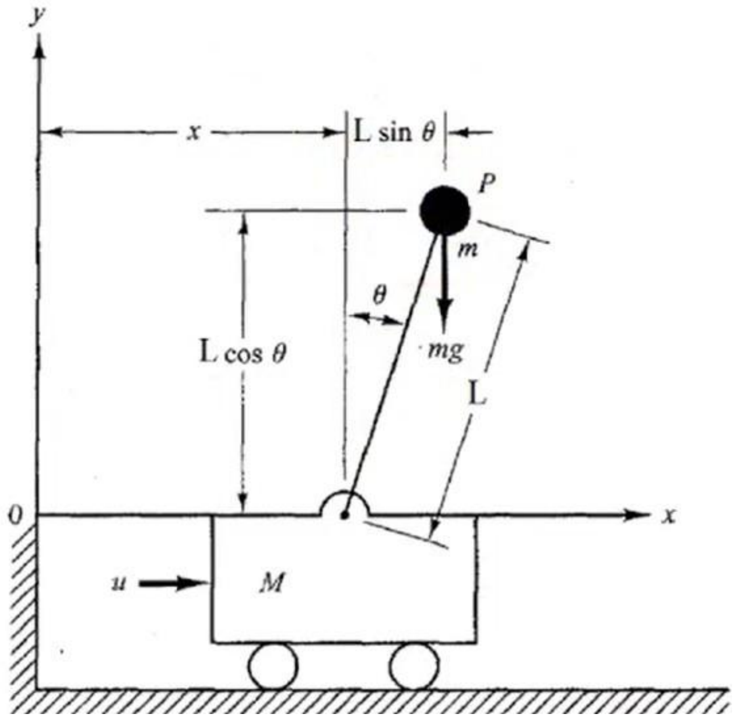
Lagrangian Calculation:

$$L = T - V$$



Part 1: Kinetic Energy(T)

1. **Pendulum:**

$$T_p = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2$$

And we have:

$$v = \frac{d}{dt}\big((x + l\sin(\theta))i + L\cos(\theta)j\big) = \big(i + L\dot{\theta}\cos(\theta)\big)i + \big(-L\dot{\theta}\sin(\theta)\big)j$$

$$\rightarrow \quad v^2 = \dot{x}^2 + \big(L\dot{\theta}\big)^2 + 2\dot{x}L\dot{\theta}\cos(\theta)$$

2. **Cart:** The cart has kinetic energy due to its horizontal motion

$$T_{cart} = \frac{1}{2} m_c \dot{x}^2$$

Thus, the total kinetic energy is equal to:

$$T = T_{pendulum} + T_{cart}$$
$$= \frac{1}{2} m_p \left[\left(\dot{x} + L\dot{\theta} \cos\theta \right)^2 + \left(L\dot{\theta} \sin\theta \right)^2 \right] + \frac{1}{2} m_c \dot{x}^2$$

Part 2: Potential Energy(V)

1. **Pendulum:** The pendulum has gravitational potential energy.

$$V_{pendulum} = m_p g L \cos\theta$$

2. **Cart:** the cart has no potential energy

$$V_{cart} = 0$$

Thus, the total potential energy is equal to:

$$V = V_{pendulum} + V_{cart} = m_p g L \cos\theta$$

Now the Lagrangian can be computed as follows:

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m_p \left[\dot{x}^2 + \left(L\dot{\theta} \right)^2 + 2\dot{x}L\dot{\theta} \cos(\theta) \right] + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m_c \dot{x}^2 - m_p g L \cos\theta$$

As we know, the Euler Lagrange Equation is as follows:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

Where, Q_i generally represents external forces.

Here, since we have two degrees of freedom, we have to solve the Euler-Lagrange equations twice.

1. **For x (cart's horizontal position)**

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F$$

1. **Computing $\frac{\partial \mathcal{L}}{\partial \dot{x}}$:**

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_c\dot{x} + m_P\left(\dot{x} + L\dot{\theta}\cos\theta\right)$$

2. **Computing $\frac{\partial \mathcal{L}}{\partial x}$:**

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

3. **Final Equation:**

$$\left(m_c + m_p\right)\ddot{x} + m_p L\ddot{\theta}\cos\theta - m_p L\dot{\theta}^2\sin\theta = F$$

2. For θ(pendulum angle)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

1. Computing $\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_p L^2 \dot{\theta} + m_p L \dot{x} \cos\theta + I\dot{\theta}$$

2. Computing $\frac{\partial \mathcal{L}}{\partial \theta}$:

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_p L \dot{x} \dot{\theta} \sin\theta + m_p g L \sin\theta$$

3. Final Equation:

$$m_p L^2 \ddot{\theta} + m_p L \ddot{x} \cos\theta - m_p g L \sin\theta + I\ddot{\theta} + m_p L \dot{x} \dot{\theta} \sin\theta = 0$$

