Bryant Har

Q29) $\int_{-1}^{1} dx\ A_0^2 = \frac{2}{1} = 2A_0^2 = 2 \Rightarrow A_0 = \pm 1$

$\int_{-1}^{1} dx\ A_1 x \cdot A_1 x = \frac{2}{3} = A_1^2 \cdot \frac{x^3}{3}\Big|_{-1}^{1} \Rightarrow A_1 = \pm 1$

$\int_{-1}^{1} dx\ A_2^2 (1-3x^2)^2 = \frac{2}{5} = A_2^2\left[x - 2x^3 + \frac{9}{5}x^5\right]\Big|_{-1}^{1} = A_2^2\left(2 - 4 + \frac{18}{5}\right) = \frac{2}{5} \Rightarrow A_2^2 = \frac{2}{8} \Rightarrow A_2 = \pm \frac{1}{2}$

$$\boxed{A_0 = \pm 1 \quad A_1 = \pm 1 \quad A_2 = \pm \frac{1}{2}}$$

Q30) $\left[\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right] P_\lambda(\theta) = -\lambda P_\lambda(\theta)$

By inspection, multiply by $\sin\theta$ and $\frac{d}{dx}\sin = \cos$, so

a) $\Leftrightarrow$ $$\boxed{\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial P_\lambda(\theta)}{\partial\theta}\right] = -\lambda \sin(\theta)\, P_\lambda(\theta)}$$

$\mathcal{L} = \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) \Rightarrow$ show $\int_0^\pi P_i^* \mathcal{L} P_j\, dx = \int_0^\pi P_j (\mathcal{L} P_i)^* dx$

From textbook eq 11.57, for Sturm-Liouville operators the Hermitian condition reduces to:

$$\left[u_n(x)p(x)\frac{du_m^*(x)}{dx}\right]\Big|_\Omega = 0 \;=\; \left[u_n \sin\theta \frac{du_m^*}{dx}\right]\Big|_0^\pi = u_n \sin(\pi)\frac{du_m^*}{dx} - u_n \sin(0)\frac{du_m^*}{dx}$$

$$= 0 - 0 = 0 \Rightarrow \boxed{\left[u_n \sin\theta \frac{du_m^*}{dx}\right]\Big|_0^\pi = 0} \rightarrow \text{Hermitian} \checkmark$$

b) By Hermitian condition

$\int d\theta\ P_m^* \mathcal{L} P_n = \int d\theta\ P_m (\mathcal{L} P_n)^* \Leftrightarrow \int d\theta\ P_m^* \lambda_n w P_n = \int d\theta\ P_n \lambda_m^* w P_m^*$

$\Rightarrow \int d\theta\ P_m^* \lambda_n \sin\theta\, P_n - P_n \lambda_m^* \sin\theta\, P_m^* = 0 \Rightarrow (\lambda_n - \lambda_m^*) \int d\theta\ P_m^* \sin\theta\, P_n = 0$

when $n \neq m$, $(\lambda_n - \lambda_m^*) \neq 0 \Rightarrow \int d\theta\ P_m^* \sin\theta\, P_n = 0$ which is of the form $\int d\theta\ g(\theta) P_n P_m = 0$

$\Rightarrow \int d\theta\ g(\theta) P_n P_m = 0 = \boxed{\int_0^\pi d\theta\ \sin\theta\, P_n P_m = 0}$ $\checkmark$ for $n \neq m$

$$\boxed{g(\theta) = \sin\theta}$$

Q33) $$\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2p_0^2\left[\cosh(2u) - \cos(2v)\right]}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\psi = 0 \qquad \psi(u,v,z) = fgh$$

$$fg\frac{\partial^2 h}{\partial z^2} + \frac{1}{2p_0^2\left[\cosh(2u) - \cos(2v)\right]}\left(gh\frac{\partial^2 f}{\partial u^2} + fh\frac{\partial^2 g}{\partial v^2}\right) = 0$$

$$\frac{1}{h}\frac{\partial^2 h}{\partial z^2} + \frac{1}{2p_0^2\left[\cosh(2u) - \cos(2v)\right]}\left(\frac{1}{f}\frac{\partial^2 f}{\partial u^2} + \frac{1}{g}\frac{\partial^2 g}{\partial v^2}\right) = 0$$

Clearly, z already separated, $\frac{1}{h}\frac{\partial^2 h}{\partial z^2} = C_z \Rightarrow \frac{1}{2p_0^2\left[\cosh(2u) - \cos(2v)\right]}\left(\frac{1}{f}\frac{\partial^2 f}{\partial u^2} + \frac{1}{g}\frac{\partial^2 g}{\partial v^2}\right) = -c_z$

