PS 5

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Q1

a

Using hooke's law $m\ddot{x} = -kx$ and inspecting both sides of each mass,

$$\boxed{\begin{cases} m_1\ddot{x}_1 = -k_1(x_1 + L_{10} - L_{1R}) + k_2(x_2 + L_{20} - x_1 - L_{2R}) \\ m_2\ddot{x}_2 = k_3(L_{30} - x_2 - L_{3R}) - k_2(x_2 + L_{20} - x_1 - L_{2R}) \end{cases}}$$

b

Since $F = 0$ at $x_1 = x_2 = 0$, the above relation becomes

$$\begin{cases} 0 = -k_1(L_{10} - L_{1R}) + k_2(L_{20} - L_{2R}) \\ 0 = k_3(L_{30} - L_{3R}) - k_2(L_{20} - L_{2R}) \end{cases}$$

$$\begin{cases} k_1(L_{10} - L_{1R}) = k_2(L_{20} - L_{2R}) \\ k_2(L_{20} - L_{2R}) = k_3(L_{30} - L_{3R}) \end{cases}$$

Resubstituting back into the complete relationship,

$$\begin{cases} m_1\ddot{x}_1 = -k_1(x_1) - k_1(L_{10} - L_{1R}) + k_2(x_2 - x_1) + k_2(L_{20} - L_{2R}) \\ m_2\ddot{x}_2 = k_3(-x_2) + k_3(L_{30} - L_{3R}) - k_2(x_2 - x_1) - k_2(L_{20} - L_{2R}) \end{cases}$$

$$\begin{cases} m_1\ddot{x}_1 = -k_1(x_1) - k_1(L_{10} - L_{1R}) + k_2(x_2 - x_1) + k_1(L_{10} - L_{1R}) \\ m_2\ddot{x}_2 = k_3(-x_2) + k_2(L_{20} - L_{2R}) - k_2(x_2 - x_1) - k_2(L_{20} - L_{2R}) \end{cases}$$

$$\boxed{\begin{cases} m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) \end{cases}}$$

We arrive at the desired answer provided in the question.

c

We substitute $x_1 = x_{1r} + ix_{1i}$, $x_2 = x_{2r} + ix_{2i}$,

$$\begin{cases} m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) \end{cases}$$

Only the first equation is of interest. Substituting,

$$m_1(\ddot{x}_{1r} + \ddot{i}x_{1i}) = -k_1(x_{1r} + ix_{1i}) + k_2((x_{2r} + ix_{2i}) - (x_{1r} + ix_{1i}))$$

$$m_1(\ddot{x}_{1r} + \ddot{i}x_{1i}) = -k_1(x_{1r} + ix_{1i}) + k_2(x_{2r} + ix_{2i} - x_{1r} - ix_{1i})$$

Isolating real components,

$$\boxed{m_1\ddot{x}_{1r} = -k_1x_{1r} + k_2(x_{2r} - x_{1r})}$$

d

The ring has radial symmetry, and each atom is identical. Further, each atom only observes identical conditions. Therefore, Each atom should experience the same types and strengths of influences internally in the ring. From each atom's perspective, it cannot distinguish itself from its neighbors. Then, nature cannot justifiably designate any one distinct atom to have a higher potential/kinetic energy than the others. $\boxed{\text{Therefore, all atoms experience the same size of amplitude}}$.

e

At relaxed state, $x_n = 0$. When $k = 0 \implies \delta_n = 0$, the cosine in x_n is at is maximum value of 1, since $\cos 0 = 1$. Therefore, x_n is at its maximum amplitude when $k = 0$. $k = 2\pi/aN$ indicates a delay, meaning that the cosine will have a nonzero initial argument and will evaluate to a value slightly less than 1 in this instance. We define the relaxed state as $x_n = 0$ (distance from equilibrium). Then, $\boxed{k = 2\pi/aN \text{ has springs closer to relaxed length than } k = 0.}$ Further, for $k = 0$, we expect every $\delta_n = 0$ and every atom to therefore oscillate in unison. By comparison, $k = 2\pi/aN$ will have different initial delays for each atom, and further has every spring closer to its relaxed length, increasing frequency. This corresponds to the ring spinning. Therefore, $\boxed{k = 0 \text{ will have a lower frequency } \omega.}$

f

Recall that $\delta_n = kna$.

$$k = 2\pi/a \implies \delta_N = 2\pi n$$

.

