COLORING OF GRAPHS

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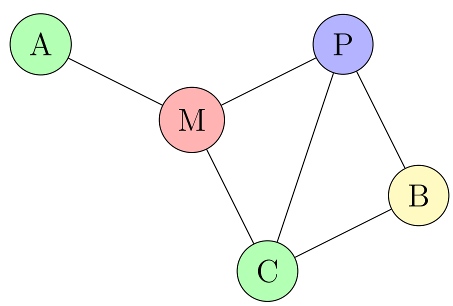
Abstract. This laboratory explores the concept of graph coloring, focusing on its applications in solving real-world problems, such as scheduling and resource allocation. Using algebraic, geometric, and combinatorial methods, we investigate how the chromatic number of a graph and various coloring techniques help us solve different types of problems.

1. Introduction

In mathematics, graph coloring is a sub-field of graph theory that has a wide range of practical applications, such as in scheduling, resource allocation and network design. The central idea is to assign colors to the vertices of a graph in a way that no two vertices connected by an edge share the same color.



To understand this better, let us consider a simple course scheduling problem with five courses: Math(M), Physics(P), Chemistry(C), Biology(B), and Anthropology(A). Each vertex represents a course, and edges indicate courses with shared student enrollment. The objective is

I thank Professor Robinson and my classmates in MATH251 for helping me learn and understand the ideas presented in this paper.

to assign time slots (represented by colors) such that no two connected courses share the same slot. In our graph:

- Math shares students with Physics, Chemistry and Anthropology
- Physics shares students with Math, Chemistry, and Biology
- Chemistry shares students with Math, Physics, and Biology
- Biology shares students with Physics and Chemistry
- Anthropology shares students with Math

Notice how non-adjacent courses (like Anthropology and Chemistry) can share the same color, demonstrating a key principle of graph coloring: vertices without direct connections can have the same color (time slot). This example also introduces us to two fundamental problems in graph coloring: the existence problem and the counting problem. The existence problem asks whether it's possible to color the graph using a given number of colors such that no vertices connected by an edge have the same color. In other words, given a set number of time slots, is it possible to schedule these courses without conflicts?

The counting problem goes a step further, asking how many different ways there are to color the graph (assign time slots) such that no two adjacent courses share the same time slot. Specifically, if we determine that three colors (time slots) are sufficient, how many unique ways can we assign these colors while ensuring no conflicting courses share a slot?

Before we explore graph coloring further, it is important to understand the concept of graph and proper coloring. Thus, we begin with the definition of a graph.

Definition 1.1. A simple graph $\mathcal{G}$ consists of a set of vertices $V = \{v_1, v_2, .., v_n\}$ and a set of edges $E = \{e_i = v_i v_j, e_2 = v_k v_l ...\}$. In a simple graph, edges have no direction, and there cannot be multiple edges between two vertices.

With this understanding of what constitutes a graph, we can now look at the concept of proper coloring.

Definition 1.2. A proper coloring of a graph $\mathcal{G}$ is a function from the vertices of $\mathcal{G}$ to a set of colors in such a way that no two vertices sharing an edge have the same color.

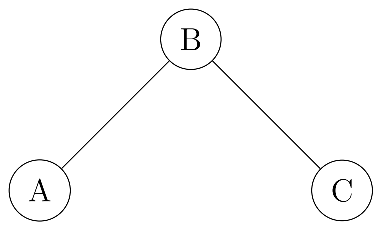
In other words, we assign colors to the vertices so that adjacent vertices (those connected directly by an edge) are colored differently. Returning to our course scheduling problem, each course is a vertex, and an edge connects two courses if they share students. A proper coloring assigns time slots (colors) to courses such that no two courses with common students are scheduled at the same time.

One fundamental question in graph coloring is the existence problem: Given a graph, can it be properly colored using a certain number of colors? This leads us to the concept of the chromatic polynomial of a graph.

Definition 1.3. The chromatic polynomial of a graph $P(\mathcal{G}; x)$ of a graph $\mathcal{G}$ is a polynomial in x (where x represents the number of colors) that evaluates the total number of proper colorings of $\mathcal{G}$ with x colors.

The chromatic polynomial helps us solve both, the existence problem and the counting problem. By evaluating $P(\mathcal{G}; x)$ at a particular value of x, we can determine whether a proper coloring with x colors exists (if $P(\mathcal{G};x) > 0$) and find out exactly how many proper colorings are possible. To better understand chromatic polynomials, let us look at an example of a simple graph with three vertices and two edges.

