Inference in Normal Regression Model

Dr. Frank Wood

Remember

- Last class we derived the sampling variance of the estimator of the slope, it being

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$

- And we made the point that an estimate of $\sigma^2\{b_1\}$ could be arrived at by substituting the MSE for the unknown error variance.

$$s^2\{b_1\} = \frac{MSE}{\sum(X_i - \bar{X})^2} = \frac{\frac{SSE}{n-2}}{\sum(X_i - \bar{X})^2}$$

Sampling Distribution of $(b_1 - \beta_1)/s\{b_1\}$

- We determined that b_1 is normally distributed so $(b_1-\beta_1)/\sigma\{b_1\}$ is a standard normal variable
- We don't know $\sigma^2\{b_1\}$ so it must be estimated from data. We have already denoted it's estimate $s\{b_1\}$
- Using this estimate we it can be shown that

$$\frac{b_1-\beta_1}{s\{b_1\}} \sim t(n-2) \qquad s\{b_1\} = \sqrt{s^2\{b_1\}}$$

Where does this come from?

- We need to rely upon the following theorem
  – For the normal error regression model

$$\frac{SSE}{\sigma^2} = \frac{\sum (Y_i - \hat{Y}_i)^2}{\sigma^2} \sim \chi^2(n-2)$$

  and is independent of b_0 and b_1

- Intuitively this follows the standard result for the sum of squared normal random variables
  – Here there are two linear constraints imposed by the regression parameter estimation that each reduce the number of degrees of freedom by one.

Another useful fact : t distribution

- Let z and $\chi^2(\nu)$ be independent random variables (standard normal (N(0,1)) and χ^2 respectively). We then *define* a t random variable as follows:

$$t(\nu) = \frac{z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}}$$

This version of the t distribution has one parameter, the degrees of freedom ν

Distribution of the studentized statistic

- To derive the distribution of this statistic using the provided theorems, first we do the following rewrite

The numerator is a N(0,1) normal variable

$$\frac{b_1 - \beta_1}{s\{b_1\}} = \frac{\boxed{\dfrac{b_1 - \beta_1}{\sigma\{b_1\}}}}{\boxed{\dfrac{s\{b_1\}}{\sigma\{b_1\}}}}$$

$$\frac{s\{b_1\}}{\sigma\{b_1\}} = \sqrt{\frac{s^2\{b_1\}}{\sigma^2\{b_1\}}}$$

Studentized statistic cont.

- And note the following

$$\frac{s^2\{b_1\}}{\sigma^2\{b_1\}} = \frac{\frac{MSE}{\sum(X_i-\bar{X})^2}}{\frac{\sigma^2}{\sum(X_i-\bar{X})^2}} = \frac{MSE}{\sigma^2} = \frac{SSE}{\sigma^2(n-2)}$$

where we know (by the given theorem) the distribution of the last term is χ^2 and indep. of b_1 and b_0

$$\frac{SSE}{\sigma^2(n-2)} \sim \frac{\chi^2(n-2)}{n-2}$$

Studentized statistic final

- But by the given definition of the t distribution we have our result

$$\frac{b_1 - \beta_1}{s\{b_1\}} \sim \frac{z}{\sqrt{\frac{\chi^2(n-2)}{n-2}}}$$

because putting everything together we can see that

$$\frac{b_1 - \beta_1}{s\{b_1\}} \sim t(n-2)$$

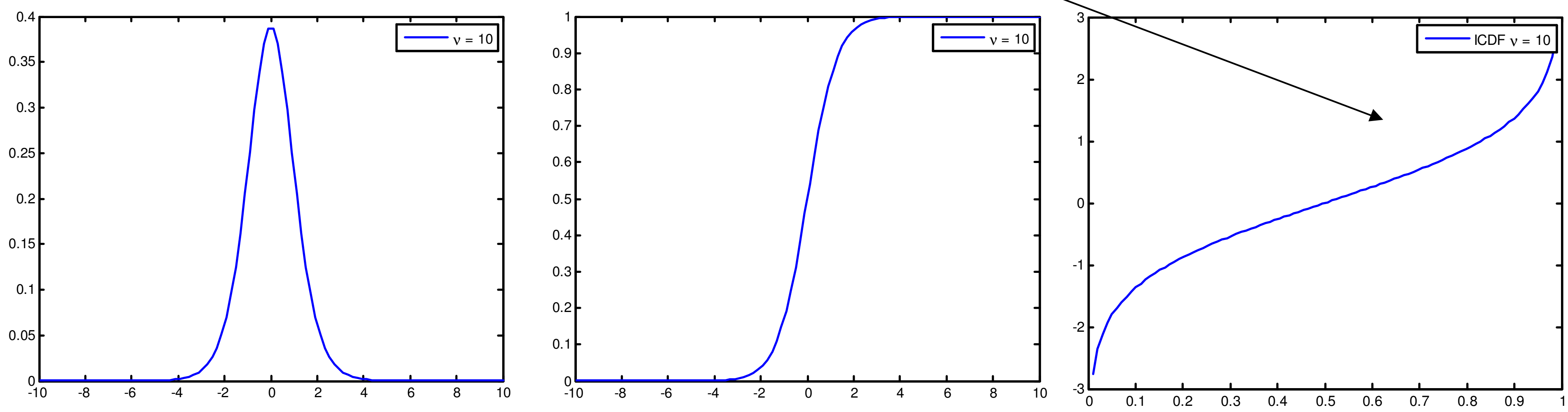
Confidence Intervals and Hypothesis Tests

