optimization in operational research:-

unit-1: part-1: Introduction and classical optimization technique:

1.1: Introduction:

optimization is the method of obtaining the best result under given circumstances.

In engineering design activities (design, construction and maintenance), engineers have to take many technological and managerial decisions at several stages.

" The main and important goal of all such decisions is either to minimize the effort required or to maximize the desired benefit.

ex: (i) Mechanical engineers design mechanical components to achieve minimum & maximum component life.

(ii) civil engineers involved in construction of building bridges, and other to minimum overall cost or maximum safety.

* minimization (or) maximization (collectively known as optimization) of an objective. All these tasks can be expressed as a function of certain (decision) variables.

optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function.

Applications of optimization:-

Typical applications from different domains.

(1) Design of aircraft and aerospace structure for minimum weight.

(2) Finding the optimal trajectories of space vehicles.

(3) optimum design of electrical networks.

(4) Selection of a site for industry.

1.2: Statement of an optimization problem:-

An optimization or a mathematical programming problem can be stated as follows:-

Find $X = (x_1, x_2, \ldots x_n)^T$ ① which minimizes $f(X)$.

subject to the constraints

$g_j(X) \leq 0 \; ; \; j = 1, 2, \ldots m.$

$l_j(X) = 0 \quad j = 1, 2, \ldots p.$

$X \rightarrow$ n-dimensional vector called design vector.

$f(X) \rightarrow$ Objective function.

$g_j(X)$ and $l_j(X) \rightarrow$ inequality & equality constraints.

eq① is called constrained optimization problem.

* Some optimization problems do not involve any constraints and can be stated as

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(X).$$

and such problems are called as "unconstrained optimization problem"

Remark: How one can solve a maximization problem as a minimization problem?

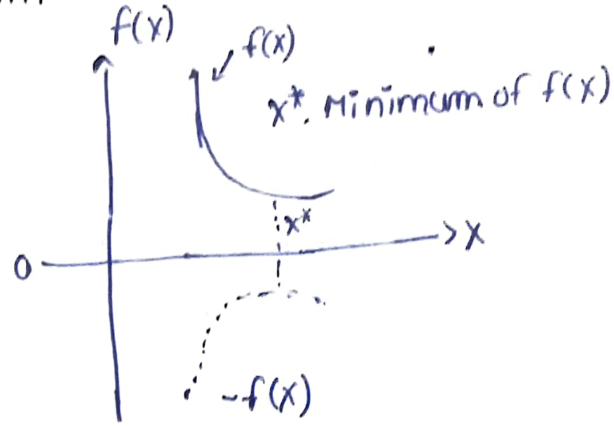


Fig: Minimum of $f(x)$ = maximum of $-f(x)$.

→ point x^* corresponds to minimum value of $f(x)$. and x^* also corresponds to max value of -ve of function $-f(x)$.

→ without loss of generality, indirectly a maximization problem can be solved as a minimization problem.

* The number of variables n and the number of constraints m and/or p need not be related in any way.

1.3 Design vector:-

Any engineering system or activity is defined by a set of quantities some of which are viewed as variables during the design process. In general, certain quantities are usually fixed and these are called preassigned parameters. All the other quantities are treated as variables in the design process and are called design or decision variables x_i, $i = 1, 2 \ldots, n$.

The design variables are collectively represented as a design vector $X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$.

If an n-dimensional cartesian space with each co-ordinate axis representing a design variable $x_i (i = 1, 2 \ldots n)$ is considered, the space is called the design variable space or design space.

-> Each point in the n-dimensional design space is called a design point.

(i) The choice of the important design variables in an optimization problem largely depends on the user and his experience

(ii) However, it is important to understand that the efficiency and speed of optimization technique depend to a large extent, on the number of chosen design variables.

(iii) Thus by selectively choosing the design variables, the efficiency of the optimization technique can be increased.

(iv) The first thumb rule of the ~~informulation~~ formulation of an optimization problem is to choose as few design variables as possible.

1.4: Design constraints:-

The, selected design variables have to satisfy certain specified functional and other requirements known as constraints (restrictions).
These constraints that must be satisfied to produce an acceptable design are collectively called design constraints.
constraints that represent limitations on the behaviour or performance of the system are termed "behaviour or functional constraints"
constraints that represent physical limitations on design variables such as availability, fabricability and transportability are known as "geometric or side constraints."

