Homework #13 **AEP 3610** **Due December 4**

1. Griffiths 4.18
2. Griffiths 4.30
3. Griffiths 4.32
4. Griffiths 4.33
5. Griffiths 4.37

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Q4.18

a) $\Psi(r,0) = \frac{1}{\sqrt{2}}\left(\psi_{211} + \psi_{21-1}\right)$ $\quad \psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}}\, e^{-r/na} \cdot \left(\frac{2r}{na}\right)^\ell \left[L^{2\ell+1}_{n-\ell-1}(2r/na)\right] Y_\ell^m$

$$\psi_{211} = \sqrt{\left(\frac{1}{a}\right)^3 \frac{1}{24}}\, e^{-r/2a}\left(\frac{r}{a}\right)\left[\cancel{L_0^3}\,1\,(r/a)\right] Y_1^1$$

$$\psi_{21-1} = \sqrt{\left(\frac{1}{a}\right)^3 \frac{1}{24}}\, e^{-r/2a}\left(\frac{r}{a}\right)\left[\cancel{L_0^3}\,1\,(r/a)\right] Y_1^{-1}$$

$$\boxed{\Psi(r,0) = \frac{r a^{-5/2}}{4\sqrt{3}}\, e^{-r/2a}\left(Y_1^1 + Y_1^{-1}\right)}$$

b) Energy only dependent on principal quantum number. By textbook

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \Rightarrow \boxed{E_\Psi = -\frac{m_e}{8\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -3.4\text{ eV}}$$

(superposition of n=2 states)

Q4.30

a) $\chi^\dagger\chi = 1 = A^2(9+16) \Rightarrow \boxed{A = \frac{1}{5}}$

b) $\langle S_x\rangle = \chi^\dagger S_x \chi = \frac{\hbar}{50}\begin{bmatrix}-3i & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3i\\ 4\end{bmatrix} = S_x = \frac{\hbar}{50}(12i - 12i) = 0$

$\langle S_y\rangle = \chi^\dagger S_y \chi = \frac{\hbar}{50}\begin{bmatrix}-3i & 4\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}3i\\ 4\end{bmatrix} = S = \frac{\hbar}{50}(-12-12) = -\frac{12\hbar}{25}$

$\langle S_z\rangle = \chi^\dagger S_z \chi = \frac{\hbar}{50}\begin{bmatrix}-3i & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3i\\ 4\end{bmatrix} = S_z = \frac{\hbar}{50}(9-16) = -\frac{7}{50}\hbar$

$$\boxed{\begin{aligned}\langle S_x\rangle &= 0\\ \langle S_y\rangle &= -\frac{12}{25}\hbar\\ \langle S_z\rangle &= -\frac{7}{50}\hbar\end{aligned}}$$

c) $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}$

$\Rightarrow \sigma_x^2 = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4} - 0 \Rightarrow \sqrt{\sigma_x^2} =$

$\sigma_y^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4} - \frac{144\hbar^2}{625} \Rightarrow \sqrt{\sigma_y^2} =$

$\sigma_z^2 = \langle S_z^2\rangle - \langle S_z\rangle^2 = \frac{\hbar^2}{4} - \frac{49\hbar^2}{2500} \Rightarrow \sqrt{\sigma_z^2} =$

$$\boxed{\begin{aligned}\sigma_x &= \frac{1}{2}\hbar\\ \sigma_y &= \frac{7}{50}\hbar\\ \sigma_z &= \frac{12}{25}\hbar\end{aligned}}$$

d)

$\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle S_z\rangle| \Rightarrow \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \geq \frac{\hbar}{2}\cdot\frac{7}{50}\hbar \Rightarrow \frac{\hbar^2}{2}\cdot\frac{7}{50} \geq \frac{7}{50}\cdot\frac{\hbar^2}{2}$ ✓

$\sigma_{S_z}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle S_x\rangle| \Rightarrow \frac{12}{25}\hbar\cdot\frac{7}{50}\hbar \geq \frac{\hbar}{2}\cdot 0 \Rightarrow \frac{12}{25}\frac{7}{50}\hbar^2 \geq 0$ ✓

