Section 4.5

Problem # 28

For an arbitrary subgroup H of the group G, the **normalizer** of H in G is the set $\mathcal{N}(H) = \{x \in G \mid xHx^{-1} = H\}$.

- Prove that $\mathcal{N}(H)$ is a subgroup of G.
- Prove that H is a normal subgroup of $\mathcal{N}(H)$.
- Prove that if K is a subgroup of G that contains H as a normal subgroup, then $K \subseteq \mathcal{N}(H)$.

Solution

Part (a)

(a) $\mathcal{N}(H)$ contains the identity element
$e \in G$ and $eHe^{-1} = H$ so $e \in \mathcal{N}(H)$

(b) $\mathcal{N}(H)$ contains inverses
For any $a \in \mathcal{N}(H)$, $aHa^{-1} = H$. This implies that $a^{-1}Ha = a^{-1}H(a^{-1})^{-1} = H$ and thus $a^{-1} \in \mathcal{N}(H)$.

(c) $\mathcal{N}(H)$ is closed
For any $a, b \in \mathcal{N}(H)$

$$\begin{aligned} abH(ab)^{-1} &= abHb^{-1}a^{-1} \\ &= aHa^{-1} \\ &= H \end{aligned} \tag{1}$$

So $ab \in \mathcal{N}(H)$.

Part (b)

H is a subset of $\mathcal{N}(H)$ because for all $h \in H$ $hHh^{-1} = H$ so $h \in \mathcal{N}(H)$.
We are told that H is a group.
By the definition of $\mathcal{N}(H)$, for all $x \in \mathcal{N}(H), xHx^{-1}$. H is a subgroup of $\mathcal{N}(H)$ so H is normal.

Part (c)

H is a normal subgroup of K. This means that $\forall k \in K, kHk^{-1} = H$. K is a subgroup of G so each k is also a member of G. The group $\mathcal{N}(H)$ is the set of all elements x in G that have the property that $xHx^{-1} = H$. Therefore $K \subseteq \mathcal{N}(H)$.

Problem # 29

Find the normalizer of the subgroup $\{(1), (1,3)(2,4)\}$ of the octic group D_4.

Solution

$$\{(1), (1,3)(2,4)\}$$

Problem # 40

Find the commutator subgroup of each of the following groups.

(a) The quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$.

(b) The symmetric group S_3.

Solution

Part (a)

$$\{1, -1\}$$

Part (b)

$$\{(1), (1,2,3), (1,3,2)\}$$

Section 4.6

Problem # 28

Assume that ϕ is an epimorphism from the group G to the group G'.

(a) Prove that the mapping $H \to \phi(H)$ is a bijection from the set of all subgroups of G that contain $\ker \phi$ to the set of all subgroups of G'.

(b) Prove that if K is a normal subgroup of G', then $\phi^{-1}(K)$ is a normal subgroup of G.

Solution

Part (a)

Let us label the mapping $H \to \phi(H)$ as f. In order to show that f is bijective we will show that the mapping $g : H' \to \phi^{-1}(H')$ is the inverse of f.

It is obvious that $\phi(\phi^{-1}(H')) = H'$. Now we need to show that $\phi^{-1}(\phi(H)) = H$. In order to do this we show that for every g in G, $\phi(g) \in \phi(H)$ implies that $g \in H$. If $\phi(g) \in \phi(H)$ then for some $h \in H$:

$$\begin{aligned} \phi(g) &= \phi(h) \\ &\implies \phi(g)\phi(h)^{-1} = e \\ &\implies \phi(g)\phi(h^{-1}) = e \\ &\implies \phi(gh^{-1}) = e \\ &\implies gh^{-1} \in \ker\phi \subseteq H \\ &\implies gh^{-1} \in H \\ &\implies g \in H \end{aligned} \tag{2}$$

Thus g is the inverse of f and f is a bijection.

Part (b)

Group homomorphisms preserve subgroups so we know that K is a subgroup of G'. For all $k \in \phi^{-1}(K)$ and all $g \in G$, $gkg^{-1} \in \phi^{-1}(K)$. This is equivalent to saying that $\phi(gkg^{-1}) \in K$ or $\phi(g)\phi(k)\phi(g)^{-1} \in K$. We know that $\phi(g)$ and $\phi(g^{-1})$ are both elements of G' and $\phi(k)$ is an element of K. We also know that K is normal in G'. Thus $\phi^{-1}(K)$ is a normal subgroup of G.

Problem # 29

Suppose ϕ is an epimorphism from the group G to the group G'. Let H be a normal subgroup of G containing $\ker\phi$, and let $H' = \phi(H)$.

(a) Prove that H' is a normal subgroup of G'.

(b) Prove that G/H is isomorphic to G'/H'.

Solution

(a) We first prove that H' is a subgroup of G'. It is clear that H' is a subset of G'.

  (i) Identity
  H contains $\ker\phi$ so it must contain e.

