Section 5.1

Problem # 38

An element x in a ring is called **idempotent** if $x^2 = x$. Find two different idempotent elements in $M_2(\mathbb{Z})$.

Solution

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem # 39

Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

Solution

Let S be the set of all idempotent elements of a commutative ring. Select arbitrary x, y from S. Then

$$(xy)(xy) = x(yx)y = x(xy)y = x^2y^2 = xy$$

Therefore xy is idempotent and thus an element of S.

Problem # 40

Let a be idempotent in a ring with unity. Prove $e - a$ is also idempotent.

Solution

$$\begin{aligned} (e-a)(e-a) &= e(e-a) - a(e-a) \\ &= ee - ea - ae - a(-a) \\ &= e - a - a + a \\ &= e - a \end{aligned} \tag{1}$$

Problem # 42

Let

$$S = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R} \right\}$$

(a) Show that S is a commutative subring of $M_2(\mathbb{R})$.

(b) Find the unity, if one exists.

(c) Describe the units in S, if any.

Solution

Part (a)

It is clear that S is a nonempty subset of $M_2(\mathbb{R})$.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \tag{2}$$

$\mathbb{R}$ is closed under addition so S is closed under addition.

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ca-bd & -(cb+da) \\ cb+da & ca-bd \end{bmatrix}$$

$\mathbb{R}$ is closed under multiplication so S is closed under multiplication.

$$-\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$$

Which is another member of S. Thus S contains additive inverses and is a subring of $M_2(\mathbb{R})$.

$$\begin{bmatrix} c & -b \\ d & c \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ca-bd & -(cb+da) \\ cb+da & ca-bd \end{bmatrix}$$

The above combined with the example demonstrating closure under multiplication shows that S is commutative with respect to multiplication and is thus a commutative subring.

Part (b)

The unity is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Part (c)

The units of S are all elements $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a^2 + b^2 \neq 0$.

Problem # 44

Consider the set T of all 2×2 matrices of the form $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$, where a and b are real numbers, with the same rules for addition and multiplication as in $M_2(\mathbb{R})$.

(a) Show that T is a ring that does not have a unity.

(b) Show that T is not a commutative ring.

Solution

Part (a)

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} c & c \\ d & d \end{bmatrix} = \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix}$$

$\mathbb{R}$ is closed under addition so T is closed under addition. Further, we know that matrix addition is both associative and commutative. Every element $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$ has an additive inverse $\begin{bmatrix} -a & -a \\ -b & -b \end{bmatrix}$ and T contains the additive identity $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We know that matrix multiplication is associative.

$$\begin{bmatrix} a & a \\ b & b \end{bmatrix} \begin{bmatrix} c & c \\ d & d \end{bmatrix} = \begin{bmatrix} ca+da & ca+da \\ cb+db & cb+db \end{bmatrix}$$

The above demonstrates that T is closed with respect to multiplication because $\mathbb{R}$ is closed with respect to multiplication and addition. Finally we show that the two distributive laws hold.

First law:

$$\begin{aligned} \begin{bmatrix} a & a \\ b & b \end{bmatrix} \left(\begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} f & f \\ g & g \end{bmatrix} \right) &= \begin{bmatrix} a(c+d+f+g) & a(c+d+f+g) \\ b(c+d+f+g) & b(c+d+f+g) \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ b & b \end{bmatrix} \begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} a & a \\ b & b \end{bmatrix} \begin{bmatrix} f & f \\ g & g \end{bmatrix} \end{aligned}$$

Second Law:

$$\begin{aligned} \left(\begin{bmatrix} c & c \\ d & d \end{bmatrix} + \begin{bmatrix} f & f \\ g & g \end{bmatrix} \right) \begin{bmatrix} a & a \\ b & b \end{bmatrix} &= \begin{bmatrix} (a+b)(c+f) & (a+b)(c+f) \\ (a+b)(d+g) & (a+b)(d+g) \end{bmatrix} &(3) \\ &= \begin{bmatrix} c & c \\ d & d \end{bmatrix} \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} f & f \\ g & g \end{bmatrix} \begin{bmatrix} a & a \\ b & b \end{bmatrix} &(4) \end{aligned}$$

Thus T is a ring. The unity with respect to multiplication of 2x2 matrices is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is not a member of T. Thus T has no unity.

