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Math Phys

Q53) $x^2 y'' + (x^2\cos(x)+2x)y' + xy = d \Rightarrow \quad y'' + \left(\cos(x)+\frac{2}{x}\right)y' + \frac{y}{x} = \frac{d}{x^2}$

a) $e^{\int \cos(x)+\frac{2}{x}dx} = x^2 e^{\sin(x)} = x^2 f(x) \Rightarrow \boxed{f(x) = e^{\sin(x)}}$

b) $x^2 y'' + (x^2\cos(x)+2x)y' + xy = d \rightarrow x^2 e^{\sin x} y'' + e^{\sin x}(x^2\cos(x)+2x)y' + xye^{\sin x} = d\,e^{\sin x}$

$x^2 e^{\sin x} y'' + e^{\sin x}(x^2\cos(x)+2x)y' + xye^{\sin x} = d\,e^{\sin x} \Rightarrow$ prod rule $\Rightarrow \boxed{\frac{d}{dx}\left(x^2 e^{\sin(x)}\frac{dy}{dx}\right) + xe^{\sin(x)}y = d(x)e^{\sin(x)}}$

$s = de^{\sin x}$
$q = xe^{\sin x}$
$p = x^2 e^{\sin x}$

Q54) a) $x^3y''+x^2y'-xy = d \Rightarrow x^3g''+x^2g'-xg = \delta(x-\xi)$. Try $g = x^m$ (CE eq) $\Rightarrow x(m^2-m+m-1) = 0 \Rightarrow m = \pm 1 \Rightarrow g = x, 1/x$

$\Rightarrow g = \begin{cases} Ax + B/x & x<\xi \\ Cx + D/x & x>\xi \end{cases}$ Homogenous BC $\Rightarrow g(0)=g(1)=0$. $\quad g(0)=0 \Rightarrow B=0 \quad g(1)=0 \Rightarrow C=-D$

Jump in derivative: $x^3g''+x^2g'-xg = \delta(x-\xi) \Rightarrow g''+g'/x - g/x^2 = \delta(x-\xi)/x^3 \Rightarrow \int_{\xi^-}^{\xi^+} dx\, g'' + \cancel{g'/x}^{0} - \cancel{g/x^2}^{0} = \int_{\xi^-}^{\xi^+} dx\, \delta(x-\xi)/x^3 = g'\Big|_{\xi^-}^{\xi^+} = 1/\xi^3$
$\Rightarrow \Delta g'(\xi) = \frac{1}{\xi^3}$

By continuity, $A\xi = C\xi - C/\xi \Rightarrow A = C - C/\xi^2$. $\Rightarrow \Delta g'(\xi) = \frac{1}{\xi^3} \Rightarrow C + C/\xi^2 - A = \frac{1}{\xi^3} \Rightarrow \frac{2C}{\xi^2} = \frac{1}{\xi^3} \Rightarrow C = 1/2\xi,\ A = \frac{1}{2\xi} - \frac{1}{2\xi^3}$

$$\boxed{g = \begin{cases} \left(\frac{1}{2\xi} - \frac{1}{2\xi^3}\right)x & 0<x<\xi \\ \frac{1}{2\xi}\left(x - \frac{1}{x}\right) & \xi<x<1 \end{cases}}$$

b) $x^3y''+x^2y'-xy = d \Rightarrow xy''+y'-x^{-1}y = dx^{-2} \Rightarrow \boxed{\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{y}{x} = \frac{d(x)}{x^2}}$

c) $\frac{d}{dx}\left(x\frac{dg}{dx}\right) - \frac{g}{x} = \delta(x-\xi) \Rightarrow \frac{d}{dx}\left(x\frac{dg}{dx}\right) - \frac{g}{x} = 0 \Rightarrow \frac{g}{x} = \frac{d}{dx}\left(x\frac{dg}{dx}\right)$

From a) $g = \begin{cases} Ax + B/x & 0<x<\xi \\ Cx + D/x & \xi<x<1 \end{cases}$ Applying continuity & BC from a, $C=-D$, $B=0$, $A = C - C/\xi^2$

