

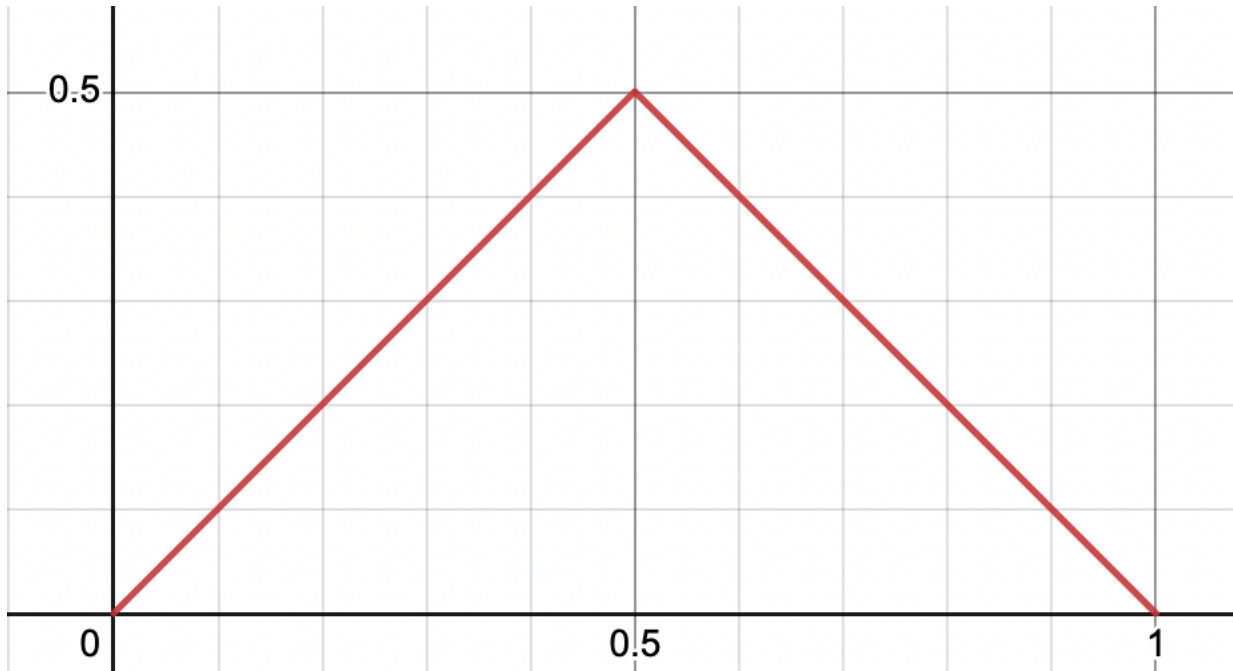
3610

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September 15, 2023

Q1: 2.7

a



Graph of $\Psi(x, 0)$ for $a = A = 1$. Integrating,

$$\begin{aligned} \int |\Psi|^2 dx &= 1 = \int_0^{a/2} dx A^2 x^2 + \int_{a/2}^a dx A^2 (a - x)^2 \\ &= \frac{A^2 a^3}{24} + \frac{A^2 a^3}{24} \implies \boxed{A = \sqrt{\frac{12}{a^3}}} \end{aligned}$$

b

$$b) \text{ Get Fourier } \Rightarrow c_n = A \sqrt{\frac{2}{a}} \left[\int_0^{\frac{a}{2}} x \sin \frac{n\pi}{a} x dx + \int_{\frac{a}{2}}^a (a-x) \sin \left(\frac{n\pi}{a} x \right) dx \right]$$

$$= \frac{4\sqrt{6}}{(n\pi)^2} \sin \frac{n\pi}{2} \text{ by calc.}$$

$\Rightarrow \Psi = \text{fourier}$

$$\Rightarrow \Psi = \frac{4\sqrt{6}}{\pi^2} \sum_{\substack{n=1,3,\dots \\ \text{odd}}} (-1)^{\frac{n-1}{2}} \frac{\sin\left(\frac{n\pi}{a} x\right)}{n^2} e^{-E_n t/\hbar}$$

Substituting calculator

results into fourier series defined at top and simplifying,

$$\Rightarrow E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad \Psi(x, t) = \frac{8}{\pi^2} \sqrt{\frac{3}{a}} \sum_{n=1,3,5,\dots} (-1)^{(n-1)/2} \frac{\sin\left(\frac{n\pi x}{a}\right)}{n^2} e^{-E_n t/\hbar}$$

