Quadratic Programming

Primal Active Set Method

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Convex Quadratic Program

Convex quadratic program

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{1a}$$

$$s.t. \quad a_i' x \geq b_i \qquad i \in \mathcal{I} = \{1, 2, \dots, m\} \tag{1b}$$

$H \in \mathbb{R}^{n \times n}$ is symmetric and positive-definite. $g \in \mathbb{R}^n$.

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{2a}$$

$$s.t. \quad A' x \geq b \tag{2b}$$

Optimality Conditions

$$\min_{x \in \mathbb{R}^n} \quad \phi = \frac{1}{2} x' H x + g' x \tag{3a}$$

$$s.t. \quad a_i' x \geq b_i \qquad i \in \mathcal{I} = \{1, 2, \ldots, m\} \tag{3b}$$

Lagrangian

$$\mathcal{L} = \frac{1}{2} x' H x + g' x - \sum_{i \in \mathcal{I}} \lambda_i (a_i' x - b_i) \tag{4}$$

$$\nabla_x \mathcal{L} = H x + g - \sum_{i \in \mathcal{I}} a_i \lambda_i = 0 \tag{5}$$

Optimality Conditions

$$H x + g - \sum_{i \in \mathcal{I}} a_i \lambda_i = 0 \tag{6a}$$

$$a_i' x - b_i \geq 0 \qquad i \in \mathcal{I} \tag{6b}$$

$$\lambda_i \geq 0 \qquad i \in \mathcal{I} \tag{6c}$$

$$\lambda_i = 0 \qquad i \in \mathcal{I} \setminus \mathcal{A}(x) \tag{6d}$$

$$\mathcal{A}(x) = \{ i \in \mathcal{I} : \, a_i' x = b_i \} \tag{6e}$$

Optimality Conditions

Active set, $\mathcal{A}(x)$, and working set, $\mathcal{W}$:

$$\mathcal{A}(x) = \left\{ i \in \mathcal{I} :\; a_i' x = b_i \right\} \tag{7a}$$

$$\mathcal{W} \subseteq \mathcal{A}(x) \tag{7b}$$

Optimality Conditions

Theorem (KKT Conditions)

$$\begin{aligned}
&Hx + g - \sum_{i \in \mathcal{I}} a_i \lambda_i = 0 && \\
&a_i' x - b_i \geq 0 && i \in \mathcal{I} \\
&\lambda_i \geq 0 && i \in \mathcal{I} \\
&\lambda_i = 0 && i \in \mathcal{I} \setminus \mathcal{A}(x)
\end{aligned}$$

$$\mathcal{A}(x) = \{ i \in \mathcal{I} :\; a_i' x = b_i \}$$

Theorem (KKT Conditions)

$$\begin{aligned}
&\lambda_i = 0 && i \in \mathcal{I} \setminus \mathcal{W} \\
&Hx + g - \sum_{i \in \mathcal{W}} a_i \lambda_i = 0 && \\
&a_i' x - b_i = 0 && i \in \mathcal{W} \\
&a_i' x - b_i \geq 0 && i \in \mathcal{I} \setminus \mathcal{W} \\
&\lambda_i \geq 0 && i \in \mathcal{W}
\end{aligned}$$

Optimality Conditions

Let

$$\bar{A} = \big[a_i\big]_{i \in \mathcal{W}} \quad \bar{b} = \big[b_i\big]_{i \in \mathcal{W}} \quad \bar{\lambda} = \big[\lambda_i\big]_{i \in \mathcal{W}}$$

The conditions

$$\begin{aligned} &Hx + g - \sum_{i \in \mathcal{W}} a_i \lambda_i = 0 \\ &a_i' x - b_i = 0 \qquad\qquad i \in \mathcal{W} \end{aligned}$$

correspond to solution of the equality constrained QP

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \phi = \frac{1}{2} x' H x + g' x \\ s.t. \quad & a_i' x = b_i \qquad i \in \mathcal{W} \end{aligned}$$

In matrix form, the conditions become

$$\begin{bmatrix} H & -\bar{A} \\ -\bar{A}' & 0 \end{bmatrix} \begin{bmatrix} x \\ \bar{\lambda} \end{bmatrix} = - \begin{bmatrix} g \\ \bar{b} \end{bmatrix}$$

and the corresponding QP is

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \phi = \frac{1}{2} x' H x + g' x \\ s.t. \quad & \bar{A}' x = \bar{b} \end{aligned}$$

Conceptual Active-Set Solution Procedure

1. Guess working set (active set), $\mathcal{W}$
2. Let $\bar{A} = \left[a_i\right]_{i \in \mathcal{W}}$, $\bar{b} = \left[b_i\right]_{i \in \mathcal{W}}$, and $\bar{\lambda} = \left[\lambda_i\right]_{i \in \mathcal{W}}$
3. Set $\lambda_i = 0 \quad i \in \mathcal{I} \setminus \mathcal{W}$
4. Solve
$$\begin{bmatrix} H & -\bar{A} \\ -\bar{A}' & 0 \end{bmatrix} \begin{bmatrix} x \\ \bar{\lambda} \end{bmatrix} = - \begin{bmatrix} g \\ \bar{b} \end{bmatrix}$$
5. Check
$$a_i' x \geq b_i \qquad i \in \mathcal{I} \setminus \mathcal{W} \tag{11a}$$
$$\lambda_i \geq 0 \qquad i \in \mathcal{W} \tag{11b}$$
6. If (11) is satisfied: Stop, the optimal solution has been found.
7. Otherwise continue at 1.