Part (b)

The following example demonstrates that T is not commutative.

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}$$

Problem # 45

Prove the following equalities in an arbitrary ring R.

(a) $(x + y)(z + w) = (xz + xw) + (yz + yw)$

(b) $(x + y)(z - w) = (xz + yz) - (xw + yw)$

(c) $(x - y)(z - w) = (xz + yw) - (xw + yz)$

(d) $(x + y)(x - y) = (x^2 - y^2) + (yx - xy)$

Solution

Part (a)

$$\begin{aligned} (x + y)(z + w) &= (x + y)z + (x + y)w \\ &= xz + yz + xw + yw \\ &= xz + xw + yz + yw \end{aligned} \tag{5}$$

Part (b)

$$\begin{aligned} (x + y)(z - w) &= (x + y)z + (x + y)(-w) \\ &= xz + yz + x(-w) + y(-w) \\ &= xz + yz - xw - yw \\ &= xz + yz - (xw + yw) \end{aligned} \tag{6}$$

Part (c)

$$\begin{aligned} (x - y)(z - w) &= (x - y)(z + (-w)) \\ &= (x - y)z + (x - y)(-w) \\ &= (x + (-y))z + (x + (-y))(-w) \\ &= xz + (-y)z + x(-w) + (-y)(-w) \\ &= xz - yz - xw + yw \\ &= xz + yw - xw - yz \\ &= (xz + yw) - (xw + yz) \end{aligned} \tag{7}$$

Part (d)

$$\begin{aligned} (x + y)(x - y) &= (x + y)(x + (-y)) \\ &= (x + y)x + (x + y)(-y) \\ &= x^2 + yx - ((x + y)y) \\ &= x^2 + yx - (xy + y^2) \\ &= x^2 - y^2 + (yx - xy) \end{aligned} \tag{8}$$

Problem # 49

An element a of a ring R is called **nilpotent** if $a^n = 0$ for some positive integer n. Prove that the set of all nilpotent elements in a commutative ring R forms a subring of R.

Solution

Let S be the set of nilpotent elements in R. It is clear that S is nonempty because $a = 0$ is an element of S. For arbitrary elements $a, b \in S$ there exist positive integers n, m such that $a^n = 0$ and $b^m = 0$. Then

$$\begin{aligned} (ab)^{nm} &= a^{nm}b^{nm} \\ &= (a^n)^m(b^m)^n \\ &= (0^m)(0^n) \\ &= 0 \end{aligned} \tag{9}$$

so S is closed under multiplication. Consider $(x+y)^q$ where q is some integer larger than $n+m$. Then using the binomial theorem it can be expanded to the sum of terms of the form ax^iy^j where $i + j = q$. Then either $i > n, j > m$,or both. Thus every term of the sum is 0 and $(x+y)^q = 0$ where q is some integer and thus S is closed under addition. For any $x \in S$ there exists a positive integer n such that $x^n = 0$. Then $(-x)^n = 0$ and $-x$ is an element of S.
Therefore S is a subring of R.

Problem # 50

Let x and y be nilpotent elements that satisfy the following conditions in a commutative ring R: Prove that $x + y$ is nilpotent.

(a) $x^2 = 0, y^3 = 0$

(b) $x^n = 0, y^m = 0$ for some $n, m \in \mathbb{Z}^+$

Solution

Part (a)

This follows from the answer to problem 49.

Part (b)

This follows from the answer to problem 49.

