Problem set 12

Applied & Engineering Physics 3330

Problem 1 only is due 5pm Tues Nov. 21.

Problems 2 & 3 are due 6pm Thursday Nov. 30.

The last part will be due the last day of lectures at 5pm.

Remember to explain your answers!

Review some things you should already know from class while skimming the first two sections of Taylor's chapter 10, and then read sections 10.3-10.9, with the exception that the second part of 10.8 entitled "Motion of a Body with Two Equal Moments: free precession" is optional. Also look at Taylor's question 10.24 and understand in your own mind why the first equation in 10.117 is equivalent to the parallel axis theorem you already know, but there is another part of the parallel axis theorem presented there for off-diagonal elements. We may get to section 10.10 by the end of the term.

Problem 1: a) In ps11 you found that inertia tensor elements found with a COM origin could be written as $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij}|\vec{r}'_\alpha|^2 - x_{\alpha i}x_{\alpha j}\right]$ if the position of each bit of that rigid object with respect to a COM origin is $\vec{r}'_\alpha = (x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3})$. Write out what should be the middle row in the matrix below to familiarize yourself with its elements. [Recall that you found the {ith component of $\underset{orig}{\vec{L}_{com}}\} = \sum_j I_{ij}\omega_j$]

$$\overleftrightarrow{I} = \begin{pmatrix} \sum_\alpha m_\alpha[x_{\alpha 2}^2 + x_{\alpha 3}^2] & -\sum_\alpha m_\alpha[x_{\alpha 1}x_{\alpha 2}] & -\sum_\alpha m_\alpha[x_{\alpha 1}x_{\alpha 3}] \\ & & \\ -\sum_\alpha m_\alpha[x_{\alpha 1}x_{\alpha 3}] & -\sum_\alpha m_\alpha[x_{\alpha 2}x_{\alpha 3}] & \sum_\alpha m_\alpha[x_{\alpha 1}^2 + x_{\alpha 2}^2] \end{pmatrix}$$

If it is possible to have a set of axes firmly attached to a rigid object while it rotates and still have **the origin of that attached axis system be stationary in the inertial frame**, then this is a "stationary pivot" case and exactly the same math steps show that {ith component of $\underset{orig}{\vec{L}_{pivot}}\} = \sum_j I_{ij}\omega_j$, but you have to use that pivot origin for positions of mass bits used in the above formula for $\overleftrightarrow{I}$.

For either origin Taylor glues on x,y,z axes and uses $\vec{r}'_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ for the position of a piece of the rigid object wrt that origin. Thus, the top left element of his matrix is $I_{xx} = \sum_\alpha m_\alpha\left[\delta_{ij}[x_\alpha^2 + y_\alpha^2 + z_\alpha^2] - x_\alpha x_\alpha\right]$ and the middle component of his matrix is $I_{yy} = \sum_\alpha m_\alpha\left[\delta_{ij}[x_\alpha^2 + y_\alpha^2 + z_\alpha^2] - y_\alpha y_\alpha\right]$

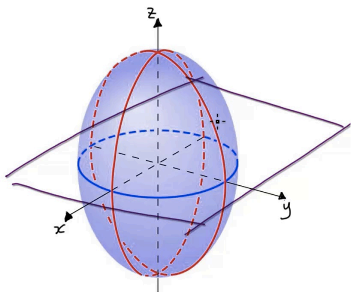
If an object has an axis of rotational symmetry, it looks exactly the same if we rotate it by an arbitrary amount about that axis.

b) If our glued-on 3 axis (same as Taylor's z axis) is an axis of rotational symmetry for a given object and if we calculate the inertia tensor using an origin on that 3 axis, give the relationship between the inertia tensor components I_{11} (Taylor's I_{xx}) and I_{22} (Taylor's I_{yy}), AND then give a corresponding relationship between I_{11} and $.5(I_{11} + I_{22})$.

c) Recall that we can switch from a sum over tiny m_α pieces to a volume integral that includes $dm = (density)dV$ if we simultaneously replace $\vec{r}_\alpha$ components with $\vec{r}$ components. Do this using Taylor's xyz notation, switch to cylindrical coordinates of position $\vec{r}$, remember that in cylindrical $dV = \rho d\phi d\rho dz$, and arrive at the following result for an object of uniform density γ that extends from zmin to zmax and from ρmin to ρmax while having rotational symmetry around the z axis we stuck to it:

$$I_{11} = I_{xx} = \frac{\gamma}{2}\int_0^{2\pi} d\phi \int_{zmin}^{zmax} dz \int_{\rho min}^{\rho max} (\rho^2 + 2z^2)\,\rho d\rho$$

