# Inference in Normal Regression Model

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# Remember

1. Last class we derived the sampling variance of the estimator of the slope, it being

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$

2. And we made the point that an estimate of $\mathrm{Var}(b_1)$ could be arrived at by substituting the MSE for the unknown error variance.

$$s^2\{b_1\} = \frac{MSE}{\sum(X_i - \bar{X})^2} = \frac{\frac{SSE}{n-2}}{\sum(X_i - \bar{X})^2}$$

# Sampling Distribution of $(b_1 - \beta_1)/s(b_1)$

1. We determined that $b_1$ is normally distributed so $(b_1 - \beta_1)/\sigma(b_1)$ is a standard normal variable
2. We don't know $\mathrm{Var}(b_1)$ so it must be estimated from data. We have already denoted it's estimate $s(b_1)$
3. Using this estimate we it can be shown that

$$\frac{b_1 - \beta_1}{s\{b_1\}} \sim t(n-2)$$

$$s\{b_1\} = \sqrt{s^2\{b_1\}}$$

# Where does this come from?

1. We need to rely upon the following theorem
   For the normal error regression model

$$\frac{SSE}{\sigma^2} = \frac{\sum(Y_i - \hat{Y}_i)^2}{\sigma^2} \sim \chi^2(n-2)$$

   and is independent of b0 and b1
2. Intuitively this follows the standard result for the sum of squared normal random variables
   Here there are two linear constraints imposed by the regression parameter estimation that each reduce the number of degrees of freedom by one.

# Another useful fact : t distribution

Let z and $\chi^2(\nu)$ be independent random variables (standard normal and $\chi^2$ respectively). We then define a t random variable as follows:

$$t(\nu) = \frac{z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}}$$

This version of the t distribution has one parameter, the degrees of freedom $\nu$

# Distribution of the studentized statistic

To derive the distribution of this statistic, first we do the following rewrite

$$\frac{b_1 - \beta_1}{\hat{S}(b_1)} = \frac{\frac{b_1 - \beta_1}{S(b_1)}}{\frac{\hat{S}(b_1)}{S(b_1)}}$$

$$\frac{\hat{S}(b_1)}{S(b_1)} = \sqrt{\frac{\hat{V}(b_1)}{V(b_1)}}$$

# Studentized statistic cont.

And note the following

$$\frac{\hat{V}(b_1)}{V(b_1)} = \frac{\frac{MSE}{\sum(X_i - \bar{X})^2}}{\frac{\sigma^2}{\sum(X_i - \bar{X})^2}} = \frac{MSE}{\sigma^2} = \frac{SSE}{\sigma^2(n-2)}$$

where we know (by the given theorem) the distribution of the last term is $\chi^2$ and indep. of $b_1$ and $b_0$

$$\frac{SSE}{\sigma^2(n-2)} \sim \frac{\chi^2(n-2)}{n-2}$$

# Studentized statistic final

But by the given definition of the t distribution we have our result

$$\frac{b_1 - \beta_1}{\hat{S}(b_1)} \sim t(n-2)$$

because putting everything together we can see that

$$\frac{b_1 - \beta_1}{\hat{S}(b_1)} \sim \frac{z}{\sqrt{\frac{\chi^2(n-2)}{n-2}}}$$

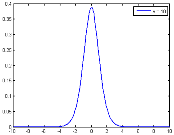
# Confidence Intervals and Hypothesis Tests

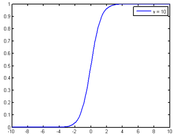
Now that we know the sampling distribution of $b_1$ (t with n-2 degrees of freedom) we can construct confidence intervals and hypothesis tests easily

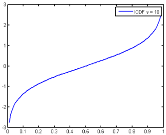
# Confidence Interval for $\beta_1$

Since the "studentized" statistic follows a t distribution we can make the following probability statement

$$P(t(\alpha/2; n-2) \leq \frac{b_1 - \beta_1}{s\{b_1\}} \leq t(1-\alpha/2; n-2)) = 1-\alpha$$







# Interval arriving from picking $\alpha$

1. Note that by symmetry

$$t(\alpha/2; n-2) = -t(1-\alpha/2; n-2)$$

2. Rearranging terms and using this fact we have
$P(b_1 - t(1-\alpha/2; n-2)s(b_1) \leq \beta_1 \leq$
$b_1 + t(1-\alpha/2; n-2)s(b_1)) = 1-\alpha$
3. And now we can use a table to look up and produce confidence intervals

# Using tables for Computing Intervals

1. The tables in the book (table B.2 in the appendix) for $t(1-\alpha/2;\nu)$ where $P\{t(\nu) \leq t(1-\alpha/2;\nu)\} = A$
2. Provides the inverse CDF of the t-distribution
3. This can be arrived at computationally as well
   Matlab: $tinv(1-\alpha/2, \nu)$

