Homework #10 **AEP 3610** **Due November 4**

1. Griffiths 3.15
2. Griffiths 3.22
3. Show $\frac{d}{dt}\langle \hat{p} \rangle = -\left\langle \frac{dV(x)}{dx} \right\rangle$. Briefly compare this to its classical analog. (Note: if you don't discuss this, you won't get credit.)
4. Griffiths 3.37
5. Griffiths 3.43

Bryant Har bjh254

Q3.15 $\left[x, \frac{p^2}{2m} + \cancel{V}^{\,0}\right] = \frac{1}{2m}[x, p^2] = \frac{1}{2m}\left(-\hbar^2 x \nabla^2 f + \hbar^2 \nabla^2 (xf)\right) = \frac{\hbar^2}{2m}\left(\nabla f + \nabla f + \cancel{x\nabla^2 f - x\nabla^2 f}\right) \Rightarrow \frac{\hbar^2}{m}\nabla = \frac{\hbar i}{m}\hat{p}$

$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2 = \left(\frac{1}{2i}\langle\frac{\hbar i}{m}\hat{p}\rangle\right)^2 \Rightarrow \boxed{\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle\hat{p}\rangle|}$

For stationary states, $\sigma_H = 0 \Rightarrow$ reduces to $0 \geq 0$ (useless)

Q3.22 $\sigma_H \sigma_t \geq \frac{\hbar}{2}$ but $\sigma_Q = \left|\frac{d\langle Q\rangle}{dt}\right| \sigma_t$. Let $Q = x \Rightarrow \left|\frac{d\langle x\rangle}{dt}\right| = \left|\frac{\langle p\rangle}{m}\right| = \frac{1}{m}|\langle p\rangle|$

$\left|\frac{d\langle x\rangle}{dt}\right| = \frac{1}{m}|\langle p\rangle| \Rightarrow \sigma_H \sigma_t = \frac{m}{|\langle\hat{p}\rangle|}\sigma_H \sigma_x \geq \frac{\hbar}{2} \Rightarrow \boxed{\sigma_H \sigma_x \geq \frac{\hbar}{2m}|\langle\hat{p}\rangle|}$ Same ✓

Q3 $\hat{p} = -i\hbar\nabla \Rightarrow \frac{\partial\langle p\rangle}{\partial t} = -i\hbar\int \frac{d}{dt}(\psi^* \nabla\psi)\,dx = -i\hbar\int dx\, \frac{d\psi^*}{\partial t}\nabla\psi + \psi^*\nabla\frac{d\psi}{\partial t} = \int -\hat{H}\psi^*(\nabla\psi) - \psi^*\nabla(\hat{H}\psi)\,dx$

$\int dx\left[-\hat{H}\psi^*(\nabla\psi) - \psi^*\nabla(\hat{H}\psi)\right] = \int dx\left[\frac{\hbar^2}{2m}(\psi^*\nabla^3\psi - \nabla^2\psi^*\nabla\psi) + (V\psi^*\nabla\psi - \psi^*\nabla(V\psi))\right] = \int dx\left[\frac{\hbar^2}{2m}(\psi^*\nabla^3\psi - \nabla^2\psi^*\nabla\psi) + (\cancel{V\psi^*\nabla\psi} - \underbrace{\psi^*\nabla V\psi}_{\to -\langle\nabla V\rangle} - \cancel{V\psi^*\nabla\psi})\right]$

$\int dx\,(\psi^*\nabla^3\psi - \nabla^2\psi^*\nabla\psi) = \nabla\psi^*\nabla^2\psi\Big|_{-\infty}^{\infty} - \nabla^2\psi^*\nabla\psi\Big|_{-\infty}^{\infty} + \cancel{\int dx\,\nabla^2\psi^*\nabla\psi} - \cancel{\int dx\,\nabla^2\psi^*}\nabla\psi = 0$ ($\psi, \psi^* \to 0$ at $x = \pm\infty$)

$= -\langle\nabla V\rangle + \frac{\hbar^2}{2m}\int dx\,(\psi^*\nabla^3\psi - \nabla^2\psi^*\nabla\psi) = -\langle\nabla V\rangle + \frac{\hbar^2}{2m}(0) = \boxed{\frac{\partial\langle\hat{p}\rangle}{\partial t} = -\left\langle\frac{dV}{dx}\right\rangle}$ ✓

This is identical to the analogous classical law $\frac{dp}{dt} = F = -\frac{dU}{dx}$.
Force is neg deriv of potential & time deriv of momentum.

