Motivating the Schrödinger Equation

Wednesday, August 30, 2023 8:38 AM

Reading : Griffiths 2.1

- Warm-up Quiz
- HW 2 due Friday
- Today : Motivating the SE
  - Wave functions
  - Expectation Values

Recall : Need a wave equation such that

$\vec{p} = \hbar\vec{k}$

$E = \hbar\omega$

$E = \frac{p^2}{2m} + V$

$\uparrow \omega$ $\quad \uparrow \hbar k^2$

$\cos[kx \mp \omega t]$

$$\omega \Psi = \frac{\hbar k^3}{2m}\Psi + \frac{V}{\hbar}\Psi$$

$\uparrow \frac{\partial}{\partial t}$ $\quad \uparrow \frac{\partial^2}{\partial x}$

Posulate:
$$\boxed{\alpha\frac{\partial\Psi}{\partial t} = \beta\frac{\partial^2\Psi}{\partial x^2} + V\Psi}$$

Try to choose α, β to satisfy our dispersion + relations

Does $\Psi = \cos(kx - \omega t)$ satisfy E, p ?
(hint: no, but it will teach us something)

Plug in: LHS $= \alpha\omega\sin(kx - \omega t)$
RHS $= [-\beta k^2 + V]\cos(kx - \omega t)$
This doesn't work [can't turn a sin into a cos]
→ different orders of derivatives

Try $\Psi = \cos(kx - \omega t) + \eta\sin(kx - \omega t)$; pick η Later

$$\frac{\partial\Psi}{\partial t} = \omega\sin(kx - \omega t) - \eta\omega\cos(\omega t)$$

$$\frac{\partial^2\Psi}{\partial x^2} = -k^2\cos(kx - \omega t) - \eta k^2\sin(kx - \omega t)$$

so

$$\alpha\omega\sin(kx - \omega t) - \alpha\eta\omega\cos(\omega t) = [-\beta k^2 + V]\cos(kx - \omega t) + \eta[-\beta k^2 + V]\sin(kx - \omega t)$$

Gather Terms

$$[\alpha\omega + \eta\beta k^2 - \eta V]\sin(kx - \omega t) = [\alpha\eta\omega - k^2\beta + V]\cos(kx - \omega t)$$

Only possible for all x, t if bracketed terms are 0

LHS : $\alpha\omega + \eta\beta k^2 - \eta V = 0 \Rightarrow \frac{\alpha\omega}{\eta} + \beta k^2 - V = 0$

RHS = $\alpha\eta\omega - k^2\beta + V = 0$

add both equation

$$\frac{\cancel{\alpha\omega}}{\eta} + \cancel{\alpha}\eta\omega = 0$$

$$\frac{1}{\eta} + \eta = 0 \rightarrow \eta = -\frac{1}{\eta} \rightarrow \eta^2 = -1$$

$\therefore \eta = \pm i$ [choose $+i$, not physically sig.]

sub η in LHS

$$\frac{\alpha\omega}{i} = (V - \beta k^2) \Rightarrow \underset{(A)}{-i\alpha\omega} = V - \underset{(B)}{\beta k^2}$$

Now choose α, β to satisfy $E = \frac{p^2}{2m} + V \Rightarrow \boxed{\hbar\omega = \frac{\hbar^2 k^3}{2m} + V}$

(A) $-i\alpha\omega = \hbar\omega \Rightarrow \alpha = i\hbar$

(B) $-\beta k^2 = \frac{\hbar^2 k^3}{2m} \Rightarrow \beta = \frac{-\hbar^2}{2m}$

So :
$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi}$$
Time dependent Schrödinger Equation (for a massive particle)

Something odd : w/ $\eta = i \Rightarrow$ for $V = 0$ solutions

$$\Psi = \cos(kx - \omega t) + i\sin(kx - \omega t) = e^{i(kx - \omega t)}$$ Complex plane wave

Solutions are complex!!
→ Theres no way around it if $\omega \propto k^2$: 1 time, 2 space derivatives.

