Inference in Regression Analysis

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Today: Normal Error Regression Model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Y_i value of the response variable in the i^{th} trial
- β_0 and β_1 are parameters
- X_i is a known constant, the value of the predictor variable in the i^{th} trial
- $\epsilon_i \sim_{iid} N(0,\sigma^2)$
- i = 1,…,n

Inferences concerning β_1

- Tests concerning β_1 (the slope) are often of interest, particularly

$$H_0 : \beta_1 = 0$$
$$H_a : \beta 1 \neq 0$$

the null hypothesis model

$$Y_i = \beta_0 + (0)X_i + \epsilon_i$$

implies that there is no relationship between Y and X

Review : Hypothesis Testing

- Elements of a statistical test
  - Null hypothesis, H_0
  - Alternative hypothesis, H_a
  - Test statistic
  - Rejection region

Review : Hypothesis Testing - Errors

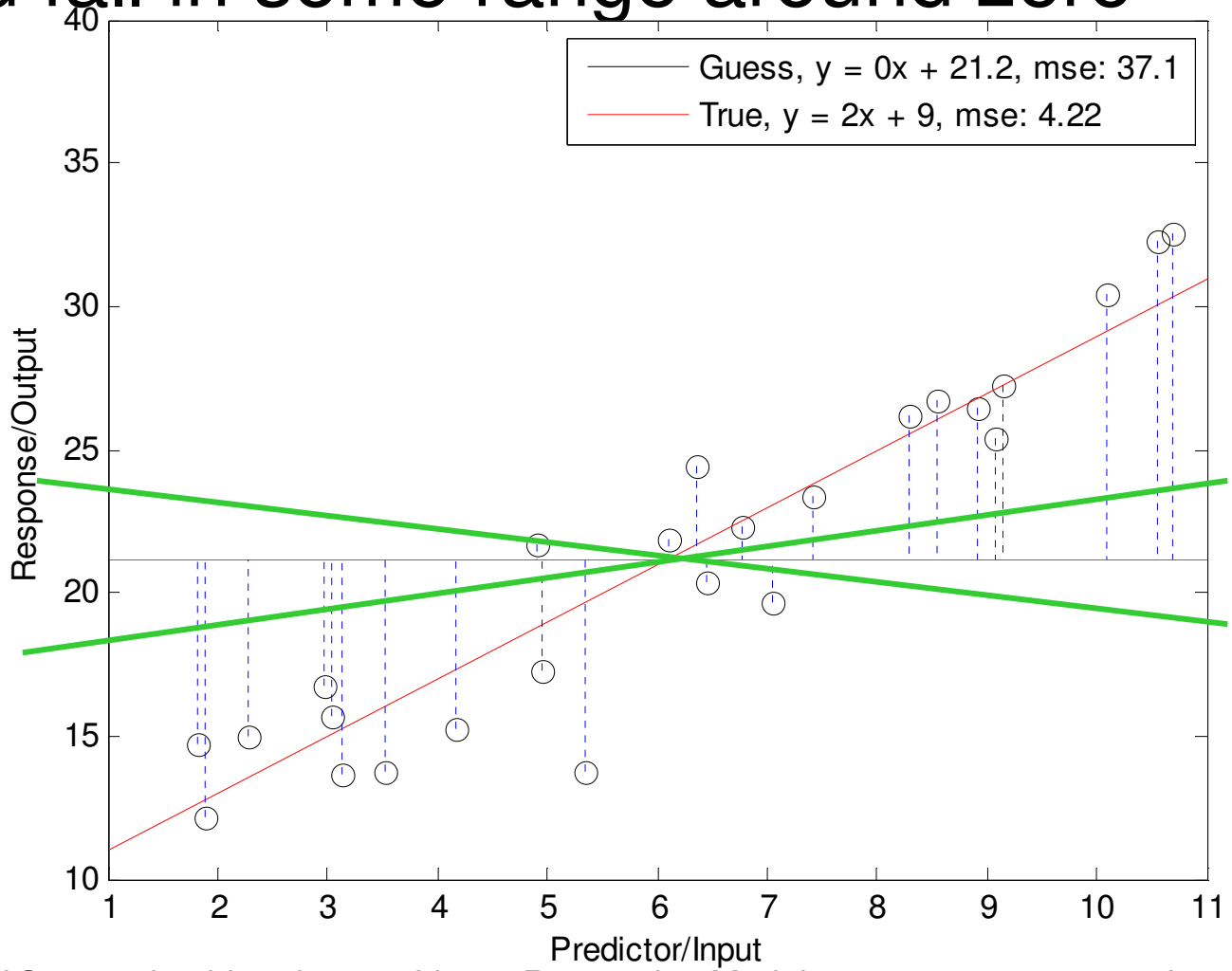
- Errors
  - A type I error is made if H_0 is rejected when H_0 is true. The probability of a type I error is denoted by α. The value of α is called the level of the test.
  - A type II error is made if H_0 is accepted when H_a is true. The probability of a type II error is denoted by β.

