Remedial Measures, Brown-Forsythe test, F test

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Remedial Measures

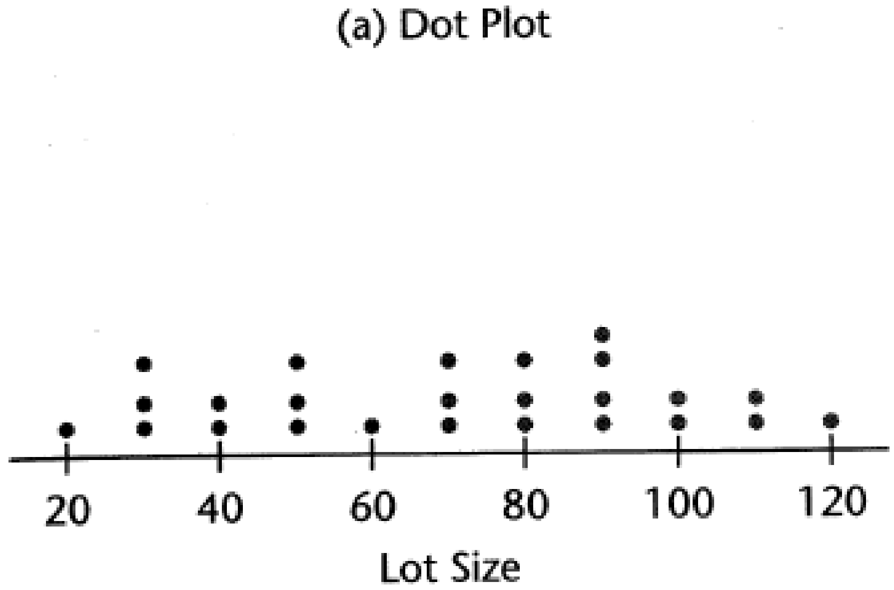
- How do we know that the regression function is a good explainer of the observed data?
  - Plotting
  - Tests
- What if it is not? What can we do about it?
  - Transformation of variables (next lecture)

Graphical Diagnostics for the Predictor Variable

- Dot Plot
  - Useful for visualizing distribution of inputs
- Sequence Plot
  - Useful for visualizing dependencies between error terms
- Box Plot
  - Useful for visualizing distribution of inputs

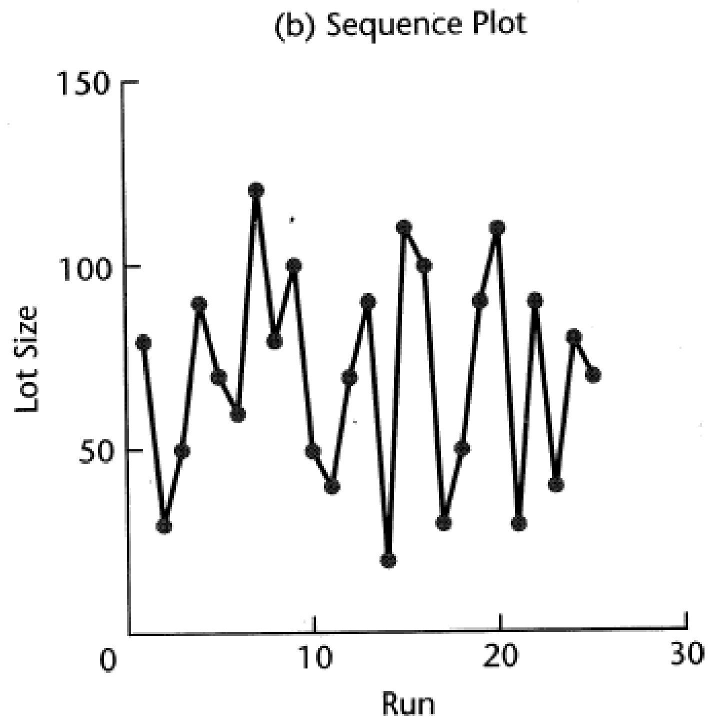
- Toluca manufacturing example again: production time vs. lot size.

Dot Plot



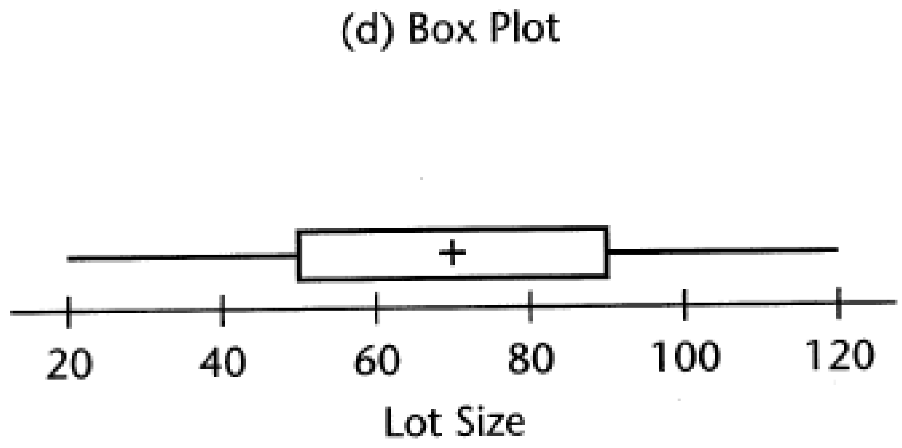
- How many observations per input value?
- Range of inputs?

Sequence Plot



- If observations are made over time, is there a correlation between input and position in observation sequence?

Box Plot



- Shows
  - Median
  - 1^{st} and 3^{rd} quartiles
  - Maximum and minimum

Residuals

- Remember, the definition of residual:

$$e_i = Y_i - \hat{Y}_i$$

- And the difference between that and the unknown true error

$$\epsilon_i = Y_i - E\{Y_i\}$$

- In a normal regression model the ϵ_i's are assumed to be iid N(0,σ^2) random variables. The observed residuals e_i should reflect these properties.

