ITERATION OF LINEAR FUNCTIONS

BHARGAVI PATIL

Abstract. This report explores the behavior of sequences generated by iterating linear functions, using algebraic, geometric, and computational methods. It examines various types of convergence and divergence that appear, and investigates how initial values, slopes, and intercepts influence the behavior of these sequences.

1. Introduction

In mathematics, the concept of iteration has many applications. One such application is identifying points where $f(x) = x$, known as fixed points, which is useful for solving equations and other mathematical problems. In simple words, iteration involves taking an initial value and repeatedly applying a function to generate a list of numbers, known as a sequence.

To better understand this idea, let us look at the function, $f(x) = \cos(x)$. Starting with an initial value of 0, we repeatedly apply the function, which produces the following values:

$$
\begin{aligned}
x_1 &= \cos(x_0) = \cos(0) = 1,\\
x_2 &= \cos(x_1) = \cos(1) = 0.5403,\\
x_3 &= \cos(x_2) \approx \cos(0.5403) \approx 0.8576,\\
&\vdots
\end{aligned}
$$

I thank Professor Robinson and my classmates for helping me learn and understand the ideas presented in this report.

This sequence converges to the limit where $\cos(x) = x$. This means that as we continue applying the cosine function to our initial value, the numbers generated get closer and closer to a value where the cosine of that value equals the value itself.

The identity line, represented by the line $y = x$ is important as it helps us determine where the output of the cosine function is equal to its input. The point of intersection between the cosine function and the identity line is where the two values are the same, introducing us to the idea of fixed points, which we will explore later in this report. The graph below shows us the intersection of the cosine function and identity line.

This example illustrates the iteration of a non-linear function, where

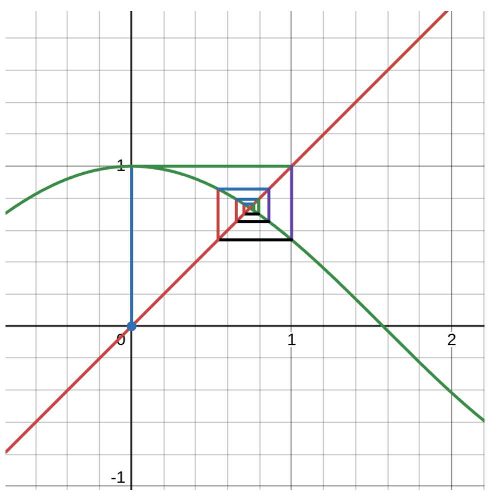

FIGURE 1. This graph gives the geometrical visualization of the the iteration of $\cos x$ with initial value $x_0 = 0$.

each output becomes the input for the next step. With the idea that we can iterate any function to produce a sequence established, we can now look at another type of iteration, that is linear iteration.

Linear iteration forms the basis of many diverse topics, from the calculation of compound interest in finance to the modeling of population growth in ecology. Before we begin to understand the process of linear iteration, it is important for us to understand the concept of a sequence. Thus, we begin with the definition of a sequence.

Definition 1.1. A sequence is a function, $x(n)$ whose domain is a special type of subset of the integers. The domain has a smallest member and includes all integers larger than its smallest member. The values produced by the function, $x(n)$ are called the terms of the sequence, and we denote these values $x(n)$ as x_n for simplicity. The special nature of the domain ensures that each term of the sequence corresponds uniquely to an index n. Sequences are often represented as $\{x_n\}_{n=m}^{\infty} = \{x_m, x_{m+1}, x_{m+2}, ...\}$, where m is the starting index, typically 0 or 1. [2]

Example 1.2. Consider the sequence defined by $x(n) = 2^{n-1}$, where the domain is a set of natural numbers. The first few terms of this sequence would be as follows:

$$\begin{aligned}
x_1 &= 2^0 = 1, \\
x_2 &= 2^1 = 2, \\
x_3 &= 2^2 = 4, \\
&\vdots
\end{aligned}$$

Here, the sequence is $\{x_n\} = \{1, 2, 4...\}$. Our example also helps us illustrate the concept of a geometric sequence, which is a sequence where each term is derived by multiplying the previous term by a constant. This constant is known as the common ratio (r). In our example, the common ratio is 2.

In general, a geometric sequence is $\{x, xr, xr^2, xr^3, ...\}$ for some particular x and r. Another important concept we need to understand in order to investigate linear iteration is that of a linear function. Linear functions are defined next.

Definition 1.3. A linear function is a function of the form $f(x) = ax + b$, where a and b are real numbers. The graph of a linear function is a straight line, where a represents the slope of the function and b represents the y-intercept, which is the point where the line intersects the y-axis.

Example 1.4. An example of a linear function is $f(x) = 2x+2$, where $a = 2$ and $b = 2$. As we substitute different real values of x into the function, it produces corresponding values of $f(x)$, which represents the y co-ordinates for the values of x. This process can be seen as follows:

$$f(1) = 2(1) + 2 = 4$$
$$f(-2) = 2(-2) + 2 = -2$$

We now have two points, $(1, 4)$ and $(-2, -2)$, that can be plotted on a Cartesian plane to visualize this linear function.

Iteration refers to repeating a process multiple times. In this context, it refers to the repeated application of a function starting with an initial value to produce a sequence of numbers. The output from one step serves as the input for the next step. [1]

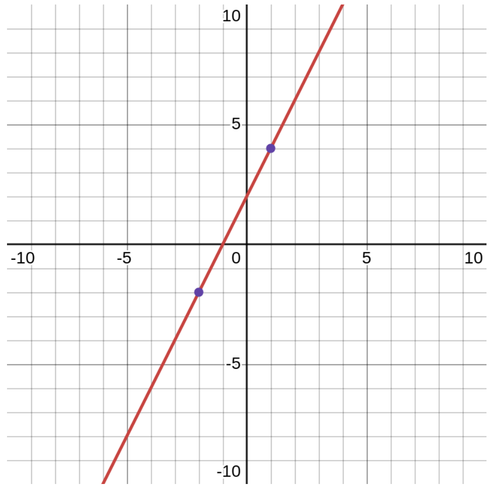

FIGURE 2. This graph gives the geometrical visualization of the linear function $f(x) = 2x + 2$.

