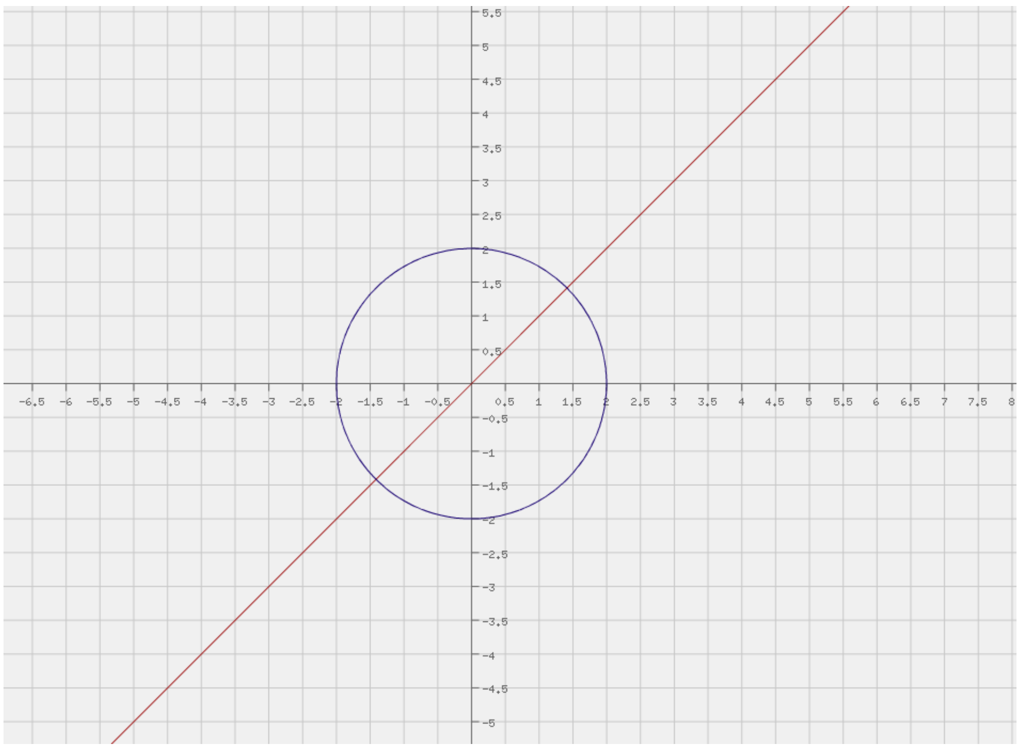
Problem #1

Define binary relations R and S from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\} \text{ and } S = \{(x, y)\mathbb{R} \times \mathbb{R} \mid x = y\}.$$

Graph $R, S, R \cup S$, and $R \cap S$ in the Cartesian plane.

Solution



In the above graph, R is represented by the blue circle, S is represented by the red line, $R \cup S$ is the line and circle taken together, and $R \cap S$ is the two points of intersection between the line and the circle.

Problem #2

Let S be the set of all strings a's and b's. Define a relation T on S as follows:

$$\text{For all}, s, t \in S, sTt \iff t = as$$

(that is, t is the concatenation of a with s)

(a) Is $abTaab$?

(b) Is $aabTab$?

(c) Is $baTaba$?

(d) Is $abaT^{-1}ba$?

(e) Is $abbT^{-1}bba$?

(f) Is $abbaT^{-1}bba$?

Solution

Part (a)

Yes.

Part (b)

No. $aabTaaab$.

Part (c)

Yes.

Part (d)

Yes.

Part (e)

No. $abbT^{-1}bb$.

Part (f)

Yes.

Problem #3

Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations R and S from A to B as follows:

$$R = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } x \mid y\} \quad S = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } y = x + 4\}$$

State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Solution

$A \times B = \{(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)\}$, $R = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$, $S = \{(2, 6), (4, 8)\}$, $R \cup S = \{(2, 6), (2, 8), (2, 10), (4, 8)\}$, $R \cap S = \{(2, 6), (4, 8)\}$

Problem #4

Suppose that $A = \{1, 2, 3\}, B = \{4, 5, 6\}, R = \{(1, 4), (1, 6), (2, 6), (3, 5)\}$, and $S = \{(4, 5), (4, 6), (6, 4), (5, 5)\}$. Note that R is a relation from A to B and S is a relation from B to B. Find the following relations, showing all work supporting how you found those relations:

(a) $S \circ R$

(b) $S \circ S^{-1}$

Solution

$A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$,
$B \times B = \{(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

Part (a)

$R \subseteq A \times B$ and $S \subseteq B \times B$. $S \circ R = \{(a, b) \in A \times B \mid \exists b_2 \in B : (a, b_2) \in R \wedge (b_2, b) \in S\}$ Using this as a guide and checking each element of $A \times B$ we find that $R \circ S = \{(1, 5), (1, 6), (2, 4), (3, 5)\}$.