Remember: residual properties

- Mean

$$\bar{e}_i = \frac{\sum e_i}{n} = 0$$

- Variance

$$s^2 = \frac{\sum (e_i - \bar{e})^2}{n-2} = \frac{SSE}{n-2} = MSE$$

Nonindependence of Residuals

- The residuals e_i are *not* independent random variables
  - The fitted values $\hat{Y}_i$ are based on the same fitted regression line.
    - The residuals are subject to two constraints
      - Sum of the e_i's equals 0
      - Sum of the products X_i e_i's equals 0

- When the sample size is large in comparison to the number of parameters in the regression model, the dependency effect among the residuals e_i can reasonably safely be ignored.

Definition: semistudentized residuals

- Like usual, since the standard deviation of ϵ_i is σ (itself estimated by MSE$^{1/2}$) a natural form of standardization to consider is

$$e_i^* = \frac{e_i - 0}{\sqrt{MSE}}$$

- This is called a semistudentized residual.

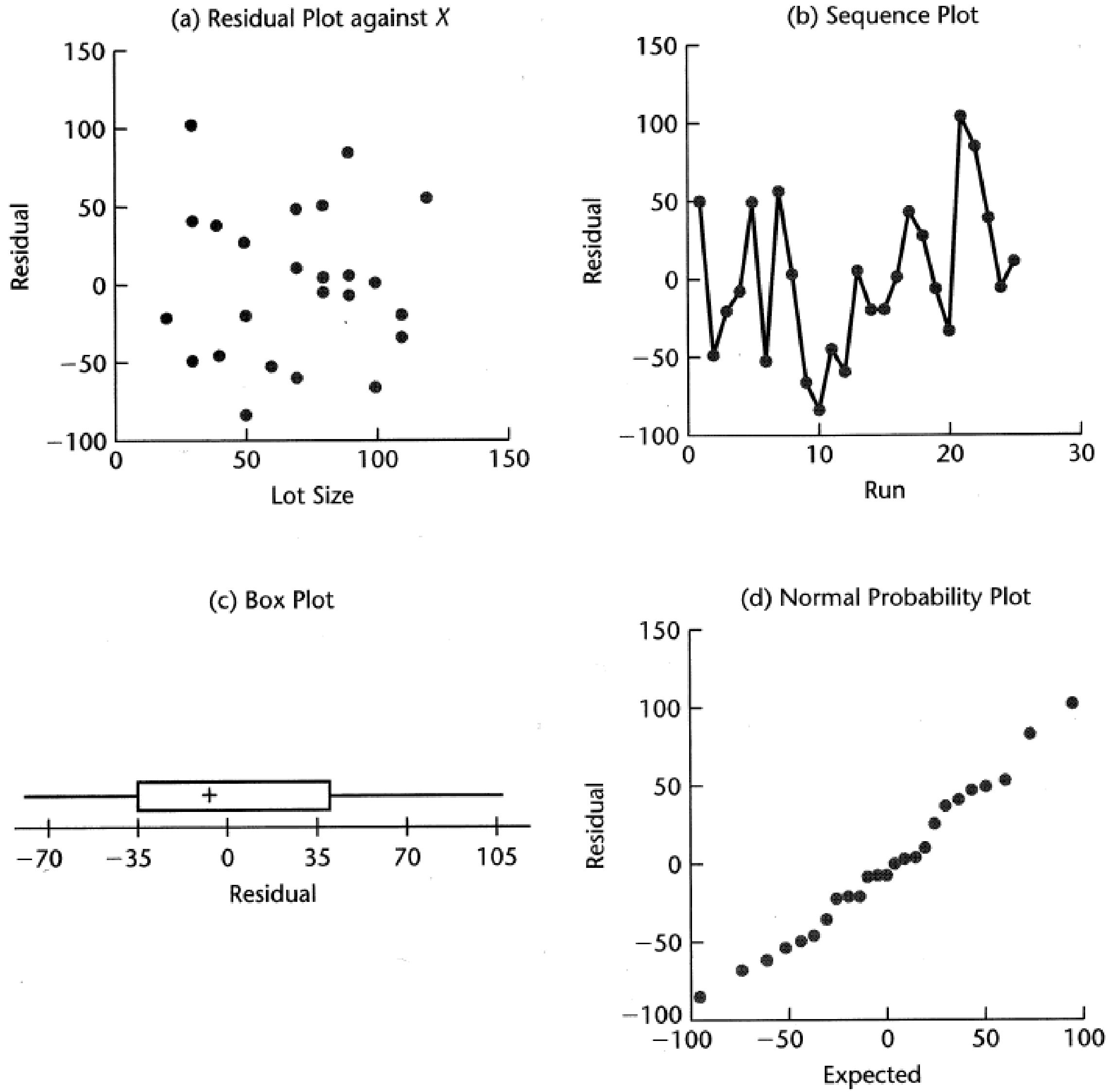
Departures from Model…

- to be studied by residuals
  - Regression function not linear
  - Error terms do not have constant variance
  - Error terms are not independent
  - Model fits all but one or a few outlier observations
  - Error terms are not normally distributed
  - One or more predictor variables have been omitted from the model

