Due: 11:59 PM Thursday Feb. 3

1. **Rotational Transformation**
   Some two dimensional vector $\vec{v}$ makes an angle ϕ with the x axis. The vector can be written in matrix representation either in terms of its components, or by its magnitude and angle:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} |\vec{v}| \cos\phi \\ |\vec{v}| \sin\phi \end{pmatrix} \tag{1}$$

   (a) Show that the rotation matrix

$$\hat{R} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{2}$$

   rotates the vector $\vec{v}$ by θ. Do this by showing that the resulting vector after rotation, $\vec{v}' = \hat{R}\vec{v}$ can be written as in (1) in terms of a new angle, $(\phi + \theta)$

   (b) Draw $\vec{v}$ and $\vec{v}'$ with components and labeled angles before and after the rotation.

   (c) Is the magnitude of the vector the same before and after rotation? Recall the magnitude of a vector can be written $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$ Show your work.

   (d) Construct the basis vectors $\hat{e}_1'$ and $\hat{e}_2'$ of $\vec{v}'$ by acting with $\hat{R}$ on the original basis vectors:

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

$$\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

   (e) Show by explicit multiplication that both sets of basis vectors satisfy the completeness relation:

$$\hat{e}_1\hat{e}_1^{\top} + \hat{e}_2\hat{e}_2^{\top} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \tag{5}$$

   note that the $^{\top}$ symbol denotes the transpose of a vector, e.g.,

$$\vec{v}^{\top} = \begin{pmatrix} v_x & v_y \end{pmatrix}. \tag{6}$$

2. **Normalization and Inner Products**
   The following discussion contains *more* than what you need to do the problem. Although complex numbers are discussed, you won't need them to do the problem. We'll use a mathematical operation called an "inner product", which is nearly identical to the dot product:

$$\langle\alpha|\alpha\rangle = \begin{pmatrix} \alpha_1^* & \alpha_2^* \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{7}$$

   Recall that to form a dot product using matrix multiplication, the row representation of one vector acted on the column vector of the other. As with the dot product, the inner product yields only a number- albeit a complex one. Taking $|\alpha\rangle$ and $|\beta\rangle$ as arbitrary vectors composed of orthonormal basis vectors $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ the action of the inner product becomes clear from several examples:

- $\langle\alpha|\, c\, |\beta\rangle = c\,\langle\alpha|\beta\rangle$ (constants commute)
- $\langle\alpha|\,(|\alpha\rangle + |\beta\rangle) = \langle\alpha|\alpha\rangle + \langle\alpha|\beta\rangle$ (distributive)
- $\langle\alpha|\beta\rangle = \alpha_0^*\beta_0 + \alpha_1^*\beta_1$, and $\langle\beta|\alpha\rangle = \beta_0^*\alpha_0 + \beta_1^*\alpha_1$, so $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$(non-commutative)
- $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$. (orthonormality of basis vectors)
- $\langle\alpha|\alpha\rangle = \alpha_0^*\alpha_0\,\langle 0|0\rangle + \alpha_0^*\alpha_1\,\langle 0|1\rangle + \alpha_1^*\alpha_0\,\langle 1|0\rangle\,\alpha_1^*\alpha_1 = \alpha_0^*\alpha_0 + \alpha_1^*\alpha_1 = |\alpha_1|^2 + |\alpha_1|^2$ (due to orthonormality)

A vector, $|\alpha'\rangle = 3\,|1\rangle + 4\,|0\rangle$ is NOT normalized. This means $\langle\alpha|\alpha\rangle \neq 1$.

(a) Find the vector $|\alpha\rangle$ that is parallel to $|\alpha'\rangle$, but is normalized. This can be done by adding a coefficient to the state

$$\left|\alpha'\right\rangle \Rightarrow c\left|\alpha'\right\rangle = |\alpha\rangle \tag{8}$$

and solving for c. The vectors representing the state of a qubit are normalized.

(b) Find the mathematical constraint on the basis vector coefficients α_0 and α_1 for a general state vector, $|\alpha\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$, if you assume that $|\alpha\rangle$ is normalized.

3. **Pauli Matrices and Eigenvalues**

An operator acting on a ket produces another ket:

$$\hat{A}\,|\alpha\rangle = |\gamma\rangle \tag{9}$$

The result, $|\gamma\rangle$, may be of different magnitude, parallel, or orthogonal to $|\alpha\rangle$, depending on $\hat{A}$. A special property of the *operator* is that some specific kets, when acted upon, yield the same ket multiplied by a constant:

$$\hat{A}\,|a\rangle = a\,|a\rangle \tag{10}$$

In this equation, the ket $|a\rangle$ is an **eigenvector** of the operator $\hat{A}$ with **eigenvalue** a. Recall that the a within the ket brackets $|a\rangle$ simply serves as an abstract label, in this case indicating that $|a\rangle$ is an eigenvector with eigenvalue a. For our purposes, an n-dimensional operator will have n independent eigenkets, each with a corresponding eigenvalue.

