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Q2

a

The up component is normal to the surface of the ground and must equal the downward force of gravity.

$$\boxed{F_{up} = Mg}$$

b

For circular motion, we require a centripetal force:

$$\boxed{F_{inward} = m(R - r\cos\theta)\Omega^2}$$

c

$\boxed{\text{None (except for gravity}}$

There is gravity, Mg, pointing downward vertically acting on the disk, but we consider contributions from the ground. In cylindrical coordinates,

$$\frac{F_\phi}{m} = 2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}$$

$\dot{\rho} = \ddot{\phi} = 0$, so there is no contributions from either type of ϕ direction force to the ϕ direction acceleration.

d

$\boxed{\text{Yes}}$ from the diagram, torque is perpendicular to the page by cross product.

e

We have the net torque, as they point in opposite directions

$$\tau = r \times F$$

$$\boxed{\tau_{into} = rF_{inward}\sin\theta - rF_{up}\cos\theta}$$

f

$\boxed{\text{Yes}}$, the disk spins as it rolls without slipping and spinning must be opposite sign to capital omega.

g

$\boxed{\text{Into the page}}$ in order to preserve right hand rule with the given 2 and 3 axes.

h

The 1 and 2 axes rotate, so they have some time dependence. The 3-axis is constant as defined before.
We then have a 1 and 2 contribution from Ω
Overall,

$$\boxed{\omega_1 = \Omega\sin(\omega' t)\sin\theta, \quad \omega_2 = \Omega\cos(\omega' t)\sin\theta, \quad \omega_3 = \Omega\cos\theta - w'}$$

i

By table, we know that along 3-axis, the moment of inertia is $\frac{1}{2}mr^2$. Along a diameter axis, it is $\frac{1}{4}mr^2$, corresponding to the 1 and 2 axes. Since the 1,2,3 axes are principal axes clearly by symmetry, the off-diagonal components are zero. Alternatively, instead of looking up in a table, integration could be done to reach the same result.

Drawing from these conclusions, we find that

$$\boxed{I = \begin{bmatrix} \frac{1}{4}Mr^2 & 0 & 0 \\ 0 & \frac{1}{4}Mr^2 & 0 \\ 0 & 0 & \frac{1}{2}Mr^2 \end{bmatrix}}$$

j

Torque is related to the body frame and inertial frame angular momenta,

$$\Gamma = \dot{L} + \omega \times L$$

$$\vec{\Gamma} = I\dot{\omega} + \omega \times (I\omega)$$

$$\vec{\Gamma} = \begin{bmatrix} \frac{1}{4}Mr^2 & 0 & 0 \\ 0 & \frac{1}{4}Mr^2 & 0 \\ 0 & 0 & \frac{1}{2}Mr^2 \end{bmatrix}\dot{\omega} + \omega \times (\begin{bmatrix} \frac{1}{4}Mr^2 & 0 & 0 \\ 0 & \frac{1}{4}Mr^2 & 0 \\ 0 & 0 & \frac{1}{2}Mr^2 \end{bmatrix}\omega)$$

$$\vec{\omega} = \begin{bmatrix} \Omega\sin(\omega' t)\sin\theta \\ \Omega\cos(\omega' t)\sin\theta \\ \Omega\cos\theta - w' \end{bmatrix}$$

$$\vec{\Gamma} = \frac{d}{dt}\begin{bmatrix} \frac{1}{4}Mr^2\Omega\sin(\omega' t)\sin\theta \\ \frac{1}{4}Mr^2\Omega\cos(\omega' t)\sin\theta \\ \frac{1}{2}Mr^2\left[\Omega\cos\theta - w'\right] \end{bmatrix} + \begin{bmatrix} \Omega\sin(\omega' t)\sin\theta \\ \Omega\cos(\omega' t)\sin\theta \\ \Omega\cos\theta - w' \end{bmatrix} \times \begin{bmatrix} \frac{1}{4}Mr^2\Omega\sin(\omega' t)\sin\theta \\ \frac{1}{4}Mr^2\Omega\cos(\omega' t)\sin\theta \\ \frac{1}{2}Mr^2\left[\Omega\cos\theta - w'\right] \end{bmatrix}$$

$$\vec{\Gamma} = \frac{1}{4}Mr^2 \begin{bmatrix} \Omega\omega'\cos(\omega' t)\sin\theta + (\Omega\cos\theta - w')(\Omega\cos(\omega' t)\sin\theta) \\ -\Omega\omega'\sin(\omega' t)\sin\theta - (\Omega\cos\theta - w')(\Omega\sin(\omega' t)\sin\theta) \\ 0 \end{bmatrix}$$

$$\vec{\Gamma} = \frac{1}{4}Mr^2\Omega(\Omega\omega' + \Omega\cos\theta - w')\begin{bmatrix} \cos(\omega' t) \\ -\sin(\omega' t) \\ 0 \end{bmatrix}\sin\theta$$

