Discussion 12 Solutions

1. Reacall that the spherical harmonics are given by

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\phi}P_\ell^m(\cos\theta)$$

where the first few are explicitly

$$\begin{aligned}
&Y_0^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
&Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \qquad && Y_1^{\pm1}(\theta,\phi) = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\pm i\phi}\sin\theta \\
&Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right) \quad && Y_2^{\pm1}(\theta,\phi) = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{\pm i\phi}\sin\theta\cos\theta \quad Y_2^{\pm2}(\theta,\phi) = \mp\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{\pm 2i\phi}\sin^2\theta
\end{aligned}$$

(a) Explicitly check that the spherical harmonics Y_1^0, Y_1^1, and Y_1^{-1} are orthogonal to each other.

Solution:

$$\int Y_1^{0*}Y_1^1\,d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta\right)\left(-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta\right)\sin\theta\,d\theta d\phi$$

$$\int Y_1^{0*}Y_1^1\,d\Omega = \left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\right)\left(-\frac{1}{2}\sqrt{\frac{3}{2\pi}}\right)\int_{\phi=0}^{2\pi}e^{i\phi}d\phi\int_{\theta=0}^{\pi}\cos\theta\sin^2\theta\,d\theta$$

$$\int Y_1^{0*}Y_1^1\,d\Omega = 0.$$

$$\int Y_1^{0*}Y_1^{-1}\,d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta\right)\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta\right)\sin\theta\,d\theta d\phi$$

$$\int Y_1^{0*}Y_1^{-1}\,d\Omega = \left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\right)\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}\right)\int_{\phi=0}^{2\pi}e^{-i\phi}d\phi\int_{\theta=0}^{\pi}\cos\theta\sin^2\theta\,d\theta$$

$$\int Y_1^{0*}Y_1^{-1}\,d\Omega = 0.$$

$$\int Y_1^{1*}Y_1^{-1}\,d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta\right)\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta\right)\sin\theta\,d\theta d\phi$$

$$\int Y_1^{1*}Y_1^{-1}\,d\Omega = \left(-\frac{1}{2}\sqrt{\frac{3}{2\pi}}\right)\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}\right)\int_{\phi=0}^{2\pi}e^{-2i\phi}d\phi\int_{\theta=0}^{\pi}\sin^3\theta\,d\theta$$

$$\int Y_1^{1*}Y_1^{-1}\,d\Omega = 0.$$

(b) Will the following integrals go to zero? Argue based on symmetry alone, do not actually do any integrals.

(i) $\int Y_0^{0*}Y_1^1\,d\Omega$

Solution:

$$\int Y_0^{0*}Y_1^1\,d\Omega = \frac{1}{2}\sqrt{\frac{1}{\pi}}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin^2\theta\,d\theta d\phi$$

Since the integrand has $e^{i\phi}$ integrated from 0 to 2π, this will be zero.

(ii) $\int Y_0^{0*}(\sin\theta)Y_0^0\, d\Omega$
Solution:

$$\int Y_0^{0*}(\sin\theta)Y_0^0\, d\Omega = \left(\frac{1}{2}\sqrt{\frac{1}{\pi}}\right)^2 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\, d\theta d\phi$$

Since $\sin^2\theta$ is always positive and there is no ϕ in the integrand, this will be nonzero.

(iii) $\int Y_1^{-1*}Y_1^1\, d\Omega$
Solution:

$$\int Y_1^{-1*}Y_1^1\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta\right)\left(-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta\right)\sin\theta\, d\theta d\phi$$

Here we again have $e^{2i\phi}$ in the integrand so this will go to zero.

(iv) $\int Y_2^{2*}(\cos\theta)Y_2^2\, d\Omega$
Solution:

$$\int Y_2^{2*}(e^{i\phi})Y_2^2\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\phi}\sin^2\theta\right)(\cos\theta)\left(-\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\phi}\sin^2\theta\right)\sin\theta\, d\theta d\phi$$

From here, we can see that we will have $\sin^4\theta$, which is always positive, and $\sin\theta\cos\theta$, which will integrate to zero on the interval 0 to π so this entire integral should go to zero.

2. The wavefunction of a particle in a spherically symmetric potential is given as

$$\psi = (x + y + 3z)\, f(r).$$

(a) Rewrite this into spherical coordinates and show that this can be separated into $\psi = Y(\theta,\phi)R(r)$.
Solution: Replacing our Cartesian coordinates with their spherical equivalents gives

$$\begin{aligned}\psi &= (r\cos\phi\sin\theta + r\sin\phi\sin\theta + 3r\cos\theta)\, f(r)\\ \psi &= (\cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta)\, rf(r)\\ \psi &= Y(\theta,\phi)R(r)\end{aligned}$$

where $Y(\theta,\phi) = \cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta$ and $R(r) = rf(r)$.

(b) Write the $Y(\theta,\phi)$ component in terms of the spherical harmonics.
Solution:

$$\begin{aligned}Y(\theta,\phi) &= \cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta\\ Y(\theta,\phi) &= \sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right) + i\sqrt{\frac{2\pi}{3}}\left(Y_1^1 + Y_1^{-1}\right) + 6\sqrt{\frac{\pi}{3}}Y_1^0\\ Y(\theta,\phi) &= \left(\sqrt{\frac{2\pi}{3}} + i\sqrt{\frac{2\pi}{3}}\right)Y_1^{-1} + \left(i\sqrt{\frac{2\pi}{3}} - \sqrt{\frac{2\pi}{3}}\right)Y_1^1 + 6\sqrt{\frac{\pi}{3}}Y_1^0\end{aligned}$$

3. For L_x, L_y, and L_z as were introduced in lecture, let $L_\pm = L_x \pm iL_y$ and recall $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$. Show that $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$. *Hint:* Start by calculating $L_\pm L_\mp$.
Solution:

$$\begin{aligned}L_\pm L_\mp &= (L_x \pm iL_y)(L_x \mp iL_y)\\ L_\pm L_\mp &= L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x)\\ L_\pm L_\mp &= L_x^2 + L_y^2 \mp i[L_x, L_y]\\ L_\pm L_\mp &= L_x^2 + L_y^2 \pm \hbar L_z\\ L_\pm L_\mp &= L^2 - L_z^2 \pm \hbar L_z\\ L_\pm L_\mp + L_z^2 \mp \hbar L_z &= L^2.\end{aligned}$$