Q28

$$\int_{-1}^{1} dx\; P_\ell'(x)P_{\ell'}'(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Part a

$$f(x) = \sum a_\ell P_\ell'(x)$$

For some i, by orthogonality,

$$\implies \int_{-1}^{1} dx\; P_i(x)f(x) = \frac{2a_i}{2i+1} \implies a_i = \frac{2i+1}{2}\int_{-1}^{1} dx\; P_i(x)f(x)$$

$$a_i = \frac{2i+1}{2}\left[\int_{-1}^{0} dx\; P_i(x)(x+1) + \int_{0}^{1} dx\; P_i(x)(1-x)\right]$$

We solve for $i = 0, 1, 2$, using desmos

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} dx\;(x+1) + \int_{0}^{1} dx\;(1-x)\right] = 0.5$$

$$a_1 = \frac{3}{2}\left[\int_{-1}^{0} dx\; x(x+1) + \int_{0}^{1} dx\; x(1-x)\right] = 0$$

$$a_2 = \frac{5}{2}\left[\int_{-1}^{0} dx\;\frac{3x^2-1}{2}(x+1) + \int_{0}^{1} dx\;\frac{3x^2-1}{2}(1-x)\right] = -0.625$$

We arrive at $\boxed{a_0 = 0.5, \quad a_1 = 0, \quad a_2 = -0.625}$

Part b

$\boxed{\text{Blows up to } +\infty \text{ or } -\infty \text{ at a polynomial rate}}$

For first 3 terms, blows up to $-\infty$. For fourth, blows up to $+\infty$.

The sign of the infinity depends on the parity of the last nonzero term's exponent, as that term will dominate as $|x| \to \infty$.

Part c

We repeat the process, but now the x term has a 1/2 in front

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} dx\;(\frac{1}{2}x+1) + \int_{0}^{1} dx\;(1-\frac{1}{2}x)\right] = 0.75$$

$$a_1 = \frac{3}{2}\left[\int_{-1}^{0} dx\; x(\frac{1}{2}x+1) + \int_{0}^{1} dx\; x(1-\frac{1}{2}x)\right] = 0$$

$$a_2 = \frac{5}{2}\left[\int_{-1}^{0} dx\;\frac{3x^2-1}{2}(\frac{1}{2}x+1) + \int_{0}^{1} dx\;\frac{3x^2-1}{2}(1-\frac{1}{2}x)\right] = -0.3125$$

Our constructed series is then

$$\boxed{a_0 = 0.75, \quad a_1 = 0, \quad a_2 = -0.3125}$$

$$\boxed{0.75 - 0.3125\left(\frac{3x^2-1}{2}\right)}$$

Part d

Repeating the process, we get

$$a_0 = \frac{1}{2}\left[\int_{-1}^{1} dx\ (1-x^2)\right] = \frac{2}{3}$$

$$a_1 = \frac{3}{2}\left[\int_{-1}^{0} dx\ x(1-x^2)\right] = 0$$

$$a_2 = \frac{5}{2}\left[\int_{-1}^{0} dx\ \frac{3x^2-1}{2}(1-x^2)\right] = -\frac{2}{3}$$

$$\boxed{a_0 = 2/3, \quad a_1 = 0, \quad a_2 = -2/3}$$

This is identical to the original function, which is to be expected.