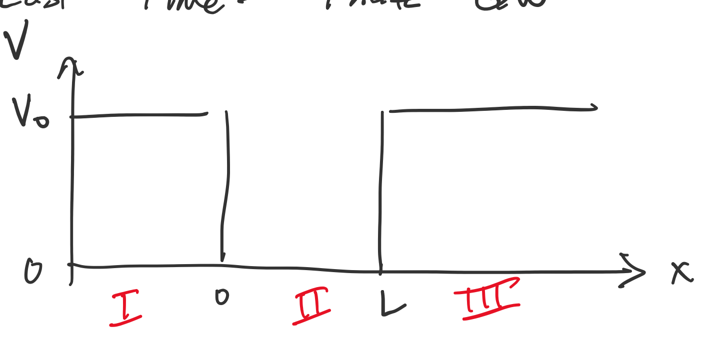
Lecture 12 - Finite Quantum Well

Monday, September 18, 2023 8:22 AM

1. Warm-up Quiz
2. HW3 due Friday
3. Today - Finite Quantum Well

Last Time: Finite QW



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x) = E\psi$$

Region II : $E>0$ and $< V_0$ {bound states}

$$\frac{d^2\psi}{dx^2} = \underbrace{\frac{-2mE}{\hbar^2}}_{k^2}\psi \Rightarrow \boxed{A\sin(kx) + B\cos(kx) = \psi_{II}}$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Region I, III $(x<0\ ;\ x>L)$ $E<V_0$ $E>0$

$$\frac{d^2\psi}{dx^2} = \overbrace{(+)\frac{2m}{\hbar^2}}^{+}\underbrace{(V_0-E)}_{+}\psi \qquad \kappa = \sqrt{\frac{2m}{\hbar^2}(V_0-E)}$$

$$\psi_I(x) = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III}(x) = Ee^{\kappa x} + Fe^{-\kappa x}$$

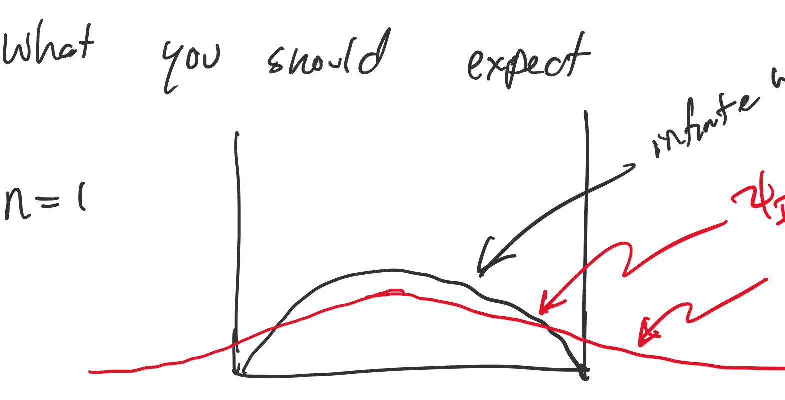
For $x>L$ if $E\neq 0$ then $\psi(x)\to\infty$ as $x\to\infty$

$\therefore E=0$; F is ok

for $x<0 \longrightarrow D=0$

$$\therefore \boxed{\begin{aligned}\psi_I &= Ce^{\kappa x} \\ \psi_{III} &= Fe^{-\kappa x}\end{aligned}} \qquad \begin{aligned}&(x<0)\\&(x>L)\end{aligned}$$

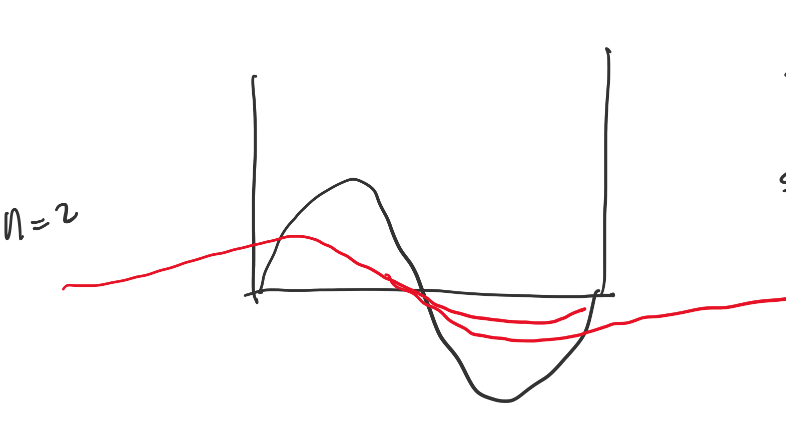
What you should expect



Finite well need longer $\lambda \Rightarrow$ less curvature

e.g. $\frac{d^2\psi}{dx^2}$ is smaller

$\Rightarrow$ less K.E.



