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Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques

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Abstract

The purpose of this paper is to compare a variety of approximation techniques for valuing contingent contracts when analytic solutions do not exist. The comparison is made with respect to the differences in both the approximation theory and the efficiency of the computation algorithms. The focus of the computational comparison is upon binomial and finite difference methods applied to option valuation models with one stochastic variable. However, many of the results would generalize to pricing corporate securities, and also to certain aspects of problems involving multiple stochastic variables.

I. Introduction

Recent advances in the area of asset pricing theory have generated many partial equilibrium conditions describing the "no arbitrage" paths for asset prices. Assumptions about the effects of information on asset price changes lead to valuation models based on a variety of stochastic processes. Examples are the constant-variance diffusion model of Black and Scholes [3], the pure jump model of Cox and Ross [10], the combined jump-diffusion model of Merton [28], and the changing variance diffusion model of Geske [17]. In these papers, analytic solutions exist.[1] However, if complex payout or exercise contingencies are present, analytic solutions are rare. One example is valuing an American put on a dividend-paying stock. The works of Parkinson [28], Brennan and Schwartz [5], and Cox, Ross, and Rubinstein [11] demonstrate examples of different approximation approaches to the American put problem.

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[1] Other analytic solutions are by Cox [9], Roll [29], Rubinstein [30], and Geske and Johnson [21].

Recently, Geske and Johnson [22] have presented an analytic solution to the American put problem, with or without dividends. However, many valuation problems have no known analytic solutions. Some examples are callable and convertible coupon bonds, insurance contracts, and the term structure of interest rate models for bond valuation. Because no arbitrage partial equilibrium conditions can be derived, many of these problems may be solved by numerical approximation. The purpose of this paper is to provide a concise comparison of the approximation techniques that financial economists have used for one-dimensional valuation problems when no analytic solutions exist. Although much of the intuition presented here would carry over for higher-dimensional problems, the exact analysis and numbers would be different.

In Section II, both valuation and approximation principles are first described. Several numerical methods, including the Monte Carlo, binomial, and finite difference techniques are discussed. The stability, rate of convergence, and accuracy conditions are described and compared.

Section III presents a comparison of the binomial and a variety of finite difference techniques based on both accuracy and efficiency criteria. *It is important for the reader to realize that results different from those presented here may arise from the use of different computer hardware and/or software.* Although comparisons and discrepancies are sensitive to the particular implementation schemes, it will be shown that fundamental differences do stand out. For example, the pure binomial approximation appears to dominate all the finite difference schemes when either there are no payouts or a small number of options are being valued. However, for fixed-cash payouts or when valuing a large number of options, the explicit finite difference approach with logarithmic transformation appears to dominate. Section IV summarizes the paper and presents conclusions.

II. Valuation and Approximation Principles

The Black-Scholes partial differential equation is valid for many significant valuation problems where no analytic solutions have been found. This has led to considerable research employing numerical methods to approximate solutions. The next few pages of this paper attempt to explain some important principles and techniques for valuation by approximation.

Although the focus of this paper is on diffusion processes, different assumptions about the effects of information arrival on the changes in asset prices imply different stochastic processes; and, hence, different partial equilibrium conditions. Efficient markets imply the rapid reflection of information in asset prices. Thus, the arrival of "new" information often will be accompanied by price changes. If the underlying asset is assumed to follow a diffusion process, then price changes are continuous. Alternatively, if the underlying asset is assumed to follow a jump process, price changes are discontinuous. In the diffusion case, information is thought to arrive in a smooth, continuous fashion; price changes can have either a constant or a changing variance. The jump process signifies that the information arrival is discontinuous. A diffusion process implies that asset price changes are either Normally or lognormally distributed, while a jump process implies a Poisson distribution. Casual empiricism leads one to suspect that a combined diffusión-jump process is generating the data.

The no-arbitrage partial equilibrium conditions have been derived for the pure diffusion, pure jump, and combined processes, and some analytic solutions have been found for each case. However, in many complex but realistic problems, numerical methods must be employed to approximate the value of the asset. There is a branch of mathematics devoted to this topic and from this work financial economists have to date employed Monte Carlo simulation [4], finite differences [5] and [6], numerical integration [28], and binomial processes [11]. This has by no means exhausted the many methods of solution available. Collocation, finite elements, and integral transform techniques are other approaches.[2] In the next subsections, some of the techniques currently in use are described and compared in terms of truncation error, stability, and convergence. The primary focus is on the binomial and finite difference approaches applied to one-dimensional, lognormal-diffusion option valuation problems.

The constant-variance diffusion approach to asset price changes has led to the now well-known parabolic partial differential equation for option valuation.[3] This equation is

$$0 = D + V_t + \frac{\sigma^2}{2} S^2 V_{ss} + (rS - D) V_s - rV \tag{1}$$

where

$V =$ Value of the option
$S =$ Value of the state variable (i.e., stock price)
$\sigma =$ Standard deviation of stock returns
$r =$ Continuously compounded risk-free rate of interest
$D =$ Dividend payout (continuous)
$t =$ Time to expiration

and subscripts denote partial derivatives. Equation (1) is subject to a variety of boundaries regarding expiration, exercise, and payout conditions. Expiration boundary conditions differentiate put options, which give the right to sell, from call options, which give the right to buy the underlying stock S for the options' exercise price X. Holders of put options receive the maximum of the exercise price minus the stock price or zero at expiration,

$$P(S,0) = \max(X - S, 0) \tag{2}$$

[2] See [13].

[3] Garman [16] has derived the fundamental partial differential equation which all assets, including derivative assets such as options, must follow under diffusion state processes. The Black-Scholes equation for options is a special case of Garman's equation. Both equations are second order because of the continuous time diffusion assumption, and specifically parabolic, rather than hyperbolic or elliptic, because their discriminant is zero. See Friedman [15] for further discussion of this.

and holders of call options receive the maximum of the stock price minus the exercise price or zero at expiration,

$$C(S,0) = \max(0, S-X). \tag{3}$$

If the option is American and can be exercised at any instant, the boundary conditions must be checked to see if for every possible stock price at each instant, the option is worth more held than exercised (i.e., dead or alive). Thus, if t^- is the instant before exercise and t^+ the instant after, then for put options,

$$P\left(S,t^-\right) = \max\left(X-S, P\left(S,t^+\right)\right) \tag{4}$$

and for call options,

$$C\left(S,t^-\right) = \max\left(C\left(S,t^+\right), S-X\right). \tag{5}$$

If the firm pays discrete cash dividends to stockholders at quarterly intervals, then at these ex-dividend dates occuring during the life of the option, the stock price must be reduced by the amount of the dividend to eliminate riskless arbitrage opportunities. If t^- is the instant before the ex-dividend date and t^+ is the instant after, then (in the absence of taxes)

$$S\left(t^-\right) = S\left(t^+\right) + D. \tag{6}$$

Merton [25] has shown that the exercise boundary condition for put options must be checked at every instant, but that call options may be exercised only at the ex-dividend dates. This implies that because more checks of exercise conditions are necessary, put options will be more expensive to value numerically than call options. Section III confirms this implication by showing that all approximation techniques are more efficient for call options than for put options.

