Lecture 10 - Momentum space and Fourier Transforms

Wednesday, September 13, 2023 8:54 AM

Reading: Griffiths 2.5

1. Warm-up Quiz
2. HW4 due Friday
3. Today
   - Localizing a wave packet
   - Momentum space representation

$$\langle \hat{T} \rangle = \left\langle \frac{\hat{p}^2}{2m} \right\rangle = \int_0^L \Psi_1^* \underbrace{\left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \Psi_1}_{H \psi_1(x) e^{-iE_1 t/\hbar}} dx \qquad \psi_1(x) e^{-iE_1 t/\hbar}$$

$$= \int_0^L \psi_1^* e^{+\frac{iE_1 t}{\hbar}} E_1 \psi_1 e^{-\frac{iE_1 t}{\hbar}} dx = E_1$$

$$\int_0^L \frac{1}{2} \left(\psi_1^* e^{\frac{iE_1 t}{\hbar}} + \psi_2^* e^{\frac{iE_2 t}{\hbar}} \right) \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \left(\psi_1 e^{-\frac{iE_1 t}{\hbar}} + \psi_2 e^{-\frac{iE_2 t}{\hbar}} \right) dx$$

cross terms

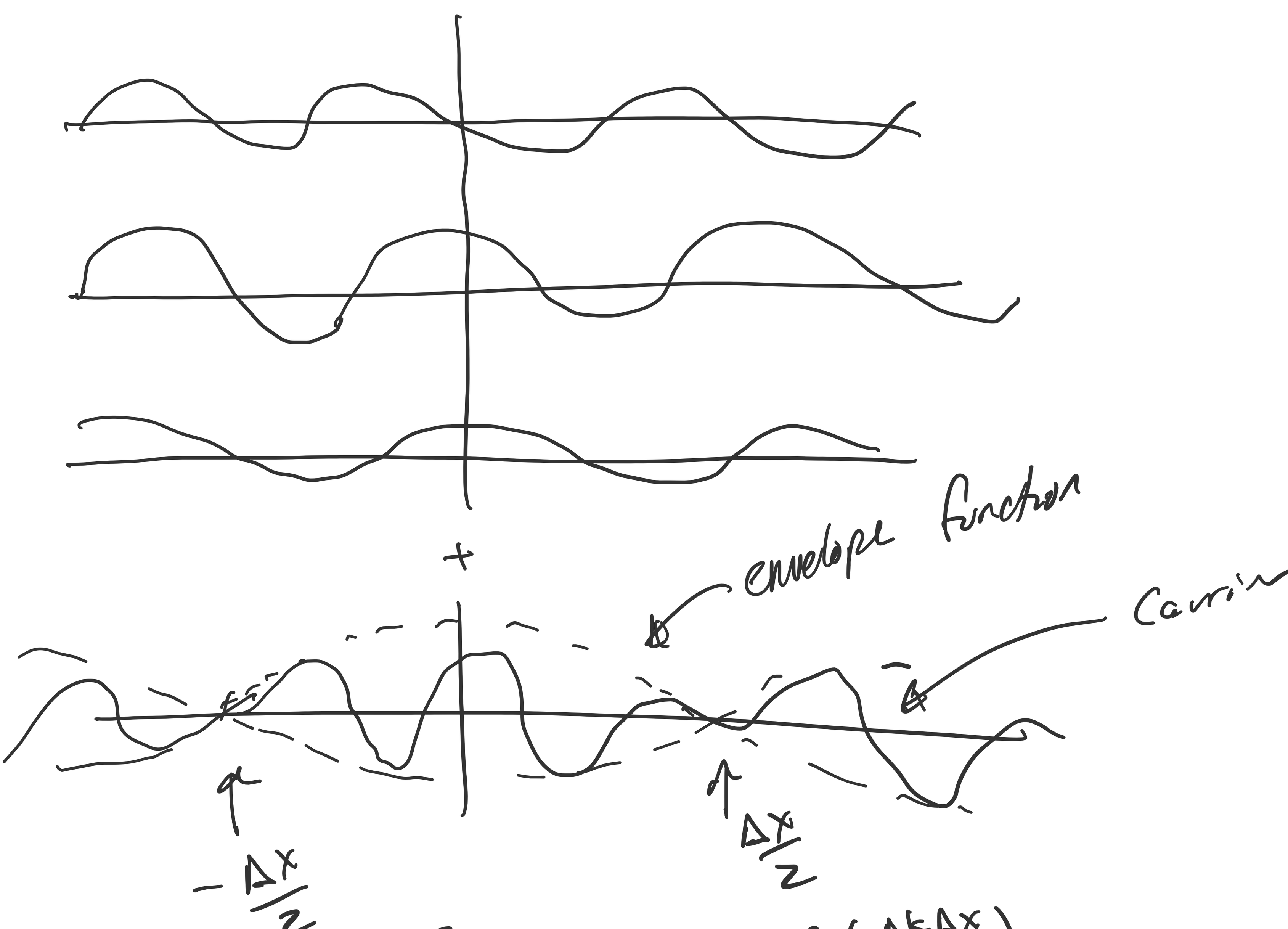
$$\frac{1}{2}\int_0^L \left(\quad \right) \left(E_1 \psi_1 e^{-\frac{iE_1 t}{\hbar}} + E_2 \psi_2 e^{-\frac{iE_2 t}{\hbar}} \right) dx = \frac{1}{2} \left(E_1 + E_2 + 0 + 0 \right)$$

