Physics 1116 Fall 2021 **Problem Set 1** Solutions

1. Given two vectors, $\vec{\mathbf{A}} = 6\hat{\mathbf{i}}+3\hat{\mathbf{j}}-9\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 9\hat{\mathbf{i}}-8\hat{\mathbf{j}}+2\hat{\mathbf{k}}$, find the following quantities
   (a) $\vec{\mathbf{A}} + \vec{\mathbf{B}}$
   (b) $\vec{\mathbf{A}} - \vec{\mathbf{B}}$
   (c) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$
   (d) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$
   (e) The angle, θ, between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
   (f) A unit vector, $\vec{\mathbf{c}}$, which is normal to the plane in which both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lie

   Solution:

   (a) As the two vectors are represented in Cartesian basis, we simply need to add the coefficients of basis vectors, $\vec{\mathbf{A}}+\vec{\mathbf{B}} = (6+9)\hat{\mathbf{i}}+(3-8)\hat{\mathbf{j}}+(-9+2)\hat{\mathbf{k}} = 15\hat{\mathbf{i}}-5\hat{\mathbf{j}}-7\hat{\mathbf{k}}$.

   (b) Similarly, we subtract the coefficients the basis vectors, $\vec{\mathbf{A}}-\vec{\mathbf{B}} = -3\hat{\mathbf{i}}+11\hat{\mathbf{j}}-11\hat{\mathbf{k}}$.

   (c) Using the matrix method for cross product

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 3 & -9 \\ 9 & -8 & 2 \end{vmatrix} = -66\hat{\mathbf{i}} - 93\hat{\mathbf{j}} - 75\hat{\mathbf{k}}$$

   We can check that this is true by taking the dot product of the final vector with the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Both of the dot products should be zero since the cross product yields a vector that is perpendicular to original vectors.
   $(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{A}} = -396 - 279 + 675 = 0$ and $(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{B}} = -594 = 744 - 150 = 0$.

   (d) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (6 \cdot 9) + (3 \cdot (-8)) + (-9 \cdot 2) = 54 - 24 - 18 = 12$

   (e) Now that we know the dot product of the two vectors, we can use the relation $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$ to find the angle between them. The magnitude of the original vectors are $A = \sqrt{\vec{\mathbf{A}} \cdot \vec{\mathbf{A}}} = \sqrt{36+9+81} = \sqrt{126}$ and $B = \sqrt{\vec{\mathbf{B}} \cdot \vec{\mathbf{B}}} = \sqrt{81+64+4} = \sqrt{149}$. Then, $\cos\theta = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB} = \frac{12}{\sqrt{149} \cdot \sqrt{126}} \approx 0.088$ and $\theta = \arccos(0.088) = 84.98^\circ = 1.48$ rad.

   (f) We know that the cross product of two vectors is perpendicular to both of them. This means the cross product will be in the normal direction to the plane spanned by these vectors. Thus, we can simply normalize (set the length 1) the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ we previously obtained.

$$\vec{\mathbf{c}} = \frac{\vec{\mathbf{A}} \times \vec{\mathbf{B}}}{|\vec{\mathbf{A}} \times \vec{\mathbf{B}}|} = \frac{-66\hat{\mathbf{i}} - 93\hat{\mathbf{j}} - 75\hat{\mathbf{k}}}{\sqrt{4356+8649+5625}} \approx -0.484\hat{\mathbf{i}} - 0.681\hat{\mathbf{j}} - 0.549\hat{\mathbf{k}}$$

2. Prove the three identities :

(a) $\frac{d}{dt}(c\vec{\mathbf{A}}) = \frac{dc}{dt}\vec{\mathbf{A}} + c\frac{d\vec{\mathbf{A}}}{dt}$

(b) $\frac{d}{dt}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \frac{d\vec{\mathbf{B}}}{dt}$

(c) $\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}$

Solution:

(a) $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$

$$\begin{aligned}
\frac{d}{dt}(c\vec{\mathbf{A}}) &= \frac{d}{dt}(cA_x\hat{\mathbf{i}} + cA_y\hat{\mathbf{j}} + cA_z\hat{\mathbf{k}}) \\
&= \frac{dc}{dt}A_x\hat{\mathbf{i}} + c\frac{dA_x}{dt}\hat{\mathbf{i}} + \frac{dc}{dt}A_y\hat{\mathbf{j}} + c\frac{dA_y}{dt}\hat{\mathbf{j}} + \frac{dc}{dt}A_z\hat{\mathbf{k}} + c\frac{dA_z}{dt}\hat{\mathbf{k}} \\
&= \frac{dc}{dt}(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) + c(\frac{dA_x}{dt}\hat{\mathbf{i}} + \frac{dA_y}{dt}\hat{\mathbf{j}} + \frac{dA_z}{dt}\hat{\mathbf{k}}) \\
&= \frac{dc}{dt}\vec{\mathbf{A}} + c\frac{d\vec{\mathbf{A}}}{dt}
\end{aligned}$$

(b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$

$$\begin{aligned}
\frac{d}{dt}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) &= \frac{dA_x}{dt}B_x + A_x\frac{dB_x}{dt} + \frac{dA_y}{dt}B_y + A_y\frac{dB_y}{dt} + \frac{dA_z}{dt}B_z + A_z\frac{dB_z}{dt} \\
&= \frac{dA_x}{dt}B_x + \frac{dA_y}{dt}B_y + \frac{dA_z}{dt}B_z + A_x\frac{dB_x}{dt} + A_y\frac{dB_y}{dt} + A_z\frac{dB_z}{dt} \\
&= \frac{d\vec{\mathbf{A}}}{dt} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \frac{d\vec{\mathbf{B}}}{dt}
\end{aligned}$$

(c) $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$

$$\begin{aligned}
\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) &= (\frac{dA_y}{dt}B_z + A_y\frac{dB_z}{dt} - \frac{dA_z}{dt}B_y - A_z\frac{dB_y}{dt})\hat{\mathbf{i}} \\
&\quad + (\frac{dA_z}{dt}B_x + A_z\frac{dB_x}{dt} - \frac{dA_x}{dt}B_z - A_x\frac{dB_z}{dt})\hat{\mathbf{j}} \\
&\quad + (\frac{dA_x}{dt}B_y + A_x\frac{dB_y}{dt} - \frac{dA_y}{dt}B_x - A_y\frac{dB_x}{dt})\hat{\mathbf{k}} \\
&= (\frac{dA_y}{dt}B_z - \frac{dA_z}{dt}B_y)\hat{\mathbf{i}} + (A_y\frac{dB_z}{dt} - A_z\frac{dB_y}{dt})\hat{\mathbf{i}} \\
&\quad + (\frac{dA_z}{dt}B_x - \frac{dA_x}{dt}B_z)\hat{\mathbf{j}} + (A_z\frac{dB_z}{dt} - A_x\frac{dB_z}{dt})\hat{\mathbf{j}} \\
&\quad + (\frac{dA_x}{dt}B_y - \frac{dA_y}{dt}B_x)\hat{\mathbf{k}} + (A_x\frac{dB_y}{dt} - A_y\frac{dB_x}{dt})\hat{\mathbf{k}} \\
&= \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}
\end{aligned}$$

3. A particle moves along the curve $y = A\sin(Cx)$ (where C has the value of 1 radian / meter). Its x position in meters as a function of time is $x(t) = Bt^2$, where B has the units of meters / sec^2.

   (a) Express the vector displacement between time t and time $t = 0$ as a function of time in the form $\vec{\mathbf{S}}(t) \equiv [x(t) - x(0)]\hat{\mathbf{i}} + [y(t) - y(0)]\hat{\mathbf{j}}$

   (b) Determine the speed (scalar) $|\vec{\mathbf{v}}(t)|$ as a function of time

   (c) Determine the average velocity (vector), $\vec{\mathbf{v}}_{\rm avg}(T)$ between time $t = 0$ and $t = T$, assuming that T is positive.

   Solution:

   (a) $\vec{\mathbf{S}}(t) = [Bt^2 - B\cdot 0^2]\hat{\mathbf{i}} + [A\sin(CBt^2) - A\sin(CB\cdot 0^2)]\hat{\mathbf{j}} = Bt^2\hat{\mathbf{i}} + A\sin(CBt^2)\hat{\mathbf{j}}$

   (b) To find the speed we first need to know the time dependence of velocity.
   $\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{S}}}{dt} = 2Bt\hat{\mathbf{i}} + 2ABCt\cos(CBt^2)\hat{\mathbf{j}}$
   Thus, we find that the speed is $|\vec{\mathbf{v}}(t)| = 2Bt\sqrt{1 + A^2C^2\cos^2(CBt^2)}$.