# $1-\alpha$ confidence limits for $\beta_1$

1. The $1-\alpha$ confidence limits for $\beta_1$ are

$$b_1 \pm t(1-\alpha/2; n-2)s\{b_1\}$$

2. Note that this quantity can be used to calculate confidence intervals given n and $\alpha$.
   2.1 Fixing $\alpha$ can guide the choice of sample size if a particular confidence interval is desired
   2.2 Give a sample size, vice versa.
3. Also useful for hypothesis testing

# Tests Concerning $\beta_1$

1. Example 1
   1.1 Two-sided test
      1.1.1 $H_0 : \beta_1 = 0$
      1.1.2 $H_a : \beta_1 \neq 0$
      1.1.3 Test statistic

$$t^* = \frac{b_1 - 0}{s\{b_1\}}$$

# Tests Concerning $\beta_1$

1. We have an estimate of the sampling distribution of $b_1$ from the data.
2. If the null hypothesis holds then the $b_1$ estimate coming from the data should be within the 95% confidence interval of the sampling distribution centered at 0 (in this case)

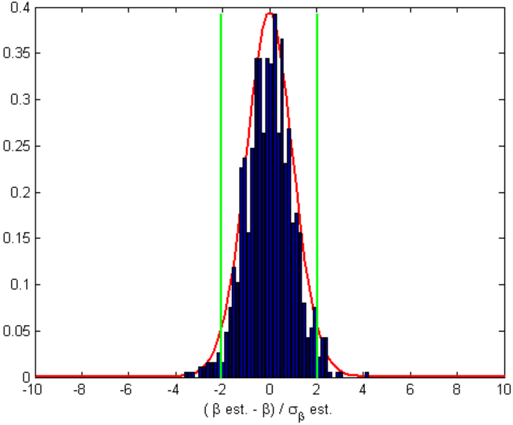
$$t^* = \frac{b_1 - 0}{s\{b_1\}}$$

# Decision rules

$$\begin{aligned} \text{if } |t^*| &\leq t(1-\alpha/2; n-2), \text{conclude } H_0 \\ \text{if } |t^*| &> t(1-\alpha/2; n-2), \text{conclude } H_\alpha \end{aligned}$$

Absolute values make the test two-sided

# Intuition




p-value is value of $\alpha$ that moves the green line to the blue line

# Calculating the p-value

1. The p-value, or attained significance level, is the smallest level of significance $\alpha$ for which the observed data indicate that the null hypothesis should be rejected.
2. This can be looked up using the CDF of the test statistic.
3. In Matlab
   Two-sided p-value
   $2 * (1 - \mathit{tcdf}(|t^*|, \nu))$

# Inferences Concerning $\beta_0$

1. Largely, inference procedures regarding $\beta_0$ can be performed in the same way as those for $\beta_1$
2. Remember the point estimator $b_0$ for $\beta_0$

$$b_0 = \bar{Y} - b_1 \bar{X}$$

# Sampling distribution of $b_0$

1. The sampling distribution of $b_0$ refers to the different values of $b_0$ that would be obtained with repeated sampling when the levels of the predictor variable X are held constant from sample to sample.
2. For the normal regression model the sampling distribution of $b_0$ is normal

# Sampling distribution of $b_0$

1. When error variance is known

$$E(b_0) = \beta_0$$

$$\sigma^2\{b_0\} = \sigma^2(\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2})$$

2. When error variance is unknown

$$s^2\{b_0\} = MSE(\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2})$$

# Confidence interval for $\beta_0$

The $1-\alpha$ confidence limits for $\beta_0$ are obtained in the same manner as those for $\beta_1$

$$b_0 \pm t(1-\alpha/2; n-2)s\{b_0\}$$

# Considerations on Inferences on $\beta_0$ and $\beta_1$

1. Effects of departures from normality
   The estimators of $\beta_0$ and $\beta_1$ have the property of asymptotic normality - their distributions approach normality as the sample size increases (under general conditions)
2. Spacing of the X levels The variances of $b_0$ and $b_1$ (for a given n and $\sigma^2$) depend strongly on the spacing of X

# Sampling distribution of point estimator of mean response

1. Let $X_h$ be the level of X for which we would like an estimate of the mean response
   Needs to be one of the observed X's
2. The mean response when $X = X_h$ is denoted by $\mathbb{E}(Y_h)$
3. The point estimator of $\mathbb{E}(Y_h)$ is

$$\hat{Y}_h = b_0 + b_1 X_h$$

   We are interested in the sampling distribution of this quantity

# Sampling Distribution of $\hat{Y}_h$

1. We have
$$\hat{Y}_h = b_0 + b_1 X_h$$
2. Since this quantity is itself a linear combination of the $Y_i's$ it's sampling distribution is itself normal.
3. The mean of the sampling distribution is
$$E\{\hat{Y}_h\} = E\{b_0\} + E\{b_1\}X_h = \beta_0 + \beta_1 X_h$$
Biased or unbiased?