However, x_n's is a cosinusoidal function in the form $x_n = A\cos(\omega t - \delta_n)$, meaning that it is 2π-periodic and adding any multiple of 2π to its argument will not affect its value. Since $\delta_N = 2\pi n$ is some multiple of 2π, the cosine argument is equivalent in value to when $\delta = 0$.

$$A\cos(\omega t) = A\cos(\omega t - 2\pi n) \implies \boxed{x_N = x_0}$$

g

$$x_n = Ae^{i(\omega t - kna)}, \quad M\ddot{x}_n = -G(x_n - x_{n-1}) + G(x_{n+1} - x_n)$$

Substituting,

$$M(i\omega)^2 x_n = -G(x_n - e^{ika}x_n) + G(e^{-ika}x_n - x_n)$$

Simplifying,

$$-M\omega^2 = G(e^{ika} + e^{-ika} - 2)$$

$$\omega^2 = -\frac{G}{M}(e^{ika} + e^{-ika} - 2)$$

Comparing the real components,

$$\omega^2 = -\frac{G}{M}(\cos(ka) + \cos(-ka) - 2)$$

Since cosine is even,

$$\omega^2 = -\frac{G}{M}(2\cos(ka) - 2)$$

Continuing to simplify,

$$\boxed{\omega^2 = \frac{2G}{M}(1 - \cos(ka))}$$

Q2

a

By hooke's law and inspection of the setup,

$$\boxed{\begin{cases} M\ddot{x}_1 = k(x_2 - x_1) \\ m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) \\ M\ddot{x}_3 = -k(x_3 - x_2) \end{cases}}$$

b

We guess solutions of the form, $x_n = A_n e^{i\omega t}$ and simplify,

$$\begin{cases} M\ddot{x}_1 = k(x_2 - x_1) \\ m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) \\ M\ddot{x}_3 = -k(x_3 - x_2) \end{cases}$$

$$\begin{cases} 0 = k(A_2 - A_1) + \omega_n^2 M A_1 \\ 0 = k(-2A_2 + A_1 + A_3) + \omega_n^2 m A_2 \\ 0 = -k(A_3 - A_2) + \omega_n^2 M A_3 \end{cases}$$

As a matrix,

$$\begin{bmatrix} -k + M\omega^2 & k & 0 \\ k & m\omega^2 - 2k & k \\ 0 & k & M\omega^2 - k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \vec{0}$$

Let $\omega'^2 = \frac{k}{m}, \quad \omega''^2 = \frac{k}{M}$

$$\begin{bmatrix} -\omega''^2 + \omega^2 & \omega''^2 & 0 \\ \omega'^2 & \omega^2 - 2\omega'^2 & \omega'^2 \\ 0 & \omega''^2 & \omega^2 - \omega''^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \vec{0}$$

To have a nontrivial solution, the determinant must be zero, since $\vec{A} \neq \vec{0}$

$$\begin{vmatrix} -\omega''^2 + \omega^2 & \omega''^2 & 0 \\ \omega'^2 & \omega^2 - 2\omega'^2 & \omega'^2 \\ 0 & \omega''^2 & \omega^2 - \omega''^2 \end{vmatrix} = (-\omega''^2 + \omega^2)((\omega^2 - 2\omega'^2)(\omega^2 - \omega''^2) - \omega'^2\omega''^2) - \omega'^2(\omega''^2)(\omega^2 - \omega''^2)$$

Simplifying,

$$0 = (-\omega''^2 + \omega^2)(\omega^2 - 2\omega'^2)(\omega^2 - \omega''^2) - \omega'^2\omega''^2(-\omega''^2 + \omega^2) - \omega'^2\omega''^2(\omega^2 - \omega''^2)$$

$$0 = (\omega^2 - \omega''^2)\left[(\omega^2 - 2\omega'^2)(\omega^2 - \omega''^2) - 2\omega'^2\omega''^2\right]$$

$$0 = (\omega^2 - \omega''^2)\left[\omega^2\omega^2 - \omega''^2\omega^2 - \omega^2 2\omega'^2 + \omega''^2 2\omega'^2 - 2\omega'^2\omega''^2\right]$$

$$0 = \omega^2(\omega^2 - \omega''^2)(\omega^2 - \omega''^2 - 2\omega'^2) \implies \boxed{\omega^2 = 0,\ \omega''^2,\ \omega''^2 + 2\omega'^2}$$