The Black-Scholes partial differential equation has variable coefficients that make it more difficult to solve numerically than one with constant coefficients. Although an equation with variable coefficients can switch from a parabolic to either an elliptic or hyperbolic form, the Black-Scholes equation remains parabolic.[4] However, because the coefficients change as the state variables change with time, the approximation equations and stability conditions become more complex. Fortunately, Black and Scholes [3], Merton [25], and others have shown that by changing variables, equation (1) can be transformed into the following equation that has received considerable analytic and numerical analysis in the physical sciences

$$a\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} k \frac{\partial u}{\partial x} \tag{7}$$

[4] In the general form of a second-order partial differential equation, if coefficients which are nonzero (or zero) can become zero (or nonzero), the form of the equation can change. Because of limited liability, this cannot happen with the Black-Scholes equation.

where a and k may be at least twice-differentiable functions of x and t. This equation may have variable coefficients and it could be nonlinear if a and k are allowed to vary with u as well as with x and t. To use established numerical schemes of high accuracy, equation (7) with variable coefficients can be converted into one with constant coefficients by making the following transformation

$$y = \int \frac{1}{k(x)} dx \tag{8}$$

whereupon (7) becomes

$$\frac{\partial u}{\partial t} = \frac{1}{ak} \frac{\partial^2 u}{\partial y^2} . \tag{9}$$

In terms of the original variable x, this transform scheme in y will have unequal spacing of the net points.[5] Brennan and Schwartz [6] and Mason [24] used a form of this transformation by substituting y-ln(S) into equation (1). Section III will reveal in dollars per option valued the efficiency gained by this transformation. The next subsections describe several alternative approximation techniques.

A. Approximation Techniques

There are a variety of techniques for approximating either the underlying stochastic process directly or the resultant partial differential equation. The classic Monte Carlo simulation or the binomial process are both approaches to approximating the stochastic process directly. It is known that a binomial distribution converges to a Normal and a Poisson distribution, depending on how the limits are taken.[6] A mixture of the two is also possible (See [12].). For the Monte Carlo method, a number of simulations can be drawn from a lognormal distribution, a Poisson distribution, or a combination of the two. Thus, both the Monte Carlo and the binomial approximations can be directly used for pure diffusion, pure jump, or jump-diffusion valuation models.

Conversely, once the underlying process is assumed, the partial equilibrium condition resulting from no-riskless-arbitrage often can be derived. If an analytic solution to the partial differential equation cannot be obtained, finite difference methods or numerical integration can be used to approximate the solution. These techniques also can be used for pure diffusion, pure jump, or combined jump-diffusion models.

All the approximation techniques are performed in a space-time hyper-

[5] The exact form of the transformed equation depends on the change of variables. See [3], [25], [32], or [2] for examples of these tranformations.

[6] See [14] Chapters VI and VII.

plane.[7] The stock price-time space is divided into a set of points in the (S, t) plane given by $S = i\Delta S$ and $t = j\Delta t$, where $i = 0, 1, 2, \ldots n$ and $j = 0, 1, 2, \ldots m$. This division results in a net (or grid or lattice) whose mesh size (or ratio) is determined by ΔS and Δt. In the transformed space, ΔS is replaced by Δx. The approximations to the put and call values given by V in equation (1) (and rewritten as P and C in equations (4) and (5)) would be $P(i\Delta S, j\Delta t)$ and $C(i\Delta S, j\Delta t)$, and are denoted by P_i^j or C_i^j. For the transformed argument in equation (9), $u(x,t) = u(i\Delta x, j\Delta t)$ is denoted u_i^j. Figure 0 depicts this (S, t) grid.

The increments ΔS(or Δx) and Δt are thought of as small and when considering limiting processes, they approach zero. In application, the step sizes are not zero, and they need not be equal. However, the step sizes must be chosen to ensure stable, accurate, and efficient convergence to the solution. The stock price and time solution space is bounded in both put and call option valuation problems. In the time dimension, the expiration date, t, determines the maximum time allowed. In the stock-price space, limited liability determines the lower absolute bound, $SMIN = 0$, and a derivative condition determines the necessary upper bound, $SMAX$. For puts, $SMAX$ is the stock price above which $\partial P/\partial S$ approaches zero, and for calls $SMAX$ is the stock price above which $\partial C/\partial S$ approaches one.[8] (See Figure 0).

In the direct approximations of the underlying stochastic process, these upper and lower stock-price bounds may or may not be reached. In the Monte Carlo simulation, achieving the bounds would depend on the number of simulations, and for the binomial process this would depend on the size of the up and down jumps and on the number of jumps or trials. The time step-size is defined as $k = \Delta t = t/m$. The range of stock prices in the binomial process is determined by the size of the up and down jumps, which depend on the estimate of the variance of the underlying stock price changes.[9] In the binomial process, the net is a cone and selecting the time step determines the number of stock price steps at any time step and thus the mesh in the stock price-time net. (See Figure 0.)

For finite difference approximations, the time step is similarly defined by expiration. The stock price step size is defined as $h = \Delta S = (SMAX\text{-}SMIN)/n$. The finite difference net is rectangular (see Figure 0) and selecting the mesh size is often critical for ensuring stable, accurate convergence to the solution. The critical mesh ratio is known to be sensitive to the type of differencing procedure

[7] The space dimension is determined by the number of stochastic variables in the problem. In the simplest valuation problems the space is one dimensional. For example, in Black-Scholes option pricing the stock price is the single underlying stochastic variable. In Merton's [25] generalization to stochastic interest rates, the state space would be two-dimensional, with both a stochastic stock price and an interest rate. The focus of this paper is on one-dimensional stochastic problems, but many of the findings would carry over to multiple dimensions.

[8] Derivative boundary conditions are necessary for the finite difference approximations but not for the direct approximations to the underlying process, such as the binomial or Monte Carlo techniques. The hedge ratio derivatives, $\partial V/\partial S$, appear in equation (1). Their convergence to either zero or one for puts or calls as S increases can be determined from the Black-Scholes European solution. A useful rule of thumb is that the stock price will be about one-and-a-half times the exercise price for these derivatives to be satisfied.

[9] The up jump, defined as the reciprocal of the down jump, is shown to be $u = \exp(\sigma\sqrt{t/m})$. See [11] for details.

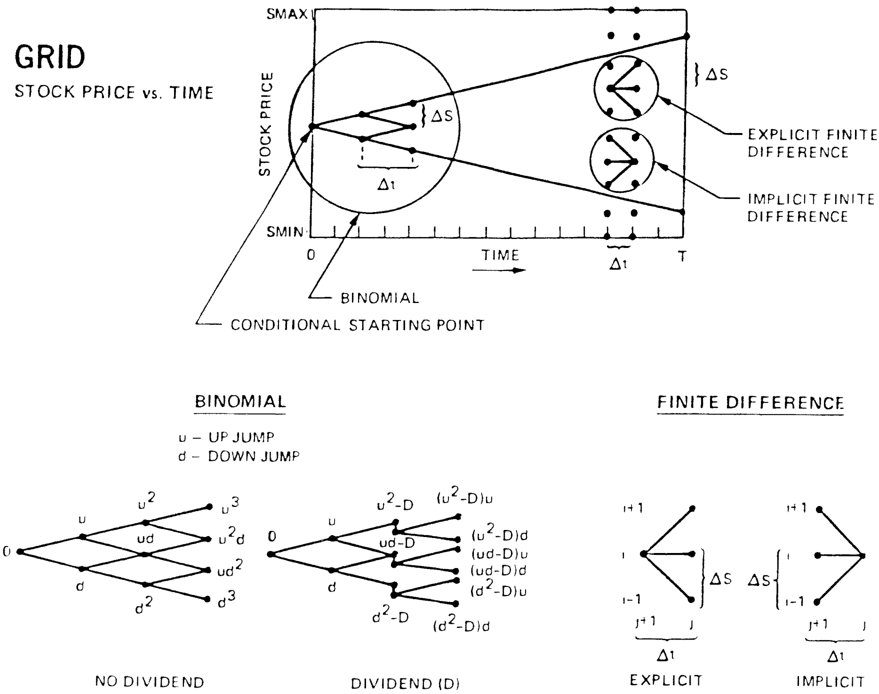

FIGURE 0

employed. Before discussing this, a brief description is presented of the Monte Carlo technique.

