Problem 1

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{(x^n + \binom{n}{1}x^{n-1}h + \cdots + h^n) - x^n}{h} \\
&= \lim_{h\to 0} \frac{\binom{n}{1}x^{n-1}h + \cdots + h^n}{h} \\
&= \lim_{h\to 0} \binom{n}{1}x^{n-1} + \cdots + h^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

Problem 2

Part (a)

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a) + f(h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(h)}{h}
\end{aligned}$$

This value is constant.

Part (b)

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a)f(h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a)(f(h) - 1)}{h} \\
&= f(a) \lim_{h\to 0} \frac{f(h) - 1}{h}
\end{aligned}$$

The value of $\lim\limits_{h\to 0} \frac{f(h)-1}{h}$ is constant so $f'(a) = c \cdot f(a)$ for some constant C.

Problem 3

Fix some $\epsilon > 0$. Assume that $|x - y| < \delta$ for some δ to be determined. Then by MVT $|f(x) - f(y)| = |f'(c)(x - y)|$ for some c. However, the value of $f'(c)$ is bounded by C so

$$|f'(c)(x - y)| \leq C|x - y| = C\delta < \epsilon$$

if $\delta < \frac{\epsilon}{C}$.

Problem 4

Part (a)

By definition of a root there are at least n values of x for which $f(x) = 0$. Order these roots in increasing order. Then we can apply Rolle's Theorem to each consecutive pair of roots to obtain a minimum of $n - 1$ values of x where $f'(x) = 0$.

Part (b)

It is trivial that a polynomial of degree one has at most one root. Now assume that polynomials with degree n have n roots for all $n \leq k$ for some k. Now consider a polynomial f of degreen k. Assume that f has more than k roots. Then by problem 1 and part a) f' has degree $< k$ but at least k roots. This contradicts our inductive hypothesis and therefore polynomials with degree $n + 1$ have at most $n + 1$ roots. By the inductive hypothesis polynomials with degree $n \in \mathbb{N}$ have n roots.

Problem 5

We assume that $\lim\limits_{x \to +\infty} f(x)$ exists. Thus we can construct a sequence x_n of real numbers such that x_n goes to infinity. To select x_n set $\epsilon = 1/n$. We know because the limit exists at infinity that there exists some M_n such that $|f(x) - f(y)| < \epsilon$ for all $x, y \geq M_n$. Note that we can always increase M_n such that it is larger than n in order to ensure that the sequence goes to infinity. For each n apply mean value theorem on the interval $[M, M + 1]$. Then $f(M+1) - f(M) = f'(x_n)(M+1-M) \implies f'(x_n) = (f(M+l) - f(M))$. This is less than $1/n$. Therefore as $n \to \infty$, $f'(x_n) \to 0$. Then by the sequential characterization of limits at infinity, $\lim\limits_{x \to +\infty} f'(x) = 0$.

Problem 6

Let $f(x) = x^{1/2} + \frac{1}{2}\log(x) - x$. Then $f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2x} - 1$ is less than 0 for all $x > 1$. This means that f is a strictly decreasing function by Theorem 4.17. Thus $\forall x > 1$:

$$f(x) = x^{1/2} + \frac{1}{2}\log(x) - x > 0 \implies f(x) = x^{1/2} + \frac{1}{2}\log(x) > x$$