   (c) To find the average velocity we can simply take the difference between the final and initial positions and divide by the time elapsed.

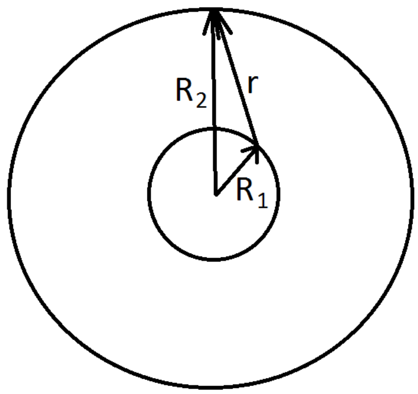
   $$\vec{\mathbf{v}}_{\rm avg}(T) = \frac{\vec{\mathbf{r}}(T) - \vec{\mathbf{r}}(0)}{T} = \frac{BT^2\hat{\mathbf{i}} + A\sin(CBT^2)\hat{\mathbf{j}}}{T}$$

   Note that if we needed to find the average speed then it is more complicated. Try it out yourself!

4. The Global Positioning System (GPS) works by using an array of orbiting satellites at precisely known locations which send out synchronized timing pulses. From measuring the time it takes for a signal to travel from a satellite to your unit, and doing this for various satellites, your position can then be precisely triangulated. Let's assume that you are at Cass Park {Latitude = 42.45°, Longitude = 76.51°}, and your unit locks onto 4 satellites which have latitude and longitude of : Sat$_a$ = {20°,70°}, Sat$_b$ = {50°,50°}, Sat$_c$ = {30°,70°}, and Sat$_d$ = {60°,90°}. **What is the delay that your GPS unit measures for each satellite's signal?**

   For this problem, assume that the earth is a perfect sphere of radius 6360 km, and that the satellites are all exactly 20,000 km above the surface of the planet. The signals are radio waves transmitted at the speed of light (3×10^8 m/s), which is very fast, so ignore the motion of the earth or satellites.

   Solution:
   Let $\vec{\mathbf{R_1}}$ be the position of Cass Park and $\vec{\mathbf{R_2}}^{a,b,c,d}$ be the positions of four satellites. $\vec{\mathbf{R_1}} = R_1\cos\theta_1\sin\phi_1\hat{\mathbf{i}} + R_1\cos\theta_1\cos\phi_1\hat{\mathbf{j}} + R_1\sin\theta_1\hat{\mathbf{k}}$ and $\vec{\mathbf{R_2}} = R_2\cos\theta_2\sin\phi_2\hat{\mathbf{i}} + R_2\cos\theta_2\cos\phi_2\hat{\mathbf{j}} + R_2\sin\theta_2\hat{\mathbf{k}}$. $r^{a,b,c,d}$ then is the distance between collegetown and a satellite, see accompanying diagram. Here R_1=6360km, R_2=26,360km, θ_1, ϕ_1 are



angular coordinates of Cass Park and θ_2, ϕ_2 are angular coordinates of the satellites. Then, $\vec{\mathbf{r}} = \vec{\mathbf{R_2}} - \vec{\mathbf{R_1}}$. We now take the dot product of this vector with itself to find its magnitude

$$\begin{aligned} r^2 &= (\vec{\mathbf{R_2}} - \vec{\mathbf{R_1}}) \cdot (\vec{\mathbf{R_2}} - \vec{\mathbf{R_1}}) \\ &= R_2^2 + R_1^2 - 2\vec{\mathbf{R_1}} \cdot \vec{\mathbf{R_2}} \\ &= R_1^2 + R_2^2 - 2R_1R_2(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2(\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2)) \\ &= R_1^2 + R_2^2 - 2R_1R_2(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos(\phi_2 - \phi_1)) \end{aligned}$$

Once this distance is found, the delay time can be calculated from $t = r/c$. The distances and delay times for the different satellites are listed in the table below.

