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Q1

Part A

We begin and add $m_1r_2 - m_1r_2$ to left side,

$$m_1r_1 + m_2r_2 = 0$$

$$m_1r_1 - m_1r_2 + (m_1+m_2)r_2 = 0 \implies r_2 = -\frac{m_1 r}{m_1+m_2}$$

By symmetry,

$$\boxed{\vec{r_1} = \frac{m_2\vec{r}}{m_1+m_2}, \quad \vec{r_2} = -\frac{m_1\vec{r}}{m_1+m_2}}$$

Part B

We begin,

$$l = r_1 \times m_1\dot{r}_1 + r_2 \times m_2\dot{r}_2$$

$$l = \frac{m_2\vec{r}}{m_1+m_2} \times \frac{m_1m_2\dot{\vec{r}}}{m_1+m_2} - \frac{m_1\vec{r}}{m_1+m_2} \times -\frac{m_1m_2\dot{\vec{r}}}{m_1+m_2}$$

Substituting reduced mass μ,

$$l = \frac{m_2\vec{r}}{m_1+m_2} \times \mu\dot{\vec{r}} + \frac{m_1\vec{r}}{m_1+m_2} \times \mu\dot{\vec{r}} = \frac{(m_1+m_2)\vec{r}}{m_1+m_2} \times \mu\dot{\vec{r}}$$

$$= \boxed{\vec{\ell} = \vec{r} \times \mu\dot{\vec{r}}}$$

This is clearly displacement cross momentum, as expected.

Part C

Referencing b, we have displacement cross momentum, where momentum is $\mu\dot{r}$. Inspecting our momentum term, we clearly see that μ is the one-particle mass analogue,

$$\boxed{\mu, \text{the reduced mass}}$$

Part D

We begin,

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Recall that $\vec{l} = r\hat{r} \times \mu\dot{\vec{r}}$. We arrive at $\dot{\theta}$ in terms of ℓ,

$$\frac{l}{\mu r} = r|\dot{\theta}| \implies |\dot{\theta}| = \frac{\ell}{\mu r^2} \implies \boxed{\dot{\theta} = \frac{\pm\ell}{\mu r^2}}$$

Q2

Part A

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

Rewriting,

$$\boxed{\dot{r} = \sqrt{\frac{2(E - U(r))}{\mu} - \frac{\ell^2}{\mu^2 r^2}}}$$

Part B

From a and d,

$$\dot{\theta} = \pm\frac{\ell}{\mu r^2}, \quad \dot{r} = \sqrt{\frac{2(E - U(r))}{\mu} - \frac{\ell^2}{\mu^2 r^2}}$$

Now, combining into the given equation in b,

$$d\theta = \frac{\dot{\theta}}{\dot{r}}dr = \frac{\pm\frac{\ell}{\mu r^2}}{\sqrt{\frac{2(E-U(r))}{\mu} - \frac{\ell^2}{\mu^2 r^2}}}dr = \frac{\pm\frac{\ell}{r^2}}{\sqrt{2\mu^2\left[\frac{E-U(r)}{\mu} - \frac{\ell^2}{2\mu^2 r^2}\right]}}dr$$

As desired, we arrive at

$$\implies \boxed{\theta = \int \frac{\pm\frac{\ell}{r^2}}{\sqrt{2\mu\left[E - U(r) - \frac{\ell^2}{2\mu r^2}\right]}}dr}$$

Part C

We replace U,

$$\theta = \int \frac{\pm\frac{\ell}{r^2}}{\sqrt{2\mu\left[E - U(r) - \frac{\ell^2}{2\mu r^2}\right]}}dr = \int \frac{\frac{\ell}{r^2}}{\sqrt{2\mu\left[E + k/r - \frac{\ell^2}{2\mu r^2}\right]}}dr$$

Note that, $u = l/r \implies du = -l/r^2 dr$. Making this sub and absorbing the numerator into du,

$$\theta = \int \frac{-du}{\sqrt{2\mu\left[E + ku/l - \frac{u^2}{2\mu}\right]}} = \boxed{\theta = \int \frac{-du}{\sqrt{2\mu E + 2\mu k u/l - u^2}}}$$