-> Mathematically, there are usually two ~~of~~ types of constraints: Equality or Inequality constraints.

* Inequality constraints state that the relationships among design variables are either greater than, smaller than or equal to a resource value.
* Equality constraints state that the relationships should exactly match a resource value.
Equality constraints are usually more difficult to handle and therefore, need to be avoided wherever possible. Thus the second thumb rule in the formulation of optimization problem is that the

number of complex equality constraints should be kept as low as possible.

1.5:- Constraint surface:

consider an optimization problem with only inequality constraints $g_j(X) \le 0$.

↳ The set of values of X that satisfy the equation $g_j(X) = 0$. forms a hyper surface in the design space and is called a constraint surface.

→ This constraint surface divides the design space into two regions: "one is $g_j(X) < 0$ and the other is $g_j(X) > 0$".

* The points lying in the region where $g_j(X) > 0$ are infeasiable or unacceptable.
* points lying in the region where $g_j(X) < 0$ are feasiable or acceptable.
* The collection of all the constraint surfaces $g_j(X) = 0$, $j = 1, 2, \ldots m$, which separates the acceptable region is called the composite constraint surface.
* A design point that lies on one or more than one constraint surface is called a bound point. and the associated constraint is called an active constraint.
* Design points that do not lie on any constraint surface are known as free points.

Depending on whether a particular design point belongs to the acceptable or unacceptable region it can be identified or classified as one of the following four types.

(i) Free and acceptable point.

(ii) Free and unacceptable point.

(iii) Bound and acceptable point.

(iv) Bound and unacceptable point.

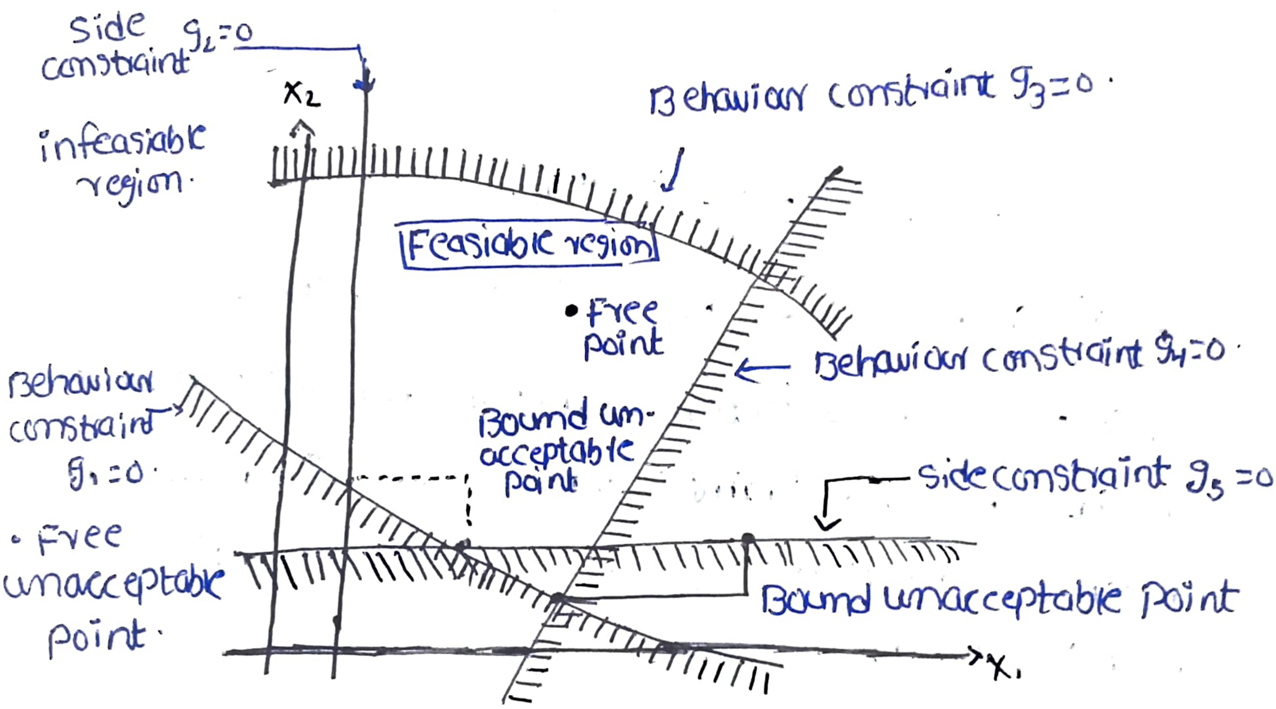


Fig:- constraint surface in 2D space.

