Section 4.1

Problem #10

Let $f = (1, 2, 3)(4, 5)$. Compute each of the following powers of f.

(a) f^{-1}

(b) f^{18}

(c) f^{23}

(d) f^{201}

Solution

Part (a)

$f^{-1} = (5, 4)(3, 2, 1)$

Part (b)

$f^{18} = (1, 2, 3)(4, 5)$

Part (c)

$f^{23} = (5, 4)(3, 2, 1)$

Part (d)

$f^{201} = (1, 2, 3)(5, 4)$

Problem #12

Compute gfg^{-1}, the conjugate of f by g, for each pair f, g.

(a) $f = (1, 2, 4, 3)$; $g = (1, 3, 2)$

(b) $f = (1, 3, 5, 6)$; $g = (2, 5, 4, 6)$

(c) $f = (2, 3, 5, 4)$; $g = (1, 3, 2)(4, 5)$

(d) $f = (1, 4)(2, 3)$; $g = (1, 2, 3)$

(e) $f = (1, 3, 5)(2, 4)$; $g = (2, 5)(3, 4)$

(f) $f = (1, 3, 5, 2)(4, 6)$; $g = (1, 3, 6)(2, 4, 5)$

Solution

Part (a)

(3, 1, 4, 2)

Part (b)

(1, 3, 4, 2)

Part (c)

(1, 2, 4, 5)

Part (d)

(2, 4)(3, 1)

Part (e)

(1, 4, 2)(5, 3)

Part (f)

(3, 6, 2, 4)(5, 1)

Problem #14

Write the permutation $f = (1, 2, 3, 4, 5, 6)$ as a product of a permutation g of order 2 and a permutation h of order 3.

Solution

$$g = (1, 5, 3)(2, 6, 4) \qquad h = (6, 3)(5, 2)(4, 1) \tag{1}$$
$$f = g \circ h \tag{2}$$

Problem #15

Write the permutation $f = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$ as a product of a permutation g of order 3 and h of order 4.

Solution

$$g = (1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12) \qquad h = (1, 10, 7, 4)(2, 11, 8, 5)(3, 12, 9, 6) \tag{3}$$
$$f = g \circ h \tag{4}$$

Problem #16

List all the elements of the alternating group A_3, written in cyclic notation.

Solution

$A_3 = \{(1), (2), (3), (1, 2, 3)\}$

Problem #18

Find all the distinct cyclic subgroups of A_4.

Solution

$(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3), (2, 3, 4), (2, 4, 3)$

Problem #19

Find cyclic subgroups of S_4 that have three different orders.

Solution

- $(1, 2)$ has order 2.
- $(1, 2, 3)$ has order 3.
- $(1, 2, 3, 4)$ has order 4.

Problem #20

Construct a multiplication table for the octic group D_4 in Example 12 of this section.

Solution

$\cdot$	e	α^2	r180	α^3	δ	β	λ	θ
e	e	α^2	r180	α^3	δ	β	λ	θ
α^2	α^2	r180	ab	b	θ	λ	δ	
r180	r180	α^3	e	b	β	δ	θ	λ
α^3	α^3	e	a	r180	λ	θ	β	δ
δ	δ	λ	β	θ	e	r180	α^2	α^3
β	β	θ	δ	λ	r180	e	α^3	α^2
λ	λ	β	θ	δ	α^3	α^2	e	r180
θ	θ	δ	λ	β	α^2	α^3	r180	e

Problem #30

Let ϕ be the mapping from S_n to the additive group $\mathbb{Z}_2$ defined by

$$\phi(f) = \begin{cases} [0] & \text{if } f \text{ is an even permutation} \\ [1] & \text{if } f \text{ is an odd permutation} \end{cases}$$

(a) Prove that ϕ is an isomorphism.

(b) Find the kernel of ϕ.

(c) Prove or disprove that ϕ is an epimorphism.

(d) Prove or disprove that ϕ is an isomorphism.

Solution

Part (a)

All $a, b \in S_n$ are either even or odd.

(a) Both a and b are even:
$\phi(a) + \phi(b) = [0] + [0] = [0] = \phi(ab)$

(b) Both a and b are odd:
$\phi(a) + \phi(b) = [1] + [1] = [0] = \phi(ab)$

(c) a is even and b is odd:
$\phi(a) + \phi(b) = [0] + [1] = [1] = \phi(ab)$

So ϕ is a homomorphism.

Part (b)

The identity element in $\mathbb{Z}_2$ is $[0]$. The kernel of ϕ is A_n.

Part (c)

For S_1 the only permutation is e. Therefore there is no even permutation and no element of S_1 maps to $[0]$. Thus ϕ is not an epimorphism.

Part (d)

ϕ is not an epimorphism and therefore it cannot be an isomorphism.

Problem #31

Let f and g be disjoint cycles in S_n. Prove that $fg = gf$.

Solution

If f and g are disjoint then any element permuted under one is invariant under the other. It is clear that the order in which the operations is applied has no impact on the result.

Problem #32

Prove that the order of A_n is $\frac{n!}{2}$.

Solution

Let us claim that in S_n there are the same number of even and odd permutations. Let O_n be the set of all odd permutations in S_n. Then select an arbitrary element of O_n denoted as x. Define the function $f : A_n \to O_n$ by $f(y) = xy$.

Part (a)

Suppose that $f(y) = f(y')$ for some y, y' in A_n. Then $xy = xy'$ and thus $y = y'$. Therefore f is one-to-one.

