

# Illusions of criticality in high-dimensional autoregressive models

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September 17, 2012

## Abstract

We look at the eigenvalue spectrum of high-dimensional autoregressive models when applied to white-noise.

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## 1 The problem

ec:theprob

Consider a model of the following type

$$x(t+1) = Ax(t) + \text{noise}, \tag{1} \quad \text{eq:model}$$

where  $x(t)$  is an  $N$ -element vector and  $A$  is an  $N \times N$  matrix.

Suppose we have a sample of  $P$  consecutive times, so  $x$  is an  $N \times P$  matrix. We can perform a least-squares estimate of  $A$  by minimising the quantity

$$\frac{1}{2} \sum_{i,\mu} \left( x_{i\mu+1} - \sum_j A_{ij} x_{j\mu} \right)^2 = \frac{1}{2} \text{Tr} (xU - Ax) (xU - Ax)^T, \quad (2) \quad \text{eq:minL}$$

where  $U$  is a shift matrix. It will be useful to use periodic boundary conditions in time, i.e.  $x_{iP+1} \sim x_{i1}$ , as this will make  $U$  orthogonal:

$$U_{\mu\nu} = \delta_{\mu\nu+1} + \delta_{\mu 1} \delta_{\nu P}. \quad (3) \quad \text{eq:Udef}$$

The estimate of  $A$  is then

$$A = (xUx^T) (xx^T)^{-1}. \quad (4) \quad \text{eq:Aest}$$

Suppose we attempted this analysis in a situation where there really is no structure, i.e. when  $x(t)$  is white noise. Then the true optimal  $A$  would be 0. However, with finite  $P$  the estimate (4) will not be zero.

We will look at the average eigenvalue distribution:

$$\rho(\omega, \bar{\omega}) = \langle \rho_A(\omega, \bar{\omega}) \rangle_x, \quad \rho_A(\omega, \bar{\omega}) = \sum_{i=1}^N \delta^{(2)}(\omega - \lambda_i), \quad (5) \quad \text{eq:eigdist}$$

where  $\lambda_i$  are the eigenvalues of  $A$  in (4) and the components of  $x$  are iid gaussian random variables with mean 0 and variance 1. This quantity is most relevant in the limit of large  $N$  and  $P$ . We will keep the quantity  $\alpha = P/N$  fixed in this limit.

Following [1], this can be computed from a potential:

$$\rho_A(\omega, \bar{\omega}) = -\nabla^2 \Phi_A(\omega, \bar{\omega}), \quad \Phi_A(\omega, \bar{\omega}) = -\frac{1}{4\pi N} \ln \det [(\bar{\omega} - A^T)(\omega - A)]. \quad (6) \quad \text{eq:potential}$$

We define a partition function

$$\Phi_A(\omega, \bar{\omega}) = \frac{1}{4\pi N} \ln Z_A(\omega, \bar{\omega}), \quad Z_A(\omega, \bar{\omega}) = \det [(\bar{\omega} - A^T)(\omega - A)]^{-1}. \quad (7) \quad \text{eq:partfn}$$

The problem is now to compute  $\langle \ln Z_A(\omega, \bar{\omega}) \rangle_x$ .

## 2 The solution

cc:solution

In §3 we will present a simplified derivation and in §4 we will fill in the gaps and justify the assumptions used in §4. The result from (34), with some constant pieces dropped, will be:

$$q = \frac{-(\alpha - 1) (1 + |\omega|^2) + \sqrt{(\alpha - 1)^2 (1 + |\omega|^2)^2 + (2\alpha - 1) (1 - |\omega|^2)^2}}{(1 - |\omega|^2)^2}, \quad (8) \quad \text{eq:phisol}$$

$$\zeta_- = \frac{1 + |\omega|^2 - q^{-1}}{2\bar{\omega}},$$

$$\Phi(\omega, \bar{\omega}) = \frac{1}{4\pi} \left[ (1 - \alpha) \ln q - \alpha \ln \left( \frac{\omega}{\zeta_-} \right) \right],$$

The potential has a rotation symmetry. Therefore, we can express the eigenvalue density in terms of

$$\rho(|\omega|) = \int \rho(\omega, \bar{\omega}) |\omega| d\phi = 2\pi |\omega| \rho(\omega, \bar{\omega}) = \frac{\partial}{\partial |\omega|} \left( |\omega| \frac{\partial(2\pi\Phi)}{\partial |\omega|} \right). \quad (9) \quad \text{eq:radialrho}$$

Let's look at two interesting limits.

