathrm{T}} u_a + \epsilon w_a^\dagger x x^{\mathrm{T}} v_a + (\text{c.c.}) \right].
\end{aligned}
\tag{38}
$$

eq:phirep

Over most of the integration domain, we expect the Hermitian overlaps $(w_a^\dagger w_b, u_a^\dagger u_b, \ldots)$ will be $\mathcal{O}(N)$, whereas the non-Hermitian overlaps $(w_a^\dagger u_b, w_a^{\mathrm{T}} w_b, \ldots)$ will be $\mathcal{O}(\sqrt{N})$. We define some new variables, ρ_a, σ_a and τ_a, which are zero mean Gaussian random vectors:

$$
\begin{aligned}
\rho_a &= x^{\mathrm{T}} w_a, \quad \langle \bar{\rho}_{\mu a} \rho_{\nu b} \rangle_x = N \delta_{\mu\nu} R_{ab}, \quad R_{ab} = \frac{w_a^\dagger w_b}{N}, \\
\sigma_a &= x^{\mathrm{T}} u_a, \quad \langle \bar{\sigma}_{\mu a} \sigma_{\nu b} \rangle_x = N \delta_{\mu\nu} S_{ab}, \quad S_{ab} = \frac{u_a^\dagger u_b}{N}, \\
\tau_a &= x^{\mathrm{T}} v_a, \quad \langle \bar{\tau}_{\mu a} \tau_{\nu b} \rangle_x = N \delta_{\mu\nu} T_{ab}, \quad T_{ab} = \frac{v_a^\dagger v_b}{N},
\end{aligned}
\tag{39}
$$

eq:reprstdef

with all other covariances negligible in the large N limit.

$$
\begin{aligned}
\Phi(\omega, \overline{\omega}) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia} \mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia} \mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia} \mathrm{d}\bar{v}_{ia}}{2\pi} \right] \left\langle \mathrm{e}^{-N \operatorname{Tr}(S+T) - \xi^\dagger A \xi} \right\rangle_x \Bigg|_{n=0}, \\
&= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[\frac{\mathrm{d}w_{ia} \mathrm{d}\overline{w}_{ia}}{2\pi} \frac{\mathrm{d}u_{ia} \mathrm{d}\bar{u}_{ia}}{2\pi} \frac{\mathrm{d}v_{ia} \mathrm{d}\bar{v}_{ia}}{2\pi} \right] \mathrm{e}^{-N E(R,S,T)} \Bigg|_{n=0},
\end{aligned}
\tag{40}
$$

eq:phirepxi

where

$$
\begin{aligned}
E(R, S, T) &= \operatorname{Tr}(S + T) + \frac{1}{N} \ln\det(I + CA), \\
\xi &= \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\
A &= -\frac{\mathrm{i}}{P} \begin{pmatrix} 0 & \overline{\omega} - \mathcal{S}^\dagger & \epsilon \\ \omega - \mathcal{S} & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \otimes I, \\
\left\langle \xi \xi^\dagger \right\rangle_x &= C = N I \otimes \begin{pmatrix} R & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix}.
\end{aligned}
\tag{41}
$$

eq:overlapen

It will be helpful to separate the integrals over w, u and v into an integral over values of w, u and v with the same overlap and an integral over values of the overlap. This can be done by inserting factors like the following into the integral:

$$
\int \mathrm{d}R_{ab} N^{n^2} \delta(N R_{ab} - w_a^\dagger w_b) = 1. \tag{42}
$$

eq:overlapde

We define

$$S(R)=\frac{1}{N}\int\prod_{ia}\left[\frac{\mathrm{d}w_{ia}\mathrm{d}\overline{w}_{ia}}{2\pi}\right]N^{n^2}\delta(NR_{ab}-w_a^\dagger w_b)=\ln\det R+\text{const.}\tag{43}$$

eq:overlapen

with the final expression valid in the large N limit. Then (40) reduces to

$$\Phi(\omega,\overline{\omega})=\frac{1}{4\pi N}\frac{\partial}{\partial n}\int\mathrm{d}R_{ab}\mathrm{d}S_{ab}\mathrm{d}T_{ab}\,\mathrm{e}^{-N(E(R,S,T)-S(R)-S(S)-S(T))}\bigg|_{n=0}.\tag{44}$$

eq:phioverla

We will perform this integral in the large N limit with the saddle point method.

