Z-Transforms

INTRODUCTION

Z-transform plays an important role in discrete analysis. Its role in discrete analysis is the same as that of Laplace and Fourier transforms in continuous system. Difference equations are also based on discrete system and their solutions and analysis are carried out by Z-transform.

SEQUENCE

This is an ordered list of real or complex numbers and is represented by $\langle u_n \rangle$ or $\langle u(n) \rangle$ or

$$\{u_0, u_1, u_2, \dots u_n \dots\}$$

$$u_n = u(n) = \text{function of } n$$

In a sequence the left end term is considered as the term corresponding to $n=0$,

For example $\{u_n\} = \{7, 3, 2, 9, 10, 0, 2, 3\}$

Zeroth term $= 7$

The other way of specifying the sequence is to define the general term of the sequence u_n as function of 'n'

For example (i) $u(n) = \frac{1}{3^n}$

This sequence represents.

$$\left\{ \cdots \frac{1}{3^{-3}} \cdots \frac{1}{3^{-2}}, \frac{1}{3^{-1}}, \underset{\uparrow n=0}{1}, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} \cdots \right\}$$

(ii) $\{u_n\} = 2^n \ \forall n.$

$$= \{ \cdots\cdots 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, 2^3 \cdots\cdots \}$$

Definition.

The Z-transform of a sequence $\langle u_n \rangle$ or $\{u_n\}$ is denoted as $Z(u_n)$

and it is defined as

$$Z(u_n) = \bar{u}(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n}.$$

$$= \cdots + u_{-2} z^2 + u_{-1} z + u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots \quad \text{— (1)}$$

since (1) z is a complex number

(2) It is a two sided z-transform as u_n is defined for $-\infty$ to $+\infty$

(3) If u_n is defined for $n \geq 0$, then,

$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$ is called one-sided z-transform.

Here u_n is also known as a Causal sequence.

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(4) $Z(u_n)$ is defined only when the infinite series $\sum u_n z^{-n}$ is absolutely convergent that is the sum is finite.

(5) Here Z is an operator of the Z-transform and $\bar{u}(z)$ is the Z-transform of $\{u_n\}$

Properties of Z-Transforms

Theorem 1: If $\{u_n\}$ and $\{v_n\}$ be any two sequences and a, b be any two constants, then $Z[a\{u_n\} \pm b\{v_n\}] = aZ[\{u_n\}] \pm bZ[\{v_n\}]$

Proof:- By definition from (1)

$$Z[a\{u_n\} \pm b\{v_n\}] = \sum_{n=-\infty}^{\infty} \{a u_n \pm b v_n\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a u_n z^{-n} \pm \sum_{n=-\infty}^{\infty} b v_n z^{-n}.$$

$$= a \sum_{n=-\infty}^{\infty} u_n z^{-n} \pm b \sum_{n=-\infty}^{\infty} v_n z^{-n}.$$

$$\therefore Z[a\{u_n\} \pm b\{v_n\}] = aZ[\{u_n\}] + bZ[\{v_n\}]$$

Proved

Theorem 2. (DAMPING RULE OR CHANGE OF SCALE PROPERTY)

If $Z[\{u_n\}] = \sum_{n=-\infty}^{\infty} u_n z^{-n} = \bar{u}(z)$, then.

(i) $Z\{a^{-n} u_n\} = \bar{u}(az)$

(ii) $Z\{a^n u_n\} = \bar{u}\left(\frac{z}{a}\right)$

Proof. (i) $Z\{a^{-n} u_n\} = \sum_{n=-\infty}^{\infty} a^{-n} u_n z^{-n} = \sum_{n=-\infty}^{\infty} u_n (az)^{-n}$.

$\therefore \quad Z\{a^{-n} u_n\} = \bar{u}(az)$ Proved

(ii) Similarly $Z\{a^n u_n\} = \sum_{n=-\infty}^{\infty} a^n u_n z^{-n}$.

$$= \sum_{n=-\infty}^{\infty} u_n \left(\frac{z}{a}\right)^{-n}.$$

$\therefore \quad Z\{a^n \cdot u_n\} = \bar{u}\left(\frac{z}{a}\right)$ Proved.

SOME STANDARD Z-TRANSFORMS

We have the following results from the definition of Z-Transforms.

