Problem 1

Let V and W be two finite-dimensional vector spaces, $S = \{x_1, \ldots, x_n\}$ a linearly independent subset of V with n distinct elements (i.e. $|S| = n$) and $(y_1, \ldots, y_n)$ an arbitrary n-tuple of vectors in W. Prove or disprove: There exists a linear map $T : V \to W$ with $T(x_i) = y_i$ for all $i = 1, \ldots, n$. Remark: "Disprove" means "give a counter-example" which shows that the claim is wrong (if that is what you think).

Solution

S is a linearly independent subset of V. Thus we can expand S to $S' = \{x_1, \ldots, x_n, \ldots, x_m\}$ where S' is a basis of V. Then define the map $R : S' \to W$ by:

$$R(x_i) = \begin{cases} y_i & \text{if } i \leq n \\ y_1 & \text{otherwise} \end{cases}$$

By proposition 2.1.5 there exists a linear transformation $T : V \to W$ with $T_{|S'} = R$. In particular, $T(x_i) = y_i$ for all $i = 1, \ldots, n$.

Problem 2

Give an alternative proof of Theorem 1.5.2 which does NOT use the second isomorphism theorem. Start with the hint for Exercise 29(a) given in [FIS] on page 57.
Statement of 1.5.2: If U, W are finite-dimensional subspaces of V, then $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

Solution

Let $B = \{r_1, \ldots, r_k\}$ be a basis for $U \cap W$. It can be extended to a basis $\{r_1, \ldots, r_k, u_1 \ldots, u_m\}$ for U and to a basis $\{r_1, \ldots, r_k, w_1, \ldots, w_p\}$ of W. The claim is that the union of these bases forms a basis for $U + W$. Assume that:

$$\sum_{i=1}^{k} a_i r_i + \sum_{i=1}^{m} b_i u_i + \sum_{i=1}^{p} c_i w_i = 0$$

Then define v by:

$$v = \sum_{i=1}^{m} b_i u_i = -\sum_{i=1}^{k} a_i r_i - \sum_{i=1}^{p} c_i w_i$$

We know that $\sum_{i=1}^{m} b_i u_i \in U$ and $-\sum_{i=1}^{k} a_i r_i - \sum_{i=1}^{p} c_i w_i \in W$. Thus $v \in U \cap W$ so we can write it as $\sum_{i=1}^{k} d_i r_i$. Then

$$\sum_{i=1}^{k} d_i r_i - \sum_{i=1}^{m} b_i u_i = 0$$

Each r_i, u_i is an element of the basis for U. If v is nonzero then clearly b_i must be nonzero for some i. However, this would imply that the sum above has a nonzero coefficient which

means that the basis for U is not linearly independent. Thus $v = 0$. Then by definition of bases we know that $a_i = b_i = c_i = 0$ for all i. Thus the proposed basis for $U + W$ is linearly independent.
Select any $v \in U + W$. v can be written as $u + w$ for some $u \in U$ and $w \in W$.

$$u = \sum_{i=1}^{k} a_i r_i + \sum_{i=1}^{m} b_i u_i \text{ and } q = \sum_{i=1}^{k} c_i r_i + \sum_{i=1}^{p} d_i w_i$$

so

$$v = \sum_{i=1}^{k} (a_i + c_i) r_i + \sum_{i=1}^{m} b_i u_i + \sum_{i=1}^{p} d_i w_i$$

Thus the proposed basis spans $U + W$ and is truly a basis. There are $k + m + p$ elements in the basis so

$$\dim(U + W) = k + m + p = (k + m) + (k + p) - k = \dim(U) + \dim(W) - \dim(U \cap W)$$

Problem 3

Prove the third isomorphism theorem, which is Theorem 2.2.10 from class. Do NOT assume that the vector spaces involved are finite-dimensional.

Solution

Consider $T : V/U \to V/W$ defined by $x + U \mapsto x + W$.

- Well Defined
  Assume that $x + U = y + U$. By equivalent conditions for equality of cosets $x - y \in U$. We know $U \subseteq W$ so $x - y \in W \implies x + W = y + W$. Thus $x + U = y + U \implies T(x) = T(y)$.

