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Q4.21 $\langle f | L_\pm g\rangle = \langle f | L_x g\rangle \pm i \langle f | L_y g\rangle \overset{\text{(Hermitian)}}{=} \langle L_x f | g\rangle \mp i \langle L_y f | g\rangle = \langle (L_x \mp i L_y) f | g\rangle = \langle L_\mp f | g\rangle$

$\Rightarrow (L_\pm)^\dagger = L_\mp \Rightarrow$ Hermitian Conjugates ✓

$L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$

$\langle f_\ell^m | L_\mp L_\pm f_\ell^m\rangle = \langle f_\ell^m | L^2 - L_z^2 \mp \hbar L_z | f_\ell^m\rangle = \langle (L^2 - L_z^2 \mp \hbar L_z) | f_\ell^m | f_\ell^m\rangle = (\hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m)\langle f_\ell^m | f_\ell^m\rangle$

$= (\hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m)$

$= \langle L_\pm f_\ell^m | L_\pm f_\ell^m\rangle \Rightarrow (\hbar^2 \ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m) = (A_\ell^m)^2 \Rightarrow A_\ell^m = \sqrt{\hbar^2 \ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m}$

$(\hbar^2 \ell(\ell+1) - \hbar^2 m^2 + \hbar^2 m) = (B_\ell^m)^2 \Rightarrow B_\ell^m = \sqrt{\hbar^2 \ell(\ell+1) - \hbar^2 m^2 + \hbar^2 m}$

$$\Rightarrow \boxed{\begin{aligned} A_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m+1)} \\ B_\ell^m &= \hbar\sqrt{\ell(\ell+1) - m(m-1)} \end{aligned}}$$

Q2 $V(r) = \begin{cases} 0 & a \le r \le b \\ \infty & \text{otherwise} \end{cases} \Rightarrow \psi = 0$ outside

$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu \Rightarrow \frac{d^2u}{dr^2} = \left[\frac{\ell(\ell+1)}{r^2} - \overbrace{\frac{2mE}{\hbar^2}}^{k^2}\right]u$

In ground state, $N=1$, $\ell=0 \Rightarrow \frac{d^2u}{dr^2} = -k^2 u \Rightarrow u = A\cos(kr) + B\sin(kr)$

We require $u(a) = u(b) = 0 \Rightarrow \begin{aligned} A\cos(ka) + B\sin(ka) &= 0 \\ A\cos(kb) + B\sin(kb) &= 0 \end{aligned} \Rightarrow u(r) = A\left[\cos(kr) - \cot(ka)\sin(kr)\right]$

$\Rightarrow \tan(ka)\cos(kb) - \sin(kb) = 0 = \sin(k(b-a)) \Rightarrow k = \frac{n\pi}{b-a}$ $n = 1, 2, \ldots$ $n=0$ is trivial case

$u(r) = A\left[\cos(kr) - \cot(ka)\sin(kr)\right] = \frac{A}{\sin ka}\sin(k(r-a)) = C\sin(k(r-a))$

$k = \frac{n\pi}{b-a} \underset{\substack{(n=1)\\ \text{ground state}}}{=} \frac{\pi}{b-a} \Rightarrow k = \frac{\sqrt{2mE_1}}{\hbar} = \frac{\pi}{b-a} \Rightarrow E_1 = \frac{\hbar^2\pi^2}{2m(b-a)^2}$

$u = C\sin\left(\frac{n\pi}{b-a}(r-a)\right) = C\sin\left(\frac{\pi}{b-a}(r-a)\right)$

Normalize wave func; $u(r) = C\sin\left(\frac{\pi}{b-a}(r-a)\right)$

$\int_a^b |u|^2 dr = \int_a^b C^2\sin^2\left(\frac{\pi}{b-a}(r-a)\right)dr = \frac{(b-a)C^2}{2} \Rightarrow C = \sqrt{\frac{2}{b-a}}$

$\Rightarrow \frac{u(r)}{r} = R \Rightarrow R(r) = \begin{cases} \frac{1}{r}\sqrt{\frac{2}{b-a}}\sin\left(\frac{\pi}{b-a}(r-a)\right) & a \le r \le b \\ 0 & \text{otherwise} \end{cases}$

$\frac{1}{\sqrt{4\pi}}\sqrt{\frac{2}{b-a}} = \frac{1}{\sqrt{2\pi(b-a)}}$

$$\psi = \frac{1}{\sqrt{4\pi}}R \Rightarrow \boxed{\psi(r) = \begin{cases} \frac{1}{\sqrt{4\pi}}\sqrt{\frac{2}{b-a}}\frac{1}{r}\sin\left(\frac{\pi}{b-a}(r-a)\right) & a \le r \le b \\ 0 & \text{otherwise} \end{cases}}$$

As stated above, $\boxed{E_1 = \frac{\hbar^2\pi^2}{2m(b-a)^2}}$

Q4.12 $R_{n\ell} = \frac{1}{r}\rho^{\ell+1} e^{-\rho} v(\rho)$ $c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} c_j$ $\rho = \frac{r}{an}$

$c_{j+1} = \frac{2(j-2)}{(j+1)(j+2)} c_j$ 30 case

$c_{j+1} = \frac{2(j-1)}{(j+1)(j+4)} c_j$ 31 case

$c_{j+1} = \frac{2j}{(j+1)(j+6)} c_j$ 32 case

R_{30}: $c_1 = \frac{2(-2)}{2} c_0 \Rightarrow c_2 = \frac{2(-1)(-2)}{2(3)} c_0 \Rightarrow c_3 = 0 \ldots$

R_{31}: $c_1 = -\frac{2}{4} \Rightarrow c_2 = 0 \ldots$

R_{32}: $c_1 = 0 \Rightarrow c_2 = 0 \ldots$

$\Rightarrow$

$$R_{30} = \frac{1}{r}\left(\frac{r}{3a}\right) e^{-r/3a} c_0 \left(1 - 2\left(\frac{r}{3a}\right) + \frac{2}{3}\left(\frac{r}{3a}\right)^2\right) = \boxed{R_{30} = \frac{c_0}{3a}\left[1 - \frac{2r}{3a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right] e^{-r/3a}}$$

$$R_{31} = \frac{1}{r}\left(\frac{r}{3a}\right)^2 e^{-r/3a} c_0 \left(1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right) = \boxed{R_{31} = \frac{c_0 r}{9a^2}\left(1 - \frac{r}{6a}\right) e^{-r/3a}}$$

$$R_{32} = \frac{1}{r}\left(\frac{r}{3a}\right)^3 e^{-r/3a} c_0 (1) = \boxed{R_{32} = \frac{c_0 r^2}{27a^3} e^{-r/3a}}$$

Q4.16 $\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \Rightarrow P_r(r_0 < r < r_0 + dr) = |\psi_{100}|^2 \cdot \underbrace{(4\pi r^2)}_{\text{(sphere surface area)}} dr = \frac{e^{-2r/a}}{\pi a^3} \cdot 4\pi r^2 dr \Rightarrow p(r) = \frac{4r^2}{a^3} e^{-2r/a}$

We seek to maximize $p(r)$: $p(r) = \frac{4r^2}{a^3} e^{-2r/a} \Rightarrow p'(r) = \frac{4}{a^3} e^{-2r/a}\left(2r - \frac{2r^2}{a}\right) \Rightarrow \boxed{r = a}$ verified maxima by graphing