First,  $\alpha \rightarrow 1$ :

$$\begin{aligned} q &\rightarrow \frac{1}{|1 - |\omega|^2|}, \\ \Rightarrow \zeta_- &\rightarrow \frac{(1 + |\omega|^2) - |1 - |\omega|^2|}{2\bar{\omega}} = \begin{cases} \omega & \text{for } |\omega| < 1, \\ \frac{1}{\bar{\omega}} & \text{for } |\omega| > 1. \end{cases} \\ \Rightarrow \Phi &\rightarrow \begin{cases} 0 & \text{for } |\omega| < 1, \\ -\frac{\ln|\omega|}{2\pi} & \text{for } |\omega| > 1. \end{cases} \end{aligned} \quad (10) \quad \text{eq:rszmphiat}$$

This is harmonic everywhere except  $|\omega| = 1$ . Applying Gauss' law to a circular loop of radius greater than 1, centred at the origin, tells us that the total charge enclosed is 1. Therefore:

$$\rho(|\omega|) \rightarrow \delta(|\omega| - 1) \quad \text{as } \alpha \rightarrow 1. \quad (11) \quad \text{eq:rhoato1}$$

Now,  $\alpha \rightarrow \infty$ :

$$q = \frac{1}{1 + |\omega|^2} \left[ 1 + \frac{2|\omega|^2}{\alpha(1 + |\omega|^2)^2} + \frac{8|\omega|^4}{\alpha^2(1 + |\omega|^2)^4} + \mathcal{O}(\alpha^{-3}) \right], \quad (12) \quad \text{eq:rsatoinft}$$

which leads to

$$\zeta_- = \frac{\omega}{\alpha(1 + |\omega|^2)} \left[ 1 + \frac{2|\omega|^2}{\alpha(1 + |\omega|^2)^2} + \mathcal{O}(\alpha^{-2}) \right]. \quad (13) \quad \text{eq:zmatoinft}$$

Dropping (infinite) constants:

$$\Phi = -\frac{\ln(1 + |\omega|^2)}{4\pi} + \mathcal{O}(\alpha^{-1}). \quad (14) \quad \text{eq:phiatoinf}$$

This results in

$$\rho(|\omega|) \rightarrow \frac{2|\omega|}{(1 + |\omega|^2)^2} \quad \text{as } \alpha \rightarrow \infty. \quad (15) \quad \text{eq:rhoatoinf}$$

This should be  $\delta(|\omega|)$ .



with all other covariances negligible in the large  $N$  limit. We can now write

$$\begin{aligned}
\langle \ln Z_A(\omega, \bar{\omega}) \rangle_x &= 2 \langle \ln \det(xx^T) \rangle_x \\
&+ \left\langle \ln \int \prod_i \left[ \frac{dw_i d\bar{w}_i}{2\pi} \frac{du_i d\bar{u}_i}{2\pi} \frac{dv_i d\bar{v}_i}{2\pi} \right] e^{-N(s+t) - \xi^\dagger A \xi} \right\rangle_x, \\
\text{where } \xi &= \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\
A &= -\frac{i}{P} \begin{pmatrix} 0 & \bar{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}, \\
\langle \xi \xi^\dagger \rangle_x &= C = N \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}.
\end{aligned} \tag{21} \quad \text{eq:potwuv}$$

As we only care about the part of  $\Phi$  that depends on  $\omega$ , we can ignore the first term. We will simplify the second term using (16), (59) and the relation

$$\int \prod_i \left[ \frac{dw_i d\bar{w}_i}{2\pi} \right] f(r) = \frac{N^N}{\Gamma(N)} \int dr r^{N-1} f(r), \tag{22} \quad \text{eq:spherical}$$

along with similar ones for  $u$  and  $v$ , to get

$$\begin{aligned}
\Phi(\omega, \bar{\omega}) &= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{dr}{r} \frac{ds}{s} \frac{dt}{t} (rst)^N e^{-N(s+t)} \langle e^{-\xi^\dagger A \xi} \rangle_x \\
&= \text{const.} + \frac{1}{4\pi N} \ln \int \frac{dr}{r} \frac{ds}{s} \frac{dt}{t} \frac{\exp[N(\ln(rst) - s - t)]}{\det(I + CA)}.
\end{aligned} \tag{23} \quad \text{eq:phiintsimple}$$

$(C^{-1} + A)$  IS NON-NORMAL!

As  $U$  is unitary, all the blocks in these matrices commute. Therefore, we can evaluate the determinant with some help from [2]. Also noting that the eigenvalues of  $U$  are  $\exp(2\pi i k/P)$ , with  $k \in \mathbb{Z}_P$ :

$$\begin{aligned}
\ln \det(I + CA) &= \ln \det \begin{bmatrix} \frac{1}{\alpha} \begin{pmatrix} \alpha & -ir(\bar{\omega} - U^\dagger) & -i\epsilon r \\ -is(\omega - U) & \alpha & 0 \\ -i\epsilon t & 0 & \alpha \end{pmatrix} \end{bmatrix} \\
&= \ln \det \left[ \frac{\alpha^2 + \epsilon^2 rt + rs(\bar{\omega} - U^\dagger)(\omega - U)}{\alpha^2} \right] \\
&= \sum_{k=0}^{P-1} \ln \left[ \frac{\alpha^2 + \epsilon^2 rt + rs(\bar{\omega} - e^{-2\pi i k/P})(\omega - e^{2\pi i k/P})}{\alpha^2} \right] \\
&= \frac{P}{2\pi} \int_0^{2\pi} d\phi \ln \left[ \frac{\alpha^2 + \epsilon^2 rt + rs(\bar{\omega} - e^{-i\phi})(\omega - e^{i\phi})}{\alpha^2} \right] \\
&= \frac{P}{2\pi i} \int \frac{d\zeta}{\zeta} \ln \left[ \frac{G(\zeta)}{\alpha^2 \zeta} \right],
\end{aligned} \tag{24} \quad \text{eq:detsimple}$$

where the function  $G(\zeta)$  is defined in §C, in particular (67). From (68), we know that only one of the zeros of  $G(\zeta)$  will lie inside the contour. We define:

$$\{\zeta_>, \zeta_<\} = \{\zeta_+, \zeta_-\}, \quad |\zeta_>| \geq 1, \quad |\zeta_<| \leq 1. \quad (25)$$

