Unit-V

Dynamic Programming problem

Introduction:

Dynamic programming determines the optimum solution of a multivariable problem by decomposing it into stages, each stage comprising of a single variable sub-problem.

The advantage of decomposition is that the optimization process at each stage involves one variable only, which is a simpler task computationally than dealing with all the variables simultaneously.

Applications of Dynamic programming:

1) Capital Budgeting problem
2) Stage Coach problem (shortest path problem)
3) optimal Sub-dividing problem
4) Linear programming problem
5) Employment Smoothening problem.

Capital Budgeting problem:-

A capital Budgeting problem is a problem in which a given amount of capital is allocated to a set of plants such that total revenue of the organization is maximized.

Ex:- An organization is planning to diversify its business with a maximum outlay of Rs. 5 crores. It has identified 3 different locations to instal plants. The organization can invest in one or more of these plants subject to availability of funds. possible alternatives, investments, returns are given in the following table. Find the optimal allocation of capital to different plants.

4.

~~Stage Com~~ Capital - Budgeting problem:

Alternative	Plant 1 Cost	Plant 1 Return	Plant 2 Cost	Plant 2 Return	Plant 3 Cost	Plant 3 Return
1	0	0	0	0	0	0
2	1	15	2	14	1	3
3	2	18	3	18	2	7
4	4	28	4	21	—	—

maximum capital amount is 5.

Sol:

Stage 1 : 0,1,2,3,4,5. (∵ max is 5)

$f_1(x_1) = R(m_1)$

State variable x_1	Alternative m_1: 1 (C 0, R 0)	2 (C 1, R 15)	3 (C 2, R 18)	4 (C 4, R 28)	Maximum $f_1^*(x_1)$	m_1^*
0	0	—	—	—	0	1
1	0	15	—	—	15	2
2	0	15	18	—	18	3
3	0	15	18	—	18	3
4	0	15	18	28	28	4
5	0	15	18	28	28	4

Stage 2:

$f_2(x_2) = R(m_2) + f_1(x_2 - C(m_2))$ ②

Alternative m_2

State Variable x_2	1 (C 0, R 0)	2 (C 2, R 14)	3 (C 3, R 18)	4 (C 4, R 21)	$f_2^*(x_2)$	m_2^*
0	0+0=0	–	–	–	0	1
1	0+15=15	–	–	–	15	1
2	0+18=18	14+0=14	–	–	18	1
3	0+18=18	14+15=29	18+0=0	–	29	2
4	0+28=28	14+18=32	18+15=33	21+0=21	33	3
5	0+28=28	14+18=32	18+18=36	21+15=36	36	3 and 4

$f_3(x_3) = R(m_3) + f_2(x_3 - C(m_3))$

Stage 3:

Alternative m_3

State Variable x_3	1 (0, 0)	2 (1, 3)	3 (2, 7)	$f_3^*(x_3)$	m_3^*
0	0+0=0	–	–	0	1
1	0+15=15	3+0=3	–	15	1
2	0+18=18	3+15=18	7+0=7	18	1, 2
3	0+29=29	3+18=21	7+15=22	29	1
4	0+33=33	3+29=32	7+18=25	33	1
5	0+36=36	3+33=36	7+29=36	36	1,2,3

Highest

Final Results

Stage 3 C^*	m_3^*	Stage 2 C^*	m_2^*	Stage 1 C^*	m_1^*
5	1	5-0=5	3	5-3=2	3
			4	5-4=1	2
5	2	5-1=4	3	4-3=1	2
5	3	5-2=3	2	3-2=1	2

∴ Optimal alternatives are

1 2 3

3–3–1

2–4–1

2–3–2

2–2–3

Stage-coach problem:- (shortest-path problem).

Stage-coach problem is a shortest path problem, in which the objective is to find the shortest distance and corresponding path from a given source Node to a distance Node in a given distance network.