Position (θ_2, ϕ_2)	Distance from Collegetown (km)	Delay time (ms)
20°, 70°	20,662	68.9
50°, 50°	20,485	68.3
30°, 70°	20,230	67.4
60°, 90°	20,470	68.2

5. K&K Problem 1.23. Smooth elevator ride. *Note:* A useful program for making plots like these and helping solve algebra problems is Mathematica. The university offers a major discount on purchasing this software. WolframAlpha can perform some Mathematica functions, but has substantial limitations. Any other programming language can also be used; python and matlab are common choices.

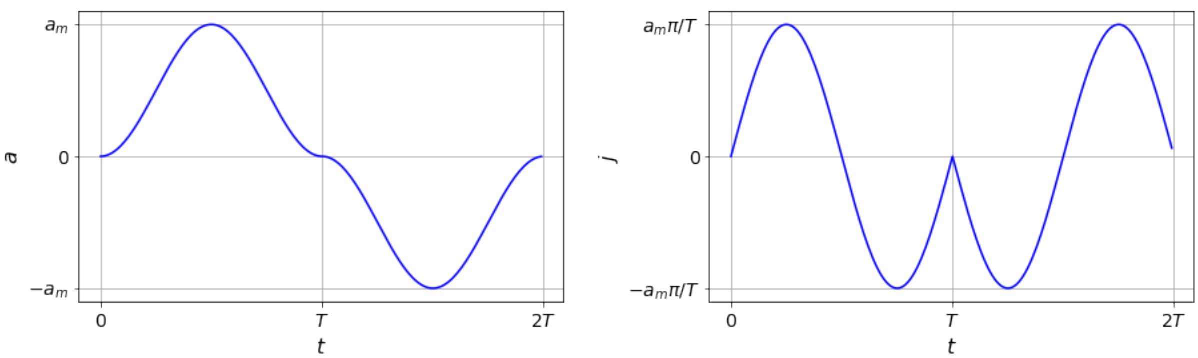
   Solution:

   (a) The acceleration and the "jerk" (i.e. the rate of change of the acceleration) are given by

$$a(t) = \begin{cases} (a_m/2)[1 - \cos(2\pi t/T)], & 0 \le t \le T \\ -(a_m/2)[1 - \cos(2\pi t/T)], & T \le t \le 2T \end{cases}$$

$$j(t) = \frac{da}{dt} = \begin{cases} a_m \pi \sin(2\pi t/T), & 0 \le t \le T \\ a_m \pi \sin(2\pi t/T), & T \le t \le 2T \end{cases}$$

So the plots are the following.



(b) From the left image above it is easy to deduce that the maximum speed of the elevator will occur midway through the trip. Thus, we can obtain it by integrating the acceleration from $t = 0$ to $t = T$.

$$\begin{aligned} v_{\max} &= \int_0^T a(t)dt = \int_0^T (a_m/2)[1 - \cos(2\pi t/T)]dt \\ &= \left[a_m t/2 - \frac{a_m T}{4\pi} \sin(2\pi t/T) \right]_0^T \\ &= a_m T/2 \end{aligned}$$

(c) First let us find an expression for the speed at $t < T$. We do this with the same integral as before.

$$\begin{aligned} v(t) &= \int_0^t a(t')dt' = \int_0^t (a_m/2)[1 - \cos(2\pi t'/T)]dt' \\ &= \left[a_m t'/2 - \frac{a_m T}{4\pi} \sin(2\pi t'/T) \right]_0^t \\ &= a_m t/2 - \frac{a_m T}{4\pi} \sin(2\pi t/T) \end{aligned}$$

Recall from the Taylor series of $\sin x$ that when $x \ll 1$ we have $\sin x = x - x^3/3! + \mathcal{O}(x^5)$, so we can approximate $\sin x \approx x - x^3/3!$ (we usually only use the linear term, but as we will see it is not enough in this case). Thus, we have

$$v(t) \approx a_m t/2 - \frac{a_m T}{4\pi}(2\pi t/T) + \frac{a_m T}{4\pi}(2\pi t/T)^3/3! = \frac{\pi^2 a_m t^3}{3T^2}$$

(d) To do this we need to express the distance traveled in terms of the total time of the trip $2T$. By symmetry we can deduce that the distance traveled in the first half is the same as in the second half. So we can integrate the expression for v that we previously found for $0 \leq t \leq T$ and multiply it by two.

$$\begin{aligned} D &= 2\int_0^T v(t)dt = 2\int_0^T \left[a_m t/2 - \frac{a_m T}{4\pi}\sin(2\pi t/T)\right] dt \\ &= 2\left[a_m t^2/4 + \frac{a_m T^2}{8\pi^2}\cos(2\pi t/T)\right]_0^T \\ &= \frac{1}{2}a_m T^2 \end{aligned}$$

Therefore, the time required for a trip of distance D is $2T = 2\sqrt{2D/a_m}$.

