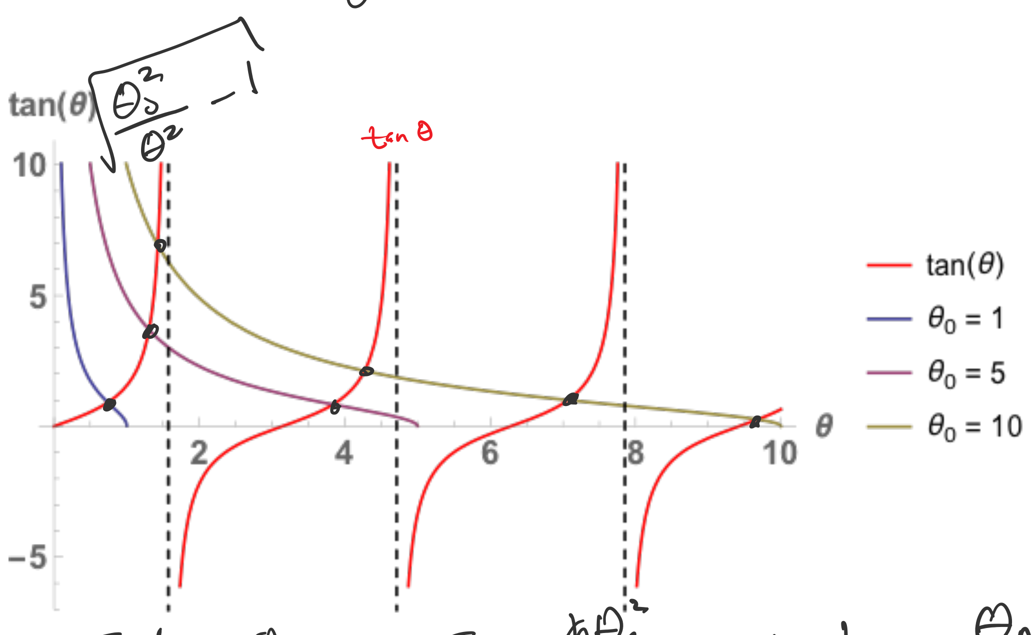
Lecture 13 – FQW + Scattering

Tuesday, September 19, 2023 8:41 AM

$$\varkappa_n = \sqrt{\frac{2m}{\hbar}\left(V_0 - E\right)}$$

1. Warm-up Quiz
2. HW5 due Friday
3. Today: • Wrap up FQW
   • Scattering



• Finding $\theta_n \rightarrow E_n = \frac{\hbar \theta_n^2}{2ma^2} \rightarrow k_n = \frac{\theta_n}{a}$

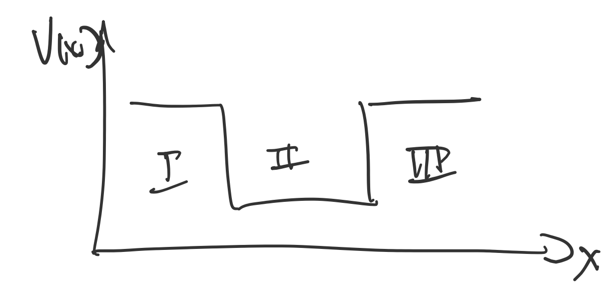
$$\varkappa_n = \frac{\sqrt{\theta_0^2 - \theta_n^2}}{a}$$

Unnormalized $\psi_n(x)$

$$\psi_n(x) = \begin{cases} A_n e^{\varkappa_n x} & x < -a \\ \left(\frac{A_n e^{-\varkappa_n x}}{\cos(ka)}\right) \cos(kx) & -a < x < a \\ A_n e^{-\varkappa_n x} & x > 0 \end{cases}$$

