LINEAR REGRESSION MODELS W4315

HOMEWORK 5 QUESTIONS

February 25, 2010

Due: 03/04/10
Instructor: Frank Wood

1. (20 points) In order to get a maximum likelihood estimate of the parameters of a Box-Cox transformed simple linear regression model ($Y_i^\lambda = \beta_0 + \beta_1 X_i + \epsilon_i$), we need to find the gradient of the likelihood with respect to its parameters (the gradient consists of the partial derivatives of the likelihood function w.r.t. all of the parameters). Derive the partial derivatives of the likelihood w.r.t all parameters assuming that $\epsilon_i \sim N(0, \sigma^2)$. (N.B. the parameters here are $\lambda, \beta_0, \beta_1, \sigma$)
(Extra Credit: Consider the transformation of the Y_i's mentioned in the lecture notes but not covered in class. Explain why this transformation is necessary in a gradient ascent procedure for λ)

2. (15 points) [1] Derive an extension of Bonferroni inequality (4.2a) which is given as

$$P(\bar{A}_1 \bigcap \bar{A}_2) \geq 1 - \alpha - \alpha = 1 - 2\alpha$$

for the case of three statements, each with statement confidence coefficient $1 - \alpha$.

3. (25 points) [2] Refer to **Consumer finance** Problems 5.5 and 5.13.
a. Using matrix methods, obtain the following: (1) vector of estimated regression coefficients, (2) vector of residuals, (3) SSR, (4) SSE, (5) estimated variance-covariance matrix of $\mathbf{b}$, (6) point estimate of $E\{Y_h\}$ when $X_h = 4$, (7) $s^2\{\text{pred}\}$ when $X_h = 4$
b. From your estimated variance-covariance matrix in part (a5), obtain the following: (1) $s\{b_0, b_1\}$; (2) $s^2\{b_0\}$; (3) $s\{b_1\}$
c. Find the hat matrix $\mathbf{H}$
d. Find $s^2\{\mathbf{e}\}$

4. (25 points) [3] In a small-scale regression study, the following data were obtained: Assume

[1] This is problem 4.22 in 'Applied Linear Regression Models(4th edition)' by Kutner etc.
[2] This is problem 5.24 in 'Applied Linear Regression Models(4th edition)' by Kutner etc.
[3] This is problem 6.27 in 'Applied Linear Regression Models(4th edition)' by Kutner etc.

i:	**1**	**2**	**3**	**4**	**5**	**6**
X_{i1}	7	4	16	3	21	8
X_{i2}	33	41	7	49	5	31
Y_i	42	33	75	28	91	55

that regression model (1) which is:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta X_{i2} + \epsilon_i \tag{1}$$

with independent normal error terms is appropriate. Using matrix methods, obtain (a) $\mathbf{b}$; (b) $\mathbf{e}$; (c) $\mathbf{H}$; (d) SSR; (e) $s^2\{\mathbf{b}\}$; (f) $\hat{Y}_h$ when $X_{h1} = 10$, $X_{h2} = 30$; (g) $s^2\{\hat{Y}_h\}$ when $X_{h1} = 10$, $X_{h2} = 30$

5. (15 points) Consider the classical matrix approach to multiple regression, i.e.

$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

where $\mathbf{X}$ is a $n \times p$ design matrix whose first column is all 1's, $\epsilon \sim N(\mathbf{0}, \mathbf{I})$ and $\mathbf{I}$ is an identity matrix. Prove the following:

a. The sum of squares error $SSE = \mathbf{e}'\mathbf{e}$ can be written in a matrix form:

$$SSE = \mathbf{y}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \tag{2}$$

b. We call the RHS of (2) a quadratic form. Prove that the matrix $\mathbf{A} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is an idempotent matrix.

c. Prove that the rank of $\mathbf{A}$ defined in part (b) is $n - p$.

N.B. p columns in design matrix means there are $p - 1$ predictors plus 1 intercept term. In your solutions please clearly notate the dimensions of all of the matrices.