We make the following, replica-symmetric ansätze for the saddle-point:

$$R_{ab}=r_0\delta_{ab}+r_1,\qquad S_{ab}=s_0\delta_{ab}+s_1,\qquad T_{ab}=t_0\delta_{ab}+t_1.\tag{45}$$

eq:RSansatze

A matrix of this form has $(n-1)$ eigenvalues equal to r_0 and one eigenvalue equal to (r_0+nr_1). Therefore,

$$S(R)=\ln\det R=(n-1)\ln r_0+\ln(r_0+nr_1)=n\ln r_0+\frac{nr_1}{r_0}+\mathcal{O}(n^2).\tag{46}$$

eq:RSentropy

The replica symmetric form of R,S,T and the unitarity of $\mathcal{S}$ means that all the blocks in (41) commute. Then, according to [2],

$$\begin{aligned}\ln\det(I+CA)&=\ln\det\left[\frac{1}{\alpha}\begin{pmatrix}\alpha & -\mathrm{i}(\overline{\omega}-\mathcal{S}^\dagger)\otimes R & -\mathrm{i}\epsilon I\otimes R\\ -\mathrm{i}(\omega-\mathcal{S})\otimes S & \alpha & 0\\ -\mathrm{i}\epsilon I\otimes T & 0 & \alpha\end{pmatrix}\right]\\ &=\ln\det\left[\frac{\alpha^2+\epsilon^2 I\otimes RT+(\overline{\omega}-\mathcal{S}^\dagger)(\omega-\mathcal{S})\otimes RS}{\alpha^2}\right].\end{aligned}\tag{47}$$

eq:detrep

Note that

$$(RS)_{ab}=r_0s_0\delta_{ab}+r_0s_1+r_1s_0+\mathcal{O}(n),\tag{48}$$

eq:overlappr

and similar for RT, so these will have a similar eigenvalue structure to R, with the same eigenvectors. Also, the eigenvalues of $\mathcal{S}$ are $\exp(2\pi\mathrm{i}k/P)$, with $k\in\mathbb{Z}_P$. Therefore:

$$\begin{aligned}\ln\det(I+CA)&=\sum_{k=0}^{P-1}n\left\{\ln\left[\frac{\alpha^2+\epsilon^2r_0t_0+r_0s_0(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2}\right]\right.\\ &\qquad\left.+\frac{\epsilon^2(r_0t_1+r_1t_0)+(r_0s_1+r_1s_0)(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}{\alpha^2+\epsilon^2r_0t_0+r_0s_0(\overline{\omega}-\mathrm{e}^{-2\pi\mathrm{i}k/P})(\omega-\mathrm{e}^{2\pi\mathrm{i}k/P})}\right\}\\ &\quad+\mathcal{O}(n^2)\\ &=\frac{nP}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\left\{\ln\left[\frac{\alpha^2+\epsilon^2r_0t_0+r_0s_0(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}{\alpha^2}\right]\right.\\ &\qquad\left.+\frac{\epsilon^2(r_0t_1+r_1t_0)+(r_0s_1+r_1s_0)(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}{\alpha^2+\epsilon^2r_0t_0+r_0s_0(\overline{\omega}-\mathrm{e}^{-\mathrm{i}\phi})(\omega-\mathrm{e}^{\mathrm{i}\phi})}\right\}\\ &=\frac{nP}{2\pi\mathrm{i}}\int\frac{\mathrm{d}\zeta}{\zeta}\left\{\ln\left[\frac{G(\zeta)}{\alpha^2\zeta}\right]\right.\\ &\qquad\left.+\frac{\epsilon^2(r_0t_1+r_1t_0)\zeta+(r_0s_1+r_1s_0)(\overline{\omega}\zeta-1)(\omega-\zeta)}{G(\zeta)}\right\},\end{aligned}\tag{49}$$

