# Unit - IV

## Nonlinear Programming

## Fibonacci search method:-

Fibonacci sequence:-

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ....

$$\therefore \quad F_n = F_{n-1} + F_{n-2} \quad , \quad n \geq 2$$

→ Aim of Fibonacci search method is to Minimize/maxime $f(x)$ over $x \in [a, b]$

### Procedure:

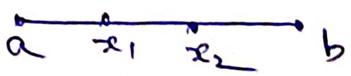
1. Determine the number of iterations.
2. Determine $L_2^* = \dfrac{F_{n-2}}{F_n} \cdot L_0$

   where $L_0 = b - a$ is the interval length.

   Find $x_1 = a + L_2^*$

   $x_2 = b - L_2^*$

   i.e




3. Set $L = a$     L - Left

   $R = b$     R - Right

**Imp. Note:** → If L is minimum, Then we keep L value as same for the Next iteration, and R value will be $x_2$ of previous iteration.

→ If R is minimum, then we keep R value as same for the next iteration and L value will be $x_1$ of previous iteration.

## Fibonacci search method:-

① Minimize $f(x) = x^2$ over $[-5, 15]$ Take $n = 7$.

Sol:

| K | $\frac{F_{n-k}}{F_{n-k+1}}$ | L | R | $x_1$ (L) | $x_2$ (R) | $f(x_1)$ (L) | $f(x_2)$ (R) | L or R |
|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{F_6}{F_7} = \frac{13}{21}$ | -5 | 15 | 2.6191 | 7.3809 | 6.8596 | 54.4776 | L (minimum) |
| 2. | $\frac{8}{13} = \frac{F_5}{F_6}$ | -5 | 7.3809 | -0.2382 | 2.6191 | 0.0567 | 6.8596 | L (minimum) |
| 3. | $\frac{5}{8} = \frac{F_4}{F_5}$ | -5 | 2.6191 | -2.1427 | -0.2382 | 4.5911 | 0.0567 | R (minimum) |
| 4. | $\frac{3}{5} = \frac{F_3}{F_4}$ | -2.1427 | 2.6191 | -0.2382 | 0.7146 | 0.0567 | 0.5106 | L |
| 5. | $\frac{2}{3} = \frac{F_2}{F_3}$ | -2.1427 | 0.7146 | -1.1899 | -0.2382 | 1.4158 | 0.0567 | R |
| 6 | $\frac{1}{2} = \frac{F_1}{F_2}$ | -1.1899 | 0.7146 | -0.2382 | -0.2377 | 0.0567 | 0.0565 | R |
| 7. | $\frac{1}{1} = \frac{F_0}{F_1}$ | -0.2382 | 0.7146 | -0.2377 | 0.741 | 0.0565 | 0.5099 | — |

| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
|---|---|---|---|---|---|---|---|
| $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ |

~~$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7$~~ - -

~~0, 1, 1, 2, 3, 5, 8, 13, 21,~~ - - -

Formula $$x_2 = L + \left(\frac{F_{n-k}}{F_{n-k+1}}\right)(R-L)$$

$$x_1 = L + R - x_2$$

Thus $x_{min} \in [-0.2382, 0.7146]$

$$\therefore x^* = \frac{-0.2382 + 0.7146}{2} = 0.2382$$

$$f(x^*) = 0.05674.$$

See (Imp. note in previous Page)

# Steepest Descent Method:- (gradient method)

**Step1:** Calculate

$S_i$ at $X_i$ by $S_i = -\nabla f_i$

**Step2:** calculate $\lambda_i$ by using $\lambda_i = \dfrac{S_i^T S_i}{S_i^T H_i S_i}$ and

the new point $X_{i+1} = X_i + \lambda_i S_i$

**Step3:** Check the optimum of $x_{i+1}$ by $\nabla f(x_{i+1}) \cong 0$

If met stop.

Otherwise implement 1 again for this newpoint $X_{i+1}$.

**Ex:** Minimize

$$f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

Starting from the point $X_1 = (0,0)$.