Deriv jump: $\int_{\xi^-}^{\xi^+} \frac{d}{dx}\left(x\frac{dg}{dx}\right) - \cancel{\frac{g}{x}}^{0} dx = \int_{\xi^-}^{\xi^+} \delta(x-\xi)dx = 1 = (xg')\Big|_{\xi} \Rightarrow C + \frac{C}{\xi^2} - A = \frac{1}{\xi} \Rightarrow \frac{2C}{\xi^2} = \frac{1}{\xi} \Rightarrow C = \frac{\xi}{2} \Rightarrow A = \frac{\xi^2-1}{2\xi}$

$$\Rightarrow \boxed{g = \begin{cases} \frac{\xi^2-1}{2\xi}x & 0<x<\xi \\ \frac{\xi}{2}\left(x-\frac{1}{x}\right) & \xi<x<1 \end{cases}}$$

d) Using a's green's func

$$g = \begin{cases} \left(\frac{1}{2\xi} - \frac{1}{2\xi^3}\right)x & 0<x<\xi \\ \frac{1}{2\xi}\left(x - \frac{1}{x}\right) & \xi<x<1 \end{cases} \qquad y(x) = \int g\, d(\xi)d\xi = \int_0^x d\xi \left(\frac{1}{2\xi} - \frac{1}{2\xi^3}\right)x\, d(\xi) + \int_x^1 d\xi \frac{1}{2\xi}\left(x - \frac{1}{x}\right)d(\xi)$$

Using c,

$$g = \begin{cases} \frac{\xi^2-1}{2\xi}x & 0<x<\xi \\ \frac{\xi}{2}\left(x-\frac{1}{x}\right) & \xi<x<1 \end{cases} \Rightarrow y(x) = \int g\frac{d(\xi)}{\xi^2}d\xi = \int_0^x d\xi \frac{\xi^2-1}{2\xi^3}x\, d(\xi) + \int_x^1 d\xi \frac{\xi}{2\xi^2}\left(x - \frac{1}{x}\right)d(\xi)$$

Comparing integrands (limits are same), $\frac{\xi^2-1}{2\xi^3}x\,d(\xi) = \left(\frac{1}{2\xi} - \frac{1}{2\xi^3}\right)x\,d(\xi)$ ✓ Match

$\frac{\xi}{2\xi^2}\left(x-\frac{1}{x}\right)d(\xi) = \frac{1}{2\xi}\left(x - \frac{1}{x}\right)d(\xi)$ ✓ Match

Same response

$\Rightarrow$ $\boxed{\text{Since 2 integrals are equal, both yield same } y(x) \text{ response}}$ ✓

Q10) $\Phi(r) = \frac{V_1 - V_2}{\ln(r_1/r_2)} \ln \rho + \frac{V_1 \ln r_2 - V_2 \ln r_1}{\ln(r_2/r_1)}$

$\Phi(r_1) = \frac{V_1 - V_2}{\ln(r_1/r_2)} \ln r_1 + \frac{V_1 \ln r_2 - V_2 \ln r_1}{\ln(r_2/r_1)} = \frac{\cancel{V_1\ln(r_1)} - \cancel{V_2\ln(r_1)} + V_1(\ln(r_1) - \ln(r_2))}{\ln(r_1/r_2)}$

$= \frac{V_1(\ln(r_1) - \ln(r_2))}{\ln(r_1/r_2)} = V_1$ ✓

$\Phi(r_2) = \frac{V_1 - V_2}{\ln(r_1/r_2)} \ln r_2 + \frac{V_1 \ln r_2 - V_2 \ln r_1}{\ln(r_2/r_1)} = \frac{\cancel{V_1\ln(r_2)} - \cancel{V_1\ln(r_2)} + V_2(\ln(r_2) - \ln(r_1))}{\ln(r_2/r_1)} = V_2$ ✓

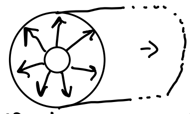
$\nabla^2 \Phi = \frac{\rho}{\epsilon_0} = 0 = \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]\Phi$

$= A\left(-\frac{1}{\rho^2} + \frac{1}{\rho^2}\right) = 0$ ✓

$\Rightarrow$ $\boxed{\text{Follows BC \& Laplace} \Rightarrow \text{Valid Solution}}$

General Solution: $\Phi = R(\rho)H(\theta)Z(z)$, $\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + c_z\rho^2 + c_\theta = 0$
textbook

No variation in θ or z



$\Rightarrow H = Z = 1 \Rightarrow C_z = C_\theta = 0$

$\Rightarrow \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{R}{R}\frac{\partial R}{\partial \rho} = 0 = \rho\frac{\partial^2 R}{\partial \rho^2} + \frac{\partial R}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{\partial R}{\partial \rho}\rho\right) = 0 \Rightarrow \frac{\partial R}{\partial \rho} = \frac{A}{\rho} \Rightarrow R = A\ln(\rho) + B$

$R = A\ln(\rho) + B$. By BC, $A\ln(r_1) + B = V_1$; $A\ln(r_2) + B = V_2 \Rightarrow A(\ln r_1 - \ln r_2) = V_1 - V_2 \Rightarrow A = \frac{V_1 - V_2}{\ln(r_1/r_2)}$