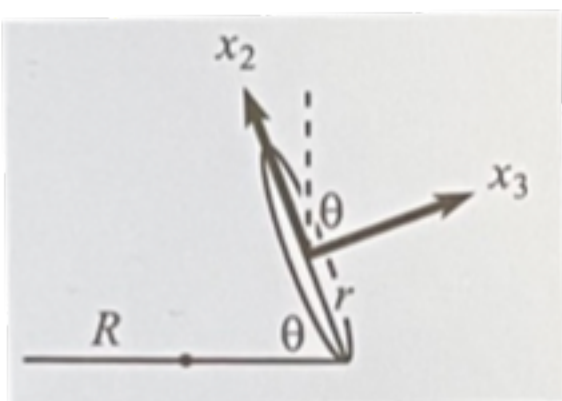
d) **Give a symmetry argument** to help you find the value of $I_{12} = I_{xy}$ for such a case. **What is the value**?



e) Recall that the equation $x = x_{scale}\sqrt{1 - \frac{z^2}{z_{sc}^2}}$ describes half of a particular ellipse in the x-z plane. (The half with positive x – the parameters x_{scale}&z_{sc} determines how fat&tall the ellipse is). Notice that the highest z value on the ellipse is z_{sc} and the lowest z value on the ellipse is $-z_{sc}$. Imagine rotating that curve around the z axis and filling the inside of that rotated shape. In cylindrical coordinates you would get an object that extends outward from the z axis to $\rho_{max} = \rho_{sc}\sqrt{1 - \frac{z^2}{z_{sc}^2}}$ at any height z, and the whole object would go from its low point $-z_{sc}$ to its top point z_{sc}. Assume that this object has uniform density $\boldsymbol{\gamma}$**,** find the components of the inertia tensor for this object, if we assume its COM is at the origin of the rotating axes stuck to the object.

[Your answer will have $\gamma = M/volume$ in it. In your mind make sure you could switch to $\overleftrightarrow{I}$ components that clearly show its M*(distance squared) units if I told you the volume of this object is $\frac{4\pi\gamma}{3}\rho_{sc}^2 z_{sc}$.]

f) If we rotate this object around that z axis glued onto its axis of rotational symmetry so $\vec{\omega} = \omega\hat{z}$, what is $\vec{L}_{\substack{com\\orig}}$ for the object you considered in e (as a function of ω)? [Notice freshman physics would give same answer.]



Problem 2: A thin uniform disk of mass M and radius r rolls without slipping so that it touches a circle of radius R on the horizontal flat ground. This problem will find torque exerted by the force of the ground on the object in 2 different ways, so that setting the 2 answers equal could find an allowed angular frequency Ω of the motion of the disk's center, where this Ω angular frequency contribution points up. A constant gravitational acceleration g points down in the picture. **Think about a stationary cylindrical coordinate system with z axis pointing up through the center of the radius R circle (at left side of picture) when answering parts a-c using pre-prelim1 knowledge.**

a) During this motion, the center of the uniform disk always stays at the same height. What is the up component of force exerted on the disk where it contacts the ground? (you can call this F_{up} in d below)

b) You already know the center is going in a horizontal circle at angular frequency Ω. What is the horizontal component of force exerted by the ground on the disk in a direction toward the center of the radius R circle on the ground traced out by the contact between disk and ground, as a function of stuff you know like R, r, M, and θ (angle of disk from horizontal) ? (you can call this F_{inward} in d below)

c) The center of the disk has a **constant** angular frequency Ω as it goes around in a horizontal circle. **Is there any other component of force on the disk besides the ones I just mentioned (F_{up} and F_{inward})? Briefly explain** your answer using the formula for ϕ component of acceleration in cylindrical coordinates.

d) Using a COM origin for the disk, is the torque exerted by the ground along a line $\perp$the page at the moment in the picture when the COM and contact point are both in the plane of the page?
yes/no

e) For the pictured moment, give the torque component into the page as a function of the names F_{up} and F_{inward} and maybe other things like r,R, θ.