1.6: Objective function:

The third task in the formulation procedure is to find a function in terms of the design or decision variables and other problem parameters. with respect to which the design is optimized. This function is known as the objective function.

(i) The choice of the optimization objective function is governed by the nature of the problem.

(ii) selection of the objective function is one of the most important decisions in the whole optimum design process.

(iii) In some situations, there may be more than one objective function to be satisfied simultaneously.

(iv) An optimization problem involving multiple objective functions is known as multi-objective programming problem.

ex:- $f(X) = a_1 f_1(X) + a_2 f_2(X)$.

$f_1(X)$ and $f_2(X)$ → two objective functions

$f(X)$ → new objective function for optimization.

→ a_1, a_2 are constraints whose value indicate the relative importance of one objective function relative to the other.

* In general, objective function can be of 2 types. either the objective function is to be maximized or to be minimized. But the optimization methods are usually either for minimization problem or for maximization problem.

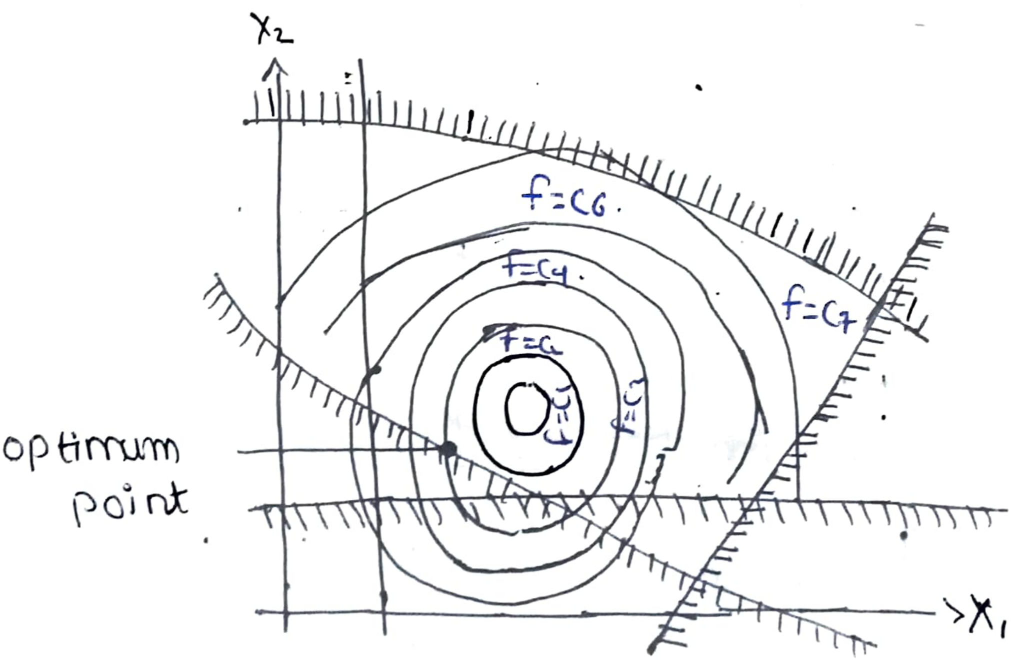
Fortunately, the duplicate principle* helps us by allowing the same method to be used for minimization or maximization with minor change in the objective function instead of a change in entire design.

* If the method/design is developed for solving a minimization problem, it can also be used to solve a maximization problem by simply multiplying the objective function by -(minus) and vice versa.

1.7:- objective function surface:-

The locus of all points satisfying $f(X) = c =$ constant forms a hyper surface in the design space, and for each value of c there corresponds a different member of a family of surfaces.

