Linear Algebra Review

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Definition of Matrix

- Rectangular array of elements arranged in rows and columns

$$\begin{bmatrix} 16000 & 23 \\ 33000 & 47 \\ 21000 & 35 \end{bmatrix}$$

- A matrix has dimensions
- The dimension of a matrix is its number of rows and columns
- It is expressed as 3×2 (in this case)

Indexing a Matrix

- Rectangular array of elements arranged in rows and columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- A matrix can also be notated

$$\mathbf{A} = [a_{ij}], a = 1, 2; j = 1, 2, 3$$

Square Matrix and Column Vector

- Square matrix has equal number of rows and columns

$$\begin{bmatrix} 4 & 7 \\ 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A column vector is a matrix with a single column

$$\begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

- All vectors (row or column) are matrices, all scalars are 1×1 matrices.

Transpose

- The transpose of ta matrix is another matrix in which the rows and columns have been interchanged

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} 2 & 7 & 3 \\ 5 & 10 & 4 \end{bmatrix}$$

Row Vector

- A row vector is the transpose of a column vector or a matrix with a single row

$$\mathbf{B} = \begin{bmatrix} 15 & 25 & 50 \end{bmatrix} \qquad \mathbf{F}' = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$$

Equality of Matrices

- Two matrices are the same if they have the same dimension and all the elements are equal

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix}$$

$A = B$ implies $a_1 = 4, a_2 = 7, a_3 = 3$

Regression Examples

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{bmatrix}$$

Regression Examples

Design matrix

$$\mathbf{X} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ . & \\ . & \\ . & \\ 1 & X_n \end{bmatrix}$$

Matrix Addition and Substraction

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

Regression Example

$$Y_i = E(Y_i) + \epsilon_i, \;\; i = 1, ..., n$$

$$E(\mathbf{Y}) = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ . \\ . \\ . \\ E(Y_n) \end{bmatrix} \qquad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ . \\ \epsilon_n \end{bmatrix}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{bmatrix} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ . \\ . \\ . \\ E(Y_n) \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ . \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} E(Y_1) + \epsilon_1 \\ E(Y_2) + \epsilon_2 \\ . \\ . \\ . \\ E(Y_n) + \epsilon_n \end{bmatrix}$$

Multiplication of a Matrix by a Scalar

$$\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix}$$

$$k\mathbf{A} = k \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 2k & 7k \\ 9k & 3k \end{bmatrix}$$

Multiplication of two Matrices

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \qquad \underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$$

$$\begin{array}{cc} & \mathbf{A} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \end{array} & \begin{bmatrix} \boxed{2 \quad 5} \\ 4 \quad 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \mathbf{B} \\ \begin{bmatrix} 4 & \boxed{\begin{matrix} 6 \\ 8 \end{matrix}} \\ 5 & \end{bmatrix} \\ \text{Col. 1 Col. 2} \end{array} \qquad \begin{array}{cc} & \mathbf{AB} \\ \text{Row 1} & \begin{bmatrix} 33 & 52 \\ & \end{bmatrix} \\ & \text{Col. 1 Col. 2} \end{array}$$

$$\begin{array}{cc} & \mathbf{A} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \end{array} & \begin{bmatrix} \boxed{2 \quad 5} \\ 4 \quad 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \mathbf{B} \\ \begin{bmatrix} 4 & \boxed{\begin{matrix} 6 \\ 8 \end{matrix}} \\ 5 & \end{bmatrix} \\ \text{Col. 1 Col. 2} \end{array} \qquad \begin{array}{cc} & \mathbf{AB} \\ \text{Row 1} & \begin{bmatrix} 33 & 52 \\ & \end{bmatrix} \\ & \text{Col. 1 Col. 2} \end{array}$$

Another Matrix Multiplication Example

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 26 \\ 41 \end{bmatrix}$$

Regression Example

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 X_1 \\ \beta_0 + \beta_1 X_2 \\ \beta_0 + \beta_1 X_3 \end{bmatrix}$$

Regression example

Sum of squares

$$
\begin{aligned}
\mathbf{Y}'\mathbf{Y} &= \mathbf{Y}'\mathbf{I}\mathbf{Y} \\
&= \begin{bmatrix} Y_1 & Y_1 & \dots & Y_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{bmatrix} \\
&= \left[Y_1^2 + Y_2^2 + \dots + Y_n^2 \right] \\
&= \left[\sum Y_i^2 \right]
\end{aligned}
$$

More Regression Examples

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ X_1 & X_1 & \dots & X_n \end{bmatrix} \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ . & \\ . & \\ . & \\ 1 & X_n \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ X_1 & X_1 & \dots & X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ . \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}$$

Special Matrices

- If $A = A'$, then A is a symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix} \qquad \mathbf{A}' = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix}$$

- If the off-diagonal elements of a matrix are all zeros it is then called a diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Identity Matrix

A diagonal matrix whose diagonal entries are all ones is an identity matrix. Multiplication by an identity matrix leaves the pre or post multiplied matrix unchanged.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Vector and matrix with all elements equal to one

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 1 & \dots & 1 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & \dots & 1 \end{bmatrix}$$

$$\mathbf{1}\mathbf{1}' = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & . & . & . & 1 \end{bmatrix} = \begin{bmatrix} 1 & \dots & 1 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & \dots & 1 \end{bmatrix} = \mathbf{J}$$

Linear Dependence and Rank of Matrix

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 2 & 10 & 6 \\ 3 & 4 & 15 & 1 \end{bmatrix}$$

and think of this as a matrix of a collection of column vectors.