A linear iteration sequence is a special type of sequence obtained by the repeated application of a linear function using an initial value. We define linear iteration below.

Definition 1.5. A linear iteration sequence is a sequence generated by the repeated application of a linear function, $f(x) = ax+b$ to an initial value, x_0. The sequence starts with x_0 and the subsequent terms are generated as follows:

$$\begin{aligned}
&x_1 = f(x_0) = ax_0 + b,\\
&x_2 = f(x_1) = ax_1 + b,\\
&x_3 = f(x_2) = ax_2 + b,\\
&\vdots\\
&x_n = f(x_{n-1}) = ax_{n-1} + b, \text{ and so on.}
\end{aligned}$$

The outputs produced at each step of the iteration process are known as iterates.

Example 1.6. We will use the linear function defined in example 1.4 to generate a linear iteration sequence. Consider the linear function, $f(n) = 2x + 2$ and an initial value, $x_0 = 1$. The iterates of the linear iteration sequence are generated as follows:

$$\begin{aligned}
&x_0 = 1,\\
&x_1 = f(x_0) = 2(1) + 2 = 4,\\
&x_2 = f(x_1) = 2(4) + 2 = 10,\\
&x_3 = f(x_2) = 2(10) + 2 = 22,\\
&\vdots
\end{aligned}$$

This linear iteration sequence can be represented as $\{x_n\} = \{1, 4, 10, 22...\}$.

In this laboratory, we will investigate and analyze the behavior of sequences generated by iterating a linear function, starting with a given initial value. We will begin by exploring the iteration of various linear functions to understand how the initial value, slope, and y-intercept affect the behavior of the sequence. Our goal is to determine when and how sequences converge or diverge, and identify the different types of convergence and divergence that may arise.

To achieve this, we will use both, geometric and algebraic methods to study the behavior of these sequences. We will learn to identify patterns and apply concepts from the finite geometric sum to construct our proofs.

We will use Python for our calculations and Desmos for graphical exploration, which will allow us to observe patterns and make sense of the behavior of sequences. This exploration will help us solidify our understanding of key mathematical concepts like limits, induction, as well as the structure of mathematical arguments, such as lemmas, propositions, theorems, and corollaries.

2. Convergence and Divergence

As we begin to explore various examples of linear iteration, an important behavior to study is convergence and divergence. A sequence is said to converge if there is a finite value (known as the limit of the sequence) which the terms of the sequence approach ever more closely as iteration progresses. A sequence is said to diverge when the sequence fails to settle down to a finite value. Divergence may occur in many ways. Let us begin with the definition of a convergent sequence.

Definition 2.1. A sequence $\{x_n\}$ is said to converge to a limit, L if for every $\epsilon > 0$ (ϵ represents an arbitrarily small number), there exists a natural number, N such that for all $n > N$, the terms of the sequence satisfy $|x_n - L| < \epsilon$.

In the case that $\{x_n\}$ converges to L, we write that $\lim_{n\to\infty} x_n = L$.

Example 2.2. An example of a convergent sequence is $\{x_n\}$ where $x_n = \frac{1}{n}$. The first few terms of this sequence are as follows:

$$\begin{aligned} &x_1 = \tfrac{1}{1} = 1, \\ &x_2 = \tfrac{1}{2} = 0.5, \\ &x_3 = \tfrac{1}{3} = 0.3\overline{3}, \\ &\vdots \end{aligned}$$

This sequence can be represented as $\{1, 0.5, 0.3\overline{3}...\}$ and we can see that as n grows, this sequence approaches the limit, 0.

Using Definition 2.1, we see that $\lim_{n\to\infty} \frac{1}{n}$ is 0 since for any $\epsilon > 0$, there exists an N such that for all $n > N$, the inequality $|\frac{1}{n} - 0| < \epsilon$ holds.

To show this, we simplify and start with:

$$|\frac{1}{n}| < \epsilon,$$

Rearranging this gives:

$$n > \frac{1}{\epsilon}.$$

Thus, for any $\epsilon > 0$, we can choose $N = \frac{1}{\epsilon}$, and for all $n > N$, we have $|x_n - 0| < \epsilon$.

Definition 2.3. A sequence is said to diverge if it does not converge. In this case, there does not exist a number, L, which the terms of the sequence approach ever more closely as n grows. There are many different ways in which a sequence can diverge.

Example 2.4. An example of a divergent sequence is $\{x_n\}$ where $x_n = n^2$. The first few terms of this sequence are as follows:

$$\begin{aligned} &x_1 = 1^2 = 1, \\ &x_2 = 2^2 = 4, \\ &x_3 = 3^2 = 9, \\ &\vdots \end{aligned}$$

This sequence can be represented as $\{1, 4, 9...\}$ and we can see that as n grows, the terms of the sequence will grow without an upper bound. To illustrate this further, for any $m > 0$, we can choose $N = \sqrt{m}$. If

$n > \sqrt{m}$ then:

$$x_n = n^2 > m.$$

This argument demonstrates that $\{x_n\}$ is unbounded.

Linear iteration sequences can converge and diverge in many different ways. We will now look at the different types of convergence and divergence exhibited by linear iteration sequences.

2.1. **Types of convergence.**

Definition 2.5. Constant convergence.