Same trends for zero crossings

sol$^{\underline{n}}$ are still orthonormal

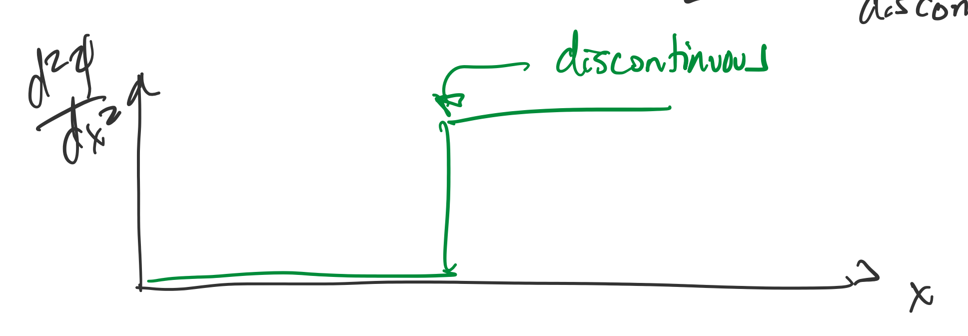
$\psi(x)$ must be continuous and smooth

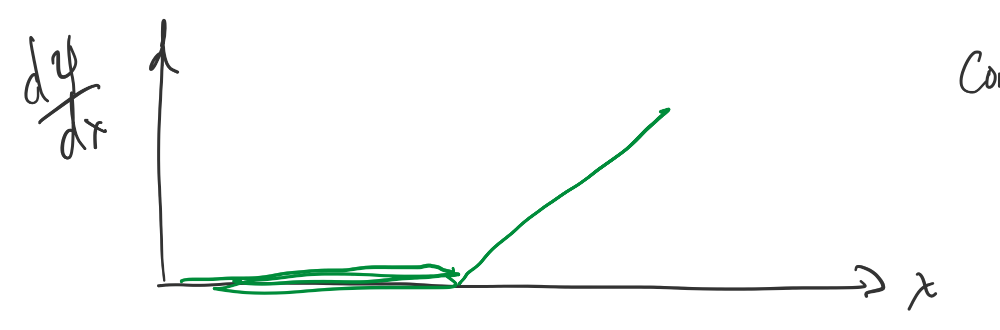
$\frac{d\psi}{dx}$ must be continuous

Why? Follows from TISE

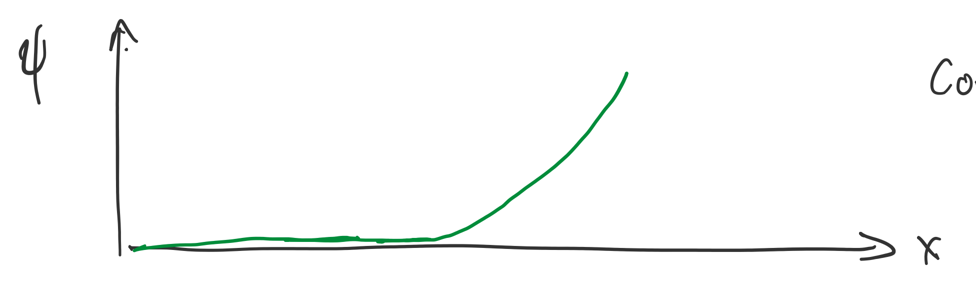
$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V(x))\psi$$

$V(x)$ ← discontinuous.





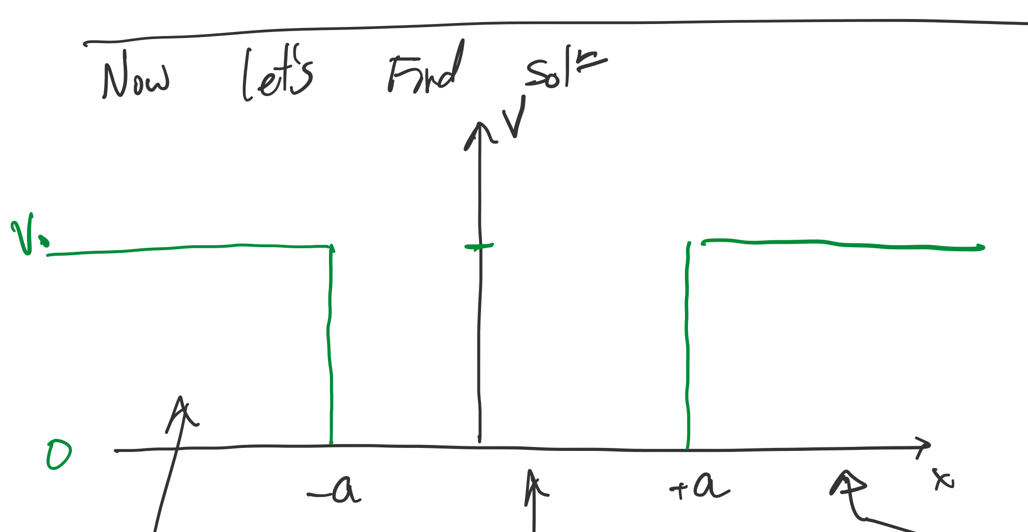
Continuous (not smooth)



Continuous, smooth

↑ automatically enforced if $\frac{d\psi}{dx}$ is cont.