Part D

We begin,

$$\theta = \int \frac{-du}{\sqrt{2\mu E + 2\mu k u/l - u^2}}$$

Applying the arcsin formula,

$$\int \frac{du}{\sqrt{au^2+bu+c}} = -\frac{1}{\sqrt{-a}}\arcsin(\frac{2au+b}{\sqrt{b^2-4ac}})$$

$$\theta = \int \frac{-du}{\sqrt{2\mu E + 2\mu k u/l - u^2}} = \arcsin(\frac{-2u+2\mu k/l}{\sqrt{(2\mu k/l)^2 + 4(2\mu E)}}) + C_1$$

Reorganizing terms, we arrive as desired

$$\boxed{\theta + C = \arcsin(\frac{-2u+2\mu k/l}{\sqrt{(2\mu k/l)^2 + 8\mu E}})}$$

Part E

$\boxed{\text{Yes}}$.
$\sin(\theta - \pi/2) = -\cos(\theta) \implies \theta - \pi/2 = \arcsin(\xi) \implies \theta = -\arccos(\xi)$.
To adjust for $\arccos$ limits, we can add a factor of π,

$$\boxed{\theta = \pi - \arccos(\frac{-2u+2\mu k/l}{\sqrt{(2\mu k/l)^2 + 8\mu E}})}$$

Part F

Letting $\epsilon \equiv \sqrt{1+\frac{2El^2}{\mu k^2}}, \quad c \equiv \frac{l^2}{\mu k}, \quad u = l/r$, starting from above,

$$\pi - \theta = \arccos(\frac{-2u+2\mu k/l}{\sqrt{(2\mu k/l)^2 + 8\mu E}}) \implies \cos(\pi-\theta) = -\cos(\theta) = \frac{-ul/\mu k + 1}{\epsilon} = \frac{-c/r+1}{\epsilon}$$

$$-\cos(\theta) = \frac{-c/r+1}{\epsilon} \implies \boxed{r = \frac{c}{1+\epsilon\cos(\theta)}}$$

Part G

$$r = \frac{c}{1+\epsilon\cos(\theta)}$$

Clearly minimized when cosine is zero, or $\boxed{\theta = \pi/2}$

Q3

Part A

$\boxed{\text{Momentum}}$

Part B

Zero when different indices, one when same.

$$\boxed{\frac{\partial \dot{q}_k}{\partial \dot{q}_i} = \delta_{ik}}$$

Part C

$$\frac{dy_a}{dt} = \sum_j \frac{dy_a}{dq_j}\dot{q}_j$$

Taking the derivative wrt $\dot{q}_i$,

$$\frac{d\dot{y}_a}{d\dot{q}_i} = \sum_j \frac{d}{d\dot{q}_i}(\frac{dy_a}{dq_j}\dot{q}_j)$$

Since y_a has no time derivative dependence,

$$\frac{d\dot{y}_a}{d\dot{q}_i} = \sum_j \frac{d\dot{q}_j}{d\dot{q}_i}\frac{dy_a}{dq_j} = \frac{dy_a}{dq_j}\delta_{ij}$$

$$\boxed{\frac{d\dot{y}_a}{d\dot{q}_j} = \frac{dy_a}{dq_j}}$$

Part D

Substitute $-\nabla U = F_{tot} - F_c$. Then, constraint is holonomic. Then, follows newton's second.

$$\delta S = \int_{t_1}^{t_2}(-m\ddot{\vec{r}} - \vec{\nabla}U)\cdot\delta\vec{r}dt = \int_{t_1}^{t_2}(-m\ddot{\vec{r}} + F_{tot} - F_c)\cdot\delta\vec{r}dt$$

$$F_c \cdot \delta r = 0 \implies \delta S = \int_{t_1}^{t_2}(-m\ddot{\vec{r}} + F_{tot})\cdot\delta\vec{r}dt$$

$$m\ddot{\vec{r}} = F_{tot} \implies \delta S = \int_{t_1}^{t_2}(-F_{tot} + F_{tot})\cdot\delta\vec{r}dt = \delta S = 0$$

$$\boxed{\delta S = 0}$$

Part E

$\boxed{\text{Yes}}$ they would be smooshing in momentum axis p_x.

