

Illusions of criticality in high-dimensional autoregressive models

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Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

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1 The problem

sec:theprob

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \quad (1) \quad \text{eq:model}$$

where $x(t)$ is an N -element vector and A is an $N \times N$ matrix.

Suppose we have a sample of P consecutive times, so x is an $N \times P$ matrix. We can perform a least-squares estimate of A by minimising

$$L = \frac{1}{2} \sum_{i,\mu} \left(x_{i\mu+1} - \sum_j A_{ij} x_{j\mu} \right)^2 = \frac{1}{2} \text{Tr} (xU - Ax) (xU - Ax)^T, \quad (2) \quad \text{eq:minL}$$

where U is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e. $x_{iP+1} \sim x_{i1}$, as this will make U orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1} \delta_{\nu P}. \quad (3) \quad \text{eq:Udef}$$

The estimate of A is then

$$A = (xUx^T) (xx^T)^{-1}. \quad (4) \quad \text{eq:Aest}$$

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when $x(t)$ is white noise. Then the true optimal A would be 0. However, with finite P the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega) = \langle \rho_A(\omega) \rangle_x, \quad \rho_A = \sum_{i=1}^N \delta(\omega - \lambda_i), \quad (5) \quad \text{eq:eigdist}$$

where λ_i are the eigenvalues of A in (4) and the components of x are iid gaussian random variables with mean 0 and variance 1.

Following [1], this can be computed from a potential:

$$\rho_A(\omega) = -\nabla^2 \Phi_A(\omega), \quad \Phi_A(\omega) = -\frac{1}{4\pi N} \ln \det [(\bar{\omega} - A^T)(\omega - A)]. \quad (6) \quad \text{eq:potential}$$

We define a partition function

$$\Phi_A(\omega) = \frac{1}{4\pi N} \ln Z_A(\omega), \quad Z_A(\omega) = \det [(\bar{\omega} - A^T)(\omega - A)]^{-1}. \quad (7) \quad \text{eq:partfn}$$

The problem is now to compute $\langle \ln Z_A(\omega) \rangle_z$.

Appendices

A Complex Gaussian integrals

sec:compgauss

Note that if we write $z = x + iy$, then $dzd\bar{z} = 2dx dy$. Let H be a positive-definite, $N \times N$ Hermitian matrix. Consider an integral of the form

$$\int \left(\prod_i dz_i d\bar{z}_i \right) \exp(-z^\dagger H z).$$

We can diagonalise H with a unitary change of variables:

$$\begin{aligned}
\int \left(\prod_i dz_i d\bar{z}_i \right) \exp(-z^\dagger H z) &= \prod_i \int dz_i d\bar{z}_i \exp(-\lambda_i |z_i|^2) \\
&= \prod_i \int dx_i dy_i 2 \exp(-\lambda_i (x_i^2 + y_i^2)) \\
&= \prod_i \frac{2\pi}{\lambda_i} \\
&= \frac{(2\pi)^N}{\det H}.
\end{aligned} \tag{8} \quad \text{eq:compgausint}$$

The proper normalisation for a gaussian distribution is

$$P(z, z^\dagger) dz dz^\dagger = \left(\prod_i \frac{dz_i d\bar{z}_i}{2\pi} \right) \frac{\exp(-z^\dagger C^{-1} z)}{\det C}. \tag{9} \quad \text{eq:compgaussnorm}$$

By completing the square, we can see that

$$\langle \exp(\zeta^\dagger z + z^\dagger \zeta) \rangle = \exp(\zeta^\dagger C \zeta). \tag{10} \quad \text{eq:compgausslin}$$

Taking partial derivatives wrt. ζ_i and $\bar{\zeta}_i$, we find

$$\langle z z^\dagger \rangle = C. \tag{11} \quad \text{eq:compgauscov}$$

Now consider an integral of the form

$$\begin{aligned}
\langle \exp(-z^\dagger A z) \rangle &= \int \left(\prod_i \frac{dz_i d\bar{z}_i}{2\pi} \right) \frac{\exp(-z^\dagger (C^{-1} + A) z)}{\det C} \\
&= (\det C \det (C^{-1} + A))^{-1} \\
&= \det (I + CA)^{-1}.
\end{aligned} \tag{12} \quad \text{eq:compgaussquad}$$

B Contour integrals

sec:contourints

Consider an integral of the form

$$I(z) = \frac{1}{2\pi i} \int \frac{d\zeta}{\zeta} \ln(z - \zeta), \tag{13} \quad \text{eq:contourint}$$

where the contour is the unit circle in a counter-clockwise direction.

References

ers1988asymmetric

- [1] H. J. Sommers, A. Crisanti, H. Sompolinsky, and Y. Stein, “Spectrum of Large Random Asymmetric Matrices,” *Phys. Rev. Lett.* **60** (May, 1988) 1895–1898.