- Now that we know the sampling distribution of b_1 (t with n-2 degrees of freedom) we can construct confidence intervals and hypothesis tests easily

Confidence Interval for β_1

- Since the "studentized" statistic follows a *t* distribution we can make the following probability statement

$$P(t(\alpha/2; n-2) \leq \frac{b_1 - \beta_1}{s\{b_1\}} \leq t(1-\alpha/2; n-2)) = 1 - \alpha$$



Interval arriving from picking α

- Note that by symmetry

$$t(\alpha/2; n-2) = -t(1-\alpha/2; n-2)$$

- Rearranging terms and using this fact we have

$$P(b_1 - t(1-\alpha/2; n-2)s\{b_1\} \leq \beta_1 \leq b_1 + t(1-\alpha/2; n-2)s\{b_1\}) = 1 - \alpha$$

- And now we can use a table to look up and produce confidence intervals

Using tables for Computing Intervals

- The tables in the book (table B.2 in the appendix) for t(1-α/2;ν) where
  - P{t(ν) $\leq$ t(1-α/2; ν)} = A
- Provides the inverse CDF of the t-distribution
- This can be arrived at computationally as well
  - Matlab: tinv(1-α/2, ν)

1-α confidence limits for β_1

- The 1-α confidence limits for β_1 are

$$b_1 \pm t(1-\alpha/2; n-2)s\{b_1\}$$

- Note that this quantity can be used to calculate confidence intervals given n and α.
  - Fixing α can guide the choice of sample size if a particular confidence interval is desired
  - Give a sample size, vice versa.
- Also useful for hypothesis testing

Show demo.m

Tests Concerning β_1

- Example 1
  - Two-sided test
    - $H_0 : \beta_1 = 0$
    - $H_a : \beta_1 \neq 0$
    - Test statistic

$$t^* = \frac{b_1 - 0}{\hat{S}(b_1)}$$

Tests Concerning β_1

- We have an estimate of the sampling distribution of b_1 from the data.
- If the null hypothesis holds then the b_1 estimate coming from the data should be within the 95% confidence interval of the sampling distribution centered at 0 (in this case)

