## **Bryant Har**

# **Q1**

## Part A

We begin and add  $m_1r_2 - m_1r_2$  to left side,

$$m_1r_1 + m_2r_2 = 0$$

$$m_1r_1-m_1r_2+(m_1+m_2)r_2=0 \implies r_2=-rac{m_1r}{m_1+m_2}$$

By symmetry,

$$oxed{ec{r_1} = rac{m_2ec{r}}{m_1 + m_2}}, \quad ec{r_2} = -rac{m_1ec{r}}{m_1 + m_2}}$$

## Part B

We begin,

$$l=r_1 imes m_1\dot{r}_1+r_2 imes m_2\dot{r}_2$$

$$l = rac{m_2 ec{r}}{m_1 + m_2} imes rac{m_1 m_2 \dot{ec{r}}}{m_1 + m_2} - rac{m_1 ec{r}}{m_1 + m_2} imes - rac{m_1 m_2 \dot{ec{r}}}{m_1 + m_2}$$

Substituting reduced mass  $\mu$ ,

$$egin{align} l &= rac{m_2ec{r}}{m_1+m_2} imes \mu \dot{ec{r}} + rac{m_1ec{r}}{m_1+m_2} imes \mu \dot{ec{r}} = rac{(m_1+m_2)ec{r}}{m_1+m_2} imes \mu \dot{ec{r}} \ &= egin{bmatrix} ec{\ell} &= ec{r} imes \mu \dot{ec{r}} \end{bmatrix} \end{split}$$

This is clearly displacement cross momentum, as expected.

#### Part C

Referencing b, we have displacement cross momentum, where momentum is  $\mu \dot{r}$ . Inspecting our momentum term, we clearly see that  $\mu$  is the one-particle mass analogue,

$$\mu$$
, the reduced mass

## Part D

We begin,

$$\dot{ec{r}}=\dot{r}\hat{r}+r\dot{ heta}\hat{ heta}$$

Recall that  $\vec{l} = r\hat{r} \times \mu \dot{\vec{r}}$ . We arrive at  $\dot{\theta}$  in terms of  $\ell$ ,

$$rac{l}{\mu r} = r |\dot{ heta}| \implies |\dot{ heta}| = rac{\ell}{\mu r^2} \implies \left| \dot{ heta} = rac{\pm \ell}{\mu r^2} 
ight|$$

# Q2

## Part A

$$E = rac{1}{2} \mu \dot{r}^2 + rac{1}{2} rac{\ell^2}{\mu r^2} + U(r)$$

Rewriting,

$$oxed{\dot{r}=\sqrt{rac{2(E-U(r))}{\mu}-rac{\ell^2}{\mu^2r^2}}}$$

## Part B

From a and d,

$$\dot{ heta}=\pmrac{\ell}{\mu r^2},~~\dot{r}=\sqrt{rac{2(E-U(r))}{\mu}-rac{\ell^2}{\mu^2 r^2}}$$

Now, combining into the given equation in b,

$$d heta = rac{\dot{ heta}}{\dot{r}}dr = rac{\pmrac{\ell}{\mu r^2}}{\sqrt{rac{2(E-U(r))}{\mu} - rac{\ell^2}{\mu^2 r^2}}}dr = rac{\pmrac{\ell}{r^2}}{\sqrt{2\mu^2igl[rac{E-U(r)}{\mu} - rac{\ell^2}{2\mu^2 r^2}igr]}}dr$$

As desired, we arrive at

$$\Longrightarrow \boxed{ heta = \int rac{\pm rac{\ell}{r^2}}{\sqrt{2\muigl[E - U(r) - rac{\ell^2}{2\mu r^2}igr]}} dr}$$

## Part C

We replace U,

$$heta = \int rac{\pmrac{\ell}{r^2}}{\sqrt{2\muigl[E-U(r)-rac{\ell^2}{2\mu r^2}igr]}}dr = \int rac{rac{\ell}{r^2}}{\sqrt{2\muigl[E+k/r-rac{\ell^2}{2\mu r^2}igr]}}dr$$

