3610

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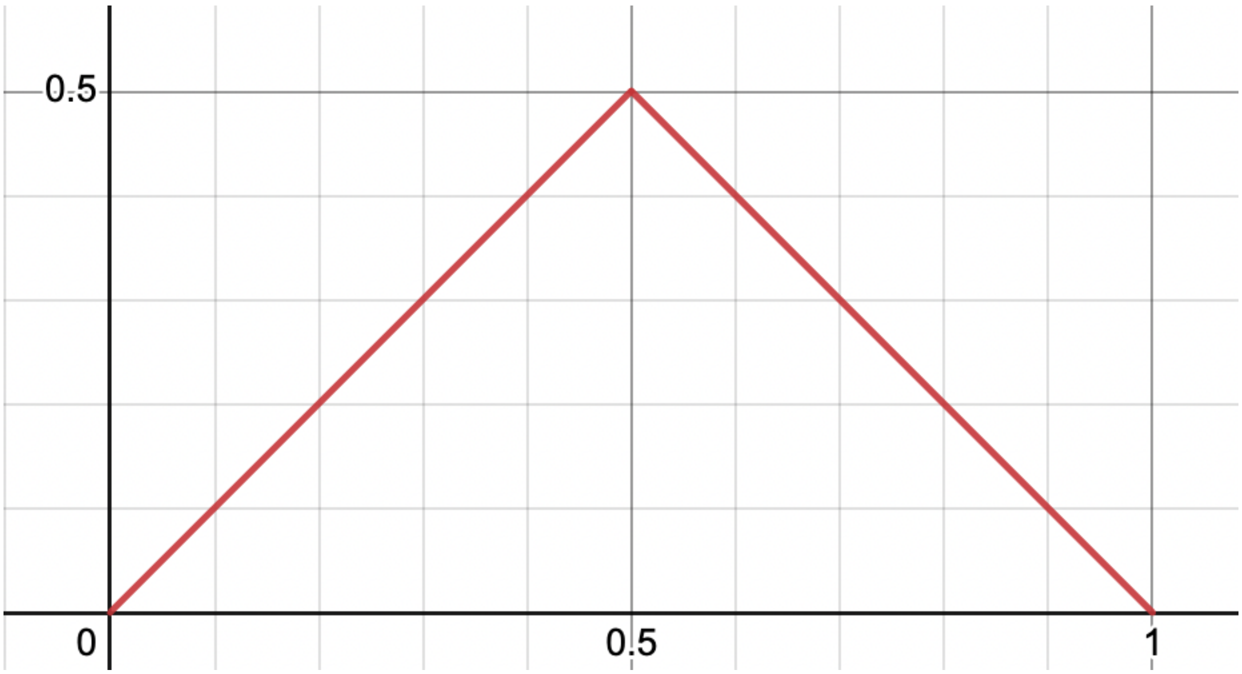
Q1: 2.7

a



Graph of $\Psi(x, 0)$ for $a = A = 1$. Integrating,

$$\int |\Psi|^2 dx = 1 = \int_0^{a/2} dx A^2 x^2 + \int_{a/2}^{a} dx A^2 (a - x)^2$$

$$= \frac{A^2 a^3}{24} + \frac{A^2 a^3}{24} \implies \boxed{A = \sqrt{\frac{12}{a^3}}}$$

b

b) Get Fourier $\Rightarrow$ $c_n = A\sqrt{\frac{2}{a}}\left[\int_0^{a/2} x \sin\frac{n\pi}{a}x\,dx + \int_{a/2}^{a}(a-x)\sin\left(\frac{n\pi}{a}x\right)dx\right]$

$= \frac{4\sqrt{6}}{(n\pi)^2}\sin\frac{n\pi}{2}$ by calc.

$\Rightarrow$ Ψ = fourier

$\Rightarrow$ $\boxed{\Psi = \frac{4\sqrt{6}}{\pi^2}\sum_{\substack{n=1,3,\dots\\ \text{odd}}} (-1)^{\frac{n-1}{2}} \frac{\sin\left(\frac{n\pi}{a}x\right)}{n^2} e^{-E_n t/\hbar}}$

Substituting calculator results into fourier series defined at top and simplifying,

$$\Longrightarrow \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad \Psi(x,t) = \frac{8}{\pi^2}\sqrt{\frac{3}{a}}\sum_{n=1,3,5\ldots}(-1)^{(n-1)/2}\frac{\sin(\frac{n\pi x}{a})}{n^2}e^{-E_n t/\hbar}}$$

