## Problem 1

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{(x^n + \binom{n}{1}x^{n-1}h + \cdots + h^n) - x^n}{h} \\
&= \lim_{h\to 0} \frac{\binom{n}{1}x^{n-1}h + \cdots + h^n}{h} \\
&= \lim_{h\to 0} \binom{n}{1}x^{n-1} + \cdots + h^{n-1} \\
&= nx^{n-1}
\end{aligned}
$$

## Problem 2

### Part (a)

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a) + f(h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(h)}{h}
\end{aligned}
$$

This value is constant.

### Part (b)

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a)f(h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a)(f(h) - 1)}{h} \\
&= f(a) \lim_{h\to 0} \frac{f(h) - 1}{h}
\end{aligned}
$$

The value of $\lim\limits_{h\to 0} \frac{f(h)-1}{h}$ is constant so $f'(a) = c \cdot f(a)$ for some constant $C$.

## Problem 3

Fix some $\epsilon > 0$. Assume that $|x - y| < \delta$ for some $\delta$ to be determined. Then by MVT $|f(x) - f(y)| = |f'(c)(x - y)|$ for some $c$. However, the value of $f'(c)$ is bounded by $C$ so

$$|f'(c)(x - y)| \leq C|x - y| = C\delta < \epsilon$$

if $\delta < \frac{\epsilon}{C}$.

## Problem 4

### Part (a)

By definition of a root there are at least $n$ values of $x$ for which $f(x) = 0$. Then we can apply Rolle's Theorem to each consecutive pair of locations to obtain a minimum of $n - 1$ values of $x$ where $f'(x) = 0$.

### Part (b)

It is trivial that a polynomial of degree one has at most one root. Now assume that polynomials with degree $n$ has $n$ roots for some $n \geq 1$. Now consider a polynomial $f$ of degreen $n + 1$. Assume that $f$ has more than $n + 1$ roots. Then by part a) $f'$ has degree $n$ but at least $n + 1$ roots. This contradicts our inductive hypothesis and therefore polynomials with degree $n + 1$ have at most $n + 1$ roots.

## Problem 5

We assume that $\lim_{x\to+\infty} f(x)$ exists. Thus we can construct a sequence $x_n$ of real numbers such that $x_n$ goes to infinity. To select $x_n$ set $\epsilon = 1/n$. We know because the limit exists at infinity that there exists some smallest $M$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \geq M$. Then set some fixed distance $l$. For each $n$ apply mean value theorem on the interval $[M, M + l]$. Then $f(M + l) - f(M) = f'(x_n)(M + l - M) \implies f'(x_n) = (f(M + l) - f(M))/(l)$. The denominator is constant and the numerator is less than $1/n$. Therefore as $n \to \infty$, $f'(x_n) \to 0$. Then by the sequential characterization of limits at infinity, $\lim_{x\to+\infty} f'(x) = 0$.

## Problem 6

Let $f(x) = x^{1/2} + \frac{1}{2}\log(x) - x$. Then $f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2x} - 1$ is less than 0 for all $x > 1$. This means that $f$ is a strictly decreasing function by Theorem 4.17. Thus $\forall x > 1$:

$$f(x) = x^{1/2} + \frac{1}{2}\log(x) - x > 0 \implies f(x) = x^{1/2} + \frac{1}{2}\log(x) > x$$