Part (b)

$S^{-1} = \{(5, 4), (6, 4), (4, 6), (5, 5)\}$. Then $S \circ S^{-1} = \{(a, b) \in B \times B \mid \exists c \in B : (a, c) \in S \wedge (c, a) \in S^{-1}\}$. Using this as a guide and checking each element in $B \times B$ we find$S \circ S^{-1} = \{(4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$.

Problem #5

For relation R and its inverse R^{-1}:

(a) What properties *must* hold for relation R if $R = R^{-1}$. Explain.

(b) What properties *might* hold if $R = R^{-1}$. Explain.

Solution

Part (a)

$R = R^{-1}$ is equivalent to saying that $(a, b) \in R \implies (b, a) \in R$. Therefore R must be symmetric.

Part (b)

R might be transitive or reflexive. There is nothing in the statement that $R = R^{-1}$ that makes either of these properties impossible, but neither is there anything that guarantees them.

Problem #6

Determine whether the given relation is an equivalence relation. Justify your answers by showing which properties of an equivalence relation are true and which are not.

(a) Let A be the set of all lines in the plane. The relation R is defined on A as follows, two lines X and Y relate to one another if they have the same slope.

(b) Let X be a nonempty set and $P(X)$ be the power set of X. Define a relation R on subsets of $P(X)$ such that A relates to B if the cardinality of A does not equal the cardinality of B.

Solution

Part (a)

This is an equivalence relation. A line must have the same slope as itself so the relation is reflexive. If line a has the same slope as line b then line b has the same slope as line a so the relation is symmetric. If line a has the same slope as line b and line b has the same slope as line c then line a has the same slope as line c so the relation is transitive.

Part (b)

This is not an equivalence relation. A set must have the same cardinality as itself so the relation is not reflexive. If set A has the same cardinality as set B then set B must have the same cardinality as set A so the relation is not symmetric. If set A has the same cardinality as set B and set B has the same cardinality as set C then set A must have the same cardinality as set C so the relation is not transitive.

Problem #7

If R is symmetric and R is transitive then R is reflexive. This can be proven by considering $(x, y) \in R$. By symmetry, $(x, y) \in R \implies (y, x) \in R$. Using transitivity, $(x, x) \in R$, thus R is reflexive. What's wrong with this proof?

Solution

The proof assumes that R contains some element (x, y). Consider the relation $R = \emptyset$ on $\mathbb{Z} \times \mathbb{Z}$. R is both transitive and symmetric but it is not reflexive.

Problem #8

Given a set, A, where $A = \{a[1], a[2], \ldots, a[n]\}$ (represented as a one dimensional array) and a relation, R, on A, design the following (use pseudocode):

(a) An algorithm to test whether R is reflexive.

(b) An algorithm to test whether R is symmetric.

(c) An algorithm to test whether R is transitive.

Solution

Part (a)

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for a in A:
    if (a, a) not in R:
        return false
return true
```

Part (b)

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for a in A:
    for b in A:
        if (a, b) in R and (b, a) not in R:
            return false
return true
```

Part (c)

```
for a in A:
    for b in A:
        if (a, b) in R:
            for c in A:
                if (b, c) in R:
                    if (a, c) not in R:
                        return false
return true
```

Problem #9

Prove or disprove the following: (Suppose that R and S are binary relations on a set A).

(a) If R and S are reflexive, then $R \cup S$ is also reflexive.

(b) If R and S are symmetric, then $R \cup S$ is also symmetric.

Solution

Part (a)

Take any element a of A. The relation R is reflexive so therefore $(a, a) \in R$. By the definition of set unions $(a, a) \in R \cup S$ for all $a \in A$ so $R \cup S$ is reflexive.

Part (b)

Select any element $(a, b) \in R \cup S$. By the definition of set unions (a, b) is either an element of R or of S. If it's an element of R then we know that $(b, a) \in R$ because R is reflexive and therefore $(b, a) \in R \cup S$. Similarly, if $(a, b) \in S$ we know that $(b, a) \in S$ because S is symmetric and therefore $(b, a) \in R \cup S$. In every case, $(a, b) \in R \cup S \implies (b, a) \in R \cup S$ and therefore $R \cup S$ is symmetric.