The normal mode frequencies are solutions to the last equation.

c

Substituting into our ansatz form,

$$x_1 = Ae^{i\omega t} = \boxed{x_1 = A}$$

d

We need the proportion with other amplitudes for this case. Substitute $\omega = 0$ into the above matrix,

$$\begin{bmatrix} -\omega''^2 & \omega''^2 & 0 \\ \omega'^2 & -2\omega'^2 & \omega'^2 \\ 0 & \omega''^2 & -\omega''^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \vec{0}$$

We find the corresponding relationships by solving rowwise,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This corresponds to $x_1 = x_2 = x_3 = A$. Then

$$\boxed{\frac{x_2}{x_1} = 1, \quad \frac{x_3}{x_1} = 1}$$

e

We repeat the above process for $\omega^2 = \omega''^2, \quad \omega''^2 + 2\omega'^2$
For $\omega = \omega''^2$, we substitute to get the below matrix and find by inspection,

$$\begin{bmatrix} 0 & \omega''^2 & 0 \\ \omega'^2 & \omega''^2 - 2\omega'^2 & \omega'^2 \\ 0 & \omega''^2 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \vec{0} \implies \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\omega^2 = \omega''^2 + 2\omega'^2$, we substitute to get the below matrix and find by inspection,

$$\begin{bmatrix} 2\omega'^2 & \omega''^2 & 0 \\ \omega'^2 & \omega''^2 & \omega'^2 \\ 0 & \omega''^2 & 2\omega'^2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \vec{0} \implies \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \propto \begin{pmatrix} \omega''^2 \\ -2\omega'^2 \\ \omega''^2 \end{pmatrix}$$

Using these ratios, we find

$$\omega = \omega''^2 \implies \boxed{\frac{B_2}{B_1} = 0, \quad \frac{B_3}{B_1} = -1}$$

$$\omega = \omega''^2 + 2\omega'^2 \implies \boxed{\frac{B_2}{B_1} = -\frac{2\omega'^2}{\omega''^2}, \quad \frac{B_3}{B_1} = 1}$$

f

We observe both ends have opposite motions in symmmetric stretching, suggesting negative amplitudes to each other. Inspecting the previous result, we see $\frac{B_3}{B_1} = -1$ for $\omega = \omega''^2$. Therefore,
$\boxed{\text{symmetric stretching corresponds to } \omega = \omega''^2 \text{ mode}}$.

We observe both ends have equivalent motions in antisymmmetric stretching, suggesting equal amplitudes. Inspecting the previous result, we see $\frac{B_3}{B_1} = 1$ for $\omega = \omega''^2 + 2\omega'^2$. Therefore,
$\boxed{\text{antisymmetric stretching corresponds to } \omega = \omega''^2 + 2\omega'^2 \text{ mode}}$.

g

A general solution involves a linear combination of these modes.

$$\boxed{\begin{cases} x_1 = A + B\cos(\omega'' t + \delta_b) + C\omega''^2 \cos(\sqrt{\omega''^2 + 2\omega'^2} t + \delta_c) \\ x_2 = A - 2C\omega'^2 \cos(\sqrt{\omega''^2 + 2\omega'^2} t + \delta_c) \\ x_3 = A - B\cos(\omega'' t + \delta_b) + C\omega''^2 \cos(\sqrt{\omega''^2 + 2\omega'^2} t + \delta_c) \end{cases}}$$

Q3

a

Converting the equations of motion into matrices,

$$\begin{bmatrix} -2C & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} y \\ \phi \end{bmatrix} = \begin{bmatrix} 2\ddot{y} + \ddot{\phi} \\ \ddot{y} + \ddot{\phi} \end{bmatrix}$$

