Problem set 9

Applied & Engineering Physics 3330

Due 6:00pm, Thursday Oct. 26, 2022 as Canvas upload

Remember to explain your answers!

Reading: After you've finished last week's reading in chapter 7, read from the start of chapter13 to the end of section 13.3 in Taylor. Please also read from the start of section 13.6 to the start of example 13.5 (which covers something you've seen) and the first 2 pages of section 13.7. Note that you should be able to start this problem set using knowledge you have at the start of this week, so no need to wait. Please ask if there is any uncertainty about a question – I am hoping many parts below are pretty easy.

Problem 1: a) We will soon be considering two particles using a reference frame whose origin is chosen to be at their center of mass, so origin at $\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$. This means $m_1\vec{r}_1+m_2\vec{r}_2=0$. (We'll be working with problems where the center of mass is not accelerated because there is no external force, so the COM can be used as the origin of an inertial frame.) We'll want to write our problem in terms of the relative vector between the two masses: $\vec{r}=\vec{r}_1-\vec{r}_2$. Here's a warmup exercise that will help you to understand something that has confused students in the past. If we enforce $m_1\vec{r}_1+m_2\vec{r}_2=0$, what is $\vec{r}_1$ in terms of $\vec{r}$? What is $\vec{r}_2$ in terms of $\vec{r}$? (easy)

b) Start with the total angular momentum for both particles $\vec{r}_1\times m_1\dot{\vec{r}}_1+\vec{r}_2\times m_2\dot{\vec{r}}_2=\vec{\ell}$. Now assume that we are using the center of mass as origin. Use (a) to rewrite this total angular momentum in terms of $\vec{r}$, $\dot{\vec{r}}$, and a thing we'll call reduced mass $\mu=\frac{m_1m_2}{m_1+m_2}$.

c) If we want to represent our 2 particle problem with an invented equivalent 1 particle problem, where that particle is at $\vec{r}$ at any time but has the same angular momentum as the two particle system, what is the mass of this imagined single particle in the equivalent 1 particle problem, written in terms of stuff I named above? The most confusing thing is that the imaginary single particle problem just puts the origin of the vector $\vec{r}$ at some fixed spot. In the real 2 particle problem, the arrow representing the vector $\vec{r}$ starts at mass 2 and goes to mass 1, and both masses are probably moving. In the real 2 particle problem the COM is the origin, which is where the vector $\vec{r}_1$ and the vector $\vec{r}_2$ both start.

d) If we describe the position of this imaginary single particle using 2D polar coordinates, give a general formula for its $\dot{\theta}=\frac{d\theta}{dt}$ as a function of the magnitude of its angular momentum $\ell=|\vec{\ell}|$ and $r=|\vec{r}|$. Recall $\dot{\vec{r}}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}\quad \vec{r}=r\hat{r},$ in 2D polar coordinates. (We'll prove the particles stay in a plane so 2D is ok.)

Problem 2: You're going to derive a general form that describes motion of two particles that interact via a gravitational interaction, which we'll describe with a potential that depends on the separation of the particles $r=|\vec{r}|=|\vec{r}_1-\vec{r}_2|$. In class, we'll write down the total energy of both particles, decide to use a coordinate system with origin at their center of mass, and discover that we can rewrite their total energy using r and $\ell = |\vec{\ell}|$ as: $E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$.

a) Solve this equation for $\dot{r}$ (easy).

b) Note that $d\theta = \frac{\dot{\theta}}{\dot{r}}dr$. Plug in your results from part (a) of this problem and part (d) of the previous problem to find $\theta(r) = \int \frac{\pm(\ell / r^2)dr}{\sqrt{2\mu[E - U(r) - (\ell^2 / 2\mu r^2)]}}$.

c) Consider the integral for the specific case of a gravitational interaction, where $U(r) = \frac{-k}{r}$. We know that the constant k is positive and we get to pick the sign in the top of the integral for convenience – remember we get to choose which way we are looking at the objects. Make the variable change $u = \ell / r$ and show you can arrive at ?

$$\theta(r) = \int \frac{-du}{\sqrt{2\mu E + 2\mu k u/l - u^2}}$$

d) I will post a page from a table of integrals. Explain how to start at the integral just above here and get the result just below. For example, are there checks on the constants you have to make to decide which answer to use? [The final check can be satisfied with a mental promise to stick to values of sin and cos that are sensibly between -1 and 1, so no need to bother with it.]

$$\theta(r) + integrationconstant = sin^{-1}\left[\frac{-2u + \frac{2\mu k}{l}}{\sqrt{\left(\frac{2\mu k}{l}\right)^2 + 8\mu E}}\right]$$

e) Can you use an identity to switch from sin to cos if you pick the integration constant to be $-\pi/2$? Use that integration constant value.

f) If define $\epsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$ and $c \equiv \frac{l^2}{\mu k}$ show whether you arrive at $\boxed{r = \frac{c}{1 + \epsilon\cos\theta}}$

g) Given our choice of integration constant, at what theta value does r have a minimum, assuming all constants in our final result are positive?

Problem 3: a) If I write $\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha} = p_{\phi_\alpha}$, then p_{ϕ_α} is the generalized _______________ associated with ϕ_α. (Fill in the blank with one word).