Note that,  $u=l/r \implies du=-l/r^2 dr.$  Making this sub and absorbing the numerator into du,

$$heta = \int rac{-du}{\sqrt{2\muigl[E+ku/l-rac{u^2}{2\mu}igr]}} = oxed{ heta = \int rac{-du}{\sqrt{2\mu E+2\mu ku/l-u^2}}}$$

## Part D

We begin,

$$heta = \int rac{-du}{\sqrt{2\mu E + 2\mu ku/l - u^2}}$$

Applying the arcsin formula,

$$\int \frac{du}{\sqrt{au^2 + bu + c}} = -\frac{1}{\sqrt{-a}} \arcsin(\frac{2au + b}{\sqrt{b^2 - 4ac}})$$

$$\theta = \int \frac{-du}{\sqrt{2\mu E + 2\mu ku/l - u^2}} = \arcsin(\frac{-2u + 2\mu k/l}{\sqrt{(2\mu k/l)^2 + 4(2\mu E)}}) + C_1$$

Reorganizing terms, we arrive as desired

$$oxed{ heta+C=rcsin(rac{-2u+2\mu k/l}{\sqrt{(2\mu k/l)^2+8\mu E}})}$$

## Part E

Yes.

$$\sin(\theta - \pi/2) = -\cos(\theta) \implies \theta - \pi/2 = \arcsin(\xi) \implies \theta = -\arccos(\xi).$$

To adjust for  $\arccos$  limits, we can add a factor of  $\pi$ ,

$$heta = \pi - rccos(rac{-2u + 2\mu k/l}{\sqrt{(2\mu k/l)^2 + 8\mu E}})$$

## Part F

Letting  $\epsilon \equiv \sqrt{1+rac{2El^2}{\mu k^2}}, ~~c \equiv rac{l^2}{\mu k}, ~~u=l/r,$  starting from above,

$$\pi - \theta = \arccos(\frac{-2u + 2\mu k/l}{\sqrt{(2\mu k/l)^2 + 8\mu E}}) \implies \cos(\pi - \theta) = -\cos(\theta) = \frac{-ul/\mu k + 1}{\epsilon} = \frac{-c/r + 1}{\epsilon}$$
$$-\cos(\theta) = \frac{-c/r + 1}{\epsilon} \implies \boxed{r = \frac{c}{1 + \epsilon \cos(\theta)}}$$

## Part G

$$r = \frac{c}{1 + \epsilon \cos(\theta)}$$

Clearly minimized when cosine is zero, or  $\overline{ heta=\pi/2}$ 

# Q3

## Part A

Momentum (or generalized momentum)

#### Part B

Zero when different indices, one when same.

$$rac{\partial \dot{q}_k}{\partial \dot{q}_i} = \delta_{ik}$$

#### Part C

$$rac{dy_a}{dt} = \sum_j rac{dy_a}{dq_j} \dot{q}_j + rac{dy_a}{dt}$$

Taking the derivative wrt  $\dot{q}_i$ ,

$$rac{d\dot{y}_a}{d\dot{q}_i} = \sum_j rac{d}{d\dot{q}_i} (rac{dy_a}{dq_j} \dot{q}_j) + rac{dy_a}{d\dot{q}dt}$$

Since  $y_a$  has no time derivative dependence, and by symmetry of second derivatives, we can pull out the inner derivative (independent wrt to  $\dot{q}$ ) and set the outer derivative term to zero,

$$egin{aligned} rac{d\dot{y}_a}{d\dot{q}_i} &= \sum_j rac{d\dot{q}_j}{d\dot{q}_i} rac{dy_a}{dq_j} + 0 = rac{dy_a}{dq_j} \delta_{ij} \ & \left[ rac{d\dot{y}_a}{d\dot{q}_j} &= rac{dy_a}{dq_j} 
ight] \end{aligned}$$

