Problem 1

Part (a)

Fix $\epsilon > 0$. Let us assume that $|x-2| < \delta$ for some $\delta > 0$ to be determined later. Then $|x^3 - 8| = |(x-2)(x^2 + 2x_4)|$. Assume that $\delta \leq 1$. Then $|(x-2)||x^2+2x+4| \leq |x-2||3^2 + 6 + 4| = 19|x-2|$.
So $|x^3 - 8| \leq 19|x-2| < 19\delta$. Choose $\delta = \min(1, \frac{\epsilon}{19})$. If $0 < |x-2| < \delta$ then because $\delta < 0$, $1 < x < 3$ so

$$|x^3 - 8| \leq 19|x-2| < 19\delta \leq 19\left(\frac{\epsilon}{19}\right) = \epsilon$$

Part (b)

Choose $\delta_0 = \frac{1}{2}$, $C = \frac{1}{2}$. Then $|x-1| < \frac{1}{2} \implies -\frac{1}{2} < x - 1 < \frac{1}{2} \implies \frac{1}{2} < x < \frac{3}{2} \implies \frac{1}{2} < |x| < \frac{3}{2}$. If $\frac{1}{2} < |x| < \frac{3}{2} \implies \frac{2}{3} < \frac{1}{|x|} < 2 \implies \frac{|x-1|}{x} < 2|x-1|$. Fix $\epsilon > 0$. Let $\delta = \min(\frac{1}{2}, \frac{\epsilon}{2})$. Then

$$\left|\frac{1}{x} - 1\right| = \left|\frac{x-1}{x}\right| = \frac{|x-1|}{|x|} < 2|x-1| < 2\frac{\epsilon}{2} = \epsilon$$

Part (c)

Fix $\epsilon > 0$. Assume $|x-a| < \delta$ for some $\delta > 0$ to be found later. Then $|x^2 - a^2| = |(x-a)(x+a)| = |x-a||x+a| < \delta|x+a|$. We know that $|x-a| < \delta \implies |x+a| = |(x-a) + 2a| \leq |x-a| + |2a| < \delta + |2a|$. Assume that $\delta \leq 1$. Then $|x+a| < 1 + |2a|$ so $|x^2 - a^2| < \delta(1 + |2a|)$. We want to have $\delta(1+|2a|) \leq \epsilon \implies \delta \leq \frac{\epsilon}{1+|2a|}$. Define $\delta = \min(1, \frac{\epsilon}{1+|2a|})$. Take $\forall x$ such that $0 < |x-a| < \delta$. Then by previous computations $|x^2 - a^2| < \delta|x+a| < \delta(1+|2a|) \leq \epsilon$. So $|x^2 - a^2| < \epsilon \; \forall x$ such that $0 < |x-a| < \delta$.

Problem 2

Part (a)

This problem is very similar to part c) from problem 1. Although our δ can't depend on a, we can use the fact that we know the maximum possible value of a. So let $\delta = \min(1, \frac{\epsilon}{1+2\max(|c|,|d|)})$. For any a in the given range this number will be smaller than the δ we already showed worked in problem 1.

Part (b)

Fix $\epsilon > 0$. Assume $\exists \delta > 0$ such that $|x^2 - a^2| < \epsilon \; \forall x, a \in \mathbb{R}$ such that $|x - a| < \delta$. Let $x = a + \frac{\delta}{2}$. Then for all a,

$$|x^2 - a^2| = |a^2 + \delta a + \frac{\delta^2}{4} - a^2| = |\delta a + \frac{\delta^2}{4}| < \epsilon$$

This implies that $\delta a < \epsilon - \frac{\delta^2}{4}$ for all a which contradicts the Archimedean Property. Thus such a δ does not exist.

Problem 3

Part (a)

$\forall \epsilon > 0 \exists N$ such that $\forall n \geq N \ |a_n - L| < \epsilon \iff \forall \epsilon > 0 \ \exists N$ such that $\forall n \geq N ||a_n - L| - 0| < \epsilon$.

Part (b)

$0 \leq |a_n - L| \leq b_n$ for all n and $\lim\limits_{n\to\infty} b_n = 0$ so by the squeeze theorem $\lim\limits_{n\to\infty} |a_n - L| = 0$. By part a) this means that $\lim\limits_{n\to\infty} a_n = L$.