For odd n

c

$$P = c_1^2 = \left(\frac{4\sqrt{6}}{1\pi^2}\right)^2 = 16\cdot 6/\pi^4 = \boxed{P = \frac{96}{\pi^4}}$$

d

$$\langle H\rangle = \sum c_n^2 E_n = \sum\left(\frac{4\sqrt{6}}{n^2\pi^2}\right)^2\frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{48\hbar^2}{\pi^2 ma^2}\sum 1/n^2 = \frac{48\hbar^2}{\pi^2 ma^2}\frac{\pi^2}{8} = \boxed{\frac{6\hbar^2}{ma^2}}$$

Q2

a

Plugging in the relevant values,

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \Longrightarrow E_1 = \frac{\pi^2\hbar^2}{2ma^2} = \boxed{204.46\text{MeV}}$$

b

i) Using $m_e = 9.11\cdot 10^{-31}$, $\boxed{E = 375914\text{MeV}}$

ii) We take distance to be half the width of the well,

$$U = \frac{kqq}{r}$$

$$U(\infty) = 0, \quad U(a/2) = \frac{2(k40e^2)}{a}$$

$$U(\infty) - U(a/2) = 1.841152\cdot 10^{-11}J = \boxed{= 115\text{MeV}}$$

iii) $\boxed{\text{No. The nucleus cannot confine the electron.}}$. $375914 > 115$. The electron has too much energy and will easily escape.

c

We seek an a such that the energies are equal.

$$\frac{80kq^2}{a} = \frac{\pi^2 h^2}{2ma^2}$$

$$\frac{80kq^2}{a} = \frac{\pi^2 h^2}{2ma^2}$$

Solving,

$$\boxed{a = 3.27 \cdot 10^{-12}\text{m}}$$

We require a nucleus diameter of $3.27 \cdot 10^{-12}$ meters.

Q3

a

The wave function is of the form

$$\Psi(x,t) = \sum_{n\geq 1} c_n \psi_n e^{-itE_n/\hbar}$$

$$\Psi(x,t) = \sum_{n\geq 1} c_n \psi_n e^{-it(n^2\hbar\pi^2/2ma^2)}$$

$$\Psi(x,0) = \sum_{n\geq 1} c_n \psi_n$$

$$\Psi(x,T) = \sum_{n\geq 1} c_n \psi_n e^{-i(4ma^2/\pi\hbar)(n^2\hbar\pi^2/2ma^2)}$$

$$\Psi(x,T) = \sum_{n\geq 1} c_n \psi_n e^{-i(2n^2\pi)}$$

Since e^{ix} is 2π-periodic, and $2n^2\pi$ is a multiple of 2π, so

$$\Psi(x,T) = \sum_{n\geq 1} c_n \psi_n = \Psi(x,0)$$

$\boxed{\text{The wave function returns to its original form after } T \text{ time.}}$

b

$$E = \frac{1}{2}mv^2 \implies v = \sqrt{2E/m}$$

The classical revival time, T_c, is given by the time to traverse the width and return,

$$T_c = 2a/v = \boxed{T_c = a\sqrt{\frac{2m}{E}}}$$

c

Equating the two,

$$T_c = a\sqrt{\frac{2m}{E}} = \frac{4ma^2}{\pi\hbar} \implies \boxed{E = \frac{\pi^2\hbar^2}{8ma^2} = \frac{E_1}{4}}$$

Q4

a

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \psi e^{-ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ e^{-\alpha|x|+i\beta x}e^{-ikx}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dx\ e^{\alpha x+i\beta x}e^{-ikx} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\ e^{-\alpha x+i\beta x}e^{-ikx}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dx\ e^{(\alpha+i(\beta-k))x} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\ e^{(-\alpha+i(\beta-k))x}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left.\frac{\mathrm{e}^{(\mathrm{i}\cdot(\beta-k)+\alpha)x}}{\mathrm{i}\cdot(\beta-k)+\alpha}\right|_{-\infty}^{0} + \left.\frac{\mathrm{e}^{(\mathrm{i}\cdot(\beta-k)-\alpha)x}}{\mathrm{i}\cdot(\beta-k)-\alpha}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\mathrm{i}k-\mathrm{i}\beta+\alpha}{k^2-2\beta k+\beta^2+\alpha^2} - \frac{\mathrm{i}k-\mathrm{i}\beta-\alpha}{k^2-2\beta k+\beta^2+\alpha^2}\right]$$

