

Problem 1

Prove Claim 3.3 from class: let S be a non-empty subset of \mathbb{R} . Then $\max(S)$ exists (that is, S has a maximal element) $\iff S$ is bounded above and $\sup(S) \in S$.

Solution

First assume that $\max(S)$ exists. Let $M = \max(S)$. By definition $\forall x \in S, x \leq M$, M is an upper bound for S , and $M \in S$. Let z be some upper bound for S and assume that $z < M$. Therefore, by the definition of an upper bound, $\forall x \in S, x < z$. In particular $M < z$. This is a contradiction so $M \leq z$. Then by definition of the supremum, $M = \sup(S) \in S$.

Now let S be bounded above and $M = \sup(S) \in S$. By the definition of the supremum, $x \leq M \forall x \in S$ so $M = \max(S)$ by definition so $\max(S)$ exists.

Problem 2

Let S be a non-empty subset of \mathbb{R} . Let $UB(S)$ be the set of all upper bounds of S (note that this set may be empty) and $LB(S)$ be the set of all lower bounds of S . Also let $-S = \{-s : s \in S\}$

- (i) Let $M \in \mathbb{R}$. Prove that $M = \sup(S)$ if and only if $M = \min(UB(S))$ (the minimal element of $UB(S)$). Also prove that $M = \inf(S)$ if and only if $M = \max(LB(S))$ (the maximal element of $LB(S)$). This is essentially a reformation of the definition of \sup and \inf .
- (ii) Let $y \in \mathbb{R}$. Prove that $y \in UB(S) \iff -y \in LB(-S)$.
- (iii) Deduce from (ii) that $UB(S)$ has a minimum $\iff LB(-S)$ has a maximum, and if they exist, then $\min(UB(S)) = -\max(LB(-S))$.
- (iv) (practice) Combine (i)-(iii) to deduce the reflection principle as formulated in Lecture 4.

Solution

- (i) Let $M = \sup(S)$. Then M is an upper bound of S and $UB(S)$ is non-empty. By definition of a supremum M is at least as small as any other upper bound of S so $\forall x \in UB(S), M \leq x$ so $M = \min(UB(S))$ by definition.

Now let $M = \min(UB(S))$. Thus $UB(S)$ is non-empty and by its presence in the set M must be an upper bound of S . By the definition of a minimum, $M \leq x \forall x \in UB(S)$. Therefore M is less than or equal to all other upper bounds of S so $M = \sup(S)$ by definition.

- (ii) $\forall x \in -S, -x \in S$. For any $y \in \mathbb{R}$, $y \in UB(S) \iff -x \leq y \iff x \geq -y \iff -y \in LB(S)$.

- (iii) Let $m = \min(UB(S))$. Then $\forall x \in UB(S), m \leq x$. By the result of (ii), $-m \in LB(-S)$. So $\forall x \in UB(S), -x \in LB(-S)$ so $m \leq x \implies -m \geq -x$ and $m = \max(LB(-S))$ by definition.
- (iv) Let $S \subseteq \mathbb{R}, S \neq \emptyset$ and let $-S = \{-s : s \in S\}$.
- (a) By (ii) if S is bounded above (and thus $UB(S)$ is non-empty) then $LB(-S)$ is also non-empty and $-S$ is bounded below. By (i) and (iii) $\inf(S) = \max(LB(S)) = -\min(UB(-S)) = -\sup(-S)$.
- (b) If S is bound below ($LB(S) \neq \emptyset$) then by (ii) $UB(-S)$ is non-empty and $-S$ is bounded below. Further, by (i) and (iii) $\inf(S) = \max(LB(S)) = -\min(UB(-S)) = -\sup(-S)$.

Problem 3

Use the Archimedean property to prove that for every real number $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Solution

By the Archimedean property $\exists n \in \mathbb{N}$ such that $n\epsilon > 1 \implies \epsilon > \frac{1}{n}$.

Problem 4

Prove the following result, which can be thought of as a converse of the Approximation Theorem (Theorem 3.2). Let S be a non-empty subset of \mathbb{R} which is bounded above. Let $M \in \mathbb{R}$ be an upper bound for S , and suppose that for all $\epsilon > 0$ there exists $x \in S$ such that $M - \epsilon < x \leq M$. Prove that $M = \sup(S)$.

Solution

In order to prove that $M = \sup(S)$ we want to show that M is an upper bound for S and that M is at least as small as any upper bound for S . The first of these conditions is given. Let z be some upper bound for S . Assume that $z < M$. Then $M - z > 0$ so we can set $\epsilon = M - z$ and it is given that

$$\begin{aligned}
 &\exists x \in S \text{ such that } M - \epsilon < x \leq M \\
 &\implies M - (M - z) < x \leq M \\
 &\implies z < x \leq M
 \end{aligned} \tag{1}$$

However, this implies that there is some element $x \in S$ that is larger than z . This is a contradiction because z is an upper bound for S . Therefore $z \geq M$ and we have shown that $M = \sup(S)$.

Problem 5

Let A and B be non-empty bounded above subsets of \mathbb{R} , and let $A + B = \{a + b : a \in A, b \in B\}$. Prove that $A + B$ is also bounded above and $\sup(A + B) = \sup(A) + \sup(B)$.

Solution

Let $M = \sup(A) + \sup(B)$. In order to prove that $M = \sup(A + B)$ we want to show that $\forall x \in (A + B), x \leq M$ and that M is at least as small as all upper bounds of S .