The Monte Carlo valuation method [4], relies on the observation by Cox and Ross [10] that when a riskless hedge can be formed, the option can be valued by discounting the expected value at expiration by the risk-free rate. The accuracy of this method depends on the number N of simulation paths used to form the distribution of stock prices at the expiration date. Generally, the accuracy increases as $1/\sqrt{N}$, so the computation cost approximately doubles as the error diminishes by about 70 percent. The Monte Carlo method can handle complex payout and exercise contingencies. However, when valuing American options, m lognormal distributions, one for each time step, must be approximated rather than just approximating one terminal distribution. Because a full set of sample paths is generated, conditioned on the starting point, multiple options for a variety of exercise prices and expiration dates can be valued. The conditional starting point makes the Monte Carlo method less efficient for valuing options for multiple stock prices.

The finite difference techniques analyze the partial differential equations (1) or (9) by using discrete estimates of the changes in the options value for small changes in time or the underlying stock price to form difference equations as approximations to the continuous partial derivatives. There are infinitely many ways to estimate the changes in the option's value with respect to time and the stock price. Forward, central, and backward differences, along with complex averages of them, can be used.[10] Although the variety of difference choices is large, all of them lead to solutions that can be classified as either explicit or implicit. In the explicit class, each unknown option price at any node can be solved explicitly in terms of previous known option price nodes, while in the implicit class a set of simultaneous equations must be solved.

An explicit finite difference approximation to equation (9) is

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = c\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} + O(\Delta x)^2 + O(\Delta t). \tag{10}$$

An implicit approximation to equation (9) is

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = c\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} + O(\Delta x)^2 + O(\Delta t) \tag{11}$$

where i and j denote stock price and time differences, respectively. Equations (10) and (11) use forward and backward time differences, respectively, relative to the time t (either j or $j+1$) that the stock price differences are expressed. In this notation, c is a transformation coefficient and $O(\cdot)$ represents the order of errors in the stock price and time approximations. Note that in the explicit equation (10), the unknown transformed option price at time $j+1$, u_i^{j+1}, can be solved for directly in terms of known prices at time j. However, in the implicit equation (11), the unknown transformed option price u_i^{j+1} depends on other adjacent and unknown prices at time $j+1$, and thus, a set of simultaneous equations must be solved. Figure 0 illustrates this distinction between explicit and implicit methods.

Figure 0 also shows that the binomial method is simply a form of the explicit finite difference scheme where the option price at stock price step i and time step $j+1$ is an explicit function of three previous option prices at time step j and at the stock price steps $i-1$, i, and $i+1$, respectively. In the coincident binomial technique, if a time step is skipped, then the current option price depends on three previous option prices (instead of two), exactly as in the explicit finite difference technique. Thus, the binomial technique is a special case of the explicit finite difference scheme with the major remaining distinction being its conditional starting point.

The conditional starting point for the binomial process allows for efficiency when a single option value is computed because the nodes in a cone rather than in a rectangle need to be evaluated. However, if option values for a variety of initial stock prices are desired, the binomial technique must be re-executed and becomes expensive. As an alternative to re-executing the binomial process for each initial stock price, two or more binomial processes could evolve simultaneously. Efficiency could be enhanced by eliminating near or overlapping stock price nodes. In the limit, as the number of initial stock prices became large, the bino-

[10] Forward differences use a "forward" differencing interval while backward differences use the opposite. For example, $(t_2 - t_1)$ where t_2 is closer to the present than t_1 could be a forward difference relative to t_1; then relative to t_1, $(t_1 - t_0)$ would be a backward difference. A central difference "centers" on t_1. For example, $(t_2 - t_0)/2$ would center on t_1.

mial technique would be exactly equivalent to the explicit finite difference scheme. However, as Section III demonstrates, the pure binomial method with its conditional starting point will not be as efficient as the explicit finite difference technique when multiple options are valued for a variety of initial stock prices.

Stability of the approximation scheme in all nets of mesh (Δt, Δx) often requires a specific ratio of time step to the square of stock price step, termed the mesh ratio R, defined as[11]

$$R \equiv \frac{\Delta t}{\Delta x^2}. \tag{12}$$

The explicit equation(10) can be written in terms of the mesh ratio as

$$u_i^{j+1} = qu_{i+1}^j + (1 - 2q)u_i^j + qu_{i-1}^j \tag{13}$$

where $q \equiv c(\Delta t/(\Delta x)^2) = cR$.
Note that if the mesh ratio R is selected to insure that $q = 1/2$, equation (13) is simplified. In matrix notation, for any stock price i, for all j,

$$u^{j+1} = Au^j \tag{14}$$

where A is a tridiagonal coefficient matrix and u^j and u^{j+1} are the successively calculated values of transformed option prices.

The implicit equation (11) can similarly be written as

$$-qu_{i+1}^{j+1} + (1+2q)u_i^{j+1} - qu_{i-1}^{j+1} = u_i^j. \tag{15}$$

In matrix notation, for any stock price i, for all j,

$$Bu^{j+1} = u^j. \tag{16}$$

To solve (16) for u^{j+1}, the matrix B must be inverted to yield $u^{j+1} = B^{-1}u^j$.

A general family of difference systems can be considered by taking a weighted average of the right-hand sides of the explicit and implicit methods in equations (10) and (11). If $g(x)$ is any function of x, define the differential operator ϕ so that $(\phi g)_i$ yields the central difference $g((i+1)\Delta x) - g((i-1)\Delta x)$; then the second differential, $(\phi^2 g)_i$, denotes $g((i+1)\Delta x)$ - $2g(i\Delta x) + g((i-1)\Delta x)$. With this notation, consider the difference system

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{\theta\left(\phi^2 u\right)_i^{j+1} + (1-\theta)\left(\phi^2 u\right)_i^j}{(\Delta x)^2} \tag{17}$$

[11] There is a voluminous amount of research on this subject. One excellent reference is [14], p. 388.

where θ is a constant in the interval $0 \leq \theta \leq 1$. When $\theta = 0$, the system is explicit in the form of equation (10); and when θ is not zero, the system is implicit, equivalent to the pure implicit form of equation (11) when $\theta = 1$. (The popular Crank-Nicholoson scheme is $\theta = 1/2$.)[12]

Equation (17) can be written as

$$\begin{aligned} &-qu_{i+1}^{j+1} + (h+2q)u_i^{j+1} - qu_{i-1}^{j+1} \\ &= q\theta h u_{i+1}^j + (h - 2q\theta h)u_i^j + q\theta h u_{i-1}^j \end{aligned} \tag{18}$$

where $h \equiv 1/(1-\theta)$. In matrix notation, for any stock price i, for all j,

$$Mu^{j+1} = Nu^j \tag{19}$$

where M and N are tridiagonal coefficient matrices of dimension $(n \times n)$. Thus, to solve a general family of finite difference schemes requires the inversion of the matrix M. If M^{-1} exists, equation (19) can be written as

$$u^{j+1} = M^{-1}Nu^j \equiv Gu^j. \tag{20}$$

The next few pages demonstrate that the truncation error, stability, and convergence properties of this general family of finite difference schemes depend upon the eigenvalues of the general $(n \times n)$ matrix G often called the amplification matrix. Note that the coefficients in equation (18) are the elements of the amplification matrix G and are dependent on q, a function of the mesh ratio. Thus, the eigenvalues of the amplification matrix will be sensitive to the mesh ratio.

B. Approximation Errors, Stability, and Convergence

If u_i^j in equation (9) is an exact solution and $\bar{u}(i\Delta x, j\Delta t)$ is an approximation, then the approximation error is termed $e_i^j = u_i^j - \bar{u}_i^j$, at the point $x = i\Delta x$, $t = j\Delta t$, $i = 1, \ldots n$, $j = 1, \ldots, m$. If the errors at each stage of the approximation grow, then the technique is not stable. Alternatively, if the errors become smaller at each time step, then the approximation converges. Larger time steps would imply fewer computations when moving from a future to a current stock price, but the errors propagated may grow and make the approximation unstable. Furthermore, the approximation may converge with finite error and the solution will not be exact. One way of evaluating the approximation is to examine the behavior of the error e_i^j as $j \to \infty$ for fixed Δx, Δt. A second and more interesting way is to examine the error e_i^j as the mesh is refined, so that Δx and $\Delta t \to 0$ for a fixed value of $m\Delta t$, $n\Delta x$. This is more valuable because the goal of an approximation is to force the error to zero in the limit so the solution becomes exact.

