#physics

Q1

Q1a

In steady state for $F = f_0 \cos(\omega t + \delta)$, it's known that

$$x_{ss} = \begin{cases} A\cos(\omega t + \delta) \\ A_{max} \approx \frac{f_0}{2\beta\omega_0} \end{cases}$$

At some later time, energy at max amplitude is given by,

$$E = \frac{1}{2}kx_{max}^2 = E = \frac{1}{2}kA^2$$

Recalling $2\beta = \frac{b}{m}$ and $F = -bv$, work over one period at ω_0, then, is

$$\Delta E = \int_t^{t+T} F\frac{dx}{dt}dt \approx \int_t^{t+T} -2m\beta(-A\omega_0\sin(\omega_0 t + \delta))^2 dt = -2\pi A^2\beta m\omega_0$$

Subbing in these approximate values (E, $|\Delta E|$) we recover the definition in Taylor,

$$Q \approx 2\pi\frac{E}{\Delta E_T} \approx 2\pi\frac{\frac{1}{2}kA^2}{2\pi A^2\beta\omega_0} = \frac{k}{2\beta m\omega_0} = \frac{\omega_0}{2\beta}$$

$$\boxed{Q = \frac{\omega_0}{2\beta} \approx 2\pi\frac{E}{\Delta E_T}}$$

Q1b

We begin,

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}$$

By difference of squares,

$$= \frac{f_0^2}{(\omega_0 + \omega)^2(\omega_0 - \omega)^2 + (2\beta\omega)^2}$$

At $\omega \approx \omega_0 \pm \beta$,

$$\approx \frac{f_0^2}{(2\omega_0)^2(\omega_0 - \omega)^2 + (2\beta\omega)^2}$$

$$\approx \frac{f_0^2}{(2\omega_0)^2(\beta)^2 + (2\beta\omega)^2}$$

$$A^2 \approx \frac{f_0^2}{2(2\beta\omega_0)^2} \ \square$$

For comparison, A^2 attains a maximum value of

$$A_{max}^2 = \frac{f_0^2}{(2\beta\omega_0)^2} \implies \frac{f_0^2}{2(2\beta\omega_0)^2} \text{ is half maximum}$$

$$\boxed{A^2\big|_{\omega\approx\omega_0\pm\beta} = \frac{f_0^2}{2(2\beta\omega_0)^2} = \text{ half maximum}}$$

Q2

Q2a

$\boxed{\textbf{iv}}$

Looking from the graph and using the hint, we arrive at this answer. Large amplitude motion in a slightly nonlinear system is a solid analogue for this, and thinking in this case, we can imagine that stepping the frequency up yields a larger effective resonant frequency than in the other direction about the same point. Therefore, in this analogous system, the same is also true. For duffing oscillators, effective k becomes the sum of k and kx^2, so since $\omega_0^2 = \frac{k}{m}$, we see that the frequency must be shifted up for higher amplitudes. From the graph, B is the high point, and A is the low point on the frequency response, and so we observe that as ω is increasing, the amplitude is higher than when the frequency is decreasing. This observation is represented by choice iv.

Q2b

We have

$$x = A\cos(\omega t - \delta)$$

Cubing and applying identity,

$$x^3 = A^3\cos^3(\omega t - \delta) = \frac{A^3}{4}\big[\cos(3\omega t - 3\delta) + 3\cos(\omega t - \delta)\big]$$

We arrive at

$$\boxed{x^3 = \frac{A^3}{4}\big[\cos(3\omega t - 3\delta) + 3\cos(\omega t - \delta)\big]}$$

Q2c

$\boxed{\text{No.}}$ $B \sim small^3$. By cubing, every instance of B is multiplied by either A or B some number of times by binomial theorem. Therefore, all terms contributed by this extra $+B\cos(3\omega t + \delta_B)$ term will at maximum be in order of $small^4$, which is less than our desired threshold of $small^3$.

