Lecture 6 – TISE, Position, Momentum

Friday, September 1, 2023 9:01 AM

Reading: Griffiths 2.2

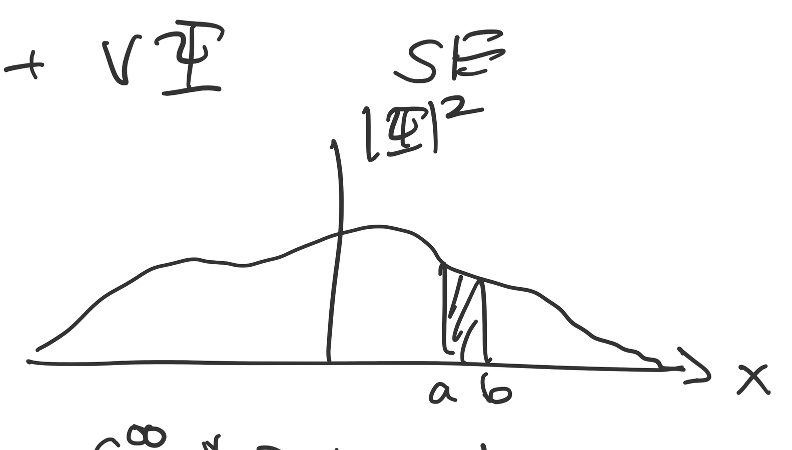
- Warm-up Quiz
- HW2 due today @ 6 PM ; HW 3 out
- Today:
  - Extracting $\langle x\rangle$, $\langle p\rangle$ from Ψ
  - Time-independent SE

What we have so far:

have a wave Equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad \text{SE}$$

$|\Psi(x,t)|^2$ is a normalized probability distribution



$$P_{ab} = \int_a^b \Psi^* \Psi \, dx \qquad \int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1 \quad \text{(normalization)}$$

We find statistical averages

Discrete $\quad \langle f(j)\rangle = \sum_j f(j)\, P(j) \quad$ weighted average

Continuous $\quad \langle f \rangle = \int_{-\infty}^{\infty} f(x)\, P(x)\, dx$

only works for normalized $P(x)$ $\quad \sum_j P(j) = 1$

$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$

Example of a particle, we can ask about $\langle x\rangle$, $\langle p\rangle$ @ time t

From statistics

$$\langle x\rangle = \int_{-\infty}^{\infty} x \, \overbrace{\underbrace{|\Psi(x,t)|^2}_{\Psi^*\Psi}}^{P(x)} dx = \boxed{\int_{-\infty}^{\infty} \Psi^* x \, \Psi \, dx}$$

what does $\langle x\rangle$ mean? statistical average → many measurements to determin.

$\boxed{\text{• Expectation Value •}}$

$\langle x\rangle$ could change w/ time, which could tell us about $\langle p\rangle$

(asside: so far I haven't written $\Psi(p,t)$, ∴ I can't just write $\int p|\Psi(x,t)|^2 dx \neq \langle p\rangle$)

so → Griffith derives $\langle p\rangle$ (Sec 1.4–1.5) which you should read.

⇒ instead use a dimensional argument

Assume a plane wave $\Psi = e^{i(kx-\omega t)}$

recall $p = \hbar k$; $E = \hbar\omega$ → we need a way to pull out a k to get p.

+ get p from k: $\boxed{-i\hbar \frac{\partial}{\partial x}}$

check $\quad -i\hbar \frac{\partial}{\partial x}\Psi = -i\hbar \cdot ik\, e^{i(kx-\omega t)} = \hbar k\, e^{i(kx-\omega t)} = p\Psi$

so

$$\boxed{\langle p\rangle = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \Psi \, dx}$$

Related dimensional analysis

for $\omega \to E$: posit $i\hbar \frac{\partial}{\partial t}$ $\hbar\omega$

$$\underbrace{i\hbar \frac{\partial}{\partial t}\Psi} = i\hbar \frac{\partial}{\partial t} e^{i(kx-\omega t)} = \overbrace{i\hbar(-i\omega)}^{\hbar\omega} e^{i(kx-\omega t)} = E\Psi$$

Not generally true

Can't use to calc $\langle E\rangle$

Looking @ SE (classically $E = \frac{p^2}{2m} + V$)

$$\underbrace{i\hbar \frac{\partial}{\partial t}\Psi}_{\to\ E\Psi} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}}_{p^2/2m\ \Psi} + \underbrace{V\Psi}_{V\Psi}$$

$$\boxed{\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + V\right]}_{\text{operator}} \Psi = E\Psi}$$

Time independent Schrödinger Eq.

This is in 1D for 3D: $\frac{\partial}{\partial x^2} \to \nabla^2$

- In QM we define operators

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$ $\quad$ momentum operator [in position basis]

use carat for operators

$$\hat{H} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}}_{KE} + \underbrace{V}_{PE}$$

Hamiltonian operator for a massive particle

I can rewrite:

number

$$\boxed{\hat{H}\Psi = E\Psi} \qquad \text{TISE}$$

This is an eigenvalue problem

→ sol^{ns} of Ψ w/ boundry conditions are the eigenfunctions of $\hat{H}$ @ const E

→ Values of E are eigenvalues

→ eigenvalues a constants of the motion. (Stationary States)

More rigorous version of this

Full SE:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

if V is time independent we can write

$$\Psi(x,t) = \psi(x)\,\phi(t)$$

So

$$i\hbar\, \psi \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}\phi + V\psi\phi$$

$$\frac{i\hbar}{\phi}\frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

only sol^{n} for each side equal to a const E

RHS: $\quad -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi \qquad$ TISE

LHS: $\quad \frac{\partial \phi}{\partial t} = -\frac{iE}{\hbar}\phi \qquad$ purely time evolution.