Q36a

We begin,

$$y_{xx} - \frac{1}{c^2} y_{tt} = 0, \quad y(x,0) = y_0 \text{ for } -x_0 < x < x_0$$

We define green's function as satisfying

$$g_{xx} - \frac{1}{c^2} g_{tt} = 0$$

$$c^2 g_{xx} = g_{tt}$$

Taking the Fourier transform wrt x,

$$-k^2 c^2 G = G_{tt}$$

This is a well known differential equation (Hooke's Law). The solution is,

$$G = G_0 e^{\pm ikct}$$

As usual,

$$G_0 = \mathcal{F}(g(x|\xi, 0)) = \mathcal{F}(\delta(x - \xi)) = \frac{1}{\sqrt{2\pi}} e^{-ik\xi}$$

$$\implies G = \frac{1}{\sqrt{2\pi}} e^{\pm ik(ct-\xi)}$$

We recover g using the inverse fourier transform (can verify below are equivalent),

$$g = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}} e^{\pm ik(ct-\xi)}\right) = \delta(x \pm (ct - \xi)) = \delta(\xi - (x \pm ct))$$

$$g_+ = \delta(\xi - (x + ct))$$

$$g_- = \delta(\xi - (x - ct))$$

We combine the retarded and advanced greens functions as a linear combination. Using the first initial condition again,

$$y(x,0) = Ay_0(x) + By_0(x)$$

$$y_0(x)_= Ay_0(x) + By_0(x) \implies A + B = 1$$

Using our second initial condition $\dot{y}(0) = 0$, we find:

$$y(x,t) = Ay_0(x - ct) + By_0(x + ct)$$
$$\dot{y}(x,0) = 0 = -cA\dot{y}_0(x) + Bc\dot{y}_0(x) \implies cB = cA \implies A = B$$

Combining the retarded and advanced greens functions as a linear combination,

$$A + B = 1, \quad A = B \implies A = B = \frac{1}{2}$$

We recover y,

$$y(x,t) = \int d\xi \; y_0(\xi)(g_+ + g_-)$$

$$y(x,t) = \int_{-\infty}^{\infty} d\xi \; y_0(\xi)(A\delta(\xi - (x - ct)) + B\delta(\xi - (x + ct)))$$

