Discrete Structures	Homework 1
CS2800	Proofs and Logic

Instructions:

- Hand in your solutions electronically on Gradescope as a single pdf file.
- You are not required to typeset your solution, but we highly recommend it; in particular, we recommend learning how to use LaTeX. Please refer to canvas (posted under Modules > Course Organization) for advice on typesetting your work.
- To help provide anonymity in your grading, do not write your name on your submission.
- Do not look for answers online! Not only is looking for answers online a violation of academic integrity, but you simply learn a lot more if you figure it out by working with other students and some guidance from the TAs.
- Collaboration (in groups of up to four students) is encouraged while solving the problems, but:
  - list the netids of those in your group at the top of your homework submission;
  - you may discuss ideas and approaches, but you should not write a detailed argument together;
  - notes of your discussions should be limited to drawings and a few keywords; you must **write up your answers on your own**.

Problems:

1. For all nonnegative real numbers a and b, the arithmetic mean is at least as large as the geometric mean:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

This fact is true, but the following is not a proper proof.

$$\begin{aligned} \frac{a+b}{2} &\overset{?}{\geq} \sqrt{ab}, && \text{so} \\ a+b &\overset{?}{\geq} 2\sqrt{ab}, && \text{so} \\ a^2+2ab+b^2 &\overset{?}{\geq} 4ab, && \text{so} \\ a^2-2ab+b^2 &\overset{?}{\geq} 0, && \text{so} \\ (a-b)^2 &\overset{?}{\geq} 0, && \text{which we know is true.} \end{aligned}$$

Explain what is wrong with the proof, and then write a proper proof.

Comments:

- *Besides a glaring problem with how the argument is structured, the author of this "improper proof" used* **so** *whenever they thought a step was obviously valid. Whenever you do this, there should be an explanation of a few words that clarifies why the step is valid – in this course, we want you to add those few words!*
- *Whenever a statement has assumptions (such as here, the assumption than a, b are* **nonnegative** *real numbers, your proof should clearly indicate in which step(s) those assumptions are used.*

2. Let a and b be real numbers.

   (a) Let n be the average (arithmetic mean) of a and b. Prove that a or b is at most n. [1]

   (b) Prove that $|a + b| \leq |a| + |b|$. [2]

   Hint: For one of these statements, proving the contrapositive is a good proof strategy; for the other, consider splitting your proof into cases.

3. An integer is called a *perfect square* if it can be expressed as the square of an integer.

   (a) Prove that every odd integer can be written as the difference of two perfect squares.

   Hint: Consider the difference between two consecutive perfect squares.

   (b) How about even integers? Prove that there exist even integers that can be written as the difference of two perfect squares, and that there also exist even integers that cannot.

   (c) Let $E(n)$ be the statement "n is even", and let $D(n)$ be the statement n is the difference of two perfect squares. Taking the domain for n to be the integers, determine whether the following statements are true or false, and give proofs of your assertions.

$$\exists n.\ E(n) \wedge D(n). \tag{1}$$
$$\exists n.\ E(n) \wedge \neg D(n). \tag{2}$$
$$\forall n.\ E(n) \Rightarrow D(n). \tag{3}$$
$$\forall n.\ \neg E(n) \Rightarrow D(n). \tag{4}$$
$$\forall n.\ \neg E(n) \Leftrightarrow D(n). \tag{5}$$

   You may of course refer to anything you proved in earlier parts without having to copy over the proof.

[1] we mean the non-exclusive "or"; in other words, a is at most $n/2$ or b is at most $n/2$ or both of them are at most $n/2$

[2] The absolute value $|a|$ equals a if a is nonnegative and $-a$ is a is negative.

4. Use truth tables to determine which of the following three logical formulas are tautologies. Make sure your truth tables show your work!

   (a) If we proved $P \Rightarrow Q$ and we know that Q is true, then we can conclude that P is true.
   $$(Q \land (P \Rightarrow Q)) \Rightarrow P.$$

   (b) If we proved $P \Rightarrow Q$ and we know that Q is false, then we can conclude that P is false.
   $$(\neg Q \land (P \Rightarrow Q)) \Rightarrow \neg P.$$

   (c) If we proved $P \Rightarrow Q, Q \Rightarrow R, R \Rightarrow P$, then we can conclude that all three of P, Q and R are true.
   $$((P \Rightarrow Q) \land (Q \Rightarrow R) \land (R \Rightarrow P)) \Rightarrow (P \land Q \land R).$$

   Hint: Read Section 3.2.1 in MCS for a convenient way of doing truth tables.