Problem set 7

Applied & Engineering Physics 3330

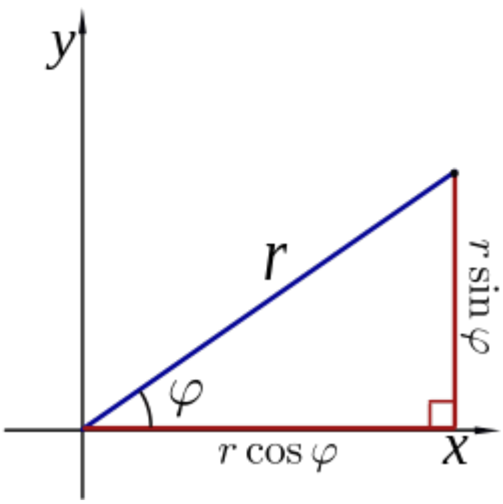
Due 6:10pm Oct. 12, 2022 as Canvas upload. Remember to explain your answers!

Reading: In Taylor, read from the start of chapter 6 until you have gotten as much as you need of the Brachistochrone example. (The heroic variable substitution toward the end is optional. It's also optional to read about the Brachistochrone challenge & how Bernoulli recognized Newton's anonymous solution "as a lion by its claw". https://www.cantorsparadise.com/the-famous-problem-of-the-brachistochrone-8b955d24bdf7)

Problem 1:

a) Taylor's problem 6.12, but you are allowed to have any accurate description of the curve that is stationary –check pages I post below this problem set. Your answer does NOT have to involve a sinh function. Make sure you have reviewed our favorite case for using Euler's equation on the slides from lecture to avoid excess pain. Since you aren't given $y(x_1), y(x_2)$, your answer will have 2 unknown constants in it.

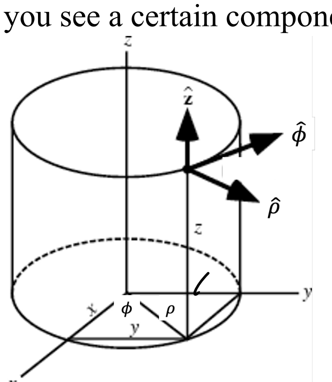
New topic:



b) A particle of mass *m* can move anywhere in a certain vertical plane, which we will describe using the coordinates *r* (distance from the origin) and ϕ (normal angle from an x axis that points horizontally to the right). There is a constant gravitational acceleration g pointing down (along $-\hat{y}$).

Find equations of motion (formulas involving $\ddot{r}$ and $\ddot{\phi}$) using Hamilton's Principle, which says that actual path taken by a particle between t_1 and t_2 is the one that gives an extremum of $\int L\,dt$, where L=T-U is the Lagrangian. Remember T is kinetic energy, which you should write using the polar coordinates I described. U is potential energy, which you should write using mg and r and ϕ. Because the particle moves freely, there is NO predetermined relationship between r and ϕ. [Hint: can you mentally switch from integrating over x earlier to integrating over t?]

c) **If** each equation you initially write in (b) corresponds to either a particular component of Newton's second law OR a component of $\vec{torque} = \dfrac{d\vec{L}}{dt}$, briefly say why you think it corresponds and what component. For this you may extend to a z cylindrical component if needed (but keep the name r not ρ.) If you see a certain component of torque on one side of equation, match the other side to what it must be.



[Here's a look ahead. We will see that the Euler-Lagrange equations are even more convenient for a particle that is constrained. If our 2D particle was forced to move only on a particular 1D curve, there would be a predetermined relationship between r and ϕ to enforce that "constraint".]

Q1 a) $S = \int_{x_1}^{x_2} \underbrace{x\sqrt{1-y'^2}}_{f}\, dx$ $\quad \mathcal{L} = \frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'}$ $\quad \frac{df}{dy} = 0$ $\quad \frac{df}{dy'} = -\frac{x y'}{\sqrt{1-y'^2}} \Rightarrow C_1 = \frac{df}{dy'}$ Bryant Har bjh254

$\Rightarrow C_2 x^2 y'^2 = 1 - y'^2 \Rightarrow 1 = (C_2 x^2 + 1) y'^2 \Rightarrow y' = \frac{1}{\sqrt{C_2 x^2 + 1}}$ $\left(\frac{d}{dx} \to \text{constant}\right)$

From table, $\frac{d}{dx} \operatorname{asinh}(\sqrt{C}x) = \frac{\sqrt{C}}{\sqrt{Cx^2+1}} \Rightarrow \boxed{y = A \sinh^{-1}(Bx)}$ A, B constants

Find partials → Solve for y' → Compare to table
(line 1) (line 2) (line 3)

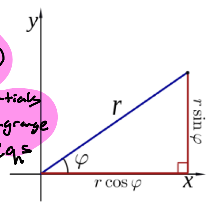
b) $T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \|\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}\|^2 = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2)$ (find T)

$U = mgh = mgr\sin\phi \Rightarrow L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\varphi}^2) - mgr\sin\varphi$ (Find U, L)

$\mathcal{L} = \frac{dL}{dr} - \frac{d}{dt}\frac{dL}{d\dot{r}} = 0 = mr\dot{\phi}^2 - mg\sin\phi - m\ddot{r}$ Sub partials into Lagrange-Euler eqn

$\frac{dL}{d\phi} - \frac{d}{dt}\frac{dL}{d\dot{\phi}} = 0 = -mgr\cos\phi - \frac{d}{dt} mr^2\dot{\phi}$
$= -mgr\cos\phi = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi}$

New topic:



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c) If each equation you initially write in (b) corresponds to either a particular component of Newton's

$$\boxed{\begin{cases} -mgr\cos\phi = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} & (1) \\ -mg\sin\phi = m\ddot{r} - mr\dot{\phi}^2 & (2) \end{cases}}$$

Collect eqs of motion

c)

In polar, $F = m\ddot{\vec{r}} = m[\underbrace{(\ddot{r} - r\dot{\phi}^2)}_{a_r,\ F_r = ma_r}\hat{r} + \underbrace{(2\dot{r}\dot{\phi} + r\ddot{\phi})}_{a_\phi,\ F_\phi = ma_\phi}\hat{\phi}]$ (Newton's 2nd)

No real explanation outside of intuition

From eq (2), $mg\sin\phi = m\ddot{r} - mr\dot{\phi}^2$ $\Rightarrow$ This corresponds to $F_r = mg\sin\phi$ or radial force is gravity × $\sin\phi$ in Newton's 2nd

From eq (1), $mgr\cos\phi = r[m(2\dot{r}\dot{\phi} + r\ddot{\phi})] = rF_\phi = \tau = rF_g\cos\phi$
$\Rightarrow$ This corresponds to torque being gravity × $\cos\phi$ × r (or equivalently, tangential force is gravity × $\cos\phi$ in Newton's 2nd)

In summary,

> I think they match because the terms match and it makes sense for gravity to be the only force acting on the particle.