1 Q20

We have

$$\Phi(r) = \frac{V_1 r_1 - r_2 V_2}{r_1 - r_2} + \frac{r_1 r_2 (V_1 - V_2)}{r_2 - r_1}\frac{1}{r}$$

$$\Phi(r_1) = \frac{r_1 - r_2}{r_1 - r_2}V_1 = V_1, \quad \Phi(r_2) = \frac{r_1 - r_2}{r_1 - r_2}V_2 = V_2 \implies \text{satisfies BCs}$$

$$\nabla^2 \Phi = 0 = \frac{d}{dr}(r\Phi) + 0 + 0 = \frac{d}{dr}(\frac{r_1 r_2 (V_1 - V_2)}{r_2 - r_1}\frac{1}{r}) = 0 \implies \text{satisfies Laplace equation}$$

$\boxed{\Phi \text{ is a valid solution}}$ We now generate it from the general solution. There's no ϕ dependence by symmetry, so $m = 0$. The general form then reduces to,

$$\Phi = \sum_\ell (A_\ell r^\ell + B_\ell r^{-\ell-1}) P'_\ell(\cos\theta)$$

By symmetry, there's also no θ dependence, so we require the legendre polynomial to be independent of θ. This is only true for the first legendre polynomial, or $P'_0 = 1$. Therefore, $\ell = 0$. Then, the general form reduces to,

$$\Phi = (A_0 r^0 + B_0 r^{0-1}) = A + Br^{-1}$$

Assert the two boundary conditions,

$$\Phi(r_1) = V_1, \quad \Phi(r_2) = V_2$$

$$V_1 = A + \frac{B}{r_1}, \quad V_2 = A + \frac{B}{r_2}$$

Subtracting,

$$V_1 - V_2 = \frac{B}{r_1} - \frac{B}{r_2} \implies B = \frac{r_1 r_2 (V_1 - V_2)}{r_2 - r_1}$$

Scaling first by r_1 and second by r_2 and subtracting yields

$$V_1 r_1 - V_2 r_2 = A r_1 - A r_2 \implies A = \frac{V_1 r_1 - V_2 r_2}{r_1 - r_2}$$

Substituting into our general form,

$$\boxed{\Phi(r, \theta, \phi) = \frac{V_1 r_1 - V_2 r_2}{r_1 - r_2} + \frac{r_1 r_2 (V_1 - V_2)}{r_2 - r_1}\frac{1}{r}, \qquad r_1 \leq r \leq r_2}$$

We recover the given solution usign the general form

2 Q21

Clearly, $m = 0$ by azimuthal symmetry. No ϕ dependence. The general solution is then,

$$\Phi = \sum_{\ell=0}^{\infty}(a_\ell r^\ell + b_\ell r^{-\ell-1})P'_\ell(\cos\theta)$$

By orthogonality, transform both sides by $\int d\theta f \sin\theta P_\ell(\cos\theta)$ to isolate a particular ℓ, this induces a $\frac{2}{2\ell+1}\delta_{\ell\ell'}$ coefficient,

$$\int_0^\pi d\theta\ \Phi \sin\theta P_\ell(\cos\theta)d\theta = (a_\ell r^\ell + b_\ell r^{-\ell-1})\frac{2}{2\ell+1}$$

By BC,

$$\Phi(R_0) = \begin{cases} -V_0 & 0 \le \theta < \pi/2 \\ V_0 & \pi/2 < \theta \le \pi \end{cases}, \qquad \Phi(2R_0) = \begin{cases} V_0 & 0 \le \theta < \pi/2 \\ -V_0 & \pi/2 < \theta \le \pi \end{cases}$$

$$\implies \begin{cases} V_0 \int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - V_0 \int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta = (a_\ell R_0^\ell + b_\ell R_0^{-\ell-1})\frac{2}{2\ell+1} \\ V_0 \int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - V_0 \int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta = (a_\ell (2R_0)^\ell + b_\ell (2R_0)^{-\ell-1})\frac{2}{2\ell+1} \end{cases}$$

$$\begin{cases} \frac{2\ell+1}{2}V_0\left[\int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - \int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta\right] = a_\ell R_0^\ell + b_\ell R_0^{-\ell-1} \\ \frac{2\ell+1}{2}V_0\left[\int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - \int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta\right] = a_\ell (2R_0)^\ell + b_\ell (2R_0)^{-\ell-1} \end{cases}$$

We arrive at our solution. Terms can be generated by numerically evaluating the integral and solving the system for the coefficients a_ℓ, b_ℓ for each ℓ,

$$\boxed{\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}(a_\ell r^\ell + b_\ell r^{-\ell-1})P'_\ell(\cos\theta) \qquad R_0 \le r \le 2R_0}$$

$$\boxed{\begin{cases} a_\ell R_0^\ell + b_\ell R_0^{-\ell-1} = \frac{2\ell+1}{2}V_0\left[\int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - \int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta\right] \\ a_\ell (2R_0)^\ell + b_\ell (2R_0)^{-\ell-1} = \frac{2\ell+1}{2}V_0\left[\int_0^{\pi/2} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta - \int_{\pi/2}^{\pi} d\theta\ \sin\theta P_\ell(\cos\theta)d\theta\right] \end{cases}}$$

The textbook example leaves coefficients defined in terms of integrals since there's no easy closed form, so I do the same. However, coefficients and partial sums can be generated by evaluating the two boundary conditions and solving the system for the two coefficients. Then, evaluate terms of the general solution sum to get the solution. Everything in the integrands is known, though evaluating it for each Legendre polynomial and solving the system is computationally intensive.

