Planar Graphs

and how to find them

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Outline

1. Definitions and Examples
2. Kuratowski's Theorem
3. DMP Algorithm
4. Other Surfaces?

What is a planar graph?

Planar embedding: drawing of a graph with no edges crossing
Planar graph: has a planar embedding

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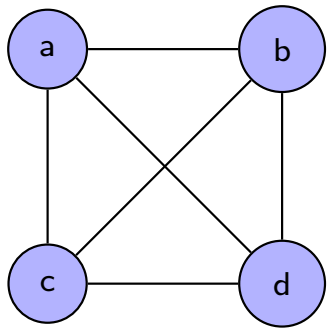

Nonplanar embedding

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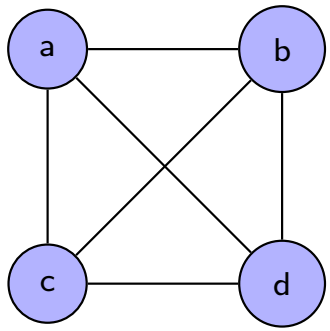

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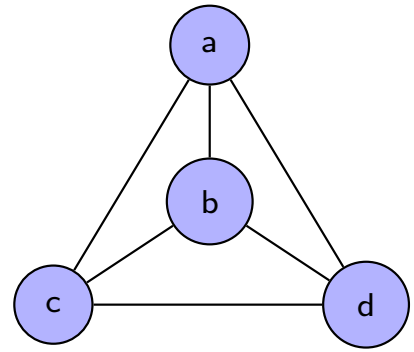

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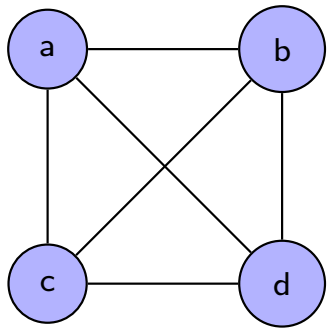

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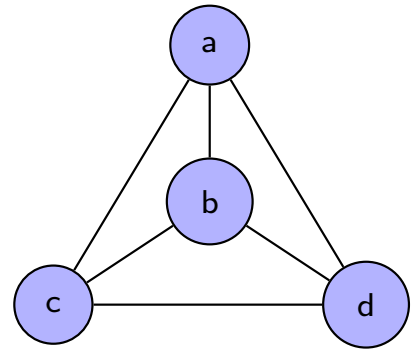
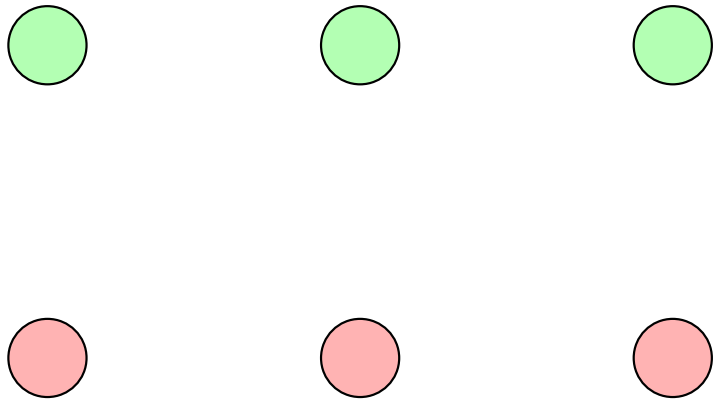


Planar embedding

K_4 is planar!

Another example

The complete bipartite graph $K_{3,3}$



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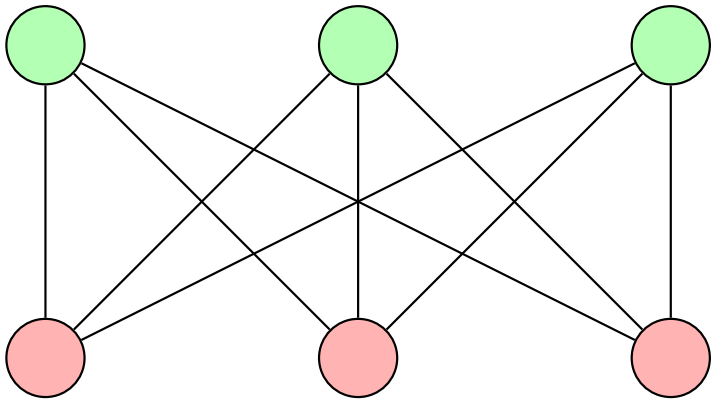

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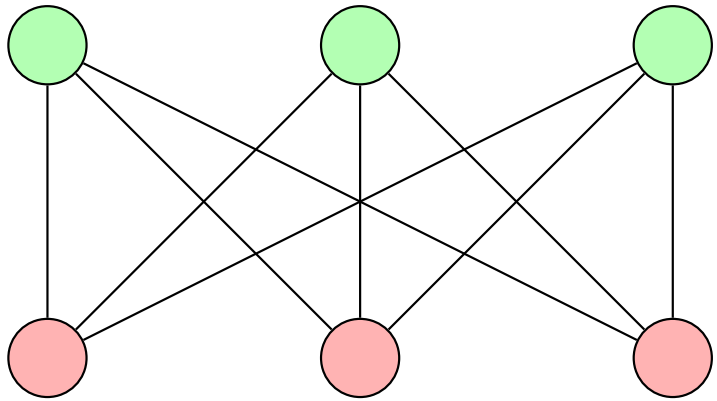
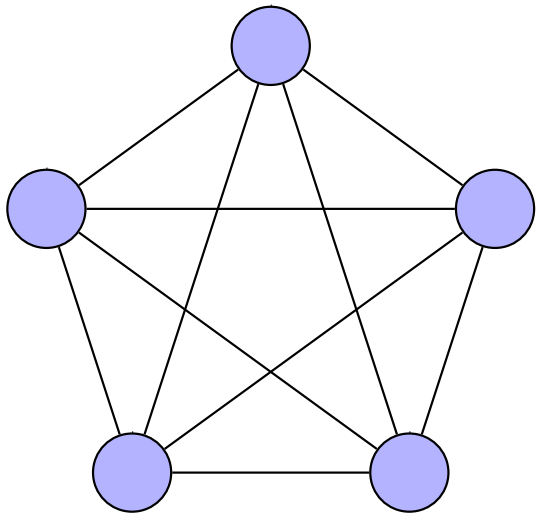


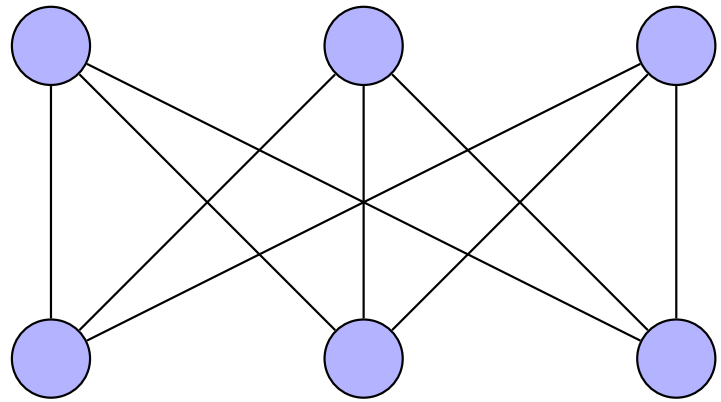
$K_{3,3}$ is **not** planar.

Kuratowski's Theorem

Theorem: A graph is planar if and only if it contains no subgraphs homeomorphic to K_5 or $K_{3,3}$.



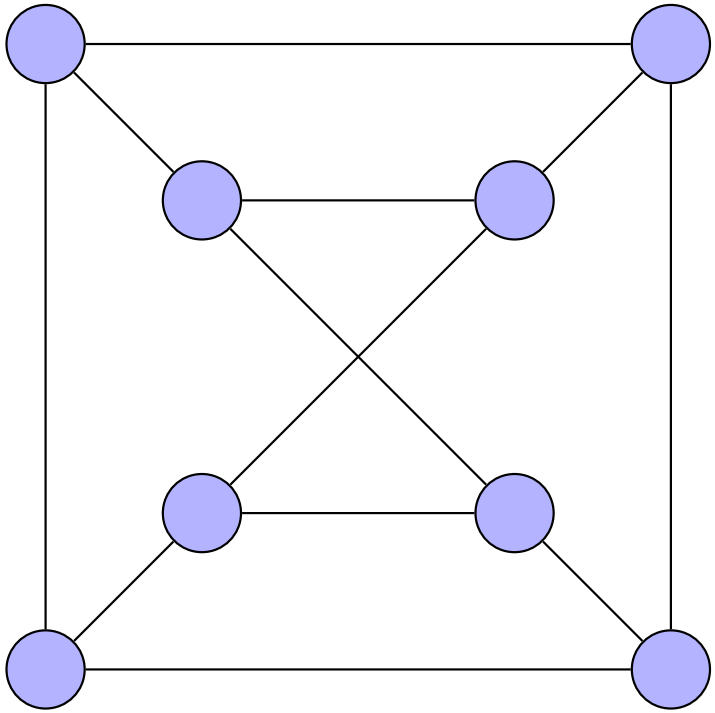
K_5



$K_{3,3}$

Homeomorphic to = has same basic shape as

Example



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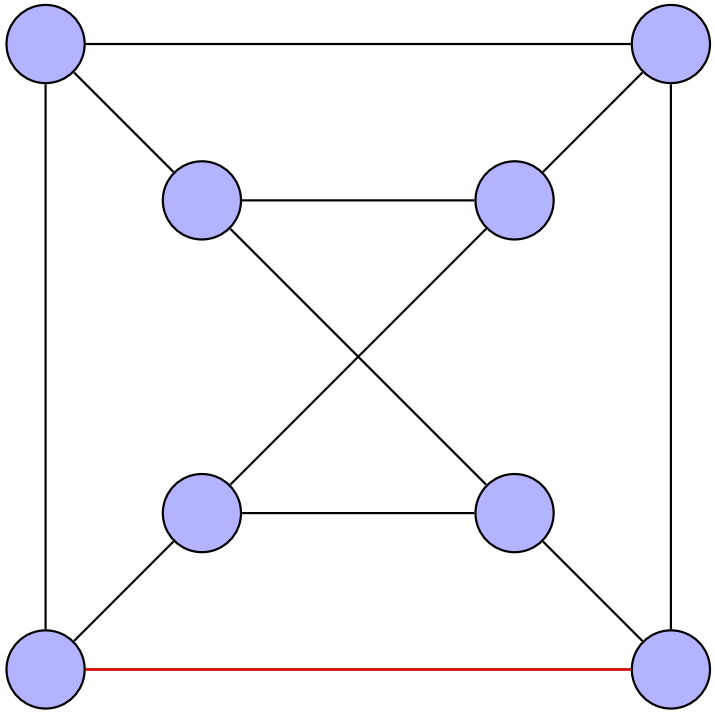