6. **Write-up Problem** - Retrograde Motion of Planets

Mars, as viewed from Earth, occasionally seems to "go backward." Visually this means that the position of Mars, viewed against the fixed background of distant stars, is usually moving eastward, but sometimes moves westward for a while. Then it resumes its eastward direction. See for example http://www.youtube.com/watch?v=72FrZz_zJFU. This strange behavior caused huge problems for astronomy before the heliocentric model of the solar system was accepted. Let's explore this with vectors in heliocentric model.

Take the origin of our coordinate system to be the sun, with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes lying in the plane of the Earth and Mars orbits. Assume for simplicity that Mars and Earth both orbit in the same plane and have circular orbits. Characterizing the motion relative to, say, the fixed $\hat{\mathbf{x}}$ axis, is equivalent to viewing against the background of fixed stars. (Think of $\hat{\mathbf{x}}$ as a unit vector pointing toward some remote star.)

Define the position of Mars and Earth by the vectors

$$\begin{aligned} \vec{R}_M(t) &= R_M\left[\cos\left(2\pi\frac{t}{T_M}\right)\hat{\mathbf{x}} + \sin\left(2\pi\frac{t}{T_M}\right)\hat{\mathbf{y}}\right] \\ \vec{R}_E(t) &= R_E\left[\cos\left(2\pi\frac{t}{T_E}\right)\hat{\mathbf{x}} + \sin\left(2\pi\frac{t}{T_E})\right)\hat{\mathbf{y}}\right]. \end{aligned}$$

Here we have set $t = 0$ to be when Mars and Earth are aligned with the sun, and have implicitly introduced the notation R_E, R_M, T_E, and T_M for the radii and orbital periods of Earth (E) and Mars (M).

Use the fact that the orbital radii are related by $R_M/R_E = 1.5$, and that the orbital periods are related by $T_M/T_E = (R_M/R_E)^{3/2}$.

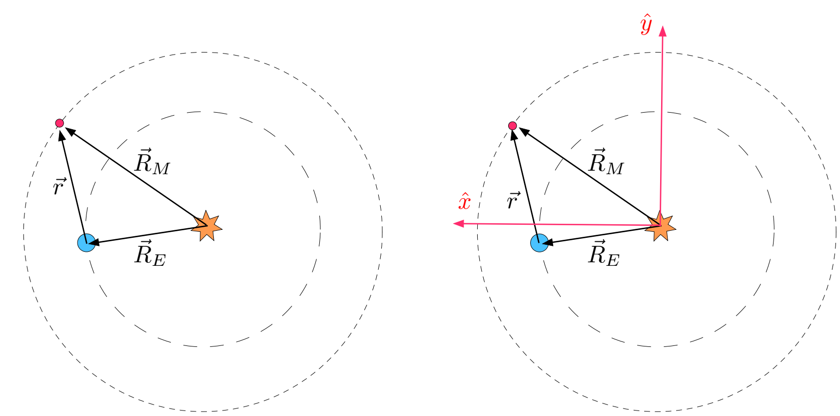
(a) **Derive a formula for the direction at which we see Mars from Earth. This should be an angle with respect to the $\hat{\mathbf{x}}$ axis.**

(b) **Plot this angle versus time and observe the apparent backward motion occuring every two Earth-years.**

Solution:

(a) The main issue here is to take motion which is easily described in the heliocentric coordinate system, and transform it into a geocentric coordinate system. A secondary, but still important, issue is to define the observable that we want to plot to reveal the retrograde motion clearly.

The problem defines already $\vec{R}_M(t)$ and $\vec{R}_E(t)$, which give the positions of Mars and Earth in the heliocentric frame. A simple diagram illustrates how we find the vector that points from Earth to Mars and thereby determine the location of Mars as viewed from Earth:



Basic definition of position vectors. Same, but with fixed coordinate axes defined.