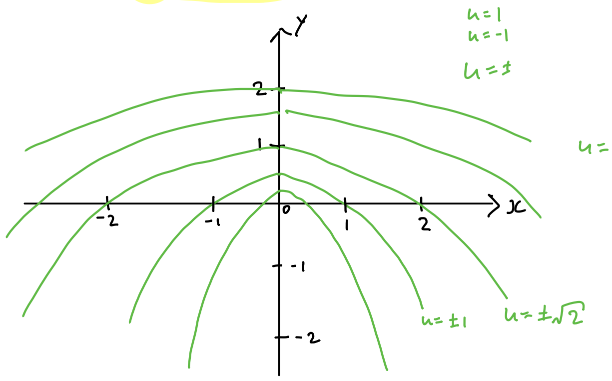
$$\frac{1}{2p_0^2\left[\cosh(2u) - \cos(2v)\right]}\left(\frac{1}{f}\frac{\partial^2 f}{\partial u^2} + \frac{1}{g}\frac{\partial^2 g}{\partial v^2}\right) = -c_z \Rightarrow 0 = \frac{1}{f}\frac{\partial^2 f}{\partial u^2} + \frac{1}{g}\frac{\partial^2 g}{\partial v^2} + c_z 2p_0^2\left[\cosh(2u) - \cos(2v)\right]$$

$$0 = \underbrace{\left[\frac{1}{f}\frac{\partial^2 f}{\partial u^2} + c_z 2p_0^2\cosh(2u)\right]}_{C} + \underbrace{\left[\frac{1}{g}\frac{\partial^2 g}{\partial v^2} - c_z 2p_0^2\cos(2v)\right]}_{-C}$$

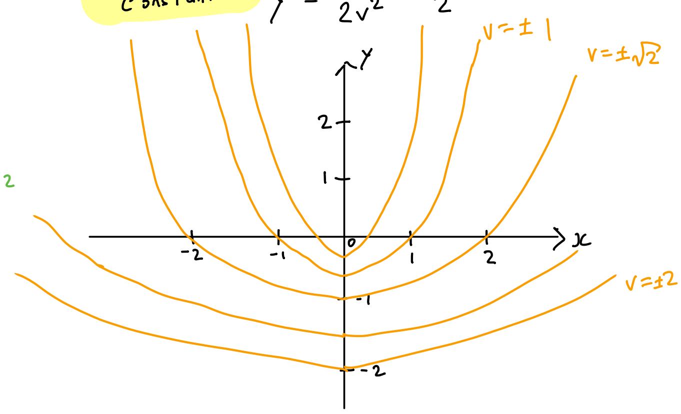
$$\Rightarrow \boxed{\begin{aligned} &\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2} = c_z \\ &\frac{1}{f(u)}\frac{\partial^2 f(u)}{\partial u^2} + c_z 2p_0^2\cosh(2u) = C \\ &\frac{1}{g(v)}\frac{\partial^2 g(v)}{\partial v^2} - c_z 2p_0^2\cos(2v) = -C \end{aligned}}$$

Q34) a) $x = uv \qquad y = \frac{u^2 - v^2}{2}$

Constant u: $y = \frac{u^2}{2} - \frac{x^2}{2u^2}$



Constant v: $y = \frac{x^2}{2v^2} - \frac{v^2}{2}$



Q 34 b) $h_v^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = u^2 + v^2 \Rightarrow h_v = h_u = \sqrt{u^2 + v^2}$

$h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = v^2 + u^2$

Plugging into formula,

$$\nabla^2 \Psi(u,v) = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}\frac{h_v}{h_u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{h_u}{h_v}\frac{\partial}{\partial v}\right]\Psi(u,v)$$

$$\boxed{\nabla^2 \Psi(u,v) = \frac{1}{u^2+v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Psi(u,v) = 0}$$

c) Let $\Psi = U(u)\,V(v)$

$$\Rightarrow \nabla^2 \Psi(u,v) = \frac{1}{u^2+v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Psi(u,v)$$

$$0 = \frac{1}{u^2+v^2}\left(V\frac{\partial^2 U}{\partial u^2} + U\frac{\partial^2 V}{\partial v^2}\right) = \frac{1}{U}\frac{\partial^2 U}{\partial u^2} + \frac{1}{V}\frac{\partial^2 V}{\partial v^2}$$

Using constant c^2,

$\Rightarrow \frac{1}{U}\frac{\partial^2 U}{\partial u^2} = c^2 \quad \frac{1}{V}\frac{\partial^2 V}{\partial v^2} = -c^2 \Rightarrow$ $V'' = -c^2 v \Rightarrow V = A_v \cos cv + B_v \sin cv$

$U'' = c^2 u \Rightarrow U = A_u e^{cu} + B_u e^{-cu}$

$\Rightarrow \Psi = UV$

$$\boxed{\Psi(u,v) = \left(A_v \cos(cv) + B_v \sin(cv)\right)\left(A_u e^{cu} + B_u e^{-cu}\right)}$$