A sequence $\{x_n\}$ is said to exhibit constant convergence if there exists a positive integer N such that $x_n = c$ for all $n \geq N$.

Example 2.6. An example of a sequence that exhibits constant convergence is $\{x_n\}$, defined by function $f(x) = -3x + 1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = 0.25$. The first few terms of this sequence are as follows:

$$\begin{aligned}
&x_0 = 0.25, \\
&x_1 = f(x_0) = -3(0.25) + 1 = 0.25, \\
&x_2 = f(x_1) = -3(0.25) + 1 = 0.25, \\
&x_3 = f(x_2) = -3(0.25) + 1 = 0.25, \\
&\vdots
\end{aligned}$$

This sequence can be represented as $\{0.25, 0.25, 0.25...\}$ and we can see that as n grows, the terms of the sequence remain constant, converging to 0.25.

Definition 2.7. Increasing convergence.

A sequence is said to exhibit increasing convergence if $x_n < x_{n+1}$ for

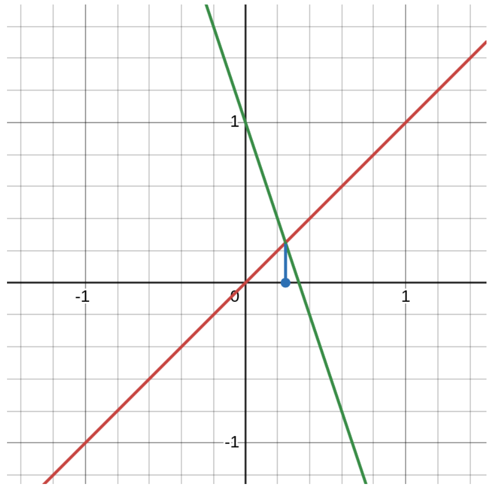

FIGURE 3. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = -3x + 1$ with initial value $x_0 = 0.25$, illustrating constant convergence.

all $n \geq 0$ and there exists a limit L such that $x_n < L$ for all n and $\lim_{n\to\infty} x_n = L$.

Example 2.8. An example of a sequence that exhibits increasing convergence is $\{x_n\}$, defined by function $f(x) = 0.5x + 1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = -5$. The first few terms of this sequence are as follows:

$$\begin{aligned}
&x_0 = -5,\\
&x_1 = f(x_0) = 0.5(-5) + 1 = -1.5,\\
&x_2 = f(x_1) = 0.5(-1.5) + 1 = 0.25,\\
&x_3 = f(x_2) = 0.5(0.25) + 1 = 1.125,\\
&\vdots
\end{aligned}$$

This sequence can be represented as $\{-5, -1.5, 0.25, 1.125...\}$ and we can see that as n grows, the terms of the sequence increase and seem to converge to the limit $L = 2$. We will show later that this limit does hold.

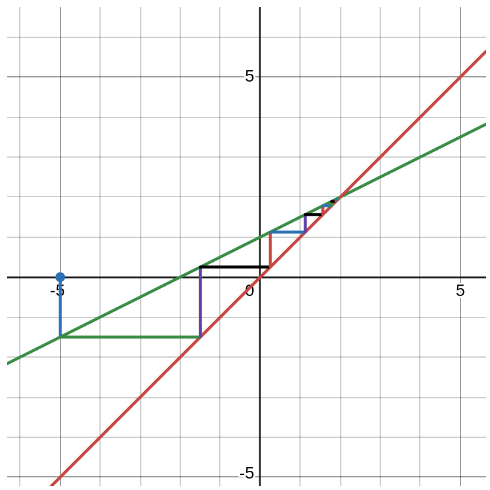

FIGURE 4. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = 0.5x + 1$ with initial value $x_0 = -5$, illustrating increasing convergence.

Definition 2.9. Decreasing convergence.

A sequence is said to exhibit decreasing convergence if $x_n > x_{n+1}$ for all $n \geq 0$ and there exists a limit L such that $x_n < L$ for all n and $\lim_{n\to\infty} x_n = L$.

Example 2.10. An example of a sequence that exhibits decreasing convergence is $\{x_n\}$, defined by function $f(x) = 0.5x+1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = 5$. The first few terms of this sequence are as follows:

$$
\begin{aligned}
&x_0 = 5, \\
&x_1 = f(x_0) = 0.5(5) + 1 = 3.5, \\
&x_2 = f(x_1) = 0.5(3.5) + 1 = 2.75, \\
&x_3 = f(x_2) = 0.5(2.75) + 1 = 2.375, \\
&\vdots
\end{aligned}
$$

This sequence can be represented as $\{5, 3.5, 2.75, 2.375...\}$ and we can see that as n grows, the terms of the sequence decrease and seem to converge to the limit $L = 2$. We will show later that this limit does hold.

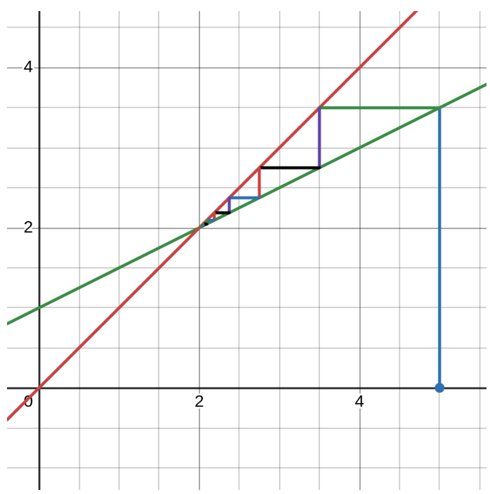

FIGURE 5. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = 0.5x + 1$ with initial value $x_0 = 5$, illustrating decreasing convergence.