(a) Consider the matrix

$$\hat{Z} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{11}$$

Show $|0\rangle$ and $|1\rangle$ are eigenvectors of this matrix. What are the eigenvalues of each vector? (In other words, act with $\hat{Z}$ on $|0\rangle$ and $|1\rangle$ using matrix multiplication. You should find that you obtain the same ket as was acted upon, multiplied by its eigenvalue.) (Note: While these vectors are the unique eigenvectors of this matrix, these vectors can be eigenvectors for other matrices.)

(b) Are $|0\rangle$ and $|1\rangle$ eigenvectors of the following matrices? Show work.

$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{12}$$

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

(c) Write the following *linear combinations* of $\hat{Z}$ eigenvectors in matrix form:

$$|\pm x\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \tag{14}$$

$$|\pm y\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle) \tag{15}$$

(d) Show that $|\pm x\rangle$ and $|\pm y\rangle$ are eigenvectors of $\hat{X}$ and $\hat{Y}$ respectively, and give the eigenvalues for each. (Note: $\hat{Y}$, and the linear combinations of $\hat{Z}$ eigenvectors above are valid *only* for quantum bits, as we will see later. Do not try to apply this to classical bits.)

4. **Number Operator Algebra**
Consider the "number" operator,

$$\hat{N} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

a related operator,

$$\tilde{N} = \mathbb{1} - \hat{N} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

and the NOT or X gate:

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

Show the following relations in two ways. First, show by explicit matrix multiplication. Then, show by using the known properties of the operators, acting directly on $|0\rangle$ and $|1\rangle$ (without using matrix representation). In this sense, when an operator is written equal to another, it means that the action of either operator on any vector is identical. Recall

$$\hat{N}\,|1\rangle = |1\rangle \qquad \hat{N}\,|0\rangle = 0 \tag{19}$$

$$\tilde{N}\,|1\rangle = 0 \qquad \tilde{N}\,|0\rangle = |0\rangle \tag{20}$$

$$\hat{X}\,|1\rangle = |0\rangle \qquad \hat{X}\,|0\rangle = |1\rangle \tag{21}$$

(a) $\hat{N}^2 = \hat{N}$

(b) $\tilde{N}\hat{N} = \hat{N}\tilde{N} = 0$

(c) $\tilde{N}\hat{X} = \hat{X}\hat{N}$

5. **CNOT and SWAP and Binary Numbers**

- The **SWAP** operator interchanges the value of the two bits upon which it acts.

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

$$S\,|10\rangle = |01\rangle \qquad S\,|01\rangle = |10\rangle \qquad S\,|00\rangle = |00\rangle \qquad S\,|11\rangle = |11\rangle\,. \tag{23}$$

  Notice that here, we don't have to define which bit S acts upon, since it acts indiscriminately on both bits. i.e., $S_{01} = S_{10}$.

- The **CNOT** (or controlled-not) gate is somewhat more complicated, in that it acts differently on particular bits within the register of bits. It has the action of modulo-2 addition. One bit (the "condition bit") is left unchanged, and the other bit ("target bit") is "XOR"-gated with the condition bit. The overall action of CNOT can be summarized as flipping the target bit if and only if the condition bit is 1. (here, commas are used to distinguish between bit states within a ket)

$$C_{10}\,|a,b\rangle = |a, b\oplus a\rangle \qquad C_{01}\,|a,b\rangle = |a\oplus b, b\rangle \tag{24}$$

  where the $\oplus$ refers to addition, modulo-2 (which is identical to the XOR gate). The **XOR** gate between two bits works as:

$$0\oplus 0 = 0 \qquad 0\oplus 1 = 1 \qquad 1\oplus 0 = 1 \qquad 1\oplus 1 = 0 \tag{25}$$

  so the XOR gate can be thought of as returning 1 if (exclusively) one of the bits is 1. There are two different CNOT gates for a two-bit system:

$$C_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad C_{01} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{26}$$

Recall how binary numbers represent integers. With n bits, one can represent a total of 2^n numbers, usually starting from 0 ending with $2^n - 1$. So, with two bits, the bit sequences $00, 01, 10, 11$ represent $0, 1, 2, 3$ respectively. The n^{th} place in a string of bits represents 2^n-times the bit value starting with $n = 0$ on the right. e.g., "110" $\Rightarrow 1 * 2^2 + 1 * 2^1 + 0 * 2^0 = 6$

(a) Make a truth table for both C_{01} and C_{10} gates. There should be 4 rows– one for each of the possible two-bit states. The 5 total columns should be for the output of C_{10}, C_{01}, and columns that indicate the numeric operation being performed (write the integer the state represented before and after applying the gates).

(b) Show that $S = C_{10}C_{01}C_{10}$ both with explicit matrix multiplication and without (see problem 4).