Problem set 10

Applied & Engineering Physics 3330

Due Thursday Nov.2 at 6pm.

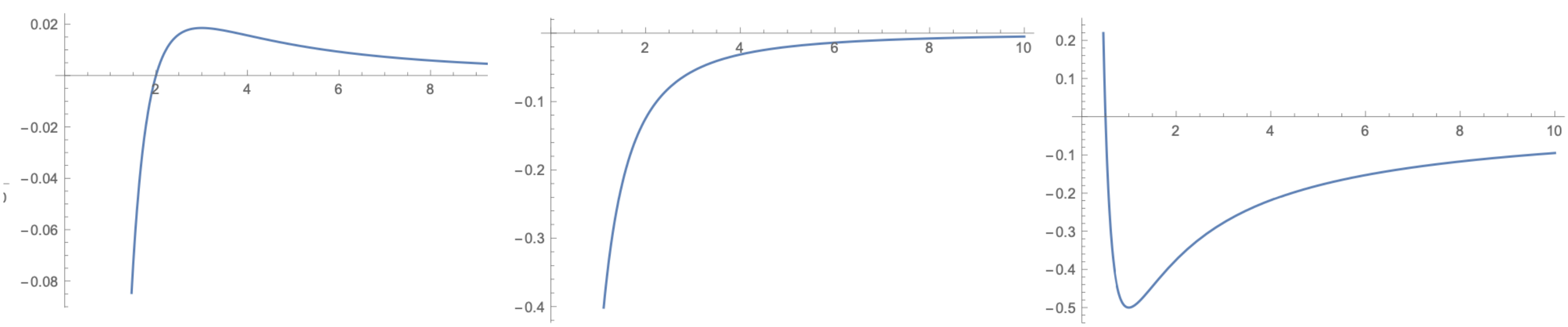
Designed to go with worksheets to make a learning package.   Remember to explain your answers. Reading: In Taylor, review lecture material from the start of chapter 8 through the end of section 8.4, but don't get bogged down in terms with $\vec{R}$ (position of COM) because we set that equal to the origin to get an inertial frame. Then you can skip to the start of "Bounded orbits" on page 309 and start reading again there because you derived the gravitational orbits a different way last week. Read from page 309 to the end of section 8.7, but before reading about Kepler's 3rd law go back and read section 3.4 on Kepler's second law. Please also read sections 12.1 and 12.2 of Taylor, and then on page 569 start reading at the "Solid Angle" subheading up to equation 14.15.

Problem 1: In lecture, we found that if we put our origin at the center of mass of an isolated 2 particle system, we could rewrite the kinetic energy by defining $\vec{r} = \vec{r}_1 - \vec{r}_2$ so $T = \frac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2}m_2\left|\dot{\vec{r}}_2\right|^2 = \frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2$, where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the same formula for reduced mass you discovered in hw8. Consider a two particle system with reduced mass μ that interacts through a central force with potential $U = k\,r^n$, where $k\,n > 0$. We are discovering that $\vec{r}$ always stays in the same plane, so we can write the Lagrangian with polar coordinates: $\mathcal{L} = \frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2 - k\,r^n = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - k\,r^n$.

a) Find the 2 Euler-Lagrange equations for this case. One just tells you that angular momentum is constant, and AFTER you have the E-L equations you may use this fact to find an equation for $\ddot{r}$ that is ONLY a function of r and constants and nothing else. Do that.

b) As a function of the size of angular momentum, find a value of r that is an equilibrium value, where the separation can stay at that constant value r if $\dot{r} = 0$. (This should look familiar. Ponder whether you see $\frac{dU_{eff}}{dr}$ in your equation – I discuss this on a Monday slide.)

c) Examine $\ddot{r}$ for a case where $r = r_{eq} + \varepsilon$ and ε is small. For what values of n is the circular orbit case stable to small perturbations in separation? Remember when we had a general stability analysis after we got an Euler-Lagrange equataion into a form $\ddot{q} = function\ (q\ and\ constants)$? Do the normal math calculation, but here are some pictures of effective potentials $U_{eff}\ (defined\ on\ sheet\ IX)$ that could help with a tricky case.



(b) $n = -3$ (c) $n = -2$ (d) $n = -1$

d) For the stable cases, find the period of small radial oscillations about the equilibrium r (separation) value, which I will call τ_{osc}. This is the time between moments when the masses are furthest apart (or moments when they are closest together) for the nearly circular orbit case. Once your answer uses given quantities you are not required to simplify.

e) For an orbit that's so close to perfect circle you can plug in r_{eq}, what would the orbital period τ_{orb} (the time it takes each mass to go around the COM once) be as a function angular momentum ℓ? [Hint: we found $\dot{\phi}$ as a function of ℓ– how long does it take to go 2π? Do NOT use the Kepler result because that is only for gravity!]

f) I will not make you check this, but it turns out $\tau_{osc} = \tau_{orb} / \sqrt{n+2}$. Give 3 values of single digit n that would make the motion repeat by starting one of the radial oscillations at the same point every time on the approximately circular orbit. Negative and positive values ok. This is a bit like avoiding quasiperiodic behavior.