Example 1.4. Consider a graph $\mathcal{G}$ with vertices A, B, and C and edges connecting A to B and B to C. This forms the following simple graph:



Our goal is to determine the number of proper colorings of this graph for different numbers of colors x and then observe the pattern to derive the chromatic polynomial of the graph. We begin by tabulating the number of proper colorings for various values of x: From the table, we

Number of colors (x)	Number of proper colorings
0	0
1	0
2	2
3	$3 \times 2 \times 2 = 12$
4	$4 \times 3 \times 3 = 36$
5	$5 \times 4 \times 4 = 80$
x	$x(x-1)^2$

observe the pattern that the number of proper colorings follows the formula:

$$x(x-1)^2.$$

This expression represents the chromatic polynomial of the graph $\mathcal{G}$:

$$P(\mathcal{G};x) = x(x-1)^2.$$

Through this example, we can see that the chromatic polynomial helps us solve the existence problem by determining whether a proper coloring with x colors is possible. If $P(\mathcal{G};x) > 0$, then a proper coloring exists. The chromatic polynomial also helps us solve the counting problem by determining the exact number of proper colorings for a given number of colors x.

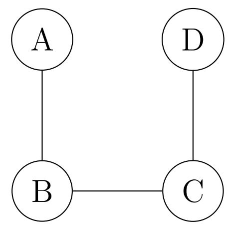
Before we move forward, let us explore the different types of graph that exist. In this paper, we will be focusing on path graphs, cycle graphs and complete graphs. Let us begin with the definition of a path graph:

Definition 1.5. A path graph with n vertices, denoted as P_n, is a graph in which vertices are connected in a sequence, such that two vertices are adjacent if and only if they are consecutive in the ordering. In an n-path graph, exactly two vertices have degree 1, and the remaining $(n-2)$ vertices have degree 2.

A n-path graph consists of a set of vertices $V = \{v_1, v_2, ..., v_n\}$, where vertices are connected in a sequence such that the set of edges $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$.

Let us look at an example to understand this better.

Example 1.6. Consider the following n-path graph with $n = 4$:



This graph consists of a set of vertices $V = \{A, B, C, D\}$ where vertices are connected in a sequence such that the set of edges $E = \{AB, BC, CD\}$. Vertices A and D have degree 1, whereas vertices B and C have degree 2.

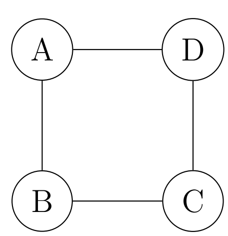
Having understood path graphs, let us now look at cycle graphs:

Definition 1.7. A cycle graph with n vertices is a graph in which vertices are connected to form a closed loop, such that each vertex has exactly two adjacent vertices. In an n-cycle graph, every vertex has degree 2.

An n-cycle graph consists of a set of vertices $V = \{v_1, v_2, ..., v_n\}$, where vertices are connected to form a closed loop such that the set of edges $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$.

Let us look at an example to understand this better.

Example 1.8. Consider the following n-cycle graph with $n = 4$:



This graph consists of a set of vertices $V = \{A, B, C, D\}$ where vertices are connected in a sequence such that the set of edges $E = \{AB, BC, CD, DA\}$. All vertices have degree 2.

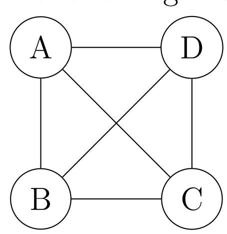
Having understood cycle graphs, let us now look at complete graphs:

Definition 1.9. A complete graph with n vertices is a graph in which every pair of distinct vertices is connected by a unique edge. In an n-complete graph, every vertex has degree $n - 1$.

An n-complete graph consists of a set of vertices $V = \{v_1, v_2, ..., v_n\}$, where the set of edges $E = \{v_i v_j | i \neq j, 1 \leq i, j \leq n\}$.

Let us look at an example to understand this better.

Example 1.10. Consider the following n-complete graph with $n = 4$:



This graph consists of a set of vertices $V = \{A, B, C, D\}$ where vertices are connected in a sequence such that the set of edges $E = \{AB, BC, CD, DA, AC, BD\}$. All vertices have degree 3.

2. Birkhoff-Lewis Reduction Algorithm

Having established the foundational concepts of graph coloring and chromatic polynomials, we now turn our attention to a systematic method for determining these polynomials for more complex graphs. This method is known as the Birkhoff-Lewis Reduction Algorithm and it allows us to compute the chromatic polynomial of any finite graph through a structured and recursive process.