The mesh ratio, $R \equiv \Delta t/\Delta x^2$, obviously is instrumental to accurate and efficient approximations. Because, in the approximation limit, the number of calculations becomes infinite and the errors may be amplified without bound, a means of establishing proper stability and convergence criteria is necessary. There are

[12] See [2].

numerous methods for approaching these problems.[13] Here an examination of the eigenvalues of the solution matrices in conjunction with the errors propagated will illustrate the proper restrictions on the mesh ratio.

1. Errors and Convergence

To find the order of the error, assume that the exact solution of equation (9) has continuous partial derivatives. Then do a Taylor's series expansions of u_i^{j+1}, u_{i+1}^j, and u_{i-1}^j. Because u_i^j satisfies equation (9), $\partial u/\partial t$ can be replaced by $c(\partial^2 u/\partial x^2)$, and solving for (10) in terms of these expansions yields

$$\tag{21} \begin{aligned} &\frac{u_i^{j+1} - u_i^j}{\Delta t} - c\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} \\ &= \left[\frac{1}{2}\Delta t\left(\frac{\partial^2 u}{\partial t^2}\right)_i^j - \frac{c}{12}(\Delta x)^2\left[\left(\frac{\partial^4 u}{\partial x^4}\right)_i\right]^j\right] + \ldots \end{aligned}$$

This result demonstrates that, as stated in (10), the error for the explicit method[14] is of order Δt and $(\Delta x)^2$ as Δt, $\Delta x \rightarrow 0$.

2. Stability

Recall equation (20) demonstrated that a family of explicit or implicit finite difference schemes could be written as $u^{j+1} = G u^j$, where G represents a general coefficient matrix and u^{j+1} and u^j are vectors whose components are successive values of transformed option prices. Using initial values u^0 to begin the solution process, the successive row calculations are $u^1 = G u^0$, $u^2 = G u^1 = G^2 u^0, \ldots,$ so that ultimately

$$u^j = Gu^{j-1} = G^j u^0$$

where the superscripts on G are exponents.

To trace the effects of errors through the calculations suppose that the approximation has an initial error so that $e^0 = u^0 - \bar{u}^0$. Then the successive approximate calculations are again $\bar{u}^j = G\bar{u}^{j-1} = G^j\bar{u}^{-0}$ and the errors would be propagated by the same algorithm as the prices, implying $e^j = u^j - \bar{u}^j = G^j e^0$. Thus, the initial error e^0 is "amplified" by the coefficient matrix G raised to the power of j.

[13] Several related methods for analyzing these concepts are fourier series, energy conservation, solution boundedness, and eigenvalues. Fourier series is more versatile, and can be used with a wide degree of analytic precision; for example, which harmonics, or wave length multiples of the mesh ratio, become amplified. However, such precise information is not currently necessary in financial economics.

[14] This approach also can be used to demonstrate an identical order of error in equation (11) for the implicit method. Sometimes the error order can be reduced to $O[(\Delta t)^2]$ which is the same as $O[(\Delta x)^4]$ if Δt and Δx go to zero so that $c\ \Delta t/(\Delta x)^2 = 1/6$. See [8] for an example of this type of accuracy improvement.

Whether this error expands or contracts depends on the eigenvalues of G. To see this, note that in general the eigenvalues of these tridiagonalized matrices are distinct so the eigenvectors $z_1, \ldots, z_j$ are independent. Thus, the error vectors are simply linear combinations of the eigenvectors, and

$$e^0 = \sum_i c_i z_i$$

where the c's are constants. After j steps

$$e^j = Ge^{j-1} = \ldots = G^j e^0 = \sum_i G^j c_i z_i = \sum_i c_i \lambda_i^j z_i \tag{22}$$

where λ_i is the i^{th} eigenvalue and $Gz_i = \lambda_i z_i$. Recall that each eigenvalue λ_i will be a function of the mesh ratio R. Obviously, if the magnitudes of all G's eigenvalues are less than or equal to 1, the errors will not grow and the approximation will be stable. Imposing this condition after solving for the eigenvalues implies that for stability, when the normalized range on X is $0 \leqslant X \leqslant 1$, the mesh ratio must be[15]

$$\begin{aligned} &0 < R \equiv \frac{\Delta t}{(\Delta x)^2} \leqslant \frac{1}{2} \frac{1}{(1-2\theta)} \quad \text{if} \quad 0 \leqslant \theta < \frac{1}{2} \\ &\qquad\qquad \text{No restriction} \quad \text{if} \quad \frac{1}{2} \leqslant \theta \leqslant 1 . \end{aligned} \tag{23}$$

The most common choices for θ are 0, 1/2, 1. The implicit schemes for $\theta \geqslant$ 1/2 are unconditionally stable. By employing such an implicit scheme, one incurs the cost of solving systems of simultaneous equations but avoids all stability worries and can choose Δt by a tradeoff based on accuracy and efficiency. In the next section, a comparison is made in terms of accuracy and efficiency between the pure explicit scheme ($\theta = 0$), equation (10), a pure implicit scheme ($\theta = 1$), equation (11), both with and without logarithmic transformations, and the binomial technique.

III. Comparison of Techniques Used For Valuing Options

In this section, two explicit finite difference methods, two implicit finite difference methods, and the binomial method are compared for both their accuracy and efficiency in valuing put and call options with and without dividends. The following notation is used throughout the tables and graphs:

Black-Scholes	Analytic Solution
Binomial	Binomial Approximation
BFCD	Binomial Fixed Cash Dividend
BFDY	Binomial Fixed Dividend Yield

[15] See [13] or [2] for reference to this stability condition.

FDE1	Finite Difference Explicit #1
FDE2	Finite Difference Explicit #2 (Logarithmic transform of #1)
FDI1	Finite Difference Implicit #1
FDI2	Finite Difference Implicit #2 (Logarithmic transform of #1)

All comparisons are made with some or all of the following parameters:[16]

S Stock Price	= \$40.00
X Exercise Price	= \$35.00, \$40.00, \$45.00
r_F Risk: Free Rate	= 5.00 percent (annual)
σ Standard Deviation of Stock Return	= 0.3 (annual)
t Time to Option Expiration	= 1, 4, 7 months
D Dividend	= \$ .50, \$1.00, \$2.00, \$3.00, \$4.00 (quarterly)
t_D Ex-Dividend Dates	= .5, 3.5, 6.5 months

Sometimes this range of stock prices and exercise prices is expanded to analyze approximation efficiency of computing multiple option values.

All dividend comparisons are made solely for the quarterly \$ .50 dividend except for the comparison between the binomial fixed dividend yield (BFDY) as an approximation to the binomial fixed cash dividend (BFCD), where a variety of dividend amounts are compared. The options with expirations in one, four, and seven months each have one, two, and three \$0.50 quarterly dividends, respectively. In every case, the last quarterly dividend is paid one-half month prior to expiration. Assume throughout that the fixed cash dividend is not suspended except for those stock prices in the approximation grid that are less than the dividend. This suspension price could be changed easily to reflect a more realistic suspension level as was done in [19], but this change would not significantly alter the conditions.

