Problem 1

Part (a)

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(x+1)^k}$$

First prove for $k = 1$.

$$f'(x) = \frac{1}{x+1} = (-1)^2\frac{0!}{(x+1)}$$

Now assume the statement is true for some $n = k$.

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(x+1)^k}$$

Then

$$\begin{aligned} f^{k+1}(x) &= \frac{d}{dx}f^k(x) \\ &= (-1)^{k+1}(k-1)!(-k(x+1)^{-(k+1)} \\ &= (-1)^{k+2}(k)!(x+1)^{-(k+1)} \end{aligned}$$

Part (b)

$$\begin{aligned} P_{n,0} &= \sum_{k=0}^{n}\frac{(-1)^{k+1}(k-1)!}{(1)^k k!}x^k \\ &= \sum_{k=0}^{n}\frac{(-1)^{k+1}}{k!}x^k \\ &= 0 + x - \frac{x^2}{2} + \cdots + (-1)^{n+1}\frac{x^n}{n} \end{aligned}$$

Part (c)

By Taylor's Theorem $|\log(1/2) - P_{7,0}(1/2)|$ is less than $\frac{f^{(8)}(c)}{(8)!}(1/2)^8$ for some $0 < c < x$.

$$\frac{f^{(8)}(c)}{(8)!}(1/2)^8 = -\frac{7!(1/2)^8}{(c+1)^8 8!} \leq -\frac{(1/2)^8}{8} = \frac{1}{2048} < \frac{1}{1000}$$

Problem 2

Part (a)

b See midterm

Part (b)

c Let $g(x) = f(x+1) - f(x)$. This function is continuous on $\mathbb{R}$ by arithmetic properties of continuity. Then

$$g(0) + g(1) = f(1) - f(0) + f(2) - f(1) = 0$$

Thus at least one of $g(0), g(1)$ is nonpositive and the other must be nonnnegative. WLOG $g(0) \leq 0$, $g(1) \geq 0$. Then by the intermediate value theorem $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$. Thus

$$f(x_0 + 1) - f(x_0) = 0 \implies f(x_0) = f(x_0 + 1)$$

Problem 3

Part (a)

$$\begin{aligned} U(P_n, f) &= \frac{1}{n}\left(\frac{1}{n}\right) + \frac{1}{n}\left(\frac{2}{n}\right) + \cdots + \frac{1}{n}\left(\frac{n}{n}\right) \\ &= \frac{1 + 2 + \cdots + n}{n^2} \\ &= \frac{n(n+1)}{2n^2} \\ &= \frac{n+1}{2n} \end{aligned}$$

$$\begin{aligned} L(P_n, f) &= \frac{1}{n}\left(\frac{0}{n}\right) + \frac{1}{n}\left(\frac{1}{n}\right) + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right) \\ &= \frac{1 + 2 + \cdots + n - 1}{n^2} \\ &= \frac{n(n-1)}{2n^2} \\ &= \frac{n-1}{2n} \end{aligned}$$

Part (b)

Fix $\epsilon > 0$. Then select n such that $\frac{1}{n} < \epsilon$. This can be done by the archimedean property. Then

$$U(P_n, f) - L(P_n, f) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n} < \epsilon$$

. Thus f is integrable on [0, 1] by definition.

Problem 4

Part (a)

Let $K = M(f, S) + M(g, S)$. Then $\forall x \in S$ $(f+g)(x) = f(x) + g(x)$ where $f(x) \leq M(f, S)$ and $g(x) \leq M(g, S)$. Thus $(f+g)(x) \leq K$ so K is an upper bound for $M(f+g, S)$ so $M(f+g, S) \leq M(f, S) + M(g, S)$.

Part (b)

$$
\begin{aligned}
U(f,P)+U(g,P) &= \sum_{k=1}^{n}(x_k - x_{k-1})M(I_k,f) + \sum_{k=1}^{n}(x_k - x_{k-1})M(I_k,g) \\
&= \sum_{k=1}^{n}(x_k - x_{k-1})(M(I_k,f)+M(I_k,g)) \\
&\le \sum_{k=1}^{n}(x_k - x_{k-1})(M(I_k,f+g)) \\
&= U(f+g,P)
\end{aligned}
$$

Part (c)

By definition of upper integrals we want to show that $\inf(\{U(P,f+g)|P \text{ a partition of } I\}) \le \inf(\{U(P,f) \mid P \text{ a partition of } I\}) + \inf(\{U(P,g) \mid P \text{ a partition of } I\})$. Assume that this was not the case. Then by the approximation property of infimum there must be some partition P such that $\inf(\{U(P,f)|P \text{ a partition of } I\})+\inf(\{U(P,g)|P \text{ a partition of } I\}) \le U(P,f) + U(P,g) < \inf(\{U(P,f+g) \mid P \text{ a partition of } I\})$. However, this implies that $U(P,f+g) < \inf(\{U(P,f+g) \mid P \text{ a partition of } I\})$ which is a contradiction.

Problem 5

Part (a)

$A' \subseteq A$ so $\sup(A) \ge \sup(A')$. For any partition P in A we can find a corresponding partition P' in A' that is identical except that it also includes the point c. Thus P' is a refinement of P and by Proposition 20.2 $L(P',f) \ge L(P,f)$. Thus by a result from the homework $\sup(A') \ge \sup(A)$. Thus $\sup(A') = \sup(A)$.

Part (b)

Every element l of A' is the lower bound of some partition P' containing c. Then divide the points of P' into two partitions P_1 and P_2 such that P_1 contains all points $x \in P'$ such that $x \le c$ and P_2 contains all points such that $x \ge c$. Then $l = L(P_1,f) + L(P_2,f)$. So $A' = A_1 + A_2$.
By the result of HW2.5 $\sup(A') = \sup(A_1) + \sup(A_2)$ and from part 1 $\sup(A) = \sup(A')$ so $\sup(A) = \sup(A_1) + \sup(A_2)$.

Part (c)

This follows directly from b) by definition.