Problem set 11

Applied & Engineering Physics 3330

Due 6pm, Thurs. Nov.16, 2023

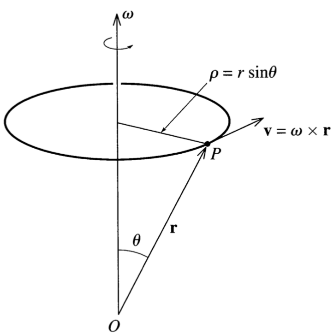
Remember to explain your answers!

Reading: In Taylor, read chapter 9, but skip the section on “the magnitude of the tides” on page 333-336 and realize that we’ve already learned about the angular velocity and how to relate that to the velocity and position vector of a bit going in a circle.

Problem 1: We said BOTH a person clinging to stationary inertial axes AND a person clinging to spinning primed axes can look at the arrow representing a vector $\vec{Q}$ and find components of that vector with respect to the spinning axes by taking dot products with unit vectors on those axes so

$Q'_x = (\vec{Q}\cdot\hat{e}'_x), \quad Q'_y = (\vec{Q}\cdot\hat{e}'_y), \quad Q'_z = (\vec{Q}\cdot\hat{e}'_z).$

The person clinging to the primed axes doesn’t think they move so if they take a time derivative of $\vec{Q}$ they get $\left(\frac{d\vec{Q}}{dt}\right)_{ROT} = \dot{Q}'_x\,\hat{e}'_x + \dot{Q}'_y\,\hat{e}'_y + \dot{Q}'_z\,\hat{e}'_z$. The person clinging to inertial axes sees those unit vectors move and uses the product rule $\left(\frac{d\vec{Q}}{dt}\right)_{IN} = \dot{Q}'_x\,\hat{e}'_x + Q'_x\left(\frac{d\hat{e}'_x}{dt}\right)_{IN} + \dot{Q}'_y\,\hat{e}'_y + Q'_y\left(\frac{d\hat{e}'_y}{dt}\right)_{IN} + \dot{Q}'_z\,\hat{e}'_z + Q'_z\left(\frac{d\hat{e}'_x}{dt}\right)_{IN}$. We argued that those unit vectors are just position vectors for the point at the end of the unit vector, so we remember



picture at left and use $\left(\frac{d\hat{e}'_x}{dt}\right)_{IN} = \vec{\omega}\times\hat{e}'_x$, etc to get

$$\left(\frac{d\vec{Q}}{dt}\right)_{IN} = \dot{Q}'_x\,\hat{e}'_x + Q'_x\,\vec{\omega}\times\hat{e}'_x \quad + \dot{Q}'_y\,\hat{e}'_y + Q'_y\,\vec{\omega}\times\hat{e}'_y \quad + \dot{Q}'_z\,\hat{e}'_z + Q'_z\vec{\omega}\times\hat{e}'_z$$

a) BRIEFLY show how you can move components inside cross products and group terms to get our Q rule: $\left(\frac{d\vec{Q}}{dt}\right)_{IN} = \left(\frac{d\vec{Q}}{dt}\right)_{ROT} + \vec{\omega}\times\vec{Q}$, true for any vector.

In class, we wrote that $\vec{r} = \vec{R} + \vec{r}'$, where $\vec{r}$ is the position of a mass m with respect to some inertial frame coordinate axes, $\vec{R}$ is a vector from the origin of the inertial axes to the origin of some potentially rotating axes and $\vec{r}'$ is a vector from the rotating axis origin to the mass m. We took derivatives of $\vec{r} = \vec{R} + \vec{r}'$ and found $\left(\frac{d\vec{r}}{dt}\right)_{IN} = \left(\frac{d\vec{R}}{dt}\right)_{IN} + \left(\frac{d\vec{r'}}{dt}\right)_{IN} = \left(\frac{d\vec{R}}{dt}\right)_{IN} + \left(\frac{d\vec{r'}}{dt}\right)_{ROT} + \vec{\omega}\times\vec{r}'$ and we defined $\vec{v}_r = \left(\frac{d\vec{r'}}{dt}\right)_{ROT}$ as velocity in rot frame.

b) Take another derivative and apply the Q rule again to verify the following result:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{IN} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{IN} + \left(\frac{d^2\vec{r'}}{dt^2}\right)_{ROT} + \dot{\vec{\omega}}\times\vec{r}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}') + 2\vec{\omega}\times\vec{v}_r$$

Note that I will be willing to give you this result on an exam formula sheet – it’s your starting point!

c) Now imagine that the **origin of the rotating axes is at the center of the Earth.** The rotating axes are stuck to the Earth, so their angular velocity is roughly constant. If a mass m is just sitting on the dirt not moving with respect to Earth, it has no velocity in the rotating frame. Which 2 terms above are now basically 0?
[Note it could be about to start moving due to acceleration.]

d) For that initially stationary mass, assume that the acceleration of m in the inertial frame has a term from the gravitational pull of Earth, which you can call $\vec{g_0}$. Thus there is a real force $m\vec{g_0}$ which is part of $\vec{F_{tot}} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{IN}$. Imagine moving the centrifugal term to the left side of the equation. **Define** an effective $\vec{g}$ equal to $\vec{g_0}$ plus that centrifugal term you just put on the left.