$$y(x,t) = \frac{1}{2}y_0(x - ct) + \frac{1}{2}y_0(x + ct)$$

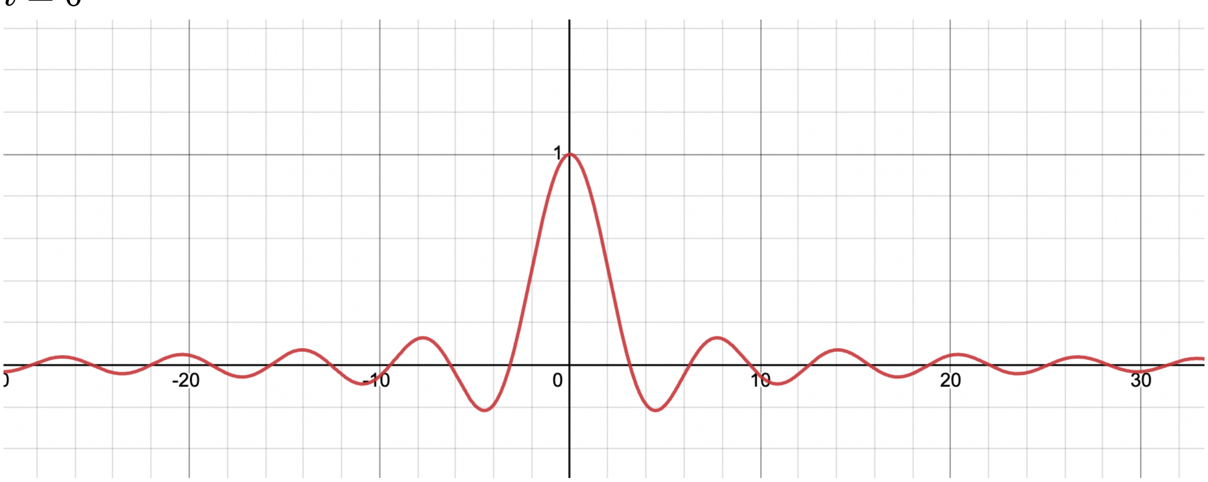
We finally arrive at,

$$\boxed{y(x,t) = \frac{y_0(x - ct) + y_0(x + ct)}{2}, \quad t > 0}$$

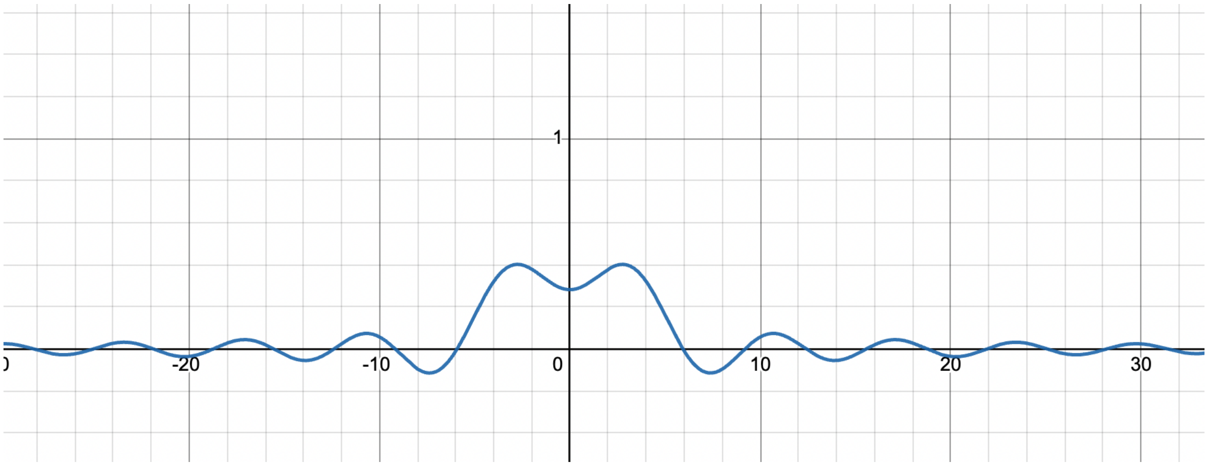
Q36b

Letting our test function be $y_0 = \sin(x)/x$,

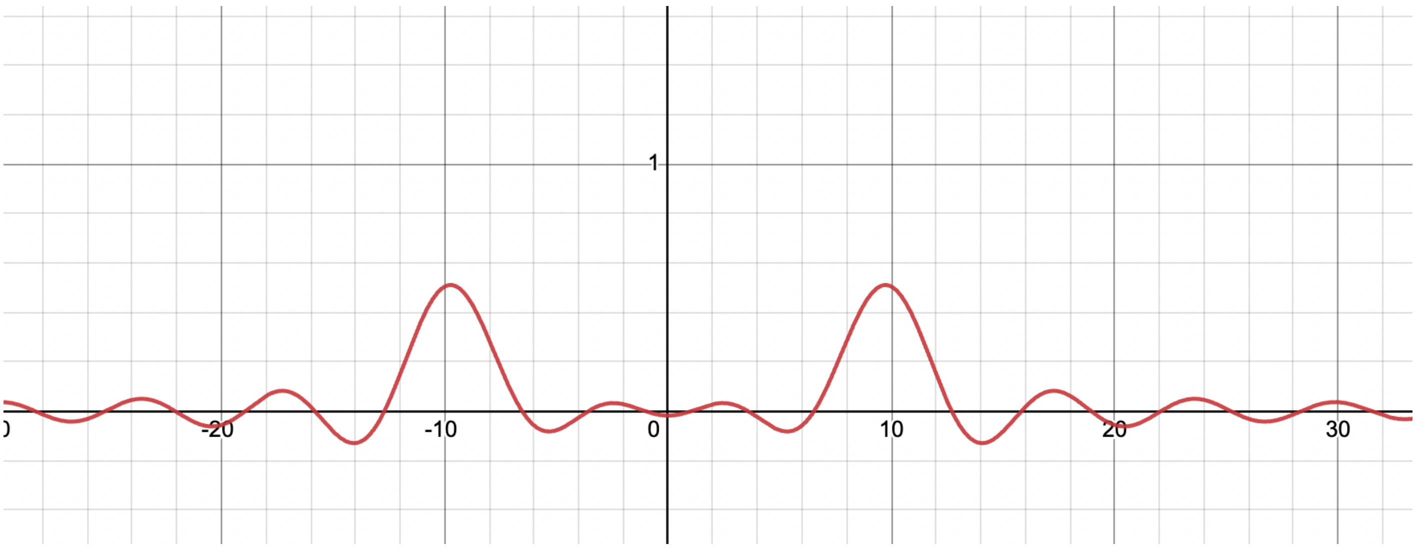
$t = 0$



$t = 2$



$t = 8$



The sinc function was plotted for t=0,2,8. Clearly, we see that this consists of the classic sinc spike that splits into two spikes that propagate outward. Animated on Desmos, this behavior is very clear.

Q36c

For

$$y_0(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

We arrive at,

$$\boxed{y(x,t) = y_1 + y_2}$$

$$y_1 = \begin{cases} 1/2, & -1 < x - ct < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1/2, & -1 < x + ct < 1 \\ 0, & \text{otherwise} \end{cases}$$

Graphically, we see that this is equivalent to the initial wave decomposing into two subwaves of half the initial height that propagate to the left and right respectively. As time increases, the two subwaves get farther apart. (would also look more wave-like as a fourier series)

Q36d

From a, we consider a linear combination of the retarded and advanced greens functions but this time by initial conditions, we require $v(x)$ due to the zero'd velocity:

$$y(x,t) = Av(x-ct) + Bv(x+ct)$$

Using the first initial condition again,

$$0 = Av(x) + Bv(x) \implies A = -B$$

Using our second initial condition $\dot{y}(0) = v(x)$, we find:

$$y(x,t) = -Bv(x-ct) + Bv(x+ct)$$

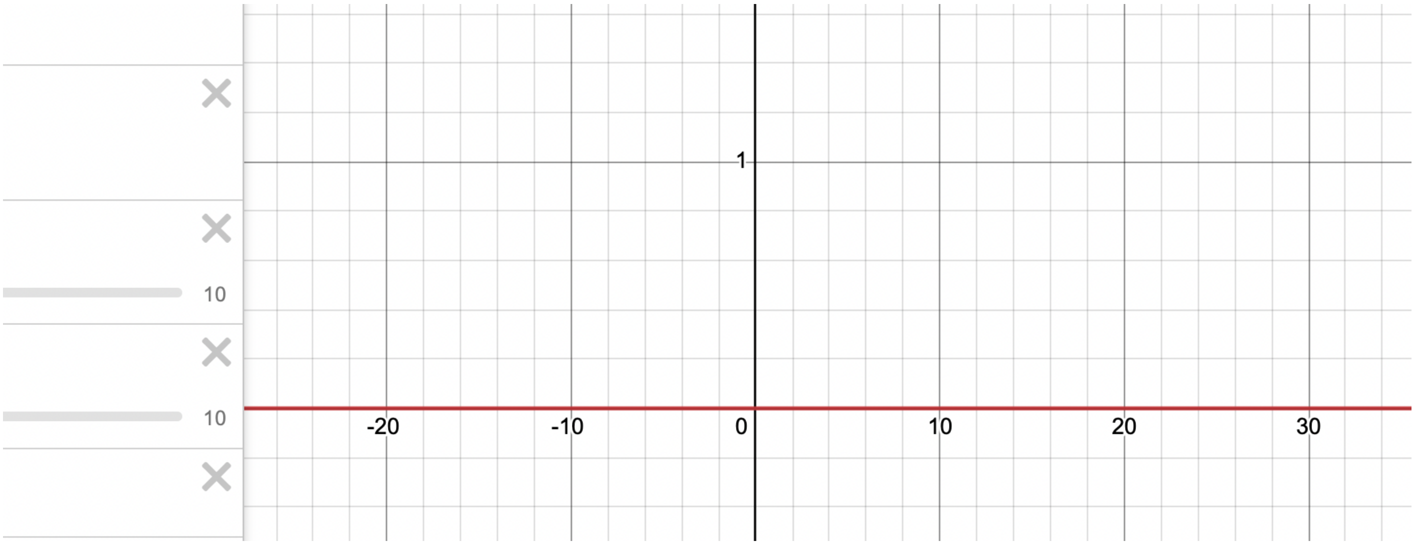
We finally arrive at,

$$\dot{y}(x,0) = v(x) = cB\dot{v}(x) + Bc\dot{v}(x) \implies 2cB = 1 \implies B = \frac{1}{2c} \implies A = -\frac{1}{2c}$$

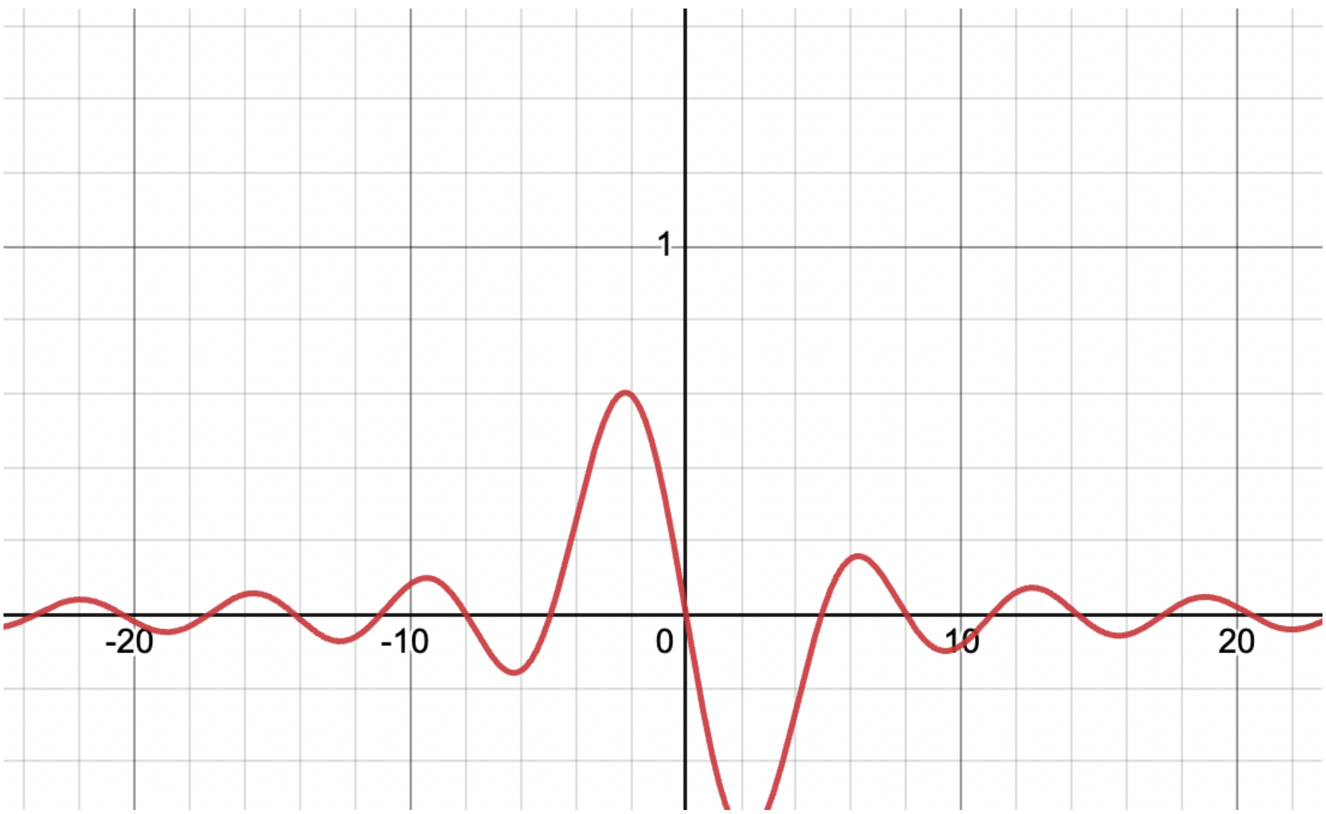
$$\boxed{y(x,t) = \frac{v(x+ct) - v(x-ct)}{2c}}, \quad t > 0$$