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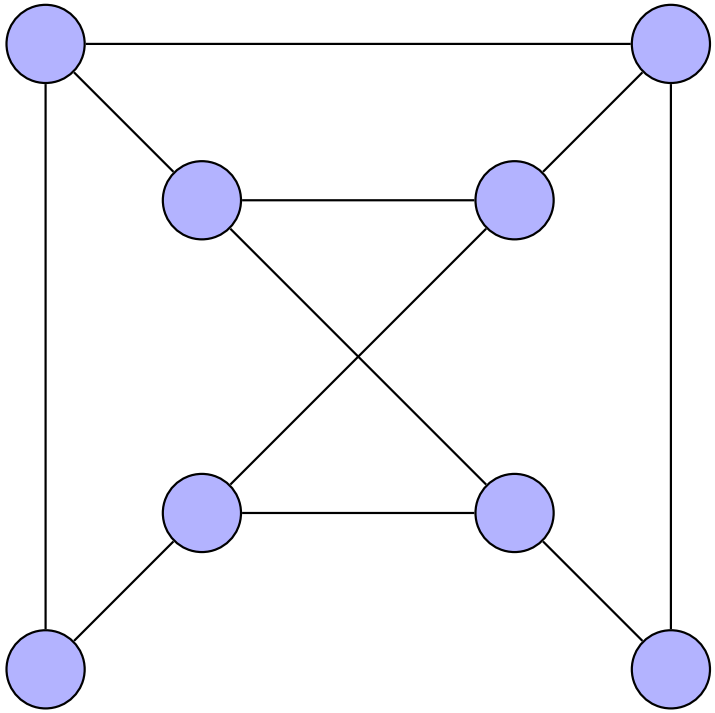

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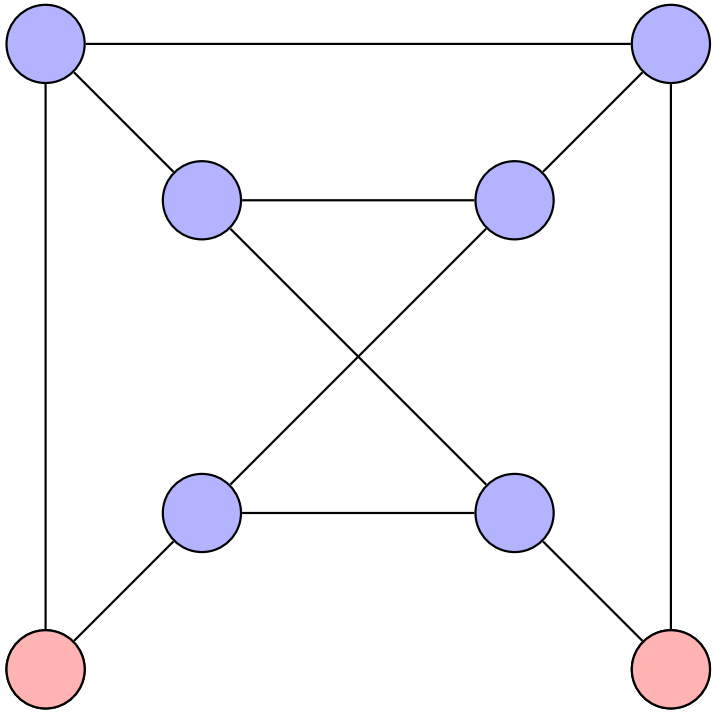

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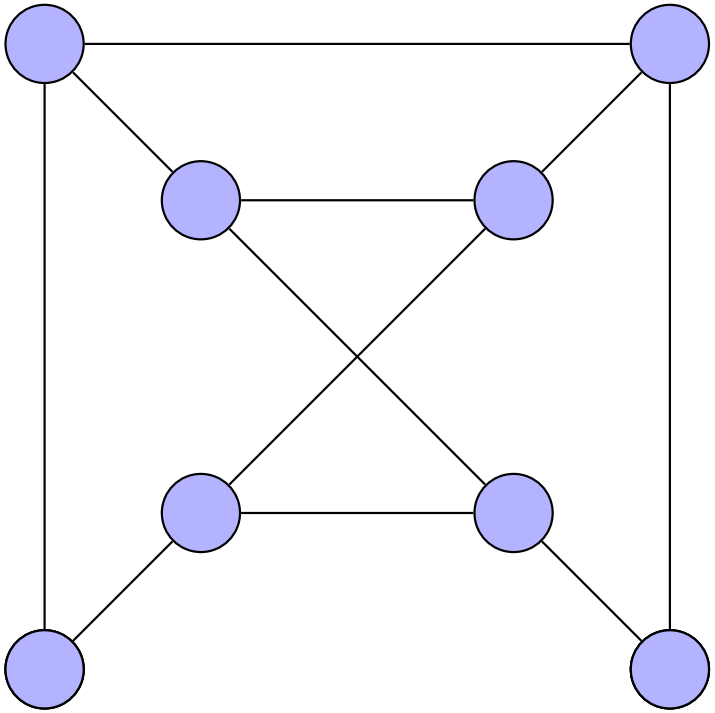

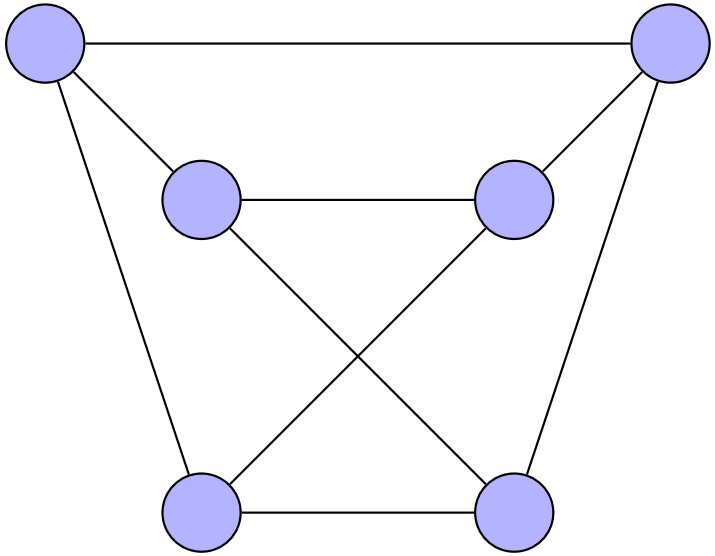

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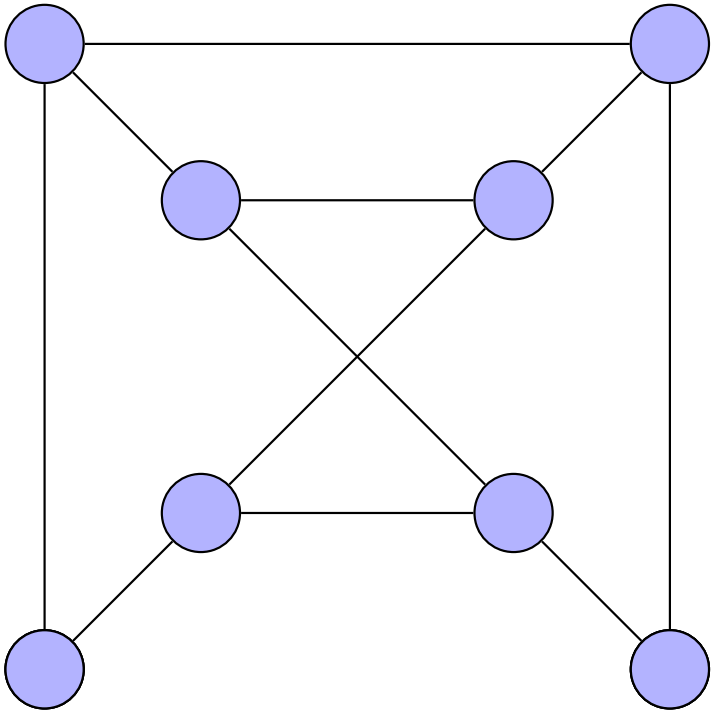