**Sol:** The gradient of $f$ is

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$\dfrac{\partial^2 f}{\partial x_1^2} = \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial f}{\partial x_1}\right) = 0 + 4 + 0 = 4$

$\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = \dfrac{\partial}{\partial x_1}\left(\dfrac{\partial f}{\partial x_2}\right) = 0 + 2 + 0 = 2$

$\dfrac{\partial^2 f}{\partial x_2 \partial x_1} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial f}{\partial x_1}\right) = 0 + 0 + 2 = 2$

$\dfrac{\partial^2 f}{\partial x_2^2} = \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial f}{\partial x_2}\right) = 0 + 0 + 2 = 2$

Now, Hessian matrix

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}.$$

iteration 1:

at $X_1(0,0)$

$$S_1 = -\nabla f(x_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \left( -\nabla f(x_1) = -\nabla f(0,0) = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$\lambda_1 = \frac{S_1^T S_1}{S_1^T \cdot H \cdot S_1} = \frac{\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \frac{2}{2} = 1$$

($\because$ matrix multiplication)

$\therefore$ New point is $X_2 = X_1 + \lambda_1 S_1$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Checking for the optimum:-

$$\nabla f(X_2) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so, go to iteration 2.

iteration 2:

$$S_2 = -\nabla f(X_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{S_2^T \cdot S_2}{S_2^T \cdot H \cdot S_2} = \frac{2}{10} = \frac{1}{5}$$

$$X_3 = X_2 + \lambda_2 S_2$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.2 \end{bmatrix}$$

$$\nabla f(X_3) = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So go to iteration 3.

iteration 3:

$$S_3 = -\nabla f(X_3) = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

$$\lambda_3 = \frac{S_3^T \cdot S_3}{S_3^T H \cdot S_3} = \frac{\begin{bmatrix} -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}}{\begin{bmatrix} -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}} = \frac{0.08}{0.08} = 1$$

$$\therefore X_4 = X_3 + \lambda_3 S_3$$

$$= \begin{bmatrix} -0.8 \\ 1.2 \end{bmatrix} + 1 \cdot \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -1.0 \\ 1.4 \end{bmatrix}$$

~~Step~~: Since $\nabla f(X_4) = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ we move to iteration 4.

iteration 4: $S_4 = -\nabla f(X_4) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$

$$\lambda_4 = \frac{S_4^T S_4}{S_4^T \cdot H \cdot S_4} = \frac{1}{5}.$$

$$X_5 = X_4 + \lambda_4 S_4$$

$$= \begin{bmatrix} -1.0 \\ 1.4 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

$$X_5 = \begin{bmatrix} -0.96 \\ 1.44 \end{bmatrix}$$

$$\therefore \nabla f(X_5) = \begin{bmatrix} 0.04 \\ -0.04 \end{bmatrix} \cong 0$$

$\therefore X_5$ is optimum.

$$\therefore X^* = \begin{bmatrix} -1.0 \\ 1.5 \end{bmatrix}$$

$$f^* = -1 - 1.5 + 2(-1)^2 + 2(-1)(1.5) + (1.5)^2$$

$$= -1.25$$

## Univariate Method:-

~~Minimize~~

(i) Choose an arbitrary starting point $X_1$

(ii) Find the search direction $S_i$ as

$$S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, S_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

(iii) $f_i = f(X_i)$

$f^+ = f(X_i + \epsilon S_i)$

$f^- = f(X_i - \epsilon S_i)$

(iv) If $f^+ < f_i$, then $S_i$ will be the correct direction of decreasing the value of $f$.

$f^- < f_i$, $-S_i$ will be the correct direction of decreasing the value of $f$.

(v) find the optimal step length $\lambda_i^*$ such that

$$f(X_i \pm \lambda_i^* S_i) = \underset{\lambda_i}{\text{Min}} (X_i \pm \lambda_i S_i)$$

where + or − sign has to be used depending upon whether $S_i$ or $-S_i$ is the direction for decreasing the function value.

vi) Set the new value of $i = i+1$ and go to step 2 continue the procedure.

pb: Minimize

$f(x_1, x_2) = 2x_1^2 + x_2^2$ from the starting point $(1,2)$ using the univariate method.

