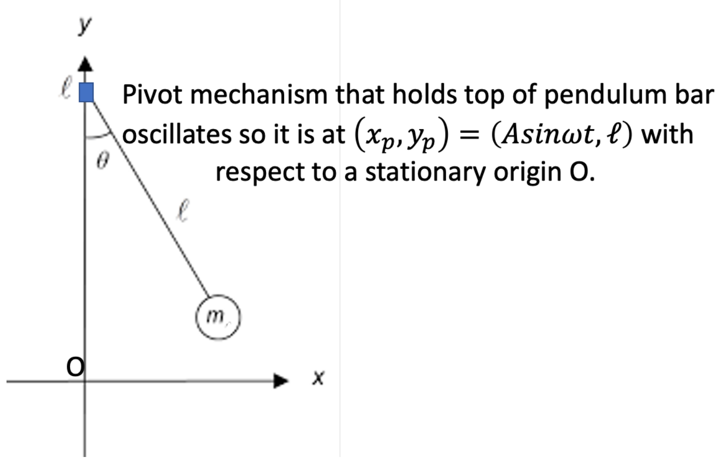
Problem set 8
Applied & Engineering Physics 3330
Due 6:00pm, Thursday Oct. 19, 2023
Remember to explain your answers!

Reading: Taylor's chapter 7 except sections 7.9&7.10, which are optional. (These pages contain lots of examples. For practice, you might try to find the Lagrangian for one or two of them before you read the solution.)

For these problems, always remember that you do not know the initial conditions – somebody probably whacked the system and now it is evolving without further external hits.



Problem 1: We will prove that if any coordinates we use automatically follow a certain kind of well-defined constraint, then we do not have to include any contribution from the constraint force when we write down U for the Lagrangian of the problem.

a) Assume there is a normal gravitational acceleration g pointing down along the negative y direction in the picture here and form your U using only the gravitational potential energy of m written with the coordinate θ, picking any convenient height for the spot where U=0. Then write a Lagrangian for the mass m in the picture, assuming it lives in the x-y plane. θ should be the only variable that's a function of time in your Lagrangian, but it can include constants like A and ω and it can include time. Sections c-e of this problem may prep you for this, but August recitation work might help too.

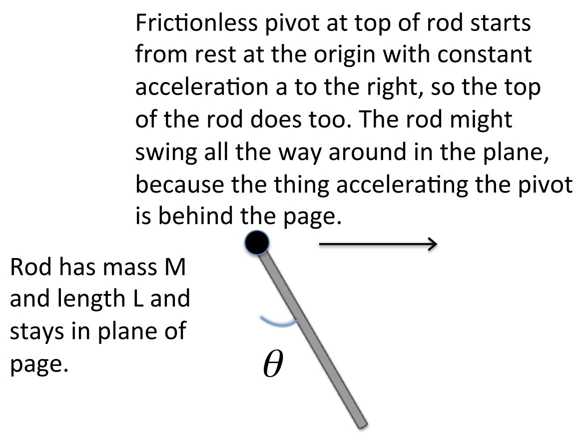
b) Find the Euler-Lagrange equation for the coordinate θ for the mass in the picture. (This should include a $\ddot{\theta}$ so it will function as an equation of motion.) [Note that θ is a clever coordinate to use in the Lagrangian, but the following discussion can reassure you on the correct NUMBER of independent coordinates to use in your Lagrangian.]

c) Now let's justify the number of independent coordinate(s) you used in that Lagrangian. If there was no constraint on a mass m moving in a plane, we could describe the position of mass m with 2 independent coordinates x and y. Forcing it to be at the end of a length ℓ pendulum with whose other end is at $(Asin\omega t, \ell)$ means that we can write a constraint equation of the form $f(x, y, A, \omega, \ell, t) - \ell^2 = 0$. Write out that equation explicitly (using a specific function in place of the name f).

d) Note that the constraint equation you wrote in c could be used to eliminate x or y in a Lagrangian. What is the correct number of degrees of freedom = number of independent coordinates for this problem? (Remember that a constant like A or ω is **<u>NOT</u>** a coordinate – you can't vary the trajectory of those things as a function of time because they don't vary. Also time is the thing we integrate over and NOT a coordinate in our calculus of variations treatment.)

e) The way you wrote the constraint equation above, function (coordinates)=0, let you see that it was **one** constraint that could eliminate **one** coordinate (and one degree of freedom). Sometimes it is helpful to write that SAME constraint in a different way. For the mass m at the end of the pendulum, write
x=(specific correct expression involving A, ω, t, ℓ ,θ)
y=(specific correct expression involving A, ω, t, ℓ ,θ)

This may help with (a) above and later in the Lagrangian unit we will use the chain rule more and it will be helpful to know how to write Cartesian coordinates for masses in a system as a function of the independent coordinates we are using in a Lagrangian and also constants and maybe t. This is a different way of writing constraint equations that is not as convenient for counting degrees of freedom.



Problem 2. Consider a thin rigid stick of <u>uniform density</u>, mass M, and length L . One end of it is connected to a frictionless pivot mechanism that allows it the possibility of swinging all the way around in a plane. (It never moves outside that 2D plane.) A constant gravitational acceleration g points down.

a) **If the frictionless pivot the stick is attached to accelerates horizontally to right (with constant acceleration a) starting from rest at t = 0, how many degrees of freedom does this system have**?

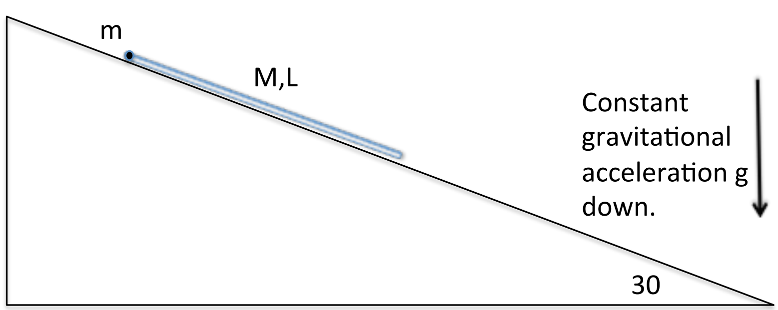
b) Find a Lagrangian for this system whose number of coordinates equals the number of degrees of freedom. (Use the angle definition in the picture – a difference from a vertical line that's positive if in the direction shown.)
Remember what you learned about writing T for a rigid object.

c) Find any Euler-Lagrange equation(s). Think about what you expect and check your math very carefully – there may be a helpful cancellation.

d) Use the results of c to find an answer for equilibrium value of θ, written using given parameters.

e) Quantitatively write the criterion for an equilibrium value θ_{eq} to be stable, and also **make a ROUGH sketch of the stick at a stable equilibrium angle to indicate** whether the stable equilibrium θ_{eq} is positive or negative (given my definition of positive θ in the picture) and whether its absolute value is less than or greater than $\pi/2$.

e) Write an expression for frequency of small oscillations around stable equilibrium that involves θ_{eq}.
If I decided to plug the stable equilibrium value of θ_{eq} into this answer, I should get a value for that ω in terms of given quantities.



Problem 3: A narrow, thin board of mass M and length L slides freely on a huge frictionless, 2D surface that is tilted 30 degrees away from being horizontal. (Maybe the surface is a hockey rink whose supports just collapsed on one side.) The board (which is small along 2 dimensions) does NOT roll because it is NOT round, but it can twist while staying on the surface and slide along the surface. At one end of the board a mass m is **glued to the board**. There is a gravitational acceleration g .

In all cases, we will call the surface an x-y plane, and assume that $\vec{g} = -g\left(\frac{\hat{y}}{2} + \frac{\sqrt{3}\,\hat{z}}{2}\right)$, because that is down. The board does not reach the bottom of the surface during the time under consideration.

a) In THIS part of the problem use X_B, Y_B, the coordinates of the center of the board, as generalized coordinates in the Lagrangian. Notice that I have chosen x to be along the horizontal, so that is a convenient reference direction that should help you give a clear picture defining any other variable you need. Find the Lagrangian.
Remember that your number of coordinates should equal the number of degrees of freedom in the problem, and that it is ok to find the kinetic energy for tiny m and the kinetic energy for sticklike M and add those. Always consider the last way of writing constraint equations in problem 1, where we wrote Cartesian coordinates for a point of interest as a function of the variables we want to use in the Lagrangian – a time derivative then gives velocity components.

b) Find the Euler-Lagrange equations for these coordinates. If any obviously say something is constant, you can stop taking derivatives in that equation and point out the constant quantity.

c) If one of those equations tells you immediately that some quantity is constant, say whether that quantity is a linear momentum or angular momentum and tell what component it is. ALSO explain what symmetry of the Lagrangian could have led you to know that this quantity is indeed constant – if it was a symmetry of translation, say translation along what direction. If it was a symmetry of rotation, say rotation around what direction.

I think the method above is the easiest way to get a valid Lagrangian and valid equations of motion for this system. However, those coordinates only made one constant quantity immediately obvious in the Euler-Lagrange equations.

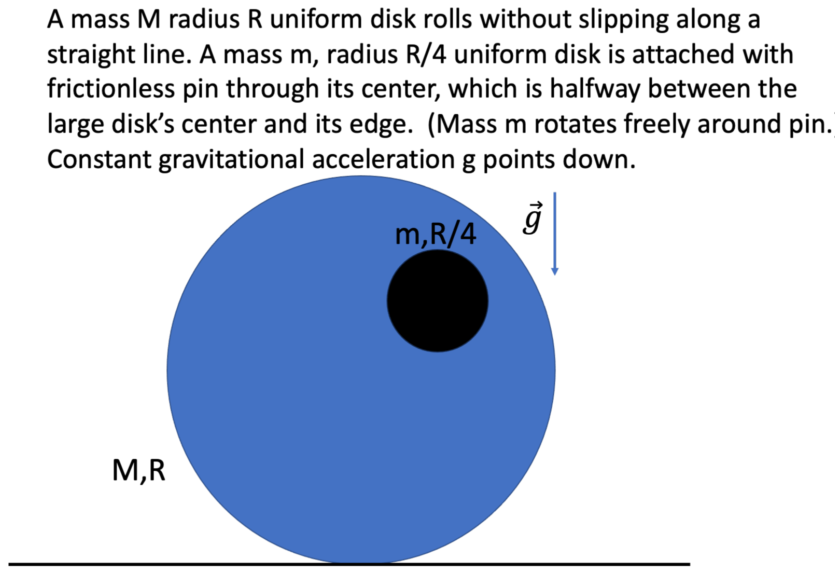
Because m is glued onto M, together they make a **rigid** object and so we are allowed to find a constant Icom for the combined object. That process starts with finding the location of the COM of the whole glued-together thing.

d) On page 102 in problem 3.20 Taylor tells how to find the location of the center of mass of a combined rigid object if you know the COM positions of its parts. **Use this to verify that the center of mass of the combined board+m object is at a distance** $\left(\frac{mL}{2(m+M)}\right)$ **from the center of just the beam**. The required problem ends here.

Optional: if you want to rewrite the Lagrangian to make another symmetry obvious, recall that $I_{COM} = \sum_\alpha m_\alpha [Distance\ of\ \alpha\ from\ a\ line \ ||\ \ \vec{\omega}\ that\ goes\ through\ COM]$ and if we calculate that for the glued-together object we can separate out the one term in that sum which has m from the rest of the sum over the parts of the board. We can get the board's part of the sum from using the parallel axis theorem to shift origin from the board's own COM to this other origin at the COM of the whole board+m thing.

You may optionally ponder whether this new Lagrangian written using COM coordinates for the combined glued-together object and I_{COM} for this combined COM would have any angle appear in the Lagrangian or would it have obvious rotation symmetry.

Problem 4: A mass M radius R uniform disk rolls without slipping along straight line. A mass m, radius R/4 uniform disk is attached with frictionless pin through its center, which is halfway between the large disk's center and its edge. (Mass m rotates freely around pin.)
A constant gravitational acceleration g points down.



a) How many degrees of freedom does this problem have?

b) Write a Lagrangian for the problem using a number of independent coordinates equal to the number of degrees of freedom.

c) Write any Euler Lagrange equations out. If any say something is constant you can stop after identifying what expression is constant, but otherwise take all derivatives as usual noticing when the derivative is total.

d) If those Euler-Lagrange equations say anything is constant, say what kind of symmetry this conserved quantity is related to (rotation? translation? of what?).

Ponder qualitatively what the static friction force might be doing at the point where big disk touches line
The torque contribution from small disk on the big disk will vary.

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