PS 12 q4-5

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Q4

a

$\boxed{\text{Yes. } \omega_3 \text{ is constant}}$
We have

$$\begin{cases} \lambda_1\dot{\omega}_1 = (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 = (\lambda_3-\lambda_1)\omega_1\omega_3 \\ \lambda_3\dot{\omega}_3 = (\lambda_1-\lambda_2)\omega_1\omega_2 \end{cases}$$

If $\lambda_2 = \lambda_1$,

$$\begin{cases} \lambda_1\dot{\omega}_1 = (\lambda_1-\lambda_3)\omega_1\omega_3 \\ \lambda_1\dot{\omega}_2 = (\lambda_3-\lambda_1)\omega_1\omega_3 = -(\lambda_1-\lambda_3)\omega_1\omega_3 \\ \lambda_3\dot{\omega}_3 = (\lambda_1-\lambda_1)\omega_1\omega_1 = 0 \end{cases} \implies \begin{cases} \dot{\omega}_1 = -\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3 = 0 \end{cases} \implies \dot{\omega}_3 = 0$$

$$\dot{\omega}_3 = 0 \implies \omega_3 = C, \quad \boxed{\omega_3 \text{ is constant}}$$

b

$$C = \frac{\lambda_2-\lambda_3}{\lambda_2}\omega_3$$

From above,

$$\begin{cases} \lambda_1\dot{\omega}_1 = (\lambda_1-\lambda_3)\omega_1\omega_3 \\ \lambda_1\dot{\omega}_2 = (\lambda_3-\lambda_1)\omega_1\omega_3 = -(\lambda_1-\lambda_3)\omega_1\omega_3 \end{cases} = \begin{cases} \lambda_2\dot{\omega}_1 = (\lambda_2-\lambda_3)\omega_2\omega_3 \\ \lambda_2\dot{\omega}_2 = -(\lambda_2-\lambda_3)\omega_1\omega_3 \end{cases}$$

$$\implies \begin{cases} \dot{\omega}_1 = \frac{\lambda_2-\lambda_3}{\lambda_2}\omega_2\omega_3 \\ \dot{\omega}_2 = -\frac{\lambda_2-\lambda_3}{\lambda_2}\omega_1\omega_3 \end{cases}$$

$$\boxed{\begin{cases} \dot{\omega}_1 = C\omega_2 \\ \dot{\omega}_2 = -C\omega_1 \end{cases} \implies \text{Yes}}$$

As desired

c

$$\boxed{\dot{\eta} = -iC\eta}$$

Doing as told,

$$\begin{cases} \dot{\omega}_1 = C\omega_2 \\ \dot{\omega}_2 = -C\omega_1 \end{cases} \implies \dot{\omega}_1 + i\dot{\omega}_2 = C\omega_2 - Ci\omega_1$$

Letting,

$$\eta = \omega_1 + i\omega_2 \implies \boxed{\dot{\eta} = -iC\eta}$$

d

$\boxed{\text{Yes}}$ Differentiating, we see that $b = 1$ and they are equivalent

$$\frac{d}{dt}\eta = \frac{d}{dt}\eta_0 e^{-iCbt} = -iCb\eta_0 e^{-iCbt} = -iCb\eta$$

This result is in the same form as above, $\implies \boxed{\text{Yes}}$. $b = 1$

e

$\boxed{\text{Yes}}$ Using the given condition,

$$\eta = \omega_0 e^{-iCt} = \omega_1 + i\omega_2 = \omega_0 \big[\cos(-Ct) + i\sin(-Ct)\big]$$
$$\omega_1 + i\omega_2 = \omega_0 \cos(Ct) - i\omega_0 \sin(Ct)$$

Comparing real and imaginary parts in isolation, we find,

$$\boxed{\begin{cases} \omega_1 = \omega_0 \cos(Ct) \\ \omega_2 = -\omega_0 \sin(Ct) \end{cases} \implies \text{Yes}}$$

f

$\boxed{|\vec{\omega}| = \sqrt{\omega_0^2 + \omega_3^2}}$ This would be

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega_0^2\big[\cos(Ct)^2 + (-\sin(Ct))^2\big] + \omega_3^2}$$

$$\boxed{|\vec{\omega}| = \sqrt{\omega_0^2 + \omega_3^2}}$$

g

$\boxed{\text{Yes.}}$
Yes we would observe precession if $\omega_0 > 0$. Therefore, the vector would rotate in a cone.