$$= \frac{1}{2} (E_1 + E_2)$$

Last Time: $V = 0$ (free space)

$$\psi(x) = \psi_1 + \psi_2 + \psi_3 = e^{ik_0 x} + \frac{1}{2} e^{i(k_0 + \frac{\Delta k}{2})x} + \frac{1}{2} e^{i(k_0 - \frac{\Delta k}{2})x}$$

$$= \underbrace{e^{ik_0 x}}_{\text{carrier}} \underbrace{\left[1 + \cos\left(\frac{\Delta k}{2} x \right) \right]}_{\text{envelope function}} = 2 e^{ik_0 x} \cos^2\left(\frac{\Delta k}{4} x \right)$$



Zero @ $\cos^2\left(\frac{\Delta k \Delta x}{4} \right) = 0$

$$\frac{\Delta x}{2} \frac{\Delta k}{2} = \frac{\pi}{2} \Rightarrow \Delta x \Delta k = 2\pi$$

$k = p/\hbar$

Larger Δk → Smaller Wavepacket in x (more localized)

Larger Δx → Small range of k needed.

⇒ underlies the uncertainty principle

How fast is our wave packet moving?

$$\Psi(x,t) = \Psi_1 + \Psi_2 + \Psi_3$$

$$= e^{i[k_0 x - \omega(k_0) t]} + \frac{1}{2} e^{i[(k_0 + \frac{\Delta k}{2})x - \omega(k_0 + \frac{\Delta k}{2})t]} + \frac{1}{2} e^{i[(k_0 - \frac{\Delta k}{2})x - \omega(k_0 - \frac{\Delta k}{2})t]}$$

if $\Delta k \ll k_0$ we can Taylor expand ω

$$\omega(k_0 \pm \Delta k/2) \simeq \underbrace{\omega(k_0)}_{\omega_0} \pm \underbrace{\frac{\partial \omega}{\partial k}\Big|_{k_0}}_{v_g} \left(\frac{\Delta k}{2} \right) + \cdots$$

So

$$\Psi(x,t) \simeq e^{i(k_0 - \omega_0 t)} \left[1 + \frac{1}{2} e^{-i \frac{\Delta k}{2}(x - v_g t)} + \frac{1}{2} e^{-i\frac{\Delta k}{2}(x - v_g t)} \right]$$

$$= \underbrace{e^{i(k_0 x - \omega_0 t)}}_{\text{carrier plane wave}} \underbrace{\left\{ 1 + \cos\left[\frac{\Delta k}{2} (x - v_g t) \right] \right\}}_{\text{envelope travels @ } v_g}$$

travels @ $v_p = \frac{\omega_0}{k_0} = \frac{\hbar k_0}{2m}$

$$v_{gr} = \frac{d\omega}{dk}\Big|_{k_0} = \frac{\hbar k_0}{m}$$

More Generally: Sum over closely space plane waves

$$\psi(x) = \sum_n C_n \psi_n e^{ik_n x} = \sum_n C_n e^{i(k_0 + \Delta k_n)x}$$

→ Go onto a continuous limit

set of $C_n \psi_n$ @ $k_0 + \Delta k$ → $\phi(k)$

With appropriate choice of $\phi(k)$, we can make $\psi(x)$ die out as $x \to \pm\infty$. ∴ Can normalize

So

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk} \qquad (1)$$

weight in k-space

plane wave

- This is a Fourier transform between $\phi(k)$ and $\psi(x)$

$$\boxed{\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx} \qquad (2)$$

Also see K+W Ch. 8

Strict def^{ns} (2) is FT
(1) is the inverse FT

- Every well-behaved $\psi(x)$ has a unique $\phi(k)$ that is an alternate description

"k-space" ⟺ "real-space"

momentum space

spatial freq — space

Usually freq — time

Well-behaved → Normalizable

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\phi(k)|^2 dk = 1$$

proof on Friday