Problem 1

(a) Prove $-(-x) = x$.

(b) Prove $-(xy) = (-x)y$.

Solution

Part (a)

$$\begin{aligned}
0 + -(-x) &= -(-x) && \text{(A3)}\\
[x + -x] + -(-x) &= -(-x) && \text{(A4)}\\
x + [-x + -(-x)] &= -(-x) && \text{(A2)}\\
x + 0 &= -(-x) && \text{(A4)}\\
x &= -(-x) && \text{(A3)}
\end{aligned}$$

Part (b)

$$\begin{aligned}
(-x)y + xy &= (-x + x)y && \text{(D)}\\
(-x)y + xy &= (0)y && \text{(A4)}
\end{aligned}$$

In class it was proved that $0 \cdot x = 0$ for all x. By this result I get

$$\begin{aligned}
(-x)y + xy &= 0 &&\\
(-x)y + xy + -(xy) &= 0 + -(xy) && \text{add} - (xy)\\
(-x)y + 0 &= 0 + -(xy) && \text{(A4)}\\
(-x)y &= -(xy) && \text{(A3)}
\end{aligned}$$

Problem 2

(a) Prove if $x > y, z < 0$ then $xz < yz$.

(b) Prove if $x > y > 0, z > w > 0$ then $xz > yw$.

(c) Prove if $x > 0$ then $x^{-1} > 0$.

Solution

Part (a)

In order to prove this I will first prove consequent 7 introduced in class, that $(-x)\cdot(y) = (-xy) = x\cdot(-y)$.

$$
\begin{aligned}
x + [(-1)\cdot x] &= [1\cdot x] + [(-1)\cdot x] && \text{(M3)}\\
x + [(-1)\cdot x] &= (1 + (-1))\cdot x && \text{(D)}\\
x + [(-1)\cdot x] &= 0\cdot x && \text{(A4)}\\
x + [(-1)\cdot x] &= 0 && \text{(consequent 3)}\\
-x + x + [(-1)\cdot x] &= -x + 0 &&\\
0 + [(-1)\cdot x] &= -x + 0 && \text{(A4)}\\
[(-1)\cdot x] &= -x && \text{(A3)}
\end{aligned}
$$

Using the equivalence established above;

$$
\begin{aligned}
(-x)(y) &= (-1\cdot x)\cdot(y) &&\\
(-x)(y) &= -1\cdot(x\cdot y) && \text{(D)}\\
(-x)(y) &= -(xy) &&
\end{aligned}
$$

This shows that $(-x)(y) = -(xy)$. The argument that $-(xy) = (x)(-y)$ has an identical structure.
$z < 0$ and $-1 < 0$ so

$$
\begin{aligned}
0\cdot -1 &< z\cdot(-1) && \text{(O7)}\\
0 &< z\cdot(-1) && \text{consequent 3 proved in class}\\
0 &< -(z\cdot 1) && \text{consequent 7}\\
0 &< -z && \text{M3}\\
-z &> 0 && \text{definition of } >
\end{aligned}
$$

Having $-z > 0$ and $x > y$ I use (O6) to get $x(-z) > y(-z)$

$$
\begin{aligned}
-(xz) &> -(yz) && \text{consequent 7}\\
-(xz) + ((xz) + (yz)) &> -(yz) + ((xz) + (yz)) && \text{(O4)}\\
-(xz) + ((xz) + (yz)) &> -(yz) + ((yz) + (xz)) && \text{(A1)}\\
(-(xz) + (xz)) + (yz) &> (-(yz) + (yz)) + (xz) && \text{(M2)}\\
0 + (yz) &> 0 + (xz) && \text{(A4)}\\
(yz) &> (xz) && \text{(A3)}
\end{aligned}
$$

By the definition of $>$ this is the same as saying $xz < yz$.

Part (b)

It is given that $z > w$ and $w > 0$. By (O2) $z > 0$. It is also given that $x > y$. Then by (O6) $xz > yz$. It is also given that $y > 0$. By (O6) again $zy > wy$. Then by (M1) $yz > yw$ and finally by (O2) $xz > yw$.

Part (c)

First assume that $x^{-1} < 0$. Then by (O7) proved in class:

$$x \cdot x^{-1} < 0 \cdot x^{-1}$$

It was also proved in class that $0 \cdot x = 0$ for all x. Thus,

$$\begin{aligned} x \cdot x^{-1} &< 0 \\ 1 &< 0 && \text{by (M4)} \end{aligned}$$

This is a contradiction with (O1) because I know that $1 > 0$, so $x^{-1} > 0$.

