4200

Q2) $\ddot{y} + \omega^2 y = 0$ Ansatz: $y = \sum a_j x^{j+r} \Rightarrow \dot{y} = (j+r)x^{-1}y$, $\ddot{y} = (j+r)(j+r-1)x^{-2}y$

$$\Rightarrow \left(\overbrace{(j+r)(j+r-1)x^{-2} + \omega^2}^{\alpha}\right)y$$

$$= a_0(\cancel{j}+r)(\cancel{j}+r-1)x^{r-2} + a_1(r+1)r\,x^{r-1} + \sum\left((j+r+2)(j+r+1)a_{j+2} + a_j\omega^2\right)x^{j+r}$$

$a_0 \neq 0 \Rightarrow a_0 r(r-1) = 0 \Rightarrow r = 0, 1$ $\quad a_1(r+1)r = 0 \Rightarrow a_1 = 0$

$$a_{j+2} = -\frac{a_j\omega^2}{(j+r+2)(j+r+1)} \quad j \geq 0 \quad ; \quad a_j = 0 \quad j \text{ odd}$$

Case: $r = 0$: $\Rightarrow a_{j+2} = -\dfrac{a_j\omega^2}{(j+2)(j+1)} \Rightarrow a_{2n} = -\dfrac{a_0\omega^{2n}}{(2n)!}$

$$y_1 = \sum_j -\frac{a_0\omega^{2j}}{(2n)!}x^{2j} = \sum_j -\frac{a_0}{(2n)!}(\omega x)^{2j} = \boxed{a_0 \cos \omega x}$$

Case $r = 1 \Rightarrow a_{j+2} = -\dfrac{a_j\omega^2}{(j+3)(j+2)}$) which is identical to above except denom

$$\Rightarrow x^{2n}a_{2n} = -\frac{a_0\omega^{2n}}{(2n+1)!}x^{2n} \Rightarrow = \sum_j \frac{a_0}{\omega}\frac{(-1)^j}{(2j+1)!}(\omega x)^{2j+1} = \boxed{a_0/\omega \, \sin(\omega x)}$$

$$\Rightarrow \boxed{Y_1 = a_0 \cos(\omega_0 t) \qquad Y_2 = \frac{a_0}{\omega}\sin(\omega_0 t)}$$

Or use linear combo form: Y = A y1 + By2

Q7) $\frac{1}{y}\dot{y} + \cot x = \frac{\cos^2}{y} \Rightarrow y' + y\cot x = \cos^2$ $\quad e^{\int \cot dx} = e^{\ln|\sin x|} = |\sin x|$

$$\frac{d}{dx}(y \sin x) = \cos^2 \sin \Rightarrow y \sin x = \int \cos^2 \sin\, dx = -\cos^3/3$$

$\Rightarrow$ Integrating factor: $\boxed{|\sin x|}$ **= integrating factor**

$$\boxed{y = -\frac{\cos^3(x) + C}{3\sin x}}$$

Q10) $W = W_0 e^{-\int x\,dx} = \quad W = Ce^{-\frac{x^2}{2}}$

$$\boxed{W = Ce^{-\frac{x^2}{2}}}$$

Q12| $W = y_1 y_2' - y_2 y_1' = Ce^{-\int \frac{1}{x}dx} = C/x = x^m y_2' - m y_2 x^{m-1}$

$\Rightarrow \frac{dy_2}{dx} - \frac{m}{x} y_2 = \frac{C}{x^{m+1}} \Rightarrow$ $e^{\int -\frac{m}{x}dx} = x^{-m}$ $(y_2 x^{-m})' = \frac{C}{x^{2m+1}}$

$y_2 x^{-m} = C\int x^{-2m-1}\,dx = C\frac{x^{-2m}}{-2m} = C x^{-2m}$

$\Rightarrow \quad y_2 = C x^{-2m+1} = \quad y_2 = C x^{-m} = \boxed{y_2 = x^{-m}}$

$m = 1 \Rightarrow y_1, y_2 = x, \frac{1}{x} \Rightarrow W = y_1 y_2' - y_2 y_1' = -\frac{2}{x}$ $g = \frac{1}{x^2(1-x)}$