eq:detrepint

where the function $G(\zeta)$ is defined in §C, in particular (71). The first integral was computed in (27). The second can be computed with the residue theorem. The integrand has poles at $\zeta \in \{0, \zeta_+, \zeta_-\}$, with only one of the last two lying inside the contour (see (73)). Using the definitions (26),

$$\begin{aligned}
\int \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta}\, & \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta + (r_0 s_1 + r_1 s_0)(\overline{\omega}\zeta - 1)(\omega - \zeta)}{G(\zeta)} \\
&= \frac{(r_0 s_1 + r_1 s_0)(-\omega)}{G(0)} + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta_< + (r_0 s_1 + r_1 s_0)(\overline{\omega}\zeta_< - 1)(\omega - \zeta_<)}{\zeta_< G'(\zeta_<)} \\
&= \frac{r_0 s_1 + r_1 s_0}{r_0 s_0}\left(1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_<)}\right) + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)}{G'(\zeta_<)}.
\end{aligned} \tag{50}$$

eq:repcontou

Combining all of this gives:

$$\begin{aligned}
\Phi(\omega, \overline{\omega}) = \frac{1}{4\pi} \max_{r_{0,1}, s_{0,1}, t_{0,1}} \Bigg\{ & \ln(r_0 s_0 t_0) + \frac{r_1}{r_0} + \frac{s_1}{s_0} + \frac{t_1}{t_0} - (s_0 + s_1 + t_0 + t_1) \\
& - \alpha \ln\left(\frac{r_0 s_0 \omega}{\alpha^2 \zeta_<}\right) - \frac{\alpha(r_0 s_1 + r_1 s_0)}{r_0 s_0}\left(1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_<)}\right) \\
& - \frac{\alpha\epsilon^2(r_0 t_1 + r_1 t_0)}{G'(\zeta_<)} \Bigg\}.
\end{aligned} \tag{51}$$

eq:phimaxrep

To find the maximum, we must set the following derivatives to zero

$$\begin{aligned}
\frac{\partial(4\pi\Phi)}{\partial r_1} &= \frac{1}{r_0} - \frac{\alpha}{r_0} + \frac{\alpha^3}{r_0 G'(\zeta_<)}, \\
\frac{\partial(4\pi\Phi)}{\partial s_1} &= \frac{1}{s_0} - 1 - \frac{\alpha}{s_0} + \frac{\alpha(\alpha^2 + \epsilon^2 r_0 t_0)}{s_0 G'(\zeta_<)}, \\
\frac{\partial(4\pi\Phi)}{\partial t_1} &= \frac{1}{t_0} - 1 - \frac{\alpha\epsilon^2 r_0}{G'(\zeta_<)},
\end{aligned} \tag{52}$$

eq:phidiff1

as well as

$$\begin{aligned}
\frac{\partial(4\pi\Phi)}{\partial r_0} &= \frac{\partial(4\pi\Phi)}{\partial r_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\
\frac{\partial(4\pi\Phi)}{\partial s_0} &= \frac{\partial(4\pi\Phi)}{\partial s_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\
\frac{\partial(4\pi\Phi)}{\partial t_0} &= \frac{\partial(4\pi\Phi)}{\partial t_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1.
\end{aligned} \tag{53}$$

eq:phidiff0

This has the solution $r_1 = s_1 = t_1 = 0$, which justifies the annealed assumption in (17). Solving (52) gives