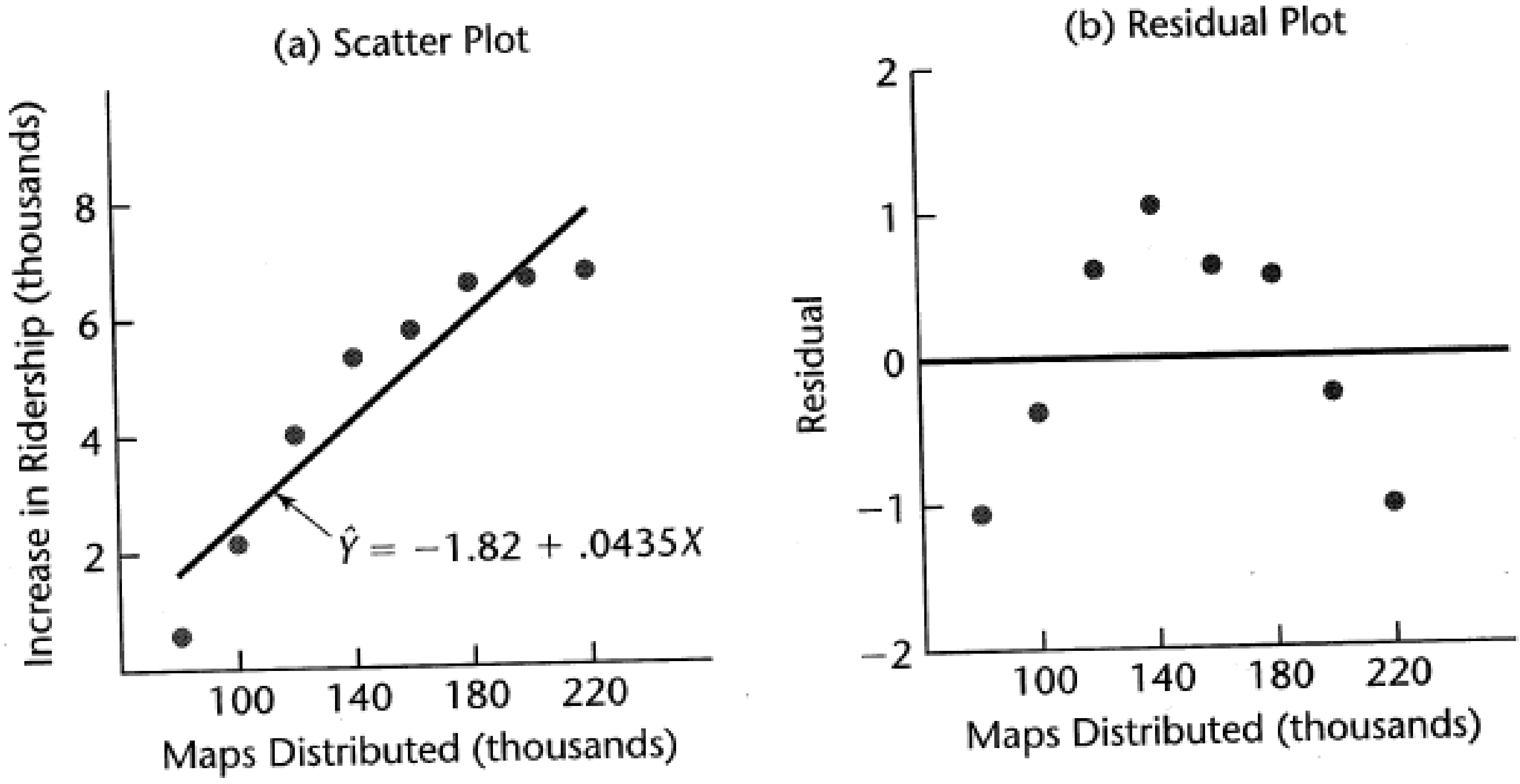
Diagnostics for Residuals

- Plot of residuals against predictor variable
- Plot of absolute or squared residuals against predictor variable
- Plot of residuals against fitted values
- Plot of residuals against time or other sequence
- Plots of residuals against omitted predictor variables
- Box plot of residuals
- Normal probability plot of residuals