$A = \frac{V_1 - V_2}{\ln(r_1/r_2)} \Rightarrow \frac{V_1 - V_2}{\ln(r_1/r_2)}\ln(r_2) + B = V_2 \Rightarrow B = \frac{V_2\ln(r_1) - \cancel{V_2\ln(r_2)} + \cancel{V_2\ln(r_2)} - V_1\ln(r_2)}{\ln(r_1/r_2)} = \frac{V_1\ln(r_2) - V_2\ln(r_1)}{\ln(r_2/r_1)}$

$\Rightarrow \Phi = RHZ = A\ln(\rho) + B = \boxed{\Phi = \frac{V_1 - V_2}{\ln(r_1/r_2)}\ln(\rho) + \frac{V_1\ln(r_2) - V_2\ln(r_1)}{\ln(r_2/r_1)}}$ Same ✓ starting from textbook solution

Q11) a) $\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}\right]H(\theta)R(\rho) = 0 \Rightarrow \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + \underbrace{\frac{1}{H}\frac{\partial^2 H}{\partial \theta^2}}_{c_\theta} = 0$

We consider when $c_\theta \le 0$. Let $c_\theta = -\nu^2$,

Then, $\frac{1}{H}\frac{\partial^2 H}{\partial \theta^2} = c_\theta \Rightarrow \frac{\partial^2 H}{\partial \theta^2} = -\nu^2 H \Rightarrow \boxed{H(\theta) = A\cos(\nu\theta) + B\sin(\nu\theta)}$

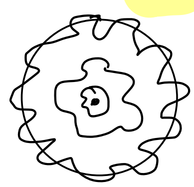
b) $\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} - \nu^2 = 0 \Rightarrow \rho^2 R_{\rho\rho} + \rho R_\rho - \nu^2 R = 0 \Rightarrow$ Cauchy-Euler Equation

R is a linear combo of ind. sols

Ansatz: $\rho^n \Rightarrow \rho^n(n^2 \cancel{-n} \cancel{+n} - \nu^2) = 0 \Rightarrow n = \pm\nu \Rightarrow \boxed{R(\rho) = A\rho^\nu + B\rho^{-\nu}}$ $\nu = 1, 2, \dots$ $\boxed{R(\rho) = A\ln(\rho) + B}$ → Double root, add $\ln r$ $\nu = 0$

c) sine Oscillations in $\theta \Rightarrow \Phi = \sum_{n=1}^{\infty} A_n \rho^n e^{in\theta} \cancel{+ B_n \rho^{-n} e^{in\theta}} \to 0$ Assume finite Φ at $\rho = 0$

odd at $\theta = 0 \Rightarrow \Phi = \sum_{n=1}^{\infty} A_n \rho^n \sin(n\theta)$.



At $r_0 = \rho$ $\sum_{n=1}^{\infty}\int_0^{2\pi} A_n r_0^n \sin(n\theta)\sin(m\theta)\,d\theta = \int_0^{2\pi}\sin(m\theta)V_0\sin\theta\,d\theta \xRightarrow{\text{by orthogonality}} A_m r_0^m = \delta_{m0}V_0$

$\Rightarrow A_n = \begin{cases} 0 & \text{otherwise} \\ V_0/r_0 & n=1 \end{cases} \Rightarrow \boxed{\Phi(\rho,\theta) = \frac{V_0}{r_0}\rho\sin\theta}$ $0 \le \rho \le r_0$ (within circle)

$\boxed{\text{Let's assume potential is finite for all } \rho : \forall \rho_0 \lim_{\rho\to\rho_0}|\Phi| \ne \infty}$ (finite as $\rho \to \infty$) discont deriv.

For $\rho > r_0$,

$\Phi = \sum_{n=1}^{\infty} \overset{0 \text{ (finite at } \infty)}{\cancel{A_n \rho^n e^{in\theta}}} + B_n \rho^{-n} e^{in\theta}$ As before $\sum\int_0^{2\pi} \overset{B_n\delta_{nm}r_0^{-n}}{\cancel{B_n r_0^{-n} \sin n\theta \sin m\theta\, d\theta}} = \int_0^{2\pi} \overset{V_0\delta_{m0}}{\cancel{V_0\sin\theta \sin m\theta\, d\theta}} \Rightarrow B_n = \begin{cases} V_0 r_0 & n=0 \\ & \text{otherwise} \end{cases}$

$\Rightarrow \boxed{\Phi(\rho,\theta) = \frac{r_0}{\rho}V_0\sin\theta}$ $r_0 < \rho < \infty$