Then, for a case where we plug in $\vec{F}_{tot} = \vec{F}_{moon} + m\vec{g_0}$, rearrange above and **define** a half-fictitious tidal force so that $m\left(\frac{d^2\vec{r}'}{dt^2}\right)_{ROT} = m\vec{g} + \vec{F}_{tidal}$, where the g on that right side is the effective g. The moon pulls on m & Earth, but you don’t yet have to plug in for the acceleration of the origin of the ROT axes. We are ignoring the sun.

e) NOW call the vector from the center of the moon to the center of the Earth $\vec{R}_{M-E}$. What is the inertial frame acceleration of the center of the Earth due to the Moon as a vector written using $\vec{R}_{M-E}$ and the mass of the moon M_m? Remember $\hat{R}_{M-E} = \frac{\overrightarrow{R_{M-E}}}{|\vec{R}_{M-E}|}$.
[Note we can set this equal to $\left(\frac{d^2\vec{R}}{dt^2}\right)_{IN}$ when we chose the rotating origin at the center of the Earth and pretend that the ONLY thing exerting a force on the Earth is the moon.]

f) Imagine the moon is directly over the point on the equator that's on the opposite side of the Earth from m, which is just sitting on the equator. You may call the radius of the earth r_E and drop the vector sign from $\vec{R}_{M-E}$ when refering to its size only (as usual). **Give the size and direction of the tidal force from the moon, if you expand the force on m using the fact the moon is far away and keep the terms that are at the level of approximation you need to have a nonzero tidal force answer? (I.e. get the leading order answer after combining terms.)** ***Hint***: $(\mathbf{1} + \boldsymbol{tiny})^{\boldsymbol{p}} \approx \mathbf{1} + \boldsymbol{p} \cdot \boldsymbol{tiny}$

In this geometry, you can figure out specific distances to put in denominators that you can write as scalars.
FYI the sun's tidal force is about half as big as the moon's and they add with same sign at full&new moon.

Problem 2: Below, note that we've glued the bits of mass to the axes so they are a rigid object together. We are preparing for a new version of torque law, so our angular momentum includes velocities taken in an inertial (or "fixed" frame). If you think of the position $\vec{r}$' of each bit as starting at the COM and going to that bit you are ready when the Q rule makes you think about whether that vector changes with respect to ROT axes glued to the whole rigid object.

a) Recall that we found the total angular momentum with respect to some fixed origin could be written $\vec{L} = \vec{R}_{COM} \, x \, M_{TOT}\dot{\vec{R}}_{COM} + \sum_\alpha \vec{r}'_\alpha \, x \, m_\alpha \dot{\vec{r}}'_\alpha$ where the $\vec{r}'_\alpha$ vectors have their origin at the center of mass, and $\vec{R}_{COM}$ is the position of the center of mass with respect to that fixed origin. Using the Q rule argue briefly that, if these bits of mass are glued to a set of rotating axes with origin at the center of mass, and the $\vec{r}'_\alpha$ vectors are the positions of those masses with respect to those axes the masses are glued to,

$$\left(\frac{d\vec{r}'_\alpha}{dt}\right)_{FIX} = \vec{\omega} \times \vec{r}'_\alpha .$$

b) Use this and the vector identity $\vec{A} \times (\vec{B} \times \vec{A}) = |\vec{A}|^2 \vec{B} - \vec{A}(\vec{A} \cdot \vec{B})$ and this description of the components of $\vec{r}'_\alpha = (x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3})$ to show that the ith component of $\sum_\alpha \vec{r}'_\alpha \, x \, m_\alpha \dot{\vec{r}}'_\alpha$ is $\sum_\alpha m_\alpha \left[\omega_i |\vec{r}_\alpha'|^2 - x_{\alpha i} \sum_j x_{\alpha j} \omega_j \right]$.

c) Rewrite your answer to (b) using the symbol I_{ij}, where I'll define $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} |\vec{r}'_\alpha|^2 - x_{\alpha i} x_{\alpha j} \right]$.

You just met the inertia tensor!

