PS 4

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Q1

a

Note that I use $\nabla_{r_a} = \nabla_{x_a,y_a,z_a} = \nabla_a$ synonymously. We are given,

$$F = \nabla U = -\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \rangle$$

We can dot both sides by a small change in the x direction, y direction, and z direction to get the changing "part" of U "that comes from the change in" each direction. As a vector, this is dr

$$F \cdot dr = -\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \rangle \cdot \langle \partial x, \partial y, \partial z \rangle$$

$$F \cdot dr = -\langle dU_x, dU_y, dU_z \rangle$$

Collecting dU from its differential components in each direction,

$$\boxed{F \cdot dr = -dU}$$

b

We begin with the given relation,

$$F_{ba} = -\nabla_b U_{ab} = -\frac{\partial U_{ab}}{\partial r_b}$$

$$\implies F_{ba} = -\frac{\partial U_{ab}}{\partial r} \frac{\partial r}{\partial r_b}$$

By definition, $r = r_a - r_b$, so $\frac{\partial r}{\partial r_b} = -1$

$$\implies F_{ba} = -\frac{\partial U_{ab}}{\partial r}(-1)$$

$$\boxed{F_{ba} = \nabla_{x,y,z} U_{ab}}$$

c

We begin with the given relation,

$$-dW = dU_a + (\nabla_a U_{ab}) \cdot dr_a + dU_b + (\nabla_b U_{ab}) \cdot dr_b$$

By multivariable chain rule,

$$\frac{\partial U_{ab}}{\partial r_a} = \frac{\partial U_{ab}}{\partial r_a} \cdot \frac{\partial r_a}{\partial r} + \frac{\partial U_{ab}}{\partial r_b} \cdot \frac{\partial r_b}{\partial r} = (\nabla_a U_{ab}) \cdot \frac{\partial r_a}{\partial r} + (\nabla_b U_{ab}) \cdot \frac{\partial r_b}{\partial r}$$

$$\implies \partial U_{ab} = (\nabla_a U_{ab}) \cdot \partial r_a + (\nabla_b U_{ab}) \cdot \partial r_b$$

Substituting this result,

$$\boxed{-dW = dU_a + dU_b + dU_{ab}}$$

d

$\boxed{\text{Yes}}$, force has an energy potential and so is conservative.

e

We're given,

$$W_{12} = -(U_2 - U_1) = T_2 - T_1$$

Simple rearranging shows,

$$\implies \boxed{U_1 + T_1 = U_2 + T_2}$$

Total energy remains constant between points 1 and 2 for conservative forces.

f

Again, we're given,

$$T_2 - T_1 = \sum_a \int_{s_1}^{s_2} dr_a \cdot F_a + \int dr_j \cdot F_j$$

We substitute $dU = -F \cdot dr$

$$T_2 - T_1 = -\sum_a \int_{s_1}^{s_2} dU + \int_{s_1}^{s_2} dr_j \cdot F_j$$

$$T_2 - T_1 = -(U_2 - U_1) + \int_{s_1}^{s_2} dr_j \cdot F_j$$

$$T_2 - T_1 + U_2 - U_1 = +\int_{s_1}^{s_2} dr_j \cdot F_j$$

Substituting E, we arrive at our desired answer,

$$\boxed{E_2 - E_1 = \int_{s_1}^{s_2} dr_j \cdot F_j}$$

g

$\boxed{\text{Negative}}$. Frictional and drag force oppose the direction of motion and are in the negative direction of the dr path differential. Therefore, the integral evaluates to a negative value.

Q2

a

We integrate force to get work,

$$\int_0^D dxF = \int_0^D dx - kx = \boxed{W = -\frac{kD^2}{2}}$$

b

By initial to final energy conservation and adding work,

$$M_2 g D = -\frac{kD^2}{2} + \frac{1}{2}M_2 v^2 + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

We know $I = \frac{1}{2}MR^2$ by table. Rearranging for v,

$$M_2 g D = \frac{kD^2}{2} + \frac{1}{2}M_2 v^2 + \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(v/R)^2$$

$$\boxed{v = \sqrt{\frac{4M_2 g D + 2kD^2}{2M_2 + 3M}}}$$

c

Using the work formula,

$$W = \int dr - cV^2 = \int_0^{\Delta t} dt - cV^3 = \boxed{W = -cS_D^3 \Delta t}$$

d

Change in energy, assuming constant velocity is the sum of the change in energy in all forms.