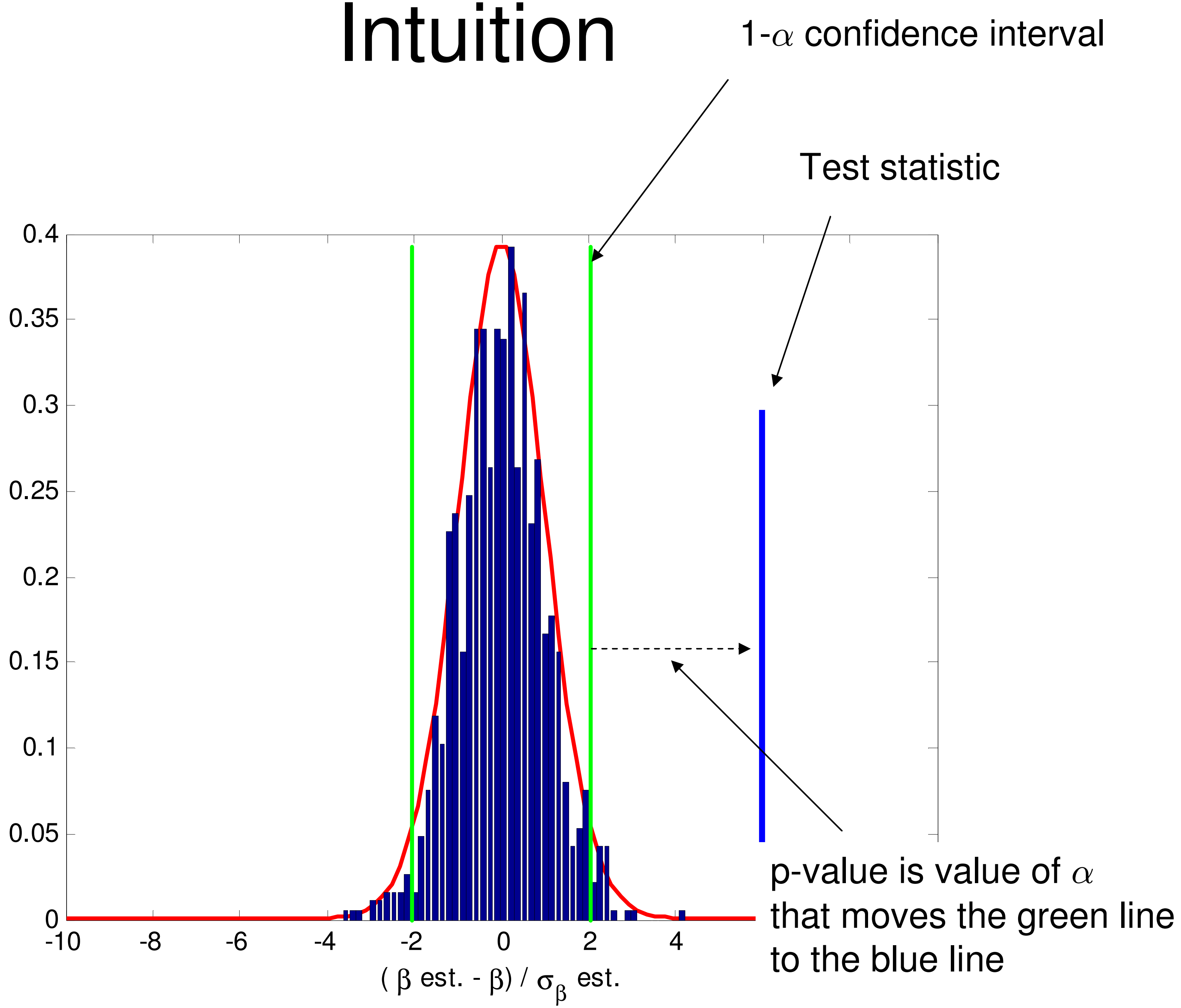
$$t^* = \frac{b_1 - 0}{s\{b_1\}}$$

Decision rules

$$\begin{aligned}\text{if } |t^*| &\leq t(1-\alpha/2; n-2), \text{conclude } H_0\\ \text{if } |t^*| &> t(1-\alpha/2; n-2), \text{conclude } H_\alpha\end{aligned}$$

- Absolute values make the test two-sided

Intuition



1-α confidence interval
Test statistic
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-10
-8
-6
-4
-2
0
2
4
(β est. - β) / σ_β est.
p-value is value of α that moves the green line to the blue line

Calculating the p-value

- The p-value, or attained significance level, is the smallest level of significance α for which the observed data indicate that the null hypothesis should be rejected.
- This can be looked up using the CDF of the test statistic.

- In Matlab
  - Two-sided p-value
    - $2*(1\text{-tcdf}(|t^*|,\nu))$

Inferences Concerning β_0

- Largely, inference procedures regarding β_0 can be performed in the same way as those for β_1
- Remember the point estimator b_0 for β_0

$$b_0 = \bar{Y} - b_1 \bar{X}$$

Sampling distribution of b_0

- The sampling distribution of b_0 refers to the different values of b_0 that would be obtained with repeated sampling when the levels of the predictor variable X are held constant from sample to sample.
- For the normal regression model the sampling distribution of b_0 is normal

Sampling distribution of b_0

- When error variance is known

$$E(b_0) = \beta_0$$

$$\sigma^2\{b_0\} = \sigma^2(\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2})$$

- When error variance is unknown

$$s^2\{b_0\} = MSE(\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2})$$

Confidence interval for β_0

- The 1-α confidence limits for β_0 are obtained in the same manner as those for β_1

$$b_0 \pm t(1-\alpha/2; n-2)s\{b_0\}$$

Considerations on Inferences on β_0 and β_1

- Effects of departures from normality
  - The estimators of β_0 and β_1 have the property of asymptotic normality – their distributions approach normality as the sample size increases (under general conditions)
- Spacing of the X levels
  - The variances of b_0 and b_1 (for a given n and σ^2) depend strongly on the spacing of X

Sampling distribution of point estimator of mean response

- Let X_h be the level of X for which we would like an estimate of the mean response
  - Needs to be one of the observed X's
- The mean response when $X=X_h$ is denoted by $E\{Y_h\}$
- The point estimator of $E\{Y_h\}$ is

$$\hat{Y}_h = b_0 + b_1 X_h$$

We are interested in the sampling distribution of this quantity

Sampling Distribution of $\hat{Y}_h$

- We have

$$\hat{Y}_h = b_0 + b_1 X_h$$

- Since this quantity is itself a linear combination of the Y_i's it's sampling distribution is itself normal.
- The mean of the sampling distribution is

$$E\{\hat{Y}_h\} = E\{b_0\} + E\{b_1\}X_h = \beta_0 + \beta_1 X_h$$

Biased or unbiased?