P-value

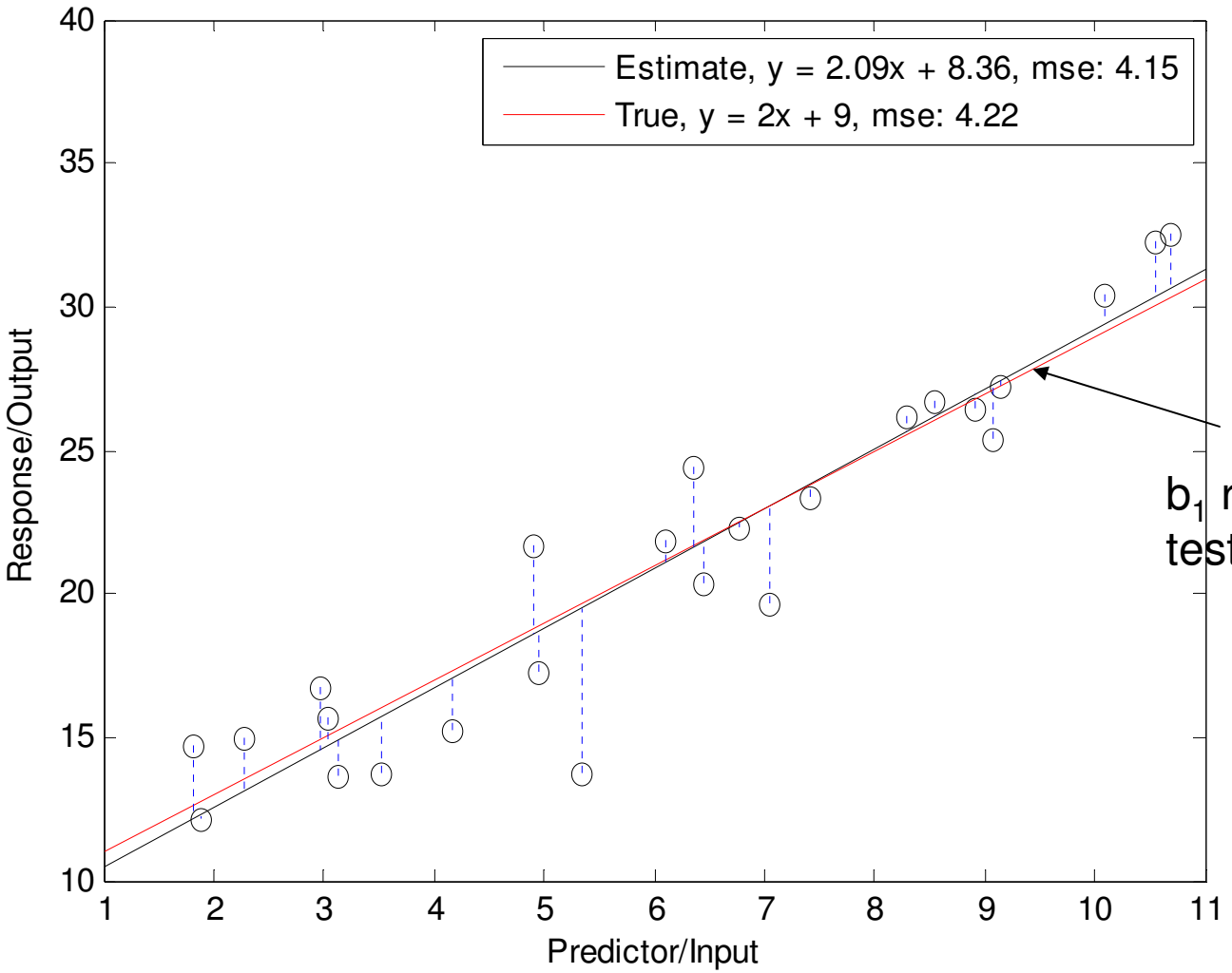
- The p-value, or attained significance level, is the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected.