Part F

$\boxed{\text{Yes}}$. Like mixing colors, points may end up with different sets of neighbors.

Q4

Part A

$\boxed{\text{No. } H \neq T + U}$. The accelerating pivot introduces energy into the system and therefore the Hamiltonian is not just the sum kinetic energy and potential energy. It must also include terms that incorporate the motion of the pivot.

Part B

Recall the Lagrangian found earlier.

$$\mathcal{L} = \frac{M}{2}(a^2t^2 + atL\cos(\theta)\dot{\theta} + \frac{L^2\dot{\theta}^2}{3}) + MgL\cos(\theta)/2$$

Finding the momenta,

$$p_\theta = \frac{d\mathcal{L}}{d\dot{\theta}} = \frac{MatL\cos(\theta)}{2} + \frac{ML^2\dot{\theta}}{3}$$

Then,

$$H = p_\theta\dot{\theta} - \mathcal{L}$$

$$\boxed{\mathcal{H} = \frac{MatL\cos(\theta)\dot{\theta}}{2} + \frac{ML^2\dot{\theta}^2}{3} - (\frac{M}{2}(a^2t^2 + atL\cos(\theta)\dot{\theta} + \frac{L^2\dot{\theta}^2}{3}) + MgL\cos(\theta)/2)}$$

Q5

Part A

We find the Hamiltonian. While we can first find the lagrangian and then derive the Hamiltonian, we can more easily consider the hamiltonian as the total energy of the inertial system

$$T = \frac{1}{2}mv^2, \quad U = mgr\cos\theta$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) - mgr\cos(\theta)$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\sin^2(\theta)\dot{\phi}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

$$p_r = \frac{\partial L}{\partial\dot{r}} = m\dot{r}$$

$$H = \frac{1}{2}mv^2 + mgr\cos(\theta) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) + mgr\cos(\theta)$$

Substituting our momenta identities,

$$\boxed{\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + mgr\cos(\theta)}$$

Part B

$$\frac{\partial H}{\partial r} = -\dot{p}_r, \quad \frac{\partial H}{\partial\theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial\phi} = -\dot{p}_\phi$$

$$\frac{\partial H}{\partial p_r} = \dot{r}, \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta}, \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi}$$

$$\boxed{\dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2(\theta)} - mg\cos\theta, \quad \dot{r} = \frac{p_r}{m}}$$

$$\boxed{\dot{p}_\phi = 0, \quad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}}$$

$$\boxed{\dot{p}_\theta = mgr\sin\theta + \frac{p_\phi^2}{mr^2\sin^2\theta}\cot\theta, \quad \dot{\theta} = \frac{p_\phi}{mr^2\sin^2\theta}}$$

The ϕ equations are equivalent to the ones ones written for the lagrangrian. (Conservation of ϕ angular momentum).

fix

fix

fix

fix

fix

Part C

$\boxed{\text{There is rotational symmetry in the } \phi \text{ direction}}$.
p_ϕ must be constant since time derivative is zero.

Part D

We desire ϕ direction momentum, so
$\boxed{p_\phi = mr^2\dot{\theta}}$

Part E

$$p_\theta = mr^2\dot{\theta} \implies \dot{p}_\theta = mr^2\ddot{\theta}$$

$$\dot{p}_\theta = mgr\sin\theta + \frac{p_\phi^2}{mr^2\sin^2\theta}\cot\theta$$

$$\boxed{mr^2\ddot{\theta} = mgr\sin\theta + \frac{p_\phi^2}{mr^2\sin^2\theta}\cot\theta}$$