If we take a closer look at examples 2.8 and 2.10, we can see that both sequences are generated by the same linear function, yet they converge in different ways. This point helps us understand that while the slope of the function determines whether a sequence converges or diverges, it is the starting value that decides whether the convergence is increasing or decreasing.

Definition 2.11. Spiral convergence.

A sequence is said to exhibit spiral convergence if the terms alternate around a limit L, while also approaching the limit ever more closely. This behaviour can happen in one of two ways:

$$1.x_0 < x_2 < ... < L... < x_3 < x_1,$$
$$2.x_1 < x_3 < ... < L... < x_2 < x_0.$$

The specific pattern depends on the starting value of the sequence. In both cases, the terms oscillate between different values but approach a finite limit ever more closely as n grows.

Example 2.12. An example of a sequence that exhibits spiral convergence is $\{x_n\}$, defined by function $f(x) = -0.5x$. The sequence is generated using the iterative process, starting with the initial value $x_0 = 1$. The first few terms of this sequence are as follows:

$$\begin{aligned}
&x_0 = 1,\\
&x_1 = f(x_0) = -0.5(1) = -0.5,\\
&x_2 = f(x_1) = -0.5(-0.5) = 0.25,\\
&x_3 = f(x_2) = -0.5(0.25) = -0.125,\\
&\vdots
\end{aligned}$$

This sequence can be represented as $\{1, -0.5, 0.25. - 0.125...\}$ and we can see that as n grows, the terms of the sequence oscillate between positive and negative values while seeming to approach the limit $L = 0$.

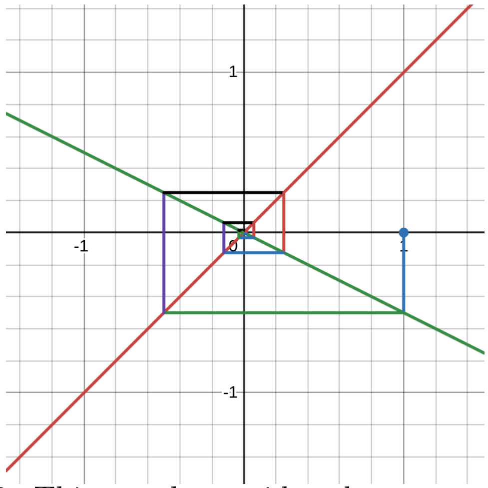

FIGURE 6. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = -0.5x$ with initial value $x_0 = 1$, illustrating spiral convergence.

2.2. **Types of divergence.**

Definition 2.13. Increasing divergence.
A sequence is said to exhibit increasing divergence if $x_n < x_{n+1}$ for all $n \geq 0$ and the terms, x_n grow without bound as n increases. There does not exist a limit L to which the terms converge.

Example 2.14. An example of a sequence that exhibits increasing divergence is $\{x_n\}$, defined by function $f(x) = 2x + 1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = 1$. The first few terms of this sequence are as follows:

$$
\begin{aligned}
&x_0 = 1,\\
&x_1 = f(x_0) = 2(1) + 1 = 3,\\
&x_2 = f(x_1) = 2(3) + 1 = 7,\\
&x_3 = f(x_2) = 2(7) + 1 = 15,\\
&\vdots
\end{aligned}
$$

This sequence can be represented as $\{1, 3, 7, 15...\}$ and we can see that as n grows, the terms of the sequence increase and grow without an upper bound.

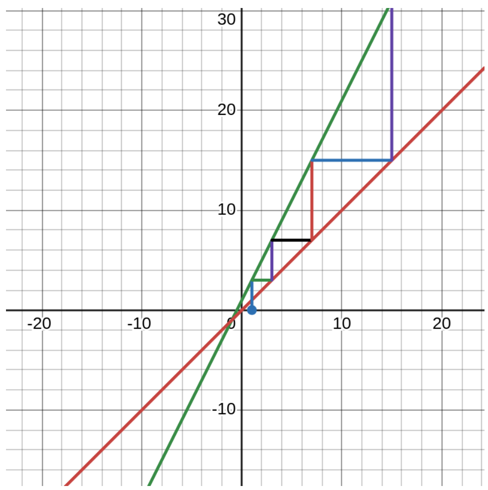

FIGURE 7. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = 2x+1$ with initial value $x_0 = 1$, illustrating increasing divergence.

Definition 2.15. Decreasing divergence.

A sequence is said to exhibit decreasing divergence if $x_n > x_{n+1}$ for all $n \geq 0$ and the terms, x_n decrease without bound as n increases. There does not exist a lower bound for the values of x_n.

Example 2.16. An example of a sequence that exhibits decreasing divergence is $\{x_n\}$, defined by function $f(x) = 2x + 1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = -2$. The first few terms of this sequence are as follows:

$$\begin{aligned} x_0 &= -2 \\ x_1 &= f(x_0) = 2(-2) + 1 = -3, \\ x_2 &= f(x_1) = 2(-3) + 1 = -5, \\ x_3 &= f(x_2) = 2(-5) + 1 = -9, \\ &\vdots \end{aligned}$$

This sequence can be represented as $\{-2, -3, -5, -9...\}$ and we can see that as n grows, the terms of the sequence decrease and grow without a lower bound.

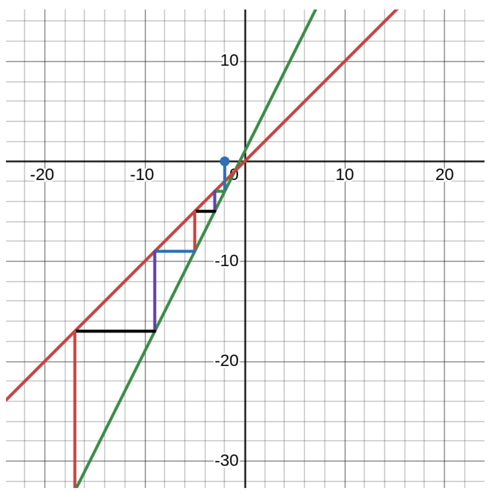

FIGURE 8. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = 2x+1$ with initial value $x_0 = -2$, illustrating decreasing divergence.