- i. Select an arbitrary element $x \in A + B$. By definition $x = a + b$ for some $a \in A, b \in B$. For any such a, b $a \leq \sup(A)$ and $b \leq \sup(B)$. Thus $x = a + b \leq \sup(A) + \sup(B) = M$.
- ii. Select some arbitrary $\epsilon > 0$. Then by the approximation property of supremum $\exists a \in A$ such that $\sup(A) - \frac{\epsilon}{2} < a \leq \sup(A)$. Similarly, $\exists b \in B$ such that $\sup(B) - \frac{\epsilon}{2} < b \leq \sup(B)$. So

$$\sup(A) + \sup(B) - \epsilon < a + b \implies \sup(A) + \sup(B) < a + b + \epsilon$$

Let z be some upper bound of S . Then $a + b \leq z \forall a \in A, b \in B$. Then $\sup(A) + \sup(B) < z + \epsilon \implies M = \sup(A) + \sup(B) \leq z$.

Thus M has satisfied the necessary conditions and $M = \sup(A + B)$.

Problem 6

This problem introduces the notions of open and closed subsets of \mathbb{R} . Let S be a subset of \mathbb{R} . We say that S is *open* if for every $x \in S$ there exists $\epsilon > 0$ (which may depend on x) such that $(x - \epsilon, x + \epsilon) \subseteq S$ (thus, for every point of S there is some open interval centered at that point which is entirely contained in S). We say that S is closed if its complement $\mathbb{R} \setminus S$ is open.

- (a) Prove that if S is an open interval (that is, $S = (a, b) = \{x \in \mathbb{R} : a < x < b\}$ for some $a < b$), then S is an open subset of \mathbb{R} . **Hint:** This is merely a reformulation of one of the results in HW#1.
- (b) Prove that if S is a closed interval ($S = [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a \leq b$), then S is a closed subset of \mathbb{R} .

Solution

Part (a)

It is clear from the definition of S that S is a subset of \mathbb{R} . In order to show that it is open we must show that, for arbitrary $x \in S$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq S$. Select any $x \in S$. By the definition of S , $a < x < b$. Then by the result of HW#5.(c) there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset S$. This proves the result.

Part (b)

It is clear from the definition of S that it is a subset of \mathbb{R} . In order to show that $S = [a, b]$ is closed we must show that $S^c \subset \mathbb{R}$ is open. $S^c \subset \mathbb{R} = \{x \in \mathbb{R} : x < a \text{ or } x > b\}$. So the complement of S can be written as the union of the two intervals $(-\infty, a)$ and (b, ∞) . We want to show that for any number x in either of those intervals, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon)$ is a subset of the particular interval. First select an x from $(-\infty, a)$. By definition, $x < a$ so let $\epsilon = \frac{a-x}{2}$. Then $x - \frac{a-x}{2} > -\infty$ and $x + \frac{a-x}{2} < a$. Now select an x from (b, ∞) . By definition $x > b$ so let $\epsilon = \frac{x-b}{2}$. Then $x - \frac{x-b}{2} > b$ and $x + \frac{x-b}{2} < \infty$. Thus $S^c \subset \mathbb{R}$ is an open subset of \mathbb{R} so S is a closed subset of \mathbb{R} .

Problem 7

- (a) Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Recall that (X, d) is a metric space by HW#1.6(b). Prove that $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$ for all $x \in X$ and $\epsilon > 0$ (thus, an open ball of radius ϵ centered at x in this case is simply the open interval of length 2ϵ centered at x). **Hint:** The result follows directly from basic properties of absolute values.
- (b) Now let $X = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, and define functions $d : X \times X \rightarrow \mathbb{R}$ and $D : X \times X \rightarrow \mathbb{R}$ by setting $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $D((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Describe the open ball $B_\epsilon((x, y))$ in each of these two metric spaces.

Solution**Part (a)**

WTS: $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ is equal to $(x - \epsilon, x + \epsilon) = \{y \in \mathbb{R} : x - \epsilon < y < x + \epsilon\}$. We know $X = \mathbb{R}$ and $d(x, y) = |x - y|$ so $B_\epsilon(x)$ can be rewritten as:

$$\{y \in \mathbb{R} : |x - y| < \epsilon\}$$

by properties 2 and 6 of absolute values:

$$\begin{aligned} \{y \in \mathbb{R} : |x - y| < \epsilon\} &= \{y \in \mathbb{R} : |y - x| < \epsilon\} \\ &= \{y \in \mathbb{R} : -\epsilon < y - x < \epsilon\} \\ &= \{y \in \mathbb{R} : x - \epsilon < y < x + \epsilon\} \end{aligned} \tag{2}$$

So the sets are equivalent.

Part (b)

In the first space it forms a disk. It selects all points less than a given cartesian distance from a central point. In the second space it forms a filled in square. These are shown in the picture below.

Problem 8

Prove that if S is any open ball in X (that is, $S = B_\alpha(y)$ for some $y \in X$ and $\alpha > 0$, then S is an open subset of X .

Solution

$S = B_\alpha(y) = \{z \in X : d(y, z) < \alpha\}$. Select an arbitrary $x \in B_\alpha$. We want to show that there exists some $\epsilon > 0$ such that $B_\epsilon(x) = \{z \in X : d(x, z) < \epsilon\} \subseteq S$.

Let $\epsilon = \alpha - d(y, x)$. By the definition of B_α $d(y, x)$ must be less than α so ϵ is positive. For any $z \in B_\epsilon$ we know that $d(x, z) < \epsilon$ and in order to show that $B_\epsilon \subseteq S$ we want to show that $d(y, z) < \alpha$. This can be done using the third property of metric spaces given in the previous homework (the triangle inequality).

$$d(x, z) < \epsilon \implies d(x, z) < \alpha - d(y, x) \quad (3)$$

$$\implies d(y, x) + d(x, z) < \alpha \quad (4)$$

$$\implies d(y, z) < \alpha \quad (5)$$

$$\implies B_\epsilon \subseteq S \quad (6)$$