For odd n

c

$$P = c_1^2 = \left(\frac{4\sqrt{6}}{1\pi^2} \right)^2 = 16 \cdot 6/\pi^4 = \boxed{P = \frac{96}{\pi^4}}$$

d

$$\langle H \rangle = \sum c_n^2 E_n = \sum \left(\frac{4\sqrt{6}}{n^2 \pi^2} \right)^2 \frac{n^2 \pi^2 \hbar^2}{2ma^2} = \frac{48\hbar^2}{\pi^2 ma^2} \sum 1/n^2 = \frac{48\hbar^2}{\pi^2 ma^2} \frac{\pi^2}{8} = \boxed{\frac{6\hbar^2}{ma^2}}$$

Q2

a

Plugging in the relevant values,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \Rightarrow E_1 = \frac{\pi^2 \hbar^2}{2ma^2} = \boxed{204.46 \text{ MeV}}$$

b

i) Using $m_e = 9.11 \cdot 10^{-31}$, $\boxed{E = 375914 \text{ MeV}}$

ii) We take distance to be half the width of the well,

$$U = \frac{kqq}{r}$$

$$U(\infty) = 0, \quad U(a/2) = \frac{2(k40e^2)}{a}$$

$$U(\infty) - U(a/2) = 1.841152 \cdot 10^{-11} \text{ J} = \boxed{= 115 \text{ MeV}}$$

iii) $\boxed{\text{No. The nucleus cannot confine the electron.}}$ $375914 > 115$. The electron has too much energy and will easily escape.

c

We seek an a such that the energies are equal.

$$\frac{80kq^2}{a} = \frac{\pi^2 \hbar^2}{2ma^2}$$
$$\frac{80kq^2}{a} = \frac{\pi^2 \hbar^2}{2ma^2}$$

Solving,

$$a = 3.27 \cdot 10^{-12} \text{ m}$$

We require a nucleus diameter of $3.27 \cdot 10^{-12}$ meters.

Q3

a

The wave function is of the form

$$\Psi(x, t) = \sum_{n \geq 1} c_n \psi_n e^{-itE_n/\hbar}$$
$$\Psi(x, t) = \sum_{n \geq 1} c_n \psi_n e^{-it(n^2 \hbar \pi^2 / 2ma^2)}$$
$$\Psi(x, 0) = \sum_{n \geq 1} c_n \psi_n$$
$$\Psi(x, T) = \sum_{n \geq 1} c_n \psi_n e^{-i(4ma^2/\pi \hbar)(n^2 \hbar \pi^2 / 2ma^2)}$$
$$\Psi(x, T) = \sum_{n \geq 1} c_n \psi_n e^{-i(2n^2 \pi)}$$

Since e^{ix} is 2π -periodic, and $2n^2\pi$ is a multiple of 2π , so

$$\Psi(x, T) = \sum_{n \geq 1} c_n \psi_n = \Psi(x, 0)$$

The wave function returns to its original form after T time.

b

$$E = \frac{1}{2}mv^2 \implies v = \sqrt{2E/m}$$

The classical revival time, T_c , is given by the time to traverse the width and return,

$$T_c = 2a/v = \boxed{T_c = a\sqrt{\frac{2m}{E}}}$$

c

Equating the two,

$$T_c = a\sqrt{\frac{2m}{E}} = \frac{4ma^2}{\pi \hbar} \implies \boxed{E = \frac{\pi^2 \hbar^2}{8ma^2} = \frac{E_1}{4}}$$