Problem # 51

Let R and S be arbitrary rings. In the Cartesian product $R \times S$ of R and S, define

$$\begin{aligned}
(r, s) = (r', s') &\qquad \text{if and only if } r = r' \text{ and } s = s', \\
(r_1, s_1) + (r_2, s_2) &\qquad = (r_1 + r_2, s_1 + s_2), \\
(r_1, s_1) \cdot (r_2, s_2) &\qquad = (r_1 r_2, s_1 s_2)
\end{aligned}$$

(a) Prove that the Cartesian product is a ring with respect to these operation. Is is called the **direct sum** of R and S and is denoted by $R \oplus S$.

(b) Prove that $R \oplus S$ is commutative if both R and S are commutative.

(c) Prove that $R \oplus S$ has a unity element if both R and S have unity elements.

(d) Give an example of rings R and S such that $R \oplus S$ does not have a unity element.

Solution

Part (a)

$R \times S$ is closed under addition and multiplication as a direct result of R and S being closed. Addition and multiplication are both associative and addition is commutative as a result of the corresponding operations in R and S having these properties. It contains the additive identity $(0, 0)$ and every element (r, s) has an additive inverse $(-r, -s)$ that is guaranteed to exist because R and S contain inverses. We now show that the distributive laws hold. For arbitrary elements $(r_1, s_1), (r_2, s_2), (r_3, s_3)$:

$$\begin{aligned}
(r_1, s_1)((r_2, s_2) + (r_3, s_3)) &= (r_1, s_1)(r_2 + r_3, s_2 + s_3) \\
&= (r_1(r_2 + r_3), s_1(s_2 + s_3) \\
&= (r_1 r_2 + r_1 r_3, s_1 s_2 + s_1 s_3) \\
&= (r_1 r_2, s_1 s_2) + (r_1 r_3, s_1 s_3) \\
&= (r_1, s_1)(r_2, s_2) + (r_1, s_1)(r_3, s_3)
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
((r_2, s_2) + (r_3, s_3))(r_1, s_1) &= (r_2 + r_3, s_2 + s_3)(r_1, s_1) \\
&= ((r_2 + r_3)r_1, (s_2 + s_3)s_1 \\
&= (r_2 r_1 + r_3 r_1, s_2 s_1 + s_3 s_1) \\
&= (r_2 r_1, s_2 s_1) + (r_3 r_1, s_3 s_1) \\
&= (r_2, s_2)(r_1, s_1) + (r_3, s_3)(r_1, s_1)
\end{aligned} \tag{11}$$

Thus $R \times S$ is a ring.

Part (b)

If R and S are commutative then for arbitrary elemtents $(r_1, s_1), (r_2, s_2) \in R \times S$:

$$\begin{aligned}(r_1, s_1)(r_2, s_2) &= (r_1 r_2, s_1 s_2) \\ &= (r_2 r_1, s_2 s_1) \\ &= (r_2, s_2)(r_1, s_2)\end{aligned} \tag{12}$$

Thus $R \times S$ is commutative when R and S are.

Part (c)

Let e_r, e_s be the unities of R and S respectively. Then the element (e_r, e_s) is the unity in $R \times S$. For an arbitrary element (r, s)

$$\begin{aligned}(e_r, e_s)(r, s) &= (e_r r, e_s s) \\ &= (r, s) \\ &= (r e_r, s e_s) \\ &= (r, s)(e_r, e_s)\end{aligned} \tag{13}$$

Part (d)

Let $R = \mathbb{Z}$ and $S = \mathbb{E}$, $\mathbb{E}$ is the set of even integers. Both R and S are rings but $R \times S$ has no unity.

Problem # 56

Suppose R is a ring in which all elements x are idempotent - that is, all x satisfy $x^2 = x$. (Such a ring is called a **Boolean Ring**).

(a) Prove that $x = -x$ for each $x \in R$. (*Hint:* Consider $(x + x)^2$.)

(b) Prove that R is commutative. (*Hint:* Consider $(x + y)^2$.)