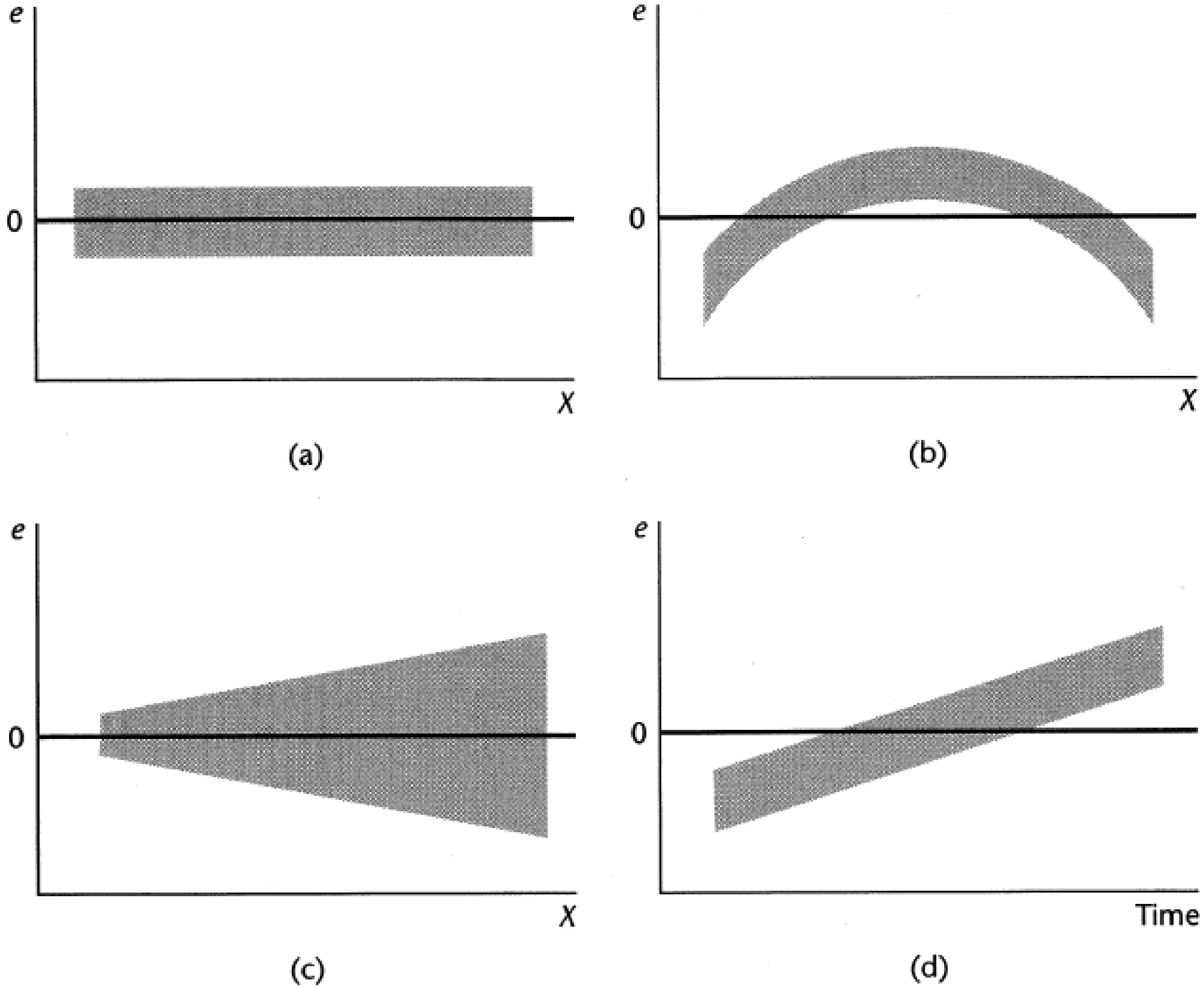
Diagnostic Residual Plots



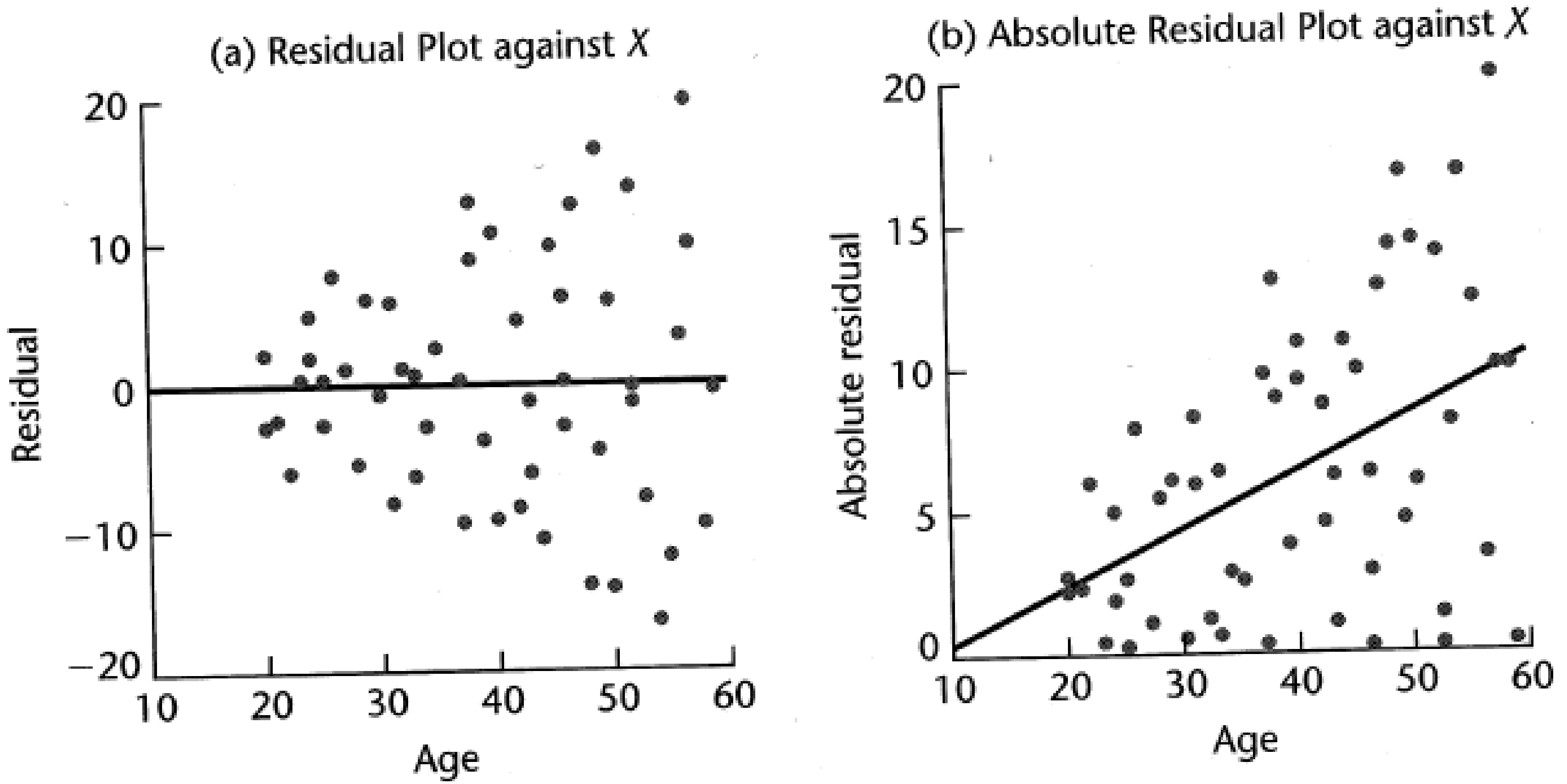
Scatter and Residual Plot