To illustrate the application of the Birkhoff-Lewis Reduction Algorithm, let's outline its key steps. Consider a graph $\mathcal{G}$. We decompose $\mathcal{G}$ recursively until we reach simple graphs whose chromatic polynomial is easy to compute. The steps for the algorithm are as follows:

1. Initial Graph Decomposition: For a given graph $\mathcal{G}$, select an edge e and derive two new graphs:

- $\mathcal{G}_1$: Obtained by deleting the edge e, which results in a graph with n vertices and $e - 1$ edges,
- $\mathcal{G}_2$: Obtained by collapsing the same edge e, effectively merging the two vertices connected by e into a single vertex, creating a graph with $n - 1$ vertices and an unknown number of edges.

The important insight here is how these transformations preserve the graph's core structure while simplifying its complexity. When we delete an edge ($\mathcal{G}_1$), we maintain the original vertex connections except for the removed edge. When we collapse an edge ($\mathcal{G}_2$), we merge two vertices while preserving their respective connections to all other vertices in the graph.

2. Recursive Decomposition: Each of these resulting graphs ($\mathcal{G}_1$ and $\mathcal{G}_2$) can be further decomposed using the same process:

- $\mathcal{G}_{1,1}$: Obtained by deleting an edge from $\mathcal{G}_1$,
- $\mathcal{G}_{1,2}$ Obtained by collapsing the same edge in $\mathcal{G}_1$,
- $\mathcal{G}_{2,1}$: Obtained by deleting an edge from $\mathcal{G}_2$,
- $\mathcal{G}_{2,2}$ Obtained by collapsing the same edge in $\mathcal{G}_2$.

3. Termination Condition: This recursion continues until we reach simple graphs with easily computable chromatic polynomials. These typically include empty graphs (graphs with no edges) and base cases such as simple path graphs and cycle graphs.

4. Simple Graph Calculation: We then calculate the chromatic polynomials for our simple graphs. For empty graphs, the chromatic polynomial is straightforward as an empty graph with k vertices can be colored in x^k ways.

5. Polynomial Computation: The chromatic polynomial of $\mathcal{G}$ follows the recursive relationship:

$$P(\mathcal{G}; x) = P(\mathcal{G}_1; x) - P(\mathcal{G}_2; x)$$

When we delete an edge, we increase the number of possible colorings. By subtracting $P(\mathcal{G}_2; x)$, we remove the colorings that would violate the key principle of graph coloring i.e. we remove the colorings where the two vertices originally connected by the edge share the same color. To better understand this, let us look at an example of how we compute the chromatic polynomial of a 4-cycle graph using the Birkhoff-Lewis Reduction Algorithm:

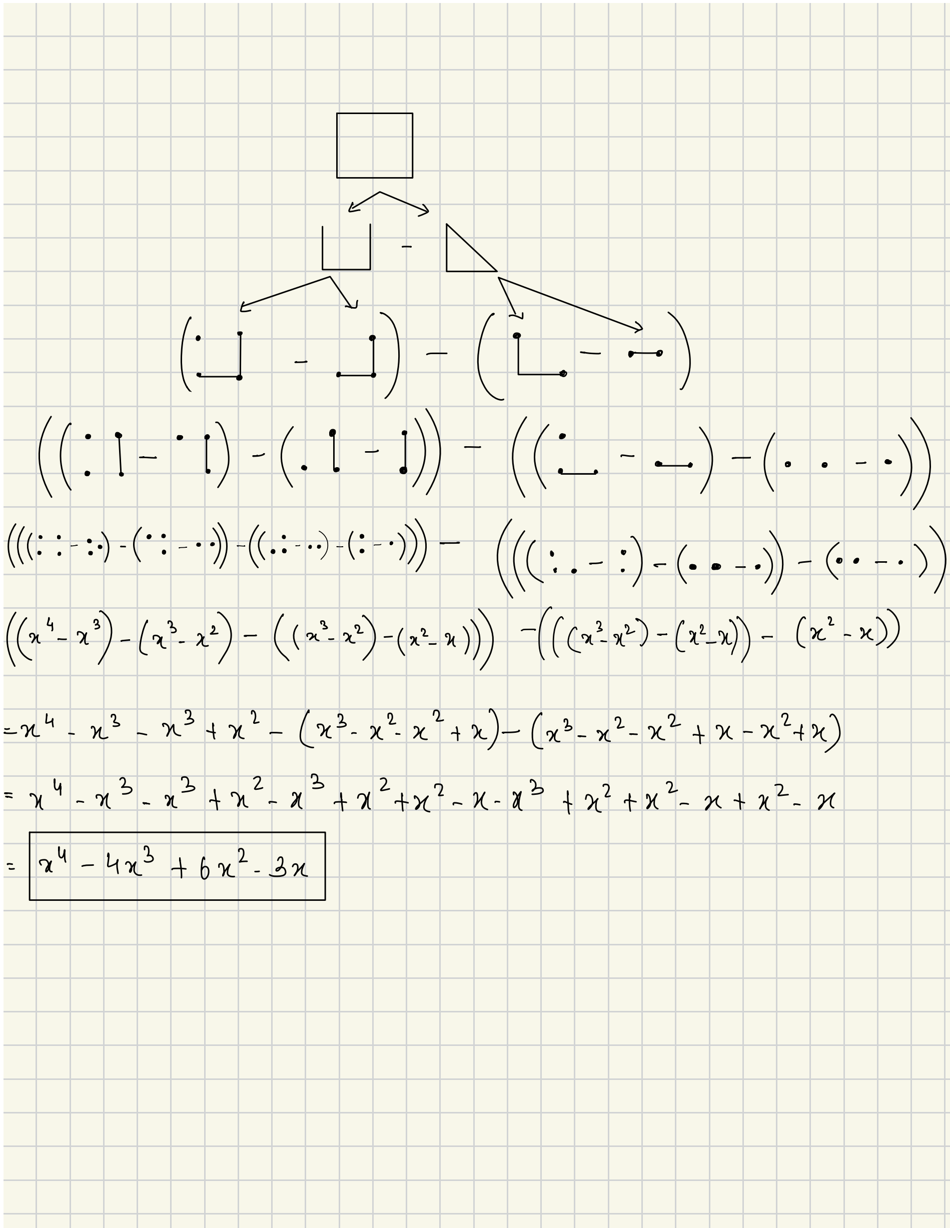


FIGURE 1. This image gives us the visualization of the Birkhoff-Lewis Reduction Algorithm for a 4-cycle graph.

Having explored the Birkhoff-Lewis Reduction Algorithm through our 4-cycle graph example, we now look at some theorems which are fundamental to graph coloring. These theorems give us insights into the structure of chromatic polynomials.

3. Theorems

We begin with the first theorem, which explores the leading term of the chromatic polynomial in relation to the number of vertices in a graph.

Theorem 3.1. *Given a graph $\mathcal{G}$ with n vertices and e edges, the chromatic polynomial $(P(\mathcal{G};x))$ begins with the leading term x^n.*

Proof. We will prove this theorem by induction on the number of edges e. For our base case, we let $e = 0$. In this case, the graph $\mathcal{G}$ consists of n vertices with no edges between them. The chromatic polynomial for this graph is:

$$P(\mathcal{G};x) = x^n.$$

Since, the leading term is x^n, our base case holds. Now, we consider the inductive hypothesis for e and assume that for any graph with n vertices and $e \geq 0$ edges, the chromatic polynomial is of the form:

$$P(\mathcal{G};x) = x^n - ex^{n-1} + ...$$

Our goal is to show that a graph with n vertices and $e + 1$ edges also has a chromatic polynomial that beings with x^n . Consider a graph $\mathcal{G}$ with n vertices and $e + 1$ edges. Using the Birkhoff-Lewis Reduction Algorithm, we select an edge e' and decompose $\mathcal{G}$ into two new graphs:

$$P(\mathcal{G};x) = P(\mathcal{G}';x) - P(\mathcal{G}'';x),$$

where:

(i) $\mathcal{G}'$ is obtained by deleting the edge e', resulting in a graph with n vertices and e edges.

(ii) $\mathcal{G}''$ is obtained by collapsing the edge e', resulting in a graph with $n-1$ vertices (the exact number of edges changes but it is not needed for the leading term).

By the inductive hypothesis:

$$P(\mathcal{G}'; x) = x^n - ex^{n-1} + ...,$$

For the collapsed graph $\mathcal{G}''$, since it has $n-1$ vertices, its chromatic polynomial is:

$$P(\mathcal{G}''; x) = x^{n-1} + ...,$$

Substituting this into the decomposition formula:

$$P(\mathcal{G}; x) = (x^n - ex^{n-1} + ...) - (x^{n-1} + ...).$$

Simplifying the expression:

$$P(\mathcal{G}; x) = x^n - (e+1)x^{n-1} + ...$$

This shows that the chromatic polynomial of $\mathcal{G}$ with n vertices and $e+1$ edges still begins with the leading term x^n. Thus, we have shown that our inductive hypothesis holds and our theorem is true . □

Having established the leading term of the chromatic polynomial, we try to better our understanding of the structure of the chromatic polynomial by examining its subsequent coefficients. Specifically, understanding the coefficient of x^{n-1} helps us understand how the number of edges in a graph affects its chromatic properties.