If we factorise  $G(\zeta)$ , we get some contour integrals of the form discussed in §B, the result of which appears in (66). We find that

$$\ln \det(I + CA) = P \ln \left( \frac{rs\bar{\omega}\zeta_>}{\alpha^2} \right) = P \ln \left( \frac{rs\omega}{\alpha^2\zeta_<} \right). \quad (26)$$

Now, if we use the saddle-point approximation of the integrals over  $r$ ,  $s$  and  $t$  in (23), which becomes exact in the limit of large  $N$  and  $P$ , we find

$$\Phi(\omega, \bar{\omega}) = \frac{1}{4\pi} \max_{r,s,t} \left[ \ln(rst) - s - t - \alpha \ln \left( \frac{rs\omega}{\alpha^2\zeta_<} \right) \right]. \quad (27)$$

One can show that (see §4, in particular (52) and (53)) the maximum has

$$r \sim \mathcal{O}(\epsilon^{-1}), \quad s \sim \mathcal{O}(\epsilon), \quad t \sim \mathcal{O}(1), \quad rs \sim \mathcal{O}(1). \quad (28)$$

If we take  $\epsilon \rightarrow 0$ , we find that  $\Phi$  depends on  $r$ ,  $s$  and  $t$  in the combinations  $rs$  and  $t$ :

$$\begin{aligned} \Phi(\omega, \bar{\omega}) &= \frac{1}{4\pi} \max_{rs,t} \left[ (1 - \alpha) \ln(rs) + \ln t - t - \alpha \ln \left( \frac{\omega}{\alpha^2\zeta_<} \right) \right] \\ \frac{\partial \Phi}{\partial(rs)} &= \frac{1 - \alpha}{rs} + \frac{\alpha^3}{rsG'(\zeta_<)}, \\ \frac{\partial \Phi}{\partial t} &= \frac{1}{t} - 1. \end{aligned} \quad (29)$$

Setting these derivatives to zero gives

$$t = 1, \quad G'(\zeta_{\pm}) = \frac{\alpha^3}{\alpha - 1}, \quad (30)$$

which can be solved for  $rs$  provided that

$$|\zeta_-| \leq 1, \quad (31)$$

in which case

$$rs = \frac{\alpha^2 \left[ -(\alpha - 1)(1 + |\omega|^2) \pm \sqrt{(\alpha - 1)^2(1 + |\omega|^2)^2 + (2\alpha - 1)(1 - |\omega|^2)^2} \right]}{(\alpha - 1)(1 - |\omega|^2)^2}, \quad (32)$$

Using (69) and (70)

$$\zeta_- = \frac{1 + |\omega|^2}{2\bar{\omega}} - \frac{\alpha^2}{2(\alpha - 1)\bar{\omega}rs}, \quad (33)$$

and (31) requires that we pick the positive root for  $rs$ . Furthermore, with this choice, (31) is satisfied everywhere. Finally, dropping some constant pieces,

$$\Phi(\omega, \bar{\omega}) = \frac{1}{4\pi} \left[ (1 - \alpha) \ln(rs) - \alpha \ln \left( \frac{\omega}{\zeta_{<}} \right) \right]. \quad (34) \quad \text{eq:saddlesol}$$

This will simplify if expressed in terms of  $q = (\alpha - 1)rs/\alpha^2$ .

## 4 Full, replica-tastic derivation

replicader

The starting point for this version of the derivation will be (19) and (21):

$$\begin{aligned} \Phi(\omega, \bar{\omega}) &= \text{const.} + \frac{1}{4\pi N} \left\langle \ln \tilde{Z} \right\rangle_x, \\ \tilde{Z} &= \int \prod_i \left[ \frac{dw_i d\bar{w}_i}{2\pi} \frac{du_i d\bar{u}_i}{2\pi} \frac{dv_i d\bar{v}_i}{2\pi} \right] e^{-F''}, \\ F'' &= u^\dagger u + v^\dagger v - \frac{i}{P} \left[ w^\dagger x(\bar{\omega} - U^\dagger) x^T u + \epsilon w^\dagger x x^T v + (\text{c.c.}) \right]. \end{aligned} \quad (35) \quad \text{eq:phiint}$$

We will use the replica trick, i.e. we rewrite the logarithm as

$$\ln \tilde{Z} = \left. \frac{\partial (\tilde{Z}^n)}{\partial n} \right|_{n=0}. \quad (36) \quad \text{eq:replicatr}$$

