Section 4.4

In Exercises 7 and 8, let G be the multiplicative group of permutation matrices $\{I_3, P_3, P_3^2, P_1, P_4, P_2\}$ in Example 6 of Section 3.5.

Problem #7

Let H be the subgroup of G given by

$$H = \{I_3, P_4\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

(a) Find the distinct left cosets of H in G, write out the elements, partition G into left cosets of G, and give $[G : H]$.

(b) Find the distinct right cosets of G in G, write out their elements, and partition G into right cosets of H.

Solution

Part (a)

The distinct left cosets of H in G are

$$I_3H = H \qquad P_1H = \{P_1, P_3^2\} \qquad P_2H = \{P_2, P_3\} \tag{1}$$

So $G = H \cup P_1H \cup P_2H$ and $[G : H] = 3$.

Part (b)

The distinct right cosets of H in G are

$$HI_3 = H \qquad HP_1 = \{P_1, P_3\} \qquad HP_2 = \{P_2, P_3^2\} \tag{2}$$

So $G = H \cup HP_1 \cup HP_2$.

Problem #8

Let H be the subgroup of G given by

$$H = \{I_3, P_3, P_3^2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

(a) Find the distinct left cosets of H in G, write out their elements, partition G into left cosets of H, and give $[G : H]$.

(b) Find the distinct right cosets of H in G, write out their elements, and partition G into right cosets of H.

Solution

Part (a)

The distinct left cosets of H in G are

$$I_3H = H \qquad P_1H = \{P_1, P_2, P_4\} \tag{3}$$

So $G = H \cup HP_1$ and $[G : H] = 2$.

Part (b)

The distinct right cosets of H in G are

$$HI_3 = H \qquad HP_1 = \{P_1, P_2, P_4\} \tag{4}$$

So $G = H \cup HP_1$.

Problem #9

Let H be a subgroup of a group G with $a, b \in G$. Prove that $aH = bH$ if and only if $a \in bH$.

Solution

Part (a)

Let $aH = bH$ and let x be some element in aH. x is also in bH. $x = ah_1$ and $x = bh_2$.

$$\begin{aligned} xh_1^{-1} &= ah_1h_1^{-1} & a &= (bh_2)h_1^{-1} \\ &= a & &= b(h_2h_1^{-1}) \end{aligned} \tag{5}$$

The product $h_2h_1^{-1}$ is in H because H is closed and therefore $a \in bH$ because it can be written in the form $a = bh$ for some $h \in H$.

Part (b)

Now let $a \in bH$. Then for some $h_1 \in H$:

$$a = bh_1 \tag{6}$$
$$b = ah_1^{-1} \tag{7}$$

Now select some arbitrary y in bH. Then for some arbitrary h_2 in H:

$$y = bh_2 \tag{8}$$
$$= (ah_1^{-1})h_2 \tag{9}$$

The product $h_1^{-1}h_2$ is in H because H is closed. Thus $bH \subseteq aH$
Take an arbitrary $z \in aH$. Then for some h_3 in H:

$$\begin{aligned} z &= ah_3 \\ &= (bh_1)h_3 \end{aligned} \tag{10}$$

The product h_1h_3 is in H because H is closed and so $aH \subseteq bH$. Thus $aH = bH$.

Problem #10

Let H be a subgroup of a group G with $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$

Solution

Part (a)

Let $aH = bH$. By the result of Problem #9 $b \in aH$. So b can be written as ah_1 for some h_1 in H.

$$\begin{aligned} a^{-1}b &= a^{-1}(ah_1) \\ &= eh_1 \\ &= h_1 \end{aligned} \tag{11}$$

So $a^{-1}b \in H$.

Part (b)

Let $a^{-1}b$ be an element of H. Then $a^{-1}b = h$ for some $h \in H$. Then $aa^{-1}b = ah$ so $b = ah$ and $b \in aH$. Then by the result of Problem #9 $aH = bH$.

Problem #18

Let G be a group of finite order n. Prove that $a^n = e$ for all a in G.

Solution

It is known from previous chapters that for finite groups every element of the group generates a subgroup and that for an element a of the group, the order of the subgroup generated by a is k where k is the smallest positive integer such that $a^k = e$. By Lagrange's Theorem we also know that the order of the subgroup, k, divides the order of the group, n. k cannot be larger than n because that would imply that the subgroup has more elements than the group of which it is a subgroup. If $k = n$ then the statement is trivially true. If $k < n$ then by Lagranges theorem $n = qk$ for some integer q. Then:

$$a^n = a^{qk} \tag{12}$$
$$= (a^k)^q \tag{13}$$
$$= e^q \tag{14}$$
$$= e \tag{15}$$

So in all cases $a^n = e$.

Problem #19

Find the order of each of the following elements in the multiplicative group of units U_p.

(a) $[2]$ for $p = 13$

(b) $[5]$ for $p = 13$

(c) $[3]$ for $p = 17$

(d) $[8]$ for $p = 17$

Solution

Part (a)

The order of $[2]$ in U_{13} is 12

Part (b)

The order of $[5]$ in U_{13} is 4

Part (c)

The order of $[3]$ in U_{17} is 16

Part (d)

The order of $[8]$ in U_{17} is 8

Problem #20

Find all subgroups of the octic group D_4.

Solution

The subgroups of D_4 are as follows:

$$\begin{array}{llll}
\{e\} & \{e,\beta\} & \{e,\gamma\}, & \{e,\delta\} \\
\{e,\theta\}, & \{e,\alpha^2\}, & \{e,\alpha,\alpha^2,\alpha^3\}, & \{e,\alpha^2,\beta,\delta\} \\
\{e,\alpha^2,\gamma,\theta\}, & D_4 & &
\end{array}$$

Problem #21

Find all subgroups of the alternating group A_4.