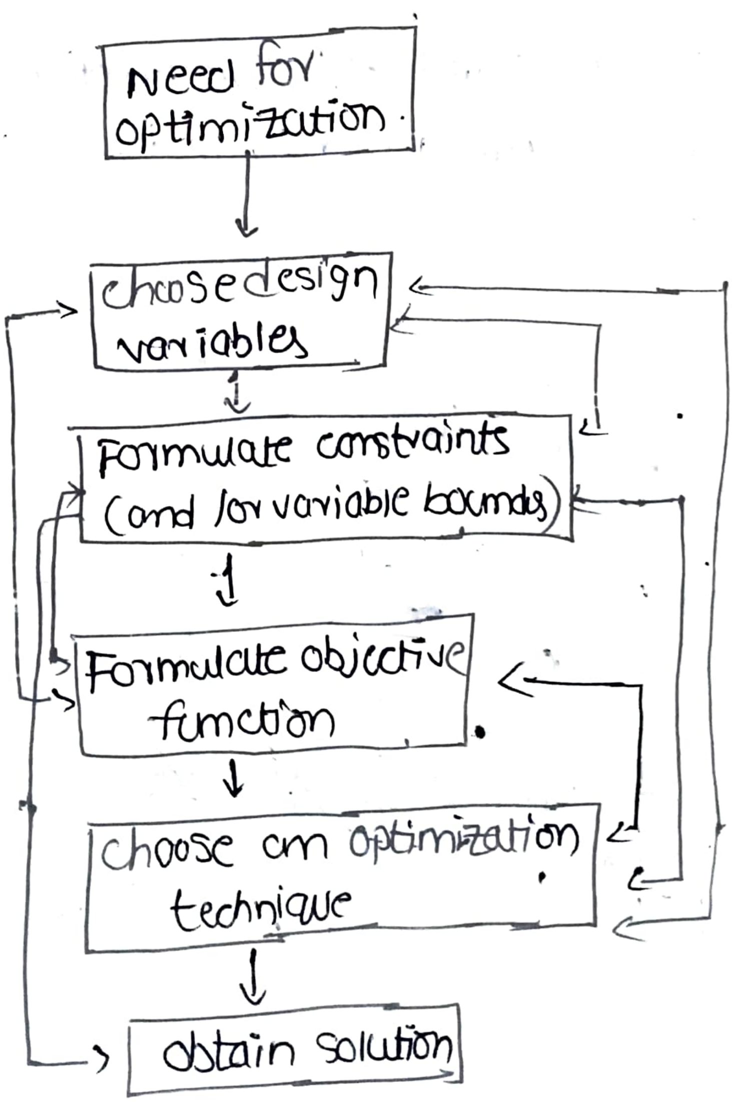
These surface called objective function surfaces can be shown in a two dimensional design space in following fig.



once the objective function surface are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty. But the main problem is that, whenever the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem has to be solved purely as a mathematical problem.

optimization problem formulation:

is used to create mathematical model of the optimal problem.



The first step is to realize the need for using optimization in a specific design problem.

Then the designer needs to choose the important design variables associated with the problem.

The formulation optimization problem involves other considerations such as constraints, objective function.

⇒ Typical optimization problem mathematical model is as follows:

Maximize or minimize (objective function).

subject to (constraints).

(or).

optimize (objective function).

subject to (constraints).

1.8: Classification of optimization problems :-

(i) classification based on the existence of constraints.

As discussed, optimization can be classified as constrained or unconstrained depending on whether or not constraints exist in the problem.

(ii) classification based on the nature of the design variables:

(i) parameter (or) static optimization:

The problem is to find values to a set of design parameters that make some prescribed function of these parameters minimum subject to certain constraints.

(ii) Trajectory (or) dynamic optimization :-

The problem is to find values to a set of design parameters which are all continous functions of some other parameter that minimizes an objective function subject to a set of constraints.

(iii) Classification based on the physical structure of the problem:-

↳ optimal control and non optimal control problems.

* An optimal control problem is a mathematical programming problem involving number of stages, where each stage evolve from the preceeding stage in a prescribed manner.

↳ Two variables describes it those are

↳ control (design) -> govern the evolution

↳ state variables.

(iv) Classifications based on the nature of the equations involved:-

↳ linear, nonlinear, geometric and quadratic programming problems.