For integer  $n$ , we can compute  $\tilde{Z}^n$  by creating  $n$  replicas of the system. We then let  $n \rightarrow 0$  after averaging over  $x$ . We index these replicas with  $a, b = 1, \dots, n$ :

$$\begin{aligned} \Phi(\omega, \bar{\omega}) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \left\langle \int \prod_{ia} \left[ \frac{dw_{ia} d\bar{w}_{ia}}{2\pi} \frac{du_{ia} d\bar{u}_{ia}}{2\pi} \frac{dv_{ia} d\bar{v}_{ia}}{2\pi} \right] e^{-F'''} \right\rangle_x \Big|_{n=0}, \\ F''' &= \sum_a u_a^\dagger u_a + v_a^\dagger v_a - \frac{i}{P} \left[ w_a^\dagger x(\bar{\omega} - U^\dagger) x^T u_a + \epsilon w_a^\dagger x x^T v_a + (\text{c.c.}) \right]. \end{aligned} \quad (37) \quad \text{eq:phirep}$$

Over most of the integration domain, we expect the Hermitian overlaps  $(w_a^\dagger w_b, u_a^\dagger u_b, \dots)$  will be  $\mathcal{O}(N)$ , whereas the non-Hermitian overlaps  $(w_a^\dagger u_b, w_a^T w_b, \dots)$  will be  $\mathcal{O}(\sqrt{N})$ . We define some new variables,  $\rho_a, \sigma_a$  and  $\tau_a$ , which are zero mean Gaussian random vectors:

$$\begin{aligned} \rho_a &= x^T w_a, & \langle \bar{\rho}_{\mu a} \rho_{\nu b} \rangle_x &= N \delta_{\mu\nu} R_{ab}, & R_{ab} &= \frac{w_a^\dagger w_b}{N}, \\ \sigma_a &= x^T u_a, & \langle \bar{\sigma}_{\mu a} \sigma_{\nu b} \rangle_x &= N \delta_{\mu\nu} S_{ab}, & S_{ab} &= \frac{u_a^\dagger u_b}{N}, \\ \tau_a &= x^T v_a, & \langle \bar{\tau}_{\mu a} \tau_{\nu b} \rangle_x &= N \delta_{\mu\nu} T_{ab}, & T_{ab} &= \frac{v_a^\dagger v_b}{N}, \end{aligned} \quad (38) \quad \text{eq:reprstdef}$$

with all other covariances negligible in the large  $N$  limit.

$$\begin{aligned}\Phi(\omega, \bar{\omega}) &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[ \frac{dw_{ia} d\bar{w}_{ia}}{2\pi} \frac{du_{ia} d\bar{u}_{ia}}{2\pi} \frac{dv_{ia} d\bar{v}_{ia}}{2\pi} \right] \left\langle e^{-N \text{Tr}(S+T) - \xi^\dagger A \xi} \right\rangle_x \Big|_{n=0}, \\ &= \frac{1}{4\pi N} \frac{\partial}{\partial n} \int \prod_{ia} \left[ \frac{dw_{ia} d\bar{w}_{ia}}{2\pi} \frac{du_{ia} d\bar{u}_{ia}}{2\pi} \frac{dv_{ia} d\bar{v}_{ia}}{2\pi} \right] e^{-NE(R,S,T)} \Big|_{n=0},\end{aligned}\tag{39}$$

where

$$\begin{aligned}E(R, S, T) &= \text{Tr}(S + T) + \frac{1}{N} \ln \det(I + CA), \\ \xi &= \begin{pmatrix} \rho \\ \sigma \\ \tau \end{pmatrix}, \\ A &= -\frac{i}{P} \begin{pmatrix} 0 & \bar{\omega} - U^\dagger & \epsilon \\ \omega - U & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \otimes I, \\ \langle \xi \xi^\dagger \rangle_x &= C = NI \otimes \begin{pmatrix} R & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & T \end{pmatrix}.\end{aligned}\tag{40}$$

It will be helpful to separate the integrals over  $w, u$  and  $v$  into an integral over values of  $w, u$  and  $v$  with the same overlap and an integral over values of the overlap. This can be done by inserting factors like the following into the integral:

$$\int dR_{ab} N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = 1.\tag{41}$$

We define

$$S(R) = \frac{1}{N} \int \prod_{ia} \left[ \frac{dw_{ia} d\bar{w}_{ia}}{2\pi} \right] N^{n^2} \delta(NR_{ab} - w_a^\dagger w_b) = \ln \det R + \text{const}.\tag{42}$$

with the final expression valid in the large  $N$  limit. Then (39) reduces to

$$\Phi(\omega, \bar{\omega}) = \frac{1}{4\pi N} \frac{\partial}{\partial n} \int dR_{ab} dS_{ab} dT_{ab} e^{-N(E(R,S,T) - S(R) - S(S) - S(T))} \Big|_{n=0}.\tag{43}$$

We will perform this integral in the large  $N$  limit with the saddle point method.