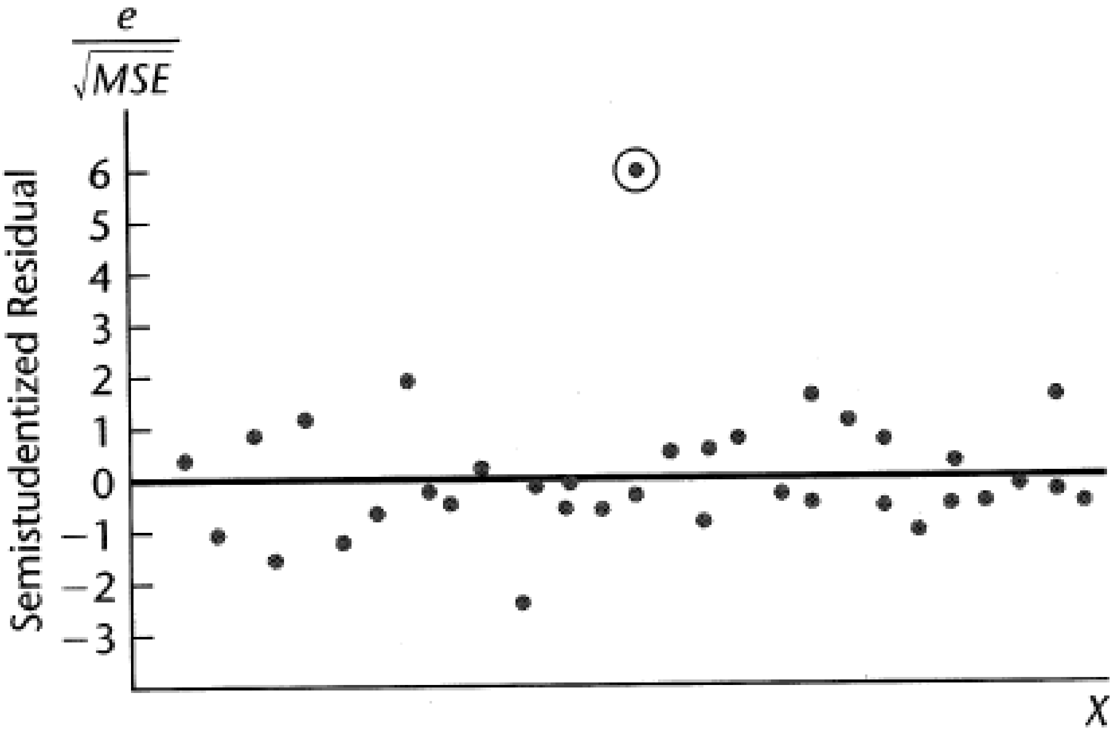
Prototype Residual Plots



Nonconstancy of Error Variance

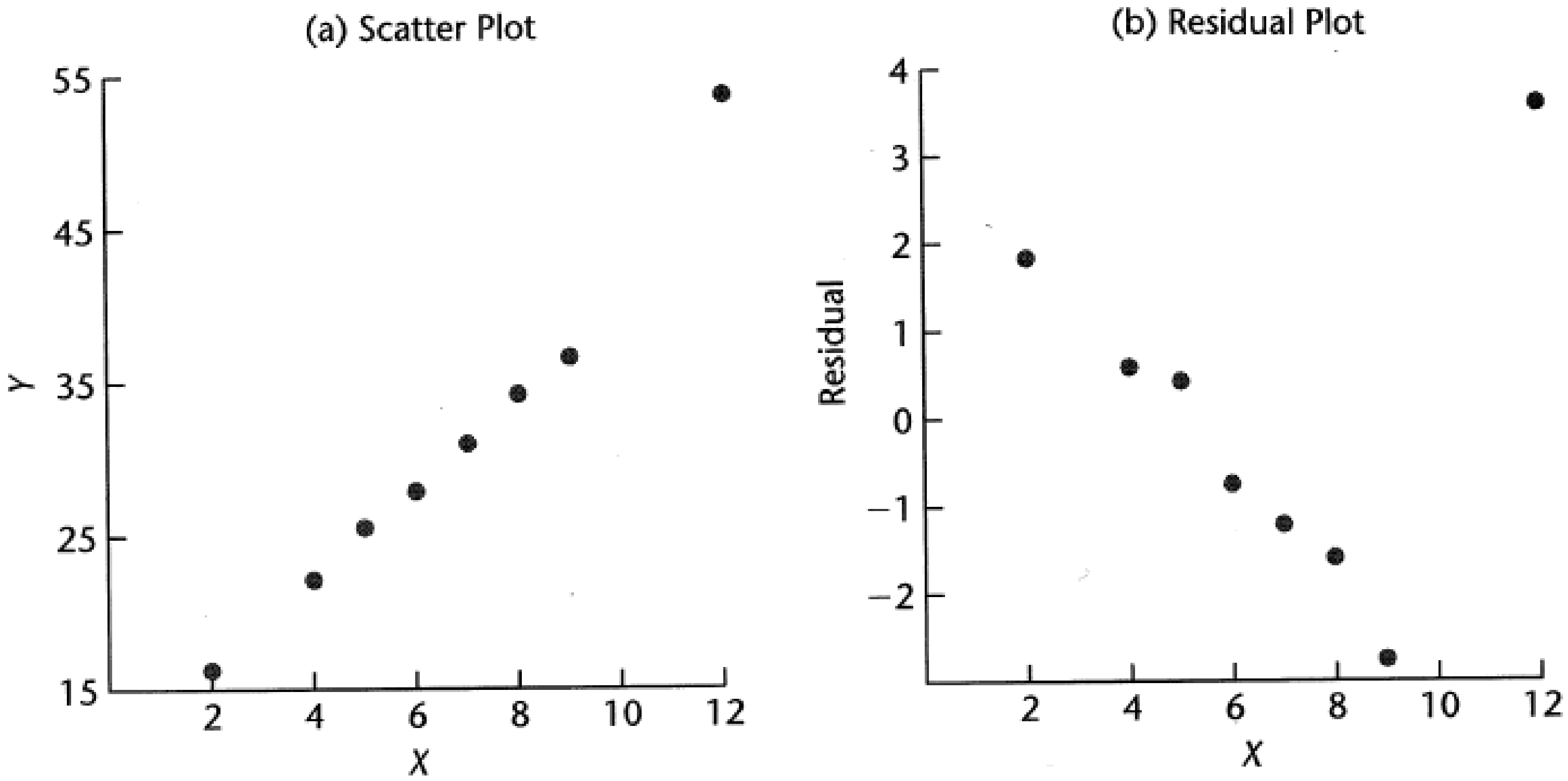


