Problem 1

Prove Claim 3.3 from class: let S be a non-empy subset of $\mathbb{R}$. Then $\max(S)$ exists (that is, S has a maximal element) $\iff$ S is bounded above and $\sup(S) \in S$.

Solution

First assume that $\max(S)$ exists. Let $M = \max(S)$. By definition $x \leq M \forall x \in S$, M is an upper bound for S, and $M \in S$. Let z be some upper bound for S and assume that $z < M$. Therefore, by the definition of an upper bound, $\forall x \in S, x < z$. In particular $M < z$. This is a contradiction so $M \leq z$. Then by definition of the supremum, $M = \sup(S) \in S$.
Now let S be bounded above and $M = \sup(S) \in S$. By the definition of the supremum, $x \leq M \forall x \in S$ so $M = \max(S)$ by definition so $\max(S)$ exists.

Problem 2

Let S be a non-empty subset of $\mathbb{R}$. Let $UB(S)$ be the set of all upper bounds of S (note that this set may be empty) and $LB(S)$ be the set of all lower bounds of S. Also let $-S = \{-s : s \in S\}$

(i) Let $M \in \mathbb{R}$. Prove that $M = \sup(S)$ if and only if $M = \min(UB(S))$ (the minimal element of $UB(S)$). Also prove that $M = \inf(S)$ if and only if $M = \max(LB(S))$ (the maximal element of $LB(S)$). This is essentially a reeformulation of the definition of sup and inf.

(ii) Let $y \in \mathbb{R}$. Prove that $y \in UB(S) \iff -y \in LB(-S)$.

(iii) Deduce from (ii) that $UB(S)$ has a minimum $\iff$ $LB(-S)$ has a maximum, and if they exist, then $\min(UB(S)) = -\max(LB(-S))$.

(iv) (practice) Combine (i)-(iii) to deduce the reflection principle as formulated in Lecture 4.

Solution

(i) Let $M = \sup(S)$. Then M is an upper bound of S and $UB(S)$ is non-empty. By definition of a suprenum M is at least as small as any other upper bound of S so $\forall x \in UB(S), M \leq x$ so $M = \min(UB(S))$ by definition.

Now let $M = \min(UB(S))$. Thus $UB(S)$ is non-empty and by its presence in the set M must be an upper bound of S. By the definition of a minimum, $M \leq x \forall x \in UB(S)$. Therefore M is less than or equal to all other upper bounds of S so $M = \sup(S)$ by definition.

(ii) $\forall x \in -S, -x \in S$. For any $y \in \mathbb{R}$, $y \in UB(S) \iff -x \leq y \iff x \geq -y \iff -y \in LB(S)$.

(iii) Let $m = \min(UB(S))$. Then $\forall x \in UB(S), m \leq x$. By the result of (ii), $-m \in LB(-S)$. So $\forall x \in UB(S), -x \in LB(-S)$ so $m \leq x \implies -m \geq -x$ and $m = \max(LB(-S))$ by definition.

(iv) Let $S \subseteq \mathbb{R}, S \neq \emptyset$ and let $-S = \{-s : s \in S\}$.

(a) By (ii) if S is bounded above (and thus $UB(S)$ is non-empty) then $LB(-S)$ is also non-empty and $-S$ is bounded below. By (i) and (iii) $\inf(S) = \max(LB(S)) = -\min(UB(-S)) = -\sup(-S)$.

(b) If S is bound below ($LB(S) \neq \emptyset$) then by (ii) $UB(-S)$ is non-empty and $-S$ is bounded below. Further, by (i) and (iii) $\inf(S) = \max(LB(S)) = -\min(UB(-S)) = -\sup(-S)$.

Problem 3

Use the Archimedean property to prove that for every real number $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Solution

By the Archimedean property $\exists n \in N$ such that $n\epsilon > 1 \implies \epsilon > \frac{1}{n}$.

Problem 4

Prove the following result, which can be thought of as a converse of the Approximation Theorem (Theorem 3.2). Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Let $M \in \mathbb{R}$ be an upper bound for S, and suppose that for all $\epsilon > 0$ there exists $x \in S$ such that $M - \epsilon < x \leq M$. Prove that $M = \sup(S)$.

Solution

In order to prove that $M = \sup(S)$ we want to show that M is an upper bound for S and that M is at least as small as any upper bound for S. The first of these conditions is given. Let z be some upper bound for S. Assume that $z < M$. Then $M - z > 0$ so we can set $\epsilon = M - z$ and it is given that

$$\begin{aligned}
&\exists x \in S \text{ such that } M - \epsilon < x \leq M \\
&\implies M - (M - z) < x \leq M \\
&\implies z < x \leq M
\end{aligned} \tag{1}$$

However, this implies that there is some element $x \in S$ that is larger than z. This is a contradiction because z is an upper bound for S. Therefore $z \geq M$ and we have shown that $M = \sup(S)$.

Problem 5

Let A and B be non-empty bounded above subsets of $\mathbb{R}$, and let $A+B = \{a+b : a \in A, b \in B\}$. Prove that $A+B$ is also bounded above and $\sup(A+B) = \sup(A) + \sup(B)$.

