Problem 1

Let V be an F-vector space, U, W two subspaces of V and X the (set-theoretic) union of U and W. Prove the following: If X is a subspace of V, then U contains W or W contains U.

Solution

I will prove the contrapositive. Assume that it is not the case that $U \subseteq W$ or $W \subseteq U$. Then there must exist some element $u \in U$ and $w \in W$ such that $u \notin W$ and $w \notin U$. By definition of X both u and w are in X. Consider the sum $u + w$. If $(u + w) \in U$ then, because U is a subspace and thus closed under an associative addition, $-u+(u+w) = (-u+u)+w = w \in U$. This is a contradiction. Similarly if $(u + w) \in W$ then by the same reasoning as above $(u + w) + -w = u + (w + -w) = u \in W$. This is also a contradiction. Therefore the sum $(u + w)$, being neither in U nor in W cannot be in X. Thus X is not a subspace.

Problem 2

Prove Lemma 1.3.4 as formulated in class for arbitrary intersections of subspaces. (You do not have to deal separately with the intersection of U and W as a special case.)

Solution

- Each W_i is a subspace and thus by definition contains zero. Thus the intersection $\bigcap_{i\in I} W_i$ contains zero.
- By definition any $w_1, w_2 \in \bigcap_{i\in I} W_i$ are elements of W_i for all $i \in I$. Because each W_i is a subspace and thus closed under addition $w_1 + w_2 \in W_i \forall i \implies w_1 + w_2 \in \bigcap_{i\in I} W_i$.
- Select arbitrary $w \in \bigcap_{i\in I} W_i$ and $a \in F$. By definition $w \in W_i \forall i \in I$. Each W_i is a subspace and thus closed under scalar multiplication so $aw \in W_i \forall i \in I$. Thus $aw \in \bigcap_{i\in I} W_i$.

Problem 3

Prove the equation in Example 1.3.6(c), i.e. show that $\text{span}\{x_1, \ldots, x_n\} = \{a_1x_1 + \ldots + a_nx_n \mid a_1, \ldots a_n \in F\}$ if $x_1, \ldots, x_n$ are elements of an F-vector space V. Keep in mind how the span is defined!

Solution

First note that $\text{span}\{x_1, \ldots, x_n\}$ by definition is the smallest subspace of V that contains $x_1, \ldots, x_n$.
Now we show that $\{a_1x_1 + \ldots + a_nx_n \mid a_i \in F\} \subseteq \text{span}\{x_1, \ldots, x_n\}$. For any $z = a_1x_1 + \ldots + a_nx_n, a_i \in F$, $z \in \text{span}\{x_1, \ldots, x_n\}$ because subspaces are closed under addition and scalar multiplication and z is simply the result of repeatedly multiplying elements of $\{x_1, \ldots, x_n\}$

by a scalar and then summing them. Thus $\{a_i x_i, \ldots, a_n x_n \mid a \in F\} \subseteq \text{span}\{x_1, \ldots, x_n\}$.

Next we show that $\text{span}\{x_1, \ldots, x_n\} \subseteq \{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$. We begin by showing that $\{x_1, \ldots, x_n\} \subseteq \{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$. Select any x_i from $\{x_1, \ldots, x_n\}$. Then $a_1 x_1 + \ldots + a_n x_n = x_i$ when $a_i = 1$ and $a_{j \neq i} = 0$.

Now we show that $\{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$ is a subspace of V. It is clearly a subset of V because we have already shown it to be a subset of $\text{span}\{x_1, \ldots, x_n\}$ which is itself a subspace of V.

- Zero element
  It contains the zero element which is when $a_i = 0$ for all $0 < i \leq n$.

- Closed under addition
  For $a_i, b_i \in F$ we have:

$$\begin{aligned} a_1 x_1 + \ldots a_n x_n + b_1 x_n + \ldots + b_n x_n &= a_1 x_1 + b_1 x_1 + \ldots + a_n x_n + b_n x_n && (1) \\ &= (a_1 + b_1) x_1 + \ldots + (a_n + b_n) x_n && (2) \end{aligned}$$

  which, because F is closed under addition, is an element of $\{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$.

Thus, because $\{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$ is a subspace of V which contains $\{x_1, \ldots, x_n\}$ and by definition $\text{span}\{x_1, \ldots, x_n\}$ is the smallest such subspace we have $\text{span}\{x_1, \ldots, x_n\} \subseteq \{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$.

