Convolution of a Gaussian with an exponential

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1 Derivation of the convolution

The convolution of a normalised (unit area) Gaussian and an exponential is derived below (special thanks to Dr Dave Forest (University of Birmingham) for the maths in Section 1.

Taking a Gaussian function, *G(x)*, of full-width at half maximum (FWHM) of $2\sigma\sqrt{2\ln(2)} = 2.35\sigma$ (see Section 2),

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}. \tag{1}$$

The exponential function is defined with decay constant, λ, as,

$$F(x) = e^{-\lambda x}. \tag{2}$$

The convolution is given by,

$$\begin{aligned} I(y) = F(x) * G(x) &\equiv \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\lambda x} e^{-\frac{(y-x)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\frac{(x^2+2\sigma^2\lambda x-2xy+y^2)}{2\sigma^2}} dx. \end{aligned} \tag{3}$$

Now use the transformation,

$$z = \frac{1}{\sqrt{2}\sigma}\left(x - (y - \sigma^2\lambda)\right). \tag{4}$$

This yields:

$$\begin{aligned} dx &= \sqrt{2}\sigma dz, \\ \Rightarrow \text{ at } x = \infty,\ z = \infty \text{ and at } x = 0, \quad z &= \frac{-(y-\sigma^2\lambda)}{\sqrt{2}\sigma}, \\ z^2 &= \frac{(x^2 + y^2 + \sigma^4\lambda^2 - 2xy + 2\sigma^2\lambda x - 2y\sigma^2\lambda}{2\sigma^2}, \\ \Rightarrow \frac{(x^2 + 2\sigma^2\lambda x - 2xy + y^2)}{2\sigma^2} &= z^2 + \lambda\left(y - \frac{\sigma^2\lambda}{2}\right). \end{aligned} \tag{5}$$

The results from Eqns. 5 above, can be substituted into Eqn. 3, leading to,

$$I(y) = \frac{1}{\sqrt{\pi}} e^{-\lambda\left(y - \frac{\sigma^2\lambda}{2}\right)} \int_{\frac{-(y-\sigma^2\lambda)}{\sqrt{2}\sigma}}^{\infty} e^{-z^2} dz. \tag{6}$$

The error function, erf(x), can be introduced as follows:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz. \tag{7}$$

and corresponds to the area under the Gaussian curve between $-x$ and x. However, erf(x) is defined for all values of x and is an odd function, such that erf$(-x) = -$erf(x). See Fig. 1. Correspondingly, the complementary error function, erfc(x), is the area outside of $-x$ and x implying,

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz. \tag{8}$$

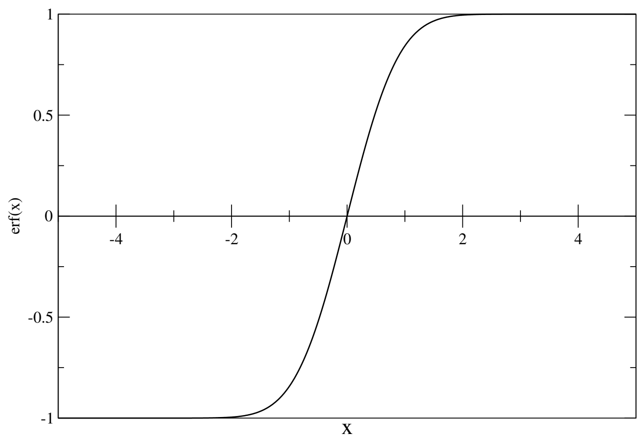

Figure 1: A plot of the error function, erf(x).

Using the above result (Eqn. 8) in Eqn. 6 leads to,

$$\begin{aligned} I(y) &= \frac{1}{\sqrt{\pi}} e^{-\lambda\left(y - \frac{\sigma^2\lambda}{2}\right)} \left[\frac{\sqrt{\pi}}{2} \mathrm{erfc}\left(\frac{-(y-\sigma^2\lambda)}{\sqrt{2}\sigma}\right)\right] \\ &= \frac{1}{2} e^{-\lambda(y - \frac{\sigma^2\lambda}{2})} \left[1 - \mathrm{erf}\left(\frac{-(y-\sigma^2\lambda)}{\sqrt{2}\sigma}\right)\right]. \end{aligned} \tag{9}$$

The final result for the convolution is, therefore,

$$\boxed{I(y) = \frac{1}{2} e^{-\lambda(y - \frac{\sigma^2\lambda}{2})} \left[1 + \mathrm{erf}\left(\frac{(y-\sigma^2\lambda)}{\sqrt{2}\sigma}\right)\right]. \qquad (10)}$$

For a non-zero offset for the Gaussian centroid, y is replaced by $y - k$, where k is the offset.

In `halflife.c` the `eval()` function calls the `gauss_exp()` function, where Eqn. 10 is evaluated, with the difference being that `gauss_exp()` expects FWHM (rather than sigma — see Section 2 below) and half-life (rather than decay constant λ or mean-life τ).

2 Gaussian FWHM

The FWHM is defined as the distance between two points either side of the Gaussian centroid at which the height falls to half of the height at the centroid, *i.e.*

$$FWHM = 2|x - y| \tag{11}$$

where the centroid lies at x and the Gaussian fall to half the centroid height at y, Such that,

$$\begin{aligned}
[G(x) - G(y)] = [G(x) - \frac{G(x)}{2}] &= \frac{1}{2}G(x), \\
\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\left[e^{\frac{-x^2}{2\sigma^2}} - e^{\frac{-y^2}{2\sigma^2}}\right] &= \frac{1}{2\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}, \\
\Rightarrow \frac{1}{2}e^{\frac{-x^2}{2\sigma^2}} - e^{\frac{-y^2}{2\sigma^2}} &= 1, \\
\Rightarrow e^{\frac{-x^2}{2\sigma^2}} &= 2e^{\frac{-y^2}{2\sigma^2}}, \\
\Rightarrow \frac{(y^2 - x^2)}{2\sigma^2} &= \ln(2), \\
\left(y^2 - x^2\right) = (y - x)(y + x) = |y - x|^2 &= 2\sigma^2 \ln(2).
\end{aligned} \tag{12}$$

Therefore,

$$\boxed{FWHM = 2\sigma\sqrt{2\ln(2)} = 2.35\sigma. \qquad (13)}$$