## Sampling Distribution of $\hat{Y}_h$

1. To derive the sampling distribution variance of the mean response we first show that $b_1$ and $(1/n)\sum Y_i$ are uncorrelated and, hence, for the normal error regression model independent
2. We start with the definitions

$$\bar{Y} = \sum (\frac{1}{n}) Y_i$$

$$b_1 \quad = \quad \sum k_i Y_i, \; k_i = \frac{(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$$

## Sampling Distribution of $\hat{Y}_h$

1. We want to show that mean response and the estimate $b_1$ are uncorrelated

$$Cov(\bar{Y}, b_1) = \sigma^2\{\bar{Y}, b_1\} = 0$$

2. To do this we need the following result (A.32)

$$\sigma^2\{\sum_{i=1}^{n} a_i Y_i, \sum_{i=1}^{n} c_i Y_i\} = \sum_{i=1}^{n} a_i c_i \sigma^2\{Y_i\}$$

when the $Y_i$ are independent

# Sampling Distribution of $\hat{Y}_h$

Using this fact we have

$$\begin{aligned}\sigma^2\{\sum_{i=1}^{n}\frac{1}{n}Y_i, \sum_{i=1}^{n}k_iY_i\} &= \sum_{i=1}^{n}\frac{1}{n}k_i\sigma^2\{Y_i\} \\ &= \sum_{i=1}^{n}\frac{1}{n}k_i\sigma^2 \\ &= \frac{\sigma^2}{n}\sum_{i=1}^{n}k_i \\ &= 0\end{aligned}$$

So the $\bar{Y}$ and $b_1$ are uncorrelated

# Sampling Distribution of $\hat{Y}_h$

1. This means that we can write down the variance

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\{\bar{Y} + b_1(X_h - \bar{X})\}$$

   alternative and equivalent form of regression function

2. But we know that the mean of Y and $b_1$ are uncorrelated so

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\{\bar{Y}\} + \sigma^2\{b_1\}(X_h - \bar{X})^2$$

# Sampling Distribution of $\hat{Y}_h$

1. We know (from last lecture)

$$\begin{aligned} \sigma^2\{b_1\} &= \frac{\sigma^2}{\sum(X_i - \bar{X})^2} \\ s^2\{b_1\} &= \frac{MSE}{\sum(X_i - \bar{X})^2} \end{aligned}$$

2. And we can find

$$\sigma^2\{\bar{Y}\} = \frac{1}{n^2}\sum \sigma^2\{Y_i\} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

# Sampling Distribution of $\hat{Y}_h$

1. So, plugging in, we get

$$\sigma^2\{\hat{Y}_h\} = \frac{\sigma^2}{n} + \frac{\sigma^2}{\sum(X_i - \bar{X})^2}(X_h - \bar{X})^2$$

2. Or

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right)$$

# Sampling Distribution of $\hat{Y}_h$

Since we often won't know $\sigma^2$ we can, as usual, plug in $s^2 = SSE/(n-2)$, our estimate for it to get our estimate of this sampling distribution variance

$$s^2\{\hat{Y}_h\} = s^2\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right)$$

# No surprise. . .

1. The sampling distribution of our point estimator for the output is distributed as a t-distribution with two degrees of freedom

$$\frac{\hat{Y}_h - E\{Y_h\}}{s\{\hat{Y}_h\}} \sim t(n-2)$$

2. This means that we can construct confidence intervals in the same manner as before.

# Confidence Intervals for $\mathbb{E}(Y_h)$

1. The $1 - \alpha$ confidence intervals for $\mathbb{E}(Y_h)$ are

$$\hat{Y}_h \pm t(1 - \alpha/2; n - 2)s\{\hat{Y}_h\}$$

2. From this hypothesis tests can be constructed as usual.

# Comments

1. The variance of the estimator for$\mathbb{E}(Y_h)$ is smallest near the mean of X. Designing studies such that the mean of X is near $X_h$ will improve inference precision
2. When $X_h$ is zero the variance of the estimator for $\mathbb{E}(Y_h)$ reduces to the variance of the estimator $b_0$ for $\beta_0$

# Prediction interval for single new observation

1. Essentially follows the sampling distribution arguments for $\mathbb{E}(Y_h)$
2. If all regression parameters are known then the $1-\alpha$ prediction interval for a new observation $Y_h$ is

$$\mathbb{E}\{Y_h\} \pm z(1-\alpha/2)\sigma$$

# Prediction interval for single new observation

1. If the regression parameters are unknown the $1-\alpha$ prediction interval for a new observation $Y_h$ is given by the following theorem
$$\hat{Y}_h \pm t(1-\alpha/2; n-2)s\{pred\}$$
2. This is very nearly the same as prediction for a known value of X but includes a correction for the fact that there is additional variability arising from the fact that the new input location was not used in the original estimates of $b_1$, $b_0$, and $s^2$

# Prediction interval for single new observation

The value of $s^2\{pred\}$ is given by

$$s^2\{pred\} = MSE\left[1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]$$