On an optional Friday slide I argued it is possible to chain rule over to a different set of coordinates used in a Lagrangian and still find a conserved quantity from a symmetry. The next 2 questions ask you to write tricks useful in chain rule calculations.

b) We assume that the $q_j's$ are independent from each other and $\dot{q}_j$'s are independent too. Write $\frac{\partial \dot{q}_k}{\partial \dot{q}_i}$ with a Kronecker delta symbol. [Hint: what's the answer if i=k? If i different from k?]

c) Imagine the Cartesian y coordinate of mass α is related to the q's by a holonomic constraint equation $y_\alpha = (q_1, q_2, q_3, \dots, t)$, <u>where there are no time derivatives of any q in that equation</u>, because it is a good holonomic constraint. Then $\frac{dy_\alpha}{dt} = \sum_j \frac{\partial y_\alpha}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial y_\alpha}{\partial t}$. **Just by looking at this form and noticing what multiplies $\dot{q}_j = \frac{dq_j}{dt}$, what is $\frac{\partial \dot{y}_\alpha}{\partial \dot{q}_j}$?** [Remember the q's are independent from each other and their time derivatives are independent from each other too. This trick was used in an optional slide Friday and is closely related to other tricks needed for such theory.]

d) Using $\mathcal{L} = T - U$ we considered a change in a single particle Lagrangian $\delta\mathcal{L}$ when the trajectory in it went from $\vec{r}(t)$ to $\vec{r}(t) + \delta\vec{r}(t)$. We threw away $\delta\dot{\vec{r}}^2$ assuming it is $tiny^2$and arrived at $\delta\text{S}=\int_{t_1}^{t_2} \delta\mathcal{L}\,dt = \int_{t_1}^{t_2} [\text{m}\dot{\vec{r}} \cdot \delta\dot{\vec{r}} - \vec{\nabla}U \cdot \delta\vec{r}]dt.$

Then integration by parts turned this into $\delta\text{S}=\int_{t_1}^{t_2} \delta\mathcal{L}\,dt = \int_{t_1}^{t_2} [-\text{m}\ddot{\vec{r}} - \vec{\nabla}U] \cdot \delta\vec{r}dt.$

Starting there show that if the trajectory $\vec{r}(t)$ follows Newton's second law, and we build any constraint into $\mathcal{L}$ so the tiny path variation $\delta\vec{r}$ automatically follows the constraint, and that any constraint is holonomic so that then $\vec{F}^{const} \cdot \delta\vec{r}(t)$=0, and we use $-\vec{\nabla}U = \vec{F}^{tot} - \vec{F}^{constraint}$ (meaning we also deal with conservative forces while ignoring constraint force), then the integral is extremal (meaning δS=0 for any allowed choice of path variation $\delta\vec{r}$(t)).

e) Make sure you've learned Liouville's theorem before finishing this problem. Imagine a single variable system that is both conservative (so it follows Hamilton's equations of motion for a particular H) and chaotic. Imagine that points representing states that can happen in this system are spreading out rapidly along the x direction for a while in a region of phase space as they represent states evolving (although this exponential growth of difference doesn't go on forever). Would points in phase space representing states have to be smooshing closer together along the p_x direction in this region while representing this same set of examples of the system during the same evolution period?   Yes or no.

f) Imagine this spreading and smooshing process happening in many areas. After spreading, many points may eventually come closer to each other along the x direction again – I think of mixing of colors in a taffy making machine. Could a point representing an evolving state typically end up with a very different set of neighboring points than it started with?   Yes or no

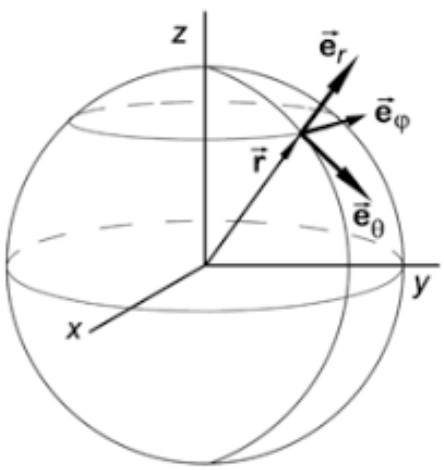
Optional: really well mixed systems can be "ergodic" https://en.wikipedia.org/wiki/Ergodicity
End optional.

Problem 4: a) Go back and look at problem 2 of problem set 8. Before carrying out calculations, do you think H=T+U for this pendulum with the accelerated pivot? Be sure to briefly explain this answer.

b) Find H for this system. Once you have a valid Hamiltonian and you have done the necessary thing to any $\dot{\theta}$ that originally appeared in it (so you now have an H that would behave correctly in Hamilton's equations), you DO NOT NEED TO SIMPLIFY.

Problem 5: a) Consider a spherical pendulum made of a mass m at the end of a roughly massless length L stick whose other end is attached to the stationary origin. Recall that velocity in spherical coordinates is $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r sin\theta\dot{\phi}\hat{\phi}$.. Assume gravity points in the -z direction. Find a Hamiltonian for this system written using the correct things you should use to make Hamilton's equations work. [Among other things, we assumed independent coordinates that would have simple Euler-Lagrange equations when arriving at Hamilton's equations.]

b) Write out Hamilton's equations for this Hamiltonian – as always you should take the appropriate partial derivatives when you do this. If some of these equations are equivalent to ones you wrote in developing your Hamiltonian, point that out.



c) If there is a rotation symmetry that causes one of your p's to be constant, say rotation symmetry around WHAT direction.

d) Note that in spherical $\vec{r} = r\hat{r}$ and angular momentum is $\vec{L}$=$\vec{r} \times m\vec{v} = m[r^2\dot{\theta}\hat{\phi} - r^2 sin\phi\ \hat{\theta}]$.
What component of angular momentum is your constant p quantity?

e) Take the expression you have for p_θ and plug it into the other Hamilton's equation where that has $\frac{d}{dt}p_\theta$ and take the indicated time derivative there. You should get a differential equation that includes $\ddot{\theta}$. **[Know that combining 2 first order differential equations from your Hamilton's equations can produce something equivalent to the Euler-Lagrange equation you would have got for that coordinate if you had used the Lagrangian method.]**