Non linear -> if constraint function is nonlinear, then problem is called non-linear programming.

geometric -> constraints are expressed as posynomials in X.

A function $h(X)$ is called posynomial if h can be expressed as the sum of power terms each of the form

$$c_1 x_1^{*i_1} x_2^{*i_2} \dots x_n^{*i_n}.$$

$$c_i > 0 \ \& \ x_j > 0.$$

Quadratic → it is a problem in a nonlinear programming problem with a quadratic objective function and linear constraints.

Linear programming problem:→ constraints are linear functions of the design variables.

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}. \text{ which minimizes}$$

$$f(X) = \sum_{i=1}^{n} c_i x_i.$$

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \quad j = 1, 2, \dots m.$$

$$x_i \geq 0 \qquad i = 1, 2, \dots, n.$$

(v) classification based on permissible values of design variables:

values permitted for the design variables, optimization problems can be classified as "integer" and "real valued" programming problems."

→ design variables $x_1, x_2 \dots x_n$ of an optimization problems are restricted to take on only integer (discrete) values, the problem is called an integer programming problem.

→ on the other hand; if all the design variables are permitted to take any real value, the optimization problem is called a real-valued programming problem.

(vi) Classification based on the deterministic nature of the variables:

Based on the deterministic nature of the variables involved, optimization problems can be classified as a deterministic and stochastic programming problems.

→ A stochastic programming problem is an optimization problem in which some or all of the parameters (design variables and/or preassigned parameters) are probabilistic (non-deterministic or stochastic).

→ According to this defination, the problem types considered previously are deterministic programming problems.

(vii) Classification based on the separability of the functions:

optimization problems can be classified as separable and non separable programming problems based on the separability of the objective and constraint functions.

→ A separable programming problem is one in which the objective function and then constraints are separable. A function $f(X)$ is said to be separable if it can be expressed as the sum of n single variable functions $f_1(x_1), f_2(x_2) \ldots f_n(x_n)$ i.e.

$$f(X) = \sum_{i=1}^{n} f_i(X_i).$$

viii. classification based on the number of objective function:

Depending on the number of objective functions to be minimized, optimization problems can be classified as single - and multi-objective programming problems.

The problems considered previously are single objective programming problems.

→ A multi objective programming problem can be stated as follow:

Find X which minimizes $f_1(X), f_2(X), \ldots f_k(X)$

Subject to $g_j(X) \leq 0$, $j = 1, 2, \ldots m$.

$f_1, f_2 \ldots f_k$ → denote objective function to be minimized simultaneously.

Part-2 :- Classical optimization technique:-

classical optimization techniques use differential calculus to determine points of maxima and minima of continous and differentiable functions

↳ optimum Solution for

→ single-variable function

→ a multi-variable function with no constraints

→ a multivariable function with equality and inequality constraints.

2.1:- Single variable optimization:

Single variable function optimization techniques of single variable function are simple and easier to understand, since single variable functions involve only one variable.

→ A single-variable optimization problem is one in which the value of $x = x^*$ in interval [a,b] such that x^* minimizes $f(x)$, where $f(x)$ is the objective function a 'x' is a real variable.

* purpose of optimization is to find a solution x. $f(x)$ is minimum.

↳ equivalent dual problem $-f(x)$, and considered it to be minimized.

* Relative (or) local minimum:

A function of one variable $f(x)$ is said to have a relative or local minimum at $x = x^*$ if $f(x^*) \leq f(x^* + h)$ for all sufficiently small positive and negative values of h.

* Relative or Local maximum:

$f(x)$ is said to have a relative or local maximum at $x = x^*$ if $f(x^*) \geq f(x^* + h)$ for all sufficiently small value of h close to zero.

* Global (or) absolute minimum:

$f(x^*) \leq f(x)$ for all x, and not Just for all x close to x^* (i.e: interval centred at x^*, (or) neighbourhood of x^*), in the domain over which $f(x)$ is defined.

* Global (or) absolute maximum:-

$f(x^*) \geq f(x)$ for all x and not Just for all x close to x^* (i.e interval centred at x^*, (or) neighbourhood of x^*) in the domain over which $f(x)$ is defined.