Verifying, this obeys $y_0(0) = 0, \quad \dot{y}_0(0) = \dot{y}(x,0)$.

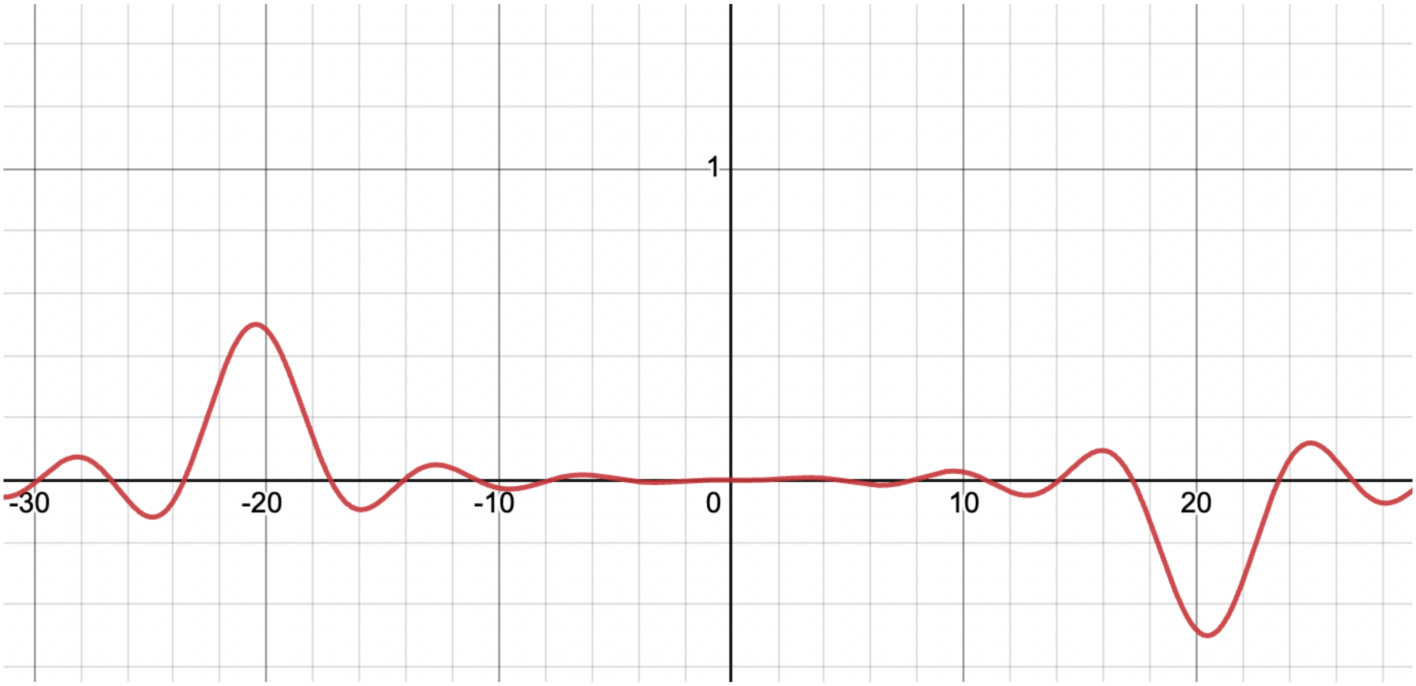
Graphing $v(x) = sinc(x)$ again as test function,
$t = 0$



$t = 1$



$t = 10$



Graphically, it appears that from nothing, two subwaves emerge and propagate in opposite directions.