Solution

Part (a)

$$\begin{aligned}x &= (-x)(-x) \\ &= (-x)^2 \\ &= -x\end{aligned} \tag{14}$$

Part (b)

$$
\begin{aligned}
(x+y)^2 &= x^2 + y^2 + xy + yx \\
x+y &= x^2 + y^2 + xy + yx \\
x+y &= x + y + xy + yx \\
0 &= xy + yx \\
-(yx) &= (xy)
\end{aligned}
\tag{15}
$$

and by the result of part one, $-(yx) = yx$ and thus R is commutative.

Section 5.2

In Exercises 4 and 5, let $U = \{a, b\}$.

Problem # 4

Is $\mathcal{P}(U)$ an integral domain? If not, find all zero divisors in $\mathcal{P}(U)$.

Solution

$\mathcal{P}(U)$ is not an integral domain. The elements $\{a\}$ and $\{b\}$ are zero divisors.

Problem # 5

Is $\mathcal{P}(U)$ an field? If not, find all nonzero elements that do not have multiplicative inverses.

Solution

$\mathcal{P}(U)$ is not a field beause every field is an integral domain and we have already shown that this is not the case. The elements $\{a\}$ and $\{b\}$ do not have multiplicative inverses.

Problem # 11

Let R be the set of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where a and b are real numbers. Assume that R is a commutative ring with unity with respect to matrix addition and multiplication. Answer the following questions and give a reason for any negative answers.

(a) Is R an integral domain?

(b) Is R a field?

Solution

Part (a)

From part b) we know R is a field. Every field is an integral domain so R is an integral domain.

Part (b)

Any nonzero element of R $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has an inverse $(a^2+b^2)\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ This matrix is always defined for nonzero elements of R because the only case when $a^2 + b^2$ is zero is when both a and b are zero. Thus R is a field.

Problem # 12

Consider the Gaussian integers modulo 3, that is, the set $S = \{a + bi \mid a, b \in \mathbb{Z}_3\} = \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$, where we write 0 for [0], 1 for [1], and 2 for [2] in $\mathbb{Z}_3$. Addition and multiplication are as in the complex numbers except that the coefficients are added and multiplied as in $\mathbb{Z}_3$. Thus $i^2 = -1$ as in the complex numbers and $-1 = 2$ in $\mathbb{Z}_3$.

(a) Is S a commutative ring?

(b) Does S have a unity?

(c) Is S an integral domain?

(d) Is S a field?

Solution

Part (a)

Using our knowledge of addition in the complex numbers and in $\mathbb{Z}_3$ it is clear that S forms an abelian group under addition with the identity element 0.
Similarly, it is clear that S is closed under multiplication and that multiplication is commutative.
The distributive laws hold for multiplication and addition defined in the complex numbers and in $\mathbb{Z}_3$.
R is a commutative ring.

Part (b)

S has the unity 1.

Part (c)

S is an integral domain, it is a commutative ring with unity and it also has no zero divisors. We show in part d) that it's a field and every field is an integral domain.

Part (d)

S is a field, every element has a multiplicative inverse.

$$1^{-1} = 1, \qquad 2^{-1} = 2, \qquad i^{-1} = 2i, \qquad (1+1)^{-1} = (2+i), \qquad (1+2i)^{-1} = (2+2i)$$

Problem # 13

Work Exercuse 12 using $S = \{a + bi \mid a, b \in \mathbb{Z}_5\}$, the Gaussian integers modulo 5.

Solution

Part (a)

See the answer to part a of the previous question with $\mathbb{Z}_5$ instead of $\mathbb{Z}_3$.

Part (b)

S is not an integral domain because it has zero divisors.

$$(2+i)(2-i) = 0$$

Part (c)

Every field is an integral domain. It has already been shown that S is not an integral domain. Thus S is not a field.

Problem # 15

Give an example of an infinite commutative ring with no zero divisors that is not an integral domain.

Solution

The set of even integers.

$\mathbb{E}$