Part (c)

(a) $\lim\limits_{x\to a} f(x) = L \iff \lim\limits_{x\to a} |f(x) - L| = 0$ This follows directly from the arithmetic properties of limits and part a) of problem 4.

(b) $f(x) - L \leq g(x) \forall x, \lim\limits_{x\to a} g(x) = 0 \implies \lim\limits_{x\to a} f(x) = L$. $0 \leq |f(x) - L| \leq g(x)$ for all x. $\lim\limits_{x\to a} g(x) = 0$. Therefore by the squeeze theorem for functions $\lim\limits_{x\to a} f(x) = L$.

Problem 4

Part (a)

We know by the definition of the limit that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - L| < \epsilon \ \forall x$ such that $0 < |x - a| < \delta$. We want to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $||f(x)| - |L|| < \epsilon \ \forall x$ such that $0 < |x - a| < \delta$. Fix some $\epsilon > 0$ then, because $||f(x)| - |L|| \leq |f(x) - L|$ we can use the same δ guaranteed by the existence of the limit of $f(x)$ and thus for all x such that $0 < |x - a| < \delta$

$$||f(x)| - |L|| < |f(x) - L| < \epsilon$$

and thus the limit of $|f|$ as x goes to a is $|L|$.

Part (b)

Assume without loss of generality that $u \geq v$. Then $u - v \geq 0$ so $|u - v| = u - v$. Thus $\frac{u+v+|u-v|}{2} = \frac{2u}{2} = u = \max(u, v)$. Similarly $\frac{u+v-|u-v|}{2} = \frac{u+v-u+v}{2} = \frac{2v}{2} = v = \min(u, v)$.

Part (c)

By the results of parts a) and b):

$$\begin{aligned}
\lim_{x\to a}(\max(f,g)) &= \lim_{x\to a}(\frac{1}{2}(f+g+|f-g|)) \\
&= \frac{1}{2}(\lim_{x\to a} f + \lim_{x\to a} g + \lim_{x\to a}(|f-g|)) \\
&= \frac{1}{2}(\lim_{x\to a} f + \lim_{x\to a} g + |\lim_{x\to a}(f-g)|) \\
&= \frac{1}{2}(L+M+|L-M|) \\
&= \max(L,M)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to a}(\min(f,g)) &= \lim_{x\to a}(\frac{1}{2}(f+g-|f-g|)) \\
&= \frac{1}{2}(\lim_{x\to a} f + \lim_{x\to a} g - \lim_{x\to a}(|f-g|)) \\
&= \frac{1}{2}(\lim_{x\to a} f + \lim_{x\to a} g - |\lim_{x\to a}(f-g)|) \\
&= \frac{1}{2}(L+M-|L-M|) \\
&= \min(L,M)
\end{aligned}$$

Problem 5

See midterm.

Problem 6

Part (a)

Let $\epsilon = \sqrt{5}$. We know by definition of the limit that for all $n > 7 + \frac{10}{\epsilon^2} = 9$, $|a_n - 4| < \sqrt{5}$. So $-\sqrt{5} < a_n - 4 < \sqrt{5} \implies 4 - \sqrt{5} < a_n < 4 + \sqrt{5}$. $-(4+\sqrt{5}) < 4 - \sqrt{5}$ so $-(4+\sqrt{5}) < a_n < (4+\sqrt{5} \implies |a_n| < 4 + \sqrt{5}$. Thus the conditions are satisfied for $N = 9, C = 4 + \sqrt{5}$.

Part (b)

$$
\begin{aligned}
|a_n^2 - 16| &= |(a_n - 4)(a_n + 4)| && (1)\\
&= |a_n - 4||a_n + 4| && (2)\\
&< \frac{\epsilon}{8+\sqrt{5}}|a_n + 4| \text{for } n > 7 + \frac{10}{(\epsilon/(8+\sqrt{5}))^2} && (3)\\
&\leq \frac{\epsilon}{8+\sqrt{5}}(|a_n| + 4) && (4)\\
&\leq \frac{\epsilon}{8+\sqrt{5}}(8+\sqrt{5}) \text{for } n \geq 9 && (5)\\
&= \epsilon && (6)
\end{aligned}
$$

So for any $\epsilon > 0$ for all $n \geq M(\epsilon) = \max\left(9, 7 + \frac{10}{(\epsilon/(8+\sqrt{5}))^2}\right)$, $|a_n^2 - 16| < \epsilon$.