Presence of Outliers

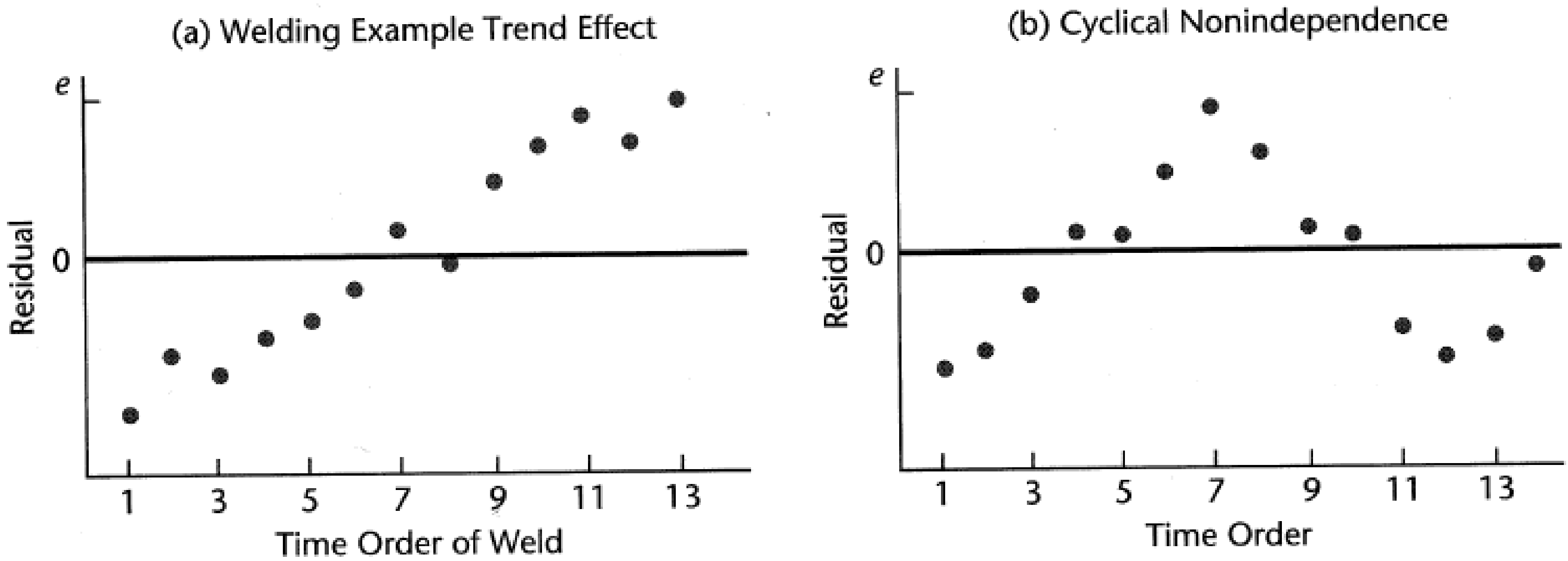


- Outliers can strongly effect the fitted values of the regression line.

