# UNIT 4

## Numerical Solution of ordinary Differential Equations:-

Introduction:- A general first order and first degree equation is

$$\frac{dy}{dx} = f(x, y) \quad \text{---} (1)$$

the initial conditions are

$$y = y_0, \quad x = x_0 \quad \text{---} (2)$$

### (1) Picard's Method of Successive Approximation:

We consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{---} (1)$$

Where $x = x_0$, $y = y_0$ --- (2)

integrating (1) and using initial conditions we have

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\text{or, } y = y_0 + \int_{x_0}^{x} f(x\, y)\, dx \quad \text{---} (3)$$

In Picard's Method, we find the approximate value of y in terms of x, we first approximation of y is given by

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \quad y \to y_0$$

P.T.O.

To get the second approximation, $y$ has been replaced by $y^{(1)}$, so that

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

Similarly, a third approximation is given by

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

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$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx \quad \text{---} (4)$$

** In Picard's Method, we solve upto third approximation only.

<u>II Method</u> <u>Euler's Method</u>:- we take the following differential equation.

$$\frac{dy}{dx} = f(x, y) \quad \text{----} (1)$$

where $y = y_0, \; x = x_0$ --- (2)

For Euler's eq$^n$. we use the following relation:-

$$y_{n+1} = y_n + h f(x_n, y_n), \quad \text{---} (3)$$

where, $n = 0, 1, 2 \ldots$

$h$ = interval.

Eq$^n$ 3) known as Euler's eq$^n$. for solving problems.

## III MODIFIED EULER'S METHOD:-

We take the eqn:

$$\frac{dy}{dx} = f(x, y) \quad \text{---- (1)}$$

Where $x = x_0$, $y = y_0$ ---(2)

We use the following two steps :-

(1) First we find

$$y^*_{n+1} = y_n + h f(x_n, y_n) \quad \text{---(3)}$$

and second, we have modified method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right] \quad \text{-(4)}$$

we find the complete solution by eqn (2).
for Euler's modified method.

## IV Runge-Kutta Method of fourth order:-

To solve the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{---- (1)}$$

Where $x = x_0$, $y = y_0$ ---(2)

By fourth order Runge-Kutta method,
we find four quantities $K_1, K_2, K_3, K_4$
by following formulae;

$$\left.\begin{aligned} K_1 &= hf(x_n, y_n) \\ K_2 &= hf\left(x_n+\frac{h}{2}, y_n+\frac{K_1}{2}\right) \\ K_3 &= hf\left(x_n+\frac{h}{2}, y_n+\frac{K_2}{2}\right) \\ K_4 &= hf(x_n+h, y_n+K_3) \end{aligned}\right] \quad -(3)$$

and the solution of the given eqn is given by

$$y_{n+1} = y_n + K \quad - (4)$$

where $K = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$ -- (5).

## V MILENE'S METHOD [PREDICTOR-CORRECTOR METHOD]:-

we have $\frac{dy}{dx} = f(x, y)$ --- (1)

where $x = x_0$, $y = y_0$ --- (2)

In meline's formulae we use the following relation:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right] - (3)$$

which is called pridictor formulae;

Exambles based on picard's method:-

(1) use Picard's method to solve

$$\frac{dy}{dx} = 1 + xy, \text{ with } x_0 = 2, \ y_0 = 0$$

(RTU, 2009)

Soln: we are given that

$$\frac{dy}{dx} = f(x, y) = 1 + xy$$

the first approximation is given by

$$y_{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 0 + \int_2^x (1 + x y_0)\, dx =$$

$$= \int_2^x dx = x - 2 \quad \text{---- (1)}$$

From the second approximation, we have

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 0 + \int_2^x (1 + x y_1)\, dx$$

$$= 0 + \int_2^x 1 + x(x-2)\, dx$$

$$= \left[ x + \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} \right]_2^x = x + \frac{x^3}{3} - x^2 - \frac{2}{3} \quad -(2)$$

similarly third approximation, we have

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$\Rightarrow y_3 = 0 + \int_2^x (1 + x y_2)\, dx$$

$$y_3 = 0 + \int_2^x (1 + x y_2)\, dx$$

$$= 0 + \int_2^x \left[1 + x\left[x + \frac{x^3}{3} - x^2 - \frac{2}{2}\right]\right] dx$$

$$= \frac{1}{15}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^2 + x - \frac{2}{15}$$