$$G'(\zeta_<) = \frac{\alpha^3}{\alpha - 1}, \qquad s_0 = \frac{(\alpha - 1)\epsilon^2 r_0}{\alpha^2 + (\alpha - 1)\epsilon^2 r_0}, \qquad t_0 = 1 - s_0. \tag{54}$$

eq:repsaddle

Expression (76) for $G'(\zeta_\pm)$ tells us that it must be ζ_- that lies inside the unit circle. Solving for r_0 is a mess, but we can see that no solutions exist for $r_0 < \mathcal{O}(\epsilon^{-1})$ or $r_0 > \mathcal{O}(\epsilon^{-1})$ as $\epsilon \to 0$, because the last equation would reduce to

$$\begin{aligned} r_0 &\sim \mathcal{O}(\epsilon^{-1+\delta}) \implies & \alpha^2 &= \frac{\alpha^3}{\alpha - 1}, \\ r_0 &\sim \mathcal{O}(\epsilon^{-1-\delta}) \implies & \frac{(\alpha-1)\left(1-|\omega|^2\right)^2 \epsilon^2 r_0^2}{\alpha^2 + (\alpha-1)\epsilon^2 r_0} &= \frac{\alpha^3}{\alpha-1}. \end{aligned} \tag{55}$$ eq:r0e

Combined with (54), this justifies (29).

5 Relation to canonical angles

sec:canonang

What would happen if we replaced the shift matrix, $\mathcal{S}$, with some other unitary matrix? Looking at (25) or (49), we see that it is only the eigenvalue density of $\mathcal{S}$ that matters. Therefore, any unitary matrix with eigenvalue uniformly distributed around the unit circle would lead the the same eigenvalue distribution for $\hat{A}$. Furthermore, following the argument from (25) to (28), or from (49) to (51), we see that the eigenvalue distribution of $\hat{A}$ depends linearly on the eigenvalue density of $\mathcal{S}$. Therefore, if $\mathcal{S}$ were chosen randomly we would just use the mean eigenvalue distribution of a random unitary matrix. As this is uniformly distributed around the unit circle, this would also lead the the same eigenvalue distribution for $\hat{A}$.

Now, let us introduce the singular-value-decomposition of x:

$$x = UDV^{\mathrm{T}}, \tag{56}$$ eq:svd

where U is an $N \times N$ orthogonal matrix, D is an $N \times N$ diagonal matrix and V is an $N \times P$ row-orthogonal matrix:

$$UU^{\mathrm{T}} = U^{\mathrm{T}}U = VV^{\mathrm{T}} = \mathbf{I}. \tag{57}$$ eq:svdorth

We can write (4) as

$$\begin{aligned} \hat{A} &= UDV^{\mathrm{T}}\mathcal{S}VDU^{\mathrm{T}}(UDV^{\mathrm{T}}VDU^{\mathrm{T}})^{-1} \\ &= UDV^{\mathrm{T}}\mathcal{S}VDU^{\mathrm{T}}UD^{-2}U^{\mathrm{T}} \\ &= UDV^{\mathrm{T}}\mathcal{S}VD^{-1}U^{\mathrm{T}} \\ &= (UD)(V^{\mathrm{T}}\mathcal{S}V)(UD)^{-1}. \end{aligned} \tag{58}$$ eq:Aestsvd

Thus, $\hat{A}$ is similar to $V^{\mathrm{T}}\mathcal{S}V$, and therefore has the same eigenvalues. Now, V is a random $N \times P$ row-orthogonal matrix. If $\mathcal{S}$ is chosen randomly, $V' = \mathcal{S}V$ is another random $N \times P$ row-orthogonal matrix, independent of V.

We can think of V and V' as basis vectors for N-dimensional subspaces of a P-dimensional space and $V^{\mathrm{T}}\mathcal{S}V = V^{\mathrm{T}}V'$ as the matrix of inner-products of these vectors. The singular values of this matrix are known as the canonical angles. They are used in high-dimensional data-analysis and the case of two random, independent subspaces is an important null-model.