As previously demonstrated, the stability of the approximation procedure may depend on the mesh ratio. If the method is stable as the number of steps used in each numerical method increases, convergence occurs when, to the nearest cent, the approximation value does not change for two successive increments. Options are valued using the binomial approximation by starting with 50 as the initial value for N time (and thus stock price) steps, and then by incrementing the number of steps until convergence occurred. Similarly, by using the finite difference approximations, the options were valued by starting with 50 steps in stock price (n) and 45 steps/month in time to maturity (m) and then incrementing each until convergence occurred. This approach to convergence was taken as a means of standardizing the criteria for efficiency comparisons. Thus, for some parame-

[16] These parameters were chosen to be consistent with those used in the forthcoming book by Cox and Rubinstein [12], and the published article by Cox, Ross, and Rubinstein [11]. The binomial approximation has been discussed and used by Cox and Rubinstein [12] with and without fixed dividend yields. The finite difference approximations have been discussed by Schwartz [31]; and Brennan and Schwartz discussed and used FDE1 [7], FDE2 [6], and FDI1 [5] in those papers, respectively. Recently, Geske and Shastri [19], [20] used FDE2 and FDI2.

ters, a small truncation error was tolerated but might have been eliminated by departure from the standardized mesh ratio adjustment. Finally, recall that, while all efficiency comparisons herein are naturally implementation-sensitive, such comparisons readily expose inherent fundamental advantages and disadvantages of each technique.

A. Call Options

Here the accuracy and efficiency of solution techniques for valuing American call options are compared, first without dividends and then with dividends. The no-dividend case for calls is presented as an initial calibration for comparing these numerical methods.[17] Merton [25] demonstrated the equivalence between American and European call values for stocks that do not pay dividends. Thus, all call-option values for stocks that do not pay dividends would usually be computed with the analytic Black-Scholes formula by using a polynomial approximation for the univariate normal distribution function.[18]

Approximating call values on stocks that pay fixed cash dividends is more complicated because of the positive probability of prematurely exercising just before each ex-dividend date. As will be shown, this reduces the binomial's time-skipping efficiency. In addition, because the binomial method is a path-dependent approximation to the possible stock price paths, its efficiency is diminished as the number of cash dividend payments, and thus paths, increases.

1. Calls without Dividends

Table 1 compares the convergence and accuracy of six solution techniques for nine call-option values when the parameters are as previously given and the underlying stock does not pay dividends. Note that all six methods first converged for the stock price and time step sizes reported. The binomial technique has no truncation error when its solution is compared to the analytic Black-Scholes solution, while all finite-difference techniques had a small truncation error of one or two cents for some parameters.

The first explicit finite difference method, FDE1, requires about seven times as many time steps per month (315 vs. 45) as the other finite difference approaches require for convergence. This disparity results because the explicit methods require a finer mesh ratio than do the implicit methods for stability. However, the log-transformed explicit method, FDE2, converges for the same mesh ratio as do these implicit methods because the transformed equation has constant coefficients.

For the no-dividend case, this binomial approximation used a large number of time steps ($n = 300$) that produced an extra fine partition of stock prices for the final stock price vector. Then, computational efficiency was enhanced by jumping backwards over time from the final vector of stock (and thus option prices) to the initial option value. This time jumping can be done because of the binomial formula, and because with no probability of premature exercise, it is

[17] All computations were done on an IBM 3033 computer at the University of California, Los Angeles. The programming language was Fortran.

[18] See [1], page 932, for examples of these polynomial approximations.

TABLE 1
Call Option Values for S = \$40.00, σ = 0.3, and r_f = 5%

Exercise Price (\$)	Solution Technique	Time to Maturity (months)		
		1.0	4.0	7.0
	Black-Scholes	5 22	6.25	7.17
	Binomial [a]	5 22	6 25	7.17
35.00	FDE1 [b]	5.22	6.26	7.19
	FDE2 [c]	5.22	6.26	7.19
	FDI1 [c]	5.22	6.26	7.19
	FDI2 [c]	5.23	6 26	7.19
	Black-Scholes	1.46	3.07	4.19
	Binomial	1.46	3.07	4.19
40.00	FDE1	1 46	3.08	4 20
	FDE2	1 47	3.08	4 20
	FDI1	1 46	3.08	4.20
	FDI2	1 46	3.08	4 20
	Black-Scholes	0.16	1.25	2.24
	Binomial	0.16	1.25	2.24
45.00	FDE1	0.16	1.26	2 24
	FDE2	0.17	1.26	2.25
	FDI1	0.16	1.26	2.24
	FDI2	0.17	1.26	2.25

a. The values for the binomial approximation are calculated for 300 steps.
b. The values for FDE1 are for 200 steps in stock-price and 315 steps/month in time.
c. The values for FDE2, FDI1, and FDI2 are for 200 steps in stock-price and 45 steps/month in time (for FDE2 and FDI2, N_1 = 3).

not necessary to calculate the intermediate option values. However, this is not true for the finite difference methods even with no probability of early exercise.

For the logarithmically transformed, finite difference methods ($y = \ln S$ in FDE2 and FDI2), a different stock price step size was used in the two stock price ranges from zero to one dollar, [0,1], and from one dollar to infinity, [1, ∞], approximated by [SMIN, 1] and [1, SMAX]. For computational accuracy and efficiency, it is not necessary to partition the stock price region of zero to one dollar as finely in the logarithmically transformed space of [$-\infty$, 0] as in the symmetric region [0, ∞]. Thus N_1, the number of stock price steps in the zero-to-one-dollar range, was set equal to three (N_1, = 3) for both the explicit and implicit methods, FDE2 and FDI2.

Table 2 presents the computing costs for the results in Table 1. Here the efficiency is compared for computing all nine option values for the approximation techniques.[19] The cost for nine options is primarily a function of the central processing unit (CPU) time.[20]

[19] Although it is very inexpensive, the analytic Black-Scholes solution is not included in this comparison because it cannot be used later for the more complex option problems.

[20] The formula for computing costs on this IBM 3033 is: Cost = (CPU TIME + (0:007)I/O

TABLE 2
Computing Costs for Results in Table 1

Solution Technique	CPU Time (secs)	I/O Requests	Core	Cost for 9 Options [a]
Binomial $N = 300$	0.26	67	145K	$ 0 21
FDE1 $n = 200, m = 2205$	39.75	104	150K	$11.90
FDE2 $n = 200, m = 315$	3.94	108	150K	$ 1.38
FDI1 $n = 200, m = 315$	7.35	123	170K	$ 2.47
FDI2 $n = 200, m = 315$	6.60	118	160K	$ 2.21

a. Costs for computer use at the UCLA Computing Facility are calculated as:
(CPU Time + (0:007) I/O Requests)(1 + 0:00135 Min (Core, 500) + 0:00015 Core)(0:24).

The least expensive solution technique is the binomial and it also uses the least CPU time (and I/O requests and core size). In fact, the ranking by least to most expensive, which is identical to the ranking by least to most CPU time, is Binomial, FDE2, FDI2, FDI1, and FDE1, respectively. Of the four finite difference techniques compared, the logarithmically transformed methods are most efficient, and the logarithmically transformed explicit (FDE2) is more efficient than the implicit (FDI2) one, because it does not require the solution of simultaneous equations.

The CPU time comprises compilation and execution times. Once either the binomial or the finite difference programs are compiled they can be executed, repetitively if necessary to compute multiple option values for a variety of time to expiration, stock, and exercise prices without being recompiled. In general, for computing multiple option values, the binomial technique with time skipping must be executed once for each option value while the finite difference methods need be executed only once for all option values.[21]

Figure 1 graphically compares four of the solution techniques in Table 2, omitting the most costly untransformed explicit method, FDE1. The graph dem-

REQUESTS) (1 + 0:00135 MIN(CORE, 500) + 0:00015 CORE) (0:24). The critical factor in this equation is CPU time. Core size and I/O requests have a small effect on cost. In fact, the standard measure is CPU time and cost alone, because the core and I/O measures have negligible impact. For example, reducing the core size by factors of 10 or 100 (i.e., from 145K to 14.5K or 1.45K) for the binomial example in Table 2 reduces cost only from $.21 to $.179 and $.175, respectively.

[21] This implies the following CPU cost equations: Binomial cost = [(Compilation Cost)/(# Options Valued)] + Execution Cost; Finite Difference Costs = (Compilation + Execution Cost)/(# Options Valued). These formulas suggest that as the number of options valued increases, the finite difference techniques should become less expensive than the binomial method on a per-option basis.

onstrates that as the number of options increases, the finite difference cost per option approaches zero while the binomial cost per option valued is asymptotic to the execution expense. When more than 300 (460, 610) options are being valued FDE2 (FDI2, FDI1), is less expensive per option than the binomial method. Thus, for individuals or firms computing a large number of option values, the finite difference methods may be more cost-efficient. These conclusions must be reconsidered when valuing the typical call option on a dividend-paying stock.

