Problem 1

(a) Prove $-(-x) = x$.

(b) Prove $-(xy) = (-x)y$.

Solution

Part (a)

$$\begin{aligned} 0 + -(-x) &= -(-x) && \text{by (A3)} \\ [x + -x] + -(-x) &= -(-x) && \text{by (A4)} \\ x + [-x + -(-x)] &= -(-x) && \text{by (A2)} \\ x + 0 &= -(-x) && \text{by (A4)} \\ x &= -(-x) && \text{by (A3)} \end{aligned}$$

Part (b)

$$\begin{aligned} (-x)y + xy &= (-x + x)y && \text{by (D)} \\ (-x)y + xy &= (0)y && \text{by (A4)} \end{aligned}$$

In class it was proved that $0 \cdot x = 0$ for all x. By this result we get

$$\begin{aligned} (-x)y + xy &= 0 \\ (-x)y + xy + -(xy) &= 0 + -(xy) && \text{add} - (xy) \\ (-x)y + 0 &= -(xy) && \text{by (A4)} \\ (-x)y &= -(xy) && \text{by (A3)} \end{aligned}$$

Problem 2

(a) Prove if $x > y, z < 0$ then $xz < yz$.

(b) Prove if $x > y > 0, z > w > 0$ then $xz > yw$.

(c) Prove if $x > 0$ then $x^{-1} > 0$.

Solution

Part (a)

In order to prove this I will first prove consequent 7 introduced in class, that $(-x)\cdot(y) = (-xy) = x\cdot(-y)$.

$$\begin{aligned}
x + [(-1)\cdot x] &= [1\cdot x] + [(-1)\cdot x] && \text{(M3)}\\
x + [(-1)\cdot x] &= (1 + (-1))\cdot x && \text{(D)}\\
x + [(-1)\cdot x] &= 0\cdot x && \text{(A4)}\\
x + [(-1)\cdot x] &= 0 && \text{(consequent 3)}\\
-x + x + [(-1)\cdot x] &= -x + 0 &&\\
0 + [(-1)\cdot x] &= -x + 0 && \text{(A4)}\\
[(-1)\cdot x] &= -x && \text{(A3)}
\end{aligned}$$

Using the equivalence established above;

$$\begin{aligned}
(-x)(y) &= (-1\cdot x)\cdot(y) &&\\
(-x)(y) &= -1\cdot(x\cdot y) && \text{(D)}\\
(-x)(y) &= -(xy) &&
\end{aligned}$$

This shows that $(-x)(y) = -(xy)$. The argument that $-(xy) = (x)(-y)$ has an identical structure.
$z < 0$ and $-1 < 0$ so

$$\begin{aligned}
0\cdot -1 &< z\cdot(-1) && \text{by (O7)}\\
0 &< z\cdot(-1) && \text{consequent 3 proved in class}\\
0 &< -(z\cdot 1) && \text{by consequent 7}\\
0 &< -z && \text{by M3}\\
-z &> 0 && \text{definition of ¡}
\end{aligned}$$

Having $-z > 0$ and $x > y$ we use (O6) to get $x(-z) > y(-z)$

$$\begin{aligned}
-(xz) &> -(yz) && \text{consequent 7}\\
-(xz) + ((xz) + (yz)) &> -(yz) + ((xz) + (yz)) && \text{(O4)}\\
-(xz) + ((xz) + (yz)) &> -(yz) + ((yz) + (xz)) && \text{(A1)}\\
(-(xz) + (xz)) + (yz) &> (-(yz) + (yz)) + (xz) && \text{(M2)}\\
0 + (yz) &> 0 + (xz) && \text{(A4)}\\
(yz) &> (xz)\text{(A3)} &&
\end{aligned}$$

By the definition of $>$ this is the same as saying $xz < yz$.

Part (b)

It is given that $z > w$ and $w > 0$. By (O2) $z > 0$. It is also given that $x > y$. Then by (O6) $xz > yz$. It is also given that $y > 0$. By (O6) again $zy > wy$. Then by (M1) $yz > yw$ and finally by (O2) $xz > yw$.