Q(2) In Euler's method with $h = 0.1$, to find the solution of equation

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 0,$$

in the range $0 \leq x \leq 0.5$

Soln: we are given that

$$\frac{dy}{dx} = f(x, y) = x^2 + y^2 \quad \text{--- (1)}$$

From Euler's formulae: we have

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \text{--- (2)}$$

putting $n = 0$, we get

$$y_1 = 0 + 0.1\left[(0)^2 + (0)^2\right] = 0.0 \text{ at } x = 0.1$$

for $n = 1$, we get

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 0 + (0.1)\left[(0.1)^2 + (0)^2\right]$$

$$= 0.001 \quad \text{at } x = 0.2$$

For $n = 2$, we have

$y_3 = y_2 + h\, f(x_2, y_2)$

$= 0.001 + (0.1)\,[(0.2)^2 + (0.001)^2]$

$= 0.001 + 0.004$

$= 0.005$ at $\underline{x = 0.3}$ (Ans).

putting $n = 3$, we get

$y_4 = y_3 + h\, f(x_3, y_3)$

$y_4 = 0.005 + (0.1)\,[(0.3)^2 + (0.005)^2]$

$= 0.014$ at $\underline{x = 0.4}$ (Ans)

putting $\underline{n = 4}$, we get

$y_5 = y_4 + h\, f(x_4, y_4)$

$= (0.014) + (0.1)\,[(0.4)^2 + (0.014)^2]$

$= 0.0300196$ at $x = 0.5$ (Ans)

Ex(3): use Euler's modified method $\underline{\text{with one}}$ step to find the value of $y$ at $x = 0.1$ taking $h = 0.1$, where

$$\frac{dy}{dx} = x^2 + y \quad \text{and } y = 0.94 \text{ when } x = 0$$

Soln. we given that

$$\frac{dy}{dx} = f(x, y) = x^2 + y \quad \text{--- (1)}$$

From modified Euler method we have

$$y_1^* = y_0 + h f(x_0, y_0) \quad \text{--- (2)}$$

and $$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right] \quad \text{--- (3)}$$

Now $y_1^* = y_0 + h f(x_0, y_0)$

$= 0.94 + (0.1) f(0, 0.94)$

$= 0.94 + (0.1)(0.94)$

$y_1^* = 1.034$

Again $y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$

$= 0.94 + 0.05\left[f(0, 0.094) + f(0.1, 1.034)\right]$ Imp.

$= 0.94 + 0.05 \times 1.9841$

$= 1.0392$

therefore $y(0.1) = 1.0392$ (Ans).

PRATICE QUESTIONS:-

Q(1) Use Picard method to solve

$\frac{dy}{dx} = x + x^4 y$ given that $y = 3$ at $x = 0$

and find $y(0.1)$, $y(0.2)$, $y(0.3)$.

(3.005) (3.02) (3.046)

Q(2) Use picard method, solve

$\frac{dy}{dx} = x + y^2$, given that $y = 0$, $x = 0$

Q(3) Use Eulers method to solve

$\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y(0) = 1$,

find $y$ for $x = 0.1$.

Q(4) Use Eulers method.

$\frac{dy}{dx} = \frac{y^2 - x}{y^2 + x}$

given $y = 1$, $x = 0$, find $y$ for $x = 0.1$, $0.2$, $0.3$.

Q(5) Use Euler's modified method to solve

$\frac{dy}{dx} = x + y$, $y(0) = 1$ and find $y$ for $x = 1$

and step size $(h) = 02$.