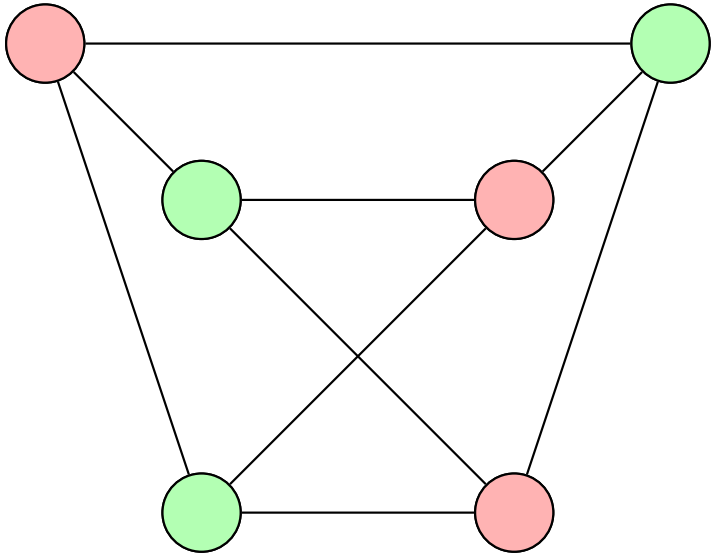

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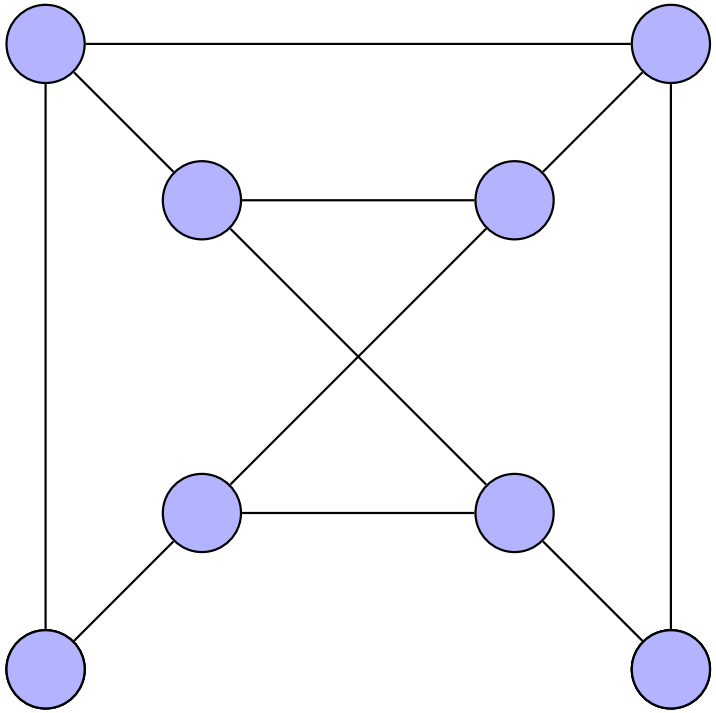

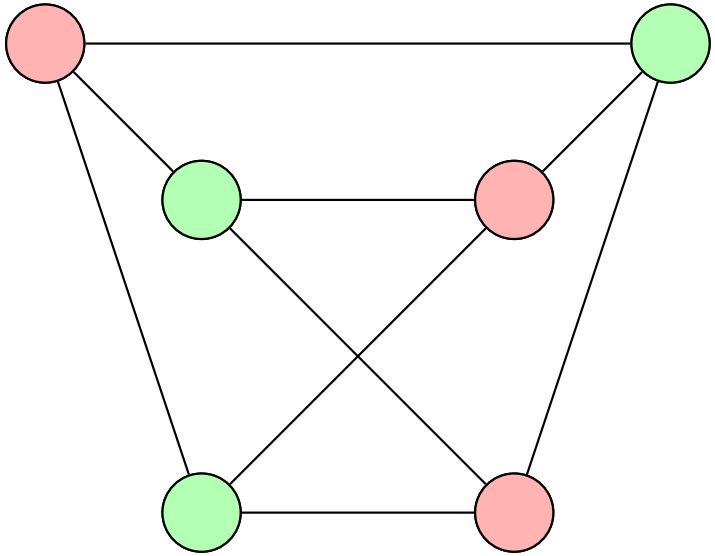

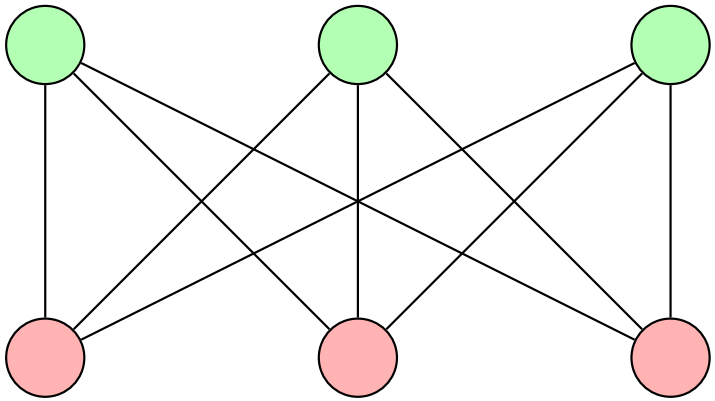

Background

Published by Demoucron, Malgrange and Pertuiset in 1964

Works by adding segments to a subgraph until graph is complete or segment is blocked

Not the most efficient, but simplest to understand

Requires pre-processing

DMP Planarity Algorithm

DMP Algorithm

Input: A 2-connected graph G
Output: A planar embedding or FALSE
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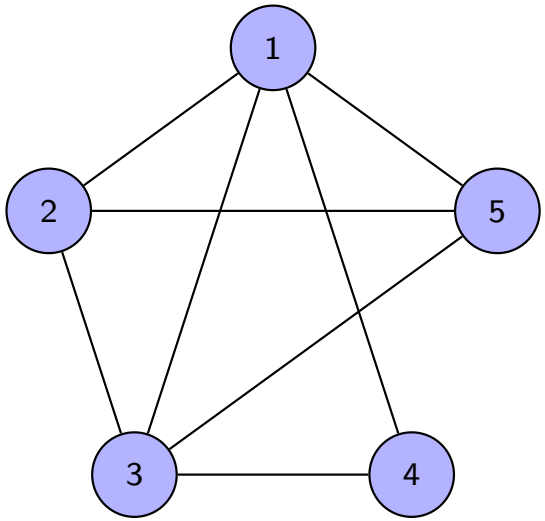

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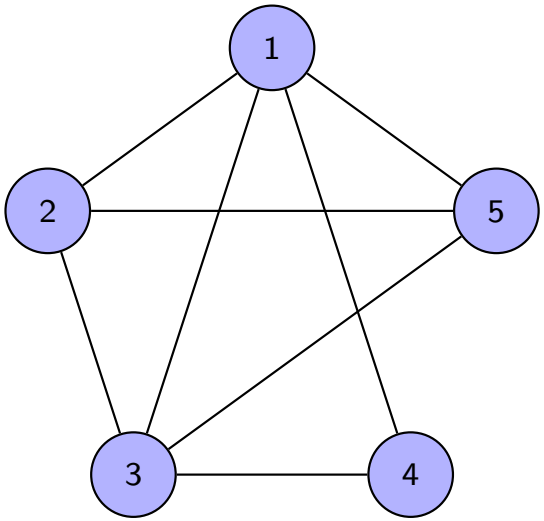
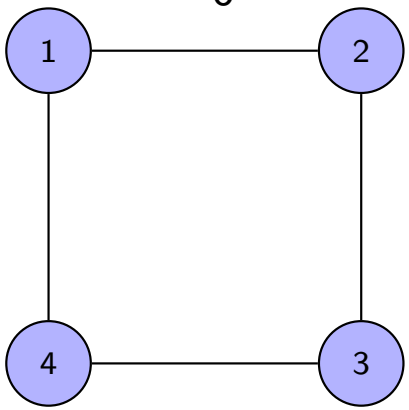
G



G_0



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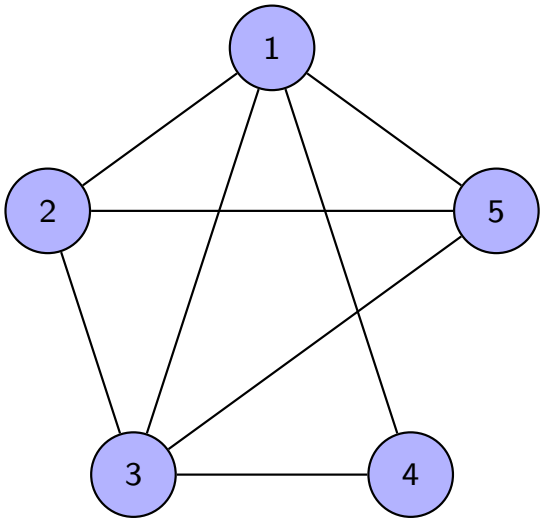