Null Hypothesis

- If $\beta_1 = 0$ then with 95% confidence the b_1 would fall in some range around zero



Alternative Hypothesis : Least Squares Fit



b_1 rescaled is test statistic

Testing This Hypothesis

- Only have a finite sample
- Different finite set of samples (from the same population / source) will (almost always) produce different estimates of β_0 and β_1 (b_0, b_1) given the same estimation procedure
- b_0 and b_1 are random variables whose *sampling distributions* can be statistically characterized
- Hypothesis tests can be constructed using these distributions.

Example : Sampling Dist. Of b_1

- The point estimator for b_1 is

$$b_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

- The sampling distribution for b_1 is the distribution over b_1 that occurs when the predictor variables X_i are held fixed and the observed outputs are repeatedly sampled

Sampling Dist. Of b_1 In Normal Regr. Model

- For a normal error regression model the sampling distribution of b_1 is normal, with mean and variance given by

$$E(b_1) = \beta_1$$

$$V(b_1) = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$

- To show this we need to go through a number of algebraic steps.

First step

- To show

$$\sum (X_i - \bar{X})(Y_i - \bar{Y}) = \sum (X_i - \bar{X}) Y_i$$

we observe

$$\begin{aligned}
\sum (X_i - \bar{X})(Y_i - \bar{Y}) &= \sum (X_i - \bar{X}) Y_i - \sum (X_i - \bar{X}) \bar{Y} \\
&= \sum (X_i - \bar{X}) Y_i - \bar{Y} \sum (X_i - \bar{X}) \\
&= \sum (X_i - \bar{X}) Y_i - \bar{Y} \sum (X_i) + \bar{Y} n \frac{\sum X_i}{n} \\
&= \sum (X_i - \bar{X}) Y_i
\end{aligned}$$

Slope as linear combination of outputs

- b_1 can be expressed as a linear combination of the Y_i's

$$\begin{aligned} b_1 &= \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} \\ &= \frac{\sum(X_i - \bar{X})Y_i}{\sum(X_i - \bar{X})^2} \\ &= \sum k_i Y_i \end{aligned}$$

where

$$k_i = \frac{\sum(X_i - \bar{X})}{\sum(X_i - \bar{X})^2}$$

Properties of the k_i's

- It can be shown that

$$\begin{aligned} \sum k_i &= 0 \\ \sum k_i X_i &= 1 \\ \sum k_i^2 &= \frac{1}{\sum (X_i - \bar{X})^2} \end{aligned}$$

(possible homework). We will use these properties to prove various properties of the sampling distributions of b_1 and b_0.