Q4

a

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi e^{-ikx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\alpha|x|+i\beta x} e^{-ikx} \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 dx e^{\alpha x+i\beta x} e^{-ikx} + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dx e^{-\alpha x+i\beta x} e^{-ikx} \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 dx e^{(\alpha+i(\beta-k))x} + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dx e^{(-\alpha+i(\beta-k))x} \\
& = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(i \cdot (\beta-k)+\alpha)x}}{i \cdot (\beta-k) + \alpha} \Big|_{-\infty}^0 + \frac{e^{(i \cdot (\beta-k)-\alpha)x}}{i \cdot (\beta-k) - \alpha} \Big|_0^{\infty} \right] \\
& = \frac{1}{\sqrt{2\pi}} \left[\frac{ik - i\beta + \alpha}{k^2 - 2\beta k + \beta^2 + \alpha^2} - \frac{ik - i\beta - \alpha}{k^2 - 2\beta k + \beta^2 + \alpha^2} \right] \\
& \mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{k^2 - 2\beta k + \beta^2 + \alpha^2} \\
& \boxed{\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{(k - \beta)^2 + \alpha^2}}
\end{aligned}$$

b

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi e^{-ikx} \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left(\sin \frac{2\pi(x-x_0)}{a} \right) e^{-ikx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left(\frac{e^{i\frac{2\pi(x-x_0)}{a}} - e^{-i\frac{2\pi(x-x_0)}{a}}}{2i} \right) e^{-ikx} \\
& = \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left(e^{i\frac{2\pi(x-x_0)}{a}} - e^{-i\frac{2\pi(x-x_0)}{a}} \right) e^{-ikx} \\
& = \frac{1}{2i\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \left(e^{i\frac{2\pi(x-x_0)}{a}} e^{-ikx} - e^{-i\frac{2\pi(x-x_0)}{a}} e^{-ikx} \right)
\end{aligned}$$

We recognize both terms as the inverse fourier transform of the complex exponential with a delay.

$$\boxed{\mathcal{F}(k) = \frac{1}{2i} \left[e^{-\frac{2\pi x_0}{a}} \delta(k - \frac{2\pi}{a}) + e^{\frac{2\pi x_0}{a}} \delta(k + \frac{2\pi}{a}) \right]}$$

c

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi e^{-ikx} \\
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \text{rect}\left(\frac{x-b}{a}\right) e^{-ikx} \\
& \frac{1}{\sqrt{2\pi}} \int_{b-a/2}^{b+a/2} dx e^{-ikx} \\
& \frac{i}{k\sqrt{2\pi}} (e^{-ik(b+a/2)} - e^{-ik(b-a/2)}) \\
& \frac{2}{k\sqrt{2\pi}} \frac{e^{-ik(b-a/2)} - e^{-ik(b+a/2)}}{2i} \\
& \frac{e^{-ibk}}{k/2\sqrt{2\pi}} \sin(ak/2) \\
& \frac{ae^{-ibk}}{\sqrt{2\pi}} \frac{\sin(ak/2)}{ak/2} \\
& \boxed{\mathcal{F}(k) = \frac{a}{\sqrt{2\pi}} e^{-ibk} \text{sinc}(ak/2)}
\end{aligned}$$

d

$$\psi^* \psi = (e^{-\alpha|x|-i\beta x})(e^{-\alpha|x|+i\beta x}) = e^{-2\alpha|x|} \quad \mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{(k-\beta)^2 + \alpha^2}$$

b is clearly a centering constant, so we can set $b = 0$ without affecting shape and width. Then, it is sufficient to prove that the product of the following functions' width is constant.

$$e^{-2\alpha|x|}, \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha}{k^2 + \alpha^2}$$

Amplitude is clearly maximized and equal to 1 and $\frac{1}{a\sqrt{2\pi}}$ respectively at $x = k = 0$. At half maximum,

$$e^{-2\alpha|x|} = \frac{1}{2}, \quad \frac{1}{\sqrt{2\pi}} \frac{\alpha}{k^2 + \alpha^2} = \frac{1}{2a\sqrt{2\pi}}$$

We solve for and double x to get width, since both functions are even.

$$\Delta x = \frac{2}{-2\alpha} \ln\left(\frac{1}{2}\right), \quad \Delta k = 2\alpha$$

Taking their product,

$$\Delta x \Delta k = \frac{2 \cdot 2\alpha}{-2\alpha} \ln\left(\frac{1}{2}\right) = \boxed{\Delta x \Delta k = 2 \ln 2}$$

Clearly, this is a constant. No variable dependence.

