Linear Algebra Review

Dr. Frank Wood

Definition of Matrix

- Rectangular array of elements arranged in rows and columns

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{Column} & \text{Column} \\ 1 & 2 \end{array} \\
\begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} & \begin{bmatrix} 16{,}000 & 23 \\ 33{,}000 & 47 \\ 21{,}000 & 35 \end{bmatrix}
\end{array}
$$

- Dimension is number of rows and columns expressed as 3x2

Indexing a Matrix

$$
\begin{array}{c c}
 & \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\
\begin{array}{c} i=1 \\ i=2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}
\end{array}
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}
$$

$$
\mathbf{A} = [a_{ij}] \qquad i = 1, 2; j = 1, 2, 3
$$

Square Matrix & Column Vector

- Square matrix has equal number of rows and columns

$$\begin{bmatrix} 4 & 7 \\ 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A (column) vector is a matrix with a single column

$$\mathbf{A} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

Transpose

- The transpose of a matrix is another matrix in which the rows and columns have been interchanged

$$\underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 3 & 4 \end{bmatrix}$$

$$\underset{2\times 3}{\mathbf{A}'} = \begin{bmatrix} 2 & 7 & 3 \\ 5 & 10 & 4 \end{bmatrix}$$

Row Vector

- A row vector is the transpose of a column vector or a matrix with a single row

$$\mathbf{B}' = [15 \quad 25 \quad 50] \qquad \mathbf{F}' = [f_1 \quad f_2]$$

Equality of Matrices

- Two matrices are the same if they have the same dimension and all of the elements are equal

$$\underset{3\times 1}{\mathbf{A}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \underset{3\times 1}{\mathbf{B}} = \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix}$$

$\mathbf{A} = \mathbf{B}$ implies:

$$a_1 = 4 \qquad a_2 = 7 \qquad a_3 = 3$$

Regression Examples

- Response matrix (vector)

$$\underset{n \times 1}{\mathbf{Y}} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

Regression Examples

- Design matrix

$$\underset{n\times 2}{\mathbf{X}} = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}} & \begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{matrix} \end{bmatrix}$$

Matrix Addition and Subtraction

$$\underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad \underset{3\times 2}{\mathbf{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$\underset{3\times 2}{\mathbf{A} + \mathbf{B}} = \begin{bmatrix} 1+1 & 4+2 \\ 2+2 & 5+3 \\ 3+3 & 6+4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

Regression Example

$$Y_i = E\{Y_i\} + \varepsilon_i \qquad i = 1, \ldots, n$$

$$\underset{n\times 1}{\mathbf{E}\{\mathbf{Y}\}} = \begin{bmatrix} E\{Y_1\} \\ E\{Y_2\} \\ \vdots \\ E\{Y_n\} \end{bmatrix} \qquad \underset{n\times 1}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \qquad \underset{n\times 1}{\mathbf{Y}} = \underset{n\times 1}{\mathbf{E}\{\mathbf{Y}\}} + \underset{n\times 1}{\boldsymbol{\varepsilon}}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} E\{Y_1\} \\ E\{Y_2\} \\ \vdots \\ E\{Y_n\} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} E\{Y_1\} + \varepsilon_1 \\ E\{Y_2\} + \varepsilon_2 \\ \vdots \\ E\{Y_n\} + \varepsilon_n \end{bmatrix}$$

Multiplication of a Matrix by a Scalar

$$\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix}$$

$$k\mathbf{A} = k \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 2k & 7k \\ 9k & 3k \end{bmatrix}$$

Multiplication of two Matrices

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \qquad \underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$$

$$\begin{array}{cc} & \mathbf{A} \\ \text{Row 1} & \\ \text{Row 2} & \end{array} \begin{bmatrix} \boxed{2 \quad 5} \\ 4 \quad 1 \end{bmatrix} \qquad \begin{array}{c} \mathbf{B} \\ \begin{bmatrix} \boxed{\begin{matrix} 4 \\ 5 \end{matrix}} & \begin{matrix} 6 \\ 8 \end{matrix} \end{bmatrix} \\ \text{Col. 1 Col. 2} \end{array} \qquad \begin{array}{cc} & \mathbf{AB} \\ \text{Row 1} & \begin{bmatrix} 33 & \\ & \end{bmatrix} \\ & \text{Col. 1} \end{array}$$

$$\begin{array}{cc} & \mathbf{A} \\ \text{Row 1} & \\ \text{Row 2} & \end{array} \begin{bmatrix} \boxed{2 \quad 5} \\ 4 \quad 1 \end{bmatrix} \qquad \begin{array}{c} \mathbf{B} \\ \begin{bmatrix} \begin{matrix} 4 \\ 5 \end{matrix} & \boxed{\begin{matrix} 6 \\ 8 \end{matrix}} \end{bmatrix} \\ \text{Col. 1 Col. 2} \end{array} \qquad \begin{array}{cc} & \mathbf{AB} \\ \text{Row 1} & \begin{bmatrix} 33 & 52 \\ & \end{bmatrix} \\ & \text{Col. 1 Col. 2} \end{array}$$