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Normality of b_1's Sampling Distribution

- Useful fact:
  - A linear combination of independent normal random variables is normally distributed
  - More formally: when $Y_1, \ldots, Y_n$ are independent normal random variables, the linear combination $a_1Y_1 + a_2Y_2 + \ldots + a_nY_n$ is normally distributed, with mean $\sum a_iE(Y_i)$ and variance $\sum a^2_iV(Y_i)$

Normality of b_1's Sampling Distribution

- Since b_1 is a linear combination of the Y_i's and each Y_i is an independent normal random variable, then b_1 is distributed normally as well

$$b_1 = \sum k_i Y_i, \; k_i = \frac{(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$$

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b_1 is an unbiased estimator

- This can be seen using two of the properties

$$\begin{aligned} E(b_1) &= E(\sum k_i Y_i) = \sum k_i E(Y_i) = \sum k_i(\beta_0 + \beta_1 X_i) \\ &= \beta_0 \sum k_i + \beta_1 \sum k_i X_i \\ &= \beta_0(0) + \beta_1(1) \\ &= \beta_1 \end{aligned}$$

Variance of b_1

- Since the Y_i are independent random variables with variance σ^2 and the k_i's are constants we get

$$\begin{aligned} V(b_1) &= V(\sum k_i Y_i) = \sum k_i^2 V(Y_i) \\ &= \sum k_i^2 \sigma^2 = \sigma^2 \sum k_i^2 \\ &= \sigma^2 \frac{1}{\sum (X_i - \bar{X})^2} \end{aligned}$$

  note that this assumes that we know σ^2.
  - Can we?

Estimated variance of b_1

- If we don't know σ^2 then we can replace it with the MSE estimate
- Remember

$$s^2 = MSE = \frac{SSE}{n-2} = \frac{\sum (Y_i - \hat{Y}_i)^2}{n-2} = \frac{\sum e_i^2}{n-2}$$

plugging in we get

$$V(b_1) = \frac{\sigma^2}{\sum (X_i - \bar{X})^2}$$

$$\hat{V}(b_1) = \frac{s^2}{\sum (X_i - \bar{X})^2}$$

Digression : Gauss-Markov Theorem

- In a regression model where E(ϵ_i) = 0 and variance V(ϵ_i) = $\sigma^2 < \infty$ and ϵ_i and ϵ_j are uncorrelated for all i and j the least squares estimators b_0 and b_1 and unbiased and have minimum variance among all unbiased linear estimators.
  - Remember

$$b_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

$$b_0 = \bar{Y} - b_1\bar{X}$$

Proof

- The theorem states that b_1 as minimum variance among all unbiased linear estimators of the form

$$\hat{\beta}_1 = \sum c_i Y_i$$

- As this estimator must be unbiased we have

$$\begin{aligned} E(\hat{\beta}_1) &= \sum c_i E(Y_i) = \beta_1 \\ &= \sum c_i (\beta_0 + \beta_1 X_i) = \beta_0 \sum c_i + \beta_1 \sum c_i X_i = \beta_1 \end{aligned}$$

Proof cont.

- Given these constraints

$$\beta_0 \sum c_i + \beta_1 \sum c_i X_i = \beta_1$$

clearly it must be the case that $\sum c_i = 0$ and $\sum c_i X_i = 1$

write these on board as conditions of unbiasedness

- The variance of this estimator is

$$V(\hat{\beta}_1) = \sum c_i^2 V(Y_i) = \sigma^2 \sum c_i^2$$

Proof cont.

- Now define $c_i = k_i + d_i$ where the k_i are the constants we already defined and the d_i are arbitrary constants. Let's look at the variance of the estimator

$$\begin{aligned} V(\hat{\beta}_1) &= \sum c_i^2 V(Y_i) = \sigma^2 \sum (k_i + d_i)^2 \\ &= \sigma^2 (\sum k_i^2 + \sum d_i^2 + 2 \sum k_i d_i) \end{aligned}$$

Note we just demonstrated that

$$\sigma^2 \sum k_i^2 = V(b_1)$$

Proof cont.