Sol:-

Step 1: $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; let $\epsilon = 0.01$

iteration 1:

Step 2: let $S_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Step 3: $f_1 = f(X_1) = f(1,2) = 2(1)^2 + (2)^2 = 2+4 = 6$

$$f^+ = f(X_1 + \epsilon S_1)$$
$$= f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (0.01)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$= f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}\right)$$
$$= f\begin{pmatrix} 1.01 \\ 2 \end{pmatrix}$$
$$= 2(1.01)^2 + (2)^2$$
$$= 6.040$$

$$f^- = f(X_1 - \epsilon S_1)$$
$$= f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - (0.01)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$= f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0.01 \\ 0 \end{pmatrix}\right)$$
$$= f\begin{pmatrix} 0.99 \\ 2 \end{pmatrix}$$
$$= 2(0.99)^2 + 2^2$$
$$= 5.960$$

$\therefore$ clearly $f^{-} < f_1$

So $-S_1$ is the correct direction for minimizing $f$ from $X_1$

<u>Step 4:</u> To find optimum step length $\lambda_1^*$, we minimize

$f(X_1 - \lambda_1 S_1)$ ($\because$ $-S_1$, we have taken)

$= f\left(\begin{bmatrix}1\\2\end{bmatrix} - \lambda_1 \begin{bmatrix}1\\0\end{bmatrix}\right)$

$= f\begin{pmatrix}1-\lambda_1\\2\end{pmatrix}$

$= 2(1-\lambda_1)^2 + 4$

$= 2(\lambda_1^2 + 1 - 2\lambda_1) + 4$

$= 2\lambda_1^2 - 4\lambda_1 + 6$.

Now let $g = 2\lambda_1^2 - 4\lambda_1 + 6$

$\Rightarrow \frac{\partial g}{\partial \lambda_1} = 0$

$\Rightarrow 4\lambda_1 - 4 = 0$

$\Rightarrow \lambda_1 = 1$

So, $\lambda_1^* = 1$

<u>Step 5:</u> take $X_2 = X_1 - \lambda_1^* S_1$ (for next iteration)

$= \begin{bmatrix}1\\2\end{bmatrix} - 1 \cdot \begin{bmatrix}1\\0\end{bmatrix}$

$= \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$

$= \begin{bmatrix}0\\2\end{bmatrix}$.

iteration 2: Step 1: $X_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Step 2: choose $S_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Step 3: $f_2 = f(X_2)$

$= f\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$= 2(0)^2 + (2)^2$

$= 4$

$f^+ = f(X_2 + \epsilon S_2)$

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + (0.01)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}\right)$

$= f\begin{pmatrix} 0 \\ 2.01 \end{pmatrix}$

$= 2(0)^2 + (2.01)^2$

$\boxed{f^+ = 4.0401}$

$f^- = f(X_2 - \epsilon S_2)$

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} - (0.01)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}\right)$

$= f\begin{pmatrix} 0 \\ 1.99 \end{pmatrix}$

$= 2(0)^2 + (1.99)^2$

$\boxed{f^- = 3.9601}$

$\therefore f^- < f_2$

$\therefore$ $-S_2$ is the correct direction for decreasing the value of $f$ from $X_2$.

Step 4: we minimize $f(X_2 - \lambda_2 S_2)$, ($\because -S_2$ taken)

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$= f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix}\right)$

$= f\begin{pmatrix} 0 \\ 2-\lambda_2 \end{pmatrix}$

$= 2(0)^2 + (2-\lambda_2)^2 = 4 + \lambda_2^2 - 4\lambda_2$

let $g = 4 + \lambda_2^2 - 4\lambda_2$

$\frac{\partial g}{\partial \lambda_2} = 2\lambda_2 - 4 = 0 \Rightarrow \boxed{\lambda_2^* = 2}$

