Problem 1.

Identify the error or errors in this argument that supposedly show that if $\forall x(P(x) \lor Q(x))$ is true then $\forall x P(x) \lor \forall Q(x)$.

(a) $\forall x(P(x) \lor Q(x))$ premise

(b) $P(c) \lor Q(c)$ universal instantiation from a

(c) $P(c)$ simplication from b

(d) $\forall x P(x)$ universal generalization from c

(e) $Q(c)$ simplification from b

(f) $\forall x Q(x)$ universal generalization from e

(g) $\forall x P(x) \lor \forall x Q(x)$ conjunction from d and f

Solution

The simplifications in step c and e, $P(c)$ and $Q(c)$, are not valid simplifications from the statement $P(c) \lor Q(c)$. They imply that $P(c) \land Q(c)$. These are obviously not equivalent.

Problem 2.

Use the following two assumptions:

(a) "Logic is difficult or not many students like logic."

(b) "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

(a) If many students like logic, then mathematics is not easy.

(b) That not many students like logic, if mathematics is not easy.

(c) That mathematics is not easy or logic is difficult.

(d) That logic is not difficult or mathematics is not easy.

(e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

Solution

Premises (a) and (b) can be written as

(a) $P \vee \neg Q$

(b) $R \implies \neg P$

where $P =$ "Logic is difficult." $Q =$"Many students like logic." $R =$"Mathematics is easy.

Part (a)

The conclusion is: $Q \implies \neg R$
This is valid. In order for the conclusion to be false then Q and R must be true. From premise a) we know that if Q is true then P is true. If P is true then by premise b) R must be false. Therefore it is impossible for the premises to be true and the conclusion false.

Part (b)

The conclusion is: $\neg R \implies \neg Q$
This is not valid. If P and Q are true and R is false the premises are true and the conclusion is false.

Part (c)

The conclusion is: $\neg R \vee P$
The conclusion is not valid. If P is false, R is true, and Q is false then the premises are all true and the conclusion is false.

Part (d)

The conclusion is: $\neg P \vee \neg R$
The conclusion is valid. If R is true then by premise b) P must be false and the conclusion is true. If R is false then the conclusion is still true. Therefore the conclusion is valid.

Part (e)

The conclusion is: $\neg Q \implies (\neg R \vee \neg P)$
This conclusion is valid. From premise a) if Q is true then P must be true. If P is true then from premise b) we know R must be false.

Problem 3.

Prove the following statement. $p \implies q \equiv (p \wedge \neg q) \implies F$.

Solution

p	q	$\neg q$	$p \wedge \neg q$	$p \Rightarrow q$	$(p \wedge \neg q) \Rightarrow F$
F	F	T	F	T	T
F	T	F	F	T	T
T	F	T	T	F	F
T	T	F	F	T	T

For all truth values of p and q, $p \implies q$ and $(p \wedge \neg q) \implies F$ have the same overall truth values. Thus the statement that they are logically equivalent is true.

Problem 4.

Given the following:
1. $\neg t$
2. $s \implies t$
3. $(\neg r \vee \neg f) \implies (s \wedge l)$

Can we conclude r? (Hint you will need to use De Morgan's Law)

Solution

(a) $\neg s$ — modus tollens from 1 and 2

(b) $\neg(\neg r \vee \neg f) \vee (s \wedge l)$ — definition of implication

(c) $(r \wedge f) \vee (s \wedge l)$ — De Morgan's from b

(d) $\neg s \vee \neg l$ — addition from a

(e) $\neg(s \wedge l)$ — De Morgan's from d

(f) $(r \wedge f)$ — disjunctive syllogism from e and c

(g) r — simplification from g

So we can conclude r.

Problem 5.

Prove or disprove the following:

(a) There exists an integer n such that $2n^2 - 5n + 2$ is a prime number.

(b) If m and n are positive integers and mn is a perfect square then m and n are perfect squares.

(c) The difference of the squares of any two consecutive integers is odd.

Solution

Part (a)

If $n = 3$ then $2n^2 - 5n + 2 = 5$ which is a prime number. Therefore the statement is true.

Part (b)

If $m = 2$ and $n = 2$ then $mn = 4$. 4 is a perfect square but 2 is not a perfect square. Therefore the statement is false.

Part (c)

For any two consecutive integers one of the integers will be odd and the other will be even. An odd number squared must be odd and an even number squared must be even. The difference between an odd number and an even number must be odd. Therefore the statement is true.

Problem 6.

Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$, ie there are infinitely many Pythagorean triples! [Hint let $x = m^2 - n^2$ and $y = 2mn$ for all integers m and n. You will need to figure out what z is in terms of m and n.]

