# Qa

Following example 10.7 from the book, we can take the laplace transform and generate these conditions,

$$M\ddot{x} + kx = \delta(t)$$

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta(t))$$

Per the book, we have the choice to use the $t = 0^-$ approach definition for our Laplace transform. Of course, by causality, everything is at rest,

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = 1$$

$$Ms^2X + kX = 1$$

$$\omega^2 = \frac{K}{M} \implies X = \frac{1/M}{s^2 + \omega^2}$$

By residue theorem, we take the inverse transform. We've done this many times already and should already know this is sinusoidal. The sum of the residue are clearly,

$$x = \frac{1}{M}\frac{1}{2i\omega}\left[e^{i\omega t} - e^{-i\omega t}\right]$$

$$x = \frac{1}{M\omega}\sin(\omega t), \quad t > 0$$

$$\dot{x} = \frac{1}{M}\cos(\omega t), \quad t > 0$$

$$\ddot{x} = -\frac{\omega}{M}\sin(\omega t), \quad t > 0$$

Then, by inspection of this solution for $x$, the initial conditions are trivially found to be

$$\boxed{\begin{cases} x(0) = \ddot{x}(0) = 0, \\ \dot{x}(0) = \frac{1}{M} \end{cases}}$$

Where as always, $\omega^2 = K/M$.

# Qb

We find the initial conditions imposed similarly,

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta'(t))$$

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = s\mathcal{L}(\delta(0^-)) - \delta(0^-)$$

$$Ms^2X + kX = s$$

$$X = \frac{s/M}{s^2 + \omega^2}$$

By residue theorem, we take the inverse transform. We've done this many times already and should already know this is sinusoidal. The sum of the residue are clearly,

$$x = \frac{1}{M}\frac{\omega i}{2i\omega}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

For $t > 0$,

$$x = \frac{1}{M}\cos(\omega t)$$

$$\dot{x} = -\frac{\omega}{M}\sin(\omega t)$$

$$\ddot{x} = -\frac{\omega^2}{M}\cos(\omega t)$$

To make this true, we look at the solution and derivatives and trivially see that

$$\boxed{x(0) = \frac{1}{M},\ \dot{x}(0) = 0,\ \ddot{x}(0) = -\frac{\omega^2}{M}}$$

Where as always, $\omega^2 = K/M$.

## Qc

We use a similar approach,

$$M\ddot{x} + kx = \delta(t) + te^{-t}, \quad t \geq 0$$

$$M\mathcal{L}(\ddot{x}) + k\mathcal{L}(x) = \mathcal{L}(\delta(t) + te^{-t})$$

Per the book, we have the choice to use the $t = 0^-$ approach definition for our Laplace transform. Of course, by causality, everything is at rest. We take the laplace transform of the exponential as well,

$$M(s^2X - sx(0^-) - \dot{x}(0^-)) + kX = 1 + \int_0^\infty dt\, te^{-t}e^{-st} = 1 + \frac{1}{(s+1)^2}$$

$$(s^2 + \omega^2)X = \frac{1}{M} + \frac{1}{M(s+1)^2}$$

$$X = \frac{1}{M(s^2+\omega^2)} + \frac{1}{M(s+1)^2(s^2+\omega^2)}$$

We already know

$$\mathcal{L}^{-1}\left(\frac{1}{M(s^2+\omega^2)}\right) = \frac{1}{M\omega}\sin(\omega t), \quad t > 0$$

We use partial fractions and residue theorem. By partial fractions calculator and laplace table, we arrive at,

$$\mathcal{L}^{-1}\left(\frac{1}{M(s+1)^2(s^2+\omega^2)}\right) = \frac{1}{M}\left[\frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} + \frac{-2\omega\cos(\omega t)+\sin(\omega t)-\omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right]$$

Collecting our results,

$$\mathcal{L}^{-1}(X) = x = \frac{1}{M}\left[\frac{1}{\omega}\sin(\omega t) + \frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} + \frac{-2\omega\cos(\omega t)+\sin(\omega t)-\omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right]$$

We find our final expression for $x$ and $\dot{x}$ (differentiating above expression by calculator),

$$\boxed{x = \begin{cases} \frac{1}{M}\left[\frac{\sin(\omega t)}{\omega} + \frac{te^{-t}}{1+\omega^2} + \frac{2e^{-t}}{(1+\omega^2)^2} - \frac{2\omega\cos(\omega t)+\sin(\omega t)-\omega^2\sin(\omega t)}{\omega(1+\omega^2)^2}\right] & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

$$\boxed{\dot{x} = \begin{cases} \frac{1}{M}\left[\frac{2\omega^2\sin(\omega t)+\omega^3\cos(\omega t)-\omega\cos(\omega t)}{\omega\cdot(\omega^2+1)^2} + \cos(\omega t) - \frac{te^{-t}}{\omega^2+1} + \frac{e^{-t}}{\omega^2+1} - \frac{2e^{-t}}{(\omega^2+1)^2}\right] & t \geq 0 \\ 0 & t < 0 \end{cases}}$$

Where as always, $\omega^2 = K/M$.