In our case, however, we are interested in the eigenvalues, not the singular values.

Appendices

A Complex Gaussian integrals

:compgauss

First, Let's get all of the factors of 2 straight. Note that if we write $z = x + \mathrm{i}y$, then $\mathrm{d}z\mathrm{d}\bar{z} = 2\mathrm{d}x\mathrm{d}y$. Let H be a positive-definite, $N \times N$ Hermitian matrix (or just a normal matrix whose eigenvalues have positive real parts). Consider an integral of the form

$$\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right).$$

We can diagonalise H with a unitary change of variables:

$$\begin{aligned}
\int \left(\prod_i \mathrm{d}z_i \mathrm{d}\bar{z}_i \right) \exp\left(-z^\dagger H z\right) &= \prod_i \int \mathrm{d}z_i \mathrm{d}\bar{z}_i \, \exp\left(-\lambda_i \left|z_i\right|^2\right) \\
&= \prod_i \int \mathrm{d}x_i \mathrm{d}y_i \, 2 \exp\left(-\lambda_i \left(x_i^2 + y_i^2\right)\right) \\
&= \prod_i \frac{2\pi}{\lambda_i} \\
&= \frac{(2\pi)^N}{\det H}.
\end{aligned} \tag{59}$$

eq:compgausi

The proper normalisation for a Gaussian distribution is

$$P(z, z^\dagger)\mathrm{d}z\mathrm{d}z^\dagger = \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger C^{-1} z\right)}{\det C}. \tag{60}$$

eq:compgauss

By completing the square, we can see that

$$\left\langle \exp\left(\zeta^\dagger z \pm z^\dagger \zeta\right) \right\rangle = \exp\left(\pm \zeta^\dagger C \zeta\right) \tag{61}$$

eq:compgauss

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\left\langle z z^\dagger \right\rangle = C. \tag{62}$$

eq:compgausc

Now consider an integral of the form

$$\begin{aligned}
\left\langle \exp\left(-z^\dagger A z\right) \right\rangle &= \int \left(\prod_i \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2\pi} \right) \frac{\exp\left(-z^\dagger (C^{-1} + A) z\right)}{\det C} \\
&= \left(\det C \det\left(C^{-1} + A\right) \right)^{-1} \\
&= \det\left(I + CA\right)^{-1}.
\end{aligned} \tag{63}$$

eq:compgauss

ONLY WORKS IF $(C^{-1}+A)$ IS POS DEF, OR AT LEAST NORMAL!

Does it matter if the matrix is normal? Suppose we diagonalise with a non-unitary transformation. What Jacobian factor would we pick up?

$$\begin{aligned} z' &= Sz, \\ \bar{z}' &= S^{-1}\bar{z}, \end{aligned} \implies \begin{aligned} x' &= \frac{z'+\bar{z}'}{2} = \frac{S+S^{-1}}{2}x - \frac{S-S^{-1}}{2\mathrm{i}}y, \\ y' &= \frac{z'-\bar{z}'}{2\mathrm{i}} = \frac{S-S^{-1}}{2\mathrm{i}}x - \frac{S+S^{-1}}{2}y, \end{aligned}$$

$$\begin{aligned} \implies \det J &= \det \begin{pmatrix} \frac{S+S^{-1}}{2} & -\frac{S-S^{-1}}{2\mathrm{i}} \\ \frac{S-S^{-1}}{2\mathrm{i}} & \frac{S+S^{-1}}{2} \end{pmatrix} \\ &= \det \left[\left(\frac{S+S^{-1}}{2}\right)^2 - \left(\frac{S-S^{-1}}{2\mathrm{i}}\right)^2 \right] \\ &= \det SS^{-1} = 1. \end{aligned}$$

The change in contours can be undone, as the integrand is analytic in x and y.