Definition 2.17. Blinking divergence.

A sequence is said to exhibit blinking divergence if the terms alternate between two distinct values, leading to an oscillation that does not approach a finite limit L as n grows.

Example 2.18. An example of a sequence that exhibits blinking divergence is $\{x_n\}$, defined by function $f(x) = -1x + 1$. The sequence is generated using the iterative process $x_{n+1} = f(x_n)$, starting with the initial value $x_0 = 2$. The first few terms of this sequence are as follows:

$$\begin{aligned}
x_0 &= 2, \\
x_1 &= f(x_0) = -1(2) + 1 = -1, \\
x_2 &= f(x_1) = -1(-1) + 1 = 2, \\
x_3 &= f(x_2) = -1(2) + 1 = -1, \\
&\vdots
\end{aligned}$$

This sequence can be represented as $\{2, -1, 2, -1...\}$. We can see that as n grows, the terms of the sequence oscillate between -1 and 2 and never converge to a finite limit L.

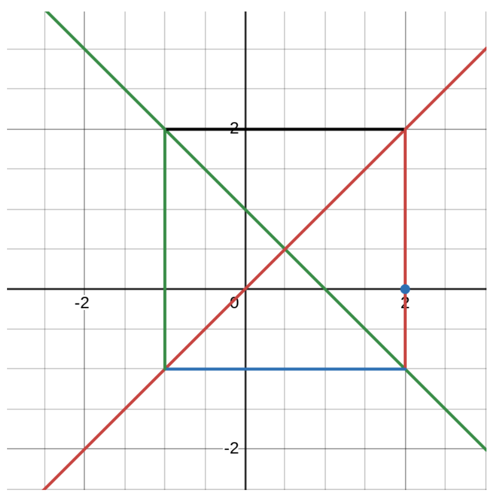

FIGURE 9. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = -1x + 1$ with initial value $x_0 = 2$, illustrating blinking divergence.

Definition 2.19. Spiral divergence.

A sequence is said to exhibit spiral divergence if the terms alternate between values while increasing in magnitude and eventually alternating between a large positive and a very small negative number. Later, we will show that this behavior occurs when $|x_n - \frac{b}{1-a}| < |x_{n+1} - \frac{b}{1-a}|$ for all n.

Example 2.20. An example of a sequence that exhibits spiral divergence is $\{x_n\}$, defined by function $f(x) = -2x + 1$. The sequence is generated using the iterative process, starting with the initial value $x_0 = 2$. The first few terms of this sequence are as follows:

$$\begin{aligned}
&x_0 = 2,\\
&x_1 = f(x_0) = -2(2) + 1 = -3,\\
&x_2 = f(x_1) = -2(-3) + 1 = 7,\\
&x_3 = f(x_2) = -2(7) + 1 = -13,\\
&\vdots
\end{aligned}$$

This sequence can be represented as $\{2, -3, 7, -13...\}$. We can see that as n grows, the terms of the sequence alternate between positive and negative values while increasing in absolute value and never converge to a finite limit L.

3. Algebraic Analysis

In Section 2, we observed various behaviors of sequences based on the relationship, $x_n = f(x_{n-1})$. To better understand the general structure of such sequences, let us derive the general form for the nth term of the sequence of form $x_{n+1} = ax_n + b$. The first few terms are as follows:

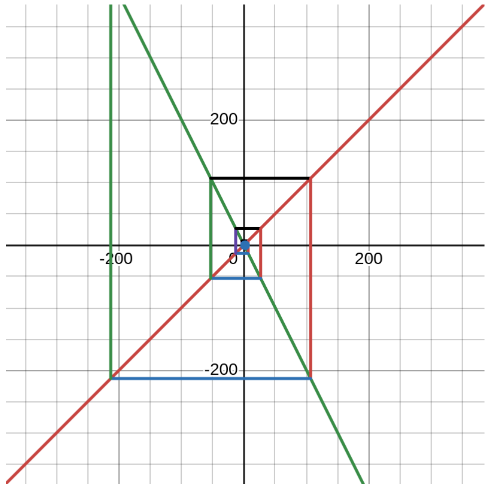

FIGURE 10. This graph provides the geometrical visualization of the the iteration of linear function $f(x) = -2x + 1$ with initial value $x_0 = 2$, illustrating spiral divergence.

$$x_1 = ax_0 + b,$$
$$x_2 = ax_1 + b = a(ax_0 + b) + b = a^2x_0 + b(1 + a),$$
$$\vdots$$

From the calculation above, we can generalize that after n steps, the value of x_n consists of two components:

1. The initial value x_0 multiplied by a^n,
2. The constant b multiplied with sum of powers of a from 1 to $n - 1$.

This work leads us to the following proposition, which provides a general formula for the nth term of the sequence:

Proposition 3.1. *If $f(x) = ax + b$ where $a, b \in \mathbb{R}$ and x_0 is the initial value, then the recursive form of the n^{th} iterate is given by:*

$$x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1})$$

for $n \geq 1$.