Q5

a

This cannot be computed in elementary functions, so by calculator,

$$\int_{-\infty}^{\infty} dx |\Psi(x, 0)|^2 = \int_{-\infty}^{\infty} dx A^2 e^{-2ax^2} = A^2 \sqrt{\frac{\pi}{2a}} = 1 \implies \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}}$$

b

Heavily using a calculator throughout,

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \Psi(x, 0) e^{-ikx} = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} dx e^{-ax^2} e^{-ikx} = \frac{e^{-\frac{k^2}{4a}}}{\sqrt[4]{2\pi a}} \\ \Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \phi e^{i(kx - \frac{\hbar k^2}{2m}t)} = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} dk e^{-k^2/4a} e^{i(kx - \frac{\hbar k^2}{2m}t)} \end{aligned}$$

Letting $\xi = 1 + 2i\hbar at/m$,

$$\Psi(x, t) = A \frac{e^{-ax^2/\xi}}{\sqrt{\xi}} = \boxed{\Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\frac{ax^2}{1+2i\hbar at/m}}}{\sqrt{1+2i\hbar at/m}}}$$

which is the same as the book provides.

c

Recall that $\xi = 1 + 2i\hbar at/m \implies \xi^* = 1 - 2i\hbar at/m$,

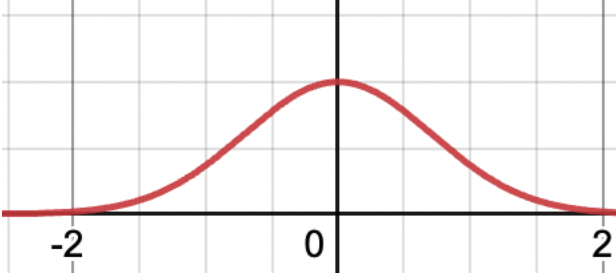
Note that $w = \sqrt{\frac{a}{\xi\xi^*}} \implies w^2 = \frac{a}{\xi\xi^*}$. Then,

$$\begin{aligned} |\Psi|^2 &= A^2 \frac{e^{-ax^2/\xi}}{\sqrt{\xi}} \frac{e^{-ax^2/\xi^*}}{\sqrt{\xi^*}} \\ |\Psi|^2 &= \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/\xi - ax^2/\xi^*}}{\sqrt{\xi\xi^*}} \\ |\Psi|^2 &= \sqrt{\frac{2}{\pi}} w e^{-ax^2(\xi+\xi^*)/\xi\xi^*} \end{aligned}$$

$$|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2ax^2/\xi\xi^*}$$

$$|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-w^2 x^2}$$

For $t = 0$, we have the first plot. For $t = 5$, we have the second plot. As t increases, $|\Psi|^2$ flattens and spreads out



d

$$\langle x \rangle = \int_{-\infty}^{\infty} dx x |\Psi|^2 = \langle x \rangle = 0$$

since odd

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx x^2 |\Psi|^2 = \int_{-\infty}^{\infty} dx x^2 |\Psi|^2 = \int_{-\infty}^{\infty} dx \sqrt{\frac{2}{\pi}} w e^{-aw^2 x^2} = \langle x^2 \rangle = \frac{1}{4w^2}$$

by calc

$$\langle p \rangle = m \langle \dot{x} \rangle = \boxed{\langle p \rangle = 0}$$

Let $C = \frac{a}{1+i2\hbar at/m}$. By calc,

$$\Psi^* \frac{d^2 \Psi}{dx^2} = -2C(1 - 2Cx^2) |\Psi|^2$$

By calc,

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} dx -2C(1 - 2Cx^2) |\Psi|^2 = \hbar^2 a$$

$$\sigma_x = \sqrt{0 - \frac{1}{4w^2}} = \frac{1}{2w}$$

$$\sigma_p = \sqrt{0 - \hbar^2 a} = \hbar \sqrt{a}$$

Collecting,

$$\boxed{\langle x \rangle = 0 \quad \langle x^2 \rangle = \frac{1}{4w^2} \quad \sigma_x = \frac{1}{2w} \quad \langle p \rangle = 0 \quad \langle p^2 \rangle = \hbar^2 a \quad \sigma_p = \hbar \sqrt{a}}$$

e

$$\sigma_x \sigma_p = \frac{\hbar \sqrt{a}}{2w} = \frac{\hbar}{2} \sqrt{1 + (2\hbar at/m)^2} \geq \frac{\hbar}{2} \implies \boxed{\text{Yes, holds}}$$

By inspection, the radicand is closest to 0 when second term zeros out at $t = 0$. Then, $\sigma_x \sigma_p = \frac{\hbar}{2}$.

$$\implies \boxed{t = 0}$$