Bryant Har

Q1

Part A

Set origin to be (0,0). Then,
$U = mgy_m = mg(l - l\cos(\theta))$
Using the image we find velocity by differentiation,

$$(x_m, y_m) = (A\sin(\omega t) + l\sin(\theta), l - l\cos(\theta))$$

$$\implies (\dot{x}_m, \dot{y}_m) = (A\omega\cos(\omega t) + l\cos(\theta)\dot{\theta}, l\sin(\theta)\dot{\theta})$$

$$\implies v^2 = (A\omega\cos(\omega t) + l\cos(\theta)\dot{\theta})^2 + (l\sin(\theta)\dot{\theta})^2$$

We then find kinetic energy,

$$T = \frac{1}{2}m\big[(A\omega\cos(\omega t) + l\cos(\theta)\dot{\theta})^2 + (l\sin(\theta)\dot{\theta})^2\big]$$

$$T = \frac{1}{2}m\big[A^2\omega^2\cos^2(\omega t) + 2A\omega l\dot{\theta}\cos(\omega t)\cos(\theta) + l^2\dot{\theta}^2\big]$$

The lagrangian, then, is

$$\implies L = T - U$$

$$\boxed{\mathcal{L} = \frac{1}{2}m\big[A^2\omega^2\cos^2(\omega t) + 2A\omega l\dot{\theta}\cos(\omega t)\cos(\theta) + l^2\dot{\theta}^2\big] - mg(l - l\cos(\theta))}$$

Part B

We begin with EL

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}}$$

Substituting L,

$$0 = -mA\omega l\dot{\theta}\cos(\omega t)\sin(\theta) - mgl\sin(\theta) - ml^2\ddot{\theta} + mA\omega l\cos(\omega t)\sin(\theta)\dot{\theta} + mA\omega^2 l\sin(\omega t)\cos(\theta)$$

We rearrange and cancel terms to arrive at our final EL equation,

$$\boxed{ml^2\ddot{\theta} = mlA\omega^2\sin(\omega t)\cos(\theta) - mlg\sin(\theta)}$$

$$\boxed{l\ddot{\theta} = A\omega^2\sin(\omega t)\cos(\theta) - g\sin(\theta)}$$

Part C

The mass must obey circular motion, so

$$(x - x_p)^2 + (y - y_p)^2 = l^2$$

$$(x - A\sin(\omega t))^2 + (y - l)^2 = l^2$$

We arrive,

$$\boxed{(x - A\sin(\omega t))^2 + (y-l)^2 - l^2 = 0}$$

Part D

We have two time-dependent coordinates, x and y, but we can eliminate one with the constraint, so

$$2 - 1 = \boxed{\text{1 degree of freedom}}$$

Part E

From part A, we found by looking at the diagram,

$$(x_m, y_m) = (A\sin(\omega t) + l\sin(\theta), l - l\cos(\theta))$$

Decomposing,

$$\boxed{\begin{cases} x = A\sin(\omega t) + l\sin(\theta) \\ y = l - l\cos(\theta) \end{cases}}$$

Q2

Part A

$\boxed{\text{One degree of freedom}}$
(θ, the constant acceleration is not free since always known)

Part B

We use the origin as the horizontal line at the pivot. x,y is at some point l away.

$$(x, y) = (\frac{a}{2}t^2 + l\sin(\theta), -l\cos(\theta))$$

Potential energy is simply gravity at com,

$$U = Mgy = -MgL/2\cos(\theta)$$

We find velocity by differentiation,

$$(\dot{x}, \dot{y}) = (at + l\cos(\theta)\dot{\theta}, l\sin(\theta)\dot{\theta})$$

$$\implies v^2 = \left[(at + l\cos(\theta)\dot{\theta})^2 + (l\sin(\theta)\dot{\theta}))^2\right] = a^2t^2 + 2atl\cos(\theta)\dot{\theta} + l^2\dot{\theta}^2$$