Q5

a

$\boxed{\text{After second rotation}}$ The first two rotations set the plane of rotation, while the last rotation sets the body axes to the appropriate orientation within that rotating plane. The $\vec{e}_1'$ lies on this plane and is rotated into position by the second rotation about $\vec{e}_2'$. So, the e_1 and e_1' vectors point along the same direction after this rotation, or the $\boxed{\text{second rotation}}$

b

By Taylor,

$$\omega = \dot{\phi}\hat{z} + \dot{\theta}e_2' + \dot{\psi}e_3$$

We seek to convert this into the body unit vectors. We already have $\hat{z} = \cos(\theta)e_3 - \sin\theta e_1'$,

$$\omega = \dot{\phi}\cos(\theta)e_3 - \dot{\phi}\sin(\theta)e_1' + \dot{\theta}e_2' + \dot{\psi}e_3$$

From Taylor's diagram, we see that

$$\begin{cases} e_1' = \cos\psi e_1 - \sin\psi e_2 \\ e_2' = \cos\psi e_2 + \sin\psi e_1 \end{cases}$$

Substituting,

$$\vec{\omega} = \dot{\phi}\cos(\theta)e_3 - \dot{\phi}\sin(\theta)(\cos\psi e_1 - \sin\psi e_2) + \dot{\theta}(\cos\psi e_2 + \sin\psi e_1) + \dot{\psi}e_3$$
$$\vec{\omega} = \dot{\phi}\cos(\theta)e_3 - \dot{\phi}\sin(\theta)\cos\psi e_1 + \dot{\phi}\sin(\theta)\sin\psi e_2 + \dot{\theta}\cos\psi e_2 + \dot{\theta}\sin\psi e_1 + \dot{\psi}e_3$$

$$\boxed{\vec{\omega} = \big[\dot{\theta}\sin\psi - \dot{\phi}\sin(\theta)\cos\psi\big]e_1 + \big[\dot{\theta}\cos\psi + \dot{\phi}\sin(\theta)\sin\psi\big]e_2 + \big[\dot{\phi}\cos(\theta) + \dot{\psi}\big]e_3}$$

$$\boxed{\vec{\omega} = \begin{bmatrix} \dot{\theta}\sin\psi - \dot{\phi}\sin(\theta)\cos\psi \\ \dot{\theta}\cos\psi + \dot{\phi}\sin(\theta)\sin\psi \\ \dot{\phi}\cos(\theta) + \dot{\psi} \end{bmatrix}}$$

c

$\boxed{\text{1: One DOF}}$ $\boxed{\theta}$

Clearly, there is only one degree of freedom, since the cylinder rotates at a constant angular velocity. Only θ can vary independently here. $\boxed{\text{One degree of freedom}}$.

d

We first find the Lagrangian,

$$\mathcal{L} = T - V$$

$$V = mgz = -mg\frac{L}{2}\cos\theta$$

$$T = \frac{1}{2}MV_{COM}^2 + \frac{1}{2}\omega^T I\omega$$

We find V_{COM} in the inertial frame using

$$\rho = a + \frac{L}{2}\sin\theta, \quad z = -\frac{L}{2}\cos\theta \quad \dot{\phi} = \Omega$$

$$V_{COM} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} = \dot{\theta}\frac{L}{2}\cos\theta\hat{\rho} + (a + \frac{L}{2}\sin\theta)\Omega\hat{\phi} + \dot{\theta}\frac{L}{2}\sin\theta\hat{z}$$

Getting squared magnitude,

$$|V_{COM}|^2 = (\dot{\theta}\frac{L}{2}\cos\theta)^2 + ((a + \frac{L}{2}\sin\theta)\Omega)^2 + (\dot{\theta}\frac{L}{2}\sin\theta)^2$$

$$|V_{COM}|^2 = \left(\frac{L\dot{\theta}}{2}\right)^2 + (a\Omega + \frac{L\Omega}{2}\sin\theta)^2$$

Now we find ω using the above formula, denoting the formula θ as θ' and the diagram θ as θ,

$$\dot{\phi} = \Omega, \quad \theta' = \frac{\pi}{2}, \quad \psi = \theta$$

$$\vec{\omega} = \begin{bmatrix} \dot{\theta}'\sin\psi - \dot{\phi}\sin(\theta')\cos\psi \\ \dot{\theta}'\cos\psi + \dot{\phi}\sin(\theta')\sin\psi \\ \dot{\phi}\cos(\theta') + \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Omega + \dot{\theta} \end{bmatrix}$$

Clearly, we only need the I_{zz} term, which is the moment about the COM or, by table,

$$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

Combining our results into $T = \frac{1}{2}MV_{COM}^2 + \frac{1}{2}\omega^T I\omega$,

$$T = \frac{1}{2}M\left(\left(\frac{L\dot{\theta}}{2}\right)^2 + (a\Omega + \frac{L\Omega}{2}\sin\theta)^2\right) + \frac{1}{2}(\frac{1}{4}Mr^2 + \frac{1}{12}ML^2)(\Omega + \dot{\theta})^2$$

$$T = \frac{1}{2}M\left(\frac{L\dot{\theta}}{2}\right)^2 + \frac{1}{2}M(a\Omega + \frac{L\Omega}{2}\sin\theta)^2 + (\frac{1}{8}Mr^2 + \frac{1}{24}ML^2)(\Omega + \dot{\theta})^2$$

Recall,

$$V = -mg\frac{L}{2}\cos\theta$$

$$\mathcal{L} = T - V$$

We arrive at our lagrangian,

$$\boxed{\mathcal{L} = \frac{1}{2}M\left(\frac{L\dot{\theta}}{2}\right)^2 + \frac{1}{2}M(a\Omega + \frac{L\Omega}{2}\sin\theta)^2 + (\frac{1}{8}Mr^2 + \frac{1}{24}ML^2)(\Omega + \dot{\theta})^2 + Mg\frac{L}{2}\cos\theta}$$