The diagram shows that Mars as viewed from Earth is given by $\vec{r}(t) = \vec{R}_M(t) - \vec{R}_E(t)$. Expanding this we can write:

$$\vec{r}(t) = \left[R_M \cos\left(2\pi \frac{t}{T_M}\right) - R_E \cos\left(2\pi \frac{t}{T_E}\right)\right] \hat{\mathbf{x}} + \left[R_M \sin\left(2\pi \frac{t}{T_M}\right) - R_E \sin\left(2\pi \frac{t}{T_E}\right)\right] \hat{\mathbf{y}}$$

As explained in the statement of the problem, the retrograde motion appears when we ask about the motion of Mars against the background of fixed stars. To do this with our equations, we need only define a coordinate system that is fixed with respect to the stars. Let's call that the $\hat{\mathbf{x}}\hat{\mathbf{y}}$ system; it is shown in the right-hand panel of the figure above. The $\hat{\mathbf{x}}$ axis points to the left in this diagram, which is a bit nonstandard but not of any deep significance: it is done only to make the diagram relate well to the youtube video linked above.

Now we can talk about the location of Mars against the background of fixed stars by looking at the angle between $\vec{r}$ and $\hat{\mathbf{x}}$. We could also choose $\hat{\mathbf{y}}$ or any other fixed axis in this plane. Let's define: $\cos\theta = \hat{r}\cdot\hat{\mathbf{x}}$, then θ is the angle to examine. Note the distinction between $\hat{r}$ and $\vec{r}$! The quantity $\vec{r}\cdot\hat{\mathbf{x}} = r\cos\theta$, while $\hat{r}\cdot\hat{\mathbf{x}} = \cos\theta$.

$$\begin{aligned}
\cos\theta &= \hat{r}\cdot\hat{\mathbf{x}} = \frac{\vec{r}\cdot\hat{\mathbf{x}}}{r} \\
&= \frac{(\vec{R}_M(t) - \vec{R}_E(t))\cdot\hat{\mathbf{x}}}{|\vec{R}_M(t) - \vec{R}_E(t)|} \\
&= \frac{R_M\cos\left(2\pi\frac{t}{T_M}\right) - R_E\cos\left(2\pi\frac{t}{T_E}\right)}{|\vec{R}_M(t) - \vec{R}_E(t)|}
\end{aligned}$$

The denominator of the equation for $\cos\theta$ can be rewritten in the form $\sqrt{r_x^2 + r_y^2}$, with

$$r_x = R_M\cos\left(2\pi\frac{t}{T_M}\right) - R_E\cos\left(2\pi\frac{t}{T_E}\right)$$

and

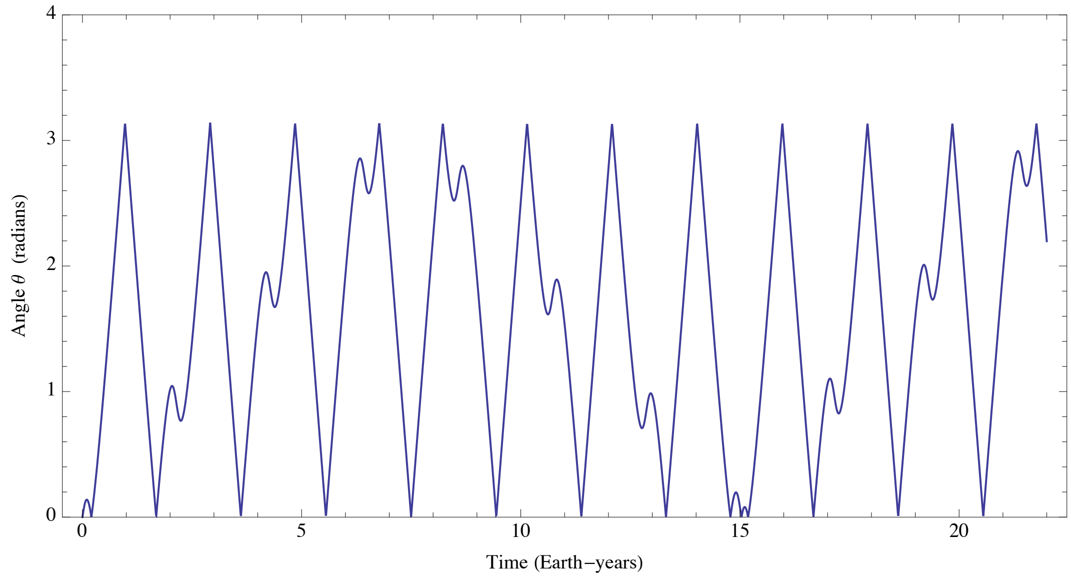
$$r_y = R_M\sin\left(2\pi\frac{t}{T_M}\right) - R_E\sin\left(2\pi\frac{t}{T_E}\right)$$

.

