Lecture 11 - Momentum space

Friday, September 15, 2023 8:12 AM

1. Warm-up Quiz
2. HW4 due HW5 out ↗ δ-function potentials
3. Reading Griffiths Sec. 2-6, 2-7 → Finite well
4. Today: • momentum space, $\hat{x}$, $\hat{p}$
   • Maybe start Finite well

Last time time: Free particle ($V=0$)
Plane waves

$$\psi(x) = e^{ik_0 x}$$ ← eigenstate but not a real state

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk$$ Integral over plane waves of different k

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx$$ Integral over plane waves w/ different positions.

What does well-behaved mean

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\phi(k)|^2 dk = 1$$

proof: $$|\psi(x)|^2 = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk \right|^2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^*(k') e^{-ik'x} dk' \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk$$

$$= \frac{1}{2\pi} \iint \phi^*(k') \phi(k) e^{i(k-k')x} dk\, dk'$$

Integrate both sides over real space

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \frac{1}{2\pi} \iint \phi^*(k') \phi(k) \underbrace{\int_{-\infty}^{\infty} e^{i(k-k')x} dx}_{2\pi\, \delta(k-k')} dk\, dk'$$

See B+W Ch. 5

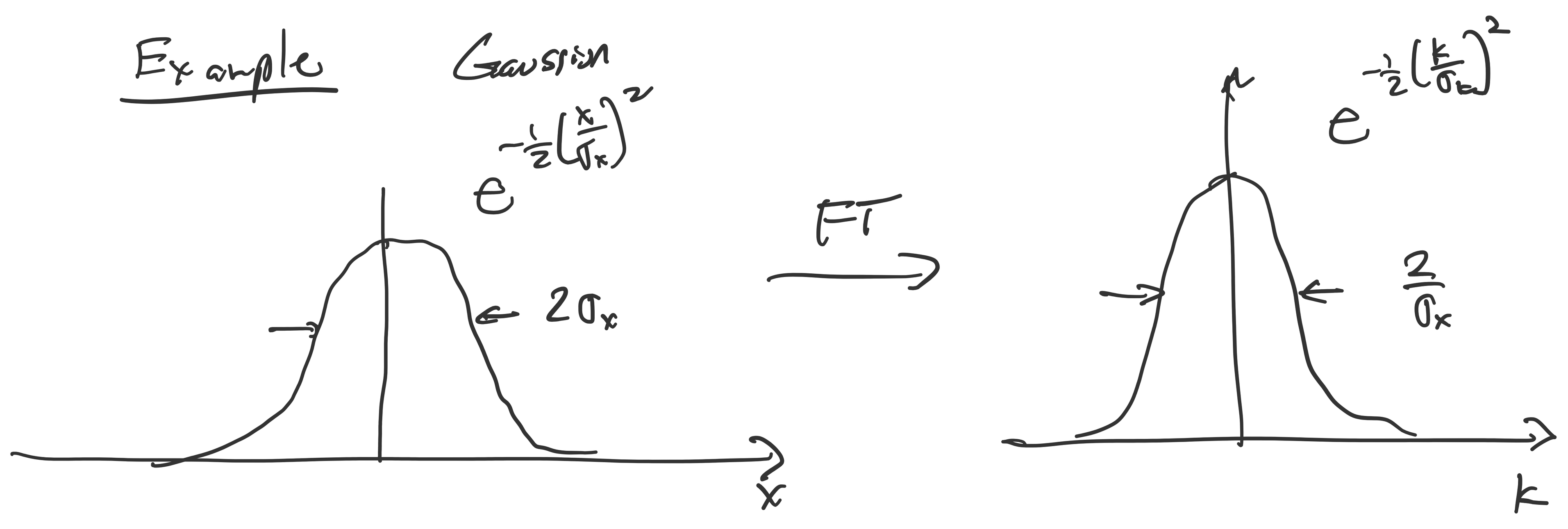
Use $$\int F(k') \delta(k-k') dk' = F(k)$$

$$= \int |\phi(k)|^2 dk \qquad \text{QED}$$

Aside: δ-function

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \int \delta(x) dx = 1$$

Example Gaussian



- Note that $\sigma_x \sigma_k = 1$
- in general $\sigma_x \sigma_k \geq 1$ for well-behaved functions
- If a function is well-localized in one space it must de-localized in the reciprocal space.

Time evolution of a wave packet w/ $\Psi(x, t=0)$
Follows Ex 2.6 in Gr.