- Now by showing that $\sum k_i d_i = 0$ we're almost done

$$\begin{aligned}
\sum k_i d_i &= \sum k_i (c_i - k_i) \\
&= \sum k_i (c_i - k_i) \\
&= \sum k_i c_i - \sum k_i^2 \\
&= \sum c_i \left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right) - \frac{1}{\sum (X_i - \bar{X})^2} \\
&= \frac{\sum c_i X_i - \bar{X} \sum c_i}{\sum (X_i - \bar{X})^2} - \frac{1}{\sum (X_i - \bar{X})^2} = 0
\end{aligned}$$

from conditions of unbiasedness

Proof end

- So we are left with

$$\begin{aligned} V(\hat{\beta}_1) &= \sigma^2(\sum k_i^2 + \sum d_i^2) \\ &= V(b_1) + \sigma^2(\sum d_i^2) \end{aligned}$$

which is minimized when the d_i's = 0.

This means that the least squares estimator b_1 has minimum variance among all unbiased linear estimators.

Sampling Distribution of (b_1 - β_1)/S(b_1)

- b_1 is normally distributed so (b_1-β_1)/(V(b_1)$^{1/2}$) is a standard normal variable
- We don't know V(b_1) so it must be estimated from data. We have already denoted it's estimate $\hat{V}(b_1)$
- Using this estimate we it can be shown that

$$\frac{b_1-\beta_1}{\hat{S}(b_1)} \sim t(n-2) \quad \hat{S}(b_1) = \sqrt{\hat{V}(b_1)}$$

Where does this come from?

- We need to rely upon the following theorem
  - For the normal error regression model

$$\frac{SSE}{\sigma^2} = \frac{\sum (Y_i - \hat{Y}_i)^2}{\sigma^2} \sim \chi^2(n-2)$$

and is independent of b_0 and b_1

- Intuitively this follows the standard result for the sum of squared normal random variables
  - Here there are two linear constraints imposed by the regression parameter estimation that each reduce the number of degrees of freedom by one.

Another useful fact : t distribution

- Let z and $\chi^2(\nu)$ be independent random variables (standard normal and χ^2 respectively). We then *define* a t random variable as follows:

$$t(\nu) = \frac{z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}}$$

This version of the t distribution has one parameter, the degrees of freedom ν

Distribution of the studentized statistic

- To derive the distribution of this statistic, first we do the following rewrite

This is a standard normal variable

$$\frac{b_1-\beta_1}{\hat{S}(b_1)} = \frac{\frac{b_1-\beta_1}{S(b_1)}}{\frac{\hat{S}(b_1)}{S(b_1)}}$$

$$\frac{\hat{S}(b_1)}{S(b_1)} = \sqrt{\frac{\hat{V}(b_1)}{V(b_1)}}$$

Studentized statistic cont.

- And note the following

$$\frac{\hat{V}(b_1)}{V(b_1)} = \frac{\frac{MSE}{\sum (X_i - \bar{X})^2}}{\frac{\sigma^2}{\sum (X_i - \bar{X})^2}} = \frac{MSE}{\sigma^2} = \frac{SSE}{\sigma^2(n-2)}$$

where we know (by the given theorem) the distribution of the last term is χ^2 and indep. of b_1 and b_0

$$\frac{SSE}{\sigma^2(n-2)} \sim \frac{\chi^2(n-2)}{n-2}$$

Studentized statistic final

- But by the given definition of the t distribution we have our result

$$\frac{b_1 - \beta_1}{\hat{S}(b_1)} \sim t(n-2)$$

because putting everything together we can see that

$$\frac{b_1 - \beta_1}{\hat{S}(b_1)} \sim \frac{z}{\sqrt{\frac{\chi^2(n-2)}{n-2}}}$$

Confidence Intervals and Hypothesis Tests

- Now that we know the sampling distribution of b_1 (t with n-2 degrees of freedom) we can construct confidence intervals and hypothesis tests easily