B Contour integrals for determinants

contourints

In evaluating determinants in §3, we will come across contour integrals of the form

$$I(\zeta_{\gtrless}) = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta} \ln\left(\frac{\gamma(\zeta_> - \zeta)(\zeta - \zeta_<)}{\zeta}\right), \tag{64}$$ eq:contourin

where the contour is the unit circle in a counter-clockwise direction and $|\zeta_{\gtrless}| \gtrless 1$. We choose the branch of the logarithm so that

$$\arg\left(\frac{\zeta-\zeta_<}{\zeta_<}\right), \arg\left(\frac{\zeta}{\zeta_<}\right) \in [0, 2\pi], \tag{65}$$ eq:branch

and we define $\theta = \arg z$. The branch cut at $\zeta = \zeta_>$ will not matter. The branch cuts and contour are shown in fig.1(a).

We write

$$I(\zeta_{\gtrless}) = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta} \ln\gamma(\zeta_> - \zeta) + \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta} \ln\left(\frac{\zeta-\zeta_<}{\zeta}\right), \tag{66}$$ eq:contoursp

For the first part, we can use the original contour C as in fig.1(b). Using the residue theorem:

$$\int \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta} \ln\gamma(\zeta_> - \zeta) = \ln\gamma\zeta_>. \tag{67}$$ eq:intout

For the second part, we can deform the contour to C' as in fig.1(c):

$$C': \quad \begin{aligned} &\zeta = \epsilon\mathrm{e}^{\mathrm{i}\phi}, && \phi \in [\theta, \theta+2\pi], \\ &\zeta = x\mathrm{e}^{\mathrm{i}(\theta+2\pi)}, && x \in [\epsilon, |\zeta_<| - \epsilon], \\ &\zeta = \zeta_< + \epsilon\mathrm{e}^{\mathrm{i}\phi}, && \phi \in [\theta-\pi, \theta+\pi], \\ &\zeta = (|\zeta_<| - x)\mathrm{e}^{\mathrm{i}(\theta+2\pi)}, && x \in [\epsilon, |\zeta_<| - \epsilon]. \end{aligned} \tag{68}$$ eq:contout

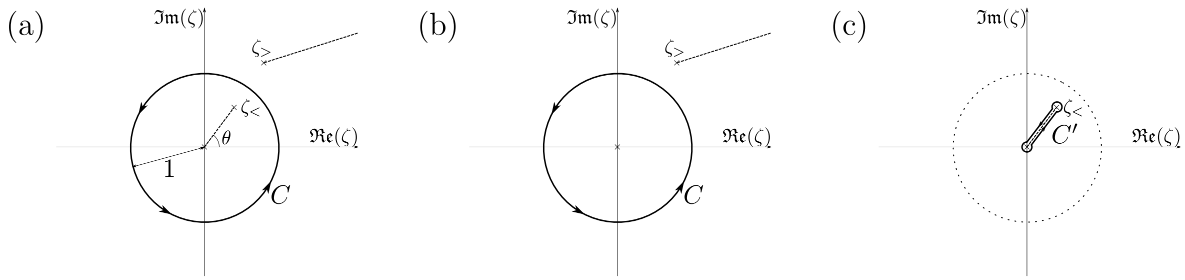

fig:branchcut

Figure 1: (a) Contours used to evaluate (64) and and branch cuts (65). (b) contour used with singularities at $\zeta_>$. (c) contour used with singularities at $\zeta_<, 0$. Branch cuts indicated by dashed line. Poles and branch points indicated by crosses.