If you write all this into one equation, it is a bit of a mess; but the statement of $\cos\theta$ above, together with the definition of the denominator which follows, contains the full answer.

(b) We can now proceed to plot the *angle*, namely, $\theta(t)$, as described in the bonus question below.

By plotting the equations for $\theta(t)$ above using a ratio of orbital radii of $A = 1.5$, we find:



Notice that in each two-Earth-year cycle Mars appears to advance, then retreat, then advance again. The 2-year cycle is neither exact nor significant; if the R_M/R_E were different the periodicity would change. The fact that the angle is limited to the range $[0, \pi)$ is a peculiarity of the way the $\cos^{-1}\theta$ function is computed.

7. **Bonus Question** *(Please work on this question by yourself.)* Batman is in hot pursuit of the Riddler - both the Batmobile and the Riddlemobilemove at the same speed, v. At all times, Batman steers towards the instantaneous position of the Riddler, and the Riddler always turns at a fixed, non-zero angle θ relative to the direction directly away from the Batmobile. Their initial separation is d. Riddle me this: Relative to their initial positions, where does Batman catch the Riddler?

Solution:

Let's pick a coordinate axis where Batman starts at the origin and the Riddler starts at $d\hat{\mathbf{i}}$. Let's denote the angle between Batman's velocity and the x-axis by $\phi(t)$. Then the velocities of Batman and the Riddler are given by

$$\vec{\mathbf{v}}_B(t) = v\cos\phi(t)\hat{\mathbf{i}} + v\sin\phi(t)\hat{\mathbf{j}} \qquad \vec{\mathbf{v}}_R(t) = v\cos(\phi(t)+\theta)\hat{\mathbf{i}} + v\sin(\phi(t)+\theta)\hat{\mathbf{j}},$$

and therefore their positions are given by

$$\begin{aligned}\vec{\mathbf{x}}_B(t) &= vC(t)\hat{\mathbf{i}} + vS(t)\hat{\mathbf{j}}\\ \vec{\mathbf{x}}_R(t) &= \left[v\cos\theta C(t) - v\sin\theta S(t) + d\right]\hat{\mathbf{i}} + v\left[\cos\theta S(t) + \sin\theta C(t)\right]\hat{\mathbf{j}},\end{aligned}$$

where we defined

$$C(t) \equiv \int_0^t \cos\phi(t')dt' \qquad S(t) \equiv \int_0^t \sin\phi(t')dt'.$$

At the time t_0 when they meet the positions are the same. First, from the $\hat{\mathbf{j}}$ components we have

$$\begin{aligned} y_B(t_0) &= y_B(t_0) \\ vS(t_0) &= v\cos\theta S(t_0) + v\sin\theta C(t_0) \\ C(t_0) &= (1-\cos\theta)S(t_0)/\sin\theta \\ &= \tan(\theta/2)S(t_0) \end{aligned}$$

And from the $\hat{\mathbf{j}}$ components we have

$$\begin{aligned} x_B(t_0) &= x_B(t_0) \\ vC(t_0) &= v\cos\theta C(t_0) - v\sin\theta S(t_0) + d \\ (1-\cos\theta)\tan(\theta/2)S(t_0) &= -\sin\theta S(t_0) + d/v \\ S(t_0) &= \frac{d\cos^2(\theta/2)}{v\sin\theta} \end{aligned}$$

Thus, we find that the point where they meet is

$$\vec{\mathbf{x}}_o = \frac{d}{2}\hat{\mathbf{i}} + \frac{d\cos^2(\theta/2)}{\sin\theta}\hat{\mathbf{j}}$$

Interestingly, the x component of the point where they meet is always $d/2$! We can check the simple case when $\theta = \pi$ and verify that it makes sense, and we also see that in the limits $\theta \to 0$ and $\theta \to \infty$ Batman will never catch the Riddler.