Problem 7

Part (a)

Sequences are infinite. Let $S =$the set of n such that $|a_n - L| \geq \epsilon$. This is a finite subset of $\mathbb{N}$ so it contains a maximum element. Let $N = \max(S) + 1$. Thus, because $n \geq N \implies n \notin S \implies |a_n = L| < \epsilon$, for every ϵ, for all $n \geq N$.

$$|a_n - L| < \epsilon$$

so a_n converges to L by definition of the limit.

Part (b)

Assume (i). It was shown in class that if a sequence converges to a number then so must all of its subsequences. Thus (ii) is impossible. Thus the two cannot hold simultaneously.

Assume (i) does not hold. By (a) there are infinitely many n such that $|a_n - L| \geq \epsilon$ for some $\epsilon > 0$. Thus you can construct a subsequence out of only a_n such that $|a_n - L| > \epsilon$. This clearly cannot converge to L or have a subsequence that converges to L. It is bounded because a_n is bounded so by Bolzano-Weierstraa it had a convergent subsequence (that doesn't converge to L). This is also a subsequence of a_n so (ii) is true. Thus one of (i) and (ii) must hold.

Problem 8

Part (a)

For this problem I will treat $\frac{1}{\infty}$ as 0.

(a) WTS: $x = y \iff d(x, y) = 0$
Assume that $x = y$. Then

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{x}\right| = 0$$

Assume that $d(x, y) = 0$. Then

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{y}\right| &= 0 \\ \implies \frac{1}{x} - \frac{1}{y} &= 0 \\ \implies \frac{1}{x} = \frac{1}{y} \\ \implies y = x \end{aligned}$$

(b) WTS: $d(x, y) = d(y, x)$ for all $x, y \in X$.
By the property of absolute values:

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = d(y, x)$$

(c) WTS: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Using the triangle inequality:

$$d(x, z) = \left|\frac{1}{x} - \frac{1}{z}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = d(x, y) + d(y, z)$$

Thus (X, d) is a metric space.

Part (b)

First assume that $\lim_{n\to\infty} = L$ in the usual sense. Fix some $\epsilon > 0$. By definition there exists some $N \in \mathbb{N}$ such that $\forall n \geq N$ $|f_n - L| < \epsilon$. We know that each f_n equals $f(x)$ for some $x \in X$. We need to find a δ such that for all $x \in B^\circ_\delta = \{x \in \mathbb{N} : d(x, \infty) < \delta\}$, $|f(x) - L| < \epsilon$. By definition $d(x, \infty) = |1/x|$ $(x \in \mathbb{N})$ so we want to find a δ such that $1/x < \delta \implies x \geq N$. Let $\delta = \frac{1}{N}$. Then for all x

$$\begin{aligned} d(x, \infty) = \frac{1}{x} < \delta &\implies \frac{1}{x} < \frac{1}{N} \\ &\implies x > N \\ &\implies |f_x - L| < \epsilon \\ &\implies |f(x) - L| < \epsilon \end{aligned}$$

So $\lim_{n\to\infty} f(n) = L$ as a limit in the metric space.

Now assume $\lim_{n\to\infty} f(n) = L$ as a limit in the metric space. Fix some $\epsilon > 0$. By definition there exists a δ such that for alll $x \in B_\delta^\circ$, $|f(x) - L| < \epsilon$. Let $N = \lceil \frac{1}{\delta} \rceil + 1$. Then for all $n \geq N$.

$$\begin{aligned}
n \geq N &\implies n \geq \lceil \frac{1}{\delta} \rceil + 1 \\
&\implies n > \frac{1}{\delta} \\
&\implies d(n, \infty) = \frac{1}{n} < \delta \\
&\implies |f(n) - L| < \epsilon \\
&\implies |f_n - L| < \epsilon
\end{aligned}$$

So $\lim_{n\to\infty} f_n = L$ in the usual sense.