Theorem 3.2. *Given a graph $\mathcal{G}$ with n vertices and $|E|$ edges, in the chromatic polynomial $(P(\mathcal{G};x))$, the coefficient of x^{n-1} is $-|E|$.*

Proof. We will prove this theorem by induction on the number of edges $e = |E|$. For our base case, we let $e = 0$. In this case, the graph $\mathcal{G}$ consists of n vertices with no edges between them. The chromatic polynomial for this graph is:

$$P(\mathcal{G};x) = x^n.$$

Since, the coefficient of x^{n-1} is 0, it matches $-|E| = -0 = 0$. Thus, our base case holds. Now, we consider the inductive hypothesis for e and assume that for any graph with n vertices and $e \geq 0$ edges, the chromatic polynomial is of the form:

$$P(\mathcal{G};x) = x^n - ex^{n-1} + ...$$

Here, the coefficient of x^{n-1} is $-e$. Our goal is to show that a graph with n vertices and $e+1$ edges has a chromatic polynomial whose coefficient of x^{n-1} is $-(e+1)$. Consider a graph $\mathcal{G}$ with n vertices and $e+1$ edges. Using the Birkhoff-Lewis Reduction Algorithm, we select an edge e' and decompose $\mathcal{G}$ into two new graphs:

$$P(\mathcal{G};x) = P(\mathcal{G}';x) - P(\mathcal{G}'';x),$$

where:

(i) $\mathcal{G}'$ is obtained by deleting the edge e', resulting in a graph with n vertices and e edges.

(ii) $\mathcal{G}''$ is obtained by collapsing the edge e', resulting in a graph with $n-1$ vertices (the exact number of edges changes but it is not needed to calculate the co-efficient of x^{n-1}).

By the inductive hypothesis:

$$P(\mathcal{G}'; x) = x^n - ex^{n-1} + ...$$

For the collapsed graph $\mathcal{G}''$, since it has $n-1$ vertices, its chromatic polynomial is:

$$P(\mathcal{G}''; x) = x^{n-1} - |E''|x^{n-2} + ...$$

Since, we want to find the coefficient of x^{n-1} in $P(G; x)$, we only consider the leading term:

$$P(\mathcal{G}''; x) = x^{n-1} + ...$$

Substituting this into the decomposition formula:

$$P(\mathcal{G}; x) = (x^n - ex^{n-1} + ...) - (x^{n-1} + ...).$$

This gives us:

$$P(\mathcal{G}; x) = x^n - ex^{n-1} - x^{n-1} + ...$$

Simplifying the expression:

$$P(\mathcal{G}; x) = x^n - (e+1)x^{n-1} + ...$$

This shows that the coefficient of x^{n-1} in $P(\mathcal{G};x)$ is $-(e+1)$, which is equal to $-|E|$. Thus, we have shown that our inductive hypothesis holds and our theorem that the chromatic polynomial of graph $\mathcal{G}$ with n vertices and $|E|$ edges has the coefficient of x^{n-1} equal to $-|E|$ is true. □

Having established fundamental properties of chromatic polynomials through Theorems 3.1 and 3.2, we now look at specific types of graphs and their respective chromatic polynomials. We begin with path graphs.

Theorem 3.3. *For a path graph P_n with n vertices, the chromatic polynomial is given by:*

$$P(P_n;x) = x(x-1)^{n-1}.$$

Proof. We will prove this theorem using two approaches: a combinatorial argument and the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction. Let us begin with the combinatorial argument.

When determining the chromatic polynomial for an n-path graph P_n, we sequntially analyze the coloring process. The first vertex v_1 can be colored in x different ways since there are no restrictions. The second vertex v_2, being adjacent to v_1, cannot have the same color as v_1, giving it $x-1$ color choices. Each subsequent vertex v_i (for $i = 3, 4, \ldots, n$) is only adjacent to its immediate predecessor v_{i-1}, and thus also has $x-1$ color choices to ensure a proper coloring. The total number of proper colorings for the entire path graph is the product of the number of color choices at each vertex. To further illustrate this pattern, consider the chromatic polynomial for path graphs with different numbers of vertices n:

Number of vertices(n)	Chromatic Polynomial ($P(P_n;x)$)
1	x
2	$x(x-1) = x^2 - x$
3	$x(x-1)^2 = x^3 - 2x^2 + x$
4	$x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x$
5	$x(x-1)^4 = x^5 - 4x^4 + 6x^3 - 4x^2 + x$