Starting from first principles, we find KE, integrating over length using uniform mass density.
Notably, this method will intrinsically incorporate **BOTH rotational and translational KE**

$$T = \frac{1}{2}\int v^2 dm = \frac{M}{2L}\int_0^L \left[a^2t^2 + atl\cos(\theta)\dot{\theta} + l^2\dot{\theta}^2\right]dl$$

$$T = \frac{M}{2}(a^2t^2 + atL\cos(\theta)\dot{\theta} + \frac{L^2\dot{\theta}^2}{3})$$

Then,

$$T - U = \boxed{\mathcal{L} = \frac{M}{2}(a^2t^2 + atL\cos(\theta)\dot{\theta} + \frac{L^2\dot{\theta}^2}{3}) + MgL\cos(\theta)/2}$$

Part C

We begin, with the general form. (I write $\mathcal{L}$ as L for convenience, not to be confused for the rod length)

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}}$$

$$\frac{dL}{d\theta} = -MgL\sin(\theta)/2 - MatL\sin(\theta)\dot{\theta}/2$$

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} = MaL\cos(\theta)/2 - MatL\sin(\theta)\dot{\theta}/2 + ML^2\ddot{\theta}/3$$

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}}$$

$$= -MgL\sin(\theta)/2 - MatL\sin(\theta)\dot{\theta}/2 - (MaL\cos(\theta)/2 - MatL\sin(\theta)\dot{\theta}/2 + ML^2\ddot{\theta}/3)$$

Canceling,

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}} = -MgL\sin(\theta)/2 - MaL\cos(\theta)/2 - ML^2\ddot{\theta}/3$$

$$\boxed{\frac{2}{3}L\ddot{\theta} = -g\sin(\theta) - a\cos(\theta)}$$

Part D

In equilibrium, $\ddot{\theta} = 0$. Then, using the above equation,

$$g\sin\theta = -a\cos(\theta)$$

$$\implies \boxed{\theta_{eq} = -\arctan(a/g)}$$

In equilibrium, the pivot dangles behind the direction of constant acceleration

Part E

To be stable, we require the potential to be convex or bowl-shaped minimum at that point.

$$\boxed{\frac{d^2U}{dt^2}(\theta_{eq}) > 0 \text{ and } \frac{dU}{dt} = 0}$$

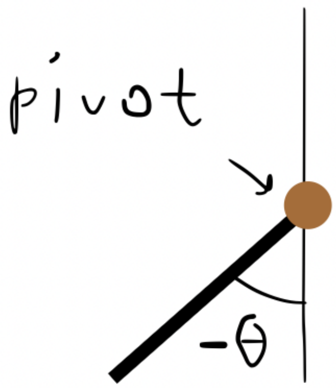
Additionally to follow hooke's law, we require the coeffiicent to be negative for stable equilibrium.
Looking at the result below, this is stable if

$$\text{consequently requires } \boxed{g\cos(\theta_{eq}) - a\sin(\theta_{eq}) > 0}$$

Equivalently for small oscillations, we can show it follows hooke's law. We do this in the next part.

$$m\ddot{q} = -kq$$

Here is a sketch of the stick at a stable equilibrium. The pivot is accelerating to the right and the tip is pulled down by gravity. In the end, it dangles behind.



We clearly see by the sketch that the bar cannot go above the horizontal in equilibrium. Gravity would pull it down. The angle absolute value must be $\boxed{\text{less than } \pi/2}$.

Part F

We seek a hooke's law relation to derive frequency. We arrived at the EOM,

$$\ddot{\theta} = -\frac{3g}{2L}\sin(\theta) - \frac{3a}{2L}\cos(\theta)$$

For small angles (oscillations) at equilibrium,

$$\theta \approx \theta_{eq} + \delta\theta$$

$$\ddot{\theta} = -\frac{3g}{2L}\sin(\theta) - \frac{3a}{2L}\cos(\theta) \approx -\frac{3g}{2L}\sin(\theta_{eq} + \delta\theta) - \frac{3a}{2L}\cos(\theta_{eq} + \delta\theta)$$

$$= -\frac{3g}{2L}\left[\sin(\theta_{eq})\cos(\delta\theta) + \cos(\theta_{eq})\sin(\delta\theta)\right] - \frac{3a}{2L}(\left[\cos(\theta_{eq})\cos(\delta\theta) - \sin(\theta_{eq})\sin(\delta\theta)\right])$$