Part (c)

First assume that $x^{-1} < 0$. Then by (O7) proved in class:

$$x \cdot x^{-1} < 0 \cdot x^{-1}$$

It was also proved in class that $0 \cdot x = 0$ for all x. Thus,

$$\begin{aligned} x \cdot x^{-1} &< 0 \\ 1 &< 0 && \text{by (M4)} \end{aligned}$$

This is a contradiction so $x^{-1} > 0$.

Problem 3

Prove that there does not exist an $x \in \mathbb{Z}$ such that $0 < x < 1$.
$\mathbb{Z} = \{x \in \mathbb{R} \mid x \in \mathbb{N} \vee x = 0 \vee -x \in \mathbb{N}\}$.

Solution

Consider any arbitrary $x \in \mathbb{R}$. There are three possible cases.

(a) Case 1: $x \in \mathbb{N}$
It was proven in class that for all x in $\mathbb{N}$, $x \geq 1$. Thus it is impossible that $x < 1$.

(b) Case 2: $x = 0$
If $x = 0$ then it is impossible that $x > 0$.

(c) Case 3: $-x \in \mathbb{N}$
By the same fact used in case 1, $-x \geq 1 \implies x \leq -1$. So it is impossible that $x > 0$.

There is no case in which it is possible that $0 < x < 1$.

Problem 4

Prove that it is impossible to define inequalities in $\mathbb{C}$ such that (O1)-(O4) hold.

Solution

The proof given in the book that for any nonzero $a \in \mathbb{R}$, $a^2 > 0$ depends only on axioms (O1)-(O4). Thus if these axioms held in $\mathbb{C}$ then it would have to be the case that the square of any nonzero element of $\mathbb{C}$ was greather than 0. However, i is defined such that $i^2 = -1$. Using the fact introduced in class that $1 > 0$ we can say

$$1 + (-1) > 0 + (-1)$$
$$0 > -1 \tag{A4}$$

By axiom (O1) it is impossible for it also to be the case that $0 < -1$. Thus this is a contradiction. Therefore it is impossible to define inequalitied in $\mathbb{C}$ in such a way that axioms (O1)-(O4) hold.

Problem 5

(a) Let $x, y \in \mathbb{R}$. Prove $x \leq y$ if and only if $x - \epsilon < y + \epsilon \forall \epsilon > 0$.

(b) Let $x, y \in \mathbb{R}$ with $x < y$. Prove there exists $z \in \mathbb{R}$ with $x < z < y$.

(c) Let $a, x, b \in \mathbb{R}$ with $a < x < b$. Prove there exists $\epsilon > 0$ such that $a < x - \epsilon < x + \epsilon < b$. Deduce that $(x - \epsilon, x + \epsilon) \subset (a, b)$.

Solution

Part (a)

By Theorem 1.9 part i proved in the book, $x < y + \epsilon$ for all $\epsilon > 0$. For any given value for $\epsilon > 0$, $0 > -\epsilon$. Then by (O5) $y + \epsilon > x - \epsilon$ for all $\epsilon > 0$.

Part (b)

Let n be the largest natural number such that $\frac{1}{n} < y - x$. Let k be the largest natural number such that $\frac{k}{n} \leq x$. Then by our selection of k, $\frac{k+1}{n} > x$.
Now assume that $y \leq \frac{k+1}{n}$. Then we have that $\frac{k+1}{n} \geq y$ and $-\frac{k}{n} \geq -x$ so by (O5)":

$$\frac{1}{n} = \frac{k+1}{n} - \frac{k}{n} \geq y - x$$

. This is a contradiction so it must be the case that $y > \frac{k+1}{n}$. Thus $z = \frac{k+1}{n}$ is a number satisfying $x < z < y$.

Part (c)

Let y be the smaller value of $b - x$ and $x - a$. Then $a \leq x - y < x < x + y \leq b$. By part b) there exists a z such that $x < z < x + y$. Let $\epsilon = z - x$. This value satisfies that desired conditions.