Q2d

$$m\ddot{x} + b\dot{x} + k_1 x + k_3 x^3$$

$$x = A\cos(\omega t - \delta) + B\cos(3\omega t - \delta_B)$$

We take the derivatives

$$\dot{x} = -3Bw\sin\left(3wt - d\right) - Aw\sin\left(wt - d\right)$$

$$\ddot{x} = -9Bw^2\cos\left(3wt - d\right) - Aw^2\cos\left(wt - d\right)$$

From before, x^3 contributes nothing. $\dot{x}$ only includes sines. We take everything else independently. The terms including $\cos(3\omega t - \delta_B)$, are then only from $\ddot{x}, x$ terms:

$$\boxed{-9\omega^2 mB\cos(3\omega t - \delta_B) + k_1 B\cos(3\omega t - \delta_B)}$$

Where we ignore all the terms including sines and different harmonics,

Q2e

$$x = A\cos(\omega t - \delta) + B\cos(3\omega t - \delta_B)$$

We simply use the larger frequency, since the faster frequency is a multiple of the larger,

$$\boxed{T = \frac{2\pi}{\omega}}$$

Q2f

$\boxed{\text{Yes.}}$ 9ω and 3ω is also a multiple of ω.

Q2g

Use the smallest frequency, or $\frac{\omega}{2}$, we get

$$T = \frac{2\pi}{\omega/2} = \boxed{T = \frac{4\pi}{\omega}}$$

Q3

Q3a

$\boxed{\text{Yes, there exists SDIC}}$

We have that

$$\ln\left(\frac{\Delta\phi}{\Delta\phi_0}\right) \sim \lambda t$$

Applying equivalent definitions,

$$\ln\left(\frac{\delta_n}{\delta_0}\right) \sim \lambda t$$

From the graph, before the absolute difference between systems stabilizes, we observe an approximate slope of

$$\lambda t = \frac{8-0}{10-0} t \implies \lambda \sim 0.8$$

Since $\lambda > 0$, $\boxed{\text{there exists SDIC}}$

Q3b

$\boxed{\text{Yes.}}$ There is SDIC and the graph clearly exhibits chaos.

Q3c

$\boxed{\text{Yes. It seems to be approaching Feigenbaum's constant.}}$
We are given the values of r,

$$r_1 = 3, r_2 = 3.4495, r_3 = 3.5441, r_4 = 3.5644, r_5 = 3.5688, r_6 = 3.5697$$

And,

$$\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

We generate the values accordingly,

$$\delta_2 = 4.752, \delta_3 = 4.660, \ldots$$

We see that $\delta_3 = 4.660$. Given that Feigenbaum's constant is ≈ 4.6692, we observe an error of only

$$\left| \frac{4.660 - 4.6692}{4.6692} \right| = 0.001970358948 \implies 0.197\%$$

$\boxed{\text{Yes. It does seem to be approaching Feigenbaum's constant.}}$

Q3d

If SDIC and chaos exist, at least one of $\lambda_x > 0$ or $\lambda_y > 0$,
$\boxed{\text{At least one of } \lambda_x, \lambda_y \text{ is greater than } 0}$

Q3e

$\boxed{\text{Yes. It is possible}}$

Q3f

$\boxed{\text{Yes. It is possible}}$

Q4

Q4a

We begin with definition,

$$\lim_{r\to 0} N(r) \approx \frac{a}{r^D}$$

Taking the log of both sides,

$$\lim_{r\to 0} \ln(N(r)) \approx \ln(\frac{a}{r^D})$$

By log rules,

$$\boxed{\lim_{r\to 0}\ \ln(N) \approx \ln(a) + D\ln(1/r)}$$

$\ln(a)$ is just an offset, so we find that $\ln(N)$ is proportional to $\ln(1/r)$ by proportionality constant D. Plotted as a line, $\boxed{\text{D would be the slope}}$.

$$\lim_{r\to 0}\ \ln(N) \propto D\ln(1/r)$$

Q4b

Counting,

$$\boxed{\begin{cases} N(1) = 14 \\ N(1/2) = 36 \\ N(1/4) = 84 \end{cases}}$$

Q4c

Using the largest/smallest N values,

$$\ln(N(1/4)) - \ln(N(1))) \approx D(\ln(4) - \ln(1))$$

Simplifying,

$$\ln(84) - \ln(14) \approx 1.3863D$$

$$\boxed{D = 1.2925}$$

Q4d

From 12.30,

$$t = t_0, t_0 + 1, t_0 + 2, \ldots$$

where period was one unit of time. Letting an arbitrary period of time τ and an arbitrary start time t_s, the equivalent more general times are,

$$\boxed{t = t_s, t_s + \tau, t_s + 2\tau, \ldots}$$

Q4e

$\boxed{\text{Yes}}$

If the properties remain the same across blow-ups, we can define such a value. The statistical

properties of the poincare map remain constant, so we can therefore define a fractal dimension as we take the limit of smaller boxes.
Yes

Q4f

Per the book, it is a **strange attractor**.

$$\boxed{\text{Strange Attractor}}$$