Problem 1

Let f and g be two real function on the subset E of $\mathbb{R}$. Let $L = \lim_{x\to a} f(x) = f(a)$ and $M = \lim_{x\to a} g(x) = g(a)$. Then by Theorem 3.6 $f(x_n) \to L$ as $n \to \infty$ and $g(x_n) \to M$ as $n \to \infty$ for every sequence $x_n \in I\backslash\{a\}$ which converges to a as $n \to \infty$. By properties of the limits of sequences $f(x_n) + g(x_n) \to L + M \forall x_n \in I\backslash\{a\}$ which converges to a as $n \to \infty$. Thus by 3.6 $\lim f(x) + \lim g(x) = L + M = \lim_{x\to a} f(x) + g(x)$.

Problem 2

If $a = 1$ then this is trivially true. Fix a and n. Consider the case when $a < 1$. Consider the function $x^n - a$. This function is continuous. $f(0) = -a < 0$ and $f(1) = 1 - a > 0$. By the IVT there exists an x such that $f(x) = 0$. Thus $x^n - a = 0 \implies x^n = a$.
Consider the case when $a > 1$ and the continuous function $x^n - a$. Then $f(0) = -a < 0$. $f(a) = a^n - a \geq 0$. So there exists an x such that $f(x) = 0$ by the IVT. So $f(x) = x^n - a = 0 \implies x^n = a$.
Assume that there are two distinct x, y such that $x^n = y^n = a$. Without loss of generality assume that $x > y$. Then $x^n > y^n$ which is a contradiction.

Problem 3

Let $g(x) = f(x) - x$. By the arithmetic properties of continuity it's continuous on $[a, b]$. Then $g(a) \geq 0, g(b) \leq 0$. By IVT there exists a c such that $g(c) = f(c) - c = 0 \implies f(c) = c$.

Problem 4

Part (a)

By definition we can divide I into three sections $I_1 = (-\infty, -c), I_2 = [-c, c], I_3 = (c, \infty)$ where c is the larger of the two absolute values of the two cs provided in the definition of the one sided limit at infinity (Definition 15.1).

I_2 is bounded by EVT.

Given any $\epsilon > 0$, there exists an M such that $\forall x \in (c, \infty)$ such that $x > M, |f(x) - L| < \epsilon$. Fix some $\epsilon = 1$. Then

$|f(x)| = |f(x) - L + L| \leq |f(x)| + L < 1 + L$ thus I_3 is bounded by $\max(f(c), f(c+1), \dots, f(M), 1+L)$

The proof that I_3 is bounded was essentially the same as the proof that any convergent sequence is bounded. This process can be repeated in almost the same way for I_1.

Part (b)

$$y = tan^{-1}(x)$$

Problem 5

Part (a)

Because $f(x)$ is increasing, S is bounded above by $f(a)$ and T is bounded below by $f(a)$.

Part (b)

Let $L = \sup(S)$. Fix some $\epsilon > 0$. By the supremum approximation theorem we can fin an N such that $L - \epsilon < f(N) \leq L$. Now let $\delta = a - N$. Then, because f is increasing, $a - \delta = a - (a - N) = N \in (c, a)$ and $N < x < a \implies$

$$\begin{aligned} & L - \epsilon < f(x) \leq L \\ \implies & L - \epsilon < f(x) < L + \epsilon \\ \implies & |f(x) - L| < \epsilon \end{aligned}$$

Thus $\lim\limits_{x \to a^-} f(x) = L = \sup(S)$.
Let $L = \inf(T)$. Fix some $\epsilon > 0$. By the infimum approximation theorem we can fin an N such that $L \leq f(N) < L + \epsilon$. Now let $\delta = N - a$. Then, because f is increasing, $a - \delta = a - (N - a) = N \in (c, a)$ and $a < x < N \implies$

$$\begin{aligned} & L \leq f(x) < L + \epsilon \\ \implies & L - \epsilon < f(x) < L + \epsilon \\ \implies & |f(x) - L| < \epsilon \end{aligned}$$

Thus $\lim\limits_{x \to a^+} f(x) = L = \inf(T)$.

Part (c)

It has already been shown that both sides of the limit at a exist. If they are not the same then by definition there is a jump dicontinuity at a. Now consider when the limits are the same. If the limits are not equal to $f(a)$ then either $f(a) < \sup(S)$ or $f(a) > \inf(T)$. This is a contradiction. Therefore the limits are equal to $f(a)$ and f is continuous by definition.

Part (d)

Consider an arbitrary discontinuity that occurs at point a. The function f is increasing and by part c) the discontinuity is a jump discontinuity. Therefore $\lim\limits_{x \to a^-} f(x) < \lim\limits_{x \to a^+} f(x)$. By the density of rationals we can select a rational number q in between the two one-sided limits. Furthermore, because f is increasing, we know that q is not located between the one-sided limits at any other discontinuity. Therefore there is a bijective map between the discontinuities and elements of a subset of $\mathbb{Q}$. It was proven previously that $\mathbb{Q}$ is countable, therefore any subset of $\mathbb{Q}$ is at most countable, and so the number of discontinuities is at most countable.

Problem 6

Part (a)

Fix $\epsilon = 0.5$. By the density of rationals and density of irrationals, for any $a \in \mathbb{R}$ for any $\delta > 0$ we can find $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{R} \backslash Q$ such that $a - \delta < x_1, x_2 < a$. However the system of equations $|f(x_i) - L| < 0.5$ has no solution. Thus by definition there is no left sided limit at a. Thus There is a type 2 discontinuity.

Part (b)

Consider any rational number a. Fix $\epsilon > 0$. By the density of rationals we can find some $\frac{1}{r}$ such that $0 < \frac{1}{r} < \epsilon$. Now consider the set of numbers that have the property that they are within 1 of a but their denominator is not greater than r. Divide it into two sets. Let S be the elements that are less than a and T be the elements greater than a. Each of these sets is finite so S has a maximum and T has a minimum. Let $\delta = \min(a - \max(S), \min(T) - a)$. Then for every number x within the punctured neighborhood $(a - \delta, a + \delta) \setminus \{a\}$ there are two cases. Either the number is irrational in which case $|f(x)| = 0 < \epsilon$ or the number is rational. If x is rational, we know by definition of the interval that the denominator of x in its simplified form is larger than r. Therefore $|f(x)| < \frac{1}{r} < \epsilon$. Thus the limit at any point a of the modified dirichlet function exists and is 0. Thus all doscontinuities are removable discontinuities.