Problem 6

Prove that each of the following are metric spaces.

(a) $X = \mathbb{R}, d(x,y) = |y-x|$

(b) $X =$ any set, $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$.

(c) Give another example of a metric space.

Solution

Part (a)

This proof will use the fact that $-1 \cdot x = -x$. This was proven as an intermediate step in problem 2.
First I will prove that $-(x-y) = y-x$.

$$
\begin{aligned}
-(x-y) &= -1 \cdot (x + (-y)) && \text{see problem 2} \\
-(x-y) &= -1 \cdot x + -1 \cdot -y && \text{(D)} \\
-(x-y) &= -x + -(-y) && \text{see problem 2} \\
-(x-y) &= -x + y && \text{proved in class} \\
-(x-y) &= y + (-x) && \text{(A1)} \\
-(x-y) &= y - x && \text{def. of -}
\end{aligned}
$$

i $d(x,y) = 0 \iff x = y$
First assume $x = y$. Then $|y-x| = |0| = 0$. Now assume that $|y-x| = 0$. Then either $y - x = 0$ or $x - y = 0$. In the first case $y - x + x = x$ so by (A4) $y = x$. In the second case $x - y + y = y$ so by (A4) $x = y$.

ii $d(x,y) = d(y,x)$
This would directly follow from a proof of property 2 of absolute valuesthat states $|y-x| = |x-y|$. There are two cases.

Case: $y - x > 0$.
By the definition of absolute value $|y-x| = y-x$. Then

$$
\begin{aligned}
y - x &> 0 \\
y - x + x &> 0 + x && \text{O4} \\
y + 0 &> 0 + x && \text{A4} \\
y &> x && \text{A3} \\
y + (-y) &> x + (-y) && \text{O4} \\
0 &> x + (-y) && \text{A4} \\
0 &> x - y && \text{def. of -}
\end{aligned}
$$

Thus by the definition of absolute value $|x - y| = -(x - y)$ which, as proved at the beginning of this problem, is equal to $y - x$.

Case: $y - x < 0$.
By the definition of absolute value $|y - x| = -(y - x)$. Using the same fact as above, this equals $x - y$.

$$\begin{aligned} y - x &< 0 & \\ y - x + x &< 0 + x & \text{O4} \\ y + 0 &< 0 + x & \text{A4} \\ y &< x & \text{A3} \\ y + (-y) &< x + (-y) & \text{O4} \\ 0 &< x + (-y) & \text{A4} \\ 0 &< x - y & \text{def. of -} \end{aligned}$$

Thus $|x - y| = x - y$ by definition.

Case: $y - x = 0$
In this case $|y - x| = y - x = 0$ by definition.

$$\begin{aligned} y - x &= 0 & \\ y - x + x &= 0 + x & \\ y + 0 &= 0 + x & \text{(A4)} \\ y &= x & \text{(A3)} \\ y + (-y) &= x + (-y) & \\ 0 &= x + (-y) & \text{(A4)} \\ 0 &= x - y & \text{def. of -} \end{aligned}$$

So $|x - y| = y - x = 0$ by definition.

iii $d(x, z) \leq d(x, y) + d(y, z)$
$|z - x| \leq |y - x| + |z - 4|$ by the triangle inequality proved in class.

Part (b)

i $d(x, y) = 0 \iff x = y$
This is true by the definition of the function d.

ii $d(x, y) = d(y, x)$
In the case when $x = y$, $d(x, y) = 0 = d(y, x)$. In the case when $x \neq y$, $d(x, y) = 1 = d(y, x)$.

iii $d(x, z) \leq d(x, y) + d(y, z)$

Case: $x = y = z$
$0 \leq 0$

Case: $x \neq y \neq z$
$1 \leq 2$

Case: $x = y \neq z$
$1 \leq 1$

Case: $x \neq y = z$
$1 \leq 1$

Case: $x = z \neq y$
$0 \leq 1$

Part (c)

$X = \mathbb{C}, d(x, y) = \sqrt{x^2 + y^2}$.