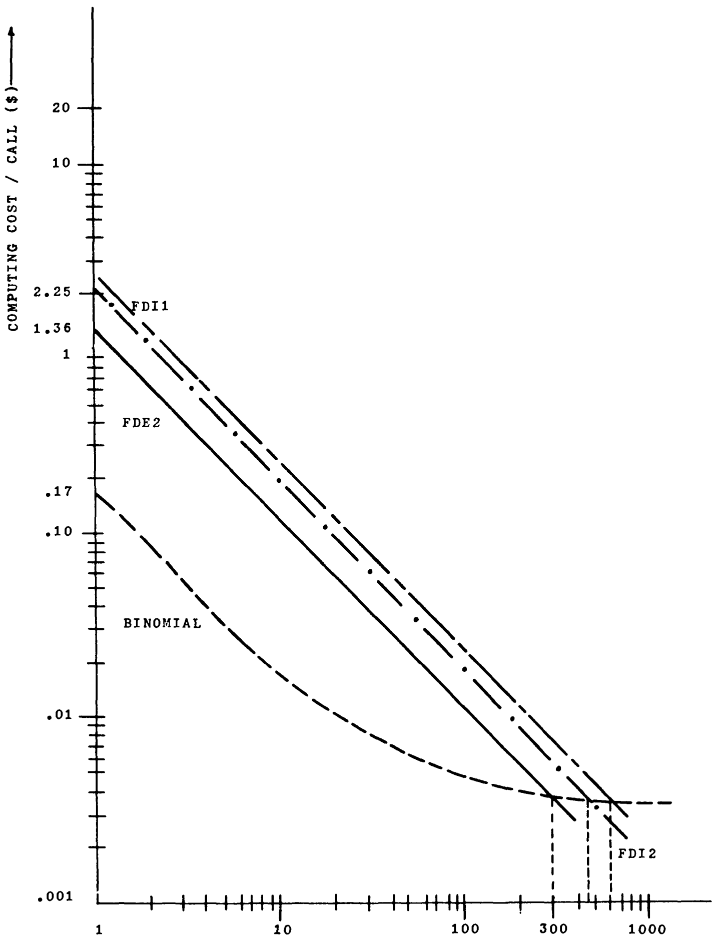

FIGURE 1

Cost Comparison of Binomial and Finite Difference Methods for American Calls on Non-Dividend Paying Stocks

2. Calls with Dividends

Most financial securities have either contracted or expected payouts. While the payouts can be discrete or continuous, and either a fixed dollar amount, a fixed yield, or stochastic, the most common type of stock dividend is an expected, discrete, fixed cash dividend. The reason is that typically, corporations maintain a stable quarterly dividend policy. All payouts complicate the valuation process, but a discrete, fixed cash dividend is one of the most difficult for continuous-time option techniques because each discrete payout requires an additional boundary condition and may alter the parameters of the stochastic process.

Here, the five approximation techniques considered in the no-dividend case are re-examined when the underlying stock pays discrete quarterly dividends. The finite difference techniques can accommodate discrete cash dividends more easily than can the binomial technique because the density of the binomial grid partition increases at each ex-dividend date. To see this, recall that in the simple coincident binomial process, the number of stock price steps grows by one for each additional time increment prior to the first ex-dividend date. However, after each dividend payment, the binomial process is no longer coincident and for each additional time step the number of stock price steps increases by $2 (n + 1)$ where n is the number of the specific time step. Figure 0 depicts the binomial with discrete dividends. For a relatively small number of discrete fixed payouts, the binomial process "explodes," becoming computationally impractical. Currently listed stock options have a maximum expiration of nine months for a maximum of three dividends during the life of any option. One discrete cash dividend seriously compromises the efficiency of the pure binomial process, and three erode it entirely.

If a stock pays a fixed dividend yield, the binomial process will remain coincident since up/down and down/up movements after the ex-dividend date will lead to coincident points. Thus, a fixed dividend yield (BFDY) will be a more efficient binomial solution technique. However, a fixed dividend yield is an approximation to the actual fixed cash dividend that most corporations pay.

If the fixed dividend yield BFDY is a good approximation to the fixed cash dividend BFCD, this accuracy may circumvent the binomial's efficiency problems caused by the exploding grid partition. Unfortunately, the fixed dividend yield assumption will produce incorrect hedge ratios. Even with a fixed dividend yield, the efficiency of the binomial process will be reduced whenever there is a positive probability of prematurely exercising an American call option at each ex-dividend date. Here we cannot achieve the economy gained by jumping time from the final expiration date to the current date. Instead, only smaller jumps are permissible to each intermediate ex-dividend date where the boundaries for early exercise must be checked.

For all finite difference methods, the fineness of the grid partition is not affected by discrete cash dividends. Instead, the grid mesh is kept constant and interpolation is performed. Previous tables for valuing call options with no dividends showed that the implicit methods were more economical than the explicit techniques due to the former's less stringent stability requirements. Furthermore, the logarithmic transformation enhanced the efficiency of both techniques and the increased economy for the explicit method was dramatic. This increased efficiency of the logarithmic transformation is still obtainable when the stock pays discrete cash dividends. However, due to the nonlinearity of the logarithmic transformation at each ex-dividend date, the grid must be transformed from the logarithm of the stock price back to stock price space, the dividend paid, and then retransformed back to the logarithm of the stock price. Fortunately, this back and forth use of the logarithmic transformation at each ex-dividend date does not destroy its efficiency. (Figure 2 demonstrates this point.)[22]

Table 3 compares the convergence and accuracy of six solution techniques for nine call option values when the parameters are as given previously in Table 1. Here the stock pays a $0.50 quarterly cash dividend. The binomial fixed yield, BFDY, is 5 percent annually or 1.25 percent quarterly, which is equivalent to a $0.50 cash dividend when the stock price is $40.

TABLE 3

Call Option Values for S = \$40.00, σ = 0.3, r = 5%, and D = \$0.50

Exercise Price ($)	Solution Technique	Time to Maturity (months)		
		1.0	4.0	7.0
	BFCD [a]	5 10	5.73	6.34
	BFDY [b]	5 10	5.74	6.34
	FDE1 [c]	5 10	5.74	6 35
35.00	FDE2 [d]	5.10	5.74	6 35
	FDI1 [d]	5.10	5 74	6.34
	FDI2 [d]	5.10	5 74	6 35
	BFCD	1.27	2 69	3 55
	BFDY	1 27	2.68	3 54
	FDE1	1.27	2 70	3 57
40.00	FDE2	1.27	2.70	3 57
	FDI1	1.26	2 69	3 56
	FDI2	1 26	2.69	3 57
	BFCD	0.12	1.04	1 82
	BFDY	0.12	1 03	1 80
	FDE1	0 12	1 04	1 83
45.00	FDE2	0 13	1.05	1 84
	FDI1	0 12	1.04	1 82
	FDI2	0.13	1.05	1 84

a. The values for the fixed dividend approach to the binomial approximation are calculated for 140, 160, and 140 steps for 1.0, 4.0, and 7 0 month maturities, respectively

b. The values for the fixed dividend yield approach to the binomial approximation are calculated for 140, 200, and 210 steps for 1 0, 4.0, and 7 0 month maturities, respectively.

c. The values for FDE1 are calculated using 200 steps in stock price and 320 steps/month in time.

d The values of FDE2, FDI1, and FDI2 are calculated using 200 steps in stock price and 80 steps/month in time.

