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HOMEWORK - 3

1) Problem 1

a) For an $m \times n$ matrix A, prove using the definition provided above that the spectral norm is given by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Answer:-

Spectral norm of a matrix says

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Now w.r.t $\|x\| = 1$

$$\|A\| = \max_{\|x\|=1} \frac{\|Ax\|}{1} = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|}$$

$$\|A\| = \max_{\|x\|=1} \frac{\|Ax\|}{\|x\|}$$

Now for an arbitrary vector norm $\|y\|$.

$y = cx$.

$$\|A\| = \max_{\|y\|=1} \frac{\|Ay\|}{\|y\|}$$

Now plugging in $y = cx$

$$\|A\| = \max_{\|y\|=1} \frac{\|Acx\|}{\|cx\|}$$

from the homogeneous matrix norm

$$\|cx\| = |c|\,\|x\|$$

when $\|y\| = 1$ and c is constant
$\underline{x \neq 0}$

$$\|A\| = \max_{\|x\| \neq 0} \frac{\cancel{|c|}\,\|Ax\|}{\cancel{|c|}\,\|x\|}$$

$$\|A\| = \max_{\|x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Spectral norm can also be given by:-
Q.E.D

b) conclude that for any $n \times 1$ vector x,

$$\|Ax\| \leq \|A\|\,\|x\|$$

Given for any $n \times 1$ vector x.
suppose that $\|Ax\| > \|A\|\,\|x\|$

$$\|Ax\| \cdot \frac{1}{\|x\|} > \|A\|$$

$$\frac{\|Ax\|}{\|x\|} > \|A\|$$

from the spectral norm, this is not possible.

$$\therefore \|Ax\| \leq \|A\| \|x\|$$

Q.E.D

c) Using the conclusion above prove that for comfortable matrices A and B (AB is well defined)

$$\|AB\| \leq \|A\| \|B\|$$

For given two matrices A & B

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

consider

$$\|AB\| = \max_{\|x\|=1} \|ABx\|$$

using the conclusion above

$$\|Ax\| \leq \|A\| \|x\|$$

$$\max_{\|x\|=1} \|ABx\| \leq \|A\| \|Bx\|$$

$\therefore \quad \|AB\| \leq \|A\| \, \|Bx\|$

Now from spectral norm

$\|Bx\| = \|B\|$ when $\|x\| = 1$

$\|x\| = 1$

So $\quad \|AB\| \leq \|A\| \, \|B\|$ $\quad$ Q.E.D

when we plug in arbitrary values which satisfy the conditions, the theorems will be proven

$\square$

2) Spectral radius of a square matrix

$$\sigma(A) = \max\{|\lambda| : \lambda \text{ is an eigen value of } A\}$$

Show that for $n \times n$ matrix A

$$\sigma(A) \leq \|A\|$$

$\|Ax\| \leq \|A\| \|x\|$ ← we know this from the conclusion from problem 1

we also know that $Ax = \lambda x$
↑ eigen vector of A

$\|Ax\| \leq \|A\| \|x\|$

$\|\lambda x\| \leq \|\lambda\| \|x\|$

w.k.t $Ax = \lambda x$

$\therefore \quad |\lambda| \|x\| \leq \|A\| \|x\|$

when $\|x\| \neq 0$ on both sides we can cancel it out

$$|\lambda| \cancel{\|x\|} \leq \|A\| \cancel{\|x\|}$$

$$|\lambda| \leq \|A\|$$

w.k.t $\sigma(A) = \max\{|\lambda|\}$

$$\therefore \quad \boxed{\sigma(A) \leq \|A\|}$$

Q.E.D

3) If $A(t)$ is a continuously differentiable $n \times n$ matrix function that is invertible at each t, show that

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t)\dot{A}(t)A^{-1}(t)$$

Since $A(t)A^{-1}(t) = I$ [non-singular]

differentiating with respect to 't' on both sides.

$$\frac{d}{dt}\left[A(t)A^{-1}(t)\right] = \frac{d}{dt} I \qquad \frac{d}{dt} I = 0$$