Solution

Let $M = \sup(A) + \sup(B)$. In order to prove that $M = \sup(A+B)$ we want to show that $\forall x \in (A+B), x \leq M$ and that M is at least as small as all upper bounds of S.

i. Select an arbitrary element $x \in A+B$. By definition $x = a+b$ for some $a \in A, b \in B$. For any such a, b $a \leq \sup(A)$ and $b \leq \sup(B)$. Thus $x = a+b \leq \sup(A) + \sup(B) = M$.

ii. Select some arbitrary $\epsilon > 0$. Then by the approximation property of supremum $\exists a \in A$ such that $\sup(A) - \frac{\epsilon}{2} < a \leq \sup(B)$. Similarly, $\exists b \in B$ such that $\sup(B) - \frac{\epsilon}{2} < b \leq \sup(B)$. So

$$\sup(A) + \sup(B) - \epsilon < a + b \implies \sup(A) + \sup(B) < a + b + \epsilon$$

Let z be some upper bound of S. Then $a+b \leq z \ \forall a \in A, b \in B$. Then $\sup(A)+\sup(B) < z + \epsilon \implies M = \sup(A) + \sup(B) \leq z$.

Thus M has satisfied the necessary conditions and $M = \sup(A+B)$.

Problem 6

This problem introduces the notions of open and closed subsets of $\mathbb{R}$. Let S be a subset of $\mathbb{R}$. We say that S is *open* if for every $x \in S$ there exists $\epsilon > 0$ (which may depend on x) such that $(x-\epsilon, x+\epsilon) \subseteq S$ (thus, for every point of S there is some open interval centered at that point which is entirely contained in S). We say that S is closed if its complement $\mathbb{R}$ S is open.

(a) Prove that if S is an open interval (that is, $S = (a, b) = \{x \in \mathbb{R} : a < x < b\}$ for some $a < b$), then S is an open subset of $\mathbb{R}$. **Hint:** This is merely a reformulation of one of the results in HW#1.

(b) Prove that if S is a closed interval ($S = [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a \leq b$), then S is a closed subset of $\mathbb{R}$.

Solution

Part (a)

It is clear from the definition of S that S is a subset of $\mathbb{R}$. In order to show that it is open we must show that, for arbitrary $x \in S$ there exists and $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq S$. Select any $x \in S$. By the definition of S, $a < x < b$. Then by the result of HW#5.(c) there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset S$. This proves the result.

Part (b)

It is clear from the definition of S that it is a subset of $\mathbb{R}$. In order to show that $S = [a, b]$ is closed we must show that $S \setminus mathbbR$ is open. $S \setminus mathbbR = \{x \in R : x < a \text{ or } x > b\}$. So the complement of S can be written as the union of the two open intervals $(-\infty, a)$ and (b, ∞). By the result of part a this is an open subset of $\mathbb{R}$ and so S is closed.

Problem 7

(a) Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Recall that (X, d) is a metric space by HW#1.6(b). Prove that $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$ for all $x \in X$ and $\epsilon > 0$ (thus, an open ball of radius ϵ centered at x in this case is simply the open interval of length 2ϵ centered at x). **Hint:** The result follows directly from basic properties of absolute values.

(b) Now let $X = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, and define functions $d : X \times X \to \mathbb{R}$ and $D : X \times X \to \mathbb{R}$ by setting $d((x_1, y_1), (x_2, y_))= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $D((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_1|$. Describe the open ball $B_\epsilon((x, y))$ in each of these two metric spaces.

Solution

Part (a)

WTS: $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ is equal to $(x - \epsilon, x + \epsilon) = \{y \in \mathbb{R} : x - \epsilon < y < x + \epsilon\}$. We know $X = \mathbb{R}$ and $d(x, y) = |x - y|$ so $B_\epsilon(x)$ can be rewritten as:

$$\{y \in \mathbb{R} : |x - y| < \epsilon\}$$

by properties 2 and 6 of absolute values:

$$\begin{aligned} \{y \in \mathbb{R} : |x - y| < \epsilon\} &= \{y \in \mathbb{R} : |y - x| < \epsilon\} \\ &= \{y \in \mathbb{R} : -\epsilon < y - x < \epsilon\} \\ &= \{y \in \mathbb{R} : x - \epsilon < y < x + \epsilon\} \end{aligned} \tag{2}$$

So the sets are equivalent.

Part (b)

In the first space it forms a disk. It selects all points less than a given cartesian distance from a central point. In the second space it forms a filled in square. These are shown in the picture below.

Problem 8

Prove that if S is any open ball in X (that is, $S = B_\alpha(y)$ for some $y \in X$ and $\alpha > 0$, then S is an open subset of X.

Solution

$S = B_\alpha(y) = \{z \in X : d(y,z) < \alpha\}$. Select an arbitrary $x \in B_\alpha$. We want to show that there exists some $\epsilon > 0$ such that $B_\epsilon(x) = \{z \in X : d(x,z) < \epsilon\} \subseteq S$.
Let $\epsilon = \alpha - d(y,x)$. By the definition of B_α $d(y,x)$ must be less than α so ϵ is positive. For any $z \in B_\epsilon$, $d(y,z) \leq d(y,x) + d(x,y)$ by the third property of a metric space.
If it were the case that $d(x,z) \geq \epsilon$ then:

$$\begin{aligned} d(x,z) \geq \epsilon &\implies d(x,z) \geq \alpha - d(y,x) \\ &\implies d(x,z) \geq d(y,z) - d(y,x) \\ &\implies d(x,z) + d(y,x) \geq d(y,z) \end{aligned} \tag{3}$$

This is a contradiction with property three so $d(x,z) < \epsilon$ for all $z \in B_\epsilon$ and S is an open subset of X.