Note that the third column vector is a multiple of the first column vector.

Linear Dependence

When c scalars $k_1, ..., k_c$ not all zero, can be found such that:

$$k_1 C_1 + ... + k_c C_c = 0$$

where 0 denotes the zero column vector and C_i is the i^{th} column of matrix C, the c column vectors are called linearly dependent. If the only set of scalars for which the equality holds is $k_1 = 0, ..., k_c = 0$, the set of c column vectors is linearly independent.

In the previous example matrix the columns are linearly dependent.

$$5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of Matrix

The rank of a matrix is defined to be the maximum number of linearly independent columns in the matrix. Rank properties include

- The rank of a matrix is unique
- The rank of a matrix can equivalently be defined as the maximum number of linearly independent rows
- The rank of an $r \times c$ matrix cannot exceed $min(r, c)$
- The row and column rank of a matrix are equal
- The rank of a matrix is preserved under nonsingular transformations., i.e. Let $\mathbf{A}$ $(n \times n)$ and $\mathbf{C}$ $(k \times k)$ be nonsingular matrices. Then for any $n \times k$ matrix $\mathbf{B}$ we have

$$rank(\mathbf{B}) = rank(\mathbf{AB}) = rank(\mathbf{BC})$$

Inverse of Matrix

- Like a reciprocal

$$6 * 1/6 = 1/6 * 6 = 1$$

$$x\frac{1}{x} = 1$$

- But for matrices

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverses of Diagonal Matrices are Easy

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Relation of Rank and Inverse

- An inverse of a square $r \times r$ matrix exists if the rank of the matrix is r.
- Such a matrix is said to be nonsingular (or full rank)
- An $r \times r$ matrix with rank less than r is said to be singular and does not have an inverse
- The inverse of an $r \times r$ matrix of full rank also has rank r

Finding the inverse

- Finding an inverse takes (for general matrices with no special structure)

$$O(n^3)$$

operations (when n is the number of rows in the matrix)
- We will assume that numerical packages can do this for us

Manual Inverse Finding

- For small matrices it is possible to find an analytical matrix inverse
- Example

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} \frac{1}{n} + \frac{\bar{X}}{\sum (X_i - \bar{X})^2} & \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \\ \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} & \frac{1}{\sum (X_i - \bar{X})^2} \end{bmatrix}$$

Uses of Inverse Matrix

- Ordinary algebra $5y = 20$
  is solved by $1/5 * (5y) = 1/5 * (20)$
- Linear algebra $\mathbf{AY} = \mathbf{C}$
  is solved by $\mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{C}, \mathbf{Y} = \mathbf{A}^{-1}\mathbf{C}$

Example

Solving a system of simultaneous equations

$$2y_1 + 4y_2 = 20$$
$$3y_1 + y_2 = 10$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

List of Useful Matrix Properties

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} \\
(\mathbf{A}+\mathbf{B})+\mathbf{C} &= \mathbf{A}+(\mathbf{B}+\mathbf{C}) \\
(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) \\
\mathbf{C}(\mathbf{A}+\mathbf{B}) &= \mathbf{CA}+\mathbf{CB} \\
k(\mathbf{A}+\mathbf{B}) &= k\mathbf{A}+k\mathbf{B} \\
(\mathbf{A}')' &= \mathbf{A} \\
(\mathbf{A}+\mathbf{B})' &= \mathbf{A}'+\mathbf{B}' \\
(\mathbf{AB})' &= \mathbf{B}'\mathbf{A}' \\
(\mathbf{ABC})' &= \mathbf{C}'\mathbf{B}'\mathbf{A}' \\
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{ABC})^{-1} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{A}^{-1})^{-1} &= \mathbf{A} \\
(\mathbf{A}')^{-1} &= (\mathbf{A}^{-1})'
\end{aligned}
$$

Derivatives

$$
\begin{aligned}
\mathbf{A}+\mathbf{B} &= \mathbf{B}+\mathbf{A} \\
(\mathbf{A}+\mathbf{B})+\mathbf{C} &= \mathbf{A}+(\mathbf{B}+\mathbf{C}) \\
(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) \\
\mathbf{C}(\mathbf{A}+\mathbf{B}) &= \mathbf{CA}+\mathbf{CB} \\
k(\mathbf{A}+\mathbf{B}) &= k\mathbf{A}+k\mathbf{B} \\
(\mathbf{A}')' &= \mathbf{A} \\
(\mathbf{A}+\mathbf{B})' &= \mathbf{A}'+\mathbf{B}' \\
(\mathbf{AB})' &= \mathbf{B}'\mathbf{A}' \\
(\mathbf{ABC})' &= \mathbf{C}'\mathbf{B}'\mathbf{A}' \\
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{ABC})^{-1} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{A}^{-1})^{-1} &= \mathbf{A} \\
(\mathbf{A}')^{-1} &= (\mathbf{A}^{-1})'
\end{aligned}
$$