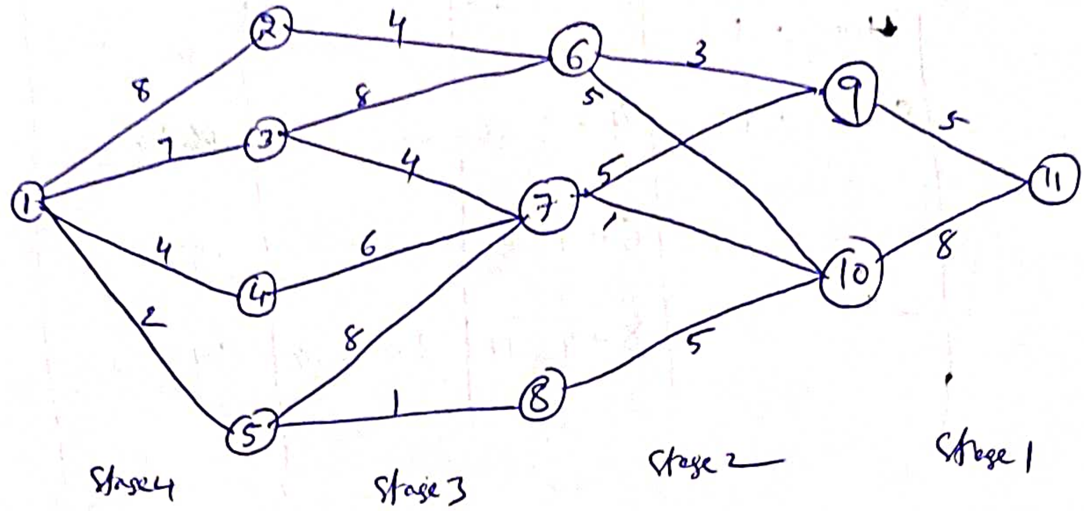
Ex:- A distance network consists of eleven nodes which are distributed as shown in the following figure. Find the shortest path from Node 1 to node 11 and also the corresponding distances.

diagram

↓

(See next page)

~~Model 2~~ Stage-coach problem:- shortest-path problem.



Sol:

$$f_1(x_1) = d(x_1, m_1)$$

Stage 1:

State variable x_1	Alternative m_1 11	$f_1^*(x_1)$	m_1^*
9	5	5	11
10	8	8	11

$$f_2(x_2) = d(x_2, m_2) + f_1(x_1 = m_2)$$

Stage 2:

State variable x_2	Alternative m_2: 9	10	$f_2^*(x_2)$	m_2^*
6	3+5 = 8	5+8 = 13	8	9
7	5+5 = 10	1+8 = 9	9	10
8	—	5+8 = 13	13	10

Stage 3: $f_3(x_3) = d(x_3, m_3) + f_2(x_2 = m_3)$.

State Variable (x_3)	Alternative (m_3) 6	7	8	$f_3^*(x_3)$	m_3^*
2	4+8 = 12	–	–	12	6
3	8+8 = 16	4+9 = 13	–	13	7
4	–	6+9 = 15	–	15	7
5	–	8+9 = 17	1+13 = 14	14	8

Stage 4: $f_4(x_4) = d(x_4, m_4) + f_3(x_3 = m_4)$

State Variable (x_4)	Alternative 2	3	4	5	$f_4^*(x_4)$	m_4^*
1	8+12 = 20	7+13 = 20	4+15 = 19	2+14 = 16	16	5

minimum

Shortest path: 1-5, 5-8, 8-10, 10-11

∴ Shortest path is 1 – 5 – 8 – 10 – 11

Shortest distance is 2+1+5+8 = 16.

Employment Smoothening problem:-

~~A firm has divided its marketing area into 3 Zones~~

Employment smoothening problem is the problem of allocating no. of employees to various zones of a firm for getting maximum profit/max. sales of the firm/company, using past data.

Question: A firm has divided its marketing area into three zones. The amount of sales depends upon the number of salesman in each zone. The firm has been collecting the data regarding sales and sales man in each area over a number of past years.

The information is summarized in the following Table. For the next year, the firm has only 9 salesmen and the problem is to allocate these salesman to 3 different zones so that the total sales are maximum.