Now using uv method

$$A^{-1}(t)\frac{d}{dt}A(t) + A(t)\frac{d}{dt}A^{-1}(t) = 0$$

$$A^{-1}(t)\dot{A}(t) + A(t)\dot{A}^{-1}(t) = 0$$

$$A^{-1}(t)\dot{A}(t) = -A(t)\dot{A}^{-1}(t)$$

multiply with $A^{-1}(t)$ on both sides

$$\boxed{\frac{d}{dt}\dot{A}^{-1}(t) = -A^{-1}(t)\mathring{A}(t)A^{-1}(t)}$$

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4) Use laplace transforms to solve $\dot{x} = ax(t) + b(t)u(t)$ with the initial condition $x(0)$. For this problem take a to be a constant and $x(t)$, $b(t)$, and $u(t)$ are real valued functions.

$$\dot{x} = ax(t) + b(t)u(t)$$

Use laplace transforms on the eqn

$$\mathcal{L}\{\dot{x}\} = s \cdot X(s) - x(0)$$

$$\mathcal{L}\{ax(t)\} = a \cdot X(s)$$

$\mathcal{L}\{b(t)u(t)\}$ ← leaving it like this

$$s \cdot X(s) - x(0) = a \cdot X(s) + \mathcal{L}\{b(t)u(t)\}$$

$$s \cdot X(s) - a \cdot X(s) = x(0) + \mathcal{L}\{b(t)u(t)\}$$

$$(s-a)X(s) = x(0) + \mathcal{L}\{b(t)u(t)\}$$

$$X(s) = \frac{x(0)}{s-a} + \frac{\mathcal{L}\{b(t)u(t)\}}{s-a}$$

$$\mathcal{L}\{b(t)u(t)\} = B(s)\,U(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{B(s)U(s)}{s-a}$$

here $u(t)$ is not a unit step function and hence convolution theorem cant be used. given $x(t)$, $b(t)$, $u(t)$ are real valued functions.

Now applying Inverse Laplace transform

$$L^{-1}[X(S)] = L^{-1}\left[\frac{x(0)}{s-a}\right] + L^{-1}\left[\frac{B(s)\,U(s)}{s-a}\right]$$

Now consider $\begin{cases} B(S)\,U(S) = D(S) \\ 1/s-a = G(S) \end{cases}$

$$L^{-1}[X(s)] = L^{-1}\left[\frac{x(0)}{s-a}\right] + L^{-1}[D(s)\,G(s)]$$

$$x(t) = x_0 e^{at} + D(t) \cdot G(t)$$

$$\boxed{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}}$$

$$D(t) \cdot G(t) = \int_{-\infty}^{\infty} D(\tau)\, G(t-\tau)\, d\tau$$

$$\therefore \quad x(t) = x_0 e^{at} + \int_{-\infty}^{\infty} B(\tau)\, U(\tau)\, e^{a(t-\tau)}\, d\tau$$

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5) $y^{n}(t) + a_{n-1} t^{-1} y^{(n-1)}(t) + a_{n-2} t^{-2} y^{(n-2)} t + \cdots$

$+ \cdots a_1 t^{-n+1} y^{(1)}(t) + a_0 t^{-n} y(t) = 0.$

where $y^{(n)}(t) = \dfrac{d^n y(t)}{dt^n}$ can be written as linear

state equation $\dot{x}(t) = t^{-1} A x(t)$

Now the possible choice of state is

$x(t) = [x_1, x_2 \ldots x_n]^T$

where $x_1 = t^{-n} y(t) \quad \cdots \quad x_n = t^{-1} y^{n-1}(t)$

$\dot{x}_1 = \dfrac{d}{dt}(t^{-n} y(t)) = -n t^{-n-1} y(t) + t^{-n} y(t)$

$= n t^{-1} x_1 + t^{-1} x_2$

$\dot{x}_2 = -(n-1) t^{-1} x_2 + t^{-1} x_3$

$\dot{x}_n = -t^{-1} x_n + t^{-1} y^{n}(t)$

$y_n = -a_{n-1} t^{-1} y^{n-1}(t) \cdots \cdots a_0 t^{n} y(t)$

$= -a_{n-1} x_n - a_{n-2} x_{n-1} \cdots \cdots - a_0 x_1$

when you write the resultant equation in the matrix form

$$\dot{x}(t) = t^{-1} \underbrace{\begin{bmatrix} -n & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1) & 1 & & & & \\ 0 & 0 & \ddots & & & & \\ \vdots & \vdots & & \ddots & & & \\ \vdots & \vdots & & & \ddots & & \\ -a_0 & -a_1 & & & & & -a_{n-1} \end{bmatrix}}_{A} x(t).$$

we get $\boxed{\dot{x}(t) = t^{-1} A x(t)}$ ID

Q.E.D

6) Prove that $\frac{\partial}{\partial \tau} \phi(t, \tau) = -\phi(t, \tau) A(\tau)$