Solution

The subgroups of A_4 are as follows:

$$\begin{array}{lll}
\{e\}, & \{e,(12)(34)\}, & \{e,(13)(24)\} \\
\{e,(14)(23)\}, & \{e,(12)(34),(13)(24),(14)(23)\} & \{e,(234),(243)\} \\
\{e,(134),(143)\}, & \{e,(124),(142)\} & \{e,(123),(132)\} \\
A_4 & &
\end{array}$$

Problem #31

A subgroup H of the group S_n is called transitive on $B = \{1, 2, \ldots, n\}$ if for each pair i, j of elements of B there exists an element $h \in H$ such that $h(i) = j$. Suppose G is a group that is transitive on $\{1, 2, \ldots, n\}$, and let H_i be the subgroup of G that leaves i fixed:

$$H_i = \{g \in G \mid g(i) = i\}$$

for $i = 1, 2 \ldots, n$. Prove that $\mid G \mid = n \cdot \mid H_i \mid$.

Solution

For H_1 consider two permutations, $\sigma, \tau \in G$, that map 1 to i. Then the composition $\tau^{-1}\sigma$ maps 1 to 1 and is an element of G (G is closed) and is thus an element of H_1. By the result of problem # 10, $\sigma H_1 = \tau H_1$. There are n possible locations for a permutation in G to map 1 to and we know, because G is transitive on B, that there is at least one element of G that maps 1 to that location. Therefore because each of these locations clearly creates a different coset and permutations that map 1 to the same element create the same coset, there are n distinct cosets. By the same arguments the above statements hold for any H_j for j in the range 1 to n. Thus $[G : H_j] = n$ for all j in that range. By Lagrange's theorem $\mid G \mid = n \cdot \mid H_i \mid$.

Section 4.5

Problem #14

Find groups H and G such that $H \subseteq G \subseteq A_4$ and the following conditions are satisfied:

(a) H is a normal subgroup of G.

(b) G is a normal subgroup of A_4.

(c) H is not a normal subgroup of A_4.

(Thus the statement "A normal subgroup of a normal subgroup is a normal subgroup" is false.)

Solution

$$G = \{e, (12)(34), (13)(24), (14)(23)\} \qquad H = \{e, (12)(34)\} \tag{16}$$

Problem #15

Find groups H and K such that the following conditions are satisfied:

(a) H is a normal subgroup of K.

(b) K is a normal subgroup of the octic group.

(c) H is a not a normal subgroup of the octic group

Solution

$$K = \{e, \gamma, \alpha^2, \theta\} \qquad H = \{e, \gamma\} \tag{17}$$

Problem #25

Find the center of the octic group D_4.

Solution

$$Z(D_4) = \{e, \alpha^2\}$$

Problem #26

Find the center of A_4.

Solution

$$Z(A_4) = \{e\}$$

Problem #27

Suppose H is a normal subgroup of order 2 of a group G. Prove that H is contained in $Z(G)$, the center of G.

Solution

We can write $H = \{e, a\}$ for $a \neq e \in G$. We know that H is normal so therefore $gHg^{-1} = H$ for all $g \in G$. For any $g \in G$, $geg^{-1} = e$ and so for H to be normal gag^{-1} must equal a. Then for all g in G:

$$gag^{-1} = a \tag{18}$$
$$gag^{-1}g = ag \tag{19}$$
$$ga = ag \tag{20}$$

a commutes with every element of G and a is an element of G so $a \in Z(G)$.

Section 4.6

Problem #18

If H is a subgroup of the group G such that $(aH)(bH) = abH$ for all left cosets aH and bH of H in G, prove that H is normal in G.

Solution

For any $x \in G$:

$$\begin{aligned} H &= (xx^{-1})H \\ &= (xH)(x^{-1}H) \end{aligned} \tag{21}$$

This means that for any $h_1, h_2 \in H$. There exists some $h \in H$ such that $h = xh_1x^{-1}h_2$. Namely, this is true for $h_2 = e$. This means that $xh_1x^{-1} \in H$ for all h_1. This is equivalent to saying that H is normal in G.

Problem #27

(a) Show that a cyclic group of order 8 has a cyclic group of order 4 as a homomorphic image.

(b) Show that cyclic group of order 6 has a cyclic group of order 2 as a homomorphic image.

Solution

Part (a)

Let G be a cyclic group of order 8. $G = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$ for some element $a \in G$. Let H be the normal subgroup generated by $a^4 = \{e, a^4\}$ of order two. Then by Lagrange's Theorem $\mid G/H \mid = 8/2 = 4$. By Theorem 4.25 the mapping $\phi : G \to G/H$ defined by $\phi(a) = aH$ is an epimorphism from G to G/H. Now it only remains to show that G/H is cyclic.
Any element $x \in G/H$ can be written as gH for some $g \in G$. G is the cyclic group generated by a so $g = a^i$ for some integer i. Then $x = a^iH = (aH)^i$. So any element of G/H can be written as $(aH)^i$ for some integer i and G/H is cyclic by definition. So a cyclic group of order 8 has a cyclic group of order 4 as a homomorphic image.

Part (b)

Let G be a cyclic group of order 6. $G = \langle a \rangle = \{e, a, a^2, a^3, a^4, a^5\}$ for some element $a \in G$. Let H be the normal subgroup generated by $a^2 = \{e, a^2, a^4\}$ of order 3. Then by Lagrange's Theorem $\mid G/H \mid = 6/3 = 2$. By theorem 4.25 the mapping $\phi : G \to G/H$ defined by $\phi(a) = aH$ is an epimorphism from G to G/H. It has already been shown that G/H is cyclic when G is cyclic. So a cyclic group of order 6 has a cyclic group of order 2 as a homomorphic image.