* Single variable optimization problem without constraints:-

Let $f'(x) = f''(x^*) = \ldots\ldots f^{n-1}(x^*) = 0$, but $f^n(x^*) \neq 0$, then $f(x^*)$ is

(i) a minimum of $f(x)$ if $f^{(n)}(x^*) > 0$ and n is even.

(ii) a maximum of $f(x)$ if $f^n(x^*) < 0$ and n is even.

(iii) neither maximum nor minimum if 'n' is odd where x^* is a stationary point.

Problem-1:

(i) Determine the maximum and minimum value.

$f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

Sol: Given $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$ — ①.

diff w.r.t to 'x'.

$f'(x) = 12(5x^4) - 45(4x^3) + 40(3x^2)$

$f'(x) = 60x^4 - 180x^3 + 120x^2$ — ②.

diff w.r.t x

$f''(x) = 240x^3 - 540x^2 + 240x$ — ③.

To find stationary points

equate $f'(x) = 0$.

$60x^4 - 180x^3 + 120x^2 = 0$.

$60x^2[x^2 - 3x + 2] = 0$.

$x^2 = 0$ (or) $x^2 - 3x + 2 = 0$.

$x = 0$ (or) $x = 1, 2$.

Case-i: put $x = 1$ in eq ③.

$f''(x) = 240 - 540 + 240 = -60 < 0$.

$f(x)$ has maximum at $x = 1$.

from ①

$f(1) = 12 - 45 + 40 + 5 = 12$

$\boxed{f_{max} = 12}$

Case-ii: put $x = 2$ in eq ③.

$f''(2) = 240(8) - 540(4) + 240(2)$.

$f''(2) = 240 > 0$.

$f(x)$ has minimum at $x = 2$.

from ①

$f(2) = 12(32) - 45(16) + 40(8) + 5$

$\boxed{f_{min} = -11}$

<u>case-3</u>: put $x=0$ in eq ③.

$f''(0) = 0$.

$\therefore$ neither maximum (nor) minimum.

② Determine extreme points of the function

$f(x) = x^3 - 3x + 6$.

<u>Sol</u>: given $f(x) = x^3 - 3x + 6$.

$f'(x) = 3x^2 - 3 = 0$.

diff w.r.t to x again

$f''(x) = 6x$.

equate $f'(x) = 0$ to find stationary points.

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x^2 - 1 = 0$

$x = \pm 1$.

at $x = 1$, $6 = f''(x) > 0 \rightarrow$ minimize point.

at $x = -1$, $-6 = f''(x) < 0 \rightarrow$ maximum point.

2.2: Multi-variable optimization with no constraints:-

consider the matrix

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The leading principle minors are given by

$$A_1 = |a_{11}|$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$A_4 = |A|.$$

→ **Nature of square matrix**:-

(1) If all the leading principle minors of a positive, then square matrix 'a' is positive definite.

(2) If the leading principle minors are alternatively negative and positive [stone-ve], then 'A' is said to be negative definite.

(3) If all the leading principle minors of A are greater than or equal to 0. then the square matrix 'A' is semi positive definite.

(4) If all the leading principle minors of 'A' are alternatively ≤ 0 and ≥ 0, then 'A' is said to be semi-negative definite.

(5) If it is a none of the above '4' types, then square matrix is indefinite.

Q: Find the nature of square matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

Sol: The leading principle minors of 'A' are

$A_1 = 3 > 0$.

$A_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8 > 0$.

$A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{vmatrix} = 3(14) - 1(4) - 1(2)$

$= 42 - 4 - 2$

$= 36 > 0$.

$\therefore$ all leading principle minors are > 0

nature of matrix is positive definite.

Hessian matrix (J):

The Hessian matrix of $f(x)$ is the matrix of second order partial derivatives of $f(x)$, where

$x = (x_1, x_2 \ldots\ldots x_n)^T$.

Hessian matrix (J) = $\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_1 x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & & \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \frac{\partial^2 f}{\partial x_m \partial x_2} & \frac{\partial^2 f}{\partial x_m \partial x_3} & \cdots & \frac{\partial^2 f_1}{\partial x_m \partial x_n} \end{bmatrix}$

case-i: If $x = (x_1, x_2)^T$, then

$$J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

case-ii: If $X = (x_1, x_2, x_3)^T$, then

$$J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

* The necessary and sufficient condition for optimization of multi-variable objective function without constraints:-

Working rule:

Step 1:- To find the maxima (or) minima of $f(x)$.
- where $X = (x_1, x_2, \dots x_n)^T$.