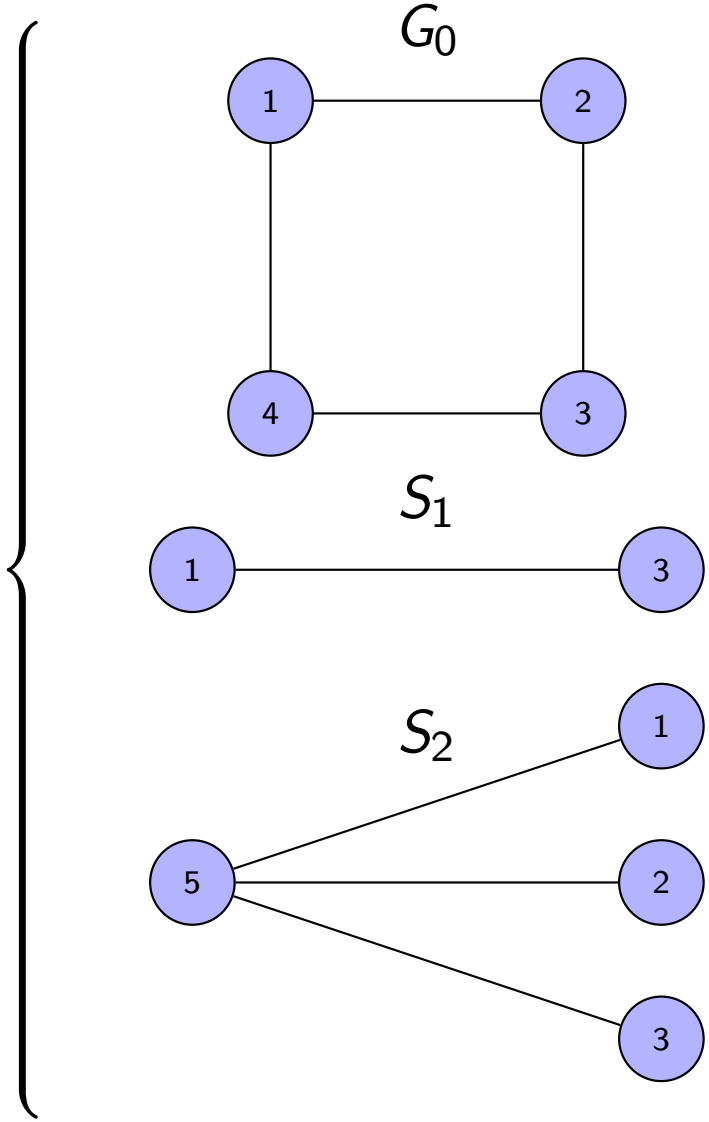

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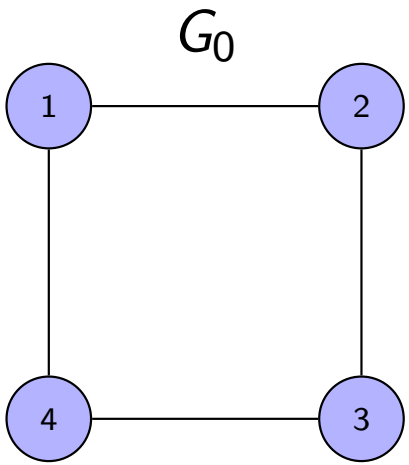

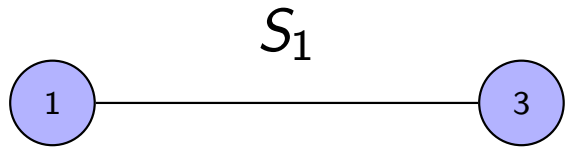

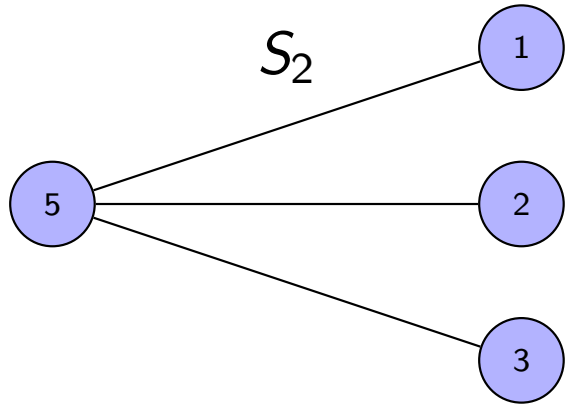

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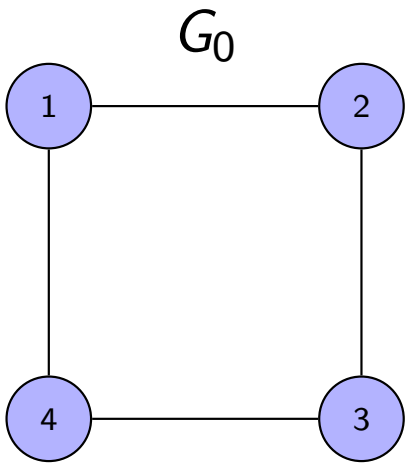

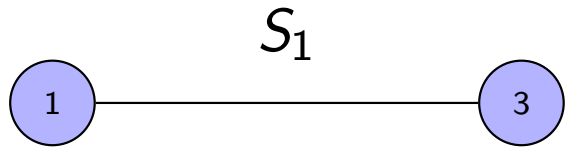

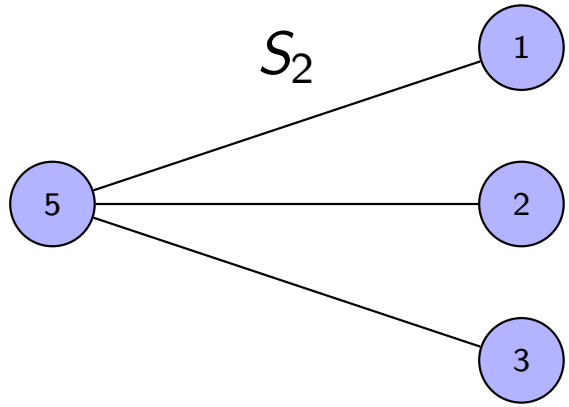

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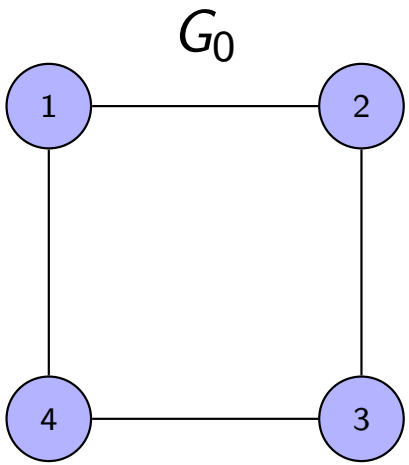

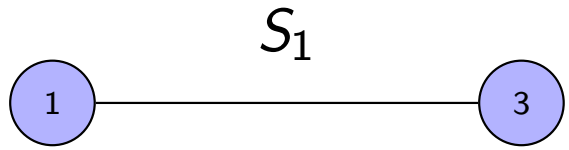

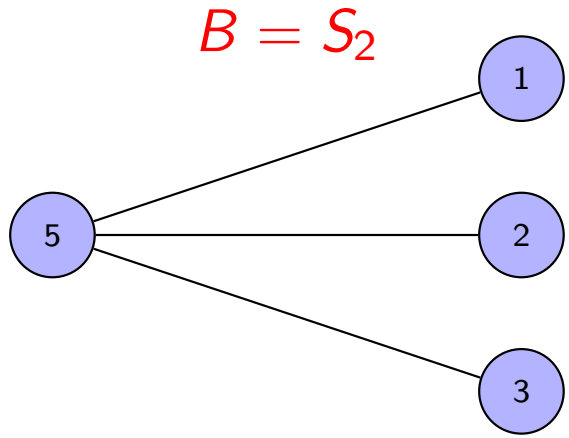

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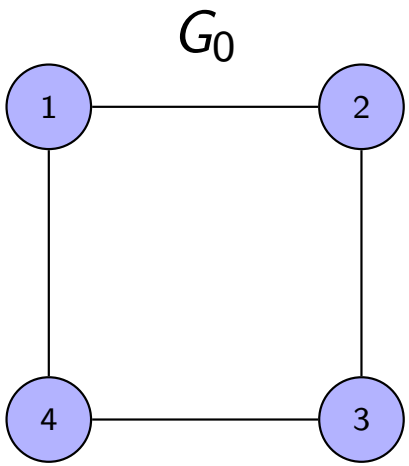

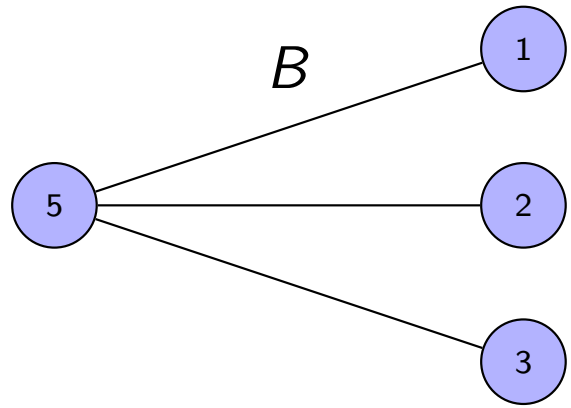

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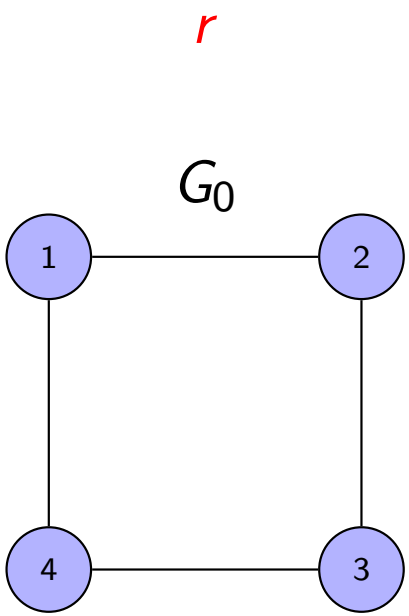