Solution

Let $x = m^2 - n^2$ and $y = 2mn$ for some positive integers m and n. Then:

$$\begin{aligned} x^2 + y^2 &= (m^2 - y^2)^2 + (2mn)^2 \\ &= m^4 - 2n^2m^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2n^2m^2 + n^4 \end{aligned} \tag{1}$$

Let $z = m^2 + n^2$. Then $z^2 = (m^4 + 2m^2n^2 + n^4) = x^2 + y^2$. This is true for any positive integers m and n and because there are infinite positive integers, there are therefore infinitely many solutions.

Problem 7.

Prove or disprove the following:

(a) If m and n are perfect squares, then $m + n + 2\sqrt{mn}$ is also a perfect square.

(b) If p is a prime number, then $2^p - 1$ is also a prime number.

Solution

Part (a)

Let $m = a^2$ and $n = b^2$ for some integers a and b. Then

$$\begin{aligned} m + n + 2\sqrt{mn} &= a^2 + b^2 + 2\sqrt{(a^2b^2)} \\ &= a^2 + b^2 + 2\sqrt{((ab)^2)} \\ &= a^2 + b^2 + 2ab \\ &= (a+b)^2 \end{aligned} \tag{2}$$

The integers are closed under multiplication so $a+b$ is an integer and therefore $m+n+2\sqrt{mn}$ is a perfect square.

Part (b)

$2^{11} - 1 = 2047 = (23)(89)$. Therefore the statement is false.

Problem 8.

Find a counterexample to the statement that every positive integer can be written as the sum of the square of three integers.

Solution

7 cannot be written in this form.

Problem 9.

For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

(a) If I play hockey, then I am sore the next day. I use the whirlpool if I am sore. I did not use the whirlpool.

(b) If I work, it is either sunny or partly sunny. I worked last Monday or I worked last Friday. It was not sunny on Tuesday. It was not partly sunny on Friday.

(c) All insects have six legs. Dragonflies are insects. Spiders do not have six legs. Spiders eat dragonflies.

(d) Every student has an Internet account. Homer does not have an Internet account. Maggie has an Internet account.

(e) All foods that are healthy to eat do not taste good. Tofu is healthy to eat. You only eat what tastes good. You do not eat tofu. Cheeseburgers are not healthy to eat.

(f) I am either dreaming or hallucinating. I am not dreaming. If I am hallucinating, I see elephants running down the road.

Solution

Part (a)

Let h =”I play hockey,” s =”I am sore the next day,” and w =”I use the whirlpool.”
We are given:
1.$h \implies s$
2.$s \implies w$
3.$\neg w$
Then

(a) $h \implies w$ hypothetical syllogism from 1 and 2

(b) $\therefore \neg h$ modus tollens from a and 3

We can conlude “I did not play hockey.”

Part (b)

Let $W(x)$= “I worked on x,” $S(x)$ = “It was sunny on x,” $P(x)$ = “It was partly sunny on x,” m =last monday, and f =last friday. We are given that
1. $W(x) \implies (S(x) \vee P(x))$
2. $W(m) \vee W(f)$
3. $\neg P(f)$
So we can reason that

(a) $W(m) \vee (S(f) \vee P(f))$ by substitution

(b) $(W(m) \vee S(f)) \vee P(f)$ by associativity of or

(c) $\therefore W(m) \vee S(f)$ disjunctive syllogism from b and 3

Our conclusion is that “Either I worked last Monday or it was sunny last Friday .”

Part (c)

Let $I(x)$ =”x is an insect,” $L(x)$ =”x has six legs,” $D(x)$ =”x is a dragonfly,” $S(x)$ =”x is a spider.”
We are given:
1. $I(x) \implies L(x)$
2. $D(x) \implies I(x)$
3. $S(x) \implies \neg L(x)$
Then:

(a) $\neg I(x) \vee L(x)$ definition of implication from 1

(b) $L(x) \vee \neg I(x)$ or is commutative

(c) $\neg L(x) \vee \neg I(x)$ definition of implication from b

(d) $D(x) \implies L(x)$ hypothetical syllogism from 1 and 2

(e) $S(x) \implies \neg I(x)$ hypothetical syllogism from 3 and c

So we can conclude both that "If something is a dragonfly then it has six legs," and that "If something is a spider then it is not an insect."

Part (d)

Let $I(x)$ = "x has an internet account," $S(x)$ = "x is a student," and h =homer. We are given that
1. $S(x) \implies I(x)$
2. $\neg I(h)$
Therefore we can conclude by modus tollens that $\neg S(h)$, "Homer is not a student."

Part (e)

Let $H(x)$ = "x is healthy to eat," $G(x)$= "x tastes good," and t =tofu. We are given that
1. $H(x) \implies \neg G(x)$
2. $H(t)$
So by modus ponens we can conclude that $\neg G(t)$, "tofu does not taste good."

Part (f)

Let d = "I am dreaming," h = "I am hallucinating," and e = "I see elephants running down the road." We are given that
1. $d \vee h$
2. $\neg d$
3. $h \implies e$
So we can conclude that h, "I am hallucinating" by disjunctive syllogism and then e, "I see elephants running down the road" by modus ponens.