If we glue 2 & 3 axes to disk as indicated at pictured moment, an additional angular velocity contribution comes from twisting of the disk around its axis of rotational symmetry (3). With $\hat{z}$ up in the inertial frame, the total angular velocity of disk is $\vec{\omega} = \Omega\hat{z} - \omega'\hat{e}_3$.
f) Is ω' positive, given disk is rolling without slipping? Yes/no

g) What is the direction of the 1 axis at t = 0 (the moment in the picture)?

h) Remember the 1,2,3 axes are glued on. Find $\omega_1(t), \omega_2(t)$, and $\omega_3(t)$, knowing they may not all have a t. [Remember you always need to have $\vec{\omega}$ components with respect to glued on axes if you are about to put them next to inertia tensor components for those axes.]

i) Give all nonzero components of the inertia tensor of disk. Use a COM origin.
[It is hard to think about any axes that could have stationary origin and be glued to rotating disk anyway.]

j) Find components of torque with respect to glued-on axes as a function of time: $\Gamma_1(t), \Gamma_2(t)$, and $\Gamma_3(t)$.
[Before using Euler's equations, be sure you know how we got them.]

[Notice whether one torque component is $\propto \cos(\omega' t)$ and another is $\propto \sin(\omega' t)$ – if true that's a sign of something that appears to be going in a circle if you are clinging to the rotating object and might have a more consistent direction in a view that does not include the ω' spin.]
k) **Give the component of torque into the page at the pictured time t = 0.**

l) I will not make you set this torque answer equal to the answer above to find allowed Ω for this motion (assuming R is also big enough as you would find), but I will make you do one last calculation necessary to make that possible. The point at which ground contacts disk goes around the R circle at the same angular velocity of the disk's COM going around its circle, so the contact point travels $2\pi R$ every Ω period ($2\pi/\Omega$). The contact point also covers a distance $2\pi r$ along its circle in every complete twist of the disk (i.e. in an ω' period=$2\pi/\omega'$). **Give an equation that relates Ω to ω'.**

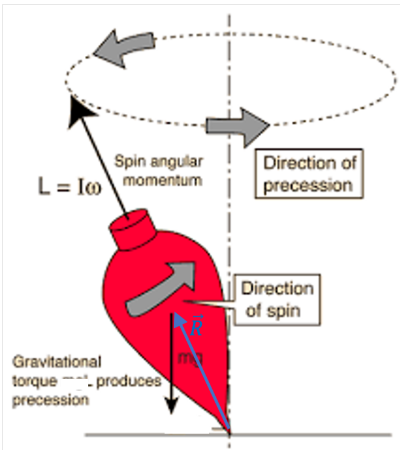
Problem 3: a) Our formula for inertia tensor elements gives real values. We argue that inertia tensors are also symmetric. Based on those facts, should their eigenvectors be perpendicular to each other? (yes or no)

b) If the matrix below represents an inertia tensor for some particular origin, find its "principal moments of inertia" (i.e. the values of "constant" that can solve $\vec{I}\vec{\omega} = (\text{constant})\vec{\omega}$ for nontrivial angular velocity.)

$$\begin{pmatrix} 5 & \sqrt{3} & 0 \\ \sqrt{3} & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} kg\, m^2$$

Take each value you found for principal moment of inertia and plug it back into the equation $\vec{I}\vec{\omega} = (\text{constant})\vec{\omega}$ to find a vector $\vec{\omega}$ which satisfies the equation for that value of principal moment, or "constant". Please choose your $\vec{\omega}$ vector to have length 1. (You are finding an eigenvector which points along a direction of a "principal axis of inertia".) Don't forget to find a vector for each of your three principal moments. These vectors represent the directions (written with components in the original axis system) along which you could glue new axes that would have a diagonal inertia tensor (for the same origin).

To do the next part, you need to know that in the physics Euler angle system we use the angle ψ to describe rotation around the 3 axis that's actually stuck to the object. Thus, using the axis system stuck to the object, rotation only around that axis gives $\omega_3 = \dot{\psi}$. Go over a physics Euler angle description (mine or 10.9 in Taylor) to see how the amount of initial Euler angle ϕ determines the plane in which the actual 3 axis stuck to the object is going to live. The second Euler angle by θ establishes the direction of that glued-on 3 axis as being θ away from the vertical, but the first Euler angle ϕ determines the plane in which the second Euler angle adjusts that 3 axis. Thus, consider what happens to the direction of the 3 axis stuck to the object if the ϕ angle becomes LARGER.