Problem 2: This uses our fuel-efficient method to plan a NASA mission. The first thing our spacecraft does is blast off and move far enough away from Earth that we can approximate that spacecraft and the sun (mass M_S) as an isolated 2 particle system, and thus ignore the mass of the Earth. Throughout this problem you may use the fact that the sun is much heavier than the spacecraft – review useful approximations at end of worksheet IX for this case.

a) If we approximate the Earth's orbit as a perfect circle of radius r_1, what would be the energy of the mass m spacecraft inhabiting Earth's orbit around the sun if we use a coordinate system with origin at the COM of the system (which is very accurately the center of the sun). Hint: a circle is an ellipse with 2a=2 (radius)

b) What would be the speed v_1 of the spacecraft in that starting orbit?

c) If the craft burns its engine briefly to switch to an elliptic orbit that will (after half a period) reach the same distance from the Sun that Venus has (called r_2 here), what would be the energy of the craft in this elliptic transfer orbit (again in the sun's reference frame)?

d) Would it be most fuel efficient for the burn to produce a change in craft velocity $\Delta\vec{v}$ that is parallel, antiparallel, or at some other angle to the original craft velocity before burn ($\vec{v}_1$). EXPLAIN your answer using an expansion of the VECTOR expression $|\vec{v}_1 + \Delta\vec{v}|^2$.

e) For this advantageous situation, find a formula for the size of $|\Delta\vec{v}|$ the burn would have to produce to get the craft into the desired elliptic transfer orbit. It is probably easiest to first calculate the desired speed JUST after the BRIEF burn at its first moment in the elliptic orbit (when it is roughly at the same distance from the sun as before the BRIEF burn).

f) How long should the craft coast in the transfer orbit, as a function of given quantities?

g) If you want to send the craft to Venus, should you plan for Venus to be near the spot where the transfer orbit reaches the orbit of Venus when the craft gets to that point? Y/N

h) Would you have to burn the engine again to roughly match the velocity of Venus if you want the craft to be near Venus for longer than a brief flyby (assuming we've ruled out an immediate collision)? Y/N

There are alternate ways to plan a mission to close objects, and if you want to plan a mission to the outer solar system, optionally Google "gravitational assist" to learn about how to steal a bit of momentum from Jupiter. It is so heavy its velocity does not have an observable change, even if the equal sized momentum given to the ship increases spacecraft speed significantly.

Problem 3: Ignore the motion of the Earth around the sun in this problem!!! Consider a rocket that has blasted off Earth and just turned off its rockets.

a) Assume at the moment engines shut off, a rocket is a distance 250 km=250,000m above the GROUND and its speed with respect to the center of the Earth is 7,900 m/s, and that we approximate the radius of the Earth as 6,371km=6,371,000m everywhere, so we model Earth as a perfect sphere. The rocket was going due East at its shutoff moment, so its velocity is perpendicular to its position vector with respect to the center of the Earth. What is the energy of the rocket with respect to an origin at the Earth's COM?

b) What is the a parameter for the rocket's orbit?

c) How long would it take for the rocket to complete one orbit if its engines stay off? (Answer this from the point of view of someone floating in space and not moving with respect to the center of the Earth, ignoring the rotation of the Earth. Hint: can you make a big mass approximation for the Earth?)

d) Give the rocket's closest distance to the center of the Earth and its furthest distance from the center of the Earth on its orbit, starting at the moment of shutoff. (Hint: is one the engine shutoff distance to center of Earth? If so, justify that assumption. You know 2a for this problem!)

e) What is the angular momentum of the rocket's orbit with respect to origin at the COM? [Ponder whether you know enough to get c and ϵ in this specific rocket problem now, but they aren't needed to answer the next part. Ponder whether you could get the perpendicular component of velocity $r\dot{\theta}$ at some other distance from COM on the orbit, using the constant angular momentum.]

f) Now I'll tell you at the engine shutoff moment (when the rocket was 250km above the ground) happened at a point that is at the latitude of Cape Canaveral (28.5 degrees north). What is the angle between the plane of the rocket's orbit (whose center you know) and the plane that contains the equator of the Earth? (easy)

Ponder how to discover how much faster THAN THE GROUND BELOW IT the rocket was going at the shutoff moment. (Remember its total velocity with respect to the Earth's center was Eastward & we pretend Earth's center stays in the same place in this problem. If you know distance from axis of rotation of a spot on the ground and you know the Earth's angular velocity (about 2pi/day) you can figure out the velocity of a spot on the ground with respect to Earth center. Think whether a spot on the ground travels toward the sunrise direction.