Problem 1

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{(x^n + \binom{n}{1} x^{n-1} h + \cdots + h^n) - x^n}{h} \\
&= \lim_{h \to 0} \frac{\binom{n}{1} x^{n-1} h + \cdots + h^n}{h} \\
&= \lim_{h \to 0} \binom{n}{1} x^{n-1} + \cdots + h^{n-1} \\
&= n x^{n-1}
\end{aligned}
$$

Problem 2

Part (a)

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a) + f(h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{f(h)}{h}
\end{aligned}
$$

This value is constant.

Part (b)

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a)f(h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a)(f(h) - 1)}{h} \\
&= f(a) \lim_{h \to 0} \frac{f(h) - 1}{h}
\end{aligned}
$$

The value of $\lim\limits_{h \to 0} \frac{f(h)-1}{h}$ is constant so $f'(a) = c \cdot f(a)$ for some constant C.

Problem 3

Fix some $\epsilon > 0$. Assume that $|x - y| < \delta$ for some δ to be determined. Then by MVT $|f(x) - f(y)| = |f'(c)(x - y)|$ for some c. However, the value of $f'(c)$ is bounded by C so

$$|f'(c)(x - y)| \leq C|x - y| = C\delta < \epsilon$$

if $\delta < \frac{\epsilon}{C}$.

Problem 4

Part (a)

By definition of a root there are at least n values of x for which $f(x) = 0$. Then we can apply Rolle's Theorem to each consecutive pair of locations to obtain a minimum of $n - 1$ values of x where $f'(x) = 0$.

Part (b)

It is trivial that a polynomial of degree one has at most one root. Now assume that polynomials with degree n has n roots for some $n \geq 1$. Now consider a polynomial f of degreen $n + 1$. Assume that f has more than $n + 1$ roots. Then by part a) f' has degree n but at least $n + 1$ roots. This contradicts our inductive hypothesis and therefore polynomials with degree $n + 1$ have at most $n + 1$ roots.

Problem 5

Real lost.

Problem 6

Let $f(x) = x^{1/2} + \frac{1}{2}\log(x) - x$. Then $f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2x} - 1$ is less than 0 for all $x > 1$. This means that f is a strictly decreasing function by Theorem 4.17. Thus $\forall x > 1$:

$$f(x) = x^{1/2} + \frac{1}{2}\log(x) - x > 0 \implies f(x) = x^{1/2} + \frac{1}{2}\log(x) > x$$