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}}\frac{2\alpha}{k^2-2\beta k+\beta^2+\alpha^2}$$

$$\boxed{\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}}\frac{2\alpha}{(k-\beta)^2+\alpha^2}}$$

b

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \psi e^{-ikx}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \left(\sin\frac{2\pi(x-x_0)}{a}\right)e^{-ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \left(\frac{e^{i\frac{2\pi(x-x_0)}{a}} - e^{-i\frac{2\pi(x-x_0)}{a}}}{2i}\right)e^{-ikx}$$

$$= \frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \left(e^{i\frac{2\pi(x-x_0)}{a}} - e^{-i\frac{2\pi(x-x_0)}{a}}\right)e^{-ikx}$$

$$= \frac{1}{2i\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \left(e^{i\frac{2\pi(x-x_0)}{a}}e^{-ikx} - e^{-i\frac{2\pi(x-x_0)}{a}}e^{-ikx}\right)$$

We recognize both terms as the inverse fourier transform of the complex exponential with a delay.

$$\boxed{\mathcal{F}(k) = \frac{1}{2i}\left[e^{-\frac{2\pi x_0}{a}}\delta(k-\frac{2\pi}{a}) + e^{\frac{2\pi x_0}{a}}\delta(k+\frac{2\pi}{a})\right]}$$

c

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \psi e^{-ikx}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \mathrm{rect}(\frac{x-b}{a})e^{-ikx}$$

$$\frac{1}{\sqrt{2\pi}}\int_{b-a/2}^{b+a/2} dx\ e^{-ikx}$$

$$\frac{i}{k\sqrt{2\pi}}\left(\mathrm{e}^{-\mathrm{i}k(b+a/2)} - \mathrm{e}^{-\mathrm{i}k(b-a/2)}\right)$$

$$\frac{2}{k\sqrt{2\pi}}\frac{\mathrm{e}^{-\mathrm{i}k(b-a/2)} - \mathrm{e}^{-\mathrm{i}k(b+a/2)}}{2i}$$

$$\frac{\mathrm{e}^{-\mathrm{i}bk}}{k/2\sqrt{2\pi}}\sin(ak/2)$$

$$\frac{a\mathrm{e}^{-\mathrm{i}bk}}{\sqrt{2\pi}}\frac{\sin(ak/2)}{ak/2}$$

$$\boxed{\mathcal{F}(k) = \frac{a}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}bk}\mathrm{sinc}(ak/2)}$$

d

$$\psi^*\psi = (e^{-\alpha|x|-i\beta x})(e^{-\alpha|x|+i\beta x}) = e^{-2\alpha|x|} \quad \mathcal{F}(k) = \frac{1}{\sqrt{2\pi}}\frac{2\alpha}{(k-\beta)^2+\alpha^2}$$

b is clearly a centering constant, so we can set $b = 0$ without affecting shape and width. Then, it is sufficient to prove that the product of the following functions' width is constant.

$$e^{-2\alpha|x|}, \quad \frac{1}{\sqrt{2\pi}}\frac{\alpha}{k^2+\alpha^2}$$

Amplitude is clearly maximized and equal to 1 and $\frac{1}{a\sqrt{2\pi}}$ respectively at $x = k = 0$. At half maximum,

$$e^{-2\alpha|x|} = \frac{1}{2}, \quad \frac{1}{\sqrt{2\pi}}\frac{\alpha}{k^2+\alpha^2} = \frac{1}{2a\sqrt{2\pi}}$$

We solve for and double x to get width, since both functions are even.

$$\Delta x = \frac{2}{-2\alpha}\ln(\frac{1}{2}), \quad \Delta k = 2\alpha$$

Taking their product,

$$\Delta x \Delta k = \frac{2 \cdot 2\alpha}{-2\alpha}\ln(\frac{1}{2}) = \boxed{\Delta x \Delta k = 2\ln 2}$$

$\boxed{\text{Clearly, this is a constant. No variable dependence.}}$

Q5

a

This cannot be computed in elementary functions, so by calculator,

$$\int_{-\infty}^{\infty} dx\ |\Psi(x,0)|^2 = \int_{-\infty}^{\infty} dx\ A^2 e^{-2ax^2} = A^2\sqrt{\frac{\pi}{2a}} = 1 \implies \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}}$$

b

Using integral table and given identity,

$$\phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ \Psi(x,0)e^{-ikx} = \frac{1}{\sqrt{2\pi}}(\frac{2a}{\pi})^{1/4}\int_{-\infty}^{\infty} dx\ e^{-ax^2}e^{-ikx} = \frac{\mathrm{e}^{-\frac{k^2}{4a}}}{\sqrt[4]{2\pi a}}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \phi e^{i(kx-\frac{\hbar k^2}{2m}t)} = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty} dk\ e^{-k^2/4a}e^{i(kx-\frac{\hbar k^2}{2m}t)}$$

