Problem #1

Give an example of a function from $\mathbb{Z}$ to $\mathbb{Z}$ with each of the following characteristics.

(a) onto but not one-to-one

(b) one-to-one but not onto

(c) both onto and one-to-one (don't use the identity functioni as your example)

(d) neither one-to-one nor onto

Solution

Part (a)

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if } x \text{ is even} \\ \frac{x-3}{2} & \text{if } x \text{ is odd} \end{cases}$$

Part (b)

Let f be the function defined by $f = \{(a, 2a+3) \mid a \in \mathbb{Z}\}$.

Part (c)

Let f be the function defined by $f = \{(a, 2-a) \mid a \in \mathbb{Z}\}$.

Part (d)

Let f be the function defined by $f(x) = 1$.

Problem #2

Determine which of the following properties are true and which are false. If the property is true give a proof and if it is false give a counter example. Let $f : X \to Y$ and $A, B \subseteq X$.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) For all subsets A and B of X, $f(A \cup B) = f(A) \cup f(B)$.

(c) For all subsets A and B of X, $f(A \cap B) = f(A) \cap f(B)$.

(d) For a subset C of Y, $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

Solution

Part (a)

Select an arbitrary element $a' \in f(A)$. Then there exists some element $a \in A$ such that $f(a) = a'$. We know $A \subseteq B$ so $a \in B$. Then $a' \in f(B)$ which implies that $f(A) \subseteq f(B)$.

Part (b)

Select an arbitrary element $x' \in f(A \cup B)$. Then there exists an element $x \in A \cup B$ such that $f(x) = x'$. It is either the case that $x \in A$ or $x \in B$ which implies that $f(x) \in f(A)$ or $f(x) \in f(B)$. Thus, in either case, $f(x) \in f(A) \cup f(B)$ and $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now select an arbitrary element $x \in f(A) \cup f(B)$. Then either $x \in f(A)$ or $x \in f(B)$. In the first case there exists an $a \in A$ such that $f(a) = x$ and $a \in A \cup B$ implies that $f(A) \cup f(B) \subseteq f(A \cup B)$. In the second case there exists a $b \in B$ such that $f(b) = x$ and $b \in A \cup B$ implies that $f(A) \cup f(B) \subseteq f(A \cup B)$. Thus $f(A \cup B) = f(A) \cup f(B)$.

Part (c)

Counterexample: Let $A = \{0, 1\}, B = \{1, 2\}$ and the function f be defined by

$$f(0) = 0, f(1) = 1, f(2) = 0$$

Part (d)

Select an arbitrary element $c' \in f^{-1}(\overline{C})$. Then c' is an element of x such that $f(x)$ is not in C. Then c' is an element of $\overline{f^{-1}(C)}$ which is the set of all elements that map to elements of Y that aren't in C. Thus $f^{-1}(\overline{C}) \subseteq \overline{f^{-1}(C)}$.

Now select an arbitrary element c of $\overline{f^{-1}(C)}$. Then c is an element of X such that that $f(c)$ is not an element of C. So $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$.

Combining the two parts, $\overline{f^{-1}(C)} = f^{-1}(\overline{C})$.

Problem #3

Prove or disprove the following statements.

(a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both 1-1 then $f + g$ is also 1-1.

(b) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both onto then $f + g$ is also onto.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ and c is any nonzero real number. If f is 1-1, then $c \cdot f$ is also 1-1.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ and c is any nonzero real number. If f is onto, then $c \cdot f$ is also onto.

Solution

Part (a)

Let f be a function that maps every element in R to itself. Let g be the function that maps every element to itself except that $g(0) = 2$ and $g(2) = 0$. Both of these functions are clearly 1-1. However, $(f + g)(0) = 2 = (f + g)(2)$. So $(f + g)$ is not 1-1.

Part (b)

Let f be the function defined by $f(x) = x$ and let g be the function defined by $g(x) = -x$. These functions are both clearly onto. However, for any $x \in \mathbb{R}$, $(f + g)(x) = f(x) + g(x) = x + (-x) = 0$. Thus $(f + g)$ is not onto.

Part (c)

Let x, y be two elements of $\mathbb{R}$ such that $(c \cdot f)(x) = (c \cdot f)(y)$. This means that

$$\begin{aligned} &c \cdot f(x) = c \cdot f(y) \\ &\implies f(x) = f(y) \\ &\implies x = y \end{aligned}$$

Thus $(c \cdot f)$ is 1-1.

Part (d)

Let y be an arbitrary element of $\mathbb{R}$ such that $(c \cdot f)(x) \neq y$ for any $x \in \mathbb{R}$. Then

$$\begin{aligned} &(c \cdot f)(x) \neq y \\ &\implies c \cdot f(x) \neq y \\ &\implies f(x) \neq y/x \end{aligned}$$

However, y/x is an element of $\mathbb{R}$ and f is onto. Then this is a contradiction and $(c \cdot f)$ must be onto.