Scattering States

So Far: Bound states of Pot. well $\underbrace{(E-V) > 0}_{\text{II}}$ $e^{\pm ikx}$ cos, sin



but $\underbrace{(E-V) < 0}_{\text{I, III}}$ $e^{\pm \varkappa x}$

$$k = \sqrt{\frac{2m}{\hbar^2}\underbrace{(E-V)}_{>0}}$$

Free / scattering states
$(E-V) > 0$ everywhere

$\Rightarrow$ wave packets to make localized

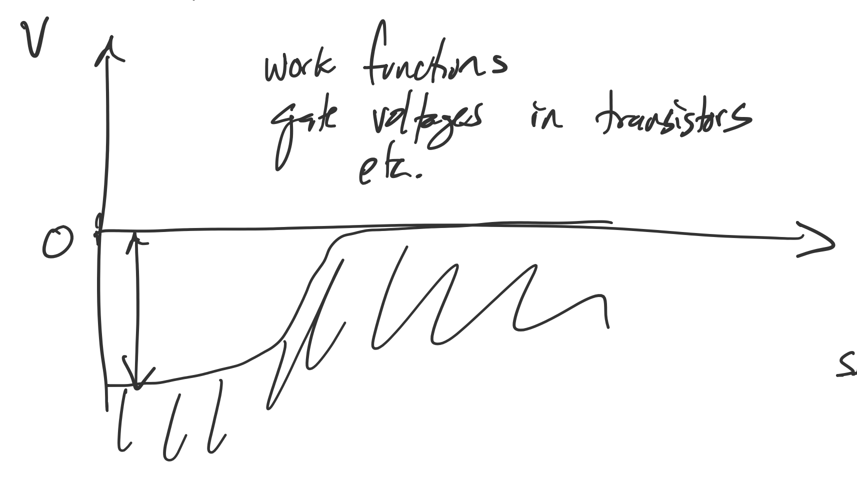
$\rightarrow$ imposes finite σ_k

$$\varkappa = \sqrt{\frac{2m}{\hbar^2}\underbrace{(V-E)}_{>0}}$$

$\rightarrow$ F.T. Convert between k-space & x-space.

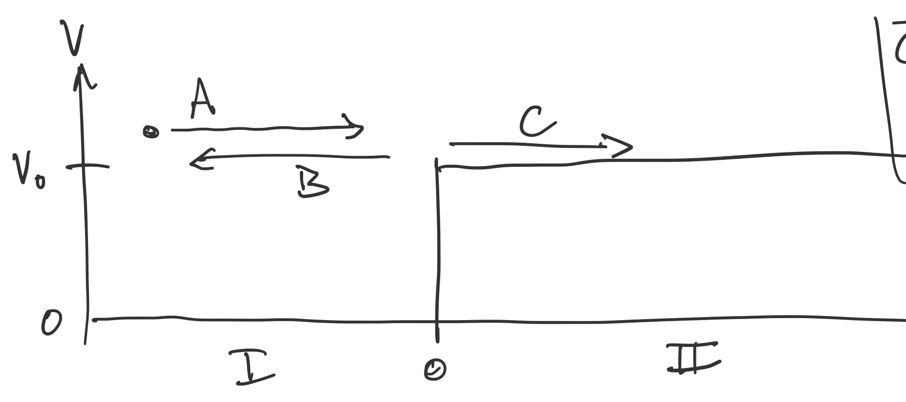
Today: How do you handle scattering states that encounter a variation in $V(x)$?

For Example



See C-T Ch I

Toy model: scattering from a 1D potential step



Classically $E < V_0$
• only reflection
$C = 0$

Region I: $\frac{d^2\psi}{dx^2} = \frac{-2m}{\hbar^2} E\psi$

Region II: $\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi$

sol$^{\underline{n}}$ $\psi_I = Ae^{ik_I x} + Be^{-ik_I x}$ $\quad k_I = \sqrt{\frac{2m(E-\cancel{V_0})}{\hbar^2}}$

