Problem 1

Part (a)

Fix $\epsilon > 0$. We want $\delta > 0$ such that $|x-a| < \delta \implies |f(x) - f(a)| < \epsilon \, \forall x, a \in [1,2]$. Assume $|x-a| < \delta$ for some δ. Then $|x^3 - a^3| = |x-a||x^2+ax+a^2 \; < \delta|x^2+ax+a^2|$. Since $x, a \in [1,2]$, $|x^2+ax+a^2| \leq 12$. Therefore $12\delta = \epsilon$ if $\delta = \frac{\epsilon}{12}$. So $\delta = \frac{\epsilon}{12}$.

Part (b)

Let $\delta = \epsilon^2$. Then, noting that $|\sqrt{x} - \sqrt{a}| \leq |\sqrt{x} + \sqrt{a}|$, we have

$$|\sqrt{x} - \sqrt{a}|^2 \leq |\sqrt{x} - \sqrt{a}||\sqrt{x} + \sqrt{a}| = |x-a| < \delta$$

implies $|\sqrt{x} - \sqrt{a}| < \epsilon$.

Problem 2

Assume that $f(x)$ is uniformly continuous on $(0,1)$. Fix $\epsilon = 1$. We can assume that $\delta < 1$. Then consider the points $x = \frac{\delta}{2}$ and $a = \delta$. $|x - a| = \left|\frac{\delta}{2} - \delta\right| = \frac{\delta}{2} < \delta \implies \left|\frac{1}{x} - \frac{1}{a}\right| < 1$. However, $\left|\frac{2}{\delta} - \frac{1}{\delta}\right| = \left|\frac{1}{\delta}\right| > 1$.

Problem 3

Part (a)

Both f and g are uniformly continuous on E so by definition, for all $\epsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that $|f(x) - f(a)| < \frac{\epsilon}{2}$ and $|g(x) - g(a)| < \frac{\epsilon}{2}$ for all $x, a \in E$ such that $|x-a| < \delta_1$ and $|x-a| < \delta_2$ respectively. Let $\delta = \min(\delta_1, \delta_2)$. Then

$$|f(x) + g(x) - (f(a) + g(a))| \leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Part (b)

Both f and g are bounded so there exist $C, D \in \mathbb{R}$ such that $\forall x \in E \; f(x) \leq C$ and $g(x) \leq D$. Then

$$\begin{aligned}
|f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\
&\leq |f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| \\
&\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\
&\leq C|g(x) - g(a)| + D|f(x) - f(a)|
\end{aligned}$$

We want this last sum to be less than ϵ. Therefore, for any given $\epsilon > 0$, select $\delta = \min(\delta_1, \delta_2)$ for δ_1 and δ_2 that satisfy the requirement of uniform continuity for f and g for $\epsilon = \min(\frac{\epsilon}{2C}, \frac{\epsilon}{2D})$.

Part (c)

Let $E = \mathbb{R}, f(x) = g(x) = x$.

Part (d)

Let $E = \mathbb{R}, f(x) = \sin(x), g(x) = x$.

Problem 4

i Pick any point $y \in [a, b]$. If $g(y) = 1$ then by the sign preservation lemma there exists $\delta > 0$ such that $g(x) > 0$ for all $x \in (a - \delta, a + \delta) \cap E$. Since there is only one possible positive value for g, $|g(x) - g(y)| = |1 - 1| = 0 < \epsilon$ for all positive ϵ. An almost identical argument can be made for the case when $g(y) = -1$. Therefore g is continuous at all points in $[a, b]$.

ii For any n let S be the set of $x_{i,n}$ such that $g(x_{i,n}) = 1$. This set is nonempty because it contains b and finite. Pick the minimum element $x_{k+1,n}$. Then $x_{k,n}$ is defined because $x_{0,n} \notin S$ and $x_{k,n} = 1$ by definition.

iii Let $\epsilon = 1$. For any proposed δ we can pick n such that $\frac{b-a}{n} < \delta$. Thus for the point $x_{k,n}$ chosen in part ii) $x_{k+1,n} \in (x_{k,n} - \delta < x_{k,n} + \delta) \cap E$ but $|f(x_{k,n}) - f(x_{k+1,n})| = 2 > \epsilon$.

Problem 5

By Lemma 3.38 from the textbook $f(x_n)$ is Cauchy. By Theorem 10.2 from the textbook, $f(x_n)$ converges.

Problem 6

(a) This follows directly from 5).

(b) Given $\epsilon > 0$, choose $\delta > 0$ such that $|x - a| < \delta and x, a \in E \implies |f(x) - f(a)| < \epsilon$ is satisfied. Since $x_n - y_n \to 0$, choose $N \in \mathbb{N}$ so that $n \geq N$ implies $|x_n - y_n| < \delta$. Then $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq N$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)| \leq \epsilon$$

for all $\epsilon > 0$. It follows from Theorem 1.9 that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)|$$

(c) If $x \in \mathbb{Q}$ we can choose the constant sequence $x_n = x$ which clearly has the limit x_n. Then $F(x) = f(x)$ and thus F is an extension of f.