From the table, we observe the pattern that the chromatic polynomial for a path graph with n vertices consistently seems to follow the formula:

$$P(P_n;x) = x(x-1)^{n-1}.$$

This expression represents the chromatic polynomial of the n-path graph P_n, showing the number of proper colorings with x colors. Next, we will prove this formula using the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction. We will prove this theorem by induction on the number of vertices n. For our base case, we let $n = 1$. In this case, the graph P_1 consists of a single vertex with no edges. When $n = 1$, since there are no adjacent vertices, the vertex can take on any of the x colors and thus, the chromatic polynomial for this graph is:

$$P(P_1;x) = x.$$

According to our formula, when $n = 1$:

$$P(P_1; x) = x(x-1)^{1-1} = x \times 1 = x.$$

Thus, our base case holds. Now, we consider the inductive hypothesis. Assume that for a path graph P_n with n vertices, the chromatic polynomial is:

$$P(P_n; x) = x(x-1)^{n-1}.$$

Our goal is to show that a graph P_{n+1} with $n+1$ vertices has a chromatic polynomial of the form:

$$P(P_{n+1}; x) = x(x-1)^n.$$

Consider a path graph P_{n+1} with $n+1$ vertices. Applying the Birkhoff-Lewis Reduction Algorithm, we select an edge $e' = v_n v_{n+1}$. and decompose P_{n+1} into two new graphs such that:

$$P(P_{n+1}; x) = P(\mathcal{G}'; x) - P(\mathcal{G}''; x),$$

where:

(i) $\mathcal{G}'$ is obtained by deleting the edge e', which gives us a path graph P_n with vertices v_1 to v_n and a single vertex v_{n+1}. Thus, the chromatic polynomial of $\mathcal{G}'$ is:

$$P(\mathcal{G}'; x) = P(P_n; x) \times P(P_1; x) = x(x-1)^{n-1} \times x = x^2(x-1)^{n-1}.$$

(ii) $\mathcal{G}''$ is obtained by collapsing the edge e', which merges vertices v_n and v_{n+1} into a single vertex. Thus, the resulting graph G'' is a path graph with n vertices. By our inductive hypothesis, the chromatic polynomial is:

$$P(\mathcal{G}''; x) = P(P_n; x) = x(x-1)^{n-1}.$$

Substituting these into the decomposition formula, we get:

$$P(P_{n+1}; x) = P(\mathcal{G}'; x) - P(\mathcal{G}''; x) = x^2(x-1)^{n-1} - x(x-1)^{n-1}.$$

This expression gives us:

$$x(x-1)^{n-1}(x-1) = x(x-1)^n.$$

Thus, our inductive step holds. Using mathematical induction and the Birkoff-Lewis Reduction Algorithm, we have shown that the chromatic polynomial for a path graph P_n with n vertices follows the formula which we observed in our combinatorial argument:

$$P(P_n; x) = x(x-1)^{n-1}.$$

$\square$

Having determined the chromatic polynomial of a path graph, we now look at complete graphs.

Theorem 3.4. *For a complete graph $\mathcal{G}_n$ with n vertices, the chromatic polynomial is given by:*

$$P(\mathcal{G}_n; x) = x(x-1)(x-2)...(x-(n-1)).$$

Proof. We will prove this theorem using two approaches: a combinatorial argument and the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction. Let us begin with the combinatorial argument. In a complete graph, every pair of distinct vertices is adjacent. This means that each vertex must be assigned a unique color to ensure a proper coloring, where no two adjacent vertices share the same color. Consider the coloring process sequentially: the first vertex can be colored in x different ways since there are no restrictions. However, the second vertex is adjacent to the first and thus cannot share its color, leaving $x-1$ color choices. As we progress through the graph, the color choices decrease further. For the third vertex, we must avoid the colors already used by the first two vertices, reducing the available choices to $x-2$. As a result, the total number of proper colorings follows a sequential reduction in color choices as we progress through the graph. To further illustrate this pattern, consider the chromatic polynomial for complete graphs with varying numbers of vertices n:

Number of vertices(n)	Chromatic Polynomial ($P(G_n; x)$)
1	x
2	$x(x-1) = x^2 - x$
3	$x(x-1)(x-2) = x^3 - 3x^2 + 2x$
4	$x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$
5	$x(x-1)(x-2)(x-3)(x-4) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$

From the table, we observe the pattern that the chromatic polynomial for a complete graph with n vertices seems to consistently follow the formula:

$$P(\mathcal{G}_n; x) = x(x-1)(x-2)...(x-(n-1)).$$

This expression represents the chromatic polynomial of the n-complete graph $\mathcal{G}_n$, showing the number of proper colorings with x colors. Next, we will prove this formula using the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction.