① Find eigenstates → Free particle → plane waves

② Fourier Transform into k-space $\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$

③ Transform back w/ Time dependence added

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \omega t)} dk$$ were $\omega = \frac{\hbar k^2}{2m}$

Operators on k-space

$\hat{x}$ in x-space → x
$\hat{p}$ in x-space → $-i\hbar \frac{\partial}{\partial x}$

$$\hat{p}\, \psi(x) = p_0 \psi(x)$$

For plane waves $$\hat{p}\, \psi(x) = -i\hbar \frac{\partial}{\partial x} e^{ik_0 x} = \underbrace{\hbar k_0}_{p_0} e^{ik_0 x}$$

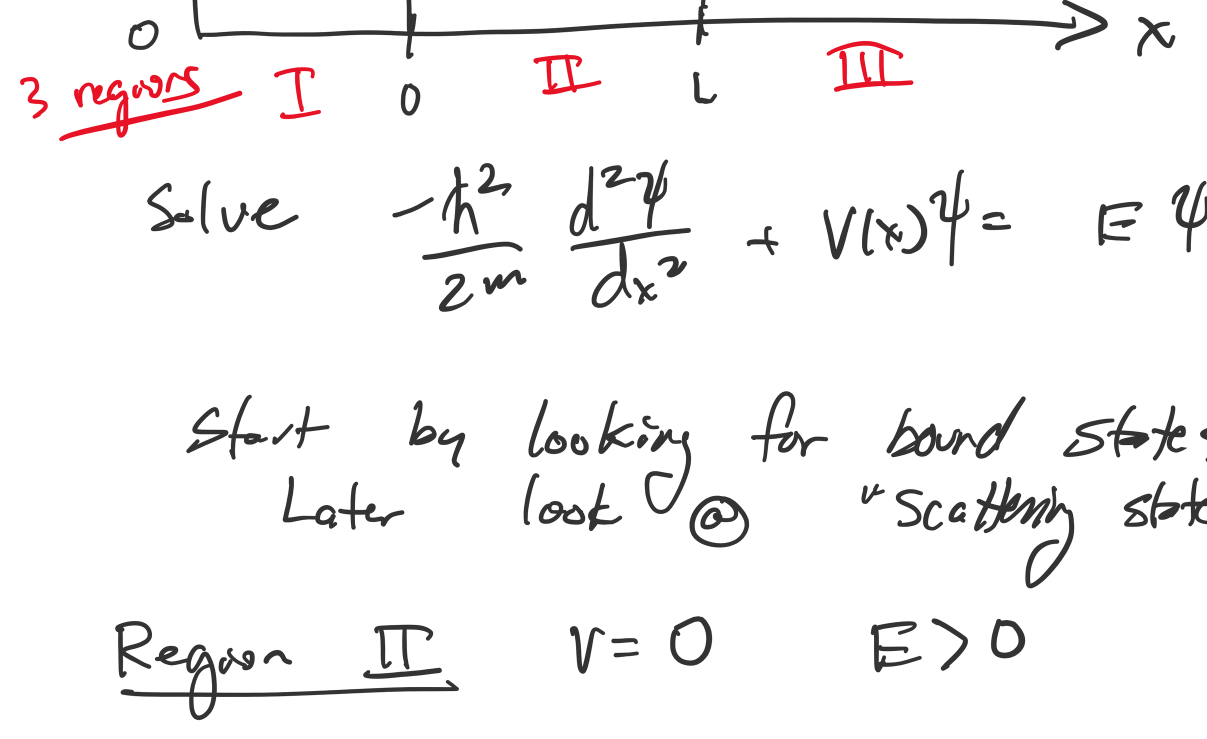
Now in k-space

$$\boxed{\hat{p} \longrightarrow \hbar k}$$

$\hat{x}$ ⟶ Plane wave $\phi(k) = e^{-ikx_0}$
Looking for $\hat{x}\, \phi(k) = x_0 \phi(k)$

so $$\frac{\partial}{\partial k} \phi = -ix_0 e^{ikx_0} \Rightarrow \boxed{\hat{x} \longrightarrow i \frac{\partial}{\partial k}}$$

Next up: Finite Quantum Well in 1-D



$$V(x) = \begin{cases} V_0 & x < 0,\ x > L \\ 0 & 0 < x < L \end{cases}$$

- Expect ψ_n that look similar to the ∞-well
- ψ_n may extend out of the well.

Solve $$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Start by looking for bound states ($E < V_0$, $E > 0$)
Later look @ "scattering states" ($E > V_0$)

Region II $V = 0$ $E > 0$ $A' e^{ik} + B' e^{-ikx}$

$$\frac{d^2\psi}{dx^2} = \underbrace{-\frac{2mE}{\hbar^2}}_{-k^2} \psi \Rightarrow A \sin(kx) + B \cos(kx) = \psi$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Region I, III ($x < 0$, $x > L$) $E > 0$, $E < V_0$

$$\frac{d^2\psi}{dx^2} = (+) \frac{2m}{\hbar^2} (V_0 - E) \psi$$