Finally, as the Euler-Lagrange model we have:

$$\begin{cases} m_p L^2 \ddot{\theta} + m_p L \ddot{x} \cos\theta - m_p g L \sin\theta + I\ddot{\theta} + m_p L \dot{x} \dot{\theta} \sin\theta = 0 \\ (m_c + m_p)\ddot{x} + m_p L \ddot{\theta} \cos\theta - m_p L \dot{\theta}^2 \sin\theta = F \end{cases}$$

Solving for $\ddot{x}$ and $\ddot{\theta}$ we get:

$$\begin{cases} \ddot{\theta} = f_1(\theta) + g_1(\theta)F \begin{cases} f(\theta) = \dfrac{(m_c + m_p) g \sin\theta - m_p L \sin\theta \cos\theta \dot{\theta}^2}{(m_c + m_p sin^2\theta)L} \\ g(\theta) = \dfrac{-\cos\theta}{(m_c + m_p sin^2\theta)L} \end{cases} \\ \ddot{x} = f_2(x) + g_2(x)F \begin{cases} f(x) = \dfrac{-m_p g \sin\theta \cos\theta + m_p L \sin\theta \dot{\theta}^2}{m_c + m_p sin^2\theta} \\ g(x) = \dfrac{1}{m_c + m_p sin^2\theta} \end{cases} \end{cases}$$

To turn it into the state space form, we define the state variables as:

$$x = [x \ \dot{x} \ \theta \ \dot{\theta}]^T$$

Having:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ f_2(x_1) + g_2(x_1) \\ x_4 \\ f_1(x_3) + g_1(x_3) \end{bmatrix}$$

And as said in the exercise instructions, the model parameters are:

$$\begin{cases} m_c = 0.5\ Kg \\ m_p = 2\ Kg \\ L = 0.2\ m \\ g = 9.81\ \dfrac{m}{s^2} \end{cases}$$

s

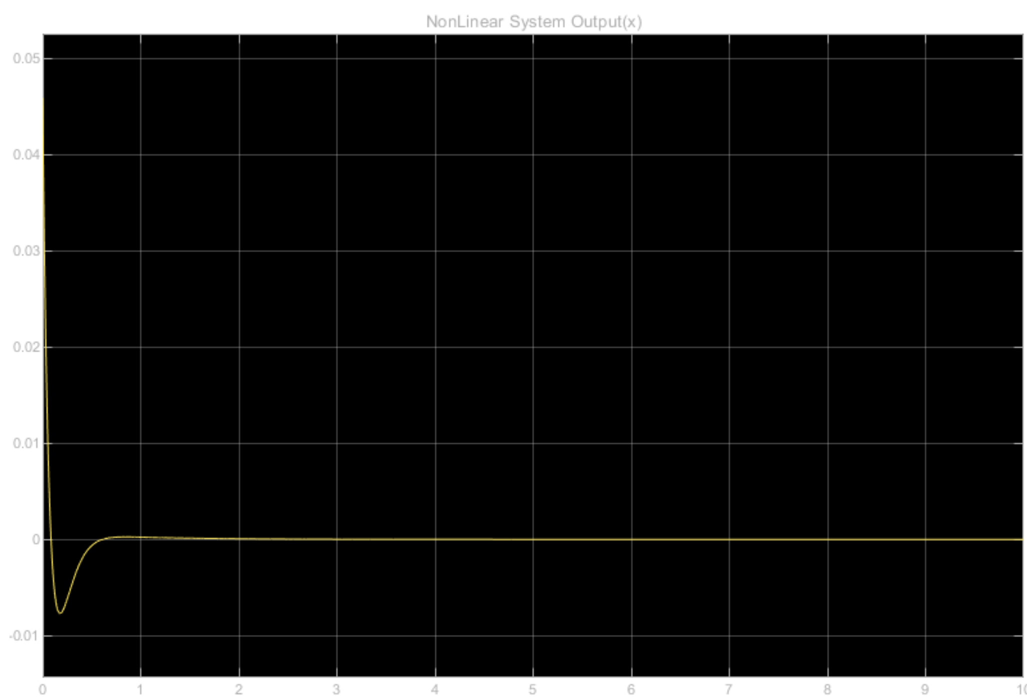
After this, the system is linearized by calculating its Jacobian and substituting the equilibrium point (which is 90 degrees because the pendulum axis is assumed to be upwards).

Then a PID Controller is designed using MATLAB PIDTuner and applied as a regulator to keep the system at zero.

Everything is done in MATLAB using Symbolic Math Toolbox and the system in simulated in Simulink.

And the results are as follows:

Non-Linear System Model:



Linearized System: