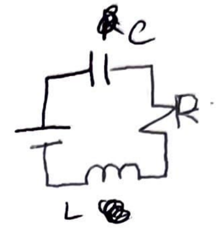

Q2) a) $v = \frac{1}{C}\int dt\, i - Ri + L\frac{di}{dt}$

$\dot{V} = \frac{1}{C} i - R\dot{i} + L\ddot{i}$

$j\omega V = \frac{1}{C} I - Rj\omega I + (j\omega)^2 L I$

$$\boxed{\left(\frac{1}{j\omega C} - R + j\omega L\right) I(\omega) = V(\omega)}$$

b) $F(V_0\,\delta(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V_0\,\delta(t)\, e^{-i\omega t}\, dt = V_0/\sqrt{2\pi} = V$

$$\Rightarrow I = \frac{-j\omega\left(V_0/\sqrt{2\pi}\,L\right)}{\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = -\frac{j\omega\left(V_0/\sqrt{2\pi}\,L\right)}{\left(\omega + j\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R_0^2}{4L^2}\right)}$$

$\Rightarrow$ Poles at $\omega_+ = j\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ $\Rightarrow$ call ω_+, ω_+^*

$\omega_+ = j\frac{R}{2L} + j\underbrace{\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}_{\xi}$

$$i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I e^{i\omega t}\, dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega V_0/L\; e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_+^*)}\, d\omega$$

$= \sum$ poles

$$\text{Resid}_{\omega_+} = \frac{\omega_+ \frac{V_0}{L} e^{i\omega_+ t}}{\omega_+ - \omega_+^*} = \frac{\frac{V_0}{L}\left(-\frac{R}{2L} + \xi\right) e^{-\left(\frac{R}{2L} + \xi\right)t}}{2\xi}$$

$$\text{Resid}_{\omega_-} = \frac{\omega_+^* \frac{V_0}{L} e^{i\omega_+^* t}}{\omega_+^* - \omega_+} = -\frac{\frac{V_0}{L}\left(\frac{R}{2L} - \xi\right) e^{\left(\frac{R}{2L} - \xi\right)t}}{-2\xi}$$

Summing residues,

$$\xi = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \omega_c = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$i = \frac{V_0}{2L\xi}\left(\omega_c e^{\omega_c t} - \omega_c^* e^{\omega_c^* t}\right)$$

for t>0, 0 otherwise

$$\Rightarrow \boxed{i = A e^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} + B e^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}}$$

$\boxed{V_0 \text{ is } V\cdot s}$

$V_0\,\delta(t) = $ volts, $\delta(t) = \frac{1}{s}$ $\Rightarrow$ $\boxed{V_0 \text{ is Volt-seconds}}$

for t>0, 0 for t<0

Q_n 20 Bryant Hor bjh254

a) $m\ddot{x} = -kx \Rightarrow \boxed{m_0(1-\alpha_0 t)\ddot{x} = -K_0 x}$

b) $\boxed{\text{Linear}}$ clearly

c) $\Rightarrow$ Let $\omega^2 = \frac{k_0}{m_0} \Rightarrow (1-\alpha_0 t)\ddot{x} = -\omega^2 x$

Try $x = \sum_{n=0}^{\infty} a_n t^{n+s}$

$\Rightarrow \sum (n+s)(n+s-1)a_n t^{n+s-2} + -a_n\alpha_0(n+s)(n+s-1)t^{n+s-1} + \omega^2 a_n t^{n+s} = 0$

$= s(s-1)a_0 t^{s-2} + [s(s+1)a_1 - \alpha_0 s(s-1)\alpha_0]t^{s-1} + \sum_{n=0}^{\infty}[\omega^2 a_n + (n+s+1)(n+s+2)a_{n+2} - \alpha_0 a_{n+1}(n+s)(n+s+1)]t^{n+s}$

Since all coeffs $=0$ & $a_0 \neq 0$, $s(s-1)=0 \Rightarrow s = 0, 1$

$\Rightarrow a_1(s(s+1)) = 0 \Rightarrow a_1 = 0$, $a_0 = a_0$

$$\Rightarrow \boxed{x = C_1 \sum_{n=0}^{\infty} a_n t^n + C_2 \sum_{n=0}^{\infty} b_n t^{n+1}}$$

$a_0 = a_0$ $b_0 = b_0$

$a_1 = 0$ $b_1 = 0$

Recurrence:

$\omega^2 = \frac{k_0}{m_0}$

$0 < t < \frac{1}{\alpha_0}$

$\alpha_0 a_{n+1}(n+s)(n+s+1) = \omega^2 a_n + (n+s+1)(n+s+2)a_{n+2}$

$s=1$ for b_ns $s=0$ for a_ns

Singular point at $t = 1/\alpha_0$ since $1-\alpha_0 t|_{t=1/\alpha_0} = 0. \Rightarrow \boxed{\text{over } 0<t<\frac{1}{\alpha_0}}$

d) Suppose $\exists$ $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\frac{t^{n+s+1}}{t^{n+s}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}t\right|$

Assuming $L>0$.

$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = \lim_{n\to\infty}\left|\frac{\alpha a_n(n+s)(n+s-1) - \omega^2 a_{n-1}}{(n+s)(n+s+1)a_n}t\right| = \lim_{n\to\infty}\left|\left[\frac{\alpha(n+s-1)}{n+s+1} - \frac{\omega^2}{O(n^2)L}\right]t\right|$

$\Rightarrow$ new condition; To converge. $L>0 \Rightarrow \frac{\omega^2}{L(n+s)(n+s+1)} \to 0$ as $n\to\infty$

$= \lim_{n\to\infty}\left|\left[\frac{\alpha(O(n))}{O(n)} - 0\right]t\right| = L = \alpha t \Rightarrow$ For $0<L<1$, $\alpha t < \frac{1}{\alpha_0}$

$\Rightarrow \boxed{\text{Converges for } 0<t<\frac{1}{\alpha_0}}$

e) At $t = 1/\alpha_0$, singular point, $\Rightarrow m_0(1-\alpha_0 t)\ddot{x} = xk_0$

$\Rightarrow (1-1)\ddot{x} = x = 0 \Rightarrow$ $\boxed{\text{Only trivial solution } x=0 \text{ exists at } t = 1/\alpha_0}$

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$\omega \leftarrow \sqrt{\frac{k_0}{m}}$

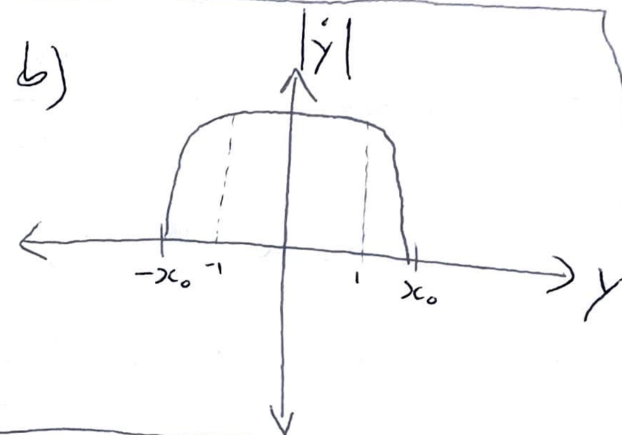
Q22) a) $m_0\ddot{y} + ky = 0 \Rightarrow m_0\ddot{y} + k_0 y^3 = 0 \Rightarrow \ddot{y} + \omega^2 y^3 = 0$

By quadrature, $2\frac{d}{dt}\dot{y}^2 = -\omega^2 y^3 \dot{y} \Rightarrow \dot{y}^2 = -\omega^2 \frac{y^4}{2} + C$

$\dot{y}(x_0) = 0 \Rightarrow C = \omega^2 \frac{x_0^4}{4} \Rightarrow$ $\cancel{\dot{y} = \pm \frac{\omega}{\sqrt{2}}\sqrt{x_0^4 - 2y^4}}$

$$\boxed{\dot{y} = \pm \frac{\omega}{\sqrt{2}}\sqrt{x_0^4 - y^4}}$$

b)



b) Including both $\pm$, superimpose flipped graph, only $+\dot{y}$ half is shown. oscillates from $[-x_0, x_0]$. Speed plateaus at relaxation about $y \in [-1, 1]$

Q23

a) $\boxed{m\ddot{x} = k_0 x + f(t)}$ where $f(t) = \begin{cases} f_0 \sin\omega t & t > 0 \\ 0 & t < 0 \end{cases}$

b) By $\mathcal{L}(\dot{x}) = sF - f(0)$ and $\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$

$\Rightarrow \mathcal{L}(m\ddot{x}) = m s \mathcal{L}\left(\frac{dx}{dt}\right) - \xrightarrow{0 \text{ by IC}} \cancel{\dot{x}(0)} = m s^2 (s X - \xrightarrow{0 \text{ by IC}} \cancel{x(0)}) = m s^2 X$

$\mathcal{L}(f) = f_0 \sin\omega t = \frac{\omega}{s^2+\omega^2}$

$\Rightarrow m s^2 X = k_0 X + \frac{\omega_0 f_0}{s^2+\omega_0^2}$ Let $\frac{k_0}{m} = \omega_0^2$

$\Rightarrow s^2 X = X\omega_0^2 + \frac{\omega_0 f_0/m}{s^2+\omega_0^2} \Rightarrow \boxed{X = \frac{\omega_0 f_0/m}{(s^2-\omega_0^2)(s^2+\omega_0^2)}}$

c) By Final Value Theorem,

$\lim_{t\to\infty} f = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{\omega_0 s f_0/m}{(s^2-\omega_0^2)(s^2+\omega_0^2)}$

$= \lim_{s\to 0} \frac{\cancel{\omega_0 s f_0/m} \rightarrow \text{zero}}{\cancel{s^4-\omega_0^4} \rightarrow \text{nonzero}} = \boxed{0}$

$\lim_{t\to\infty} x'(t) = \boxed{0}$

$\boxed{x(t) \text{ goes to } 0}$

q25

Qa

Following example 10.7 from the book, we can take the laplace transform and generate these conditions,

$$M\ddot{x} + kx = \delta(t)$$

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta(t))$$

Per the book, we have the choice to use the $t = 0^-$ approach definition for our Laplace transform. Of course, by causality, everything is at rest,

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = 1$$

$$Ms^2X + kX = 1$$

$$\omega^2 = \frac{K}{M} \implies X = \frac{1/M}{s^2 + \omega^2}$$

By residue theorem, we take the inverse transform. We've done this many times already and should already know this is sinusoidal. The sum of the residue are clearly,

$$x = \frac{1}{M}\frac{1}{2i\omega}\left[e^{i\omega t} - e^{-i\omega t}\right]$$

$$x = \frac{1}{M\omega}\sin(\omega t), \quad t > 0$$

$$\dot{x} = \frac{1}{M}\cos(\omega t), \quad t > 0$$

$$\ddot{x} = -\frac{\omega}{M}\sin(\omega t), \quad t > 0$$

Then, by inspection of this solution for x, the initial conditions are trivially found to be

$$\boxed{\begin{cases} x(0) = \ddot{x}(0) = 0, \\ \dot{x}(0) = \frac{1}{M} \end{cases}}$$

Where as always, $\omega^2 = K/M$.

Qb

We find the initial conditions imposed similarly,

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta'(t))$$

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = s\mathcal{L}(\delta(0^-)) - \delta(0^-)$$

$$Ms^2X + kX = s$$

$$X = \frac{s/M}{s^2 + \omega^2}$$

By residue theorem, we take the inverse transform. We've done this many times already and should already know this is sinusoidal. The sum of the residue are clearly,

$$x = \frac{1}{M}\frac{\omega i}{2i\omega}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

For $t > 0$,

$$x = \frac{1}{M}\cos(\omega t)$$

$$\dot{x} = -\frac{\omega}{M}\sin(\omega t)$$

$$\ddot{x} = -\frac{\omega^2}{M}\cos(\omega t)$$

To make this true, we look at the solution and derivatives and trivially see that

$$\boxed{x(0) = \frac{1}{M},\ \dot{x}(0) = 0,\ \ddot{x}(0) = -\frac{\omega^2}{M}}$$

Where as always, $\omega^2 = K/M$.

Qc

We use a similar approach,

$$M\ddot{x} + kx = \delta(t) + te^{-t}, \quad t \geq 0$$

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta(t) + te^{-t})$$

Per the book, we have the choice to use the $t = 0^-$ approach definition for our Laplace transform. Of course, by causality, everything is at rest. We take the laplace transform of the exponential as well,

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = 1 + \int_0^\infty dt\ te^{-t}e^{-st} = 1 + \frac{1}{(s+1)^2}$$

$$(s^2 + \omega^2)X = \frac{1}{M} + \frac{1}{M(s+1)^2}$$

$$X = \frac{1}{M(s^2+\omega^2)} + \frac{1}{M(s+1)^2(s^2+\omega^2)}$$

We already know

$$\mathcal{L}^{-1}\left(\frac{1}{M(s^2+\omega^2)}\right) = \frac{1}{M\omega}\sin(\omega t), \quad t > 0$$

We use partial fractions and residue theorem. By partial fractions calculator and laplace table, we arrive at,

$$\mathcal{L}^{-1}\left(\frac{1}{M(s+1)^2(s^2+\omega^2)}\right) = \frac{1}{M}\left[\frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} + \frac{-2\omega\cos(\omega t) + \sin(\omega t) - \omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right]$$

Collecting our results,

$$\mathcal{L}^{-1}(X) = x = \frac{1}{M}\left[\frac{1}{\omega}\sin(\omega t) + \frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} + \frac{-2\omega\cos(\omega t) + \sin(\omega t) - \omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right]$$

We find our final expression for x and $\dot{x}$ (differentiating above expression by calculator),

$$\boxed{x = \begin{cases} \frac{1}{M}\left[\frac{\sin(\omega t)}{\omega} + \frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} - \frac{2\omega\cos(\omega t)+\sin(\omega t)-\omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right] & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

$$\boxed{\dot{x} = \begin{cases} \frac{1}{M}\left[\frac{2\omega^2\sin(\omega t)+\omega^3\cos(\omega t)-\omega\cos(\omega t)}{\omega\cdot(\omega^2+1)^2} + \cos(\omega t) - \frac{\mathrm{t}e^{-t}}{\omega^2+1} + \frac{\mathrm{e}^{-t}}{\omega^2+1} - \frac{2\mathrm{e}^{-t}}{(\omega^2+1)^2}\right] & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

Where as always, $\omega^2 = K/M$.