We make the following, replica-symmetric ansätze for the saddle-point:

$$R_{ab} = r_0 \delta_{ab} + r_1, \quad S_{ab} = s_0 \delta_{ab} + s_1, \quad T_{ab} = t_0 \delta_{ab} + t_1.\tag{44}$$

A matrix of this form has  $(n-1)$  eigenvalues equal to  $r_0$  and one eigenvalue equal to  $(r_0 + nr_1)$ . Therefore,

$$S(R) = \ln \det R = (n-1) \ln r_0 + \ln(r_0 + nr_1) = n \ln r_0 + \frac{nr_1}{r_0} + \mathcal{O}(n^2).\tag{45}$$



The replica symmetric form of  $R, S, T$  and the unitarity of  $U$  means that all the blocks in (40) commute. Then, according to [2],

$$\begin{aligned} \ln \det(I + CA) &= \ln \det \begin{bmatrix} \frac{1}{\alpha} \begin{pmatrix} \alpha & -i(\bar{\omega} - U^\dagger) \otimes R & -i\epsilon I \otimes R \\ -i(\omega - U) \otimes S & \alpha & 0 \\ -i\epsilon I \otimes T & 0 & \alpha \end{pmatrix} \end{bmatrix} \\ &= \ln \det \left[ \frac{\alpha^2 + \epsilon^2 I \otimes RT + (\bar{\omega} - U^\dagger)(\omega - U) \otimes RS}{\alpha^2} \right]. \end{aligned} \quad (46) \quad \text{eq:detrep}$$

Note that

$$(RS)_{ab} = r_0 s_0 \delta_{ab} + r_0 s_1 + r_1 s_0 + \mathcal{O}(n), \quad (47) \quad \text{eq:overlap}$$

and similar for  $RT$ , so these will have a similar eigenvalue structure to  $R$ , with the same eigenvectors. Also, the eigenvalues of  $U$  are  $\exp(2\pi i k/P)$ , with  $k \in \mathbb{Z}_P$ . Therefore:

$$\begin{aligned} \ln \det(I + CA) &= \sum_{k=0}^{P-1} n \left\{ \ln \left[ \frac{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0 (\bar{\omega} - e^{-2\pi i k/P})(\omega - e^{2\pi i k/P})}{\alpha^2} \right] \right. \\ &\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0) + (r_0 s_1 + r_1 s_0)(\bar{\omega} - e^{-2\pi i k/P})(\omega - e^{2\pi i k/P})}{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0 (\bar{\omega} - e^{-2\pi i k/P})(\omega - e^{2\pi i k/P})} \right\} \\ &\quad + \mathcal{O}(n^2) \\ &= \frac{nP}{2\pi} \int_0^{2\pi} d\phi \left\{ \ln \left[ \frac{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0 (\bar{\omega} - e^{-i\phi})(\omega - e^{i\phi})}{\alpha^2} \right] \right. \\ &\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0) + (r_0 s_1 + r_1 s_0)(\bar{\omega} - e^{-i\phi})(\omega - e^{i\phi})}{\alpha^2 + \epsilon^2 r_0 t_0 + r_0 s_0 (\bar{\omega} - e^{-i\phi})(\omega - e^{i\phi})} \right\} \\ &= \frac{nP}{2\pi i} \int \frac{d\zeta}{\zeta} \left\{ \ln \left[ \frac{G(\zeta)}{\alpha^2 \zeta} \right] \right. \\ &\quad \left. + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta + (r_0 s_1 + r_1 s_0)(\bar{\omega}\zeta - 1)(\omega - \zeta)}{G(\zeta)} \right\}, \end{aligned} \quad (48) \quad \text{eq:detrepint}$$

where the function  $G(\zeta)$  is defined in §C, in particular (67). The first integral was computed in (26). The second can be computed with the residue theorem. The integrand has poles at  $\zeta \in \{0, \zeta_+, \zeta_-\}$ , with only one of the last two lying inside the contour (see (69)). Using the definitions (25),

$$\begin{aligned} &\int \frac{d\zeta}{2\pi i \zeta} \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta + (r_0 s_1 + r_1 s_0)(\bar{\omega}\zeta - 1)(\omega - \zeta)}{G(\zeta)} \\ &= \frac{(r_0 s_1 + r_1 s_0)(-\omega)}{G(0)} + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)\zeta_< + (r_0 s_1 + r_1 s_0)(\bar{\omega}\zeta_< - 1)(\omega - \zeta_<)}{\zeta_< G'(\zeta_<)} \\ &= \frac{r_0 s_1 + r_1 s_0}{r_0 s_0} \left( 1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_<)} \right) + \frac{\epsilon^2(r_0 t_1 + r_1 t_0)}{G'(\zeta_<)}. \end{aligned} \quad (49) \quad \text{eq:repcontour}$$