Problem #4

Prove the following. Given any set X and given any functions $f : X \to X$, $g : X \to X$, and $h : X \to X$, if h is 1-1 and $h \circ f = h \circ g$, then $f = g$.

Solution

h is 1-1 so it is simple to define a left inverse h^{-1} such that $h^{-1} \circ h$ is the function that maps every element to itself. Define h^{-1} as follows. Let a_0 be an arbitrary fixed element in X. For each x in X, h^{-1} is defined by:

(a) If there is an element y in X such that $h(y) = x$, then $h^{-1}(x) = y$.

(b) If no such element y exists in X, then $h^{-1}(x) = a_0$.

Then

$$\begin{aligned} h \circ f &= h \circ g \\ h^{-1} \circ h \circ f &= h^{-1} \circ h \circ g \\ f &= g \end{aligned}$$

Problem #5

Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) If $g \circ f$ is 1-1, must f and g be 1-1 as well? Prove or give a counter example.

(b) If $g \circ f$ is onto, must f and g be onto as well? Prove or give a counter example.

Solution

Part (a)

Assume that f is not one-to-one. This implies that there exist elements $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. If this is the case then $g(f(a_1)) = g(f(a_2))$ and $(g \circ f)(a_1) = (g \circ f)(a_2)$. This contradicts the given that $(g \circ f)$ is one-to-one and thus f must also be one-to-one.

It is not necessary that g be one-to-one. For example consider the case when $X = \{0, 1\}$ and $Y = Z = \{0\}$. Then defining $f(x) = 0$ and $g(x) = 0$ means that $g \circ f$ is one-to-one but g is not one-to-one.

Part (b)

By definition $(g \circ f)$ being onto means that $\forall c \in C$ there exists an element $a \in A$ such that

$$\begin{aligned} c &= (g \circ f)(a) \\ &= g(f(a)) \end{aligned} \tag{1}$$

Let b be the result of $f(a)$. It is given that $b \in B$.
Therefore for any element $c \in C$ there exists an element $b \in B$ such that $g(b) = c$ and g is onto by definition.

It is not necessary that f be onto. Suppose that X is a proper subset of Y and that $X = Z$. Then letting both f and g be the identity mapping means that $g \circ f$ is onto but f is not onto.

Problem #6

Suppose that f is a function from A to B, where A and B are finite sets with $\mid A \mid = \mid B \mid$. Show that f is one-to-one if and only if it is onto.

Solution

From the problem statement we know that $f(A) \subseteq B$.

First assume that f is one-to-one. This means that for any x and y in A it is not the case that $f(x) = f(y)$. Then $\mid f(A) \mid = \mid A \mid = \mid B \mid$ which implies that $f(A) = B$ and thus f is onto.

Now assume that f is onto. Then $f(A) = B$. Assume that there exist two elements $x, y \in A$ such that $f(x) = f(y)$ and $x \neq y$. Then $\mid f(A) \mid < \mid A \mid$. However, this is impossible because $\mid A \mid = \mid B \mid$. Thus f is one-to-one.

Problem #7

Show that a set S is infinite if and only if there is a proper subset A of S such that there is a one-to-one correspondence between A and S.

Solution

Assume that S is an infinite set containing the elements $s_1, s_2, \ldots$. Then let A be the subset $S \setminus \{s_1\}$. A is a proper subset of S. Define a function $f : S \to A$ by $f(s_x) = s_{x+1}$. It is clear from inspection that this mapping is both 1-1 and onto. Thus it is a one-to-one correspondence.

Now assume that there is a one-to-one correspondence between A and S. If A is a finite set then its cardinality must be less than the cardinality of S. By the pigeonhole principle there can be no one to one correspondence between the two. Thus A must be infinite. It's impossible for a finite set to contain an infinite subset so S must also be infinite.

Problem #8

Let S be a subset of a universal set U. The **characteristic function** f_s of S is the function from U to the set $\{0, 1\}$ such that $f_s(x) = 1$ if x belongs to S and $f_s(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all $x \in U$:

(a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$

(b) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$

(c) $f_{\overline{A}}(x) = 1 - f_A(x)$

(d) $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x)$

Solution

Part (a)

There are three cases:

(a) x is in A and B
$1 = 1 \cdot 1$

(b) x is in one of the two (assume A wlog)
$0 = 1 \cdot 0$

(c) x is in neither
$0 = 0 \cdot 0$

Part (b)

There are three cases:

(a) x is in A and B
$1 = 1 + 1 - 1 \cdot 1$

(b) x is in one of the two (assume A wlog)
$1 = 1 + 0 - 1 \cdot 0$

(c) x is in neither A nor B
$0 = 0 + 0 - 0 \cdot 0$

There are two cases:

(a) x is in A
$0 = 1 - 1$

(b) x is not in A
$1 = 1 - 0$

Part (c)

There are three cases:

(a) x is in A and B
$0 = 1 + 1 - 2 \cdot 1$

(b) x is in one of the two (assume A wlog)
$1 = 1 + 0 - 2 \cdot 0$

(c) x is in neither A nor B
$0 = 0 + 0 - 0 \cdot 0$