  (ii) Closed
  ϕ is onto so for any $a', b' \in H'$ there exist $a, b \in H$ such that $a' = \phi(a)$ and $b' = \phi(b)$. Then:

$$\begin{aligned} a'b' &= \phi(a)\phi(b) \\ &= \phi(ab) \end{aligned} \tag{3}$$

  H is closed so $ab \in H$ and $\phi(ab) \in H'$ so H' is closed.

(iii) Inverses
ϕ is onto so for any $a' \in H'$, there exists an $a \in H$ such that $a' = \phi(a)$. Then:

$$\begin{aligned} a'^{-1} &= \phi(a)^{-1} \\ &= \phi(a^{-1}) \end{aligned} \tag{4}$$

H is a group so $a^{-1} \in H$ and thus $a'^{-1} \in H'$

Thus H' is a subgroup of G'.

We now prove that H' is normal in G'. Let $g' \in G'$ and $h' \in \phi(H)$. So there exists an $h \in H$ such that $\phi(h) = h'$ and because ϕ is an epimorphism there exists $g \in G$ such that $\phi(g) = g'$. Then:

$$\begin{aligned} g'h'g'^{-1} &= \phi(g)\phi(h)\phi(g)^{-1} \\ &= \phi(ghg^{-1}). \end{aligned} \tag{5}$$

H is normal in G so $ghg^{-1} \in H$ and thus $\phi(ghg^{-1} \in H'$. So H' is normal in G'.

(b) Define the mapping $\sigma : G/H \to G'/H'$ by $\sigma(gH) = \phi(g)H'$. First we show that σ is well defined. Assume that for $g_1, g_2 \in G$, $g_1H = g_2H$. Then $g_1^{-1}g_2 = h$ for some $h \in H$. Then:

$$\begin{aligned} \phi(g_2)^{-1}\phi(g_1) &= \phi(g_2^{-1}g_1) \\ &= \phi(h) \end{aligned} \tag{6}$$

$\phi(h) \in \phi(H) = H'$ so $\phi(g_1)H' = \phi(g_2)H'$. Thus:

$$\sigma(g_1H) = \phi(g_1)H' = \phi(g_2)H' = \sigma(g_2H)$$

so σ is well defined. Now we show σ is a homomorphism.

$$\begin{aligned} \sigma(g_1H)\sigma(g_2H) &= \phi(g_1)H'\phi(g_2)H' \\ &= \phi(g_1)\phi(g_2)H' \\ &= \phi(g_1g_2)H' \\ &= \sigma(g_1g_2H) \\ &= \sigma(g_1Hg_2H) \end{aligned} \tag{7}$$

In order to show that σ is one-to-one, assume that for some $g_1, g_2 \in G$, $\sigma(g_1H) = \sigma(g_2H)$. Then $\phi(g_1)H' = \phi(g_2)H'$ which implies that $\phi(g_1)^{-1}\phi(g_2) \in H'$ and $\phi(g_1)^{-1}\phi(g_2) = h'$ for some $h' \in H'$. $H' = \phi(H)$ so there exists an $h \in H$ such that $h' = \sigma(h)$. Thus:

$$\phi(g_1)\phi(g_2) = \phi(h) \tag{8}$$
$$\phi(g_1)\phi(g_2)\phi(h)^{-1} = e' \tag{9}$$
$$\phi(g_1g_2h^{-1}) = e' \tag{10}$$

Therefore, $g_1g_2h^{-1} \in \ker\phi \subseteq H$. Multiplying by h we get $g_1g_2 \in H$ and thus $g_1H = g_2H$ and σ is one-to-one.

Let $g'h'$ be an arbitrary element in G'/H'. σ is onto so there exists an element $g \in G$ such that $\phi(g) = g'$. Then $\sigma(gH) = \phi(g)H' = g'H'$ and σ is onto.

Being a bijective homomorphism, σ is an isomorphism and G/H is isomorphic to G'/H'.

Problem # 30

Let G be a group with center $Z(G) = C$. Prove that if G/C is cyclic, then G is abelian.

Solution

$G/C = \{gC \mid g \in G\}$. G/C is cyclic so there is some element gC that generates G/C. For $a, b \in G$, $a \in (gC)^i = g^iC$, and $b \in (gC)^j = g^jC$. Then for some $c_1, c_2 \in C$, $a = g^ic_1$ and $b = g^jc_2$. Then

$$\begin{aligned}
ab &= g^ic_1g^jc_2 \\
&= g^ig^jc_1c_2 \\
&= g^{i+j}c_1c_2 \\
&= g^{j+1}c_1c_2 \\
&= g^jg^ic_1c_2 \\
&= g^jc_1g^ic_2 \\
&= ba
\end{aligned} \tag{11}$$

Therefore G is abelian.

Problem # 32

Let a be a fixed element of the group G. According to Exercise 20 of Section 3.5, the mapping $t_a : G \to G$ defined by $t_a(x) = axa^{-1}$ is an automorphism of G. Each of these automorphisms t_a is called an **inner automorphism** of G. Prove that the set $\mathrm{Inn}(G) = \{t_a \mid a \in G\}$ forms a normal subgroup of the group of all automorphisms of G.