→ Physically only real values can be measured
→ what is the physical signifance of a wave function?
The formalism tells us probabilities
Recall double slit experiment: detector → prob. of detecting a photon
→ detects intensity : $I \propto (\text{amplitude})^2$
→ we need something similar

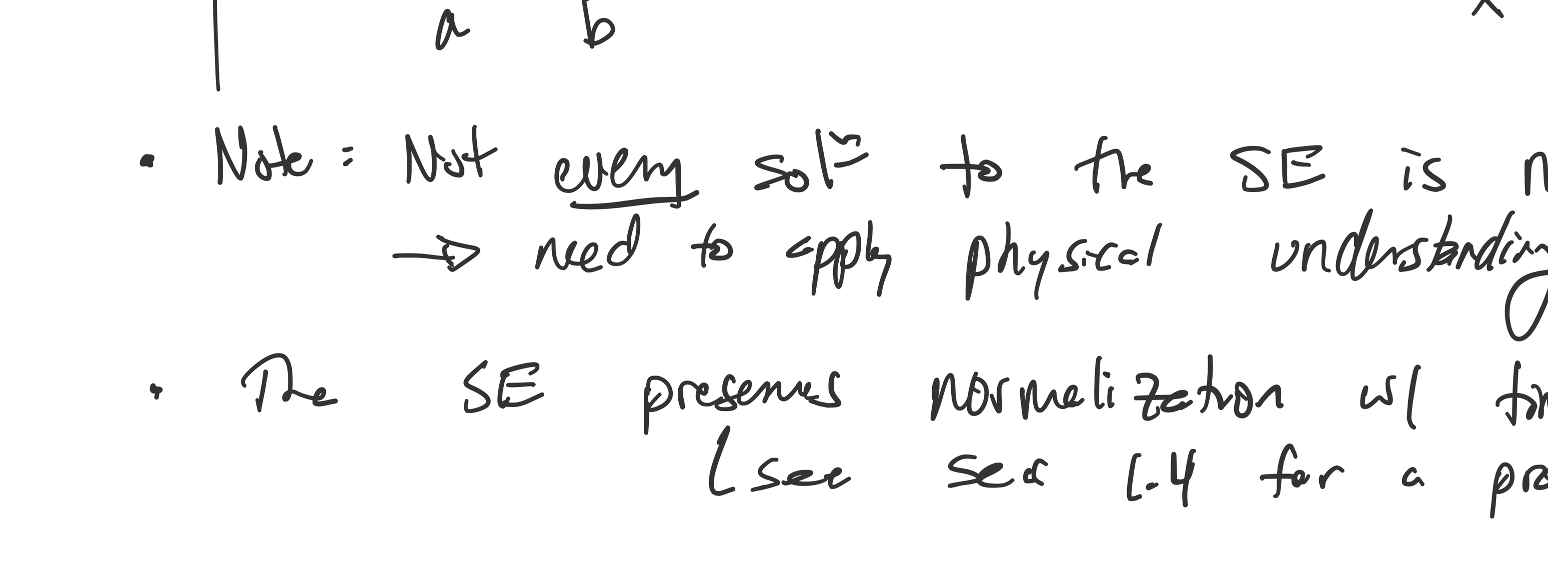
Postulate : Probability of finding a particle is

$$P \propto |\Psi|^2 = \Psi^*\Psi$$
← Max Born: this is a prob. density

$$P = \int_{-\infty}^{\infty} P(x)\,dx = 1$$

$$\therefore P = \int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx = 1 \quad @ \text{ time } t$$

We must always normalize $\Psi(x,t)$ to make this correct



$$P(\text{between } a \text{ and } b) = \int_a^b |\Psi(x,t)|^2\,dx$$

- Note : Not <u>every</u> sol$^{\underline{n}}$ to the SE is normalizable
  → need to apply physical understanding
- The SE preserves normalization w/ time evolution (see sec 1.4 for a proof)

Like any prob. distribution, we can find statistical quantities

Discrete distribution $\langle f(i)\rangle = \sum_j^{\infty} f(j)P(j)$

Continuous distribution $\langle f\rangle = \int_{-\infty}^{\infty} f(x)P(x)\,dx$

only works if $\sum_i P(i) = 1$

or $\int_{-\infty}^{\infty} P(x)\,dx = 1$