Another Matrix Multiplication Example

$$\underset{2\times 3}{\mathbf{A}} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 8 \end{bmatrix} \qquad \underset{3\times 1}{\mathbf{B}} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\underset{2\times 1}{\mathbf{AB}} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 26 \\ 41 \end{bmatrix}$$

Regression Examples

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & X_3 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 X_1 \\ \beta_0 + \beta_1 X_2 \\ \beta_0 + \beta_1 X_3 \end{bmatrix}$$

More Regression Examples

$$\underset{1\times 1}{\mathbf{Y}'\mathbf{Y}} = [Y_1 \quad Y_2 \quad \cdots \quad Y_n] \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = [Y_1^2 + Y_2^2 + \cdots + Y_n^2] = \left[\sum Y_i^2\right]$$

$$\underset{2\times 2}{\mathbf{X}'\mathbf{X}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

$$\underset{2\times 1}{\mathbf{X}'\mathbf{Y}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}$$

Special Matrices

- If A = A' then A is a symmetric matrix

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix} \qquad \underset{3\times 3}{\mathbf{A}'} = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix}$$

- If the off-diagonal elements of a matrix are all zeros it is called a diagonal matrix

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \qquad \underset{4\times 4}{\mathbf{B}} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Indentity Matrix

- A diagonal matrix whose diagonal entries are all one is an identity matrix. Multiplication by an identity matrix leaves the (pre or post) multiplied matrix unchanged.

$$\mathbf{IA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Vector and Matrix with all Elements = 1

$$\underset{r\times 1}{\mathbf{1}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \underset{r\times r}{\mathbf{J}} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

$$\underset{n\times n}{\mathbf{1}\mathbf{1}'} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [1 \quad \cdots \quad 1] = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \underset{n\times n}{\mathbf{J}}$$

Linear Dependence and Rank of Matrix

- Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 2 & 10 & 6 \\ 3 & 4 & 15 & 1 \end{bmatrix}$$

and think of this as a matrix of a collection of column vectors. Note that the third column vector is a multiple of the first column vector

Linear Dependence

When c scalars $k_1, \ldots, k_c$, not all zero, can be found such that:

$$k_1\mathbf{C}_1 + k_2\mathbf{C}_2 + \cdots + k_c\mathbf{C}_c = \mathbf{0}$$

where $\mathbf{0}$ denotes the zero column vector, the c column vectors are *linearly dependent*. If the only set of scalars for which the equality holds is $k_1 = 0, \ldots, k_c = 0$, the set of c column vectors is *linearly independent*.

$$5\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}2\\2\\4\end{bmatrix} - 1\begin{bmatrix}5\\10\\15\end{bmatrix} + 0\begin{bmatrix}1\\6\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Rank of Matrix

- The rank of a matrix is defined to be the maximum number of linearly independent columns in the matrix.
  - The rank of a matrix is unique
  - Can equivalently be defined as the maximum number of linearly independent rows
  - The rank of an r x c matrix cannot exceed min(r,c)

Inverse of a Matrix

- Like a reciprocal

$$6 \cdot \frac{1}{6} = \frac{1}{6} \cdot 6 = 1$$

$$x \cdot \frac{1}{x} = x \cdot x^{-1} = x^{-1} \cdot x = 1$$

- But for matrices

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

Example

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\underset{2\times 2}{\mathbf{A}}^{-1} = \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Inverses of Diagonal Matrices are Easy

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \end{bmatrix}$$

$$\underset{3\times}{\mathbf{A}}$$

$$\underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Relation of Rank and Inverse

- An inverse of a square r x r matrix exists if the rank of the matrix is r.
- Such a matrix is said to nonsingular (or full rank)
- An r x r matrix with rank less than r is said to be singular and does not have an inverse
- The inverse of an r x r matrix of full rank also has rank r.

Finding the Inverse

- Finding an inverse takes (for general matrices with no special structure)

$$O(n^3)$$

  operations (where n is the number of rows in the matrix)
- We will assume that numerical packages can do this for us.

Manual Inverse Finding

- For small matrices it is possible to find an analytic matrix inverse
- Example

$$\underset{2\times 2}{\mathbf{X}'\mathbf{X}} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}$$

$$\underset{2\times 2}{(\mathbf{X}'\mathbf{X})^{-1}} = \begin{bmatrix} \frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2} & \frac{-\bar{X}}{\sum(X_i - \bar{X})^2} \\ \frac{-\bar{X}}{\sum(X_i - \bar{X})^2} & \frac{1}{\sum(X_i - \bar{X})^2} \end{bmatrix}$$

Uses of Inverse Matrix

- Ordinary algebra $5y = 20$

  is solved by $\frac{1}{5}(5y) = \frac{1}{5}(20)$

- Linear algebra $\mathbf{AY} = \mathbf{C}$

  is solved by
  $$\mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{C}$$
  $$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{C}$$

Example

- Solving a system of simultaneous equations

$$2y_1 + 4y_2 = 20$$

$$3y_1 + y_2 = 10$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

List of Useful Matrix Properties

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$$

$$(\mathbf{A}')' = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$$

$$(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$$

$$(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$$