$$\Delta E = -cS_D^3 - mg\Delta tV + \frac{1}{2}k(\Delta tV)$$

$$\boxed{\Delta E = -cS_D^3 - mg\Delta tS_D + \frac{1}{2}k(\Delta tS_D^2)}$$

Q3

a

By definition, that is the $\boxed{\text{y-coordinate of the COM}}$.

b

By moment table for a thin hoop, $\boxed{I = MR^2}$

c

Energy initial,

$$E_0 = Mg(y_1 - y_2) + \frac{1}{2}I\omega_1^2 + \frac{1}{2}M(R\omega_1)^2$$

Energy at bottom, and by conservation,

$$E_0 = E_b = Mg(y_1 - y_2) + \frac{1}{2}I\omega_1^2 + \frac{1}{2}M(R\omega_1)^2 = \frac{1}{2}I\omega_2^2 + \frac{1}{2}M(R\omega_2)^2$$

$$Mg(y_1 - y_2) + I\omega_1^2 = I\omega_2^2$$

We are given $R\omega_2 = (R_c - R)\dot{\theta}$,

$$Mg(y_1 - y_2) + I\omega_1^2 = I\left(\frac{R_c - R}{R}\dot{\theta}\right)^2$$

$$\implies \boxed{\dot{\theta} = \frac{R}{R_c - R}\sqrt{\frac{Mg}{I}(y_1 - y_2) + \omega_1^2}}$$

d

Normal force is equal to downward force which is equal to the centripetal force.

$$N = v^2/R_c = R_c\dot{\theta}^2$$

Downward force is equal, or

$$\implies \boxed{F = \frac{R_c R^2}{(R_c - R)^2}\left(\frac{Mg}{I}(y_1 - y_2) + \omega_1^2\right)}$$

Q4

a

Applying the formula and differentiating,

$$-\frac{dU}{dx} = -F \implies \boxed{F = U_0\left(\frac{a}{x^2} - \frac{1}{a}\right)}$$

b

To find equilibrium points, $F = 0$. By inspection, $x = \pm a$ are the only such points. We get second derivative,

$$U'' = aU_0 \frac{2}{x^3}$$

$F'(a) > 0 \implies \boxed{x = a \text{ is stable equilibrium}}$
$F'(-a) < 0 \implies \boxed{x = -a \text{ is unstable equilibrium}}$

c

$\boxed{\text{No, cannot make it to } x = 0}$. Initial energy is finite, but potential energy approaches infinity at $x = 0$.

d

For small oscillations, we approximate F and use Hooke's Law.

$$F'(a) = -aU_0 \frac{2}{x^3}|_{x=a} = -\frac{2U_0}{a^2}$$

$$m\ddot{x} = -\frac{2U_0}{a^2}(x-a) \implies k = \frac{2U_0}{a^2}$$

We desire $\omega = \sqrt{k/m}$. Substituting,

$$\boxed{\omega = \sqrt{\frac{2U_0}{ma^2}}}$$

e

Since we approximate with hooke's law, we know it's in the form:

$$x = A\cos(\omega t) + B\sin(\omega t) + x_0$$

We know $x_0 = a$, since x_0 is center of oscillation. We know $\dot{x}(0) = -V_0$. Differentiating,

$$\dot{x}(0) = \omega(-A\sin\omega t + B\cos\omega t)|_{t=0} \implies \omega B = -V_0 \implies B = -V_0/\omega$$

We also know amplitude is given by $A^2 + B^2 = C^2$, so by energy conservation,

$$\frac{1}{2}mV_0^2 = \frac{1}{2}k(A^2 + B^2) \implies V_0^2 = \omega^2(V_0^2/\omega^2 + A^2) \implies A = 0$$

$$\implies \boxed{x = -\omega V_0 \sin(\omega t) + a}$$

f

Cosine is $-\pi/2$ phase delay from sine. $e^{i\omega t}$ has a real cosine component. We induce the appropriate phase delay:

$$\implies \boxed{C = -\omega V_0 e^{-i\frac{\pi}{2}}}$$

$$x - a = Ce^{i\omega t}$$

g

Given the form $e^{-i\delta}$,

$$\boxed{\delta = \pi/2 \ \text{ per above}}$$