(i) $Z.[\{a^n\}] = \frac{z}{z-a}$ if $n \geq 0$ and $z \neq a$

Proof:→: We know that

$$Z[\{a^n\}] = \sum_{n=0}^{\infty} a^n z^{-n} \quad \text{(by def.)}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots + a^n z^{-n} + \cdots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots + \left(\frac{a}{z}\right)^n + \cdots$$

$$= \frac{1}{1-\left(\frac{a}{z}\right)} = \frac{z}{z-a}.$$

$$\therefore Z(a^n) = \frac{z}{z-a} \quad \text{if } z \neq a$$

$$\boxed{Z(a^n) = \frac{z}{z-a}}$$

since $\{a^n\} = \{1, a, a^2, a^3, \cdots a^n, \cdots\}$

Note :- If $a=1$, then $\boxed{Z(1) = \frac{z}{z-1}}$ if $z \neq 1$

(ii) $Z[\{n^p\}] = -z \frac{d}{dz} Z[\{n^{p-1}\}]$

$n \geq 0$, p is a +tive Integer That is

$\{n^p\} = \{0, 1^p, 2^p, 3^p, \cdots, n^p, \cdots\}$

Proof:→ changing p to $(p-1)$ we get

$$Z[\{n^{p-1}\}] = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

Diff. both sides w.r.t. z, we get

$$\frac{d}{dz} Z[\{n^{p-1}\}] = \sum_{n=0}^{\infty} n^{p-1}(-n) z^{-n-1}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n^{p} z^{-n}$$

$$= -\frac{1}{z} Z[\{n^p\}]$$

$$\Rightarrow \quad Z[\{n^p\}] = -z \frac{d}{dz} Z[(n^{p-1})] \quad \text{Proved}$$

Some Particular Cases :-

When $p=1$:- $Z[\{n\}] = -z \frac{d}{dz} Z[\{n^0\}]$

$$= -z \frac{d}{dz} Z[\{1\}]$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right) \quad \left(\because Z\{1\} = \frac{z}{z-1}\right)$$

$$= -z \left\{\frac{(z-1)\cdot 1 - z(1)}{(z-1)^2}\right\}$$

$$= -z \left\{\frac{z-1-z}{(z-1)^2}\right\}$$

$$\therefore \quad Z[\{n\}] = \frac{z}{(z-1)^2}$$

When $p=2$, $Z[\{n^2\}] = -z \frac{d}{dz}\{Z\{n\}\}$

$$= -z \frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$\therefore \quad Z[\{n^2\}] = \frac{z^2+z}{(z-1)^2}$$

Similarly $Z[\{n^3\}] = \frac{z^3+4z^2+z}{(z-1)^4}$

(iii) $$Z\{a^n \cdot n\} = \frac{az}{(z-a)^2}$$

Proof:→ by change of scale Prop, we know that

$$Z[\{n\}] = \frac{z}{(z-1)^2}$$

$$\therefore \quad Z[a^n \cdot n] = \frac{z/a}{(z/a-1)^2}$$

$$= \frac{z}{a} \times \frac{a^2}{(z-a)^2}$$

$$Z[a^n \cdot n] = \frac{az}{(z-a)^2}$$ Proved

(4) $$Z\{a^n \cdot n^2\} = \frac{az^2+a^2z}{(z-a)^3}$$

Proof:→ we know that

$$Z[\{n^2\}] = \frac{z^2+z}{(z-1)^3}$$

$$\therefore \quad Z[\{a^n \cdot n^2\}] = \frac{\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)}{\left(\frac{z}{a}-1\right)^3}$$ [by change of scale Prop]

$$= \frac{a(z^2+az)}{(z-a)^3}$$

(5) $$Z[\{\cos n\theta\}] = \frac{z(z-\cos\theta)}{(z^2-2z\cos\theta+1)}$$

(6) $$Z[\{\sin n\theta\}] = \frac{z\sin\theta}{(z^2-2z\cos\theta+1)}$$

<u>Proof:</u> → we know that

$$Z(a^n) = \frac{z}{z-a}$$

putting $a = e^{i\theta}$, we get

$$Z[\{e^{i\theta}\}^n] = \frac{z}{z-e^{i\theta}}$$

$$\Rightarrow Z[\{e^{in\theta}\}] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$\Rightarrow Z[\{\cos n\theta + i\sin n\theta\}] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z[\{z-\cos\theta\} + i\sin\theta]}{(z-\cos\theta)^2 + \sin^2\theta} \qquad \because i^2=-1$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\therefore Z(\cos n\theta) + iZ(\sin n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Separating real and Imagi. Parts, we get

(5) $$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

(6) $$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}.$$

(7) $$Z\{a^n \cos n\theta\} = \frac{z(z-a\cos\theta)}{(z^2-2z\cos\theta+a^2)}$$

(8) $$Z\{a^n . \sin n\theta\} = \frac{az\sin\theta}{z^2-2z\cos\theta+a^2}$$

Proof:- From result (5)

(7) Let $$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} = \bar{u}(z)$$

$$\therefore \quad Z(a^n \cos n\theta) = \bar{u}\left(\frac{z}{a}\right)$$ (Using change of scale Prop.)

$$= \frac{\frac{z}{a}\left(\frac{z}{a}-\cos\theta\right)}{\frac{z^2}{a^2}-2\frac{z}{a}\cos\theta+1}$$

$$= \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$ Proved

(8) Again, by assuming

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1} = \bar{u}(z)$$

$$\Rightarrow Z(a^n \sin n\theta) = \bar{u}\left(\frac{z}{a}\right)$$ {by change of Scale Property}

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2}-2\frac{z}{a}\cos\theta+1}$$

$$= \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$$ Proved