Using an ansatz of $y = Ae^{i\omega t}$ and $\phi = Be^{i\omega t}$, we seek the solutions as before by solving for zero and finding determinant,

$$\begin{bmatrix} 2\omega^2 - 2C & \omega^2 \\ \omega^2 & \omega^2 - C \end{bmatrix} \begin{bmatrix} y \\ \phi \end{bmatrix} = \vec{0} \implies \begin{vmatrix} 2\omega^2 - 2C & \omega^2 \\ \omega^2 & \omega^2 - C \end{vmatrix} = (2\omega^2 - 2C)(\omega^2 - C) - \omega^4$$

Completing the square,

$$\omega^2 = 2C \pm \sqrt{2}C \implies \boxed{\omega^2 = 2C + \sqrt{2}C, \quad 2C - \sqrt{2}C}$$

b

We find normal modes again using the smaller $\omega^2 = 2C - \sqrt{2}C$. Substituting,

$$\begin{bmatrix} 2(2C-\sqrt{2}C)-2C & 2C-\sqrt{2}C \\ 2C-\sqrt{2}C & 2C-\sqrt{2}C-C \end{bmatrix} = \begin{bmatrix} 2(2C-\sqrt{2}C)-2C & 2C-\sqrt{2}C \\ 2C-\sqrt{2}C & 2C-\sqrt{2}C-C \end{bmatrix} = \begin{bmatrix} 2(1-\sqrt{2})C & (2-\sqrt{2})C \\ (2-\sqrt{2})C & (1-\sqrt{2})C \end{bmatrix}$$

Through some algebra and inspection, we find the null space as,

$$\implies \begin{bmatrix} y \\ \phi \end{bmatrix} = \begin{bmatrix} 2-\sqrt{2} \\ 2\sqrt{2}-2 \end{bmatrix}$$

This gives a ratio of amplitudes, and given y, we find ϕ as

$$\boxed{\phi = \frac{2\sqrt{2}-2}{2-\sqrt{2}} A_L \cos(\omega_L t - \delta_L)}$$

c

We find the other mode by same method as above. For $\omega^2 = 2C + \sqrt{2}C$,

$$\begin{bmatrix} 2\omega^2 - 2C & \omega^2 \\ \omega^2 & \omega^2 - C \end{bmatrix} \begin{bmatrix} y \\ \phi \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 2(2+\sqrt{2})C-2C & (2+\sqrt{2})C \\ (2+\sqrt{2})C & (2+\sqrt{2})C-C \end{bmatrix} = \begin{bmatrix} (2+2\sqrt{2})C & (2+\sqrt{2})C \\ (2+\sqrt{2})C & (1+\sqrt{2})C \end{bmatrix}$$

By some algebra and inspection, null space is,

$$\implies \begin{bmatrix} y \\ \phi \end{bmatrix} = \begin{bmatrix} -2-\sqrt{2} \\ 2+2\sqrt{2} \end{bmatrix}$$

Combining this with the previous result as a linear combination, we arrive at the general equation,

$$\boxed{\begin{cases} y = (2-\sqrt{2})A\cos(\omega_A t + \delta_A) + (-2-\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \phi = (2\sqrt{2}-2)A\cos(\omega_A t + \delta_A) + (2+2\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \omega_A = \sqrt{2C - \sqrt{2C}} \\ \omega_B = \sqrt{2C + \sqrt{2C}} \end{cases}}$$

Some minor further simplification can be done, but I leave it as is, so expressions are clearly sourced.

d

We now use $\nu_1 = \sqrt{2}y + \phi$. We can use come clever inspection,

$$\begin{cases} 2\ddot{y} + \ddot{\phi} = -2Cy \\ \ddot{y} + \\ dot\phi = -C\phi \end{cases}$$

Dividing first line by $\sqrt{2}$

$$\begin{cases} \sqrt{2}\ddot{y} + \frac{1}{\sqrt{2}}\ddot{\phi} = -\sqrt{2}Cy \\ \ddot{y} + \ddot{\phi} = -C\phi \end{cases}$$

