Problem 1 (5 points)

Find the solution (x^*, y^*) to the following problem.

$$\text{optimize } xy$$

$$\text{subject to } x + y = 10$$

Problem 2 (5 points)

Define the Lagrangian function for the following optimization problem.

$$\text{minimize } 7x_1^3 + 8x_2^2 + 9x_1$$

$$\text{subject to } 6x_1 + 11x_2^2 = 13x_2$$

$$5x_2^2 \leq 4x_1 - 7$$

$$20x_1 + 5 \geq x_2$$

Do not solve for the variables, i.e., do not try to compute the partial derivatives. Make sure the constraints are in standard form.

Problem 3 (5 points)

The SVM optimization can be defined by the primal form:

$$\min_w \frac{1}{2}\|\boldsymbol{w}\|^2$$

$$\text{subject to } y_i(\boldsymbol{w}^T\boldsymbol{x}_i + b) \geq 1, \qquad i = 1, \ldots, N$$

Or by its the dual form:

$$\max_\alpha J(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j y_i y_j \left(\boldsymbol{x}_i^T\boldsymbol{x}_j\right)$$

$$\text{subject to } \alpha_i \geq 0, i = 1, \ldots N \text{ and } \sum_{i=1}^{N} \alpha_i y_i = 0$$

Show the objective function $J(\alpha)$ is the Lagrangian function $L(\boldsymbol{w}, b, \alpha)$ evaluated at $\boldsymbol{w}$ that minimizes that function.

Hints:

1. Write the primal problem in standard form
2. Form the Lagrangian function $L(\boldsymbol{w}, b, \alpha)$
3. Find $\boldsymbol{w}$ and b that minimize $L(\boldsymbol{w}, b, \alpha)$
4. Plug the results back into $L(\boldsymbol{w}, b, \alpha)$

Problem 4 (5 points)

Consider a dataset with 2 points in 1D: $(x_1 = 0, y_1 = -1)$ and $(x_2 = \sqrt{2}, y_2 = 1)$. Map each point to 3D using the feature vector $\phi(x) = [1, \sqrt{2}x, x^2]^T$. This is equivalent to using a second order polynomial kernel. The SVM classifier has the form

$$\min \|x\|^2 \; s.t.$$

$$y_1(\boldsymbol{w}^T\phi(x_1) + w_0) \geq 1$$

$$y_2(\boldsymbol{w}^T\phi(x_2) + w_0) \geq 1$$

(a). Find the corresponding points in 3D. That is, $\phi(x_1)$ and $\phi(x_2)$.

(b). What is the value of the margin? Notice since there are only 2 points in the dataset, those points are the support vectors. Hence, the margin is the distance between each of them in 3D to the decision boundary, which lies in the middle.

(c). Assume the margin obtained from part (b) is equal to $1/\|\boldsymbol{w}\|$. Determine the vector $\boldsymbol{w}$. Recall this vector is the line through $\phi(x_1)$ and $\phi(x_2)$, which is perpendicular to the decision boundary.

(d). Solve for w_0 using your value for $\boldsymbol{w}$ and the above equations. Since the points will on the decision boundary, the inequalities will become equalities.

(e). Write the function $f(x) = w_0 + \boldsymbol{w}^T\phi(x)$ as an explicit function of x.