Combining all of this gives:

$$\Phi(\omega, \bar{\omega}) = \frac{1}{4\pi} \max_{r_{0,1}, s_{0,1}, t_{0,1}} \left\{ \ln(r_0 s_0 t_0) + \frac{r_1}{r_0} + \frac{s_1}{s_0} + \frac{t_1}{t_0} - (s_0 + s_1 + t_0 + t_1) \right. \\ \left. - \alpha \ln \left( \frac{r_0 s_0 \omega}{\alpha^2 \zeta_{<}} \right) - \frac{\alpha(r_0 s_1 + r_1 s_0)}{r_0 s_0} \left( 1 - \frac{\alpha^2 + \epsilon^2 r_0 t_0}{G'(\zeta_{<})} \right) \right. \\ \left. - \frac{\alpha \epsilon^2 (r_0 t_1 + r_1 t_0)}{G'(\zeta_{<})} \right\}. \quad (50) \quad \text{eq:phimaxrep}$$

To find the maximum, we must set the following derivatives to zero

$$\begin{aligned} \frac{\partial(4\pi\Phi)}{\partial r_1} &= \frac{1}{r_0} - \frac{\alpha}{r_0} + \frac{\alpha^3}{r_0 G'(\zeta_{<})}, \\ \frac{\partial(4\pi\Phi)}{\partial s_1} &= \frac{1}{s_0} - 1 - \frac{\alpha}{s_0} + \frac{\alpha(\alpha^2 + \epsilon^2 r_0 t_0)}{s_0 G'(\zeta_{<})}, \\ \frac{\partial(4\pi\Phi)}{\partial t_1} &= \frac{1}{t_0} - 1 - \frac{\alpha \epsilon^2 r_0}{G'(\zeta_{<})}. \end{aligned} \quad (51) \quad \text{eq:phidiff1}$$

Solving these equations gives

$$s_0 = \frac{(\alpha - 1)\epsilon^2 r_0}{\alpha^2 + (\alpha - 1)\epsilon^2 r_0}, \quad t_0 = 1 - s_0, \quad G'(\zeta_{<}) = \frac{\alpha^3}{\alpha - 1}. \quad (52) \quad \text{eq:repsaddle}$$

Expression (71) for  $G'(\zeta_{\pm})$  tells us that it must be  $\zeta_{-}$  that lies inside the unit circle. Solving for  $r_0$  is a mess, but we can see that no solutions exist for  $r_0 < \mathcal{O}(\epsilon^{-1})$  or  $r_0 > \mathcal{O}(\epsilon^{-1})$  as  $\epsilon \rightarrow 0$ , because the last equation would reduce to

$$\begin{aligned} r_0 \sim \mathcal{O}(\epsilon^{-1+\delta}) &\implies \alpha^2 = \frac{\alpha^3}{\alpha - 1}, \\ r_0 \sim \mathcal{O}(\epsilon^{-1-\delta}) &\implies \frac{(\alpha - 1)(1 - |\omega|^2)^2 \epsilon^2 r_0^2}{\alpha^2 + (\alpha - 1)\epsilon^2 r_0} = \frac{\alpha^3}{\alpha - 1}. \end{aligned} \quad (53) \quad \text{eq:r0e}$$

Combined with (52), this justifies (28).

We must also set the following derivatives to zero

$$\begin{aligned} \frac{\partial(4\pi\Phi)}{\partial r_0} &= \frac{\partial(4\pi\Phi)}{\partial r_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\ \frac{\partial(4\pi\Phi)}{\partial s_0} &= \frac{\partial(4\pi\Phi)}{\partial s_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1, \\ \frac{\partial(4\pi\Phi)}{\partial t_0} &= \frac{\partial(4\pi\Phi)}{\partial t_1} + (\cdots)r_1 + (\cdots)s_1 + (\cdots)t_1. \end{aligned} \quad (54) \quad \text{eq:phidiff0}$$

Therefore the saddle-point will have  $r_1 = s_1 = t_1 = 0$ . This justifies the annealed assumption in (16).

## Appendices

### A Complex Gaussian integrals

eq:compgauss

First, Let's get all of the factors of 2 straight. Note that if we write  $z = x + iy$ , then  $dzd\bar{z} = 2dxdy$ . Let  $H$  be a positive-definite,  $N \times N$  Hermitian matrix (or just a normal matrix whose eigenvalues have positive real parts). Consider an integral of the form

$$\int \left( \prod_i dz_i d\bar{z}_i \right) \exp(-z^\dagger H z).$$

We can diagonalise  $H$  with a unitary change of variables:

$$\begin{aligned} \int \left( \prod_i dz_i d\bar{z}_i \right) \exp(-z^\dagger H z) &= \prod_i \int dz_i d\bar{z}_i \exp(-\lambda_i |z_i|^2) \\ &= \prod_i \int dx_i dy_i 2 \exp(-\lambda_i (x_i^2 + y_i^2)) \\ &= \prod_i \frac{2\pi}{\lambda_i} \\ &= \frac{(2\pi)^N}{\det H}. \end{aligned} \tag{55} \quad \text{eq:compgauss}$$

The proper normalisation for a Gaussian distribution is

$$P(z, z^\dagger) dz dz^\dagger = \left( \prod_i \frac{dz_i d\bar{z}_i}{2\pi} \right) \frac{\exp(-z^\dagger C^{-1} z)}{\det C}. \tag{56} \quad \text{eq:compgauss}$$