First note that all six methods converged for the stock price and time step sizes reported. It was necessary to make the step sizes in Table 3 different from

[22] An analytic solution to the problem of valuing an American call option on a dividend paying stock (see [29], [17], and [33]) might be more efficient than an approximation. The purpose of this paper is to compare alternative approximation techniques, so these analytic solutions are not considered.

those in Table 1 because of dividends. The steps were chosen to insure convergence subject to the restriction that the dividend payments and expiration dates fall on and not between time steps. This choice avoids interpolation errors that would diminish accuracy. All the methods are accurate and are within one or two cents of each other. The binomial fixed dividend yield closely approximates the fixed cash dividend for these nine call option values. The quality of this fixed dividend yield approximation is examined in Table 4.

A graphical summary of the efficiency of these solution techniques is provided by Figure 2 (other graphs are available upon request). Here the cost per call option computed versus the number of options valued is presented for two dividends. The graph shows that the logarithmically transformed finite difference (FDE2) is less expensive than the binomial fixed cash dividend (BFCD) in every case. BFDY is quickly dominated by FDE2 for valuing multiple call options with many stock prices due to the binomial's conditional starting point that necessitates re-execution for each value. However, FDE2 is dominated by BFDY when computing multiple call options with many different exercise prices.

Table 4 compares the binomial fixed dividend yield as an approximation to the fixed cash dividend for quarterly cash dividends ranging from \$0.50 to \$4 and equivalent quarterly yields ranging from 1.25 to 10 percent when the stock price is \$40. It is surprising that the biases previously mentioned approximately cancel. BFDY is a very accurate approximation to BFCD for the one- and two-dividend cases (i.e., expirations of one and four months). Even in the three-dividend case, the maximum error was 5 cents for a cash dividend of \$2 when the exercise price and option's expiration date were \$35 and seven months, respectively.

B. Put Options

At every instant, American put options have a positive probability of premature exercise regardless of whether the stock pays dividends [26]. Thus, there is always a critical stock price, independent of the current stock price, below which it is optimal to exercise the American put. Because of this possibility of early exercise, an exercise condition comparing the value of the put if held to the value if exercised must be checked at every instant.

A main advantage of the binomial method over all finite difference techniques for valuing American call options is attributable to the binomial formula that allows jumping over many time steps and computing values at only the ex-dividend dates. This computational advantage is obviously not plausible when valuing American puts, even if the stock pays no dividends. Discrete cash dividend payments will further complicate the binomial technique because of the "exploding" tree problem.

In the next few pages, the accuracy and efficiency of several solution techniques for valuing American put options are compared, first with no dividends in subsection 1, and then with dividends in subsection 2. The binomial fixed cash dividend (BFCD) approach is not presented because it is always dominated by the finite difference techniques. Only the binomial fixed dividend yield (BFDY) is presented for put valuation. To check the convergence and approximation error of the binomial fixed dividend yield, put values for a logarithmically transformed

TABLE 4

Comparison of Binomial Approximations: Fixed Cash Dividend vs. Fixed Dividend Yield
$S = \$40.00, \sigma = 0.3, r = 5\%$

Exercise Price ($)	Time to Maturity (Mos)	Dividend Amount/Dividend Yield									
		$0 50/1 25%		$1 0/2 5%		$2 0/5 0%		$3 0/7 5%		$4 0/10%	
		BFCD	BFDY	BFCD	BFDY	BFCD	BFDY	BFCD	BFDY	BFCD	BFDY
	1 0	5.10	5 10	5 09	5 09	5 08	5 09	5 08	5 08	5 08	5.08
35 00	4.0	5.73	5 74	5 40	5 41	5 17	5 19	5 11	5 12	5 10	5.10
	7.0	6 34	6 34	5 76	5 79	5 24	5 29	5 12	5 15	5 10	5.11
	1 0	1 27	1 27	1 17	1 17	1 07	1 08	1.04	1 04	1 02	1 02
40.00	4 0	2 69	2 68	2 39	2 38	1 92	1 91	1 58	1 60	1 38	1 39
	7.0	3 55	3 54	3 06	3 05	2 32	2 31	1 81	1 83	1 48	1 51
	1 0	0 12	0 12	0 09	0 09	0 05	0 05	0 04	0 04	0 03	0 03
45 00	4 0	1 04	1 03	0 88	0 87	0 64	0 62	0 46	0 44	0 32	0 31
	7 0	1 82	1 80	1 50	1 47	1 02	0 99	0 69	0 66	0 46	0 43

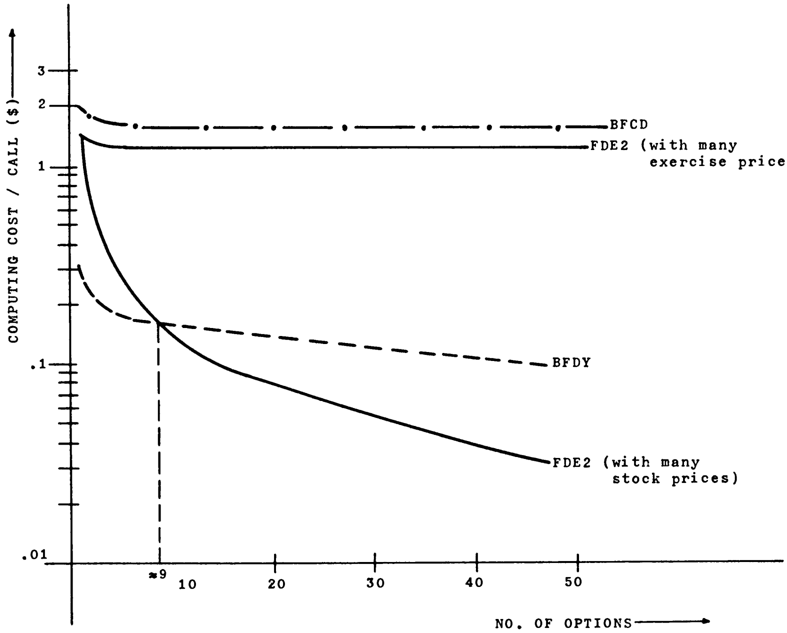

FIGURE 2

Cost Comparison of Binomial and Finite Difference Approximations for American Calls on Dividend Paying Stocks (Two Dividend Payments)

explicit finite difference method with a fixed dividend yield, (FDE2DY), are also presented in tables but not in the graphs.

1. Puts without Dividends

Table 5 compares the convergence and accuracy of four solution techniques for nine put option values when the parameters are as previously given and the underlying stock does not pay dividends. All four methods converged to values within one cent of each other for the time and stock price step sizes reported.

Figure 5 presents the computational efficiency for the four solution techniques of Table 5 for valuing multiple options. We considered a variety of exercise prices and expiration dates to value a range of put options. The binomial technique is more efficient for valuing a small number of puts; but at nine options valued, the finite difference methods become competitive and are dominant thereafter. As with calls, the logarithmically transformed explicit finite difference method is most efficient. The binomial computation cost per option approaches an asymptote as before because it is necessary to re-execute the program for each option valued. This re-execution is not necessary with the finite difference techniques. The binomial execution costs are much greater when valuing American puts rather than calls because time jumping is not feasible. The finite difference methods become more economical when only about ten American put options are being valued, whereas the finite differences did not dominate