Then finally this implies $\text{span}\{x_1, \ldots, x_n\} = \{a_1 x_1 + \ldots + a_n x_n \mid a \in F\}$.

Problem 4

Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operation? Justify your answer.

Solution

V is not a vector space. For arbitrary $a, b \in \mathbb{R}$, $x \in V$ it is not the case that $(a+b)x = ax + bx$. For example, let $a = 1, b = 1, x = (1, 1)$. Then:

$$\begin{aligned} (1+1)(1,1) &= 2(1,1) && (3) \\ &= (2, \frac{1}{2}) && (4) \end{aligned}$$

but

$$\begin{aligned} 1(1,1) + 1(1,1) &= (1,1) + (1,1) && (5) \\ &= (2,2) && (6) \end{aligned}$$

Problem 5

Let S be a nonempty set and F a field. Let $C(S,F)$ denote the set of all functions $f \in \mathcal{F}(S,F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $C(S,F)$ is a subspace of $\mathcal{F}(S,F)$.

Solution

It is clear that $C(S,F)$ is a subset of $\mathcal{F}(S,F)$.

- The zero element in $\mathcal{F}(S,F)$ is the function that maps s to 0 for all $s \in S$. There are thus no s for which $f(s)$ is nonzero so $f \in C(S,F)$.
- Select arbitrary $f, g \in C(S,F)$. Then (f+g)(s) = f(s) + g(s)$.Then f(s) \neq 0$ for a finite number n of s in S and $g(s) \neq 0$ for a finite number m of s in S. Then the sum $f(s)+g(s)$ can only be nonzero for at most $n+m$ elements of S. The sum of two finite numbers is finite so $(f+g) \in C(S,F)$.
- Select arbitrary $f \in C(S,F)$ and $a \in F$. For all $s \in S$, $(cf)(s) = c(f(s))$. $c0 = 0$ regardless of the value of c and $f(s)$ is only nonzero for finitely many s. Thus $(cf)(s)$ is nonzero for finitely many s and $(cf) \in C(S,F)$.

Thus $C(S,F)$ is a subspace of $\mathcal{F}(S,F)$.

Problem 6

A matrix M is called **skew-symmetric** if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$.

Solution

Part (a)

It is clear that $W_1 \subseteq M_{n\times n}(F)$.

(a) Zero element
The zero matrix (whose elements are all 0) is an element of W_1.

(b) Closed under +
Select arbitrary $A, B \in W_1$. Then:

$$
\begin{aligned}
(A+B)^T &= A^T + B^T && (7)\\
&= -A - B && (8)\\
&= -(A+B) && (9)
\end{aligned}
$$

(c) Closed under scalar multiplication
Select arbitrary $A \in W_1, c \in F$. Then:

$$
\begin{aligned}
(cA)^T &= c(A^T) && (10)\\
&= c(-A) && (11)\\
&= -(cA) && (12)
\end{aligned}
$$

Thus W_1 is a subspace of $M_{n\times n}(F)$.

Part (b)

The sum of any two $n \times n$ matrix must be another $n \times n$ matrix. It is clear then that $W_1+W_2 \subseteq M_{n\times n}(F)$. To prove that $M_{n\times n}(F) \subseteq W_1+W_2$ select an arbitrary $M \in M_{n\times n}(F)$. Then $(M+M^T)^T = (M^T+M^{M^T}) = (M^T+M) = (M+M^T)$. Thus $M+M^T \in W_2$. Similarly $(M - M^T)^T = (M^T - M^{T^T}) = (M^T - M) = -(M - M^T)$. So $M^T - M \in W_1$. Then M can be written as $M = \frac{1}{2}(M + M + (M^T - M^T)) = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$. W_1 and W_2 are subspaces and closed under scalar multiplication so this is the sum of an element of W_1 and an element of W_2. Thus $M_{n\times n}(F) \subseteq W_1 + W_2$. This implies that $M_{n\times n}(F) = W_1 + W_2$.
Additionally, if $A \in W_1 \cap W_2$ then $A^T = -A$ and $A^T = A$. Thus $A = 0$ and $W_1 \cap W_2 = \{0\}$.
Therefore $M_{n\times n}(F) = W_1 \oplus W_2$.