Lecture worksheet III: coupled oscillators – finding normal mode solutions

Key: seek special same ω "normal mode" solutions $x_1 = C_1 e^{i\omega t}$ $x_2 = C_2 e^{i\omega t}$ and plug into

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \qquad \text{to get} \qquad m_1(-\omega^2 C_1) = -k_1 C_1 + k_2(C_2 - C_1)$$
$$m_2\ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) \qquad\qquad\qquad m_2(-\omega^2 C_2) = -k_3 C_2 - k_2(C_2 - C_1)$$

IV a) Are these equivalent to $\begin{pmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_3 + k_2 - m_2\omega^2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$? (yes or no)

b) What must be true about the matrix to have a chance of finding solutions with nonzero $\frac{C_1}{C_2}$?

c) For the special case $k_1 = k_3 = k, \quad m_1 = m_2 = m$ $\begin{pmatrix} k + k_2 - m\omega^2 & -k_2 \\ -k_2 & k + k_2 - m\omega^2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

what specific values of ω^2 can give solutions with nonzero C_1 and C_2 here?

d) I'll call the higher value of ω you just found ω_H. Plug that larger ω^2 back in to find the allowed ratio $\frac{C_2}{C_1}$ for that frequency. [Should you get same answer from both equations?]

e) If I pick a very specific set of initial conditions that makes the masses only oscillate at frequency ω_H and in that case $x_1(t) = A_H cos(\omega_H t - \delta_H)$ for a specific A_H value and δ_H value, what should $x_2(t)$ be then in the real world

[Hint1: set $C_1 = A_H e^{-i\,\delta_H}$. If $x_1 = C_1 e^{i\omega t}$ $x_2 = C_2 e^{i\omega t}$ are solutions, linear equations mean $x_1 = Re[C_1 e^{i\omega t}]$ $x_2 = Re[C_2 e^{i\omega t}]$ are solutions too -- ones appropriate for REAL world.

Hint2: What is the required ratio $\frac{C_2}{C_1}$? Feel free to make an easy choice for phase and then make the amplitude ratio right. Then take a real part to get a real solution for x_2 for higher frequency.]

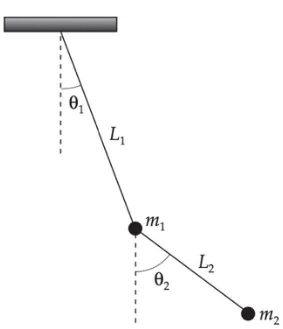
f) I'll call the lower value of ω you found in (c) ω_L. We plugged that lower ω^2 back in to find $C_1 = C_2$ for THAT frequency.
If I pick a specific set of initial conditions that makes the masses only oscillate at frequency ω_L and in that case $x_1(t) = A_L cos(\omega_L t - \delta_L)$ for a specific A_L value and δ_L value, what should $x_2(t)$ be then in the real world?

h) To make a general solution, I try $x_1(t) = A_L cos(\omega_L t - \delta_L) + A_H cos(\omega_H t - \delta_H)$. What solution for $x_2(t)$ has to go with it? [Plugging $x_1(t) = A_L cos(\omega_L t - \delta_L) + A_H cos(\omega_H t - \delta_H)$ AND your answer for $x_2(t)$ here into the equations at top has to work -- we found 2 pairs of solutions that work by themselves.]

i) Are there any OTHER frequencies that can have NONzero solutions for C_1, C_2 above? Y or N

j) Will any trial solution where each mass moves at a single frequency DIFFERENT from the other's frequency be able to fit the equations at top of page 1 at **ALL** times? Y or N

k) For (h) answer, write down a set of possible δ_L, δ_H values that could fit initial conditions where the starting velocity of both masses is 0.



l) For SMALL angles this double pendulum follows $\ddot{\theta}_1 + A\ddot{\theta}_2 + B\theta_1 = 0$ and (with $A = \frac{m_2 L_2}{(m_1+m_2)L_1}$ $B = \frac{g}{L_1}$ $D = \frac{L_2}{L_1}$) $\ddot{\theta}_1 + D\ddot{\theta}_2 + B\theta_2 = 0$
To find a normal mode where both angles oscillate at the same ω, what matrix has to have a nonzero null space (and thus a special determinant value)?