We can use the general form of State transformation matrix :-

$$\frac{\partial}{\partial t} \phi(t, t_0) = A(t) \phi(t, t_0)$$

From this we can write

$$\frac{\partial}{\partial t} \phi(\tau, t_0) = A(\tau) \phi(\tau, t)$$

Now here we use the theorem that if $A(t)$ is continuously differentiable $n \times n$ matrix function that is invertible at t

then

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t) \dot{A}(t) A^{-1}(t)$$

similiarly.

$$\frac{\partial}{\partial t} \phi^{-1}(\tau, t) = -\phi^{-1}(\tau, t) \dot{\phi}(\tau, t) \phi^{-1}(\tau, t).$$

we know from the properties of transition matrices that

$$\phi^{-1}(t_0, t) = \phi(t, t_0).$$

Using that property,

$$\frac{\partial}{\partial t}\phi(t,\tau) = -\phi(t,\tau)\dot{\phi}(\tau,t)\phi(t,\tau)$$

w.k.t $\quad \dot{\phi}(\tau,t) = A(\tau)\phi(\tau,t).$

$$\frac{\partial}{\partial t}\phi(t,\tau) = -\phi(t,\tau)A(\tau)\cancel{\phi(\tau,t)\phi(t,\tau)}$$

| D

$$\therefore \boxed{\frac{\partial}{\partial t}\phi(t,\tau) = -\phi(t,\tau)A(\tau)}$$

7) Given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}$$

considering :-

$$\dot{x}(t) = A(t)x(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \eta(t) \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Now for $\dot{x}_1(t)$

$$\dot{x}_1(t) = x_1(t)$$

$$\frac{dx_1}{dx} = x_1(t)$$

$$dx_1 = x_1(t)\,dt$$

$$x_1 = \int_{t_0}^{t_1} x_1(t)\,dt$$

$$x_1 = x_1(t_0)\,e^{t-t_0}$$

Now for x_2

$$\dot{x}_2(t) = x_1(t) + \eta(t)\,x_2(t)$$

$$\dot{x}_2(t) - \eta(t)\,x_2(t) = x_1(t)$$

$$= x_1(t_0)\,e^{t-t_0}$$

$\Rightarrow$ $y' + Py = Q$ is the general eqn form.

$$\therefore x_2 = \frac{1}{I.F}\left[\int x_1(t_0)\,e^{t-t_0}\,I.F\,dt + C\right]$$

$$I.f = e^{\int -\eta(t)\,dt}$$

$$= e^{-\int \eta(t)\,dt}$$

$$\therefore x_2 = \frac{1}{e^{-\int \eta(t)\,dt}}\left[\int x_1(t_0)\,e^{t-t_0}\cdot e^{-\int \eta(t)\,dt}\,dt + C\right]$$

$$= e^{\int \eta(t)\,dt}\;x_1(t_0)\int e^{t-t_0}\,e^{-\int \eta(t)\,dt} + C\,e^{\int \eta\,dt}$$

$$\Rightarrow \int e^{t-t_0}\,e^{-\int \eta(t)\,dt}\,dt = \frac{e^{t-t_0}\,e^{-\int \eta(t)\,dt}}{1-\eta(t)}$$

$$x_2(t) = e^{-\int \eta(t)dt} x_1(t_0) \frac{e^{t-t_0} e^{\int \eta(t)dt}}{1-\eta(t)} + c e^{\int \eta(t)dt}$$

$$\Rightarrow x_1(t_0) \frac{e^{t-t_0}}{1-\eta(t)} + c e^{-\int \eta(t)dt}$$

Now for $t = t_0$

$$c = \left[x_2(t_0) - x_1(t_0) \frac{1}{1-\eta(t_0)} \right] e^{\int \eta(t)dt}$$