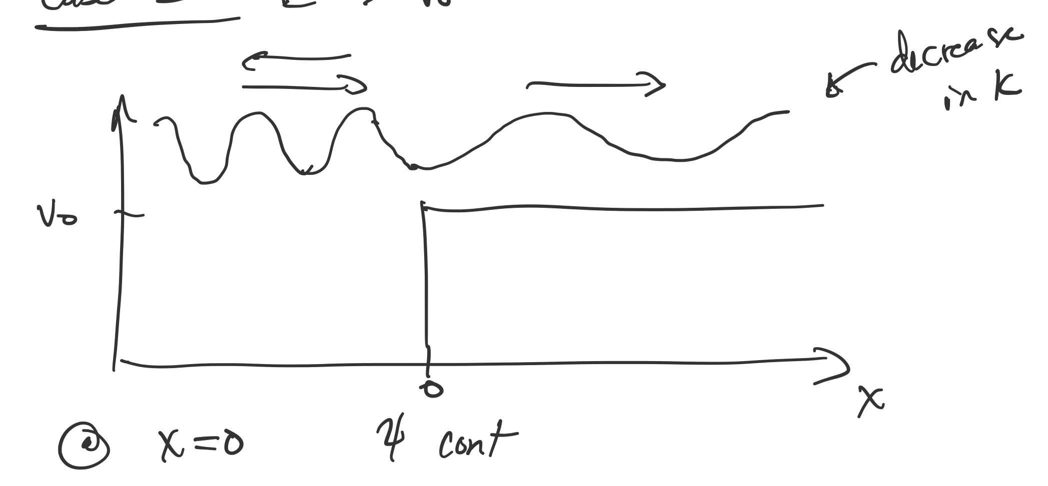
time dep $\psi(x)\phi(t) = Ae^{i(k_I x - \omega t)} + Be^{-i(k_I x + \omega t)}$

$\psi_{II} = Ce^{ik_{II} x}$ $\quad k_{II} = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

imag. if $E < V_0$
real if $E > V_0$

3 unknowns (A, B, C)
2 boundary conditions ψ con, ψ' cont.

Case 1 $E > V_0$



@ $x = 0$ ψ cont

$$\psi_I(0) = \psi_{II}(0) \Rightarrow A + B = C \quad (1)$$

ψ' cont

$$\left.\frac{d\psi_I}{dx}\right|_0 = \left.\frac{d\psi_{II}}{dx}\right|_0 \Rightarrow \cancel{i}k_I(A-B) = \cancel{i}k_{II}C \quad (2)$$

Combine: $A - (C - A) = \frac{k_{II}}{k_I} C \Rightarrow \boxed{C = \left(\frac{2k_I}{k_I + k_{II}}\right) A}$

$$B = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right) A$$

Can't fully solve but can look @ Ratios

$\frac{\text{incident}}{\text{Trans.}}$ $\quad \frac{\text{incident}}{\text{Refl.}}$

Probability current $\rightarrow$ Transmitted flux

$$j_{inc} = j_{refl} + j_{trans}$$

$$j_{an} = \frac{i\hbar}{2m}\left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi\right)$$

prob. 1.14 in Gr.

For plane waves w/ energy E $\sigma_E = 0$ $\Psi = \psi(x)e^{-i\omega t}$

$\Rightarrow$ only spacial part, in 1D

$$j = \frac{i\hbar}{2m}\left(\psi \frac{d\psi^*}{dx} - \psi^* \frac{d\psi}{dx}\right)$$

so

$$j_{inc} = \frac{i\hbar}{2m}|A|^2\left[\cancel{e^{ik_I x}}\left(-ik_I \cancel{e^{-ik_I x}}\right) - \cancel{e^{-ik_I x}}\left(ik_I \cancel{e^{ik_I x}}\right)\right]$$

$$= \frac{\hbar k_I}{m}|A|^2$$

Similarly $j_{refl} = \frac{\hbar k_I}{m}|B|^2$; $j_{trans} = \frac{\hbar k_{II}}{m}|C|^2$

Calculate Transmission / Reflection coefficients
w/ $R + T = 1$

$$T = \frac{j_{trans}}{j_{inc}} = \frac{k_{II}|C|^2}{k_I|A|^2} = \frac{4k_I k_{II}}{(k_I + k_{II})^2}$$

Can expand for E, V_0 $\quad T = \frac{4\sqrt{E(E-V_0)}}{\left(E - V_0 + E + 2\sqrt{E(E-V_0)}\right)}$

$$R = \frac{j_{left}}{j_{right}} = \frac{\cancel{k_I}|B|^2}{\cancel{k_I}|A|^2} = \left(\frac{k_I - k_{II}}{k_I + k_{II}}\right)^2$$

in the Limit $E \gg V_0$ $\quad T \rightarrow 1$
$R \rightarrow 0$

as $E \rightarrow V_0$ $\quad T \rightarrow 0$
$R \rightarrow 1$