Section 1.1

Problem #38

Prove or disprove that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution

$$\begin{aligned} \mathcal{P}(A \cup B) &= \{X \mid X \subseteq (A \cup B)\} \\ &= \{X \mid X \subseteq A\} \cup \{X \mid X \subseteq B\} \\ &= \mathcal{P}(A) \cup \mathcal{P}(B) \end{aligned} \tag{1}$$

Problem #40

Prove or disprove that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

Solution

$$\begin{aligned} \mathcal{P}(A - B) &= \{X \mid X \subseteq (A - B)\} \\ &= \{X \mid X \subseteq \{x \in U \mid x \in A \text{ and } x \notin B\}\} \\ &= \{X \mid X \subseteq (\{x \in U \mid x \in A\} \cap \{x \in U \mid x \notin B\})\} \\ &= \{X \mid X \subseteq (A \cap B')\} \\ &= \{X \mid X \in (\mathcal{P}(A) \cap \mathcal{P}(B'))\} \\ &= \{X \mid X \in \mathcal{P}(A) \text{ and } X \notin \mathcal{P}(B)\} \\ &= \{X \mid X \in \{Y \mid Y \subseteq A\} \text{ and } X \notin \{Z \mid Z \subseteq B\}\} \\ &= \mathcal{P}(A) - \mathcal{P}(B) \end{aligned} \tag{2}$$

Section 1.2

Problem #14

Let $f : Z \to \{-1, 1\}$ be given by

$$f(x) = \begin{cases} 1 & \text{if x is even} \\ \text{-1} & \text{if x is odd} \end{cases}$$

(a) Prove or disprove that f is onto.

(b) Prove or disprove that f is one-to-one.

(c) Prove or disprove that $f(x_1 + x_2) = f(x_1)f(x_2)$.

(d) Prove or disprove that $f(x_1 x_2) = f(x_1)f(x_2)$.

Solution

Part (a)

$$1, 2 \in \mathbb{Z} \tag{3}$$
$$f(1) = -1 \tag{4}$$
$$f(2) = 1 \tag{5}$$

This demonstrates that for all $x \in \{-1, 1\}$ there exists a $y \in \mathbb{Z}$ such that $f(y) = x$ and f is onto by definition.

Part (b)

$$2, 4 \in \mathbb{Z} \tag{6}$$
$$f(2) = 1 \tag{7}$$
$$f(4) = 1 \tag{8}$$

f is not one-to-one because for two values $x, y \in \mathbb{Z}$ where $x \neq y$, $f(x) = f(y)$.

Part (c)

Case 1: x_1 and x_2 are even.
The sum of any two even numbers is also an even number.

$$f(x_1 + x_2) = 1 \tag{9}$$
$$f(x_1)f(x_2) = (1)(1) = 1 \tag{10}$$

in this case $f(x_1 + x_2) = f(x_1)f(x_2)$.

Case 2: x_1 and x_2 are odd.
The sum of any two odd numbers is also an odd number.

$$f(x_1 + x_2) = 1 \tag{11}$$
$$f(x_1)f(x_2) = (-1)(-1) = 1 \tag{12}$$

in this case $f(x_1 + x_2) = f(x_1)f(x_2)$.

Case 3: One of x_1 and x_2 is odd and the other is even.
The sum of any odd numbers and any even number is an odd number.

$$f(x_1 + x_2) = -1 \tag{13}$$
$$f(x_1)f(x_2) = (1)(-1) = -1 \tag{14}$$

in this case $f(x_1 + x_2) = f(x_1)f(x_2)$.
In all cases $f(x_1 + x_2) = f(x_1)f(x_2)$

Part (d)

For $x_1 = 1, x_2 = 1$

$$f(x_1 + x_2) = 1 \tag{15}$$
$$f(x_1) + f(x_2) = -1 + (-1) = -2 \tag{16}$$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$

In Exercises 20-22 , suppose m and n are positive integers, A is a set with exactly m elements, and B is a set with exactly n elements.