$\therefore X_3 = X_2 + \lambda_2^* S_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$\therefore f_3(X_3) = f_3\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 2(0)^2 + 4^2 = 16.$

Interior penalty function method:-

Pb: ~~$f(x_1,x_2) = (x_1+2)^{3} + 3x_2+1$ subject to $x_1$~~

Minimize $f(x_1,x_2) = \frac{1}{3}(x_1+1)^3 + x_2$

subject to $x_1 \geq 1$

$x_2 \geq 0$

Sol:-

let $f(x_1,x_2) = \frac{1}{3}(x_1+1)^3 + x_2$ —— (1)

$x_1 \geq 1 \Rightarrow x_1 - 1 \geq 0 \Rightarrow 1 - x_1 \leq 0$

$x_2 \geq 0 \Rightarrow -x_2 \leq 0$

$\therefore$ let $g_1 = 1 - x_1$

$g_2 = -x_2$

Now construct

$$\phi = f - \epsilon\left(\frac{1}{g_1} + \frac{1}{g_2}\right)$$

$$\phi = \frac{1}{3}(x_1+1)^3 + x_2 - \epsilon\left(\frac{1}{1-x_1} - \frac{1}{x_2}\right)$$ —— (2)

Now we find $x_1$ by taking $\frac{\partial \phi}{\partial x_1} = 0$

and $x_2$ by taking $\frac{\partial \phi}{\partial x_2} = 0$

$$\therefore \frac{\partial \phi}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{1}{3}(x_1+1)^3 + x_2 - \epsilon\left(\frac{1}{1-x_1} - \frac{1}{x_2}\right)\right)$$

$$= \frac{1}{\cancel{3}}\cdot\cancel{3}(x_1+1)^2 + 0 - \epsilon\left(\frac{-1}{(1-x_1)^2}(-1) - 0\right)$$

$$\frac{\partial \phi}{\partial x_1} = (x_1+1)^2 - \epsilon \cdot \frac{1}{(1-x_1)^2}$$

$$= \frac{(x_1+1)^2(x_1-1)^2 - \epsilon}{(x_1-1)^2}$$

$$\therefore \frac{\partial \phi}{\partial x_1} = 0 \Rightarrow (x_1+1)^2(x_1-1)^2 - \epsilon = 0 \Rightarrow (x_1^2-1)^2 - \epsilon = 0 \Rightarrow x_1^2 - 1 = \sqrt{\epsilon}$$

$$\Rightarrow \boxed{x_1 = \sqrt{1+\sqrt{\epsilon}}}$$

Now $\dfrac{\partial \phi}{\partial x_2} = 0$

$$\Rightarrow \frac{\partial}{\partial x_2}\left[ \frac{1}{3}(x_1+1)^3 + x_2 - \epsilon\left(\frac{1}{1-x_1} - \frac{1}{x_2}\right)\right] = 0$$

$$\Rightarrow \quad 0 + 1 - \epsilon\left(0 - \left(\frac{-1}{x_2^2}\right)\right) = 0$$

$$\Rightarrow \quad 1 - \frac{\epsilon}{x_2^2} = 0$$

$$\Rightarrow \quad 1 = \frac{\epsilon}{x_2^2}$$

$$\Rightarrow \quad x_2 = \sqrt{\epsilon}$$

eq(1) eq(2)

| Value of $\epsilon$ | $x_1 = \sqrt{1+\sqrt{\epsilon}}$ | $x_2 = \sqrt{\epsilon}$ | $f(x_1, x_2)$ | $\phi(x_1, x_2, \epsilon)$ |
|---|---|---|---|---|
| 0.01 | 1.0488 | 0.1 | 2.9666 | 3.0671 |
| 0.001 | 1.0156 | 0.0316 | 2.7611 | 2.8568 |
| 0.0001 | 1.0049 | 0.01 | ~~2.8568~~ 2.6963 | 2.7267 |
| 0.00001 | 1.0015 | 0.0031 | 2.6960 | 2.6856 |
| 0.000001 | 1.0005 | 0.001 | 2.672 | 2.672 |

equal

optimality is reached

$\therefore f^*$ (optimum) $= 2.672$

# Exterior Penalty method:-

Minimize $f(x_1,x_2)=\frac{1}{3}(x_1+1)^3+x_2$

Subject to $x_1\geq 1$

$x_2\geq 0$.

