## Problem #1

### Solution

If the power sets of natural numbers is countable then all of its elements can be written down as $A_1, A_2, A_3, \ldots$. Now construct a set $S = \{i \in \mathbb{N} \mid i \notin A_i\}$. It is clear that $S$ must be different from every $A_i$ listed. Thus the power set of natural numbers is not countable because there is no one-to-one mapping to it from the natural numbers.

## Problem #2

### Solution

### Part (a)

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

### Part (b)

$$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$$

### Part (c)

$$a_n = 3 \cdot 2^{n-1}$$

## Problem #3

### Solution

### Part (a)

Let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

First we show that it is true for $n = 1$.

$$\begin{aligned} 1 &= 1^3 \\ &= (1(1+1)/2)^2 \qquad (1) \\ &= 1 \end{aligned}$$

Assume that $P(k)$ is true for some arbitrarily chosen but fixed integer $k \geq 1$. Now we show that $P(k) \implies P(k+1)$

$$\begin{aligned}
P(k+1) &= 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4(k^3 + 3k^2 + 3k + 1)}{4} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2
\end{aligned} \tag{2}$$

Thus $P(n)$ is true for all integers $n \geq 1$.

**Part (b)**

Let $P(n)$ be the statement that

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

First we show that it is true for $n = 0$.

$$\begin{aligned}
2 &= 1 \cdot 2^1 \\
&= 1 \cdot 0 + 2 \\
&= 2
\end{aligned} \tag{3}$$

Assume that $P(k)$ is true for some arbitrarily chosen but fixed integer $k \geq 1$. Now we show that $P(k) \implies P(k+1)$

$$\begin{aligned}
P(k+1) &= 2^1 + 2 \cdot 2^2 + \ldots + k \cdot 2^k + (k+2)(2^{k+2}) \\
&= k \cdot 2^{k+2} + 2 + (k+2)2^{k+2} \\
&= k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2 \cdot 2^{k+2} \\
&= 2k \cdot 2^{k+2} + 2 + 2^{k+3} \\
&= k \cdot 2^{k+3} + 2^{k+3} + 2 \\
&= (k+1)(2^{(k+1)+2}) + 2
\end{aligned} \tag{4}$$

Thus $P(n)$ is true for all integers $n \geq 1$.

**Part (c)**

Let $P(n)$ be the statement that

$$n! < n^n$$

First we show that it is true for $n = 2$.

$$2 = 2! < 4 = 2^2 \tag{5}$$

Assume that $P(k)$ is true for some arbitrarily chosen but fixed integer $k > 2$. Now we show that $P(k) \implies P(k+1)$. To do this we need to show that $(k+1)! < (k+1)^{k+1}$.

$$\begin{aligned} &(k+1)! < (k+1)^{k+1} \\ \implies\ & k!(k+1) < (k+1)(k+1)^k \\ \implies\ & k! < (k+1)^k \end{aligned} \tag{6}$$

By induction we know that $k! < k^k$ so clearly $k! < (k+1)^k$. Thus $P(n)$ is true for all integers $n \geq 1$.

### Part (d)

Let $P(n)$ be the statement that

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

We know by De Morgan's law that this is true for $n = 2$. Assume that it is true for all $n < k$ for some arbitrary integer $k \geq 2$ and consider the case when $n = k$.

$$\begin{aligned} \overline{\bigcup_{k=1}^{n} A_k} &= \overline{A_1 \cup A_2 \cup \ldots A_k \cup A_{k+1}} \\ &= \overline{(A_1 \cup A_2 \cup \ldots A_k) \cup A_{k+1}} \\ &= \overline{(A_1 \cup A_2 \cup \ldots \cup A_k)} \cap \overline{A_{k+1}} \\ &= \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_k} \cap \overline{A_{k+1}} \\ &= \bigcap_{k=1}^{n} \overline{A_k} \end{aligned} \tag{7}$$

Thus we have shown by induction that $\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$ for all integers $n$ greater than 2.

## Problem #4

### Solution

I claim that in order to obtain $n$ seperate square requires $n-1$ breaks. The case when $n = 1$ is clearly true, no breaks are required. Now assume that the statement is true for all $2 \leq n < k$ for some arbitrarily chosen integer $k > 2$. Now consider the case when the chocolate bar is made of $k$ squares. First break it into two pieces, one with $a$ pieces and the other with $k - a$ pieces. Then by the inductive hypothesis the first piece requires $a - 1$ breaks to obtain $a$ pieces and the second piece requires $k - a - 1$ breaks to obtain $k - a$ pieces. Then the total number of breaks is $1 + (a-1) + (k-a-1) = k-1$. So the claim is true for all positive integers greather than or equal to 1.

## Problem #5

### Solution

The flaw occurs when considering the case that $j = 0$. Then the reference to the term $a^{j-1}$ is invalid because the inductive hypothesis only tells us that $a^k = 1$ for nonnegative integers $k$.

## Problem #6

### Solution

There are 72 different types of this shirt.

## Problem #7

### Solution

(a) Numbers divisible by seven. $\lfloor 999/7 \rfloor = 142$

(b) Numbers divisible by both 7 and 11 are divisible by 77. $142 - \lfloor 999/77 \rfloor = 130$

(c) This is from the part above. $\lfloor 999/77 \rfloor = 12$

(d) There are $\lfloor 999/11 \rfloor = 90$ multiples of 11 less than 1000. Then the number of numbers divisible by 7 or 11 but not both is $142 + 90 - 12 = 220$

(e) We know there are 130 things divisible by 7 and not by 11 and $90 - 12 = 78$ things divisible by 11 but not 7. Then $130 + 78 = 208$.

(f) There are $9 \cdot (9 \cdot 9) \cdot (9 \cdot 9 \cdot 8) = 738$ such numbers.

(g) There are $4 + (4 \cdot 4 + 5 \cdot 5) + (4 \cdot 4 \cdot 8 + 5 \cdot 5 \cdot 8) = 373$.

## Problem #8

### Solution

(a) $2^8 = 256$

(b) $8!/(3!5!) = 56$

(c) $256 - 8!/(0!8!) - 8!/(1!7!) - 8!/(2!6!) = 219$

(d) $8!/(4!4!) = 80$

## Problem #9

### Solution

There are $45!/(3!42!)$ ways to select the three countries from the block of 45. There are $57!/(4!53!)$ ways to select the four countris from the block of 57. There are $69!/(5!64!)$ ways to select the five countries from the block of 69. So in total there are $6.29940220356447 \times 10^{16}$ different ways to select the countries.

## Problem #10

### Solution

The first possibility is that there is one man and five women. The number of ways to do this is $10!/(9!1!) \cdot 15!/(10!5!) = 30030$.
The second possibility is that there are two men and four women. The number of ways to do this $10!/(8!2!) \cdot 15!/(11!4!) = 61425$.
So in total there are $30030 + 61425 = 91455$ different ways to form such a committee.

## Problem #11

### Solution

There are $25!/22! = 13,800$ ways to distribute the awards.