Outlier effect on residuals



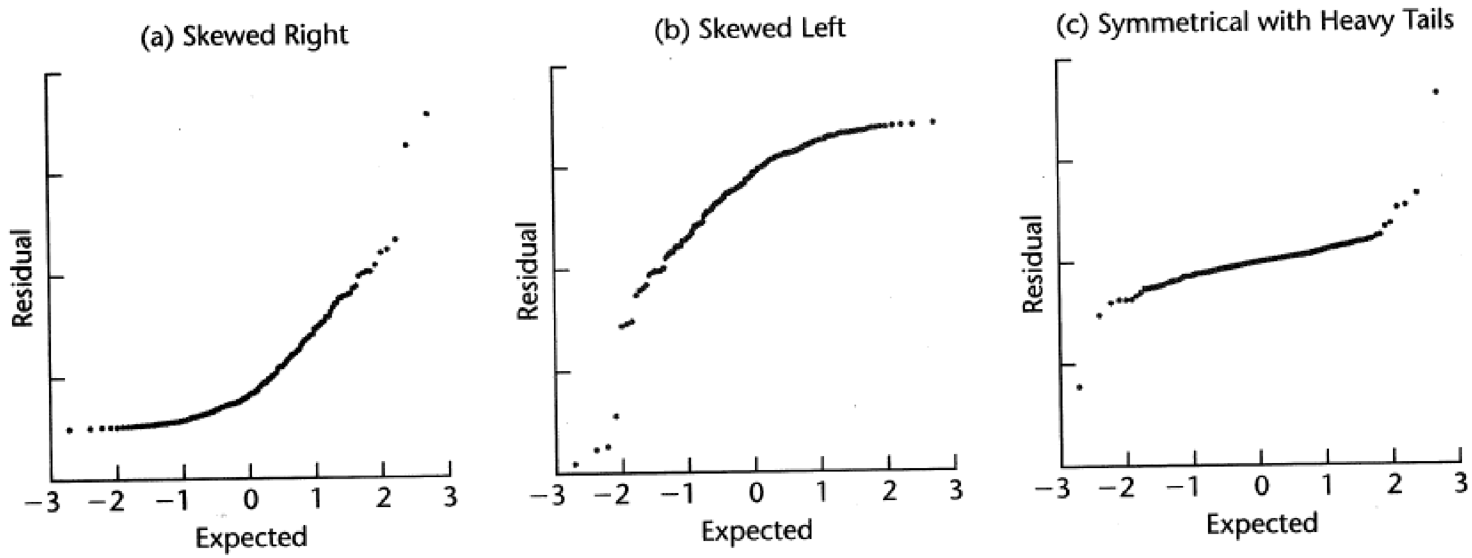
Nonindependence of Error Terms



- Sequential observations

Non-normality of Error Terms

- Distribution plots
- Comparison of Frequencies
- Normal probability plot



Normal probability plot

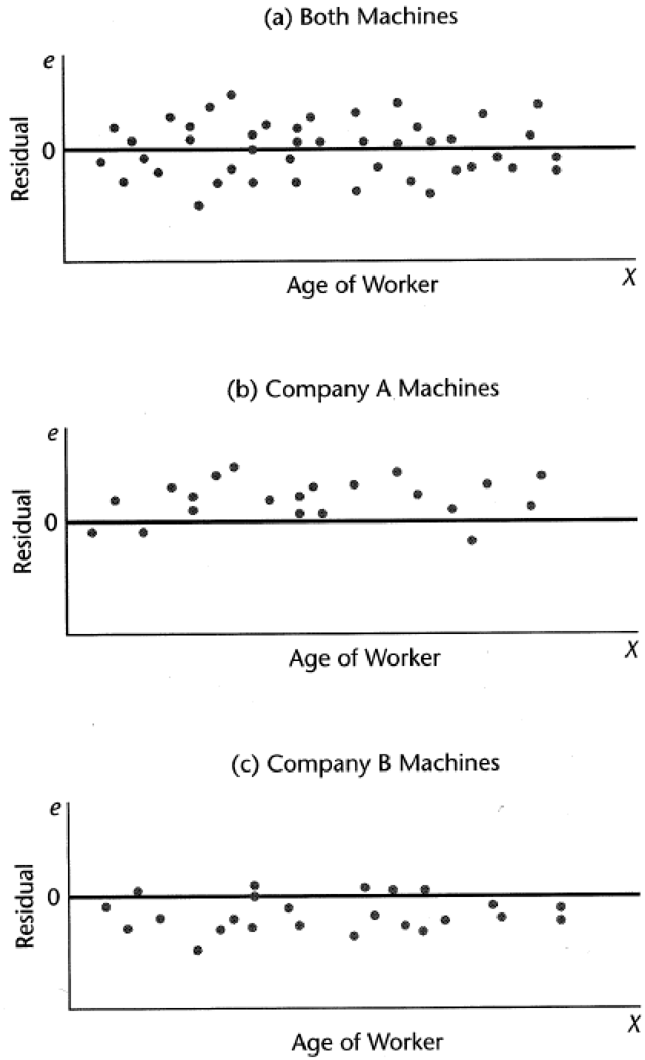
- For a N(0,MSE$^{1/2}$) random variable, a good approximation of the expected value of the kth smallest observation in a random sample of size n is

$$\sqrt{MSE}\left[z\left(\frac{k-.375}{n+.25}\right)\right]$$

- A normal probability plot consists of plotting the expected value of the kth smallest observation against the *observed* kth smallest observation

Omission of Important Predictor Variables

- Example
  - Qualitative variable
    - Type of machine
- Partitioning data can reveal dependence on omitted variable(s)
- Works for quantitative variables as well
- Can suggest that inclusion of other inputs is important



Tests Involving Residuals

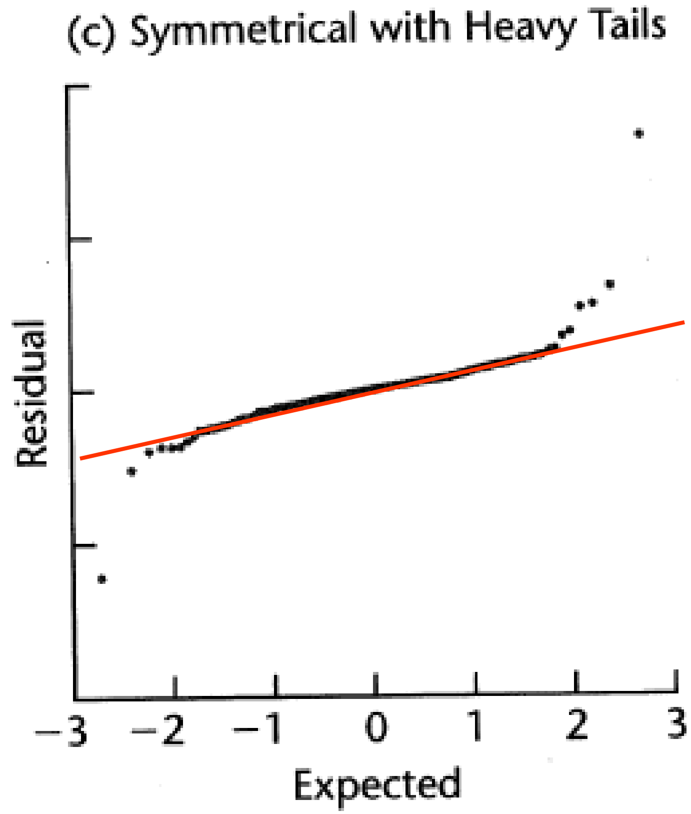
- Tests for randomness
- Tests for constancy of variance
- Tests for outliers
- Tests for normality

Correlation Test for Normality

- Calculated the coefficient of correlation between residuals e_i and their expected values under normality

$$r = \sqrt{R^2}$$

- Tables (B.6 in the book) given critical values for the null hypothesis (normally distributed errors) holding



Tests for Constancy of Error Variance

- Brown-Forsythe test does not depend on normality of error terms.
  - The Brown-Forsythe test is applicable to simple linear regression when
    - The variance of the error terms either increases or decreases with X
    - Sample size is large enough to ignore dependencies between the residuals
- Basically a t-test for testing whether the means of two normally distributed populations are the same

Brown-Forsythe Test

- Divide X into X_1 (the low values of X) and X_2 (the high values of X)
- Let e_{i1} be the error terms for X_1 and vice versa
- Let $n = n_1 + n_2$
- The Brown-Forsythe test uses the absolute deviations of the residuals around their group median

$$d_{1i} = |e_{1i} - \tilde{e}_1|$$

Brown-Forsythe Test

- The test statistic for comparing the means of the absolute deviations of the residuals around the group medians is

$$t^*_{BF} = \frac{\bar{d}_1 - \bar{d}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where

$$s^2 = \frac{\sum (d_{i1} - \bar{d}_1)^2 + \sum (d_{i2} - \bar{d}_2)^2}{n-2}$$

Brown-Forsythe Test

- If n_1 and n_2 are not extremely small

$$t^*_{BF} \sim t(n-2)$$

approximately

- From this confidence intervals and tests can be constructed.