Problem 3

Prove that there does not exist an $x \in \mathbb{Z}$ such that $0 < x < 1$.
$\mathbb{Z} = \{x \in \mathbb{R} \mid x \in \mathbb{N} \vee x = 0 \vee -x \in \mathbb{N}\}$.

Solution

Consider any arbitrary $x \in \mathbb{R}$. There are three possible cases.

(a) Case 1: $x \in \mathbb{N}$
It was proven in class that for all x in $\mathbb{N}$, $x \geq 1$. Thus it is impossible that $x < 1$.

(b) Case 2: $x = 0$
If $x = 0$ then it is impossible that $x > 0$.

(c) Case 3: $-x \in \mathbb{N}$
By the same fact used in case 1,

$$\begin{aligned} -x &\geq 1 \\ -x + (-1) &\geq 1 + (-1) \\ -x + (-1) &\geq 0 && \text{(A4)} \\ x + (-x) + (-1) &\geq 0 + x \\ 0 + (-1) &\geq 0 + x && \text{(A4)} \\ -1 &\geq x && \text{(A3)} \end{aligned}$$

If $x = -1$ it is shown in problem 4 that $-1 < 0$. If $x < -1$ then by (O2) $x < 0$. So it is impossible that $x > 0$.

There is no case in which it is possible that $0 < x < 1$.

Problem 4

Prove that it is impossible to define inequalities in $\mathbb{C}$ such that (O1)-(O4) hold.

Solution

The proof given in the book that for any nonzero $a \in \mathbb{R}$, $a^2 > 0$ depends only on axioms (O1)-(O4). Thus if these axioms held in $\mathbb{C}$ then it would have to be the case that the square of any nonzero element of $\mathbb{C}$ was greather than 0. However, i is defined such that $i^2 = -1$. Using the fact introduced in class that $1 > 0$ I can say

$$\begin{aligned} 1 + (-1) &> 0 + (-1) && \text{(O4)} \\ 0 &> -1 && \text{(A4)} \end{aligned}$$

By axiom (O1) it is impossible for it also to be the case that $0 < -1$. Thus this is a contradiction. Therefore it is impossible to define inequalities in $\mathbb{C}$ in such a way that axioms (O1)-(O4) hold.

Problem 5

(a) Let $x, y \in \mathbb{R}$. Prove $x \leq y$ if and only if $x - \epsilon < y + \epsilon \ \forall \epsilon > 0$.

(b) Let $x, y \in \mathbb{R}$ with $x < y$. Prove there exists $z \in \mathbb{R}$ with $x < z < y$.

(c) Let $a, x, b \in \mathbb{R}$ with $a < x < b$. Prove there exists $\epsilon > 0$ such that $a < x - \epsilon < x + \epsilon < b$. Deduce that $(x - \epsilon, x + \epsilon) \subset (a, b)$.

Solution

Part (a)

First let $x \leq y$. By part (i) of theorem 1.9 proved in the textbook I know that $x < y + \epsilon$ for all $\epsilon > 0$. As done with z in problem 2.a I can show that for any ϵ, $-\epsilon < 0$. Thus by (O5) $x - \epsilon > y + \epsilon$.
Now let $x - \epsilon < y + \epsilon$ for all $\epsilon > 0$. Assume that $x > y$. Then $x - y > 0$ so I can set $\epsilon_0 = \frac{x-y}{3}$. Then plugging in I get $x - \frac{x-y}{3} < y + \frac{x-y}{3}$. $\epsilon_0 > 0$ so by (O5)

$$\begin{aligned} x &< y + \epsilon_0 + \epsilon_0 \\ x + \epsilon_0 &< y + \epsilon_0 + \epsilon_0 + \epsilon_0 && \text{(O5)} \\ x + \epsilon_0 &< y + (x + (-y)) \\ x + \epsilon_0 &< y + ((-y) + x) && \text{(A1)} \\ x + \epsilon_0 &< (y + (-y)) + x && \text{(A2)} \\ x + \epsilon_0 &< 0 + x && \text{(A4)} \\ x + \epsilon_0 &< x && \text{(A3)} \\ (-x) + x + \epsilon_0 &< (-x) + x && \text{(O4)} \\ 0 + \epsilon_0 &< 0 && \text{(A4)} \\ \epsilon_0 &< 0 && \text{(A3)} \end{aligned}$$

This is a contradiction with (O1) because I know that $\epsilon_0 > 0$. Therefore $x \leq y$.