$$x_2(t) = x_2(t_0) \frac{e^{t-t_0}}{1-\eta(t)} + \left[x_2(t_0) - \frac{x_1(t_0)}{1-\eta(t_0)} \right] e^{\int \eta(t) - \eta(t_0)}$$

$$= x_1(t_0) \left[\frac{e^{t-t_0}}{1-\eta(t)} - \frac{e^{-\int \eta(t) - \eta(t_0)}}{1-\eta(t_0)} \right]$$

$$+ x_2(t_0) e^{\int \eta(t) dt + \int \eta(t_0) dt}$$

Now we can say that $\phi(t, t_0)$

$$\phi(t, t_0) = \begin{bmatrix} e^{t-t_0} & 0 \\ \frac{e^{t-t_0}}{1-\eta(t)} - \frac{e^{\int \eta(t) \eta(t_0) dt}}{1-\eta(t_0)} & e^{-\int \eta(t) dt + \int \eta(t_0) dt} \end{bmatrix}$$

$\therefore \boxed{x(t) = \phi(t, t_0)\, x(t_0)}$

8) compute the matrix exponential e^{At}

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix}$$

Eigen values and Eigen vectors

$|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & 5 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$$

after solving we get 3 eigen values

$\lambda = -3, -2, -1$

Now we also have to find eigen vectors

$(A - \lambda I)X = 0$

Substituting eigen values each ~~and~~ in the above eqn will result in a system of eqn's which upon solving will give us three eigen vectors.

$\lambda_1 = 2$ considering $x_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow X_1 = \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$ considering $x_3 = 1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \lambda_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_3 = -3$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \lambda_3 = \begin{bmatrix} 1/9 \\ -1/3 \\ 1 \end{bmatrix}$$

$$P = [\lambda_1 \; \lambda_2 \; \lambda_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix}$$

The Diagonalized matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1/5 & 4/15 & 1/15 \\ 1 & -1/6 & -1/6 \\ -1/5 & -1/10 & 1/10 \end{bmatrix}$$

w.k.t that exponential series is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\rightarrow e^A = I + A + \frac{A^2}{2!} + \cdots \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$A = PDP^{-1}$$

$$A^2 = PDP^{-1}\, PDP^{-1} = PD^2P^{-1}$$

$$A^N = PD^NP^{-1}$$

$$e^A = \sum_{N=0}^{\infty} \frac{A^N}{N!} = P\left(\sum_{N=0}^{\infty} \frac{D^N}{N!}\right)P^{-1}$$

$$= Pe^DP^{-1}$$

$$D^N = \begin{bmatrix} 2^N & 0 & 0 \\ 0 & -1^N & 0 \\ 0 & 0 & -3^N \end{bmatrix}$$

$$\rightarrow e^D = \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-3} \end{bmatrix}$$

$$\underline{e^A} = Pe^DP^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-3} \end{bmatrix} \begin{bmatrix} 1/5 & 4/15 & 1/15 \\ 1 & -1/6 & -1/6 \\ -1/5 & -1/10 & 1/10 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-3} \end{bmatrix} \begin{bmatrix} 6 & 8 & 2 \\ 30 & -5 & -5 \\ -6 & -3 & -3 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 4 & 1 & 9 \end{bmatrix} \begin{bmatrix} 6e^2 & 8e^2 & 2e^2 \\ 30e^{-1} & -5e^{-1} & -5e^{-1} \\ -6e^{-3} & -3e^{-3} & -3e^{-3} \end{bmatrix}$$

$$= \frac{e^{-3}}{30} \begin{bmatrix} 6e^5+30e^2-6 & 8e^5-5e^2-3 & 2e^5-5e^2-3 \\ 12e^5-30e^2+18 & 16e^5+5e^2+9 & 4e^5+5e^2+9 \\ 24e^5+30e^2-54 & 32e^5-5e^2-27 & 8e^5-5e^2-27 \end{bmatrix}$$

consider this to be B.

$$e^A = \frac{e^{-3}}{30} B$$

$$\boxed{e^A = \frac{1}{30e^3} B}$$