Adding the two together,

$$(\sqrt{2}\ddot{y} + \ddot{\phi}) + (\ddot{y} + \frac{1}{\sqrt{2}}\ddot{\phi}) = -C(\sqrt{2}y + \phi)$$

$$\ddot{\eta} + \frac{\ddot{\eta}}{\sqrt{2}} = -C\eta$$

$$\ddot{\eta} = -\frac{C}{(1+\frac{1}{\sqrt{2}})}\eta = -C(2-\sqrt{2})\eta$$

Using the form $\ddot{\eta}_1 = -\omega^2\eta_1$, we compare and see that

$$\boxed{\omega^2 = C(2-\sqrt{2}) \implies \text{matches above result}}$$

e

From before,

$$\begin{cases} y = (2-\sqrt{2})A\cos(\omega_A t + \delta_A) + (-2-\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \phi = (2\sqrt{2}-2)A\cos(\omega_A t + \delta_A) + (2+2\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \omega_A = \sqrt{2C - \sqrt{2}C} \\ \omega_B = \sqrt{2C + \sqrt{2}C} \end{cases}$$

Then, we evaluate $\sqrt{2}y + \phi$

$\sqrt{2}y+\phi = \sqrt{2}\big[(2-\sqrt{2})A\cos(\omega_A t+\delta_A)+(-2-\sqrt{2})B\cos(\omega_B t+\delta_B)\big]+(2\sqrt{2}-2)A\cos(\omega_A t+\delta_A)+(2+2\sqrt{2})B\cos(\omega_B t+\delta_B)$

Simplifying and canceling like terms,

$$\sqrt{2}y + \phi = (2\sqrt{2}-2)A\cos(\omega_A t + \delta_A) + (2\sqrt{2}-2)A\cos(\omega_A t + \delta_A)$$

$$\sqrt{2}y + \phi = (\sqrt{2}-1)4A\cos(\omega_A t + \delta_A)$$

This result oscillates at ω_A. Recall that $\omega_A^2 = 2C - \sqrt{2}C$, which matches the ω^2 result found above in d. Therefore, $\boxed{\text{Matches. Oscillates at } \omega_A = \sqrt{2C - \sqrt{2}C}}$.

f

We again recall but with additional constraints,

$$\begin{cases} y = (2-\sqrt{2})A\cos(\omega_A t + \delta_A) + (-2-\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \phi = (2\sqrt{2}-2)A\cos(\omega_A t + \delta_A) + (2+2\sqrt{2})B\cos(\omega_B t + \delta_B) \\ \omega_A = \sqrt{2C - \sqrt{2}C} \\ \omega_B = \sqrt{2C + \sqrt{2}C} \\ y(0) = \dot{y}(0) = \dot{\phi}(0) = 0 \\ \phi(0) = \phi_0 \end{cases}$$

Applying boundary conditions,

$$y(0) = 0 \implies (2+\sqrt{2})B = (2-\sqrt{2})A$$

$$\phi(0) = \phi_0 \implies \phi_0 = (2\sqrt{2}-2)A + (2+2\sqrt{2})B$$

Solving for $A,\ B$, we find,

$$(2+\sqrt{2})B = (2-\sqrt{2})A \implies (2+2\sqrt{2})B = (2\sqrt{2}-2)A$$

$$\phi_0 = (2\sqrt{2}-2)A + (2+2\sqrt{2})B$$

$$\begin{cases} A = \frac{\phi_0}{4(1+\sqrt{2})} \\ B = \frac{\phi_0}{4(\sqrt{2}-1)} \end{cases}$$

Applying velocity boundary conditions, and letting ξ represent various unimportant coefficients,

$$\dot{y}(0) = 0 \implies 0 = \xi\sin(\delta_A) + \xi\sin(\delta_B)$$

$$\dot{\phi}(0) = 0 \implies 0 = \xi\sin(\delta_A) + \xi\sin(\delta_B)$$

Clearly, this requires the argument of sin to be zero,

$$\delta_A = \delta_B = 0$$

We collect the particular solution,

$$\boxed{\begin{cases} y = \frac{\phi_0(2-\sqrt{2})}{4(1+\sqrt{2})}\cos(\omega_A t) + \frac{\phi_0(-2-\sqrt{2})}{4(\sqrt{2}-1)}\cos(\omega_B t) \\ \phi = \frac{\phi_0(2\sqrt{2}-2)}{4(1+\sqrt{2})}\cos(\omega_A t) + \frac{\phi_0(2+2\sqrt{2})}{4(\sqrt{2}-1)}\cos(\omega_B t) \\ \omega_A = \sqrt{2C - \sqrt{2}C} \\ \omega_B = \sqrt{2C + \sqrt{2}C} \end{cases}}$$