Solution

Part (a)

It is clear that $\mathrm{Inn}(G)$ is a subset of the group containing all automorphisms of G. First we show that it is a subgroup.

- Identity
  $t_e \in Inn(G)$

- Closed
  For $t_{a_1}, t_{a_2} \in \text{Inn}(G)$:

$$\begin{aligned} t_{a_1} \circ t_{a_2}(g) &= t_{a_1}(t_{a_2}(g)) \\ &= t_{a_1}(a_2 g a_2^{-1}) \\ &= (a_1 a_2 g a_2^{-1} a_1^{-1}) \\ &= (a_1 a_2) g (a_1 a_2)^{-1} \\ &= t_{a_1 a_2}(g) \end{aligned} \tag{12}$$

  G is closed so $a_1 a_2 \in G$ and $t_{a_1 a_2}$ is another inner automorphism.

- Inverses
  For $a \in G$, $t_a(G) = aga^{-1}$. Then:

$$\begin{aligned} t_{a^{-1}} \circ t_a(g) &= t_{a^{-1}}(t_a(g)) \\ &= t_{a^{-1}a}(g) \\ &= t_e(g) \end{aligned} \tag{13}$$

  and

$$\begin{aligned} t_a \circ t_{a^{-1}}(g) &= t_a(t_{a^{-1}}(g)) \\ &= t_{aa^{-1}}(g) \\ &= t_e(g) \end{aligned} \tag{14}$$

  So $\text{Inn}(G)$ contains inverses.

Therefore $\text{Inn}(G)$ is a subgroup of the group of all automorphisms of G.

Part (b)

Now we show that $\text{Inn}(G)$ is normal by demonstrating that for any automorphism $\phi : G \to G$ and any $t_a \in \text{Inn}(G)$, $\phi t_a \phi^{-1} \in \text{Inn}(G)$.

$$\begin{aligned} \phi \circ t_a \circ \phi^{-1}(g) &= \phi(t_a(\phi^{-1}(g))) \\ &= \phi(a\phi^{-1}(g)a^{-1}) \\ &= \phi(a)\phi(\phi^{-1}(g))\phi(a^{-1}) \\ &= \phi(a)h\phi(a^{-1}) \\ &= \phi(a)h[\phi(a)]^{-1} \\ &= t_{\phi(a)} \end{aligned} \tag{15}$$

ϕ is an automorphism so $\phi(a) \in G$ and therefore $\phi t_a \phi^{-1} = t_{\phi(a)} \in \text{Inn}(G)$.

Problem # 34

If H and K are normal subgroups of the group G such that $G = HK$ and $H \cap K = \{e\}$, then G is said to be the **internal direct product** of H and K, and we write $G = H \times K$ to denote this. If $G = H \times K$, prove that $\phi : H \to G/K$ defined by $\phi(h) = hK$ is an ismorphism from H to G/K.

Solution

First we show that ϕ is a homomorphism. For $h_1, h_2 \in H$:

$$\begin{aligned} \phi(h_1 h_2) &= h_1 h_2 K \\ &= h_1 K h_2 K \\ &= \phi(h_1)\phi(h_2) \end{aligned} \tag{16}$$

Now we show that ϕ is one-to-one. For $h_1, h_2 \in H$ assume that $\phi(h_1) = \phi(h_2$. Then $h_1 K = h_2 K$ and $h_2^{-1} h_1 = K$. This implies that $h_2^{-1} h_2 \in K$ and because H is closed it is clear that $h_2^{-1} h_1 \in H$. Therefore $h_2^{-1} h_1 = e$. This implies that $h_1 = h_2$ and thus ϕ is one-to-one. Now we show that ϕ is onto. For any $g \in G$, $g = hk$ for some $h \in H, k \in K$. Thus $gK = hkK = hK = \phi(h)$ and ϕ is onto.

Being a homomorphism that is both one-to-one and onto, ϕ is an isomorphism.

Section 4.7

Problem # 18

(a) Find all subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

(b) Find all subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_6$.

Solution

Part (a)

$$\{(0,0)\}, \{(0,2),(0,0)\}, \{(1,0),(0,0)\}, \{(1,2),(0,0)\}, \{(0,2),(1,0),(1,2),(0,0)\}, \\ \{(0,1),(0,2),(0,3),(0,0)\}, \{(1,1),(0,2),(1,3),(0,0)\}, \mathbb{Z}_2 \oplus \mathbb{Z}_4$$

Part (b)

$\{(0,0)\}, \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\}, \{(0,0),(0,3)\}, \{(0,0),(1,0)\}, \{(0,0),(1,3)\}$
$\{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(1,0),(1,1),(1,2),(1,3),(1,4),(1,5)\}, \{(0,0),(0,3),(1,0),(1,4)\}$
$\{(0,0),(1,0),(1,1),(1,2),(1,3),(1,4),(1,5)\}$