Contents

CS508: Assignment-1

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1 Linear Program Formulation

We are given n tasks and n employees. Each employee does each job with some cost. We have to assign at most one task to each employee and at most one employee to each task. Our goal is to minimize the total cost of jobs done by the employees. This problem can be formulated as **Integer Linear Program** as follows.
Consider the bipartite graph $G(V, E)$ where $V = X \cup Y$ is the vertex set and E is the edge set that connects a vertex of X to a vertex of Y. For each edge e, there is a positive cost assigned w_e

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{e \in E : v \in e} x_e = 1 \text{ for each vertex } v \in V \\ & x_e \in \{0, 1\} \text{ for each edge } e \in E \end{aligned}$$

The **LP-relaxation** for the above **ILP** will be as follows.

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to} \quad & \sum_{e \in E : v \in e} x_e = 1 \text{ for each vertex } v \in V \\ & 0 \leq x_e \leq 1 \text{ for each edge } e \in E \end{aligned}$$

2 Implementation of the LP and execution steps

The above two programs are implemented in python using cplex.

1. Open command prompt / terminal.
2. Type **python JobAllocation.py** and press enter.
3. It will create two pdfs (namely JobAllocationILP.pdf and JobAllocationLP.pdf) for ILP and LP respectively which will show the exact job-employee assignment which is optimal.
4. In terminal, we can see the objective value and other details.

3 Observation and Proof

For any possible example of $N \times N$ matrix of job-employee allocation costs, the solution of **ILP** and **LP** will be the same and hence integral. The argument goes as follows.

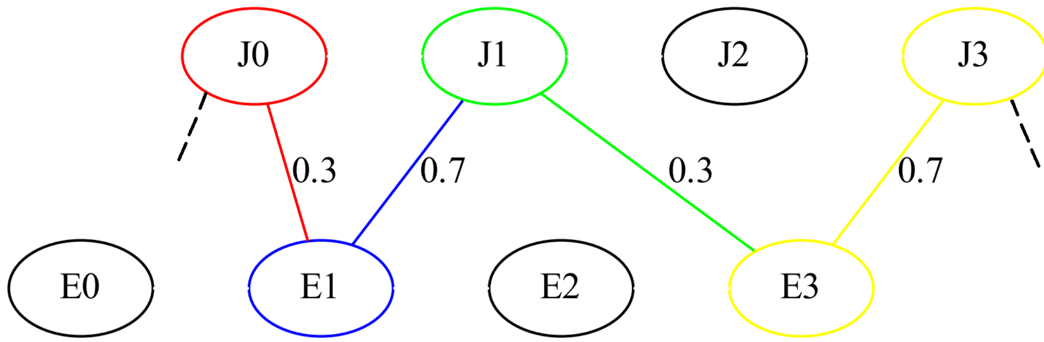

Figure 1: Example of fractional x_e values

As described in the slides, if the value of x_e for some edge e is not integral, then to compensate the fractional value (say n), there will be another edge from the same vertex for some edge e' where x'_e will be equal to $1 - n$. This will go on till all the fractions are compensated. So it will form a cycle. Figure 1 shows a small example.

For bipartite matching, all the vertices will be integral. That means, for each edge incident on the vertex, either the corresponding edge will be considered or not considered (i.e. 0 or 1 for each edge incident on the vertex). Mathematically, all vertices are in $\{0, 1\}^k$ where $k = |E|$. Hence, **LP** cannot give non-integral solutions and the solution of **LP** and **ILP** will be equal.