#### Part D

Substitute  $-\nabla U = F_{tot} - F_c$ . Then, constraint is holonomic. Then, follows newton's second. We apply these constraints in that order below, line by line,

$$egin{aligned} \delta S &= \int_{t_1}^{t_2} (-m\ddot{ec r} - ec 
abla U) \cdot \delta ec r dt = \int_{t_1}^{t_2} (-m\ddot{ec r} + F_{tot} - F_c) \cdot \delta ec r dt \ & F_c \cdot \delta r = 0 \implies \delta S = \int_{t_1}^{t_2} (-m\ddot{ec r} + F_{tot}) \cdot \delta ec r dt \ & m\ddot{ec r} = F_{tot} \implies \delta S = \int_{t_1}^{t_2} (-F_{tot} + F_{tot}) \cdot \delta ec r dt = \int_{t_1}^{t_2} (0) dt = \delta S = 0 \end{aligned}$$

If these above conditions hold, we arrive that  $\delta S$  must be zero as the integrand is zero and by fundamental theorem of calculus.

$$\delta S = 0$$
 Q.E.D.

## Part E

 $\overline{\mathrm{Yes}}$  they would be smooshing in momentum axis  $p_x$ .

#### Part F

Yes. Like mixing colors, points may end up with different sets of neighbors as the phase space evolves.

# Q4

#### Part A

 $\boxed{\mathrm{No.}\ H 
eq T + U}$ . The accelerating pivot introduces energy into the system and therefore the Hamiltonian is not just the sum kinetic energy and potential energy. It must also include terms that incorporate the motion of the pivot.

### Part B

Recall the Lagrangian found earlier.

$$\mathcal{L} = rac{M}{2}(a^2t^2 + atL\cos( heta)\dot{ heta} + rac{L^2\dot{ heta}^2}{3}) + MgL\cos( heta)/2$$

Finding the momenta,

$$p_{ heta} = rac{d\mathcal{L}}{d\dot{ heta}} = rac{MatL\cos( heta)}{2} + rac{ML^2\dot{ heta}}{3} \implies \dot{ heta} = rac{3p_{ heta}}{ML^2} - rac{3at\cos( heta)}{2L}$$

Then,

$$\mathcal{H}=\dot{ heta}p_{ heta}-\mathcal{L}=rac{MatL\cos( heta)\dot{ heta}}{2}+rac{ML^2\dot{ heta}^2}{3}-rac{M}{2}(a^2t^2+atL\cos( heta)\dot{ heta}+rac{L^2\dot{ heta}^2}{3})-MgL\cos( heta)/2 \ \mathcal{H}=\dot{ heta}p_{ heta}-\mathcal{L}=rac{ML^2\dot{ heta}^2}{6}-rac{Ma^2t^2}{2}-MgL\cos( heta)/2 \ \mathcal{H}=rac{ML^2}{6}\left(rac{3p_{ heta}}{ML^2}-rac{3at\cos( heta)}{2L}
ight)^2-rac{Ma^2t^2}{2}-MgL\cos( heta)/2 \ \mathcal{H}=rac{ML^2}{6}\left(rac{3p_{ heta}}{ML^2}-rac{3at\cos( heta)}{2L}
ight)^2-rac{Ma^2t^2}{2}-MgL\cos( heta)/2 \ \mathcal{H}=\frac{ML^2}{6}\left(rac{3p_{ heta}}{ML^2}-rac{3at\cos( heta)}{2L}
ight)^2-rac{Ma^2t^2}{2}-MgL\cos( heta)/2 \ \mathcal{H}=\frac{ML^2}{6}\left(rac{3p_{ heta}}{ML^2}-rac{3at\cos( heta)}{2L}
ight)^2$$

# Q5

## Part A

We find the Hamiltonian. While we can first find the lagrangian and then derive the Hamiltonian, we can more easily consider the hamiltonian as the total energy of the inertial system. Notice that r = L by constrains r, so we only require two coordinates.