Proof. We will prove this formula by induction on n. For our base case, we let $n = 1$. Here, we need to show that the proposition holds for $n = 1$. The meaning of our proposition when $n = 1$ is that:

$$x_1 = ax_0 + b(1) = ax_0 + b.$$

This equation is true by the definition of linear iteration because:

$$x_1 = f(x_0) = ax_0 + b.$$

Thus, our base case holds. Now, we consider inductive hypothesis for n and assume that the statement is true for some $n \geq 1$. Our goal is to show that the statement holds for $n+1$. We assume for the sake of induction that:

$$x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1}).$$

Next, we must show that:

$$x_{n+1} = a^{n+1} x_0 + b(1 + a + \ldots + a^n).$$

By the definition of iteration, we know that:

$$x_{n+1} = f(x_n) = ax_n + b.$$

Next, we use the inductive hypothesis for the form of x_n and substitute that into our formula for x_{n+1} to get:

$$x_{n+1} = a(a^n x_0 + b(1 + a + \ldots + a^{n-1})) + b.$$

We then distribute a, which gives us:

$$x_{n+1} = a^{n+1}x_0 + ab(1 + a + \ldots + a^{n-1}) + b.$$

Next, we factor out b and get the formula:

$$x_{n+1} = a^{n+1}x_0 + b(a(1 + a + \ldots + a^{n-1}) + 1).$$

Finally, we distribute the variable a to get:

$$x_{n+1} = a^{n+1}x_0 + b((a + a^2 + \ldots + a^n) + 1),$$

and our formula for x_{n+1} becomes:

$$x_{n+1} = a^{n+1}x_0 + b(1 + a + a^2 + \ldots + a^n).$$

Thus, we have shown that our inductive step holds and our proposition is true by induction for all $n \geq 1$. □

As we can see above, the expression $(1+a+...+a^{n-1})$ appears in our formula for the nth term of the sequence. This expression is a finite geometric sum and so we will evaluate it. To sum the finite geometric sum, we denote it as follows:

$$S = 1 + a + a^2 + ... + a^{n-1}.$$

Our goal is to derive a closed form expression for S that will allow us to simplify our formula.

Lemma 3.2. *If a is any real number then*

$$1 + a + ... + a^{n-1} = \begin{cases} \frac{1-a^n}{1-a} & \text{if } a \neq 1, \\ \\ n & \text{if } a = 1. \end{cases}$$

Proof. We will prove this expression by considering two cases: when $a \neq 1$ and when $a = 1$.

For our first case, we consider $a \neq 1$.

To derive the closed form expression for S when $a \neq 1$, we start with its definition:

$$S = 1 + a + a^2 + ... + a^{n-1}.$$

Next, we multiply both sides by a and get:

$$aS = a + a^2 + ... + a^n.$$

Subtracting the second equation from the first, we get that:

$$S - aS = (1 + a + a^2 + ... + a^{n-1}) - (a + a^2 + ... + a^n).$$

Cancelling and simplifying the expression above, we get:

$$S - aS = 1 - a^n.$$

Factoring out S, we get that:

$$S(1 - a) = 1 - a^n.$$

Finally, this expression simplifies to the general formula for the finite geometric sum when $a \neq 1$ and can be written as:

$$S = \frac{1 - a^n}{1 - a}.$$

In the case where $a = 1$, the sum becomes:

$$S = 1 + 1 + 1 + ... + 1.$$

This sum consists of n 1's and gives us the formula:

$$S = n.$$

Thus, our lemma is proved for all a. □

Now that we have proven the lemma that gives us the general form of the finite geometric sum, we can use it to derive the closed form expression of the nth iterate of the sequence defined by the function $f(x) = ax + b$.

Having a closed form expression is important for us as it helps us understand how different components affect the behavior of the sequence. Now, we will work on the theorem which formalizes our findings related to the closed form expression for the nth iterate of the sequence.

Theorem 3.3. *If $f(x) = ax+b$ and we take an initial value x_0 then for each integer $n \geq 1$, we have that the members of the iteration sequence* *are given by the closed form expression below:*

$$x_n = \begin{cases} a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a} & \textit{for } a \neq 1, \\ \\ x_0 + bn & \textit{for } a = 1. \end{cases}$$

Proof. We will prove this theorem by considering two cases: when $a \neq 1$ and when $a = 1$. Let us start by looking at the expression that defines the sequence:

$$x_n = f(x_{n-1}) = ax_{n-1} + b.$$

Using Proposition 3.1, we can write the recursive formula for the nth iterate as follows:

$$x_n = a^n x_0 + b(1 + a + \ldots + a^{n-1}).$$

First, we consider the case where $a \neq 1$. We use Lemma 3.2 and substitute the formula for the finite geometric sum when $a \neq 1$ into the formula for the nth iterate to get the following expression for x_n:

$$x_n = a^n x_0 + b\frac{1 - a^n}{1 - a}.$$

Next, focusing on the second term, we can split the fraction as follows:

$$x_n = a^n x_0 + b(\frac{1}{1 - a} - \frac{a^n}{1 - a}).$$

We can then group the terms with a^n and rewrite the expression as follows:

$$x_n = a^n(x_0 - \frac{b}{1 - a}) + \frac{b}{1 - a}.$$

Next, we consider the second case where $a = 1$. We use Lemma 3.2 and substitute the formula for the finite geometric sum when $a = 1$ into the formula for the nth iterate to get the following expression for x_n:

$$x_n = x_0 + bn.$$

The above two cases give us the general form of closed form expression of the nth iterate of a linear function. □

4. Corollaries

Corollary 4.1. *Given the hypothesis of Theorem 3.3, if in addition $a < -1$, then there are two cases for the behavior of the iteration sequence. This behavior depends on the value of x_0:*
Case 1: *If $x_0 = \frac{b}{1-a}$, then $\{x_n\}$ exhibits constant convergence to $\frac{b}{1-a}$,*
Case 2: *If $x_0 \neq \frac{b}{1-a}$, then $\{x_n\}$ exhibits oscillating divergence where $|x_n|$ grows without bound.*

Proof. By Theorem 3.3, we know that the general form of the iteration sequence when $a \neq 1$ is:

$$x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}.$$

In the **first case**, when $x_0 = \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to zero:

$$x_n = a^n(0) + \frac{b}{1-a} = \frac{b}{1-a}.$$

This result means that for every n, x_n is equal to $\frac{b}{1-a}$. Since, the value stays constant for all n, the sequence exhibits constant convergence.