Problem 3; For this type of problem we ignore the existence of sun and moon in 3330. A chunk of liquid that's unaccelerated in Earth's frame has $\left(\frac{d^2\vec{r}\prime}{dt^2}\right)_{rot}$=0. There's typically one real force other than gravity acting in this unaccelerated case -- it comes from a difference in pressure pushing on opposite sides of the chunk of liquid and can be called a "buoyant force". What's interesting for us is this real $\vec{F}_{bourant}$ is perpendicular to the surface of such unaccelerated liquid, so if you can figure out the direction of that vector (by plugging values in what it is equal to) and think about a surface perpendicular to that vector you can figure out the tilt of the surface.

Typically we put an origin on the surface of the Earth near the liquid and hide the centrifugal effect (which comes from a large $-\left(\frac{d^2\vec{R}}{dt^2}\right)_{in}$ and a small $-\vec{\omega}\times(\vec{\omega}\times\vec{r'})$ in this case) inside an effective $\vec{g}$. Then, liquid moving at constant velocity has $\vec{g}+\frac{\vec{F}_{bouyant}}{m}=2\vec{\omega}\times\vec{v}_r$.

a) Imagine the Mississippi river flowing straight south at a particular latitude angle 30 degrees North (near New Orleans). Is the east or west side of the river surface further from the center of the Earth?

b) Assume the river is W wide there and flow speed is locally measured to be S. **By how much does the Coriolis effect raise the water level on one side then, compared to the other side**? (I.e. how much further from the center of the Earth is one side of the surface than the other?) Ok to ignore curvature of Earth in that width (considering surface of river as flat), to approximate $\vec{g}$ as roughly $\vec{g}_0$ (throwing away centrifugal effects), and to assume the tilt angle is small enough $tan\theta \approx sin\theta$.

I found these numbers in the area if you optionally want to plug in: width about 700m and speed 1.3 meters/second, although the speed varies a lot.

c) In the Hadley cell atmospheric circulation just north and south of the equator, air travels along the surface toward the equator in response to the fact a lot of hot air rises at equator. Think about whether that air started with the tangential velocity of the ground under it if it started in a place with no wind, and **then briefly explain whether, after traveling toward the equator, that air also travels east or west with respect to the ground below it (on average). Use a verbal conceptual argument here that does not reference a vector cross product.** [Note that at the equator north/south motion is opposed by air coming from the other side of the equator, but east/west motion is not.] Optional: Google Hadley cell, trade winds.

d) Imagine a FAST rocket headed Eastward barely above the equator. Is its angular velocity bigger or smaller than that of the ground below?
Is its Coriolis fictitious force in the same direction as the centrifugal fictitious force on something sitting at a rotating coordinate origin on the equator below it? (yes/no)
Is its speed relative to ground just below it given by $|\vec{v}_r| = \Delta\omega r_E$, where
$\Delta\omega = [rocket\ angular\ velocity - angular\ velocity\ of\ ground\ just\ below]$ & R_E= radius of Earth? (yes/no)
[Notice that $\Delta[\omega(\omega R_E)]$ would have more than 1 term and the Coriolis fictitious force has a factor of 2.]

Problem 4: A projectile is fired from a point on the Earth's surface with latitude angle λ. (This is a normal latitude measured from the equatorial plane – Taylor uses a different angle.) To make the grader's life easier glue with origin at that launch spot an x' axis pointing East and a z' axis pointing up as is typical. (Since firing east.)

a) Give x', y', z' components of the Earth's angular velocity, using the symbol ω for its size of roughly $2\frac{\pi}{day}$.

b) If we fire mass m East&up with initial speed V_0 and initial angle from horizontal α, give the freshman physics answer (which I called the 0th order answer) for rotating frame velocity components along x' ($v_{x'}^{oth}$) and along z'($v_{z'}^{oth}$). Pay attention to time.

c) Give x', y', z' components of the Coriolis rot frame acceleration term, $2\vec{v}_r\times\vec{\omega}$, using just 1 factor of ω in that (so to 1st order). Remember this is typically a function of time.

d) Measured with respect to those axes we stuck to Earth near the launch site, find the height of the projectile as a function of time, z'(t), to 1st order in ω.

e) Assuming the ground is flat, find the landing time to 1st order in ω. (Note that if ω is in a denominator, you can expand that denominator for small ω and clearly see the term that is linear in ω. $Recall\ \text{"}tiny\text{"}\ trick.$)

f) State whether y' points in the north or south direction (using a right handed system as always) and then give y'(t) to 1st order in ω.

g) Is your answer to (f) already proportional to ω ? Yes/no

h) If your answer for position as a function of time is ALREADY proportional to ω (with an ω in every term), then you can find a 1st order answer for how far you get at landing time in that direction by plugging in a freshman physics landing time that ignores Coriolis and has no ω in it. Find how far to the north or south (on Earth) the projectile is at landing time, to linear order in ω, compared to an origin at its launch spot.