Problem # 0

MATH 3354 - Fall 2014

Problem # 1

Assume that $\sqrt{12}$ is rational. Then it can be written in the form $\frac{P}{Q}$ for some integers P and Q. We can assume that this fraction is in lowest terms without loss of generality. Then

$$\sqrt{12} = \frac{P}{Q} = \frac{12}{\sqrt{12}} = \frac{12Q}{P}$$

It is clear that 12 is not a perfect square so each of the above representations has a nonzero fractional component. For $\frac{P}{Q}$ it can be represented as $\frac{q}{Q}$ where $q < Q, q \neq 0$. For $\frac{12Q}{P}$ it can be represented as $\frac{p}{P}$ where $p < P, p \neq 0$. These fractional components must be equal to each other, therefore

$$\frac{p}{P} = \frac{q}{Q} \implies \frac{p}{q} = \frac{P}{Q}$$

This is a contradiction with the assumption that $\frac{P}{Q}$ was in lowest terms, therefore $\sqrt{12}$ is irrational.

Problem # 2

First we show that the statement is true for $n = 2$.

$$a_2 = 2a_{n-1} - 3 = 2(4) - 3 = 5 = 2^1 + 3$$

Now assume that the statement is true for some $k > 1$.

$$a_k = 2^{k-1} + 3$$

and show that this implies truth for $k + 1$.

$$a_{k+1} = 2a_k - 3 = 2(2^{k-1} + 3) - 3 = 2^k - 3$$

Therefore the statement is true for all integers $n > 1$.

Problem # 3

(a) Injective, surjective.

(b) Injective, not surjective.

(c) Not injective, not surjective.

(d) Not injective, surjective.

Problem # 4

(a) True.
By definition of $f^{-1}(B)$, the mappings of each of the elements in that set must be in B so $f(f^{-1}(A)) \subseteq A$.

(b) False.
Let $X = Y = \mathbb{Z}$ and $B = \{1, 2, 3\}$. Let f be defined by $f(x) = 1$. Then $f(f^{-1}(B)) = \{1\}$. B is not a subset of this set.

(c) False.
Let $X = \{0, 1\}$, $A = \{0\}$, and $C = \{1\}$. Let f be defined by $f(x) = 1$. Then $f(A \cap C) = \emptyset$ while $f(A) \cap f(C) = \{1\}$.

(d) True.
For any element $x \in f^{-1}(B \cap D)$, $f(x) \in B$ and $f(x) \in D$ implies that $x \in f^{-1}(B) \cap f^{-1}(D)$. Thus $f^{-1}(B \cap D) \subseteq f^{-1}(B) \cap f^{-1}(D)$.

For any x in $f^{-1}(B) \cap f^{-1}(D)$, $f(x) \in D$ and $f(x) \in B$ implies that $f(x) \in B \cap D$ and s $f^{-1}(B) \cap f^{-1}(D) \subseteq f^{-1}(B \cap D)$.

Therefore $f^{-1}(B \cap D) = f^{-1}(B) \cap f^{-1}(D)$