$$T=rac{1}{2}mv^2,\;\;U=mgL\cos heta \ \mathcal{L}=T-U=rac{1}{2}mL^2(\dot{ heta}^2+\sin^2( heta)\dot{\phi}^2)-mgL\cos( heta) \ p_\phi=rac{\partial \mathcal{L}}{\partial \dot{\phi}}=mL^2\sin^2( heta)\dot{\phi} \ p_ heta=rac{\partial \mathcal{L}}{\partial \dot{ heta}}=mL^2\dot{ heta} \ H=rac{1}{2}mv^2+mgL\cos( heta)=rac{1}{2}m(L^2\dot{ heta}^2+L^2\sin^2( heta)\dot{\phi}^2)+mgL\cos( heta) \ H=rac{1}{2}mv^2+mgL\cos( heta)=rac{1}{2}m(L^2\dot{ heta}^2+L^2\sin^2( heta)\dot{\phi}^2)+mgL\cos( heta) \ H=\frac{1}{2}mv^2+mgL\cos( heta)=\frac{1}{2}m(L^2\dot{ heta}^2+L^2\sin^2( heta)\dot{\phi}^2)+mgL\cos( heta) \ H=\frac{1}{2}mv^2+mgL\cos( heta)=\frac{1}{2}m(L^2\dot{ heta}^2+L^2\sin^2( heta)\dot{\phi}^2)$$

Substituting our momenta identities,

$$oxed{\mathcal{H} = rac{p_{ heta}^2}{2mL^2} + rac{p_{\phi}^2}{2mL^2\sin^2 heta} + mgL\cos( heta)}$$

## Part B

$$rac{\partial H}{\partial heta} = -\dot{p}_{ heta} \;\; rac{\partial H}{\partial \phi} = -\dot{p}_{\phi} \;\; rac{\partial H}{\partial p_{ heta}} = \dot{ heta}, \;\;\; rac{\partial H}{\partial p_{\phi}} = \dot{\phi}$$

We substitute and take the partial derivatives as necessary to generate the above equations. We arrive at four relevant equations,

$$egin{aligned} \dot{p}_{\phi} = 0, \qquad \dot{\phi} = rac{p_{\phi}}{mL^2\sin^2 heta} \end{aligned}$$

$$egin{split} egin{split} \dot{p}_{\phi} &= 0, \qquad \dot{\phi} = rac{p_{\phi}}{mL^2\sin^2{ heta}} \ \ \dot{p}_{ heta} &= mgL\sin{ heta} + rac{p_{ heta}^2}{mL^2\sin^2{ heta}}\cot{ heta}, \qquad \dot{ heta} = rac{p_{ heta}}{mL^2} \end{split}$$

The momentum equations relating  $\dot{q}$  and  $p_q$  (second equations in each pair) are equivalent to the ones ones written when developing.

## Part C

 $|\phi|$  direction |.

There is rotational symmetry in the  $\phi$  direction.

 $p_{\phi}$  must be constant since time derivative is zero.

## Part D

 $-\hat{ heta} \ ext{direction}$ 

Recall we defined  $p_{\phi}=mL^2\sin^2(\theta)\dot{\phi}$ .

Inspecting the definition given, this corresponds to  $L \cdot (-\hat{\theta})$ , or the  $-\hat{\theta}$  direction

## Part E

$$p_{ heta} = m L^2 \dot{ heta} \implies \dot{p}_{ heta} = m L^2 \ddot{ heta}$$

$$\dot{p}_{ heta}=mgL\sin heta+rac{p_{\phi}^{2}}{mL^{2}\sin^{2} heta}\cot heta$$

$$mL^2\sin^2 heta \ mL^2\ddot{ heta} = rac{p_\phi^2}{mL^2\sin^2 heta}\cot heta + mgL\sin heta$$