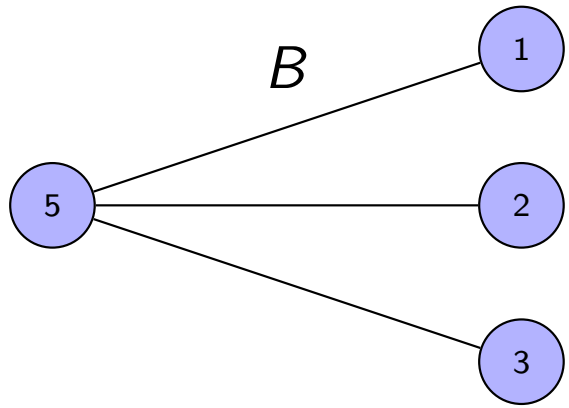

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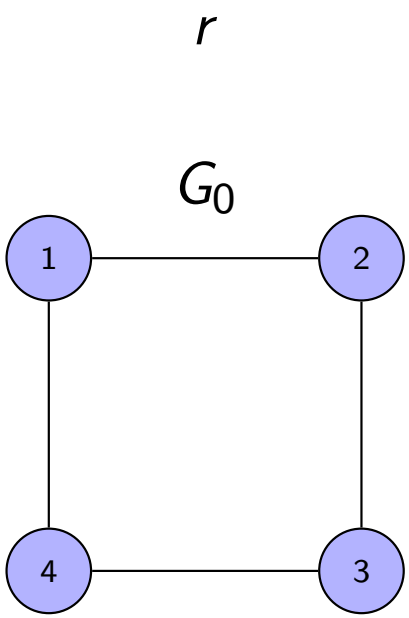

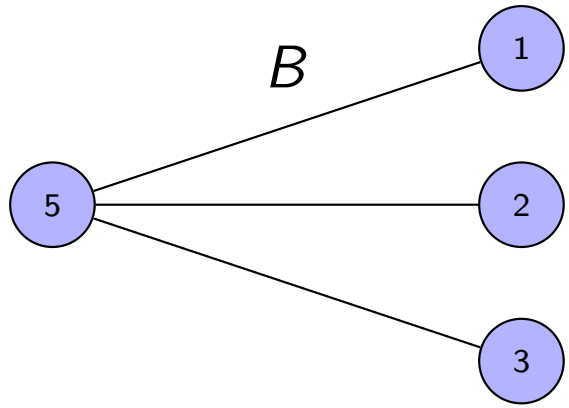

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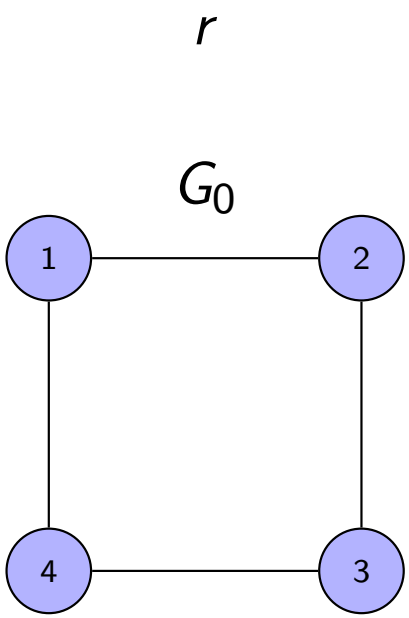

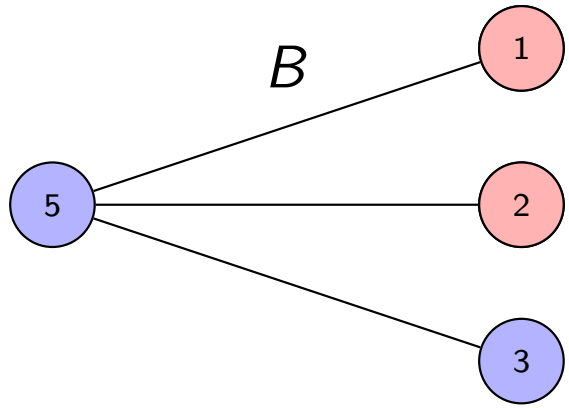

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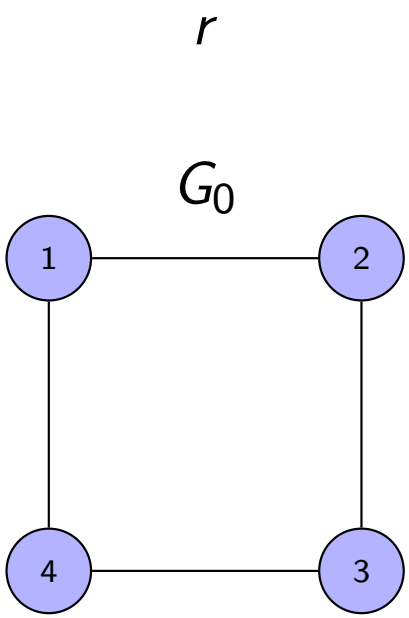

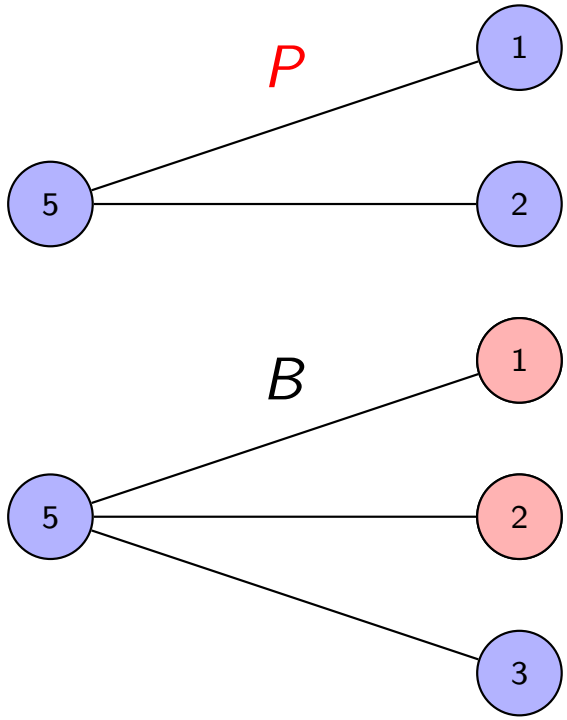

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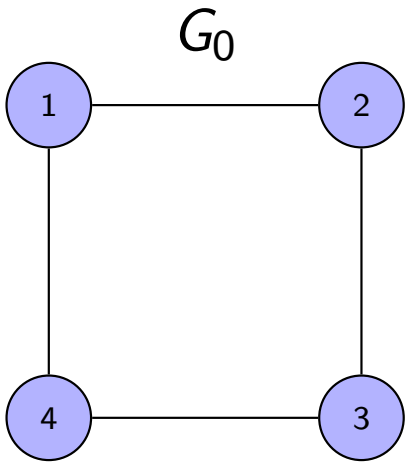

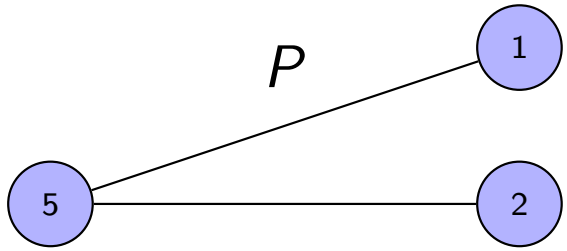

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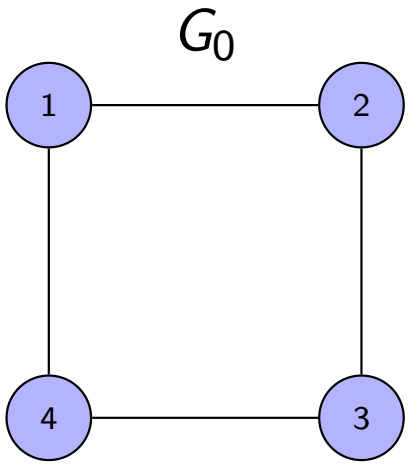