Problem #20

How many mappings are there from A to B?

Solution

For each of the n positions in B there are m possible values. The total number of mappings is m^n.

Problem #22

If $m \leq n$, how many one-to-one mappings are there from A to B?

Solution

$$\frac{n!}{(n-m)!} \tag{17}$$

Section 1.3

Problem #10

Let $g : A \to B$ and $f : B \to C$. Prove that f is onto if $f \circ g$ is onto.

Solution

By definition $(f \circ g)$ being onto means that $\forall c \in C$ there exists an element $a \in A$ such that

$$\begin{aligned} c &= (f \circ g)(a) \\ &= f(g(a)) \end{aligned} \tag{18}$$

Let b be the result of $g(a)$. It is given that $b \in B$.
Therefore for any element $c \in C$ there exists an element $b \in B$ such that $f(b) = c$ and f is onto by definition.

Problem #11

Let $g : A \to B$ and $f : B \to C$. Prove that f is one-to-one if $f \circ g$ is one-to-one.

Solution

Assume that g is not one-to-one. This implies that there exist elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $g(a_1) = g(a_2)$. If this is the case then $f(g(a_1)) = f(g(a_2))$ and $(f \circ g)(a_1) = (f \circ g)(a_2)$. This contradicts the given that $(f \circ g)$ is one-to-one and thus g must also be one-to-one.

Problem #12

Let $f : A \to B$ and $g : B \to A$. Prove that f is one-to-one and onto if $f \circ g$ is one-to-one and $g \circ f$ is onto.

Solution

This is a specific case of the situation in problems #10 and #11 where $A = C$. Thus we can use the same steps from those proofs to show that g is both one-to-one and onto.

Part (a)

$$(f \circ g) \text{is one-to-one so by definition if } f(g(b_1)) = f(g(b_2)) \text{ then } b_1 = b_2 \tag{19}$$

Let us assume that f is not one-to-one. This implies that there exist elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and $a_1 \neq a_2$.

Define b_1, b_2 such that $g(b_1) = a_1$ and $g(b_2) = a_2$. We know that b_1 and b_2 exist because g is onto and because $a_1 \neq a_2$ we know from the definition of a mapping that $b_1 \neq b_2$. This contradicts (19). Therefore f must be one-to-one.

Part (b)

It has been shown that bth g and f are one-to-one. By Theorem 1.17 (pg. 27) the composition $(g \circ f)$ is also one-to-one. We are told that $(g \circ f)$ is onto. The composition is thus bijective. By definition the following is then true

$$\forall a_2 \in A \text{ there must exist a unique element } a_1 \in A \text{ such that } a_2 = (g \circ f)(a_1) \tag{20}$$

Let us assume that f is not onto. Therefore

$$\text{there exists an element } x \in X \text{ such that for all } y \in A, f(y) \neq x \tag{21}$$

Let $a = g(x)$ for the x indicated above. By (20) we know that there must exist some value $n \in A$ such that

$$\begin{aligned} a &= g(x) \\ &= (g \circ f)(n) \\ &= g(f(n)) \end{aligned} \tag{22}$$

by (20) $f(n) = x$. This contradicts (21). The function f must be onto.

Section 1.4

Problem #11b

Prove or disprove that the set $B = \{z^3 \mid z \in \mathbb{Z}\}$ is closed with respect to multiplication defined on $\mathbb{Z}$.

