Problem 1

(a) Prove $-(-x) = x$.

(b) Prove $-(xy) = (-x)y$.

Solution

Part (a)

$$\begin{aligned} 0 + -(-x) &= -(-x) && \text{by (A3)} \\ [x + -x] + -(-x) &= -(-x) && \text{by (A4)} \\ x + [-x + -(-x)] &= -(-x) && \text{by (A2)} \\ x + 0 &= -(-x) && \text{by (A4)} \\ x &= -(-x) && \text{by (A3)} \end{aligned}$$

Part (b)

$$\begin{aligned} (-x)y + xy &= (-x + x)y && \text{by (D)} \\ (-x)y + xy &= (0)y && \text{by (A4)} \end{aligned}$$

In class it was proved that $0 \cdot x = 0$ for all x. By this result we get

$$\begin{aligned} (-x)y + xy &= 0 \\ (-x)y + xy + -(xy) &= 0 + -(xy) && \text{add} - (xy) \\ (-x)y + 0 &= -(xy) && \text{by (A4)} \\ (-x)y &= -(xy) && \text{by (A3)} \end{aligned}$$

Problem 2

(a) Prove if $x > y, z < 0$ then $xz < yz$.

(b) Prove if $x > y > 0, z > w > 0$ then $xz > yw$.

(c) Prove if $x > 0$ then $x^{-1} > 0$.

Solution

Part (a)

$z < 0$ so $-z > 0$. By (O6) which was proved in class $x(-z) > y(-z) \implies xz < yz$.

Part (b)

DO THIS YOU DIDN'T DO IT

Part (c)

First assume that $x^{-1} < 0$. Then by (O7) proved in class:

$$x \cdot x^{-1} < 0 \cdot x^{-1}$$

It was also proved in class that $0 \cdot x = 0$ for all x. Thus,

$$\begin{aligned} x \cdot x^{-1} &< 0 \\ 1 &< 0 && \text{by (M4)} \end{aligned}$$

This is a contradiction so $x^{-1} > 0$.

Problem 3

Prove that there does not exist an $x \in \mathbb{Z}$ such that $0 < x < 1$.
$\mathbb{Z} = \{x \in \mathbb{R} \mid x \in \mathbb{N} \vee x = 0 \vee -x \in \mathbb{N}\}$.

Solution

Consider any arbitrary $x \in \mathbb{R}$. There are three possible cases.

(a) Case 1: $x \in \mathbb{N}$
It was proven in class that for all x in $\mathbb{N}$, $x \geq 1$. Thus it is impossible that $x < 1$.

(b) Case 2: $x = 0$
If $x = 0$ then it is impossible that $x > 0$.

(c) Case 3: $-x \in \mathbb{N}$
By the same fact used in case 1, $-x \geq 1 \implies x \leq -1$. So it is impossible that $x > 0$.

There is no case in which it is possible that $0 < x < 1$.

Problem 4

Prove that it is impossible to define inequalities in $\mathbb{C}$ such that (O1)-(O4) hold.

Solution

The proof given in the book that for any nonzero $a \in \mathbb{R}$, $a^2 > 0$ depends only on axioms (O1)-(O4). Thus if these axioms held in $\mathbb{C}$ then it would have to be the case that the square of any nonzero element of $\mathbb{C}$ was greather than 0. However, i is defined such that $i^2 = -1$ which is less than 0. Thus is is impossible to define inequalitied in $\mathbb{C}$ in such a way that axioms (O1)-(O4) hold.

Problem 5

(a) Let $x, y \in \mathbb{R}$. Prove $x \leq y$ if and only if $x - \epsilon < y + \epsilon \forall \epsilon > 0$.

(b) Let $x, y \in \mathbb{R}$ with $x < y$. Prove there exists $z \in \mathbb{R}$ with $x < z < y$.

(c) Let $a, x, b \in \mathbb{R}$ with $a < x < b$. Prove there exists $\epsilon > 0$ such that $a < x - \epsilon < x + \epsilon < b$. Deduce that $(x - \epsilon, x + \epsilon) \subset (a, b)$.

Solution

Part (a)

By Theorem 1.9 part i proved in the book, $x < y + \epsilon$ for all $\epsilon > 0$. For any given value for $\epsilon > 0$, $0 > -\epsilon$. Then by (O5) $y + \epsilon > x - \epsilon$ for all $\epsilon > 0$.

Part (b)

Let n be the largest natural number such that $\frac{1}{n} < y - x$. Let k be the largest natural number such that $\frac{k}{n} \leq x$. Then by our selection of k, $\frac{k+1}{n} > x$.
Now assume that $y \leq \frac{k+1}{n}$. Then we have that $\frac{k+1}{n} \geq y$ and $-\frac{k}{n} \geq -x$ so by (O5)":

$$\frac{1}{n} = \frac{k+1}{n} - \frac{k}{n} \geq y - x$$

. This is a contradiction so it must be the case that $y > \frac{k+1}{n}$. Thus $z = \frac{k+1}{n}$ is a number satisfying $x < z < y$.

Part (c)

Let y be the smaller value of $b - x$ and $x - a$. Then $a \leq x - y < x < x + y \leq b$. By part b) there exists a z such that $x < z < x + y$. Let $\epsilon = z - x$. This value satisfies that desired conditions.

Problem 6

Prove that each of the following are metric spaces.

(a) $X = \mathbb{R}, d(x, y) = |y - x|$

(b) $X = \text{any set}, d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

(c) Give another example of a metric space.

Solution

Part (a)

i $d(x, y) = 0 \iff x = y$
First assume $x = y$. Then $|y - x| = |0| = 0$. Now assume that $|y - x| = 0$. Then either $y - x = 0$ or $x - y = 0$. In the first case $y - x + x = x$ so by (A4) $y = x$. In the second case $x - y + y = y$ so by (A4) $x = y$.

ii $d(x, y) = d(y, x)$
By property 2 of absolute values, $|y - x| = |x - y|$.

iii $d(x, z) \leq d(x, y) + d(y, z)$
$|z - x| \leq |y - x| + |z - 4|$ by the triangle inequality proved in class.

Part (b)

i $d(x, y) = 0 \iff x = y$
This is true by the definition of the function d.

ii $d(x, y) = d(y, x)$
In the case when $x = y$, $d(x, y) = 0 = d(y, x)$. In the case when $x \neq y$, $d(x, y) = 1 = d(y, x)$.

iii $d(x, z) \leq d(x, y) + d(y, z)$

Case: $x = y = z$
$0 \leq 0$

Case: $x \neq y \neq z$
$1 \leq 2$

Case: $x = y \neq z$
$1 \leq 1$

Case: $x \neq y = z$
$1 \leq 1$

Case: $x = z \neq y$
$0 \leq 1$

Part (c)

$X = \mathbb{C}, d(x, y) = \sqrt{x^2 + y^2}$.