3 Q22

By method of images, consider an extension of the problem. Let the top half remain charged $+V_0$, but consider a $-V_0$ bottom hemi-sphere extension. Then, by method of images, the original problem is still valid above the central plane by symmetry for this new problem. Clearly, under the central plane will have zero field in our original problem due to the conducting ground plane. The new boundary condition is then,

$$\Phi(R_0) = \begin{cases} V_0 & 0 \leq \theta < \pi/2 \\ -V_0 & \pi/2 < \theta \leq \pi \end{cases}$$

Clearly, $m = 0$ by azimuthal symmetry. No ϕ dependence. The general solution is then,

$$\Phi = \sum_{\ell=0}^{\infty} (a_\ell r^\ell + b_\ell r^{-\ell-1}) P_\ell'(\cos\theta)$$

Inside the sphere, potential must be finite at $r = 0$, so $b_\ell = 0$.

$$\Phi = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell'(\cos\theta)$$

Asserting BC at $r = R_0$, depending on theta, we have

$$\pm V_0 = \sum_{\ell=0}^{\infty} a_\ell R_0^\ell P_\ell'(\cos\theta)$$

By orthogonality, transform both sides by $\int d\theta f \sin\theta P_\ell(\cos\theta)$ to isolate a particular ℓ, this induces a $\frac{2}{2\ell+1}\delta_{\ell\ell'}$ coefficient,

$$\int_0^{\pi/2} d\theta V_0 \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta V_0 \sin\theta P_\ell(\cos\theta) = a_\ell R_0^\ell \frac{2}{2\ell+1}$$

$$\implies a_\ell = \frac{2\ell+1}{2R_0^\ell} V_0 \left[\int_0^{\pi/2} d\theta \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta P_\ell(\cos\theta) \right]$$

Outside the sphere, potential must be finite at $r = \infty$, so $a_\ell = 0$.

$$\Phi = \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell'(\cos\theta)$$

Asserting BC at $r = R_0$, depending on theta, we have

$$\pm V_0 = \sum_{\ell=0}^{\infty} b_\ell R_0^{-\ell-1} P_\ell'(\cos\theta)$$

By orthogonality, transform both sides by $\int d\theta f \sin\theta P_\ell(\cos\theta)$ to isolate a particular ℓ, this induces a $\frac{2}{2\ell+1}\delta_{\ell\ell'}$ coefficient,

$$\int_0^{\pi/2} d\theta V_0 \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta V_0 \sin\theta P_\ell(\cos\theta) = b_\ell R_0^{-\ell-1} \frac{2}{2\ell+1}$$

$$\implies b_\ell = \frac{2\ell+1}{2R_0^{-\ell-1}} V_0 \left[\int_0^{\pi/2} d\theta \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta P_\ell(\cos\theta) \right]$$

We have a valid solution inside and outside the sphere. Below the central plane, $\Phi = 0$ by the conducting plate. Therefore,

$$\boxed{\Phi(r,\theta,\phi) = \begin{cases} \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell'(\cos\theta) & 0 \leq r < R_0, 0 \leq \theta < \pi/2 \\ \sum_{\ell=0}^{\infty} b_\ell r^{-\ell-1} P_\ell'(\cos\theta) & R_0 \leq r < \infty, \pi/2 < \theta \leq \pi \\ 0 & \pi/2 < \theta \leq \pi \end{cases}}$$

$$\boxed{\begin{cases} a_\ell = \frac{2\ell+1}{2R_0^\ell} V_0 \left[\int_0^{\pi/2} d\theta \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta P_\ell(\cos\theta) \right] \\ b_\ell = \frac{2\ell+1}{2R_0^{-\ell-1}} V_0 \left[\int_0^{\pi/2} d\theta \sin\theta P_\ell(\cos\theta) - \int_{\pi/2}^{\pi} d\theta \sin\theta P_\ell(\cos\theta) \right] \end{cases}}$$

The textbook example leaves coefficients defined in terms of integrals since there's no easy closed form, so I do the same. However, coefficients and partial sums can be generated by solving coefficients and evaluating terms of the sum. Everything in the integrand is known, though evaluating it for each Legendre polynomial is computationally intensive.