Part (b)

Let n be the largest natural number such that $\frac{1}{n} < y - x$. Let k be the largest natural number such that $\frac{k}{n} \leq x$. Then by our selection of k, $\frac{k+1}{n} > x$.
Now assume that $y \leq \frac{k+1}{n}$. Then I have that $\frac{k+1}{n} \geq y$ and $-\frac{k}{n} \geq -x$ so by (O5)":

$$\frac{1}{n} = \frac{k+1}{n} - \frac{k}{n} \geq y - x$$

. This is a contradiction so it must be the case that $y > \frac{k+1}{n}$. Thus $z = \frac{k+1}{n}$ is a number satisfying $x < z < y$.

Part (c)

First I prove consequent 5 introduced in class that $-(x-y) = y - x$.

$$\begin{aligned}
-(x-y) &= -(x+(-y)) && \text{def. of -}\\
-(x-y) &= -1 \cdot (x+(-y)) && \text{see prob. 2}\\
-(x-y) &= -1 \cdot x + -1 \cdot (-y)) && \text{(D)}\\
-(x-y) &= -x + -(-y)) && \text{see prob. 2}\\
-(x-y) &= -x + y && \text{proved in class}\\
-(x-y) &= y + (-x) && \text{(A1)}\\
-(x-y) &= y - x && \text{def. of -}
\end{aligned}$$

Let y be the smaller value of $b - x$ and $x - a$, both of which are positive. If $y = x - a$ then

$$\begin{aligned}
x - y &= x + (-(x-a)) &&\\
x - y &= x + (a + (-x)) && \text{consequent 5}\\
x - y &= x + ((-x) + a) && \text{(A1)}\\
x - y &= (x + (-x)) + a && \text{(A2)}\\
x - y &= 0 + a && \text{(A4)}\\
x - y &= a && \text{(A3)}
\end{aligned}$$

$a \leq a$ by definition so in this case $a \leq x - y$.
The other case is when $y = b - x$. By our selection of y I know $x - a > y$ so $(x-a) - y > 0$ and I also know from the first case that $x - (x-a) \geq a$. So by (O5)'

$$\begin{aligned}
x + (-(x-a)) + ((x-a) + (-y)) &\geq a &&\\
x + ((-(x-a)) + (x-a)) + (-y) &\geq a && \text{(A2)}\\
x + 0 + (-y) &\geq a && \text{(A4)}\\
x + (-y) &\geq a && \text{(A3)}
\end{aligned}$$

so in both cases $a \leq x - y$. It is given that both $b - x$ and $x - a$ are positive so in either case $y > 0$. By (O4) $x + y > x$. Application of (O5) obtains $x > x - y$.

Now I want to show that $x + y \leq b$. In the case when $y = b - x$

$$\begin{aligned} x + y &= x + (b + (-x)) \\ x + y &= x + ((-x) + b) && \text{(A1)} \\ x + y &= (x + (-x)) + b && \text{(A2)} \\ x + y &= 0 + b && \text{(A4)} \\ x + y &= b && \text{(A3)} \end{aligned}$$

so $x + y \leq b$ by definition.
In the case when $y = x - a$ I know by our selection of y that $y < (b - x)$ and thus $0 > y + (-(b - x))$ by (O4). I also know from the first case that $b \geq x + (b - x)$. Then by (O5)'

$$\begin{aligned} b + 0 &> x + (b - x) + (y + (-(b - x))) \\ b + 0 &> x + (b - x) + ((-(b - x)) + y) && \text{(A1)} \\ b + 0 &> x + ((b - x) + (-(b - x))) + y && \text{(A2)} \\ b + 0 &> x + 0 + y && \text{(A4)} \\ b &> x + y && \text{(A3)} \end{aligned}$$

So therefore $x + y \leq b$ by the definition of $\leq$.
Having shown that $x < x + y$ I can use the result of part b) to produce some number z such that $x < z < x + y$. Let $\epsilon = z - x$. To show that ϵ satisfies the desired characteristics it must be shown that $\epsilon > 0$, and that $a < x - \epsilon < x + \epsilon < b$. It is known that $x < z$ so by(O4) $\epsilon = z - x > 0$. By the same process as before, $0 > -\epsilon$. Then by (O2) $-\epsilon < \epsilon$ and by (O4) $x - \epsilon < x + \epsilon$.

