#physics

Q1

Q1a

We know that $I_{ij} = m_\alpha(\delta_{ij}|r'_\alpha|^2 - x_{\alpha i}x_{\alpha j})$,
Then, the middle row is

$$\boxed{-m_\alpha\left[x_{\alpha 2}x_{\alpha 1}\right] \quad -m_\alpha\left[x_{\alpha 1}^2 + x_{\alpha 3}^2\right] \quad -m_\alpha\left[x_{\alpha 2}x_{\alpha 3}\right]}$$

Q1b

In such a case, by symmetry

$$\boxed{I_{11} = I_{22}}$$

Similarly, substituting the above in, we clearly see

$$\boxed{\frac{1}{2}(I_{11} + I_{22}) = I_{11}}$$

Q1c

We begin

$$I_{xx} = \int\int\int dxdydz\ \gamma(y^2 + z^2) = \int\int\int dV\ \gamma(y^2 + z^2)$$

Recall that from previous question,

$$\frac{1}{2}(I_{xx} + I_{yy}) = I_{xx} \implies I_{xx} = \frac{1}{2}\int\int\int dV\ \gamma(x^2 + y^2 + 2z^2)$$

Converting the cartesian variables to cylindrical,

$$dV = \rho d\rho d\phi dz \implies I_{xx} = \frac{1}{2}\int\int\int d\rho d\phi dz\ \gamma\rho(\rho^2 + 2z^2)$$

$$\boxed{I_{xx} = \frac{\gamma}{2}\int_0^{2\pi} d\phi \int_{z_{min}}^{z_{max}} dz \int_{\rho_{min}}^{\rho_{max}} (\rho^2 + 2z^2)\rho\, d\rho}$$

Q1d

Since the object has rotational symmetry about the z-axis, we find that

$$\boxed{I_{xy} = 0}$$

This is by axial symmetry. I_{xy} is the product of the x and y coordinates summed over all the masses dm. Since there is axial symmetry about the z axis, this means that mass is symmetrically distributed across the x and y directions. Then, the object's mass distribution is mirrored about the z axis. Therefore, the product of the x and y for each point of mass is mirrored by a corresponding mass on the other side, and therefore the contribution is canceled out by its corresponding mirrored point mass. Contributions from one side of the z axis will be canceled out by equal contributions on the other side of the axis.

Q1e

By the above argument, since there is axial symmetry, all the off diagonal elements are zero.

$$I_{ij} \propto \delta_{ij}$$

Further, since z is an axis of symmetry for the object, we observe that $I_{11} = I_{22}$ as above in part b,
We now find the diagonal elements using the result from c,

$$I_{xx} = I_{yy} = \frac{\gamma}{2} \int_0^{2\pi} d\phi \int_{z_{min}}^{z_{max}} dz \int_{\rho_{min}}^{\rho_{max}} (\rho^2 + 2z^2)\rho \, d\rho$$

By calculator,

$$= \frac{\gamma}{2} \int_0^{2\pi} d\phi \int_{-z_{sc}}^{z_{sc}} dz \int_0^{\rho_{sc}\sqrt{1-\frac{z^2}{z_{sc}^2}}} (\rho^2 + 2z^2)\rho \, d\rho = \frac{4\gamma\pi\rho_{\mathrm{sc}}^2 z_{\mathrm{sc}} \left(z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2\right)}{15}$$

We solve I_{zz} the same way by calculating this integral using calculator,

$$I_{zz} = \gamma \int_0^{2\pi} d\phi \int_{-z_{sc}}^{z_{sc}} dz \int_0^{\rho_{sc}\sqrt{1-\frac{z^2}{z_{sc}^2}}} (\rho^2)\rho \, d\rho = \gamma\pi \frac{8\rho_{\mathrm{sc}}^4 z_{\mathrm{sc}}}{15}$$

Our final tensor is,

$$I = \begin{pmatrix} \frac{4\gamma\pi\rho_{\mathrm{sc}}^2 z_{\mathrm{sc}} \left(z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2\right)}{15} & 0 & 0 \\ 0 & \frac{4\gamma\pi\rho_{\mathrm{sc}}^2 z_{\mathrm{sc}} \left(z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2\right)}{15} & 0 \\ 0 & 0 & \frac{\gamma\pi 8\rho_{\mathrm{sc}}^4 z_{\mathrm{sc}}}{15} \end{pmatrix}$$

Using $V = \frac{4\pi\gamma}{3}\rho_{sc}^2 z_{sc} \implies \gamma = \frac{M}{V}$,

$$\boxed{I = M \begin{pmatrix} \frac{z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2}{5} & 0 & 0 \\ 0 & \frac{z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2}{5} & 0 \\ 0 & 0 & \frac{2\rho_{\mathrm{sc}}^2}{5} \end{pmatrix}}$$

Q1f

We do the matrix multiplication,

$$\vec{L} = I\vec{\omega} = M \begin{bmatrix} \frac{z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2}{5} & 0 & 0 \\ 0 & \frac{z_{\mathrm{sc}}^2 + \rho_{\mathrm{sc}}^2}{5} & 0 \\ 0 & 0 & \frac{2\rho_{\mathrm{sc}}^2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \frac{2M\rho_{\mathrm{sc}}^2\omega}{5}$$

$$\boxed{\vec{L} = \frac{2M\rho_{\mathrm{sc}}^2\omega}{5}\hat{e}_z}$$