Employment Smoothening problem:-

No. of Salesmen	Profits in thousands of Rupees ZONE 1	ZONE 2	ZONE 3
0	30	35	42
1	45	45	54
2	60	52	60
3	70	64	70
4	79	72	82
5	90	82	95
6	98	93	102
7	105	98	110
8	100	100	110
9	90	100	110

Sol: ~~Zone~~

Stage -1:

No. of Salesmen	0	1	2	3	4	5	6	7	8	9
Profit	30	45	60	70	79	90	98	105	100	90

Stage 2.

ZONE(1) x_1, $f_1(x_1)$ →		0	1	2	3	4	5	6	7	8	9
ZONE(2) x_2 ↓	$f_2(x_2)$	30	45	60	70	79	90	98	105	100	90
0	35	65	80	95	105	114	125	133	140	195	125
1	45	75	90	105	115	124	135	143	150	145	
2	52	82	97	112	122	131	142	150	157		
3	64	94	109	124	134	143	154	162			
4	72	102	117	132	142	151	162				
5	82	112	127	142	152	161					
6	93	123	138	152	163						
7	98	128	143	158							
8	100	130	145								
9	100	130									

Here maximum values in the diagonal are taken.

Stage 3:

No. of salsman:	0	1	2	3	4	5	6	7	8	9
Total profit:	65	80	95	105	115	125	135	143	154	163
$f_2(x_2)+f_1(x_1)$	0+0	0+1	0+2	1+2 0+3	1+3	0+5	1+5	3+4 1+6	3+5	6+3

~~No. of Salesman in Zone-3~~ 9 8 7 6 5 4 3 2 1 0

42 54 60 70 82 95 102 110 110 110

~~Total profit $f_3(x_3)+f_2(x_2)+f_1(x_1)$~~ 205 208

No. of salesman in Zone 3	9	8	7	6	5	4	3	2	1	0
profit $f_3(x_3)$	110	110	110	102	95	82	70	60	54	42
Total profit $f_3(x_3)+f_2(x_2)+f_1(x_1)$	175	190	205	207	~~207~~ 210	~~210~~ 207	~~207~~ 205	203	208	205

maximum

Now Consider the distribution of salesmen in three zones. 1,2,3.

∴ No. of Salesman allocated at Zone 3 : 5 (maximum 210)

No. of " " " Zone 2 : 1

No. of Salesmen allocated at Zone 1 : 3

Linear Programming problem as dynamic programming Problem:-

Ex: Solve the following LPP using dynamic programming Technique. Subject to

$$\text{Maximize } Z = 10x_1 + 30x_2$$

$$3x_1 + 6x_2 \leq 168 \quad \text{—} ①$$

$$12x_2 \leq 240. \quad \text{—} ②$$

$$x_1 \text{ and } x_2 \geq 0$$

Sol:- The number of decision variables in the given problem is equal to 2.

So there will be two stages.

Stage 1 is assigned to variable x_1

Stage 2 is assigned to variable x_2.

We use backward recursion to solve the problem.

Stage j	Decision Variable	Set of States
2	x_2	b_{12}, b_{22}
1	x_1	b_{11}, b_{21}

$$\therefore f_2(b_{12}, b_{22}) = \max 30x_2$$

$$f_1(b_{11}, b_{21}) = \max\left(10x_1 + 30 \cdot \min\left(\frac{168-3x_1}{6}, \frac{240}{12}\right)\right)$$

Since from eq ①, $x_2 = \dfrac{168-3x_1}{6}$

from eq ②, $x_2 = \dfrac{240}{12}$

$$= \max\left[10x_1 + 30 \cdot \min\left(\frac{168-3x_1}{6}, 20\right)\right]$$

Now take $\frac{168-3x_1}{6} = 20$

$\Rightarrow 168 - 3x_1 = 120$

$\Rightarrow x_1 = 16.$

take $\frac{168-3x_1}{6} = 0$

$168 - 3x_1 = 0$

$\Rightarrow x_1 = 56.$

$\therefore$ The ranges for x_1 are

$$0 \le x_1 \le 16, \quad 16 \le x_1 \le 56.$$

$$\therefore f_1(b_{11}, b_{21}) = \max\begin{bmatrix} 10x_1 + 30\min\left(\frac{168-3x_1}{6}, 20\right), & 0 \le x_1 \le 16 \\ 10x_1 + 30\min\left(\frac{168-3x_1}{6}, 20\right), & 16 \le x_1 \le 56 \end{bmatrix}$$