It is known that $z < x + y$. Then by (O4) $z - x < y$ so $\epsilon < y$. It is also known $a \leq x - y$. Then by (O5)' $a + \epsilon < x$ and by (O4) $a < x - \epsilon_0$.

From above it is known that $x + y \leq b$ and that $\epsilon < y$. Then by (O5)' $x + y + \epsilon < b + y$ and by (O4) $x + \epsilon < b$.

Thus ϵ satisfies the desired properties and by definition $(x - \epsilon, x + \epsilon) \subset (a, b)$.

Problem 6

Prove that each of the following are metric spaces.

(a) $X = \mathbb{R}, d(x, y) = |y - x|$

(b) $X =$ any set, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

(c) Give another example of a metric space.

Solution

Part (a)

This proof will use the fact that $-1 \cdot x = -x$. This was proven as an intermediate step in problem 2.
First I will prove that $-(x-y) = y - x$.

$$
\begin{aligned}
-(x-y) &= -1 \cdot (x + (-y)) && \text{see problem 2} \\
-(x-y) &= -1 \cdot x + -1 \cdot -y && \text{(D)} \\
-(x-y) &= -x + -(-y) && \text{see problem 2} \\
-(x-y) &= -x + y && \text{proved in class} \\
-(x-y) &= y + (-x) && \text{(A1)} \\
-(x-y) &= y - x && \text{def. of -}
\end{aligned}
$$

i $d(x,y) = 0 \iff x = y$
First assume $x = y$. Then $|y - x| = |0| = 0$. Now assume that $|y - x| = 0$. Then either $y - x = 0$ or $x - y = 0$. In the first case $y - x + x = x$ so by (A4) $y = x$. In the second case $x - y + y = y$ so by (A4) $x = y$.

ii $d(x,y) = d(y,x)$
This would directly follow from a proof of property 2 of absolute valuesthat states $|y-x| = |x-y|$. There are two cases.

(i) Case: $y - x > 0$.
By the definition of absolute value $|y - x| = y - x$. Then

$$
\begin{aligned}
y - x &> 0 \\
y - x + x &> 0 + x && \text{O4} \\
y + 0 &> 0 + x && \text{A4} \\
y &> x && \text{A3} \\
y + (-y) &> x + (-y) && \text{O4} \\
0 &> x + (-y) && \text{A4} \\
0 &> x - y && \text{def. of -}
\end{aligned}
$$

Thus by the definition of absolute value $|x - y| = -(x - y)$ which, as proved at the beginning of this problem, is equal to $y - x$.

i. Case: $y - x < 0$.
By the definition of absolute value $|y - x| = -(y - x)$. Using the same fact as

above, this equals $x - y$.

$$\begin{aligned} y - x &< 0 & \\ y - x + x &< 0 + x & \text{O4} \\ y + 0 &< 0 + x & \text{A4} \\ y &< x & \text{A3} \\ y + (-y) &< x + (-y) & \text{O4} \\ 0 &< x + (-y) & \text{A4} \\ 0 &< x - y & \text{def. of -} \end{aligned}$$

Thus $|x - y| = x - y$ by definition.

ii. Case: $y - x = 0$
In this case $|y - x| = y - x = 0$ by definition.

$$\begin{aligned} y - x &= 0 & \\ y - x + x &= 0 + x & \\ y + 0 &= 0 + x & \text{(A4)} \\ y &= x & \text{(A3)} \\ y + (-y) &= x + (-y) & \\ 0 &= x + (-y) & \text{(A4)} \\ 0 &= x - y & \text{def. of -} \end{aligned}$$

So $|x - y| = y - x = 0$ by definition.

iii $d(x, z) \leq d(x, y) + d(y, z)$
$|z - x| \leq |y - x| + |z - y|$ by the triangle inequality proved in class.

Part (b)

i $d(x, y) = 0 \iff x = y$
This is true by the definition of the function d.

ii $d(x, y) = d(y, x)$
In the case when $x = y$, $d(x, y) = 0 = d(y, x)$. In the case when $x \neq y$, $d(x, y) = 1 = d(y, x)$.

iii $d(x, z) \leq d(x, y) + d(y, z)$

(i) Case: $x = y = z$
$0 \leq 0$

(ii) Case: $x \neq y \neq z$
$1 \leq 2$

(iii) Case: $x = y \neq z$
$1 \leq 1$

(iv) Case: $x \neq y = z$
$1 \leq 1$

(v) Case: $x = z \neq y$
$0 \leq 1$

Part (c)

$X = \mathbb{C}, d(x, y) = \sqrt{x^2 + y^2}$.