

Compute PCA by hand for the following data matrix

$$X = \begin{bmatrix} -3 & -2 \\ -2 & -1 \\ -1 & -1 \\ 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$$

1). Find the mean vector  $\mu^T$  (size = 1x2)

$$0 \quad -0.5$$

2). Form a 6x2 mean matrix:  $M = \begin{bmatrix} \mu^T \\ \mu^T \\ \mu^T \\ \mu^T \\ \mu^T \\ \mu^T \end{bmatrix}$

$$\begin{bmatrix} 0 & -0.5000 \\ 0 & -0.5000 \\ 0 & -0.5000 \\ 0 & -0.5000 \\ 0 & -0.5000 \\ 0 & -0.5000 \end{bmatrix}$$

3). Create a 6x2 zero-mean data matrix:  $XM = X - M$

$$\begin{bmatrix} -3.0000 & -1.5000 \\ -2.0000 & -0.5000 \\ -1.0000 & -0.5000 \\ 1.0000 & 1.5000 \\ 2.0000 & 1.5000 \\ 3.0000 & -0.5000 \end{bmatrix}$$

4). Compute the 2x2 unbiased covariance matrix:  $\Sigma = \frac{1}{N-1} XM^T \cdot XM$

- $N$  = the number of observations

$$\begin{bmatrix} 5.6000 & 1.8000 \\ 1.8000 & 1.5000 \end{bmatrix}$$

5). Find the eigenvalues  $\lambda_1, \lambda_2$  and the eigenvectors  $v_1, v_2$  of the covariance matrix  $\Sigma$

- $\lambda_1 > \lambda_2$  and  $v_1, v_2$  are 2x1 vectors

Eigenvalues	Eigenvectors
$\lambda_1 = 6.2781$	$v_1 =$ -0.9358 -0.3525
$\lambda_2 = 0.8219$	$v_2 =$ 0.3525 -0.9358

6). Form a 2x2 eigen matrix consisting of the two eigenvectors:  $\Phi = [v_1 \ v_2]$

-0.9358 0.3525  
-0.3525 -0.9358

7). Compute the principal components by projecting  $XM$  onto  $\Phi$ :  $Y = XM \cdot \Phi$

- $Y$  is a 6x2 matrix. The first column is PC1 and the second column is PC2

3.3362 0.3461  
2.0479 -0.2372  
1.1121 0.1154  
-1.4646 -1.0512  
-2.4004 -0.6986  
-2.6311 1.5255

8). Compute the reconstructed data matrix:  $\hat{X} = \bar{Y} \cdot V^T + M$

(a). Use only PC1. Hence,  $\bar{Y}$  is a 6x1 vector which is the first column of  $Y$  and  $V$  is the eigenvector  $v_1$

-3.1220 -1.6761  
 -1.9164 -1.2219  
 -1.0407 -0.8920  
 1.3706 0.0163  
 2.2463 0.3462  
 2.4622 0.4276

(b). Compute the reconstruction error

$$Error = \frac{\|X - \hat{X}\|_F}{\|X\|_F} = \frac{\begin{vmatrix} 0.1220 & -0.3239 \\ -0.0836 & 0.2219 \\ 0.0407 & -0.1080 \\ -0.3706 & 0.9837 \\ -0.2463 & 0.6538 \\ 0.5378 & -1.4276 \end{vmatrix}_F}{\begin{vmatrix} -3 & -2 \\ -2 & -1 \\ -1 & -1 \\ 1 & 1 \\ 2 & 1 \\ 3 & -1 \end{vmatrix}_F} = \frac{2.0272}{6.0828} = 0.3333$$

(c). Use both PC1 and PC2. Hence,  $\bar{Y} = Y$  and  $V = \Phi$

-3.0000 -2.0000  
 -2.0000 -1.0000  
 -1.0000 -1.0000  
 1.0000 1.0000  
 2.0000 1.0000  
 3.0000 -1.0000

(d). Compute the reconstruction error

$$Error = \frac{\|X - \hat{X}\|_F}{\|X\|_F} = \frac{0}{6.0828} = 0$$