In the **second case**, when $x_0 \neq \frac{b}{1-a}$, the expression $(x_0 - \frac{b}{1-a})$ evaluates to a non-zero value. Since, $|a| > 1$, a^n grows in magnitude as n increases. Additionally, the value of a^n alternates in sign:

1. When n is even, a^n is positive,
2. When n is odd, a^n is negative.

This means that x_n alternates in sign and the magnitude of $|x_n|$ increases without bound. As a result, the sequence exhibits oscillating divergence. □

Corollary 4.2. *Given the hypothesis of Theorem 3.3, if in addition $a = -1$, then there are two cases for the behavior of the iteration sequence. This behavior depends on the value of x_0:*

Case 1: *If $x_0 = \frac{b}{1-a}$, then $\{x_n\}$ exhibits constant convergence to $\frac{b}{2}$,*

Case 2: *If $x_0 \neq \frac{b}{1-a}$, then $\{x_n\}$ exhibits blinking divergence.*

Proof. In the **first case**, when $x_0 = \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to zero and $\frac{b}{1-a}$ evaluates to $\frac{b}{2}$:

$$x_n = a^n(0) + \frac{b}{1-(-1)} = \frac{b}{2}.$$

This expression means that for every n, x_n is equal to $\frac{b}{2}$. Since, the value stays constant for all n, the sequence exhibits constant convergence to $\frac{b}{2}$.

In the **second case**, when $x_0 \neq \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to a non-zero value. The term, $(-1)^n$ causes the sequence to behave in the following manner:

1. When n is even, x_n evaluates to:

$$x_n = (-1)^n (x_0 - \frac{b}{2}) + \frac{b}{2} = (x_0 - \frac{b}{2}) + \frac{b}{2} = x_0,$$

2. When n is odd, x_n evaluates to:

$$x_n = (-1)^n (x_0 - \frac{b}{2}) + \frac{b}{2} = -(x_0 - \frac{b}{2}) + \frac{b}{2} = b - x_0.$$

Thus, for all n, the sequence alternates between two distinct values, x_0 and $b - x_0$, and does not converge to a single value. As a result, the sequence exhibits blinking divergence.

□

Corollary 4.3. *Given the hypothesis of Theorem 3.3, if in addition* $-1 < a < 0$*, then there are two cases for the behavior of the iteration sequence. This behavior depends on the value of* x_0*:*

Case 1: *If* $x_0 = \frac{b}{1-a}$*, then* $\{x_n\}$ *exhibits constant convergence to* $\frac{b}{1-a}$*,*

Case 2: *If* $x_0 \neq \frac{b}{1-a}$*, then* $\{x_n\}$ *exhibits oscillating convergence to* $\frac{b}{1-a}$*.*

Proof. In the **first case**, when $x_0 = \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to zero:

$$x_n = a^n(0) + \frac{b}{1-a} = \frac{b}{1-a}.$$

This expression means that for every n, x_n is equal to $\frac{b}{1-a}$. Since, the value stays constant for all n, the sequence exhibits constant convergence to $\frac{b}{1-a}$.

In the **second case**, when $x_0 \neq \frac{b}{1-a}$, the expression $(x_0 - \frac{b}{1-a})$ evaluates to a non-zero value. As n increases, the term, a^n decreases in magnitude and approaches 0, while also alternating in sign (depending on whether n is odd or even). Due to this behavior, x_n moves back and forth around $\frac{b}{1-a}$, while getting closer with each iteration. As a result, the sequence exhibits oscillating convergence to $\frac{b}{1-a}$. □

Corollary 4.4. *Given the hypothesis of Theorem 3.3, if in addition $a = 0$, then there is only one case for the behavior of the iteration sequence. The sequence, $\{x_n\}$ exhibits constant convergence to b.*

Proof. Since, $a = 0$, we have:

$$x_n = 0.(x_0 - \frac{b}{1-a}) + \frac{b}{1-0} = b.$$

This expression means that for every n, x_n is equal to b. Since, the value stays constant for all n, the sequence exhibits constant convergence to b. □

Corollary 4.5. *Given the hypothesis of Theorem 3.3, if in addition $0 < a < 1$, then there are three cases for the behavior of the iteration sequence. This behavior depends on the value of x_0:*

Case 1: *If $x_0 = \frac{b}{1-a}$, then $\{x_n\}$ exhibits constant convergence to $\frac{b}{1-a}$,*

Case 2: *If $x_0 < \frac{b}{1-a}$, then $\{x_n\}$ exhibits increasing convergence to $\frac{b}{1-a}$,*

Case 3: *If $x_0 > \frac{b}{1-a}$, then $\{x_n\}$ exhibits decreasing convergence to $\frac{b}{1-a}$.*

Proof. In the **first case**, where $x_0 = \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to zero:

$$x_n = a^n(0) + \frac{b}{1-a} = \frac{b}{1-a}.$$

This expression means that for every n, x_n is equal to $\frac{b}{1-a}$. Since, the value stays constant for all n, the sequence exhibits constant convergence to $\frac{b}{1-a}$.