$$= \max\begin{bmatrix} 10x_1 + 30\cdot(20) & , 0 \le x_1 \le 16 \\ 10x_1 + 30\cdot\left(\frac{168-3x_1}{6}\right) & , 16 \le x_1 \le 56 \end{bmatrix}$$

$$= \max\begin{bmatrix} 10x_1 + 600 & , 0 \le x_1 \le 16 \\ 10x_1 + 840 - 15x_1 & , 16 \le x_1 \le 56 \end{bmatrix}$$

Substitute $x_1 = 16$. (Common in both inequalities)

$\therefore f_1 = \max(760, 760) = 760.$

$x_1^* = 16.$

$x_2^* = \min\left(\frac{120}{6}, \frac{240}{12}\right) = 20.$

$\therefore x_1^* = 16, \; x_2^* = 20, \; Z(\text{optimum}) = 760$

modl 5

(ip)

optimal sub-dividing problem:

Find the value of $Max(y_1 \cdot y_2 \cdot y_3)$

subject to $y_1 + y_2 + y_3 = 5$;

$y_1, y_2, y_3 \geq 0$.

sol: let $S_3 = y_1 + y_2 + y_3 = 5$

and $S_2 = y_1 + y_2 = S_3 - y_3$

$S_1 = y_1 = S_2 - y_2$, $max(y_j \cdot f_{j-1}(S_{j-1}))$

also $f_3(S_3) = Max(y_3 \cdot f_2(S_2))$

$f_2(S_2) = Max(y_2 \cdot f_1(S_1))$

$f_1(S_1) = Max(y_1) = y_1 = S_2 - y_2$

and $f_2(S_2) = max(y_2 \cdot (S_2 - y_2))$ //

By differential calculus let $f = y_2(S_2 - y_2)$

$\frac{\partial f}{\partial y_2} = 0$

$y_2(S_2 - y_2) = 0$

$\Rightarrow y_2 S_2 - y_2^2 = 0$

$\Rightarrow S_2 - 2y_2 = 0$ ($\because$ Differentiation w.r.t y_2)

$\Rightarrow y_2 = \frac{S_2}{2}$ ✓ y_2 will have maximum at $\frac{S_2}{2}$.

$\Rightarrow f_2(S_2) = max\left(\frac{S_2}{2} \cdot \left(S_2 - \frac{S_2}{2}\right)\right) = \frac{S_2^2}{4}$

$\therefore f_3(S_3) = Max\left(y_3 \cdot \frac{S_2^2}{4}\right)$ $\left(\because f_2(S_2) = \frac{S_2^2}{4}\right)$

$= Max\left(y_3 \cdot \frac{(S_3 - y_3)^2}{4}\right)$

again by Maxima and Minima ~~y_3~~

$y_3 = \frac{S_3}{3} = \frac{5}{3}$ $(\because S_3 = y_1 + y_2 + y_3 = 5)$

$$\boxed{\therefore y_3 = \frac{5}{3}}$$

Since

$$S_2 = y_1 + y_2$$

$$= S_3 - y_3$$

$$= 5 - \frac{5}{3}$$

$$\boxed{\cancel{S_2 = \frac{10}{3}}} \qquad \therefore S_2 = \frac{10}{3}.$$

But since, $y_2 = \frac{S_2}{2}$

$$\Rightarrow y_2 = \frac{\left(\frac{10}{3}\right)}{2} = \frac{5}{3} \qquad \boxed{\therefore y_2 = \frac{5}{3}}$$

Now, $y_1 + y_2 + y_3 = 5$

$$\Rightarrow y_1 = 5 - (y_2 + y_3)$$

$$= 5 - \left(\frac{5}{3} + \frac{5}{3}\right)$$

$$\boxed{y_1 = \frac{5}{3}}$$

$$\therefore \text{Max}\,(y_1 \cdot y_2 \cdot y_3) = \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{5}{3} = \frac{125}{27}.$$