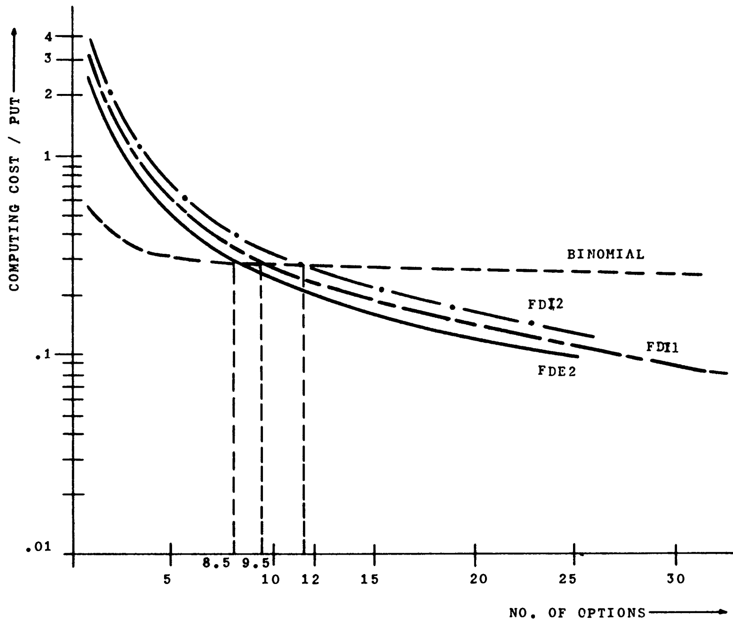

FIGURE 3

Cost Comparison of Binomial and Finite Difference Approximations for American Puts on Non-Dividend Paying Stocks

the binomial method until more than about 300 American call values were computed. Thus, practitioners interested in efficiently valuing put options would reduce their computation costs by using finite difference techniques, even if dividend payments were not considered.[24]

2. Puts with Dividends

Table 6 compares the binomial fixed dividend yield as an approximation to the fixed cash dividend computed by the logarithmically transformed explicit finite difference technique FDE2CD. In addition, this explicit finite difference technique modified for a fixed dividend yield FDE2DY is also computed to check the binomial put scheme BFDY. All the parameters are as given in Table 5, and the cash dividend is $0.50, which together imply a quarterly yield of 1.25 per-

[23] The fixed dividend yield assumption maintains linear homogeneity in S and X for both the binomial and finite difference processes. Thus, if this assumption were employed for the finite difference methods, they would also dominate the binomial method for fixed dividend yield.

[24] In a recent paper, Johnson [23] demonstrated that American puts without dividends can be analytically approximated. This analytic approximation is both accurate and highly efficient for reasonable parameters. Furthermore, Geske and Johnson [21] have recently presented an exact analytic solution to the partial differential equation that is computationally efficient because its evaluation requires very few critical stock price computations. It can also be used for valuing puts on stocks with dividends.

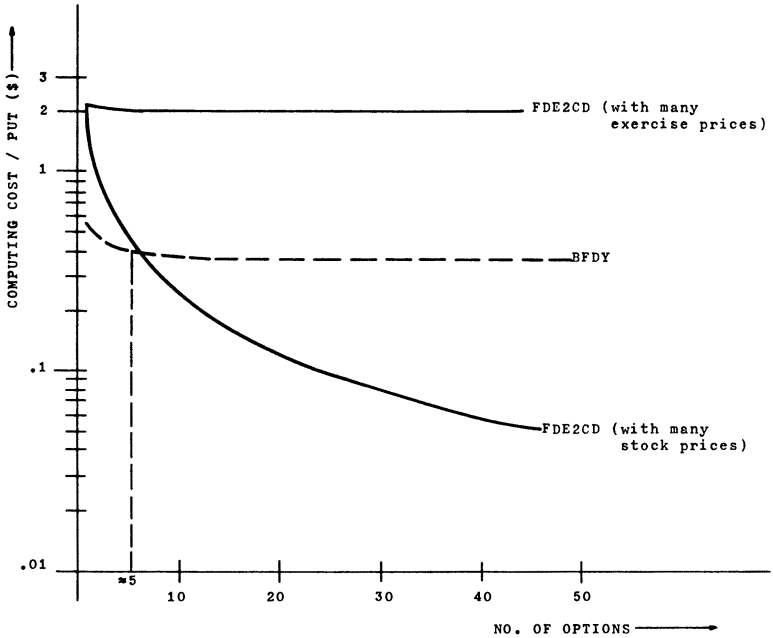

FIGURE 4

Cost Comparisons of Binomial and Finite Difference Approximations for American Puts on Dividend Paying Stock (Two Dividend Case)

TABLE 5

Put Option Values for S = \$40.00, σ = 0.3, and r = 5%

Exercise Price ($)	Solution Technique	Time to Maturity (months)		
		1.0	4.0	7.0
35.00	Binomial[a]	0.08	0.70	1.22
	FDE2[b]	0.08	0.70	1.21
	FDI1[b]	0.08	0 69	1.21
	FDI2[b]	0 08	0 69	1.21
40.00	Binomial	1.31	2 48	3 17
	FDE2	1.31	2 48	3 16
	FDI1	1 30	2.47	3 16
	FDI2	1 30	2.47	3.16
45.00	Binomial	5.06	5 71	6.24
	FDE2	5.06	5.70	6.23
	FDI1	5.06	5 70	5.23
	FDI2	5.06	5.70	6.23

a. The values reported are for N = 150.

b. The values reported are for 200 steps in stock price and 45 steps/month in time (for FDE2, N_1 = 3).

cent. At this point of comparison, the implicit methods are dropped because they are less efficient than the explicit technique.

TABLE 6

Put Option Values for S = \$40, σ = 0.3, and r = 5%, and D = \$0.50

Exercise Price (\$)	Solution Technique	Time to Maturity (months)		
		1.0	4 0	7.0
	BFDY[a]	0.11	0.88	1 55
35.00	FDE2DY[b]	0.11	0.88	1 54
	FDE2CD[c]	0.11	0.91	1 59
	BFDY	1 56	2.91	3 75
40 00	FDE2DY	1.55	2 90	3 74
	FDE2CD	1 56	2.93	3.80
	BFDY	5.50	6.30	7 00
45.00	FDE2DY	5.50	6 29	6 98
	FDE2CD	5 50	6 31	7 02

a. The values for the fixed dividend yield approach to the binomial approximation BFDY are calculated for 140, 160, and 140 steps for 1 0, 4.0, and 7 0 month maturities, respectively.
b. The values for the fixed dividend yield FDE2DY are calculated for 200 steps in stock price and 80 steps/month in time.
c. The values for the fixed cash dividend FDE2CD are calculated for 200 steps in stock price and 80 steps/month in time.

First, note that all three methods converge. Again, the fixed dividend yield is shown to be an accurate approximation to the fixed cash dividend. The maximum discrepancy between FDE2CD and BFDY is five cents for an at-the-money put with a seven-month expiration, and thus three scheduled dividend payments.

Figure 6 summarizes the cost comparisons for the three solution techniques in Table 6 for valuing multiple put options on stocks for two dividend payments. Again the binomial fixed dividend yield is more efficient than the logarithmically transformed explicit finite difference method when the multiple options are generated by varying the exercise price for a single stock price. Conversely, when the multiple options are valued for multiple stock prices and a single exercise price, the explicit finite difference scheme asserts its efficiency after only a few options are valued. As before, if the fixed dividend yield were used for the finite difference schemes, thus maintaining linear homogeneity in S and X, they would dominate the pure binomial process for both multiple exercise and multiple stock prices.

IV. Summary and Conclusion

In summary, the results for puts and calls are similiar. All approximation methods analyzed converge and are accurate. The binomial technique, the implicit and the logarithmically transformed explicit, finite difference methods are "best," and each has its strong points. The binomial method works well for computing a small number of options on stocks without dividends, but is inefficient when effects of cash dividends must be analyzed. However, the assumption of a fixed dividend yield is shown to be a reasonably accurate and efficient approximation. Unfortunately, the fixed dividend yield produces an incorrect hedge ratio. The binomial technique also loses efficiency when valuing American options. Furthermore, because the binomial process has a conditional starting point it is less efficient than the two finite difference methods for valuing multiple options.

The explicit finite difference method should not be discarded for stability problems because these can be readily overcome. In addition, when transformed logarithmically, the explicit method is more efficient than the implicit method because it does not require the solution of a set of simultaneous equations. The binomial technique is more intuitive and also may be more readily implemented than the finite difference methods. Thus, it is pedagogically superior. In conclusion, researchers computing a smaller number of option values may prefer the binomial approximation, while practitioners in the business of computing a larger number of option values will generally find that the finite difference approximations are more efficient.

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