By completing the square, we can see that

$$\langle \exp(\zeta^\dagger z \pm z^\dagger \zeta) \rangle = \exp(\pm \zeta^\dagger C \zeta) \tag{57} \quad \text{eq:compgauss}$$

Taking partial derivatives wrt.  $\zeta_i$  and  $\bar{\zeta}_i$ , we find

$$\langle z z^\dagger \rangle = C. \tag{58} \quad \text{eq:compgauss}$$

Now consider an integral of the form

$$\begin{aligned} \langle \exp(-z^\dagger A z) \rangle &= \int \left( \prod_i \frac{dz_i d\bar{z}_i}{2\pi} \right) \frac{\exp(-z^\dagger (C^{-1} + A) z)}{\det C} \\ &= (\det C \det (C^{-1} + A))^{-1} \\ &= \det (I + CA)^{-1}. \end{aligned} \tag{59} \quad \text{eq:compgauss}$$

ONLY WORKS IF  $(C^{-1} + A)$  IS POS DEF, OR AT LEAST NORMAL!

Does it matter if the matrix is normal? Suppose we diagonalise with a non-unitary transformation. What Jacobian factor would we pick up?

$$\begin{aligned}
z' = Sz, \quad \bar{z}' = S^{-1}\bar{z}, \quad \implies \quad & x' = \frac{z' + \bar{z}'}{2} = \frac{S + S^{-1}}{2}x - \frac{S - S^{-1}}{2i}y, \\
& y' = \frac{z' - \bar{z}'}{2i} = \frac{S - S^{-1}}{2i}x - \frac{S + S^{-1}}{2}y, \\
\implies \det J = \det & \begin{pmatrix} \frac{S+S^{-1}}{2} & -\frac{S-S^{-1}}{2i} \\ \frac{S-S^{-1}}{2i} & -\frac{S+S^{-1}}{2} \end{pmatrix} \\
& = \det \left[ \left( \frac{S + S^{-1}}{2} \right)^2 - \left( \frac{S - S^{-1}}{2i} \right)^2 \right] \\
& = \det SS^{-1} = 1.
\end{aligned}$$

The change in contours can be undone, as the integrand is analytic in  $x$  and  $y$ .

## B Contour integrals for determinants

contourints

In evaluating determinants in §3, we will come across contour integrals of the form

$$I(\zeta_{\geq}) = \oint_C \frac{d\zeta}{2\pi i \zeta} \ln \left( \frac{\gamma(\zeta_{>} - \zeta)(\zeta - \zeta_{<})}{\zeta} \right), \quad (60) \quad \text{eq:contourin}$$

where the contour is the unit circle in a counter-clockwise direction and  $|\zeta_{\geq}| \geq 1$ . We choose the branch of the logarithm so that

$$\arg \left( \frac{\zeta - \zeta_{<}}{\zeta_{<}} \right), \arg \left( \frac{\zeta}{\zeta_{<}} \right) \in [0, 2\pi], \quad (61) \quad \text{eq:branch}$$

and we define  $\theta = \arg z$ . The branch cut at  $\zeta = \zeta_{>}$  will not matter. The branch cuts and contour are shown in fig.1(a).

We write

$$I(\zeta_{\geq}) = \oint_C \frac{d\zeta}{2\pi i \zeta} \ln \gamma(\zeta_{>} - \zeta) + \oint_C \frac{d\zeta}{2\pi i \zeta} \ln \left( \frac{\zeta - \zeta_{<}}{\zeta} \right), \quad (62) \quad \text{eq:contoursp}$$

For the first part, we can use the original contour  $C$  as in fig.1(b). Using the residue theorem:

$$\int \frac{d\zeta}{2\pi i \zeta} \ln \gamma(\zeta_{>} - \zeta) = \ln \gamma \zeta_{>}. \quad (63) \quad \text{eq:intout}$$

For the second part, we can deform the contour to  $C'$  as in fig.1(c):

$$\begin{aligned}
C' : \quad & \zeta = \epsilon e^{i\phi}, & \phi \in [\theta, \theta + 2\pi], \\
& \zeta = x e^{i(\theta+2\pi)}, & x \in [\epsilon, |\zeta_{<}| - \epsilon], \\
& \zeta = \zeta_{<} + \epsilon e^{i\phi}, & \phi \in [\theta - \pi, \theta + \pi], \\
& \zeta = (|\zeta_{<}| - x) e^{i(\theta+2\pi)}, & x \in [\epsilon, |\zeta_{<}| - \epsilon].
\end{aligned} \quad (64) \quad \text{eq:contout}$$