F test for lack of fit

- Formal test for determining whether a specific type of regression function adequately fits the data.
- Assumptions (usual) :
  - Y | X
    - iid
    - normally distributed
    - same variance σ^2
- Requires: repeat observations at one or more X levels (called replicates)

Example

- 12 similar branches of a bank offered gifts for setting up money market accounts
- Minimum initial deposits were specified to qualify for the gift
- Value of gift was proportional to the specified minimum deposit
- Interested in: relationship between specified minimum deposit and number of new accounts opened

F Test Example Data and ANOVA Table

(a) Data

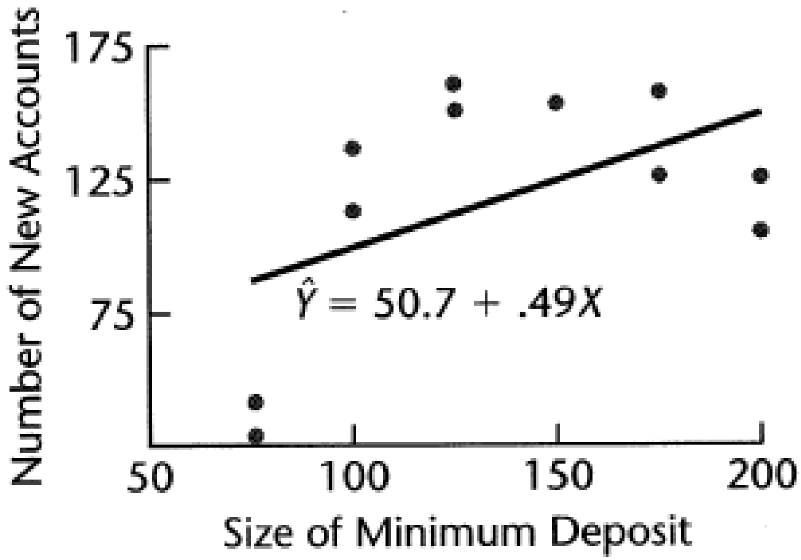
Branch i	Size of Minimum Deposit (dollars) X_i	Number of New Accounts Y_i	Branch i	Size of Minimum Deposit (dollars) X_i	Number of New Accounts Y_i
1	125	160	7	75	42
2	100	112	8	175	124
3	200	124	9	125	150
4	75	28	10	200	104
5	150	152	11	100	136
6	175	156			

(b) ANOVA Table

Source of Variation	SS	df	MS
Regression	5,141.3	1	5,141.3
Error	14,741.6	9	1,638.0
Total	19,882.9	10	

Fit

$$\hat{Y} = 50.72251 + .48670X$$



Data Arranged To Highlight Replicates

	Size of Minimum Deposit (dollars)					
Replicate	$j=1$ $X_1=75$	$j=2$ $X_2=100$	$j=3$ $X_3=125$	$j=4$ $X_4=150$	$j=5$ $X_5=175$	$j=6$ $X_6=200$
$i=1$	28	112	160	152	156	124
$i=2$	42	136	150		124	104
Mean $\bar{Y}_j$	35	124	155	152	140	114

- The observed value of the response variable for the i^{th} replicate for the j^{th} level of X is Y_{ij}.
- The mean of the Y observations at the level $X = X_j$ is $\bar{Y}_j$

Full Model vs. Regression Model

- The full model is

$$Y_{ij} = \mu_j + \varepsilon_{ij} \qquad \text{Full model}$$

  where
  - μ_j are parameters j = 1,…,c
  - ϵ_{ij} are iid N(0,σ^2)
- Since the error terms have expectation zero

$$E\{Y_{ij}\} = \mu_j$$

Full Model

- In the full model there is a different mean (a free parameter) for each X_i
- In the regression model the mean responses are constrained to lie on a line

$$E\{Y\} = \beta_0 + \beta_1 X$$

Fitting the Full Model

- The estimators of μ_j are simply

$$\hat{\mu}_j = \bar{Y}_j$$

- The error sum of squares for the full model therefore is

$$SSE(F) = \sum_j \sum_i (Y_{ij} - \bar{Y}_j)^2 = SSPE$$

Degrees of Freedom

- Ordinary total sum of squares has n-1 degrees of freedom.
- Each of the j terms is a ordinary total sum of squares
  - Each then has n_j -1 degrees of freedom
- The number of degrees of freedom of SSPE is the sum of the component degrees of freedom

$$df_F = \sum_j (n_j - 1) = \sum_j n_j - c = n - c$$

General Linear Test

- Remember: the general linear test proposes a reduced model null hypothesis
  - this will be our normal regression model
- The full model will be as described (one independent mean for each level of X)

$$H_0\colon E\{Y\} = \beta_0 + \beta_1 X$$
$$H_a\colon E\{Y\} \neq \beta_0 + \beta_1 X$$

SSE For Reduced Model

- The SSE for the reduced model is as before
  - remember

$$SSE(R) = \sum \sum [Y_{ij} - (b_0 + b_1 X_j)]^2$$
$$= \sum \sum (Y_{ij} - \hat{Y}_{ij})^2 = SSE$$

  - and has n-2 degrees of freedom

$$df_R = n - 2$$

SSE(R)

(a) Data

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F Test Statistic

- From the general linear test approach

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F}$$

$$F^* = \underbrace{\frac{SSE - SSPE}{(n-2) - (n-c)} \div \frac{SSPE}{n-c}}$$

a little algebra takes us to the next slide

F Test Rule

- From the F test we know that large values of F^* lead us to reject the null hypothesis

$$\text{If } F^* \leq F(1-\alpha; c-2, n-c), \text{ conclude } H_0$$

$$\text{If } F^* > F(1-\alpha; c-2, n-c), \text{ conclude } H_a$$

- For this example we have

$$SSPE = 1{,}148.0 \qquad n - c = 11 - 6 = 5$$

$$SSE = 14{,}741.6$$

$$SSLF = 14{,}741.6 - 1{,}148.0 = 13{,}593.6 \qquad c - 2 = 6 - 2 = 4$$

$$F^* = \frac{13{,}593.6}{4} \div \frac{1{,}148.0}{5}$$

$$= \frac{3{,}398.4}{229.6} = 14.80$$

Example Conclusion

- If we set the significance level to

$$\alpha = .01$$

- And look up the value of the F inv-cdf

$$F(.99; 4, 5) = 11.4$$

- We can conclude that the null hypothesis should be rejected.