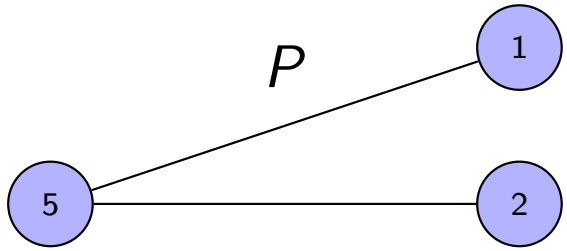

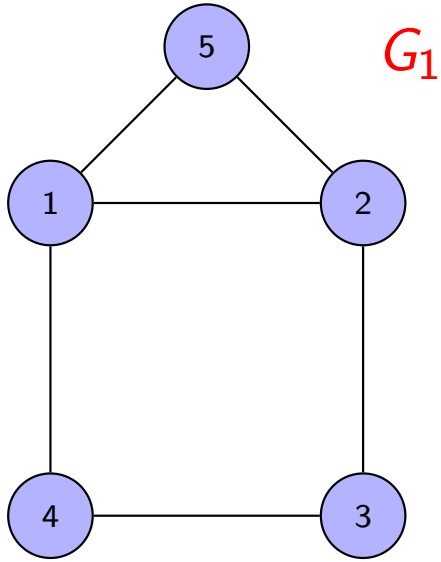

DMP Planarity Algorithm

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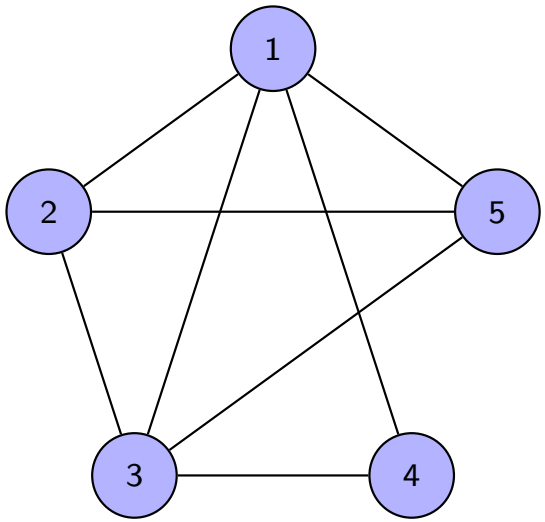
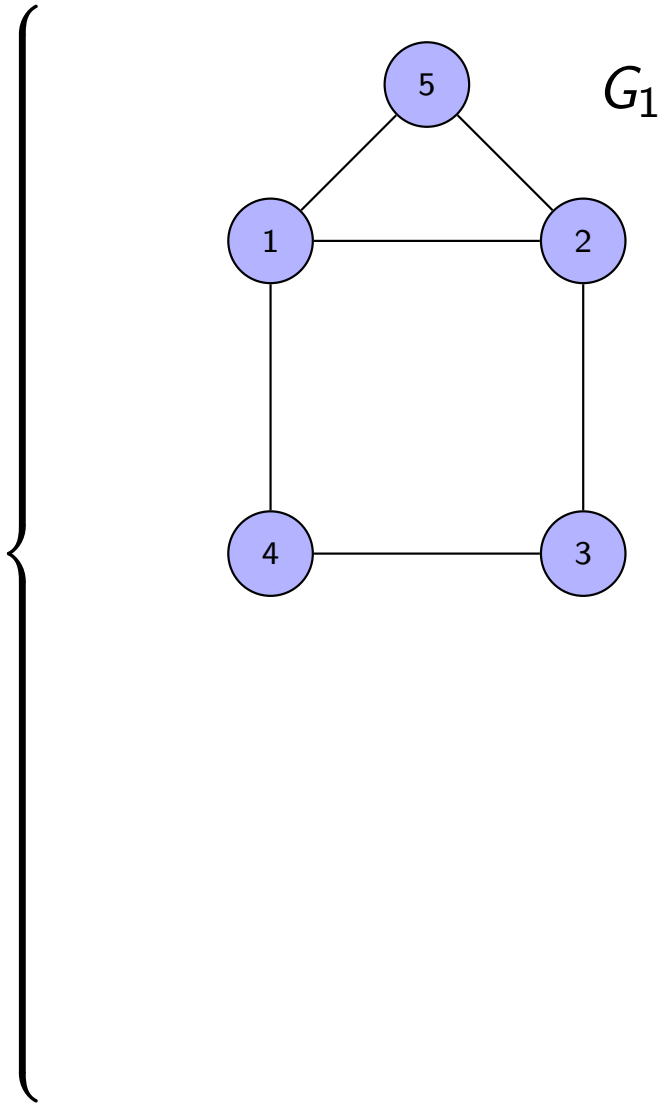
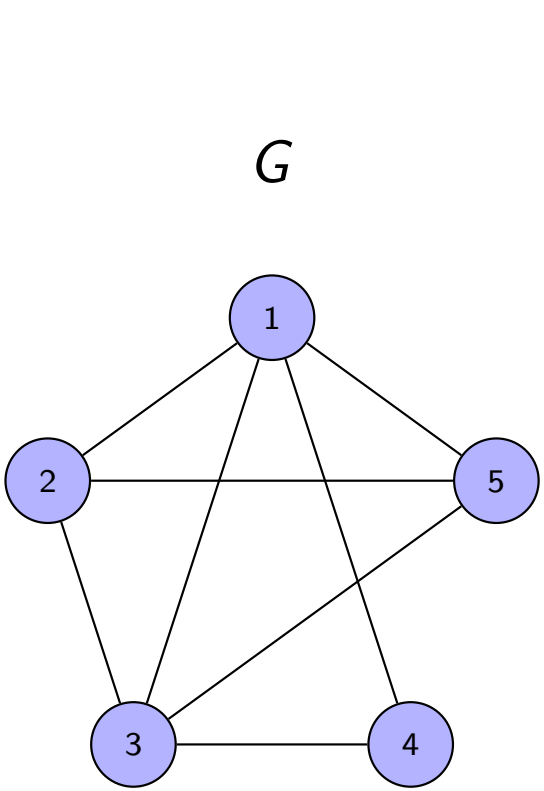
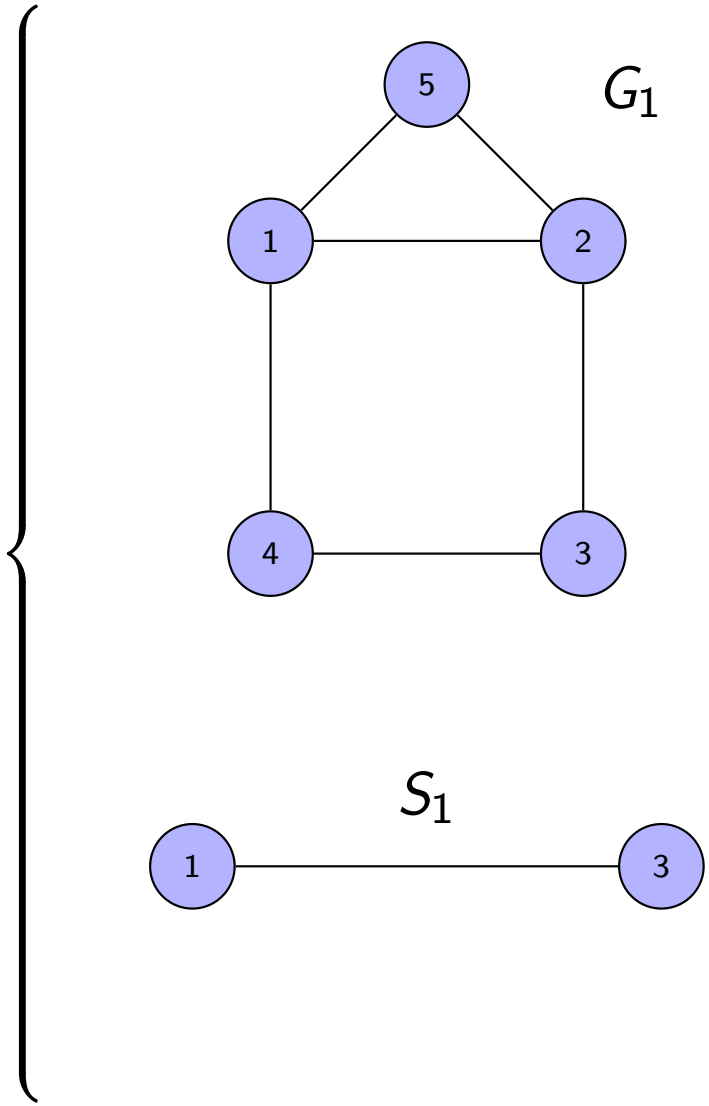


