Quadratic Programming

Equality Constrained QP

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Convex Equality Constrained Quadratic Program

Convex equality constrained quadratic program

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{1a}$$

$$s.t. \qquad a_i' x = b_i \qquad\qquad i \in \mathcal{E} = \{1, 2, \ldots, m\} \tag{1b}$$

$H \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite. $g \in \mathbb{R}^n$. $m < n$.
$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_m \end{bmatrix}$ has full column rank.

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{2a}$$

$$s.t. \qquad A' x = b \tag{2b}$$

Optimality Conditions

$$\min_{x\in\mathbb{R}^n} \quad \phi = \frac{1}{2}x'Hx + g'x \tag{3a}$$

$$s.t. \quad a_i'x = b_i \qquad i \in \mathcal{E} = \{1, 2, \dots, m\} \tag{3b}$$

Lagrangian

$$\mathcal{L} = \frac{1}{2}x'Hx + g'x - \sum_{i\in\mathcal{E}} \lambda_i(a_i'x - b_i) \tag{4}$$

$$\nabla_x \mathcal{L} = Hx + g - \sum_{i\in\mathcal{E}} a_i\lambda_i \tag{5}$$

Optimality Conditions

$$Hx + g - \sum_{i\in\mathcal{E}} a_i\lambda_i = 0 \tag{6a}$$

$$a_i'x - b_i = 0 \qquad i \in \mathcal{E} \tag{6b}$$

Optimality Conditions

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{7a}$$

$$s.t. \qquad A' x = b \tag{7b}$$

Lagrangian

$$\mathcal{L} = \frac{1}{2} x' H x + g' x - \lambda' (A x - b) \tag{8}$$

$$\nabla_x \mathcal{L} = H x + g - A \lambda \tag{9}$$

Optimality Conditions

$$H x + g - A \lambda = 0 \tag{10a}$$

$$A' x - b = 0 \tag{10b}$$

KKT System

$$\begin{bmatrix} H & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \tag{11}$$

Convex Equality Constrained QP & KKT System

Convex equality constrained quadratic program

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{12a}$$

$$s.t. \quad a_i' x = b_i \qquad i \in \mathcal{E} = \{1, 2, \dots, m\} \tag{12b}$$

$H \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite. $g \in \mathbb{R}^n$. $m < n$. $A = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix}$ has full column rank.

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{13a}$$

$$s.t. \quad A' x = b \tag{13b}$$

The convex equality constrained QP is solved by solution of the KKT system

$$\begin{bmatrix} H & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \tag{14}$$

Direct Methods

$$\overbrace{\begin{bmatrix} H & -A \\ -A' & 0 \end{bmatrix}}^{=K} \overbrace{\begin{bmatrix} x \\ \lambda \end{bmatrix}}^{=z} = -\overbrace{\begin{bmatrix} g \\ b \end{bmatrix}}^{=d} \tag{15}$$

$x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$, $g \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $H \in \mathbb{R}^{n \times n}, A \in \mathbb{R}^{n \times m}$

Dense Methods – The KKT matrix is represented as a dense matrix

Sparse Methods – The KKT matrix is represented as a sparse matrix

- backslash: `z = K \ d;`
- LU factorization: $PK = LU$
  1. `[L,U,p] = lu(K,'vector');`
  2. `z = U \ ( L \ d(p) );`
- LDL factorization: $P'KP = LDL'$
  1. `z = zeros(n+m,1);`
  2. `[L,D,p] = ldl(K,'lower','vector');`
  3. `z(p) = L' \ ( D \ ( L \ d(p) ) );`

Range-Space Method / Schur-Complement Method

$$\begin{bmatrix} H & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \qquad H \in \mathbb{S}^{n\times n}_{++} \tag{16}$$

$$Hx - A\lambda = -g \quad \Leftrightarrow \quad x = H^{-1}A\lambda - H^{-1}g \tag{17}$$

$$b = A'x = \left(A'H^{-1}A\right)\lambda - A'H^{-1}g \tag{18}$$

Procedure

1. Cholesky factorize $H = LL'$.
2. Solve $Hv = g$ for v.
3. Form $H_A = A'H^{-1}A = L_A L_A'$ and its factorization (do not form H^{-1}).
4. Solve $H_A\lambda = b + A'v$ for λ.
5. Solve $Hx = A\lambda - g$ for x.

Useful when

- H is well-conditioned and easy to invert (H is diagonal or block-diagonal)
- H^{-1} is known explicitly, e.g. through quasi-Newton update formulas
- The number of equality constraints (m) is small, i.e. $H_A = A'H^{-1}A \in \mathbb{R}^{m\times m}$ is small.

Null-Space Method

$$\begin{bmatrix} H & -A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = - \begin{bmatrix} g \\ b \end{bmatrix} \qquad A \in \mathbb{R}^{n \times m} \tag{19}$$

Define the non-singular matrix $\begin{bmatrix} Y & Z \end{bmatrix} \in \mathbb{R}^{n \times n}$ with $Y \in \mathbb{R}^{n \times m}$ and $Z \in \mathbb{R}^{n \times (n-m)}$ such that

$$A' \begin{bmatrix} Y & Z \end{bmatrix} = \begin{bmatrix} A'Y & A'Z \end{bmatrix} = \begin{bmatrix} A'Y & 0 \end{bmatrix} \quad A'Y \in \mathbb{R}^{m \times m} \text{ non-singular}$$

$$\begin{aligned} x &= Y x_Y + Z x_Z \\ -g &= Hx - A\lambda = HYx_Y + HZx_Z - A\lambda \end{aligned}$$

$$\begin{aligned} b = A'x = A'Yx_y + A'Zx_Z &= A'Yx_Y \\ \left(Z'HZ\right) x_Z &= -Z'(HYx_Y + g) \\ \left(A'Y\right)' \lambda &= Y'(Hx + g) \end{aligned}$$

Procedure

1. Solve: $(A'Y)x_Y = b$
2. Solve: $(Z'HZ)x_Z = -Z'(HYx_Y + g)$
3. Compute: $x = Yx_Y + Zx_Z$
4. Solve: $(A'Y)'\lambda = Y'(Hx + g)$

Properties

- Useful when the number of degrees of freedom, $n - m$, is small.
- Main drawback is its need for the null-space matrix, Z, which may be expensive to compute.

Null-Space Method: Orthonormal basis

$$A = Q\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R \tag{20}$$

$$A' = \begin{bmatrix} R' & 0 \end{bmatrix} Q' \tag{21}$$

$$A'Q = A'\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} A'Q_1 & A'Q_2 \end{bmatrix} = \begin{bmatrix} R' & 0 \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} Y & Z \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \quad A'Y = A'Q_1 = R' \tag{23}$$

QR factorization in Matlab ($A \in \mathbb{R}^{n\times m}$)

1. `[Q,Rbar] = qr(A);`
2. `m1 = size(Rbar,2);`
3. `Q1 = Q(:,1:m1); Q2 = Q(:,m1+1:n); R = Rbar(1:m1,1:m1)`

If $A \in \mathbb{R}^{n\times m}$ has full column rank $m_1 = m$.
Procedure

1. Solve: $R' x_Y = b$
2. Solve: $(Q_2' H Q_2) x_Z = -Q_2'(H Q_1 x_Y + g)$
3. Compute: $x = Q_1 x_Y + Q_2 x_Z$
4. Solve: $R\lambda = Q_1'(Hx + g)$