$y_p = -y_1 \int \frac{y_2 g}{W}\,dx + y_2 \int dx\, \frac{y_1 g}{W} = \frac{x}{2}\int g\,dx + \frac{1}{2x}\int \frac{1}{(1-x)}\,dx$

$-xI_1 = \frac{x}{2}(\ln x - \ln(x-1) - \frac{1}{x})$ $-\frac{1}{2x}I_2 = \frac{1}{2x}(\ln(x-1)$

$\Rightarrow \boxed{y_p = \frac{x}{2}\left(\ln\left|\frac{x}{x-1}\right|\right) - \frac{1}{2} + \frac{1}{2x}\ln(x-1)}$

$\Rightarrow \boxed{y = C_1 x + C_2 \frac{1}{x} + \frac{x}{2}\ln\left|\frac{x}{x-1}\right| + \frac{1}{2x}\ln|x-1| - \frac{1}{2}}$

Q13| $x^2 y'' + 4xy' + xy = 0 \Rightarrow y = \sum_j a_j x^{j+r}$

$\Rightarrow \sum_j a_j (j+r)(j+r-1)x^{j+r} + a_j 4(j+r)x^{j+r} + a_j x^{j+r+1}$

$= x^r a_0 [r(r-1) + 4r] + \sum_{j=0}^{\infty} a_{j+1}(j+r)(j+r+1)x^{j+r+1} + a_{j+1}4(j+r+1)x^{j+r+1}$
$+ a_j x^{j+r+1}$

$r(r-1) + 4r = 0 \Rightarrow r = 0, -3$ $a_{j+1}(j+r)(j+r+1) + a_{j+1}4(j+r+1) + a_j = 0$

$\Rightarrow -(j+r+4)(j+r+1)a_{j+1} = a_j$

$r=0$:

$y_1 = \sum_{j=0}^{\infty} a_j x^j$ $a_{j+1} = -\frac{a_j}{(j+4)(j+1)}$ $a_0 = a_0$, $a_1 = -\frac{a_0}{4}$, $a_2 = \frac{a_0}{40}$ ...

$r=-3$:

$y_2 = \sum_{j=0}^{\infty} a_j x^{j-3}$ $a_{j+1} = -\frac{a_j}{(j+1)(j-2)}$ $a_0 = a_0$, $a_1 = a_0/2$, $a_2 = a_0/4$

Both sums should converge, since each coefficient is proportional to 1/j^2.

Q 16 $y'' - xy = 0 \Rightarrow a_j(j+r)(j+r-1)x^{r+j-2} - a_j x^{r+j+1} = 0$

$$a_0 r(r-1)x^{r-2} + a_1(r+1)(r)x^{r-1} + a_2(r+2)(r+1)x^r$$
$$+ \sum_j \left(a_{j+3}(j+r+3)(j+r+2) - a_j\right) x^{j+r+1} = 0$$

From x^{r-2}, $r = 0, 1$. If $r=0$, a_1 = anything. If 1, $a_1 = 0$ $\Rightarrow a_1 = 0$

If $r = 1$, $a_2 = 0$, $r=0 \Rightarrow a_2 = 0$

$$\Rightarrow a_{j+3} = \frac{a_j}{(j+r+3)(j+r+2)}$$

$r=0$

$$Y_1 = \sum_{j=0}^{\infty} a_{3j} x^{3j}$$

$a_0 = a_0$
$a_3 = a_0/6$
$a_6 = a_0/180$

$$\boxed{Y_1 = a_0 + \frac{a_0 x^3}{2 \cdot 3} + \frac{a_0 x^6}{2 \cdot 3 \cdot 5 \cdot 6} \cdots}$$

$\boxed{a(j+3) = a(j)/ (j+3)(j+2)}$

$\boxed{a1=a2 = 0}$

$r=1$

$$Y_2 = \sum_{j=0}^{\infty} a_{3j} x^{3j+1}$$

$a_0 = a_0$
$a_3 = a_0/12$
$a_6 = a_0/504$

$\boxed{a(j+3) = a(j)/ (j+4)(j+3)}$

$$\Rightarrow \quad \boxed{Y_2 = a_0 + \frac{a_0 x^4}{3 \cdot 4} + \frac{a_0 x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots}$$

By Abel's, $W = Ce^{-\int p\,dx} = Ce^{-\int 0\,dx} = C \Rightarrow \boxed{W(y_1, y_2) = C}$