Applying small angle,

$$\ddot{\theta} \approx -\frac{3g}{2L}\left[\sin(\theta_{eq}) + \cos(\theta_{eq})\delta\theta\right] - \frac{3a}{2L}(\left[\cos(\theta_{eq}) - \sin(\theta_{eq})\delta\theta\right])$$

Comparing the linear term coefficient to hooke's law $\ddot{q} = -\omega^2 q$, we arrive at

$$\omega^2 \approx \frac{g\cos(\theta_{eq}) - a\sin(\theta_{eq})}{2L/3}$$

$$\implies \boxed{\omega = \sqrt{\frac{g\cos(\theta_{eq}) - a\sin(\theta_{eq})}{2L/3}}}$$

Q3

Part A

$$g = -g\langle 0, 1/2, \sqrt{3}/2\rangle$$

We have three degrees of freedom. We define coordinate $x = X_b, y = Y_b, \theta$. x, y is the center of the stick, and θ is the angle of the stick taken CCW from the positive x direction. We also assume all coordinates start at 0 as origin. We also let the board decline in the y direction. We now find the Lagrangian.

We find kinetic energy. This is rotational and translational.

$$T_M = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2\cdot 12}ML^2\dot{\theta}^2$$

Letting $R = L/2$,

$$(x_m, y_m) = (x + R\cos(\theta), y + R\sin(\theta))$$

$$T_m = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m(\dot{x} - R\sin(\theta)\dot{\theta})^2 + \frac{1}{2}m(\dot{y} + R\cos(\theta)\dot{\theta})^2$$

We can make the calculations more tractable by simply considering their translational velocities to be the same.

$$T_m = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

Potential is simply gravity. We define it as zero at origin.

$$U(y) = -Mgy\frac{\sqrt{3}/2}{1/2} = -\frac{Mgy}{\sqrt{3}}$$

$$\implies U = -\frac{Mgy}{\sqrt{3}} - \frac{mg(y + R\sin(\theta))}{\sqrt{3}}$$

Together, we get

$$\mathcal{L} = T - U =$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{24}ML^2\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{Mgy}{\sqrt{3}} + \frac{mg(y + R\sin(\theta))}{\sqrt{3}}$$

Replacing R,

$$\boxed{\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{24}ML^2\dot{\theta}^2 + \frac{1}{8}mL^2\dot{\theta}^2 + \frac{Mgy}{\sqrt{3}} + \frac{mg(y + L\sin(\theta)/2)}{\sqrt{3}}}$$

Part B

We again use

$$0 = \frac{dL}{dx} - \frac{d}{dt}\frac{dL}{d\dot{x}}$$

X derivative is zero. We compute the rest

$$\implies (M+m)\ddot{x} = 0$$

In y direction, we compute all the partials as none are zero

$$0 = \frac{dL}{dy} - \frac{d}{dt}\frac{dL}{d\dot{y}}$$

$$\frac{(M+m)g}{\sqrt{3}} - (M+m)\ddot{y} = 0$$

In theta direction,

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}}$$

The theta derivative is zero, we compute the rest

$$0 = (\frac{1}{12}ML^2 + \frac{1}{4}mL^2)\ddot{\theta}$$

Collecting our EOM,

$$\boxed{\begin{cases} (\frac{1}{12}ML^2 + \frac{1}{4}mL^2)\ddot{\theta} = 0 \\ (M+m)\ddot{x} = 0 \\ \ddot{y} = \frac{g}{\sqrt{3}} \end{cases}}$$

We can remove the constant terms for the values that equal zero if desired, though some meaning is lost.

$$\boxed{\begin{cases} \ddot{\theta} = 0 \\ \ddot{x} = 0 \\ \ddot{y} = \frac{g}{\sqrt{3}} \end{cases}}$$