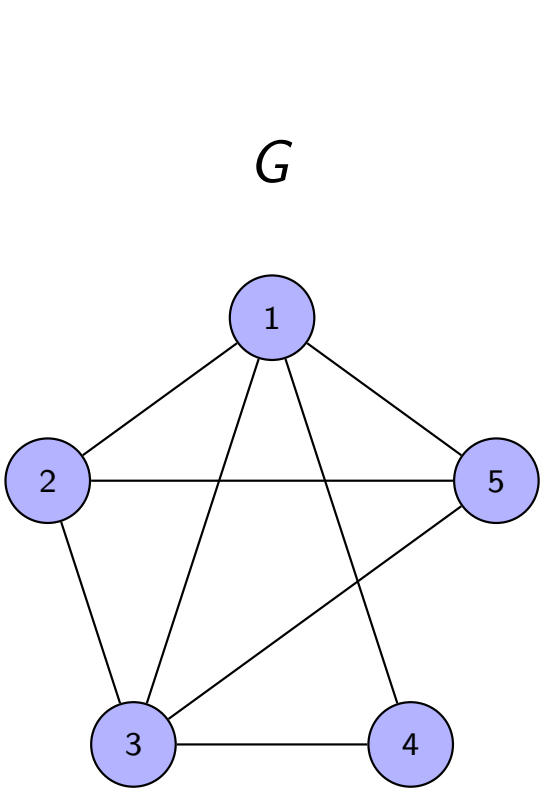


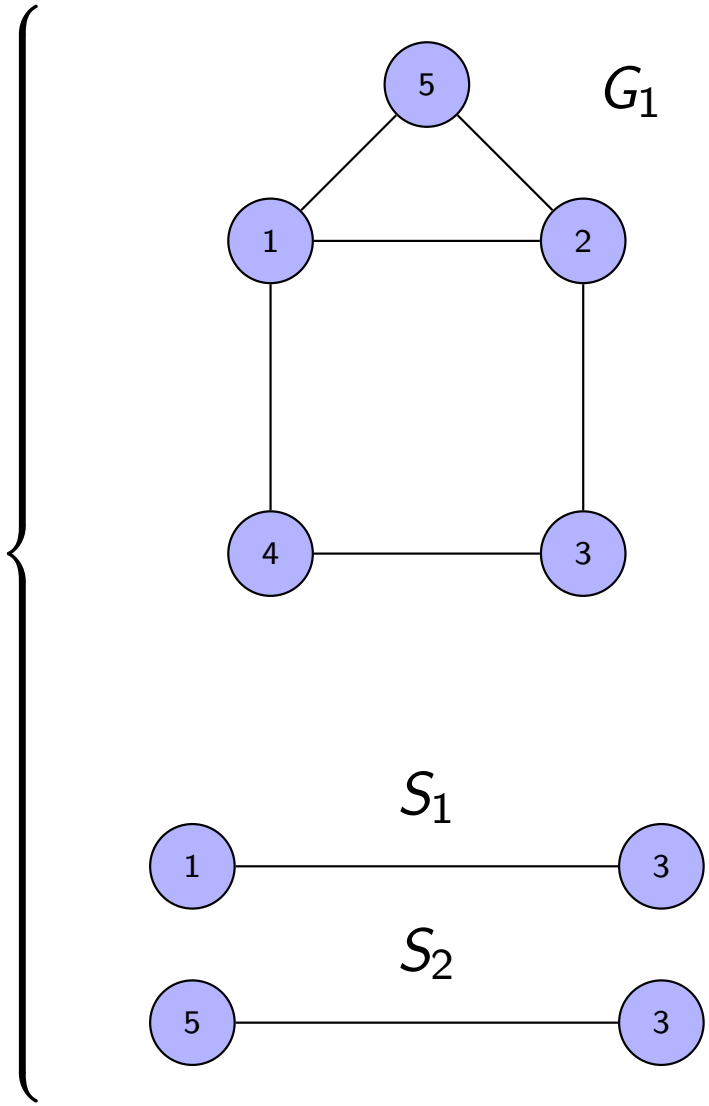
DMP Planarity Algorithm





DMP Planarity Algorithm





DMP Planarity Algorithm

DMP Algorithm

Input: A 2-connected graph G
Output: A planar embedding or FALSE
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while $G_j \neq G$ **do**
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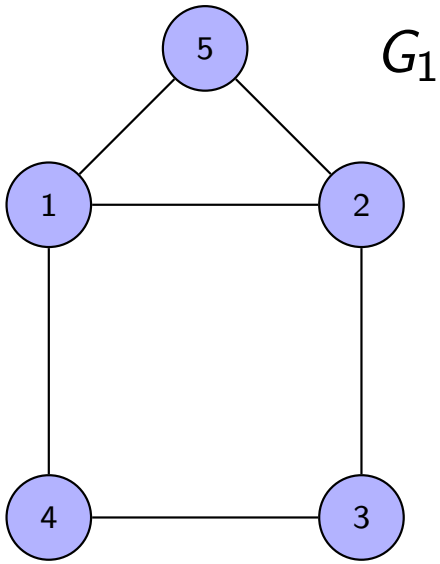
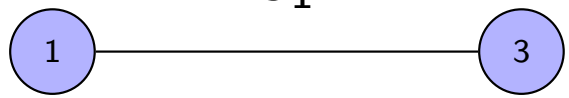
        return FALSE
    end
    if *a segment is forced* **then**
        B := that segment
    end
    else
        B := any segment
    end
    r := region whose boundary contacts B
    P := path between two contact points of B
    G_{j+1} := drawing of P in r
end

DMP Planarity Algorithm

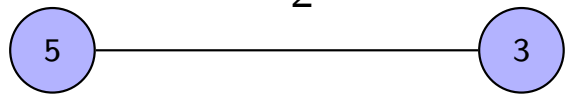


G_1

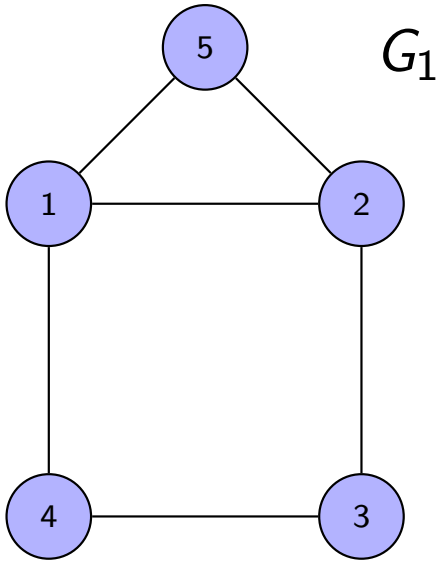
S_1



S_2

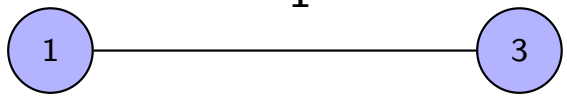


DMP Planarity Algorithm

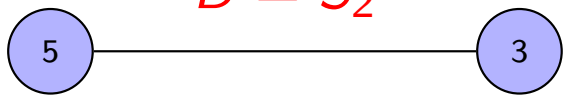


G_1

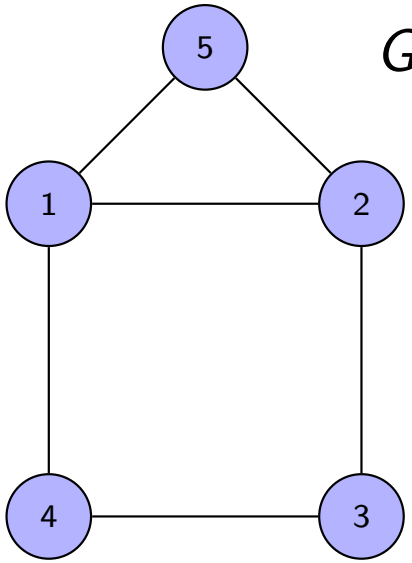
S_1



$B = S_2$

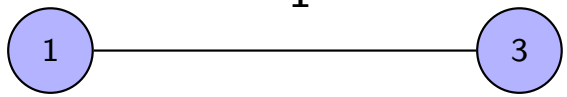


DMP Planarity Algorithm



G_1 r

S_1



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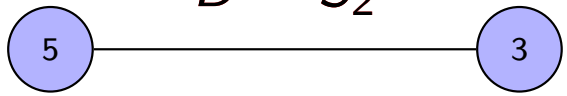

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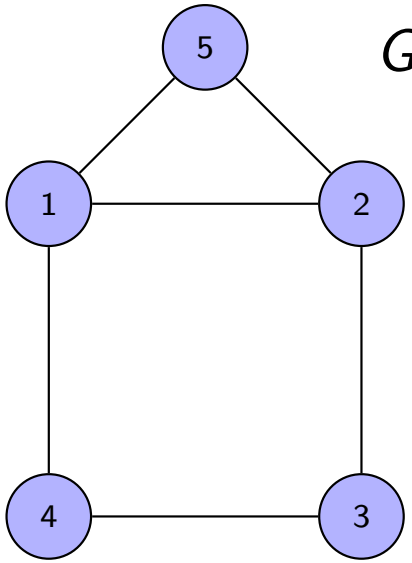
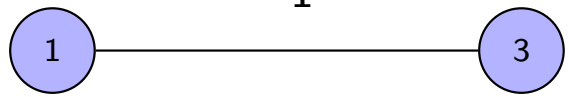
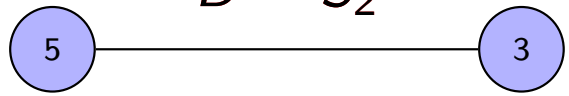


G_1 r

S_1

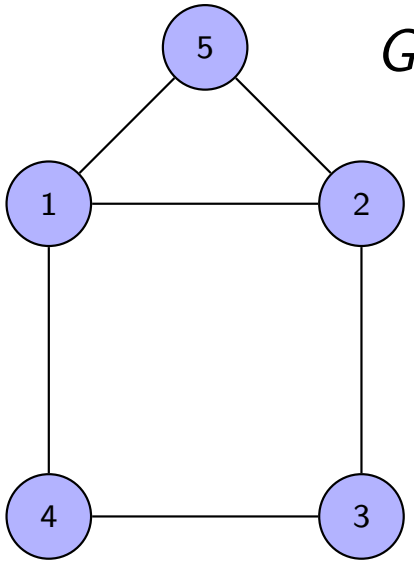


$B = S_2$

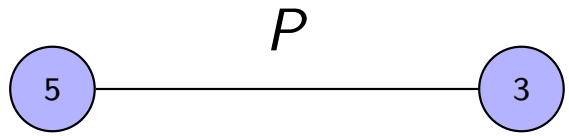


P

DMP Planarity Algorithm



G_1 r



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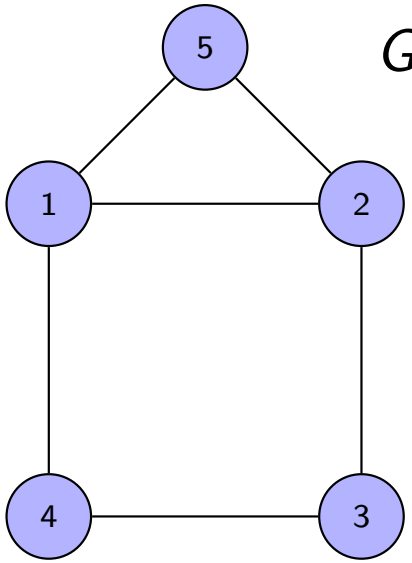
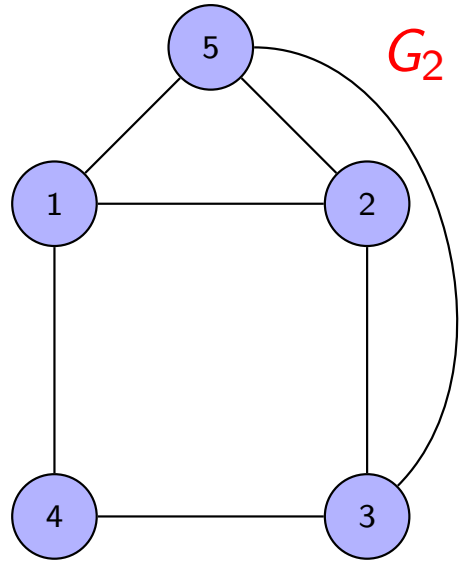