We will prove this theorem by induction on the number of vertices n. For our base case, we let $n = 1$. In this case, the graph $\mathcal{G}_1$ consists of single vertex with no edges. From, the table we know that the chromatic polynomial for the graph is:

$$P(\mathcal{G}_1; x) = x.$$

According to our formula, when $n = 1$:

$$P(\mathcal{G}_1; x) = (x-0) = x.$$

Thus, our base case holds. Now, we consider the inductive hypothesis for a complete graph $\mathcal{G}_n$ with n vertices and assume that chromatic polynomial is:

$$P(\mathcal{G}_n; x) = x(x-1)(x-2)...(x-(n-1)).$$

Our goal is to show that a complete graph $\mathcal{G}_{n+1}$ with $n+1$ vertices has a chromatic polynomial of the form:

$$P(\mathcal{G}_{n+1}; x) = x(x-1)(x-2)...(x-n).$$

Consider a complete graph G_{n+1} with $n+1$ vertices. Applying the Birkhoff Lewis Reduction Algorithm, we select an edge e' connecting the vertex v_{n+1} to v_n and decompose $\mathcal{G}_{n+1}$ into two new graphs:

$$P(\mathcal{G}_{n+1};x) = P(\mathcal{G}';x) - P(\mathcal{G}'';x),$$

where:

(i) $\mathcal{G}'$ is obtained by deleting the edge e', resulting in a graph with $n+1$ vertices and $(\frac{n(n+1)}{2} - 1)$ edges.

(ii) $\mathcal{G}''$ is obtained by collapsing the edge e', which merges v_n andv_{n+1} into one vertex. This graph is a complete graph on n vertices.

Thus, by our inductive hypothesis:

$$P(\mathcal{G}'';x) = P(\mathcal{G}_n;x) = x(x-1)(x-2)...(x-(n-1)).$$

Next, we need to determine $P(\mathcal{G}';x)$. In this graph, all vertices are connected except v_n and v_{n+1}. Therefore, when we calculate the total number of proper colorings, vertices v_n and v_{n+1} can either share the same color or have distinct colors.

First, let us consider colorings where v_n and v_{n+1} share the same color. This is allowed as they are not connected by an edge. Since, both vertices take the same color, we are we effectively reducing the coloring problem to that of a complete graph with n vertices. Using the inductive hypothesis, this gives us:

$$P(\mathcal{G}_n;x) = x(x-1)(x-2)...(x-(n-1)).$$

Second, let us consider colorings where v_n and v_{n+1} have different colors. This means that we need to assign v_{n+1} a color different from the n vertices. There are $(x - n)$ choices for the color of v_{n+1}. Therefore, the number of colorings this graph is:

$$P(\mathcal{G}_n; x) \times (x - n).$$

Combining these two cases, the total number of proper colorings of $\mathcal{G}'$ emerges as:

$$P(\mathcal{G}'; x) = P(\mathcal{G}_n; x) + P(\mathcal{G}_n; x)(x - n) = P(\mathcal{G}_n; x)(1 + (x - n)).$$

This expression can be written as:

$$P(\mathcal{G}'; x) = P(\mathcal{G}_n; x)(x - n + 1).$$

Substituting our findings related to $\mathcal{G}'$ and $\mathcal{G}''$ into the Birkhoff-Lewis Algorithm, we get:

$$P(\mathcal{G}_{n+1}; x) = P(\mathcal{G}'; x) - P(\mathcal{G}''; x) = P(\mathcal{G}_n; x)(x - n + 1) - P(\mathcal{G}_n; x).$$

Factoring out the chromatic polynomial, we get:

$$P(\mathcal{G}_{n+1}; x) = P(\mathcal{G}_n; x)(x - n + 1 - 1) = P(\mathcal{G}_n; x)(x - n).$$

Substituting the inductive hypothesis, we get:

$$P(\mathcal{G}_{n+1}; x) = [x(x - 1)(x - 2)...(x - (n - 1))](x - n).$$

This expression can be written as:

$$P(\mathcal{G}_{n+1}; x) = x(x-1)(x-2)...(x-n).$$

Thus, our inductive step holds. Using mathematical induction and the Birkoff-Lewis Reduction Algorithm, we have shown that the chromatic polynomial for a complete graph $\mathcal{G}_n$ with n vertices follows the formula which we observed in our combinatorial argument:

$$P(\mathcal{G}_n; x) = x(x-1)(x-2)...(x-(n-1)).$$

□

Having determined the chromatic polynomial of complete graphs, we now calculate the chromatic polynomial for cycle graphs.

Theorem 3.5. *For a cycle graph $\mathcal{G}_n$ with n vertices, the chromatic polynomial is given by:*

$$P(\mathcal{G}_n; x) = (x-1)^n + (-1)^n(x-1).$$

Proof. We will prove this theorem using the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction. To understand the chromatic polynomial of a cycle graph, we encounter a more complex coloring problem compared to path or complete graphs. In a cycle graph, each vertex is connected to exactly two neighbors, creating a circular structure that introduces unique colorings. Unlike path graphs with a linear progression or complete graphs with full connectivity, cycle graphs have a much more complex structure. For small cycles, the combinatorial argument can be seen as follows:

(i) In $\mathcal{G}_2$, there are two vertices and they can have $x(x-1)$ total colorings.