Letting $\xi = 1 + 2i\hbar at/m$ and using tables and tabular integration,

$$\Psi(x,t) = A\frac{e^{-ax^2/\xi}}{\sqrt{\xi}} = \boxed{\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{-\frac{ax^2}{1+2i\hbar at/m}}}{\sqrt{1+2i\hbar at/m}}}$$

which is the same as the book provides.

c

Recall that $\xi = 1 + 2i\hbar at/m \implies \xi^* = 1 - 2i\hbar at/m$,
Note that $w = \sqrt{\frac{a}{\xi\xi^*}} \implies w^2 = \frac{a}{\xi\xi^*}$. Then,

$$|\Psi|^2 = A^2\frac{e^{-ax^2/\xi}}{\sqrt{\xi}}\frac{e^{-ax^2/\xi^*}}{\sqrt{\xi^*}}$$

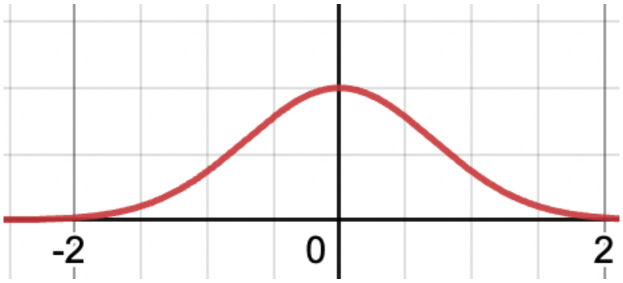
$$|\Psi|^2 = \sqrt{\frac{2a}{\pi}}\frac{e^{-ax^2/\xi-ax^2/\xi^*}}{\sqrt{\xi\xi^*}}$$

$$|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-ax^2(\xi+\xi^*)/\xi\xi^*}$$

$$|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2ax^2/\xi\xi^*}$$

$$\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-w^2x^2}}$$

For $t = 0$, we have the first plot. For $t = 5$, we have the second plot. $\boxed{\text{As } t \text{ increases, } |\Psi|^2 \text{ flattens and spreads out}}$



d

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\ x|\Psi|^2 = \langle x \rangle = 0$$

since odd

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\ x^2|\Psi|^2 = \int_{-\infty}^{\infty} dx\ x^2|\Psi|^2 = \int_{-\infty}^{\infty} dx\ \sqrt{\frac{2}{\pi}} w e^{-aw^2x^2} = \langle x^2 \rangle = \frac{1}{4w^2}$$

by calc

$$\langle p \rangle = m\langle \dot{x} \rangle = \boxed{\langle p \rangle = 0}$$

Let $C = \frac{a}{1+i2\hbar at/m}$. By calc,

$$\Psi^* \frac{d^2\Psi}{dx^2} = -2C(1 - 2Cx^2)|\Psi|^2$$

By calc,

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} dx\ \ -2C(1 - 2Cx^2)|\Psi|^2 = \hbar^2 a$$

$$\sigma_x = \sqrt{0 - \frac{1}{4w^2}} = \frac{1}{2w}$$

$$\sigma_p = \sqrt{0 - \hbar^2 a} = \hbar\sqrt{a}$$

Collecting,

$$\boxed{\langle x \rangle = 0 \quad \langle x^2 \rangle = \frac{1}{4w^2} \quad \sigma_x = \frac{1}{2w} \quad \langle p \rangle = 0 \quad \langle p^2 \rangle = \hbar^2 a \quad \sigma_p = \hbar\sqrt{a}}$$

e

$$\sigma_x \sigma p = \frac{\hbar\sqrt{a}}{2w} = \frac{\hbar}{2}\sqrt{1 + (2\hbar a t/m)^2} \geq \frac{\hbar}{2} \implies \boxed{\text{Yes, holds}}$$

By inspection, the radicand is closest to 0 when second term zeros out at $t = 0$. Then, $\sigma_x \sigma p = \frac{\hbar}{2}$.

$$\implies \boxed{t = 0}$$