Problem Set # 3

Due: Sat Oct 30 (11:59pm) in ELMS

Instructions:

- Please write your solutions clearly and show all your work.
- Please number all the pages of your submission.
- The assignment will be submitted online via ELMS.

Problem 1:

First we provide some axioms and definitions of matrix norms and specifically the spectral norm of a matrix. We use the notation $|||.|||$ for a general matrix norm and the notation $||.||$ for the specific matrix norm called the spectral norm of a matrix. Also, when we use the notation $||.||$ for a vector norm we mean the euclidean norm of a vector.

Matrix Norms: Let $\mathbb{M}$ denote the field of real or complex numbers. Let $\mathbb{M}^{m\times n}$ denote the vector space containing all matrices with m rows and n columns with entries in the field $\mathbb{M}$. A function $|||.||| : \mathbb{M}^{m\times n} \to \mathbb{R}$ is a matrix norm if, for any $A, B \in \mathbb{M}^{m\times n}$, it satisfies the following axioms.

(i) $|||A||| \geq 0$ **(Nonnegative)**

(ii) $|||A||| = 0$ if and only if $A = 0$ **(Positive)**

(iii) $|||cA||| = |c|\ |||A|||$ for all $c \in \mathbb{M}$ **(Homogeneous)**

(iv) $|||A + B||| \leq |||A||| + |||B|||$ **Triangle Inequality**

(v) For the case $m = n$, $|||AB||| \leq |||A|||\ |||B|||$ **(Submultiplicativity)**

Spectral Norm of a Matrix: The spectral norm of a $m \times n$ matrix A can be defined in terms of a constrained maximization problem as follows:

$$||A|| = \max_{||x||=1} ||Ax||$$

It should be noted that the spectral norm of a matrix is a matrix norm and satsfies the axioms states above. Also, $||x||$ and $||Ax||$ are euclidean norms of the vectors x and Ax where

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Now we state the problem that you have to solve.

Problem Statement: For an $m \times n$ matrix A, prove using the definition provided above that the spectral norm is given by

$$||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$$

Conclude that for any $n \times 1$ vector x,

$$||Ax|| \;\leq\; ||A||\,||x||$$

Using the conclusion above prove that for conformable matrices A and B (AB is well defined)

$$||AB|| \;\leq\; ||A||\,||B||$$

Problem 2: First we provide a definition of the Spectral Radius of a Matrix.

Spectral Radius of a Matrix: The spectral radius of a square matrix A, denoted by $\sigma(A)$, is defined as the largest absolute value of the eigen values of A. Mathematically we can write it as follows:

$$\sigma(A) = \max\left\{|\lambda| : \lambda \text{ is an eigen value of } A\right\}$$

Problem Statement: Show that for the $n \times n$ matrix A

$$\sigma(A) \leq \|A\|$$

where $\|A\|$ is the spectral norm of the matrix A.

Problem 3: If $A(t)$ is a continuously-differentiable $n \times n$ matrix function that is invertible at each t, show that

$$\frac{d}{dt}A^{-1}(t) \;=\; -\,A^{-1}(t)\,\dot{A}(t)\,A^{-1}(t)$$

Problem 4: Use Laplace transforms to solve $\dot{x} = ax(t) + b(t)u(t)$, with the initial condition $x(0)$. For this problem take a to be a contant and $x(t), b(t)$, and $u(t)$ are real valued functions.

Problem 5: Define state variables such that the n^{th}-order differential equation

$$y^{(n)}(t) + a_{n-1}t^{-1}y^{(n-1)}(t) + a_{n-2}t^{-2}y^{(n-2)}(t) + \cdots + a_1 t^{-n+1}y^{(1)}(t) + a_0 t^{-n}y(t) = 0$$

where $y^{(n)}(t) = \dfrac{d^n y(t)}{dt^n}$, can be written as the linear state equation

$$\dot{x}(t) \;=\; t^{-1}Ax(t)$$

where A is a constant $n \times n$ matrix

Problem 6: Prove that

$$\frac{\partial}{\partial\tau}\Phi(t,\tau) \;=\; -\,\Phi(t,\tau)A(\tau)$$

Problem 7: Compute the state-transition matrix $\Phi(t, t_0)$ for the following matrix $A(t)$:

$$A(t) = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$$

where η is a bounded and continuous function of t.

Problem 8: Compute the matrix exponential e^{At} for the following 3×3 matrix A:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$