Sampling Distribution of $\hat{Y}_h$

- To derive the sampling distribution variance of the mean response we first show that b_1 and $(1/n) \sum Y_i$ are uncorrelated and, hence, for the normal error regression model independent
- We start with the definitions

$$\bar{Y} = \sum (\frac{1}{n}) Y_i$$

$$b_1 \quad = \quad \sum k_i Y_i, \, k_i = \frac{(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$$

Sampling Distribution of $\hat{Y}_h$

- We want to show that mean response and the estimate b_1 are uncorrelated

$$Cov(\bar{Y}, b_1) = \sigma^2\{\bar{Y}, b_1\} = 0$$

- To do this we need the following result (A.32)

$$\sigma^2\{\sum_{i=1}^n a_i Y_i, \sum_{i=1}^n c_i Y_i\} = \sum_{i=1}^n a_i c_i \sigma^2\{Y_i\}$$

when the Y_i are independent

Sampling Distribution of $\hat{Y}_h$

- Using this fact we have

$$\begin{aligned}
\sigma^2\{\sum_{i=1}^n \frac{1}{n} Y_i, \sum_{i=1}^n k_i Y_i\} &= \sum_{i=1}^n \frac{1}{n} k_i \sigma^2\{Y_i\} && \text{from appendix} \\
&= \sum_{i=1}^n \frac{1}{n} k_i \sigma^2 \\
&= \frac{\sigma^2}{n} \sum_{i=1}^n k_i && \sum_i k_i = 0 \\
&= 0
\end{aligned}$$

So the mean of Y and b_1 are uncorrelated

Sampling Distribution of $\hat{Y}_h$

- This means that we can write down the variance

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\{\bar{Y} + b_1(X_h - \bar{X})\}$$

alternative and equivalent form of regression function

- But we know that the mean of Y and b_1 are uncorrelated so

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\{\bar{Y}\} + \sigma^2\{b_1\}(X_h - \bar{X})^2$$

Sampling Distribution of $\hat{Y}_h$

- We know (from last lecture)

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}$$
$$s^2\{b_1\} = \frac{MSE}{\sum(X_i - \bar{X})^2}$$

- And we can find

$$\sigma^2\{\bar{Y}\} = \frac{1}{n^2}\sum \sigma^2\{\bar{Y}\} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Sampling Distribution of $\hat{Y}_h$

- So, plugging in, we get

$$\sigma^2\{\hat{Y}_h\} = \frac{\sigma^2}{n} + \frac{\sigma^2}{\sum(X_i - \bar{X})^2}(X_h - \bar{X})^2$$

- Or

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right)$$

Sampling Distribution of $\hat{Y}_h$

- Since we often won't know σ^2 we can, as usual, plug in s^2 = SSE/(n-2), our estimate for it to get our estimate of this sampling distribution variance

$$s^2\{\hat{Y}_h\} = s^2\left(\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right)$$

No surprise…

- The sampling distribution of our point estimator for the output is distributed as a t-distribution with two degrees of freedom

$$\frac{\hat{Y}_h - E\{Y_h\}}{s\{\hat{Y}_h\}} \sim t(n-2)$$

- This means that we can construct confidence intervals in the same manner as before.

Confidence Intervals for E{Y_h}

- The 1-α confidence intervals for E{Y_h} are

$$\hat{Y}_h \pm t(1-\alpha/2; n-2)s\{\hat{Y}_h\}$$

- From this hypothesis tests can be constructed as usual.

Comments

- The variance of the estimator for $E\{Y_h\}$ is smallest near the mean of X. Designing studies such that the mean of X is near X_h will improve inference precision
- When X_h is zero the variance of the estimator fo $E\{Y_h\}$ reduces to the variance of the estimator b_0 for β_o

Prediction interval for single new observation

- Essentially follows the sampling distribution arguments for $E\{Y_h\}$
- If all regression parameters are known then the 1-α prediction interval for a new observation Y_h is

$$E\{Y_h\} \pm z(1 - \alpha/2)\sigma$$

Prediction interval for single new observation

- If the regression parameters are unknown the 1-α prediction interval for a new observation Y_h is given by the following theorem

$$\hat{Y}_h \pm t(1-\alpha/2; n-2)s\{pred\}$$

- This is very nearly the same as prediction for a known value of X but includes a correction for the fact that there is additional variability arising from the fact that the new input location was not used in the orginal estimates of b_1, b_0, and s^2

Prediction interval for single new observation

- The value of $s^2\{pred\}$ is given by

$$s^2\{pred\} = MSE\left[1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right]$$