- Linear
  $T((x+U)+(y+U)) = T(x+y+U) = x+y+W = x+W+y+W = T(x+U)+T(y+U)$
  $T(c(x + U)) = T(cx + U) = cx + W = c(x + W) = cT(x + U)$

Now we determine $N(T)$. $T(x + U) = 0 + W \iff x \in W$ so $N(T) = W/U$. Next we find $R(T)$. By definition $R(T) \subseteq V/W$. If $v + W$ is an arbitrary element of V/W then $v + W = T(v + U)$ where $v + U \in V/U$. Thus $R(T) = V/W$. Apply the first isomorphism theorem to conclude that $V/U/W/U \cong V/W$.

Problem 4

Part (a)

Prove that there exists a linear transformation $T : R^2 \to R^3$ such that $T(l, l) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?

Part (b)

Is there a linear transformation $T : R^3 \to R^2$ such that $T(l, 0, 3) = (l, 1)$ and $T(-2, 0, -6) = (2, l)$?

Solution

(a) First show that $\{b_1 = (1, 1), b_2 = (2, 3)\}$ is a basis for R^2. They are not multiples of each other so they are linearly independent. The dimension of R^2 is 2 so they must be a basis. By proposition 2.1.5 there does exist a map T with the desired properties. We can see that $(8, 11) = 2b_1 + 3b_2$. Thus by the construction of T from the proof of 2.1.5 we know that $T(8, 11) = 2T(b_1) + 3T(b_2) = 2(1, 0, 2) + 3(1, -1, 4) = (5, -3, 16)$.

(b) It is not possible. Assume T is some linear map with the properties given above. Then $T(0) = T(2(1, 0, 3) + (-2, 0, -6)) = 2T(1, 0, 3) + T(-2, 0, 6) = (4, 2) \neq (0, 0)$ Thus T cannot be linear.

Problem 5

Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W, If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

Solution

Assume that $a_1T + a_2U = 0$. This implies that for any $v \in V$:

$$a_1T(v) = -a_2U(v) \implies T(a_iv) = U(-a_2u)$$

By definition $T(a_iv) \in R(T)$ and $U(-a_2u) \in R(U)$. We know $R(T) \cap R(U) = \{0\}$ so $T(a_iv) = U(-a_2u) = 0$. If a_1 and a_2 are not equal to 0 then $\forall v \in V, T(v) = U(v) = 0$. However, we know that T and U are nonzero. Thus $a_1 = a_2 = 0$ and so U and T are linearly independent.

Problem 6

For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $R^2 \to R^3$ defined by $T(a_1, a_2) = (a_12a_2, a2, 3a_1 + 4a_2)$.

(c) $R^2 \to R^3$ defined by $T(a_1, a_2, a_3) = (3a_12a_3, a2, 3a_1 + 4a_2)$.

(d) $P_3(R) \to P_3(R)$ defined by $T(p(x)) = p'(x)$.

(f) $M_{2\times2}(R) \to M_{2\times2}(R)$ defined by $T\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+b & a \\ c & c+d \end{smallmatrix}\right)$.

Solution

(a) T is not invertible because it is not surjective. If $T(a_1, a_2) = (1, 0, 1)$ this implies $a_2 = 0 \implies a_1 = 1$ and $a_1 = \frac{1}{3}$ which is impossible.

(c) Assume that $T(x) = 0$. Then $a_2 = 0 \implies a_3 = 0 \implies a_2 = 0 \implies T$ is injective. For any $(x, y, z) \in R^3$ let $a_1 = \frac{z-4y}{3}, a_2 = y, a_3 = \frac{z-4y-x}{2}$. Then $T(a_1, a_2, a_3) = (x, y, z)$. Thus T is onto which means that T is bijective and thus invertible.

(d) $T(3) = T(4)$. Thus T is not injective and so it cannot be invertible.

(f) $T(x) = 0 \implies c = a = 0 \implies b = d = 0$. Thus T is injective. For any $M = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in M_{2\times 2}(R)$ we have $T\left(\begin{smallmatrix} x & w-x \\ y & z-y \end{smallmatrix}\right) = M$. Thus T is onto which means that it is bijective and thus invertible.