c) Imagine an object is spinning really fast around its 3 axis, so that $\dot{\psi}$ is big. Also imagine its COM is along that 3 axis a distance R from a fixed pivot origin at the base of the object. At the moment in this picture, assume the 3 axis is in the plane of the page and gravity points down. For a stationary pivot origin at the bottom of the object (which doesn't move but is in a frictionless pivot holder), does the torque from gravity point into the page or out of the page?

d) If the object starts out just spinning fast around its 3 axis so that angular velocity along 3 gives the only contribution to $\vec{L}$, is the gravitational torque perpendicular to that $\vec{L}$? (yes or no). Given that $\overrightarrow{Torque} = \frac{d\vec{L}}{dt}$, is that torque likely to have a bigger change on the direction or the length of $\vec{L}$? [We are going to assume a fairly small torque in parts d, e, and f, and look for a solution in which the torque is acting to move the tip of the $\vec{L}$ vector while moving the 3 axis of the object without introducing other noticeable changes to the angular momentum.]

e) Because we use glued-on axes, elements of the inertia tensor don't change. Given the assumptions above, $\vec{R} \times M\vec{g} = \overrightarrow{Torque} = \frac{d\vec{L}}{dt} \approx \frac{d}{dt}\left[\lambda_3 \dot{\psi} \hat{e}_3\right]$. If we hypothesize that the top mostly keeps spinning around the 3 axis at about the same rate and approximate $\dot{\psi}$ as a constant and bring it to the other side of the equation, does the torque law give:

$$\frac{d\hat{e}_3}{dt} \approx \frac{1}{\lambda_3 \dot{\psi}} \vec{R} \times M\vec{g} = \frac{1}{\lambda_3 \dot{\psi}} R\hat{e}_3 \times Mg(-\hat{z})? \quad (yes\ or\ no)$$

[We will see below whether there is a solution that follows the assumption in this part.]

f) Remember that a unit vector just gives the position of the point at the end of the unit vector. Does

$$\frac{d\hat{e}_3}{dt} = \left[\frac{MgR}{\lambda_3 \dot{\psi}} \hat{z}\right] \times \hat{e}_3$$

have the same type of form as $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$? (yes or no)

g) In $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, $\vec{\omega}$ is the angular velocity at which the tip of the $\vec{r}$ vector goes around the $\vec{\omega}$ direction.

If $\frac{d\hat{e}_3}{dt} = \left[\frac{MgR}{\lambda_3 \dot{\psi}} \hat{z}\right] \times \hat{e}_3$ accurately describes the top above, **what is the size of $\dot{\phi}$ for this motion if we use a physics Euler angle system to describe it**? (Remember that the $\hat{z}$ axis doesn't move in the inertial frame. Think about the direction that the 3 axis of the object is going around due to the description in this equation. Then think about how increasing ϕ makes the 3 axis that's stuck to the object move.)

Problem 4: a) Consider a case with NO torque and rotational symmetry axis along 3 axis glued to object, so $\boldsymbol{\lambda_1 = \lambda_2}$. Notice no-torque Euler is $\begin{pmatrix} \lambda_1\dot{\omega}_1 \\ \lambda_2\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} (\lambda_2-\lambda_3)\omega_2\omega_3 \\ (\lambda_3-\lambda_1)\omega_3\omega_1 \\ (\lambda_1-\lambda_2)\omega_1\omega_2 \end{pmatrix}$. **Is $\boldsymbol{\omega_3}$ constant?** Yes/no

b) IF we define $C = \frac{\lambda_2-\lambda_3}{\lambda_2}\omega_3$ do above become $\dot{\omega}_1 = \frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\omega_2 = C\omega_2$

and (remembering $\boldsymbol{\lambda_1 = \lambda_2}$) $\dot{\omega}_2 = -\frac{\lambda_1-\lambda_3}{\lambda_2}\omega_3\omega_1 = -C\omega_1$? Yes/no

c) Multiply 2nd equation by i & add like this:

$$\begin{aligned} \dot{\omega}_1 &= C\omega_2 \\ \underline{+i\,[\dot{\omega}_2} &\underline{= -C\omega_1]} \end{aligned}$$

and then show you can rewrite result using $\eta = \omega_1 + i\omega_2$. (Hint: remind you of Foucault pendulum trick?).

d) Is $\eta(t) = \eta_0 e^{-i\,C\,bt}$ a solution of the equation you found in c? Yes/no

e) If initial conditions cause real $\eta_0 = \omega_0$ then does matching real & imaginary parts in $\omega_1 + i\omega_2 = \eta_0 e^{-i\,C\,t}$ give $\omega_1 = \omega_0 \cos Ct$ $\omega_2 = -\omega_0 \sin Ct$ yes/no

f) What is the size $|\vec{\omega}|$ as a function of just ω_0 and ω_3?