Sol: $f(x_1,x_2)=\frac{1}{3}(x_1+1)^3+x_2$

$\therefore\ x_1\geq 1 \Rightarrow x_1-1\geq 0 \Rightarrow 1-x_1\leq 0$

$x_2\geq 0 \Rightarrow -x_2\leq 0$

Let $g_1=1-x_1,\ g_2=-x_2$.

Now construct

$$\phi=f+\epsilon\left[\max(0,g_1)\right]^2+\epsilon\left[\max(0,g_2)\right]^2$$

$$\phi=\frac{1}{3}(x_1+1)^3+x_2+\epsilon\left[\max(0,1-x_1)\right]^2+\epsilon\left[\max(0,-x_2)\right]^2$$

Now, $\frac{\partial\phi}{\partial x_1}=0$

$\Rightarrow \frac{1}{3}(3).(x_1+1)^2-2\epsilon\max(0,1-x_1)=0$

which can be written as

$$\min\left[(x_1+1)^2,\ (x_1+1)^2-2\epsilon(1-x_1)\right]=0$$

Since $(x_1+1)^2=0\Rightarrow x_1=-1$ which does not satisfy $x_1\geq 1$ given constraint, we ignore this case.

$\therefore\ (x_1+1)^2-2\epsilon(1-x_1)=0$

$\Rightarrow x_1^2+1+2x_1-2\epsilon+2\epsilon x_1=0$

$\Rightarrow x_1^2+2x_1(1+\epsilon)+(1-2\epsilon)=0$ $\quad(\because ax^2+bx+c=0)$

$\Rightarrow x_1=\dfrac{-2(1+\epsilon)\pm\sqrt{4(1+\epsilon)^2-4.1.(1-2\epsilon)}}{2}$ $\quad\left(\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}\right.$ roots$\left.\right)$

$$\therefore x_1 = \cancel{2}\left[\frac{-1-\epsilon+\sqrt{(1+\epsilon)^2 - 1+2\epsilon}}{\cancel{2}}\right]$$

$$x_1 = -1-\epsilon+\sqrt{\cancel{1}+\epsilon^2+2\epsilon-\cancel{1}+2\epsilon}$$

$$\boxed{\therefore x_1 = -1-\epsilon+\sqrt{\epsilon^2+4\epsilon}} \quad \text{—} \; 1$$

Now $\dfrac{\partial \phi}{\partial x_2} = 0$

$$\Rightarrow 0+1+0-2\epsilon\left(\max\left(0,-x_2\right)\right) = 0$$

$$\Rightarrow 1-2\gamma\left(\max\left(0,-x_2\right)\right) = 0$$

This can be written as

$$\text{Min}\left(1,\ 1+2\epsilon x_2\right) = 0$$

$$1+2\epsilon x_2 = 0$$

$$\Rightarrow \boxed{x_2 = \frac{-1}{2\epsilon}} \quad \text{—} \; 2$$

from eq ①

$$x_1 = -1-\epsilon+\sqrt{\epsilon^2\left(1+\frac{4}{\epsilon}\right)}$$

$$\therefore x_1 = -1-\epsilon+\epsilon\sqrt{\left(1+\frac{4}{\epsilon}\right)}$$

$\Rightarrow$ if $\epsilon \to \infty$ then $x_1 = -1$

also if $\epsilon \to \infty$ then $x_2 = 0$ $\left(\because x_2 = \frac{-1}{2\epsilon}\right)$

$\therefore$ Optimal point is $(x_1, x_2) = (-1, 0)$

Optimal solution is

$$\min\left(\frac{1}{3}(x_1+1)^3 + x_2\right)$$

$$= \frac{1}{3}(-1+1)^3 + 0$$

$$= \underline{0}$$