This makes sense. Over expectation, quantum systems follow classical laws.

(ehrenfests theorem)

Q 3.37

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$$

Let $Q = xp$ in Eq. 3.73, $\frac{d}{dt}\langle xp\rangle = \frac{i}{\hbar}\langle[\hat{H},xp]\rangle + \cancel{\langle\frac{\partial xp}{\partial t}\rangle} \to 0$ (no operator t dependence)

$$[H,xp] = [T+V,xp] = [T,x]p + x[T,p] + [V,x]p + x[V,p]$$

$$[T,x] = Txf - xTf = -\frac{\hbar^2}{2m}x\nabla f - \frac{\hbar^2}{2m}x\nabla f + \cancel{xTf - xTf} = -\frac{i\hbar p}{m} \qquad [T,p] = Tpf - pTf = Tpf - Tpf = 0$$

$$[V,x] = Vxf - xVf = Vxf - Vxf = 0 \qquad [V,p] = Vpf - pVf = \cancel{Vpf} - (pV)f - \cancel{Vpf} = -pV = i\hbar\nabla V$$

$$\Rightarrow \frac{i}{\hbar}\langle H,xp\rangle = \frac{i}{\hbar}\left[-\frac{i\hbar}{m}\langle p^2\rangle + i\hbar\langle x\nabla V\rangle\right] = 2\langle T\rangle - \langle x\frac{\partial V}{\partial x}\rangle$$

$$\boxed{\frac{\partial}{\partial t}\langle xp\rangle = 2\langle T\rangle - \langle x\frac{\partial V}{\partial x}\rangle}$$

In stationary state, $\frac{\partial}{\partial t}\langle xp\rangle = 0$ since all expectations are independent of time unless otherwise stated.

$$V = \frac{1}{2}m(\omega x)^2 \Rightarrow V' = m\omega^2 x \Rightarrow x\frac{\partial V}{\partial x} = m\omega^2 x^2 = 2\langle V\rangle$$

$$\Rightarrow 2\langle T\rangle = 2\langle V\rangle \Rightarrow \boxed{\langle T\rangle = \langle V\rangle}$$ as desired

Q 3.43

$$|z|^2 = Re(z)^2 + Im(z)^2 = \frac{1}{4}[(z+z^*)^2 - (z-z^*)^2]$$

a) $\Rightarrow \sigma_A^2\sigma_B^2 \geq |\langle f|g\rangle|^2 = \frac{1}{4}[(z+z^*)^2 - (z-z^*)^2]$

Per hint, we retain & examine the Re term (other goes to $\langle C\rangle^2$)

$$\langle f|g\rangle = \langle(\hat{A}-\langle A\rangle)(\hat{B}-\langle B\rangle)\Psi|\Psi\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle$$

$$\Rightarrow \langle f|g\rangle + \langle g|f\rangle = \langle AB\rangle - \langle A\rangle\langle B\rangle + \langle BA\rangle - \langle A\rangle\langle B\rangle = \langle AB\rangle + \langle BA\rangle - 2\langle A\rangle\langle B\rangle = \hat{D}$$

$$\Rightarrow \sigma_A^2\sigma_B^2 \geq |\langle f|g\rangle|^2 = \frac{1}{4}[(\overset{\hat{D}}{\cancel{z+z^*}})^2 - (\overset{\hat{C}}{\cancel{z-z^*}})^2] \Rightarrow \boxed{\sigma_A^2\sigma_B^2 \geq \frac{1}{4}(\langle\hat{C}\rangle^2 + \langle\hat{D}\rangle^2)}$$ ✓

b) $A = B \Rightarrow \hat{C} = 0$, $\sigma_A^4 \geq \frac{1}{4}(\langle A^2\rangle + \langle A^2\rangle - 2\langle A\rangle\langle A\rangle)^2 = \frac{4}{4}(\langle A^2\rangle - \langle A\rangle^2)^2 = Var(\hat{A})^2 = \sigma_A^4$

$$\Rightarrow \boxed{\sigma_A^2\sigma_A^2 \geq \sigma_A^4}$$ They are in fact always equal