g) If you could watch this $\vec{\omega}$ vector while clinging to the rotating object, would you see the tip of $\vec{\omega}$ going around circle, and the whole vector going around a cone centered on the 3 axis glued to the object's rotational symmetry axis? Yes/no

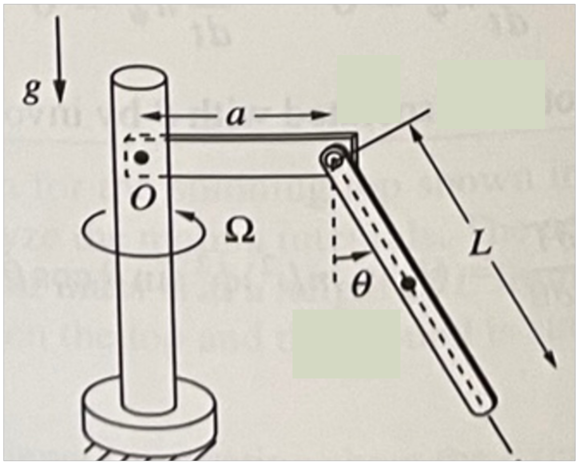
Optional: if you want to understand what's going on with $\vec{\omega}$ in the inertial frame, look at optional pages 399-400 in Taylor. [No torque means the angular momentum vector is constant in the intertial frame.]

Problem 5: a) Look at equation 10.98 where Taylor breaks a unit vector along the stationary z direction into components along the 3 axis stuck to the object and along an intermediate 1' direction. This 1' direction is along one of the directions the mobile 1 axis pointed in the process of going through Euler angle rotations. [The mobile axes started along inertial x,y,z axes and went through 3 rotations to arrive at the directions of the axes stuck to the object at some moment.] After which Euler axis rotation did the mobile axis point along the 1' direction mentioned in this equation 10.98 (first, second, or third rotation)?

b) Notice that Taylor made use of equation 19.98 to start from 10.97, which writes a general angular velocity using unit vectors along the directions of each of its Euler angle contributions, and to turn that into equation 10.99. The recitation 12 sheet gave practice breaking the $\dot{\theta}$ contribution to $\vec{\omega}$ into components along the 1 and 2 axes stuck to the object and you can use that knowledge here, because **in this part you should write the total general angular velocity using only 3 unit vectors: those along the 1, 2 and 3 axes stuck to the object at this instant.** It may help to look at my pictures as you break $\hat{e}_1$, into components along glued-on 1,2,3 axes, but Taylor has pictures too. Beware random pictures from the internet, because some use a different convention for Euler angles.

We'll show that it is generally true that $T = \frac{1}{2}MV_{COM}^2 + \frac{1}{2}\vec{\omega}^T \overleftrightarrow{I}\vec{\omega}$ if COM origin of $\overleftrightarrow{I}$.

IF one can have a STATIONARY origin of axes GLUED to ROTATING object also $T = \vec{\omega}^T \overleftrightarrow{I} \vec{\omega}$ with a stationary pivot origin for $\overleftrightarrow{I}$ should be true and equal to the T above.



A round uniform cylinder (of mass M, FINITE but modest radius r, and length L) hangs from a frictionless nail at the end of a length *a* horizontal bar whose other end is attached to the middle of a vertical pole spinning with constant upward angular velocity $\vec{\Omega}$ as shown (thus everything spins and has that angular velocity contribution). The axis of rotational symmetry of the cylinder (along its length L) is always in the plane that has the central axis of the pole and of the bar at any instant – they are forced to spin together. The angle of that symmetry axis from vertical, θ , can change freely without friction, but the cylinder is NOT able to twist around that symmetry axis because the hole snuggly fits the nail inside it. Gravity points down.

c) How many degrees of freedom does the length L cylinder have?

d) Find a Lagrangian for the just the cylinder, using a number of generalized coordinates (that are functions of time) equal to the number of degrees of freedom and any other necessary constants as usual. I will help by pointing out that the COM of the cylinder has distance from the central axis of the vertical pole $\rho = a + \frac{L}{2} sin\theta$ and is at a height $z = -\frac{L}{2} cos\,\theta$ if the length a bar is at height 0.

In an inertial frame cylindrical system $\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}.$