Breaking it up into components

$$\boxed{\begin{cases} \Gamma_1 = \frac{1}{4}Mr^2\Omega(\Omega\omega' + \Omega\cos\theta - w')\cos(\omega' t)\sin\theta \\ \Gamma_2 = -\frac{1}{4}Mr^2\Omega(\Omega\omega' + \Omega\cos\theta - w')\sin(\omega' t)\sin\theta \\ \Gamma_3 = 0 \end{cases}}$$

As expected, one is proportional to $\cos(\omega' t)$ and another is to $\sin(\omega' t)$

k

At $t = 0$, the 1 axis points into the page. Using the above expression,

$$\Gamma_1(t) = \frac{1}{4}Mr^2\Omega(\Omega\omega' + \Omega\cos\theta - w')\cos(\omega' t)\sin\theta$$

$$\boxed{\Gamma_1(t=0) = \frac{1}{4}Mr^2\Omega(\Omega\omega' + \Omega\cos\theta - w')\sin\theta}$$

l

We know, the speeds must match

$$\frac{2\pi R}{2\pi/\Omega} = \frac{2\pi r}{2\pi/\omega'}$$

$$R\Omega = r\omega' \implies \boxed{\Omega = \frac{r}{R}\omega'}$$

Q3

a

$\boxed{\text{Yes}}$. Eigenvectors should be perpendicular

b

We find eigenvectors.

$$\begin{vmatrix} 5-\lambda & \sqrt{3} & 0 \\ \sqrt{3} & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)((5-\lambda)(3-\lambda)-3) = 0$$

$$\implies \lambda = 1, 2, 6$$

Plugging them back in,
$\lambda = 1$:

$$\begin{bmatrix} 4 & \sqrt{3} & 0 \\ \sqrt{3} & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{\omega} = 0 \implies \vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$\begin{bmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\vec{\omega} = 0 \implies \vec{\omega} = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \\ 0 \end{bmatrix}$$

$\lambda = 6$:

$$\begin{bmatrix} -1 & \sqrt{3} & 0 \\ \sqrt{3} & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \vec{\omega} = 0 \implies \vec{\omega} = \frac{1}{2} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{Principle moments} = 1, 2, 6}$$

$$\text{Eigenvectors: } \boxed{\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} -1 \\ \sqrt{3} \\ 0 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix}}$$

c

$\boxed{\text{Out of the page}}$ by handedness of cross product and torque

d

$\boxed{\text{Yes}}$, since L points along 3 axis.
$\boxed{\text{Direction}}$ of L since the torque is perpendicular to L, only direction should be affected

e

$\boxed{\text{Yes}}$
We have

$$\vec{R} \times M\vec{g} = \frac{d}{dt}\left[\lambda_3 \dot{\phi} \hat{e}_3\right]$$

Moving constants,

$$R\hat{e}_3 \times Mg(-\hat{z}) = \lambda_3 \dot{\phi} \frac{d\hat{e}_3}{dt}$$

$$\boxed{\frac{d\hat{e}_3}{dt} = \frac{1}{\lambda_3 \dot{\phi}} R\hat{e}_3 \times Mg(-\hat{z})} \implies \boxed{\text{Yes}}$$

f

$\boxed{\text{Yes}}$. If $\hat{e}_3$ gives the position vector, and omega is equivalent as above, this is true by substituting values and observing.

g

$\frac{d\hat{e}_3}{dt}$ is in the same direction as $\dot{\phi}$, since it sweeps a circle in its precession movement. So,

$$|\dot{\phi}| = \left|\frac{d\hat{e}_3}{dt}\right| = \frac{MgR}{\lambda_3 \dot{\psi}} |\hat{z} \times \hat{e}_3| = |\vec{\omega} \times \vec{r}|$$

But,

$$|\vec{\omega} \times \vec{r}| = R|\dot{\phi}||\hat{z} \times \hat{e}_3| \implies \frac{MgR}{\lambda_3 \dot{\psi}} = |\dot{\phi}|R$$

$$\boxed{|\dot{\phi}| = \frac{Mg}{\lambda_3 \dot{\psi}}}$$