Step 2: Find the Stationary points by equating
$$\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \frac{\partial f}{\partial x_3} = 0 \dots \frac{\partial f}{\partial x_n} = 0.$$

Step 3: Form the Hassian matrix (J) at each Stationary point.

Step 4: If Nature of the matrix (J) is positive definite
- Then f(x) has minimum.

Step 5: If the Nature of the matrix (J) is negative definite, then f(x) has maximum.

Step 6: The Nature of the matrix (J) is indefinite then f(x) has no extreme points.

Problem:

(1) Find the extreme point of the function (or) find the maxima (or) minima of $f(x) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$

Sol: Given

$$f(x) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6 \quad \text{—(1)}$$

diff eq(1) partially w.r.t to x_1

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 \quad \text{— (2)}$$

diff eq(2) w.r.t 'x_1'.

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4$$

diff eq(2) w.r.t 'x_2'.

$$\frac{\partial}{\partial x_2}\left[\frac{\partial f}{\partial x_1}\right] = \frac{\partial}{\partial x_2}[3x_1^2 + 4x_1]$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

diff eq(1) partially w.r.t to 'x_2'.

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2 = 0 \quad \text{— (3)}$$

diff eq(3) partially w.r.t to x_2.

$$\frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

diff eq(3) partially w.r.t to 'x_1'.

$$\frac{\partial}{\partial x_1}\left[\frac{\partial f}{\partial x_2}\right] = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

To find stationary points

equate $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 0$.

$3x_1^2 + 4x_1 = 0$

$= x_1 [3x_1 + 4] = 0$.

$x_1 = 0$ (or) $x_1 = -4/3$.

$3x_2^2 + 8x_2 = 0$

$x_2(3x_2 + 8) = 0$.

$x_2 = 0$ (or) $x_2 = -8/3$.

$\therefore$ The stationary points are

$(0,0), (0, -\frac{8}{3}), (-\frac{4}{3}, 0)$ and $(-\frac{4}{3}, -\frac{8}{3})$.

→ From the Hessian matrix (J)

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$J = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix}$$

case-i: put $(x_1, x_2) = (0,0)$ in J.

$$J = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 + 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} = \cancel{32} > 0.$$

$J_1 = 4 > 0$

$J_2 = 32 > 0$.

J is +ve definite

$\therefore$ f has minimum at $(0,0)$

$f_{min} = 6$ $\boxed{\text{from eq (1)}}$

case-ii: put $(x_1, x_2) = (0, -8/3)$

$$J = \begin{bmatrix} 6x_1+4 & 0 \\ 0 & 6x_2+8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6(-\frac{8}{3})+8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}$$

leading principle $= J_1 = 4 > 0$.

$$J_2 = \begin{vmatrix} 4 & 0 \\ 0 & -8 \end{vmatrix} = -32 < 0.$$

$\therefore$ matrix J is indefinite.

$\therefore$ f has neither maximum nor minimum.

case-iii: put $(x_1, x_2) = (-4/3, 0)$ in J.

$$J = \begin{bmatrix} 6x_1+4 & 0 \\ 0 & 6x_2+8 \end{bmatrix} = \begin{bmatrix} 6(-4/3)+4 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix}$$

leading principle $J_1 = -4 < 0$

$$J_2 = \begin{vmatrix} -4 & 0 \\ 0 & 8 \end{vmatrix} = -32 < 0.$$

$\therefore$ matrix J is indefinite.

$\therefore$ f has neither maximum (nor) minimum.

case-iv: put $(x_1, x_2) = (-4/3, -8/3)$ in J.

$$J = \begin{bmatrix} 6x_1+4 & 0 \\ 0 & 6x_2+8 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$$

leading principle $J_1 = -4 < 0$.

$J_2 = 32 > 0$.

Matrix J is -ve definite.

f has maximum at $(-4/3, -8/3)$.

Now, $f_{max} = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$.

$\boxed{f_{max} = 16.67}$ from eq(1).