Part C

We see that the theta and x direction EOM are zero. This means there is symmetry along theta and x. Specifically, these refer to

$$\boxed{\text{Linear momentum conserved along x direction. Symmetry of translation along x direction.}}$$

$$\boxed{\text{Angular momentum conserved. Symmetry along theta direction (stick spinning)}}$$

Part D

From taylor,

$$R = \frac{M_1R_1 + M_2R_2}{M_1 + M_2}$$

Let 1 be the large stick and 2 be the point mass. R_1 averages out to 0. R_2 is $L/2$.
Then, we have

$$R = \frac{0 + mL/2}{M+m}$$

$$\boxed{R = \frac{mL}{2(M+m)}}$$

as desired

Q4

Part A

$\boxed{\text{Two degrees of freedom}}$. One for small and large disk. Let the small disk be ϕ and the large disk be θ, both measured CCW from the downward vertical.

Part B

We define kinetic energy,

$$T = \frac{1}{2}M(R\dot{\theta})^2 + \frac{1}{2}\frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}m(\frac{R\dot{\theta}}{2})^2 + \frac{1}{2}\frac{1}{2}m(\frac{R}{4})^2\dot{\phi}^2$$

$$T = \frac{1}{2}M(R\dot{\theta})^2 + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{8}m(R\dot{\theta})^2 + \frac{1}{64}mR^2\dot{\phi}^2$$

$$T = (\frac{3}{4}MR^2 + \frac{1}{8}mR^2)\dot{\theta}^2 + \frac{1}{64}mR^2\dot{\phi}^2$$

We find potential energy,

$$U = MgR + mg(R - \frac{R}{2}\cos(\theta))$$

The lagrangian, then, is

$$\mathcal{L} = T - U = \boxed{\mathcal{L} = (\frac{3}{4}MR^2 + \frac{1}{8}mR^2)\dot{\theta}^2 + \frac{1}{64}mR^2\dot{\phi}^2 - (MgR + mg(R - \frac{R}{2}\cos(\theta)))}$$

Part C

We write out the EL for the first equation wrt to the little disk angle ϕ using the above lagrangian and computing partials,

$$0 = \frac{dL}{d\phi} - \frac{d}{dt}\frac{dL}{d\dot{\phi}}$$

$$\frac{dL}{d\phi} = 0, \quad \frac{d}{dt}\frac{dL}{d\dot{\phi}} = \frac{1}{32}mR^2\ddot{\phi}$$

$$0 = 0 - \frac{1}{32}mR^2\ddot{\phi} \implies \ddot{\phi} = 0$$

We repeat wrt to the big disk angle θ using the above lagrangian and computing partials,

$$0 = \frac{dL}{d\theta} - \frac{d}{dt}\frac{dL}{d\dot{\theta}}$$

$$\frac{dL}{d\theta} = -\frac{mgR}{2}\sin(\theta), \quad \frac{d}{dt}\frac{dL}{d\dot{\theta}} = (\frac{3}{2}MR^2 + \frac{1}{4}mR^2)\ddot{\theta}$$

Combining,

$$0 = -\frac{mgR}{2}\sin(\theta) - (\frac{3}{2}MR^2 + \frac{1}{4}mR^2)\ddot{\theta} \implies \ddot{\theta} = \frac{-\frac{mgR}{2}\sin(\theta)}{(\frac{3}{2}MR^2 + \frac{1}{4}mR^2)} = \ddot{\theta} = \frac{-mgR\sin(\theta)}{3MR^2 + \frac{1}{2}mR^2}$$

Collecting our EOM,

$$\boxed{\begin{cases} \ddot{\phi} = 0 \\ \ddot{\theta} = \frac{-mgR\sin(\theta)}{3MR^2 + \frac{1}{2}mR^2} \end{cases}}$$

Part D

This says that the rotational speed of the little disk $\dot{\phi}$ is constant. This says that there is symmetry in the energy and momentum in the rotation of the little disk, and that $\boxed{\text{Rotational Energy and Momentum is conserved in the little disk}}$. No torques act on the little disk.