Again, some minor further simplification can be done, but I leave it as is, so expressions are clearly sourced.

g

$\boxed{\text{No. Motion will never repeat itself.}}$ Find the ratio between frequencies, supposing $C = 1$:

$$\frac{\omega_A}{\omega_B} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{2} - 1$$

Clearly, $\sqrt{2} - 1$ is irrational. By Taylor, when the ratio of frequencies is irrational, the motion is quasiperiodic and never repeats itself. That is because the ratio between the two corresponding frequencies must also be irrational, and therefore do not have a least common multiple.

Q4

a

Substituting,

$$\beta = \omega \implies x = ye^{-\omega_0 t}, \quad \dot{x} = y'e^{-\omega_0 t} - \omega_0 y e^{-\omega_0 t}, \quad \ddot{x} = y''e^{-\omega_0 t} - 2\omega_0 y' e^{-\omega_0 t} + \omega_0^2 y e^{-\omega_0 t}$$

Substituting,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \implies y''e^{-\omega_0 t} - 2\omega_0 y'e^{-\omega_0 t} + \omega_0^2 y e^{-\omega_0 t} + 2\omega_0(y'e^{-\omega_0 t} - \omega_0 y e^{-\omega_0 t}) + \omega_0^2(ye^{-\omega_0 t}) = 0$$

Canceling,

$$\boxed{\ddot{y}e^{-\omega_0 t} = 0}$$

We try the ansatz $y = A + Bt$.

$$\ddot{y} = 0 \implies 0 = 0$$

$\boxed{\text{Yes } y = A + Bt \text{ fits the differential equation.}}$

b

Recall that $\omega^2 = \omega_0^2 - \beta^2$. Then assuming $\beta = 0$,

$$T = 1/f = 2\pi/\omega$$

$$\beta = 0, \quad T = 1 \implies \omega_0 = 2\pi$$

Now assuming nonzero damping,

$$T = 1.001, \omega_0 = 2\pi \implies T/1/f = 2\pi/\sqrt{\omega_0^2 - \beta^2} = 2\pi/\sqrt{(2\pi)^2 - \beta^2} = 1.001$$

$$\implies \boxed{\beta = 0.281}$$

After 10 cycles, $10(1.001) = 10.01$ seconds have passed. Amplitude A follows,

$$A \propto e^{-\beta t}$$

After 10 cycles, it has decayed to $e^{-\beta(10.01)} = 0.06$ The amplitude will $\boxed{\text{decrease to 0.06 of its original size}}$ over 10 cycles. $\boxed{\text{The amplitude change would be much more noticeable.}}$ A difference of period of 0.001s would be almost imperceptible, while falling to 6% of the original amplitude over 10 seconds would be very noticeable.

c

$\boxed{\text{Yes. } \beta > \omega_0}$

$\boxed{\text{Yes. } e^{-\beta t} \text{ dominates for high } t \text{ and } x \to 0}$

d

$\boxed{\text{Yes}}$. Derivatives are linear operators, so their linear combination will yield a valid solution, as the 0 term will not affect the result.

e

$\boxed{\text{Yes}}$. The real Re operator is also linear. Further, the derivative of a real function cannot be imaginary, so since $\forall_t x(t), y(t) \in \mathbb{R}$, we can isolate the real part from the imaginary part safely.

f

Clearly, there's an equilibrium at $x = 0$. Treating y as constant along this axis, we find force and find hooke's law relation:

$$-\frac{dU}{dx} = Bx = M\ddot{x}$$
$$\implies \omega_x^2 = \frac{B}{M}$$
$$\implies \boxed{\omega_x = \sqrt{\frac{B}{M}}}$$