Now let's find sol$^{\underline{n}}$



$\psi_L = Ae^{\kappa x}$

$\psi_c = B\cos(kx) + D\sin(kx)$

$\psi_R = Fe^{-\kappa x}$

For even states $D=0$ (odd states $B=0 \Rightarrow$ HW)

Need $\psi, \frac{d\psi}{dx}$ to be continuous @ $\pm a$

$\therefore A=F$ (by even symmetry)

$\Rightarrow$ $\boxed{\text{Energy},\ A/B}$

So $\psi_c = B\cos(kx)$ $\qquad \psi_R = Ae^{-\kappa x}$

$\frac{d\psi_c}{dx} = -Bk\sin(kx)$ $\qquad \frac{d\psi_R}{dx} = -\kappa Ae^{-\kappa x}$

Continuity ψ @ $x=a$ $\qquad B\cos(ka) = Ae^{-\kappa a}$

" $\frac{d\psi}{dx}$ @ $x=a$ $\qquad \cancel{-}Bk\sin(ka) = \cancel{-}\kappa Ae^{-\kappa a}$

divide the two $\qquad k\tan(ka) = \kappa$

$$\Rightarrow \tan(ka) = \kappa/k$$

expand for κ, k

$$(\star)\quad \tan\left(a\sqrt{\frac{2mE}{\hbar^2}}\right) = \sqrt{\frac{V-E}{E}} = \sqrt{\frac{V_0}{E}-1}$$

Equation determines Energy but but no analytical sol$^{\underline{n}}$

Can find graphical (or numerically)

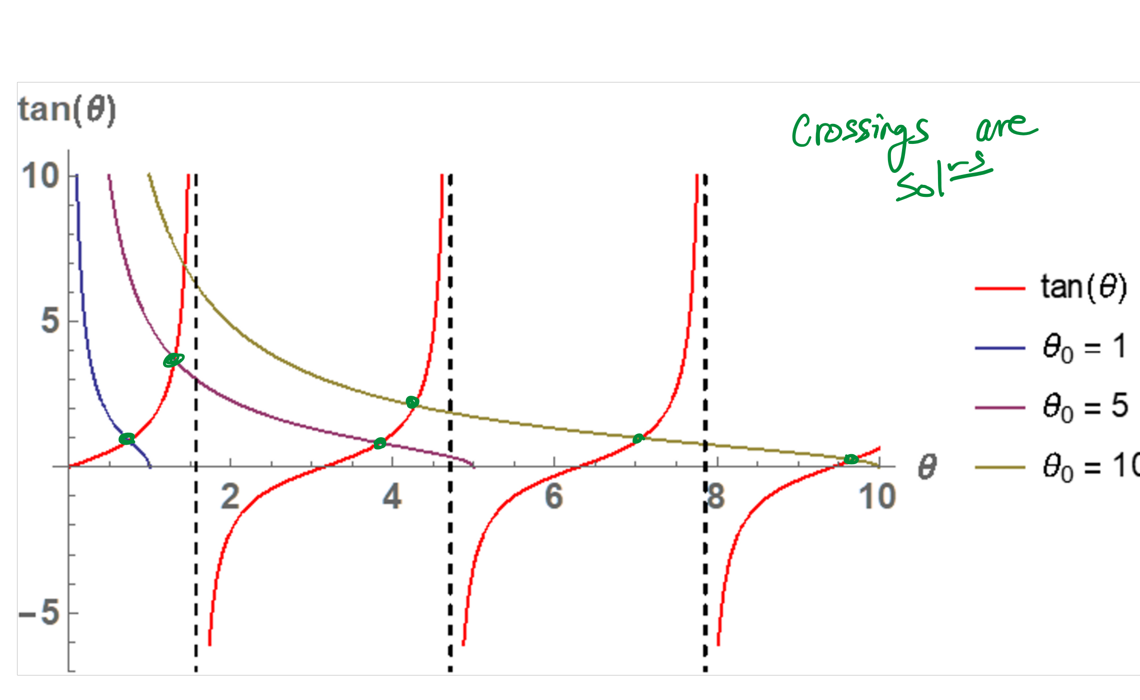
$$\Rightarrow \theta \equiv ka = \frac{a}{\hbar}\sqrt{2mE} \Rightarrow E = \frac{\hbar^2\theta^2}{2ma^2}$$

LHS of $(\star)$ $= \tan\theta$

RHS of $(\star)$ $= \sqrt{\frac{V_0}{E}-1} = \sqrt{\frac{V_0}{\frac{\hbar^2\theta^2}{2ma^2}}-1} = \sqrt{\frac{\theta_0^2}{\theta^2}-1}$

where $\theta_0 = a\sqrt{\frac{2mV_0}{\hbar^2}}$

Plot both as a function of θ



$\theta_0 = 1 \longrightarrow$ 1 even bound state

$\theta_0 = 5 \longrightarrow$ 2 even bound states

$\theta_0 = 10 \longrightarrow$ 4 even bound states