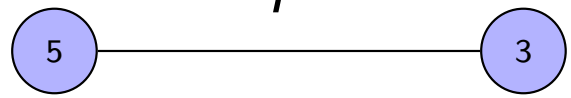
G_1

r



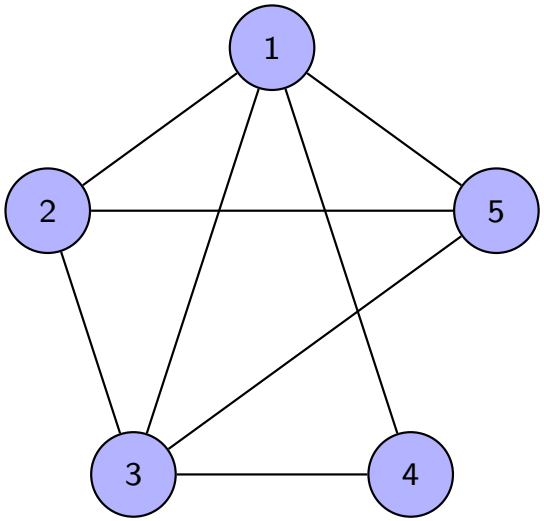
G_2

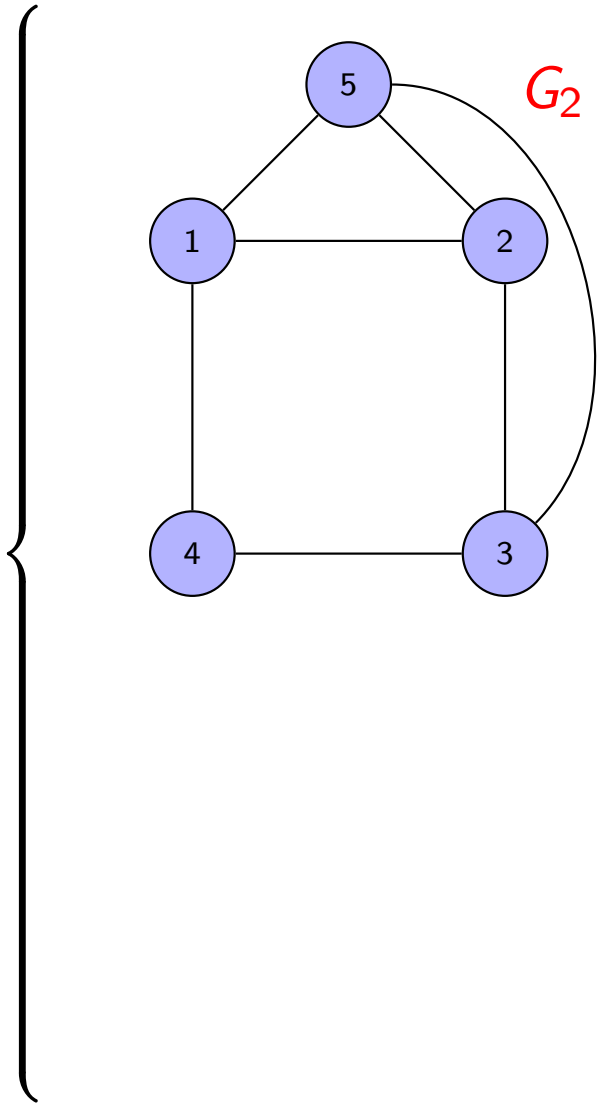
P



DMP Planarity Algorithm

G





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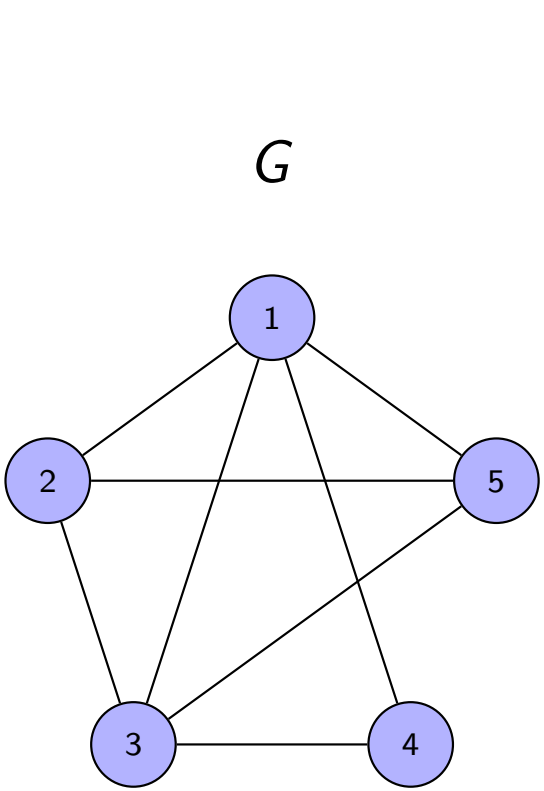

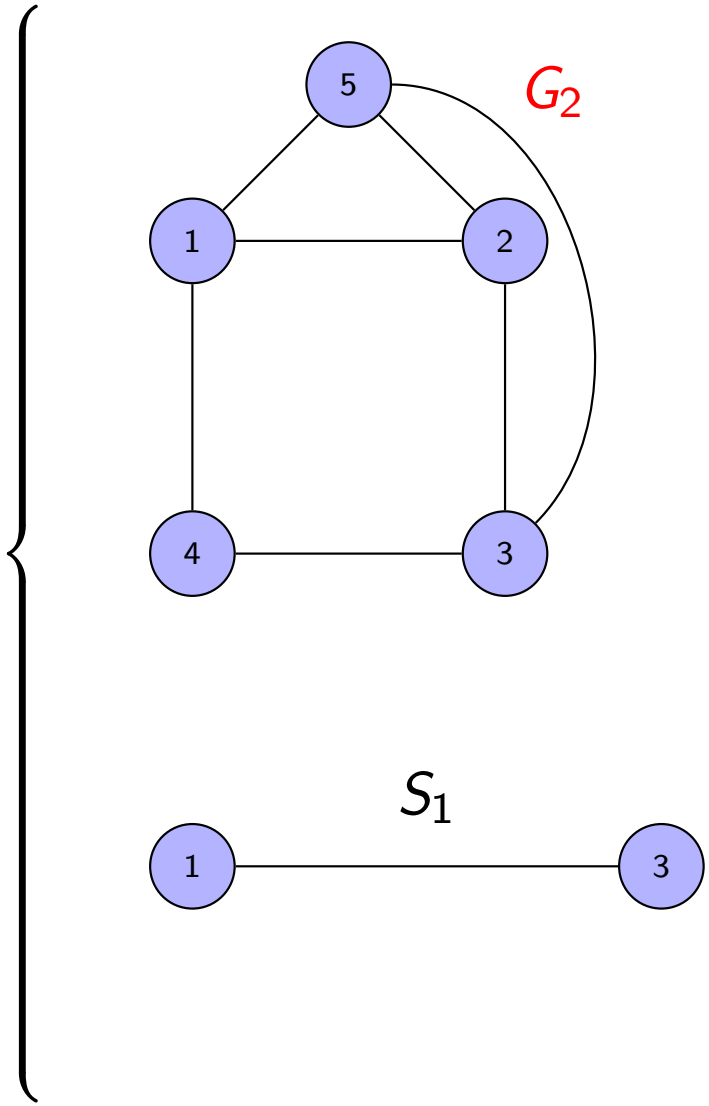

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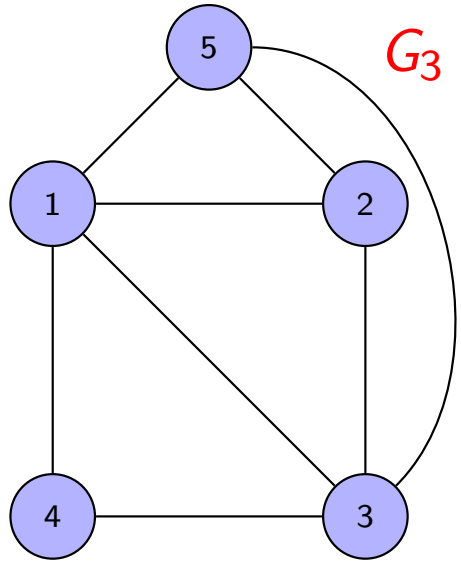

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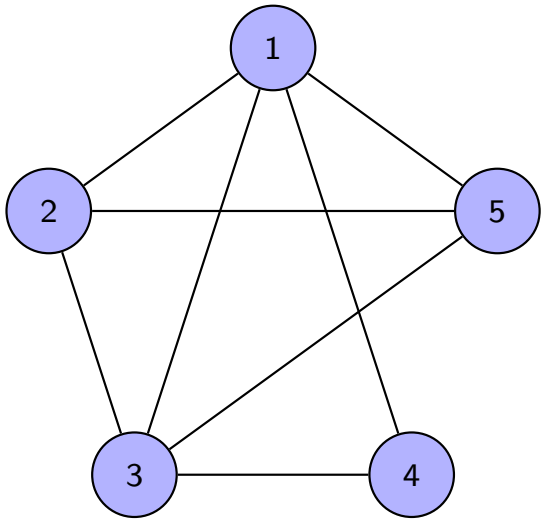

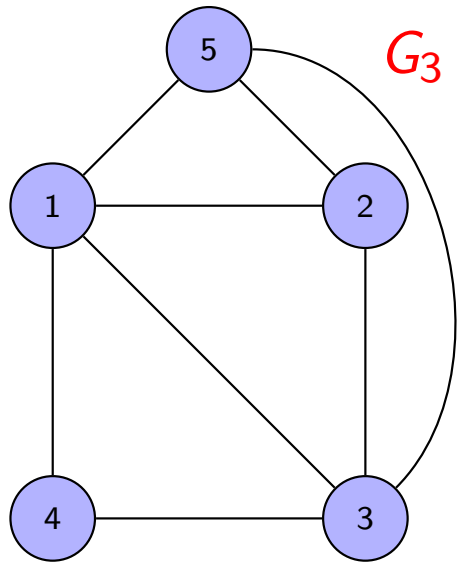

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return G_j

What about other surfaces?

Sphere?

What about other surfaces?

Sphere? same as plane

Annulus?

What about other surfaces?

Sphere? same as plane

Annulus? same as plane

Torus?

What about other surfaces?

Sphere? same as plane

Annulus? same as plane

Torus? different!