#physics

Q1

Q1a

We begin,

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - kr^n$$

By Euler-Lagrange,

$$0 = \frac{d\mathcal{L}}{d\phi} - \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{\phi}} = \boxed{-\mu r^2\ddot{\phi} = 0}$$

$$0 = \frac{d\mathcal{L}}{dr} - \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{r}} = \mu r\dot{\phi}^2 - knr^{n-1} - \mu\ddot{r} \implies \boxed{\mu\ddot{r} = \mu r\dot{\phi}^2 - knr^{n-1}}$$

Combining,

$$\dot{\phi} = \frac{L}{\mu r^2} \implies \boxed{\mu\ddot{r} = \frac{L^2}{\mu r^3} - knr^{n-1}}$$

Q1b

If $\dot{r} = 0$, then

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - knr^{n-1} = 0 \implies \boxed{r_{eq} = \left(\frac{L^2}{kn\mu}\right)^{1/(n+2)}}$$

Q1c

We have

$$\mu\ddot{r} = \frac{L^2}{\mu r^3} - knr^{n-1}, \quad kn > 0$$

To be stable, this derivative must be negative at equilibrium.

$$-\frac{3L^2}{\mu r_{eq}^4} - kn(n-1)r_{eq}^{n-2} < 0, \quad kn > 0$$

$$-\frac{3L^2}{\mu} < kn(n-1)r_{eq}^{n+2} = kn(n-1)\frac{L^2}{kn\mu} \implies -3 < n-1$$

$$-3 < n-1 \implies \boxed{-2 < n} \implies \boxed{\text{stable for } n \geq -1}$$

Q1d

Using the above derivative as $-k$ for hooke's law,

$$\tau_{osc} = 2\pi/\omega, \quad \omega^2 = k/\mu = \frac{3L^2}{\mu^2 r_{eq}^4} + \frac{kn(n-1)}{\mu} r_{eq}^{n-2}$$

$$\boxed{\tau_{osc} = \frac{2\pi}{\sqrt{\frac{3L^2}{\mu^2 r_{eq}^4} + \frac{kn(n-1)}{\mu} r_{eq}^{n-2}}}}$$

Q1e

Recall $\dot{\phi} = \frac{L}{\mu r^2}$. Period is $2\pi/\dot{\phi}$

$$\implies \boxed{\tau_{orb} = \frac{2\pi\mu}{L}\left(\frac{L^2}{kn\mu}\right)^{2/(n+2)}}$$

Q1f

We require rational square root values. We use $\sqrt{n+2} = 1, 2, 3$. Then,

$$\boxed{n = -1, 2, 7}$$

Q2

Big sun mass and radius approximations used throughout.

Q2a

By formula, we have

$$E = -\frac{GM_s m}{2a} \implies \boxed{E = -\frac{GM_S m}{2r_1}}$$

Q2b

Kinetic energy is

$$T = E - U = -\frac{GM_s}{2r_1} + \frac{GM_s}{r_1} = \frac{GM_s m}{2r_1} = \frac{1}{2}mv_1^2$$

$$\implies \boxed{v_1 = \sqrt{\frac{GM_s}{r_1}}}$$

Q2c

The energy of the orbital transfer is the change in energy:

$$\boxed{\Delta E = \frac{GM_S m}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

Q2d

$\boxed{\text{Parallel to original velocity (tangential)}}$

We expand,

$$|v_1 + \Delta v|^2 = |v_1|^2 + 2v_1 \cdot \Delta v + |\Delta v|^2$$

To maximize the post-impulse speed, we must maximize the $2v_1 \cdot \Delta v$ term. Clearly, this dot product is maximized when Δv is tangential or parallel to v_1.

Q2e

At maximum, $a = \frac{r_1+r_2}{2}$. By energy conservation,

$$-\frac{GM_s m}{r_1} + \frac{1}{2}mv_f^2 = -\frac{GM_s m}{r_1 + r_2} \implies v_f = \sqrt{2GM_s(\frac{1}{r_1} - \frac{1}{r_1 + r_2})}$$

$$v_f - v_1 = \boxed{\Delta v = \sqrt{\frac{2GM_s}{r_1} - \frac{2GM_s}{r_1 + r_2}} - \sqrt{\frac{GM_s}{r_1}}}$$

Q2f

By Kepler's third,

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2} \approx \frac{GM_s}{4\pi^2} \implies T = \sqrt{\frac{4\pi^2(r_1+r_2)^3}{2^3 GM_s}} = \sqrt{\frac{\pi^2(r_1+r_2)^3}{2GM_s}}$$

We only complete a half period, so we coast for

$$\boxed{t = \sqrt{\frac{\pi^2(r_1+r_2)^3}{8GM_s}}} \text{ units of time}$$

Q2g

$\boxed{\text{Yes}}$. Otherwise you'll have to catch up/slow down to Venus.

Q2h

$\boxed{\text{Yes}}$. Otherwise you'll continue moving in the elliptical non-Venus orbit.

Q3

Q3a

By formula, we have for rocket mass m_r,

$$E = \frac{1}{2}m_r(7900)^2 - \frac{GMm_r}{6371000 + 250000} = -28997770.6691 \cdot m_r$$

$$\boxed{E = m_r \cdot -2.9 \cdot 10^7} \text{ Joules}$$

where m_r is numerical rocket mass in kgs but carries no units.

Q3b

We have,

$$E = -\frac{GMm}{2a} = m \cdot -2.9 \cdot 10^7$$

$$\implies a = 6872986m \approx \boxed{a = 6.87 \cdot 10^6 \text{ meters}}$$

Q3c

By Kepler's Third,

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2} \approx \frac{GM_s}{4\pi^2} \implies T = 2\pi\sqrt{\frac{a^3}{GM_E}} = 5666.892s$$

$$\boxed{T = 5667 \text{ seconds}}$$

Q3d

We already found a is 6872986m. The closest distance is the initial distance or $6371000 + 250000 = 6621000$ meters. This must be either the closest or the farthest, since the rocket is perpendicular to the earth at this point. Since it's less than a, it is the closest. The farthest, then, must be at a distance

$$r_{close} = 6621000 \implies r_{far} = (a - r_c) + a = 7119000$$

$$\boxed{r_{close} = 6.62 \cdot 10^6 \text{ meters}, \quad r_{far} = 7.12 \cdot 10^6 \text{ meters}}$$

Q3e

By formula,

$$\epsilon = \frac{r_{far} - r_{close}}{r_{far} + r_{close}} = 0.0363901018923$$

$$\boxed{\epsilon = 0.0364}$$

So it is fairly circular.

Q3f

$\boxed{28.5 \text{ degrees}}$

They are the same.