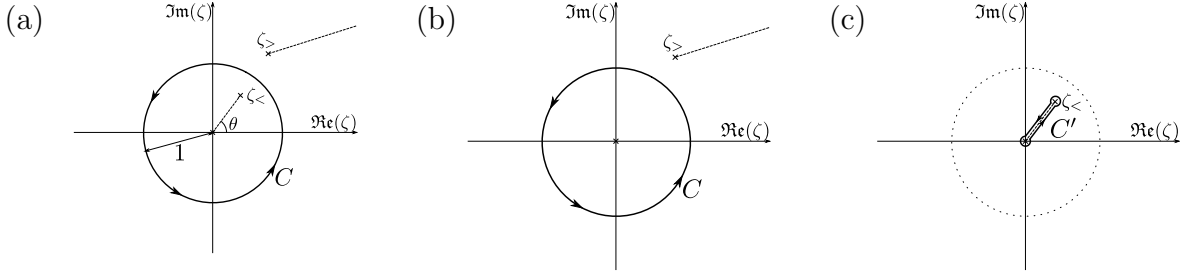


Figure 1: (a) Contours used to evaluate (60) and branch cuts (61). (b) contour used with singularities at  $\zeta_>$ . (c) contour used with singularities at  $\zeta_<, 0$ . Branch cuts indicated by dashed line. Poles and branch points indicated by crosses.

fig:contours

The integral over the third part vanishes as  $\epsilon \rightarrow 0$ . The second and fourth parts would cancel, if not for the discontinuity in the logarithm of the denominator (the logarithm of the numerator has no discontinuity, due to (61)). This leaves:

$$\begin{aligned}
 \oint_{C'} \frac{d\zeta}{2\pi i \zeta} \ln \left( \frac{\zeta - \zeta_<}{\zeta} \right) &= \int_{\theta}^{\theta+2\pi} \frac{d\phi}{2\pi} \ln \left( \frac{\epsilon e^{i\phi} - \zeta_<}{\epsilon e^{i\phi}} \right) + \int_{\epsilon}^{|\zeta_<|-\epsilon} \frac{dx}{2\pi i x} \text{disc} \ln \left( \frac{1}{\zeta} \right) \\
 &= \ln \left( \frac{e^{i\pi} \zeta_<}{\epsilon} \right) - \int_{\theta}^{\theta+2\pi} \frac{i\phi d\phi}{2\pi} - \int_{\epsilon}^{|\zeta_<|} \frac{dx}{x} \\
 &= \ln \left( \frac{\zeta_<}{\epsilon} \right) + i\pi - i(\theta + \pi) - \ln \left( \frac{|\zeta_<|}{\epsilon} \right) \\
 &= 0.
 \end{aligned} \tag{65}$$

eq:intin

Therefore:

$$I(\zeta_{\geq}) = \ln \gamma \zeta_>. \tag{66}$$

eq:countouri

## C The quadratic function $G(\zeta)$

In evaluating determinants in §3 and §4, we come across the function

$$G(\zeta) = (\alpha^2 + \epsilon^2 r t) \zeta + r s (\bar{\omega} \zeta - 1)(\omega - \zeta) = -r s \bar{\omega} (\zeta - \zeta_+)(\zeta - \zeta_-). \tag{67}$$

eq:Gammadef

We will collect some useful features of  $\zeta_{\pm}$  here.

First, by comparing the two forms of  $G(\zeta)$ , we see that:

$$\zeta_+ \zeta_- = \frac{\omega}{\bar{\omega}}, \tag{68}$$

eq:zpzm

$$\zeta_+ + \zeta_- = \frac{\alpha^2 + \epsilon^2 r t + r s (1 + |\omega|^2)}{r s \bar{\omega}}, \tag{69}$$

eq:zppzm

$$G'(\zeta_{\pm}) = \mp r s \bar{\omega} (\zeta_+ - \zeta_-), \tag{70}$$

eq:Gprime

and (68) tells us that  $|\zeta_+| |\zeta_-| = 1$ . Solving the equation  $G(\zeta_{\pm}) = 0$  gives

$$G'(\zeta_{\pm}) = \mp \sqrt{[\alpha^2 + \epsilon^2 r t + r s (1 + |\omega|^2)]^2 - 4(rs)^2 |\omega|^2}. \quad (71) \quad \text{eq:Gprimesol}$$

Differentiating the equation  $G(\zeta_{\pm}) = 0$  gives

$$\begin{aligned} \frac{\partial \zeta_{\pm}}{\partial r} &= -\frac{\epsilon^2 t \zeta_{\pm} + s(\bar{\omega} \zeta_{\pm} - 1)(\omega - \zeta_{\pm})}{G'(\zeta_{\pm})} = \frac{\alpha^2 \zeta_{\pm}}{r G'(\zeta_{\pm})}, \\ \frac{\partial \zeta_{\pm}}{\partial s} &= -\frac{r(\bar{\omega} \zeta_{\pm} - 1)(\omega - \zeta_{\pm})}{G'(\zeta_{\pm})} = \frac{(\alpha^2 + \epsilon^2 r t) \zeta_{\pm}}{s G'(\zeta_{\pm})}, \\ \frac{\partial \zeta_{\pm}}{\partial r} &= -\frac{\epsilon^2 r \zeta_{\pm}}{G'(\zeta_{\pm})}. \end{aligned} \quad (72) \quad \text{eq:dzpmdrst}$$

## References

- asymmetric
- determinants
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