Part (b)

The composition of and two odd cycles is an even cycle. For any $b \in O_n$ $x^{-1}b \in A_n$ and

$$\begin{aligned} f(x^{-1}b) &= xx^{-1}b \\ &= eb \\ &= b. \end{aligned} \tag{5}$$

The claim is proved. Knowing that the total number of permutations is equal to the magnitued of $S_n = n!$ and that there are an equal number of even and odd permutations we can say that $\mid A_n \mid = \frac{n!}{2}$.

Section 4.2

In Exercises 1-4, let G be the given group. Write out the elements of a group of permutations that is isomorphic to G, and exhibit an isomorphism from G to this group.

Problem #1

Let G be the additive group $\mathbb{Z}_3$.

Solution

Part (a)

For all $a \in G$ let the permutation $f_a : G \to G$ be defined by $f_a(x) = ax$. So:

$$f_{[0]} = e f_{[1]} = ([0], [1], [2]) \qquad f_{[2]} = ([0], [2], [1]) \tag{6}$$

Then the set $G' = \{f_{[0]}, f_{[1]}, f_{[2]}\}$ is a group of permutations isomorphic to G according to the proof of Cayley's Theorem.

Part (b)

Then the mapping $\phi : G \to G'$ defined by $\phi(x) = f_x$ is an isomorphism between G and G' according to the proof of Cayley's Theorem.

Problem #2

Let G be the cyclic group $\langle a \rangle$ of order 5.

Solution

Part (a)

For all $a \in G$ let the permutation $f_a : G \to G$ be defined by $f_a(x) = ax$. So:

$$f_a = (a, a^2, a^3, a^4, e) f_{a^2} = (a^2, a^4, a, a^3, e) \quad f_{a^3} = (a^3, a, a^4, a^2, e) f_{a^4} = (a^4, a^3, a^2, a, e) \quad f_e = e \tag{7}$$

Then the set $G' = \{f_a, f_{a^2}, f_{a^3}, f_{a^4}, f_e\}$ is a group of permutations isomorphic to G according to the proof of Cayley's Theorem.

Part (b)

Then the mapping $\phi : G \to G'$ defined by $\phi(x) = f_x$ is an isomorphism between G and G' according to the proof of Cayley's Theorem.

Problem #3

Let G be the klein four group $\{e, a, b, ab\}$ with its multiplication table as given:

·	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	b
ab	ab	a	a	e

Solution

Part (a)

For all $a \in G$ let the permutation $f_a : G \to G$ be defined by $f_a(x) = ax$. So:

$$f_e = ef_a = (a, e)(b, ab) \qquad f_b = (b, e)(a, ab)f_{ab} = (e, ab)(b, a) \tag{8}$$

Then the set $G' = \{f_e, f_a, f_b, f_{ab}\}$ is a group of permutations isomorphic to G according to the proof of Cayley's Theorem.

Part (b)

Then the mapping $\phi : G \to G'$ defined by $\phi(x) = f_x$ is an isomorphism between G and G' according to the proof of Cayley's Theorem.

Problem #4

Let G be the multiplicative group of units $\mathbb{U}_5 = \{[1], [2], [3], [4]\} \subseteq \mathbb{Z}_{10}$

Solution

Problem #11

For each element a in the group G, define a mapping $k_a : G \to G$ by $k_a(x) = xa^{-1}$ for all x in G.

(a) Prove that each k_a is a permutation on the set of elements of G.

(b) Prove that $K = \{k_a \mid a \in G\}$ is a group with respect to mapping composition.

(c) Define $\phi : G \to K$ by $\phi(a) = k_a$ for each a in G. Determine whether ϕ is always an isomorphism. This mapping ϕ is known as the **right regular representation** of G.

Solution

Part (a)

Assume that for some $x, y \in G$ $k_a(x) = k_a(y)$. This implies

$$xa = ya \tag{9}$$
$$x = y \tag{10}$$

Therefore k_a is one-to-one. Let b be an arbitrary element in G. Let $x = ba^{-1}$. We know that x is in G because G is a group. Then

$$\begin{aligned} k_a(x) &= xa \\ &= (ba^{-1})a \\ &= b \end{aligned} \tag{11}$$

and k_a is onto. k_a is a bijection from G to G so k_a is a permutation on the elements of G.

Part (b)

(a) For any $k_a, k_b \in K$.

$$\begin{aligned} k_a k_b(x) &= k_a(k_b(x)) \\ &= k_a(xb) \\ &= (xb)a \\ &= k_{ba}(x) \end{aligned} \tag{12}$$

So K is closed under composition of mappings.

(b) $k_e(x) = xe = x$ for all x in G so k_e is the identity element in K.

(c)

$$\begin{aligned} k_a k_{a^{-1}} &= k_{a^{-1}a} \\ &= k_e \end{aligned} \tag{13}$$

and

$$\begin{aligned} k_{a^{-1}} k_a &= k_{aa^{-1}} \\ &= k_e \end{aligned} \tag{14}$$

So K is a group with respect to composition of mappings.

Part (c)

$\phi : \phi(x) = k_a$ is clearly onto.

$$\begin{aligned} \phi(a) = \phi(b) &\implies k_a = k_b \\ &\implies k_a(x) = k_b(x) \forall x \in G \\ &\implies xa = xb \qquad \forall x \in G \\ &\implies a = b \end{aligned}$$

so ϕ is one-to-one.

$$\phi(a)\phi(b) = k_a k_b = k_b a = \phi(ba) \tag{15}$$

Thus ϕ is isomorphic when $\phi(ab) = \phi(ba)$.