$\sigma_{S_x}\sigma_{S_z} \geq \frac{\hbar}{2}|\langle S_y\rangle| \Rightarrow \frac{\hbar}{2}\cdot\frac{12}{25}\hbar \geq \frac{\hbar}{2}\cdot\frac{12}{25}\hbar \Rightarrow \frac{6}{25}\hbar^2 \geq \frac{6}{25}\hbar^2$ ✓

All true inequalities (z, y exactly equal to uncertainty limit)

Q4.32

a) $S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \Rightarrow \begin{bmatrix}-\lambda & -\frac{\hbar i}{2}\\ \frac{\hbar i}{2} & -\lambda\end{bmatrix} \Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0 \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$

$\begin{bmatrix}-\hbar/2 & -\hbar i/2\\ -\hbar i/2 & -\hbar/2\end{bmatrix} \Rightarrow \boxed{\chi_+ = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}}$ $\quad \begin{bmatrix}\hbar/2 & -\hbar i/2\\ -\hbar i/2 & \hbar/2\end{bmatrix} \Rightarrow \boxed{\chi_- = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}}$ $\quad$ clear from λ-added matrix what eigenvecs are

b) Project onto eigenvectors...

$\lambda = \frac{\hbar}{2}: \chi_+^\dagger \begin{bmatrix}a\\ b\end{bmatrix} = \frac{1}{\sqrt{2}}(a - ib) \Rightarrow$ ← dagger

$\lambda = -\frac{\hbar}{2}: \chi_-^\dagger \begin{bmatrix}a\\ b\end{bmatrix} = \frac{1}{\sqrt{2}}(a + ib) \Rightarrow$

$$\boxed{\begin{aligned}&+\frac{\hbar}{2}, \text{ probability } \frac{1}{\sqrt{2}}|a - ib|^2\\ &-\frac{\hbar}{2}, \text{ probability } \frac{1}{\sqrt{2}}|a + ib|^2\end{aligned}}$$

c) $S_y^2 = 50\%\cdot\left(\frac{\hbar}{2}\right)^2 + 50\%\left(-\frac{\hbar}{2}\right)^2 = S_y^2 \to \boxed{\frac{\hbar^2}{4} \quad \text{Probability } 1}$

Q4.33

$$S_r = S_x(\overbrace{S_r\cdot i}^{\sin\theta\cos\phi}) + S_y(\overbrace{S_r\cdot j}^{\sin\theta\sin\phi}) + S_z(\overbrace{S_r\cdot k}^{\cos\theta})$$

$$S_r = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{bmatrix} = \boxed{S_r = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{bmatrix}}$$

$$\frac{\hbar}{2}\begin{vmatrix} \cos\theta-\lambda & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\lambda-\cos\theta \end{vmatrix} = \frac{\hbar}{2}[-\cos^2\theta+\lambda^2-\sin^2\theta]=0 \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}}$$

$$\Rightarrow \begin{bmatrix} \cos\theta-1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -1-\cos\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \vec{0} \Rightarrow \begin{matrix} a[\cos\theta-1] = -be^{-i\phi}\sin\theta \\ ae^{i\phi}\sin\theta = b[1+\cos\theta] \end{matrix} \xRightarrow{\text{by system solver}} \begin{bmatrix} a \\ b \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta+1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \vec{0} \Rightarrow \begin{matrix} a[\cos\theta+1] = -be^{-i\phi}\sin\theta \\ ae^{i\phi}\sin\theta = b[1-\cos\theta] \end{matrix} \xRightarrow{\text{by system solver}} \begin{bmatrix} a \\ b \end{bmatrix} =$$

$$\boxed{\begin{matrix} \lambda = \hbar/2 \quad \chi_+ = \begin{bmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix} \\ \lambda = -\hbar/2 \quad \chi_- = \begin{bmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{bmatrix} \end{matrix}}$$