Solution

Select arbitrary elements $x, y \in B$. By the definition of B, $x = z_1^3$ and $y = z_2^3$ for some $z_1, z_2 \in \mathbb{Z}$. Then

$$\begin{aligned} xy &= (z_1 \cdot z_1 \cdot z_1) \cdot (z_2 \cdot z_2 \cdot z_2) \\ &= (z_1 \cdot z_2) \cdot (z_1 \cdot z_2) \cdot (z_1 \cdot z_2) \\ &= (z_1 \cdot z_2)^3 \end{aligned} \tag{23}$$

Let $z_3 = (z_1 \cdot z_2)$. $\mathbb{Z}$ is closed under multiplication and therefore $z_3 \in \mathbb{Z}$. It has been shown that for any elements $x, y \in B$ there exists an element $z_3 \in \mathbb{Z}$ such that $x * y = z_3^3$. Therefore for all $x, y \in B$ the product $x * y$ is also in $B.B$ is closed under multiplication defined on $\mathbb{Z}$.

Problem #12b

Prove or disprove that the set $\mathbb{Q} - \{0\}$ of nonzero rational numbers is closed with respect to division defined on the set $\mathbb{R} - \{0\}$ of nonzero real numbers.

Solution

$\mathbb{Q}$ is defined as $\{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ Choose two arbitrary elements $x, y \in \mathbb{Q}$. Then $x = \frac{m_1}{n_1}, y = \frac{m_2}{n_2}$ for some elements $m_1, m_2, n_1, n_2 \in \mathbb{Z} \neq 0$.

$$\frac{x}{y} = \frac{m_1 \cdot n_2}{n_1 \cdot m_2} \tag{24}$$

$\mathbb{Z}$ is closed under multiplication so the products $m_1 \cdot n_2$ and $n_1 \cdot m_2$ are both elements of $\mathbb{Z}$ and $n_1 \cdot m_2 \neq 0$ because neither n_1 nor n_2 is equal to 0. Thus $\frac{x}{y}$ is by definition an element of $\mathbb{Q}$. This means that $\mathbb{Q}$ is closed under division defined on $\mathbb{Z}$.

Problem #13

Assume that * is an associative binary operation on the nonempty set A. Prove that

$$a * [b * (c * d)] = [a * (b * c)] * d \tag{25}$$

for all $a, b, c,$ and d in A.

Solution

Let $e = (b * c)$. Using the fact that * is associative on A

$$\begin{aligned} a * [b * (c * d)] &= a * [(b * c) * d] \\ &= a * [e * d] \\ &= [a * e] * d \\ &= [a * (b * c)] * d \end{aligned} \tag{26}$$

Section 1.5

Problem #4

Let $f : A \to A$, where A is nonempty. Prove that f has a left inverse if and only if $f^{-1}(f(S)) = S$ for every subset S of A.

Solution

Part (a)

Assume that f has a left inverse. By definition every element of S is an element of A and $1_A(a) = a$ for every element of A. Then

$$\begin{aligned} f^{-1}(f(S)) &= (f^{-1} \circ f)(S) \\ &= 1_A(S) \\ &= S \end{aligned} \tag{27}$$

Part (b)

Assume $f^{-1}(f(S)) = S$ for every subset S of A. A is a possible subset of A. Therefore for all elements $a \in A$

$$\begin{aligned} a &= f^{-1}(f(a)) \\ &= (f^{-1} \circ f)(a) \end{aligned} \tag{28}$$

and f^{-1} must be the left inverse of f.

Problem #6

Prove that if f is a permutation on A, then f^{-1} is a permutation on A.

Solution

By Theorem 1.26 (pg. 42) we know that, because f is a permutation, it must be invertible. So there exists a single function $g : A \to A$ such that

$$(g \circ f) = 1_A = (f \circ g) \tag{29}$$

The function g has both a left and right inverse and is thus invertible. By Theorem 1.26 we know that g must be a permutation.

Problem #7

Prove that if f is a permutation on A, then $(f^{-1})^{-1} = f$.

Solution

For a permutation f, f^{-1} is defined such that

$$(f^{-1} \circ f) = 1_A = (f \circ f^{-1}) \tag{30}$$

It can be seen from the above equation that f is both the left and right inverse of f^{-1} and thus $\left(f^{-1}\right)^{-1} = f$.