(ii) In $\mathcal{G}_3$, three vertices form a triangle, requiring three distinct colors, leading to $x(x-1)(x-2)$ colorings.

However, as the cycle length increases beyond four vertices, the combinatorial argument becomes significantly more nuanced. To illustrate this pattern for small cycles, consider the chromatic polynomial for cycle graphs with numbers of vertices $n = 2, 3, 4$:

Number of vertices(n)	Chromatic Polynomial ($P(P_n; x)$)
2	$x(x-1) = x^2 - x.$
3	$x(x-1)(x-2) = x^3 - 3x^2 + 2x$
4	$x(x-1)(x-1) + x(x-1)(x-2) = x^4 - 4x^3 + 6x^2 - 3x$

We will prove this formula using the Birkhoff-Lewis Reduction Algorithm combined with mathematical induction. For our base case, let us assume number of vertices $n = 2$. In this case the $\mathcal{G}_2$ consists of two vertices connected by an edge. From our table, the chromatic polynomial for this graph is:

$$P(\mathcal{G}_2; x) = x^2 - x.$$

According to our formula, when $n = 2$:

$$P(\mathcal{G}_2; x) = (x-1)^2 + (-1)^2(x-1) = (x^2 - 2x + 1) + (x-1) = x^2 - x.$$

Thus, our base case holds. Now, we consider the inductive hypothesis and assume that for a graph $\mathcal{G}_n$ with n vertices, the chromatic polynomial is:

$$P(\mathcal{G}_n; x) = (x-1)^n + (-1)^n(x-1).$$

Our goal is to show that for a cycle graph $\mathcal{G}_{n+1}$ with $n+1$ vertices has a chromatic polynomial of the form:

$$P(\mathcal{G}_{n+1}; x) = (x-1)^{n+1} + (-1)^{n+1}(x-1).$$

Consider a cycle graph $\mathcal{G}_{n+1}$ with $n+1$ vertices. Applying the Birkhoff-Lewis Reduction Algorithm, we select an edge $e = v_n v_{n+1}$ and decompose G_{n+1} into two new graphs:

$$P(\mathcal{G}_{n+1}; x) = P(\mathcal{G}'; x) - P(\mathcal{G}''; x),$$

where:

(i) $\mathcal{G}'$ is obtained by deleting the edge e', resulting in a path graph P_{n+1} with $n+1$ vertices. Thus, using Theorem 3.3, the chromatic polynomial of $\mathcal{G}'$ is:

$$P(\mathcal{G}'; x) = P(P_{n+1}; x) = x(x-1)^n.$$

(ii) $\mathcal{G}''$ is obtained by collapsing the edge e', merging vertices v_n and v_{n+1} into a single vertex. The resulting graph is a cycle graph with n vertices. By our inductive hypothesis:

$$P(\mathcal{G}''; x) = (x-1)^n + (-1)^n(x-1).$$

Substituting these into the decomposition formula, we get:

$$P(\mathcal{G}_{n+1};x) = P(\mathcal{G}';x) - P(\mathcal{G}'';x) = x(x-1)^n - [(x-1)^n + (-1)^n(x-1)].$$

By simplifying the expression and factoring out $(x-1)^n$ we get:

$$P(\mathcal{G}_{n+1};x) = (x-1)^n(x-1) + (-1)^{n+1}(x-1).$$

This expression gives us:

$$P(\mathcal{G}_{n+1};x) = (x-1)^{n+1} + (-1)^{n+1}(x-1).$$

Thus, our inductive step holds. Using mathematical induction and the Birkoff-Lewis Reduction Algorithm, we have shown that the chromatic polynomial for a cycle graph $\mathcal{G}_n$ with n vertices follows the formula:

$$P(\mathcal{G}_n;x) = (x-1)^n + (-1)^n(x-1).$$

□

4. Conclusion

This study of graph coloring gives us insights into the foundations of graph theory and explores how graphs and color assignments can solve complex real-world problems such as scheduling, resource allocation, and network design.

A key part of this laboratory was the relationship between combinatorial reasoning and rigorous mathematical proof. For each type of graph, we initially used a combinatorial approach by analyzing small graphs. This approach allowed us to observe patterns in how vertices could be colored and generalize that to form hypotheses about the form of the chromatic polynomial. The transition from these combinatorial observations to rigorous proof was done using mathematical induction and the Birkhoff-Lewis Reduction Algorithm. This algorithm gave us a really powerful computational method by which we could systematically break down complex graphs into simpler sub-graphs and use their chromatic polynomial for our calculation.

Looking forward, future research directions include exploring chromatic polynomials for more types of graphs and investigating practical applications of graph coloring.

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