Q4.37

a) $$S_-|10\rangle = (S_-^{(1)} + S_-^{(2)})\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{1}{\sqrt{2}}\left[(S_-^{(1)}\uparrow)\downarrow + \uparrow\cancelto{0}{(S_-^{(2)}\downarrow)} + \cancelto{0}{(S_-^{(1)}\downarrow)}\uparrow + \downarrow(S_-^{(2)}\uparrow)\right] = \frac{1}{\sqrt{2}}\left[(S_-^{(1)}\uparrow)\downarrow + \downarrow(S_-^{(2)}\uparrow)\right]$$

$$\frac{1}{\sqrt{2}}\left[(S_-^{(1)}\uparrow)\downarrow + \downarrow(S_-^{(2)}\uparrow)\right] = \frac{1}{\sqrt{2}}\left[\hbar\downarrow\downarrow + \hbar\downarrow\downarrow\right] = \frac{2\hbar}{\sqrt{2}}|1-1\rangle = \boxed{S_-|10\rangle = \hbar\sqrt{2}|1-1\rangle} \checkmark$$

b) $$S_\pm|00\rangle = (S_\pm^1 + S_\pm^2)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}\left[(S_\pm\uparrow)\downarrow - (S_\pm\downarrow)\uparrow + \uparrow(S_\pm\downarrow) - \downarrow(S_\pm\uparrow)\right]$$

$$S_+|00\rangle = \frac{1}{\sqrt{2}}\left[\cancel{(S_+\uparrow)}\downarrow - (S_+\downarrow)\uparrow + \uparrow(S_+\downarrow) - \cancel{\downarrow(S_+\uparrow)}\right] = \frac{1}{\sqrt{2}}\left[\hbar\uparrow\uparrow - \hbar\uparrow\uparrow\right] = \boxed{0 = S_+|00\rangle} \checkmark$$

$$S_-|00\rangle = \frac{1}{\sqrt{2}}\left[(S_-\uparrow)\downarrow - \cancel{(S_-\downarrow)\uparrow} + \cancel{\uparrow(S_-\downarrow)} - \downarrow(S_-\uparrow)\right] = \frac{1}{\sqrt{2}}\left[\hbar\downarrow\downarrow - \hbar\downarrow\downarrow\right] = \boxed{0 = S_-|00\rangle} \checkmark$$

c) $$S^2|11\rangle = \left[(S^1)^2 + (S^2)^2 + 2S^1\cdot S^2\right]|\uparrow\uparrow\rangle = (S^2\uparrow)\uparrow + \uparrow(S^2\uparrow) + 2[S_x\uparrow S_x\uparrow + S_y\uparrow S_y\uparrow + S_z\uparrow S_z\uparrow]$$

$$= \frac{3}{4}\hbar^2\uparrow\uparrow + \frac{3}{4}\hbar^2\uparrow\uparrow + 2\left[\frac{\hbar^2}{4}\uparrow\uparrow\right] = S^2|11\rangle = 2\hbar^2\uparrow\uparrow = 2\hbar^2|11\rangle$$

$$\boxed{S^2|11\rangle = \underbrace{2\hbar^2}_{\lambda}|11\rangle}$$

$$S^2|1-1\rangle = \left[(S^1)^2 + (S^2)^2 + 2S^1\cdot S^2\right]|\downarrow\downarrow\rangle = (S^2\downarrow)\downarrow + \downarrow(S^2\downarrow) + 2[S_x\downarrow S_x\downarrow + S_y\downarrow S_y\downarrow + S_z\downarrow S_z\downarrow]$$

$$= \frac{3}{4}\hbar^2\downarrow\downarrow + \frac{3}{4}\hbar^2\downarrow\downarrow + 2\left[\frac{\hbar^2}{4}\downarrow\downarrow\right] = S^2|1-1\rangle = 2\hbar^2\downarrow\downarrow = 2\hbar^2|1-1\rangle$$

$$\boxed{S^2|1-1\rangle = \underbrace{2\hbar^2}_{\lambda}|1-1\rangle}$$