In the **second case**, where $x_0 < \frac{b}{1-a}$, the term $(x_0 - \frac{b}{1-a})$ is negative. As n increases, a^n decreases towards zero since $0 < a < 1$. When we multiply a^n with the negative value $(x_0 - \frac{b}{1-a})$, we obtain a small negative term. As n increases, $a^n(x_0 - \frac{b}{1-a})$ approaches zero from the negative side, causing x_n to increase and approach $\frac{b}{1-a}$ ever more closely. As a result, the sequence exhibits increasing convergence to $\frac{b}{1-a}$.

In the **third case**, where $x_0 > \frac{b}{1-a}$, the term $(x_0 - \frac{b}{1-a})$ is positive. As n increases, a^n decreases towards zero since $0 < a < 1$. When we multiply a^n with $(x_0 - \frac{b}{1-a})$, we obtain a small positive term. As n increases, $a^n(x_0 - \frac{b}{1-a})$ approaches zero from the positive side, causing x_n to decrease and approach $\frac{b}{1-a}$ ever more closely. As a result, the sequence exhibits decreasing convergence to $\frac{b}{1-a}$.

$\square$

Corollary 4.6. *Given the hypothesis of Theorem 3.3, if in addition $a = 1$, then there are three cases for the behavior of the iteration sequence.*

This behavior depends on the value of b:

Case 1: *If $b = 0$, then $\{x_n\}$ exhibits constant convergence to x_0,*

Case 2: *If $b < 0$, then $\{x_n\}$ exhibits decreasing divergence,*

Case 3: *If $b > 0$, then $\{x_n\}$ exhibits increasing divergence.*

Proof. By Theorem 3.3, we know that the general form of the iteration sequence when $a = 1$ is:

$$x_n = x_0 + bn.$$

In the **first case**, when $b = 0$, the expression evaluates to:

$$x_n = x_0.$$

This expression means that for every n, x_n is equal to x_0. Since, the value stays constant for all n, the sequence exhibits constant convergence to x_0.

In the **second case**, when $b < 0$, the term nb becomes increasingly negative as n increases, causing the terms of the sequence to decrease from x_0 in steps of size,b without bound. Thus, the sequence exhibits decreasing divergence.

In the **third case**, when $b > 0$, the term nb becomes increasingly positive as n increases, causing the terms of the sequence to increase from x_0 in steps of b without bound. Thus, the sequence exhibits increasing divergence. □

Corollary 4.7. *Given the hypothesis of Theorem 3.3, if in addition* $a > 1$ *then there are three cases for the behavior of the iteration sequence. This behavior depends on the value of* x_0*:*

Case 1: *If* $x_0 = \frac{b}{1-a}$*, then* $\{x_n\}$ *exhibits constant convergence to* $\frac{b}{1-a}$*,*

Case 2: *If* $x_0 < \frac{b}{1-a}$*, then* $\{x_n\}$ *exhibits decreasing divergence,*

Case 3: *If* $x_0 > \frac{b}{1-a}$*, then* $\{x_n\}$ *exhibits increasing divergence.*

Proof. In the **first case**, where $x_0 = \frac{b}{1-a}$, the expression, $(x_0 - \frac{b}{1-a})$ evaluates to zero:

$$x_n = a^n(0) + \frac{b}{1-a} = \frac{b}{1-a}.$$

This means that for every n, x_n is equal to $\frac{b}{1-a}$. Since, the value stays constant for all n, the sequence exhibits constant convergence.

In the **second case**, where $x_0 < \frac{b}{1-a}$, the term $(x_0 - \frac{b}{1-a})$ is negative. Even though a^n grows larger as n increases, multiplying it by this negative value, makes x_n more negative with each step. As a result, the sequence exhibits decreasing divergence.

In the **third case**, where $x_0 > \frac{b}{1-a}$, the term $(x_0 - \frac{b}{1-a})$ is positive. As a^n grows larger as n increases, multiplying it by this positive value causes x_n to increase in the positive direction with each step. As a result, the sequence exhibits increasing divergence. □

5. Conclusion

This study on the iteration of linear functions gives us key insights into the behavior of sequences generated by these functions. Firstly, the convergence or divergence of a sequence depends on the value of the slope, a. The type of convergence or divergence, however, depends on the initial value x_0, and in the special case where $a = 0$, it depends on the value of b. Additionally, the magnitude of $|a|$ determines the speed at which sequences converge or diverge.

From our geometric analysis, we find that when these sequences converge, they approach $\frac{b}{1-a}$, which is known as the fixed point. This fixed point occurs at the intersection of the linear function, $f(x)$ with the identity line, $y = x$. For a sequence, $\{x_n\}$ to converge, the terms, x_n and x_{n+1} must get closer with each iteration. Since $f(x_n)$ is the value of the next iteration, this means that $x_n \approx f(x_n)$.

The identity line $y = x$ represents all points where the input equals the output. For a linear iteration sequence to converge, the sequence must approach a point where $x_n = f(x_n)$, which is the definition of a fixed point. The only such fixed point for linear functions, when the slope is not 1 is $\frac{b}{1-a}$. There is only one fixed point because both, linear functions and the identity lines are straight lines, and two straight lines can only intersect at a single point (unless they are parallel).

In summary, our study of linear iterations shows us that the behavior of sequences depends on the slope and the initial value. We also learn about the concept of fixed points and their importance in understanding convergence. The insights gained from this study provide us with a strong foundation for exploring other iterative systems.

References

[1] J.W. Bruce, G. Cobb, G. Davidoff, C. Dugaw, A. Durfee, A. Duval, J. Gifford, H. Knaust, D. O'Shea, M. Peterson, et al. *Laboratories in Mathematical Experimentation: A Bridge to Higher Mathematics*. Press at Cal Poly Humboldt, 2024.

[2] J. Rogawski. *Calculus*. W. H. Freeman, 2011.

Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075