fig:contours

The integral over the third part vanishes as $\epsilon \to 0$. The second and fourth parts would cancel, if not for the discontinuity in the logarithm of the denominator (the logarithm of the numerator has no discontinuity, due to (65)). This leaves:

$$\begin{aligned}
\oint_{C'} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,\zeta} \ln\left(\frac{\zeta-\zeta_<}{\zeta}\right) &= \int_\theta^{\theta+2\pi} \frac{\mathrm{d}\phi}{2\pi} \ln\left(\frac{\epsilon\mathrm{e}^{\mathrm{i}\phi}-\zeta_<}{\epsilon\mathrm{e}^{\mathrm{i}\phi}}\right) + \int_\epsilon^{|\zeta_<|-\epsilon} \frac{\mathrm{d}x}{2\pi\mathrm{i}\,x} \operatorname{disc}\ln\left(\frac{1}{\zeta}\right) \\
&= \ln\left(\frac{\mathrm{e}^{\mathrm{i}\pi}\zeta_<}{\epsilon}\right) - \int_\theta^{\theta+2\pi} \frac{\mathrm{i}\phi\,\mathrm{d}\phi}{2\pi} - \int_\epsilon^{|\zeta_<|} \frac{\mathrm{d}x}{x} \\
&= \ln\left(\frac{\zeta_<}{\epsilon}\right) + \mathrm{i}\pi - \mathrm{i}(\theta+\pi) - \ln\left(\frac{|\zeta_<|}{\epsilon}\right) \\
&= 0.
\end{aligned} \tag{69}$$

eq:intin

Therefore:

$$I(\zeta_\gtrless) = \ln\gamma\zeta_>. \tag{70}$$

eq:countouri

C The quadratic function $G(\zeta)$

sec:Gamma

In evaluating determinants in §3 and §4, we come across the function

$$G(\zeta) = (\alpha^2+\epsilon^2 rt)\zeta + rs(\overline{\omega}\zeta-1)(\omega-\zeta) = -rs\overline{\omega}(\zeta-\zeta_+)(\zeta-\zeta_-). \tag{71}$$

eq:Gammadef

We will collect some useful features of $\zeta_\pm$ here.

First, by comparing the two forms of $G(\zeta)$, we see that:

$$\zeta_+\zeta_- = \frac{\omega}{\overline{\omega}}, \tag{72}$$

eq:zpzm

$$\zeta_+ + \zeta_- = \frac{\alpha^2+\epsilon^2 rt + rs\left(1+|\omega|^2\right)}{rs\overline{\omega}}, \tag{73}$$

eq:zppzm

$$G'(\zeta_\pm) = \mp rs\overline{\omega}(\zeta_+-\zeta_-), \tag{74}$$

eq:Gprime

and (72) tells us that $|\zeta_+|\,|\zeta_-| = 1$. Solving the equation $G(\zeta_\pm) = 0$ gives

$$\zeta_\pm = \frac{\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2 \pm \sqrt{[\alpha^2 + \epsilon^2 rt + rs(1+|\omega|)^2] - 4(rs)^2\,|\omega|^2}}{2rs\overline{\omega}}, \tag{75}$$

eq:zetapm

$$G'(\zeta_\pm) = \mp\sqrt{\left[\alpha^2 + \epsilon^2 rt + rs\left(1+|\omega|^2\right)\right]^2 - 4(rs)^2\,|\omega|^2}. \tag{76}$$

eq:Gprimesol

Differentiating the equation $G(\zeta_\pm) = 0$ gives

$$\begin{aligned}
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 t\zeta_\pm + s(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{\alpha^2\zeta_\pm}{rG'(\zeta_\pm)},\\
\frac{\partial\zeta_\pm}{\partial s} &= -\frac{r(\overline{\omega}\zeta_\pm - 1)(\omega - \zeta_\pm)}{G'(\zeta_\pm)} &&= \frac{(\alpha^2 + \epsilon^2 rt)\zeta_\pm}{sG'(\zeta_\pm)},\\
\frac{\partial\zeta_\pm}{\partial r} &= -\frac{\epsilon^2 r\zeta_\pm}{G'(\zeta_\pm)}.
\end{aligned} \tag{77}$$

eq:dzpmdrst

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