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PROBLEM SET-4

Problem 1:-

Consider the following state eqn:-

$$\dot{x}(t) = \frac{1}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} x(t) + e^{-t/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

To determine whether this system is controllable or not.

Answer:-

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} e^{-t/2} \\ e^{-t/2} \end{bmatrix}}_{B} u(t)$$

$\dot{x}(t) = Ax + Bu \leftarrow$ form.

According to the controllability theorem, the state eqn of the linear time varying system is controllable if and only if the nxn Grammian matrix of controllability is invertible

then again, the Grammian matrix of controllability is invertible if and only if the 'nxm' controllability matrix satisfies

Grammian matrix $\longrightarrow W_c(0,t)$

$$\text{rank}([B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots \,\vdots\, A^{n-1}B]) = n$$

In this case n will be 2.

$$B = \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix} \qquad AB = \begin{bmatrix} \frac{5}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} e^{t/2} \\ e^{t/2} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} e^{t/2} + \frac{1}{12} e^{t/2} \\ \frac{1}{12} e^{-t/2} + \frac{5}{12} e^{t/2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} e^{t/2} \\ \frac{1}{2} e^{t/2} \end{bmatrix}$$

$$[B \;\; AB] = \begin{bmatrix} e^{t/2} & \frac{1}{2} e^{t/2} \\ e^{t/2} & \frac{1}{2} e^{t/2} \end{bmatrix}$$

If you observe Grammian, the 2nd column is an integer multiple of 1st column, therefore the columns are not linearly independent.

$\therefore$ $\text{rank}([B \;\; AB]) < n$ where $n < 2$

Hence, the system is "UNCONTROLLABLE"

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Problem 2 :

$$\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The given eqn of the system is in the form of

$$\dot{x}(t) = A x(t)$$

To find the state transition matrix the eqn is

$$\phi(t_0, t_f) = P\, e^{D(t-t_0)} P^{-1} \quad — \text{①}$$

So Eigenvalues will be $\rightarrow$ $|A - \lambda I|$

$$A = \begin{bmatrix} \gamma_1(t) & \gamma_2(-t) \\ -\gamma_2(t) & \gamma_1(t) \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} \gamma_1(t) - \lambda & \gamma_2(t) \\ -\gamma_2(t) & \gamma_1(t) - \lambda \end{vmatrix}$$

$$(\gamma_1 - \lambda)^2 + \gamma_2^2 = 0$$

$$\gamma_1^2 + \lambda^2 - 2\gamma_1\lambda + \gamma_2^2 = 0 \Rightarrow \lambda^2 - 2\lambda(\gamma_1) + (\gamma_1^2 + \gamma_2^2) = 0$$

finding roots $\dfrac{2b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{2\gamma_1 \pm \sqrt{4\gamma_1^2 - 4(\gamma_1^2 + \gamma_2^2)}}{2}$$

$$= \frac{\cancel{2}\left(\gamma_1 \pm \sqrt{\gamma_1^2 - (\gamma_1^2 + \gamma_2^2)}\right)}{\cancel{2}}$$

$$= \gamma_1 \pm \sqrt{-\gamma_2^2}$$

$$= \gamma_1 \pm \gamma_2 i$$

$$\boxed{\lambda_1 = \gamma_1 + \gamma_2 i} \qquad \boxed{\lambda_2 = \gamma_1 - \gamma_2 i}$$

Now to find Eigen vectors

for $\lambda_1 = \gamma_1 + i\gamma_2$

$$\Rightarrow \begin{bmatrix} -i\gamma_2 & \gamma_2 \\ -\gamma_2 & -i\gamma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 i + x_2 = 0$$

$$x_1 - x_2 i = 0$$

$$\therefore \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

for $\lambda_2 = \gamma_1 - i\gamma_2$

$$\Rightarrow \begin{bmatrix} i\gamma_2 & \gamma_2 \\ -\gamma_2 & i\gamma_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-ix_1 + x_2 = 0$$
$$-x_1 + ix_2 = 0$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\therefore$ from eqn ①

$$P = [X_1 \; X_2] = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } P}{|P|} = \frac{1}{-2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$\therefore$ The state transition matrix is

$$\phi(t_0, t_f) = P e^{D(t-t_0)} P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \cdot \begin{bmatrix} e^{\int_{t}^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt} & 0 \\ 0 & e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

where

$$D_1 = e^{\int_{t_0}^{t_f} \gamma_1(t) + i\gamma_2(t)\,dt}$$

$$D_2 = e^{\int_{t_0}^{t_f} \gamma_1(t) - i\gamma_2(t)\,dt}$$

So

$$\Rightarrow \frac{1}{2}\begin{bmatrix} D_1 + D_2 & i(D_2 - D_1) \\ i(D_1 - D_2) & D_1 + D_2 \end{bmatrix} = \phi(t_0, t_t).$$

The state transition matrix is $\phi(t_0, t_t)$ so therefore the solution of the system is given by

$$x(t) = \phi(t_0, t_t)\, x t_0.$$

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Problem 3 :

$J_1 \rightarrow$ assumed to represent the space shuttle robot arm inertia

$J_2 \rightarrow$ inertia of the shuttle itself

The Equations of motion are :-

$$J_1 \ddot{q}_1 = \tau$$

$$J_2 \ddot{q}_2 = \tau$$

$$J_1 \ddot{q}_1 = \tau = J_2 \ddot{q}_2$$

Now let the state variables be :-

$$x_1 = q_1 \quad -(1)$$
$$x_2 = \dot{x}_1 = \dot{q}_1 \quad -(2)$$
$$x_3 = q_2 \quad -(3)$$
$$x_4 = \dot{x}_3 = \dot{q}_2 \quad -(4)$$

$$x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$$

State Space Eqn's

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = Ax + Bu$$

where :-

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} \qquad u = [\tau]$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} [\tau]$$

Now to check for controllability :-

(Theorem used in problem 1)

$$\text{rank}\left([B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B] \right) = n$$

for this case $n=4$

$$B = \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} = \begin{bmatrix} 1/J_1 \\ 0 \\ 1/J_2 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/J_1 \\ 0 \\ 1/J_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A^3B \Rightarrow$ when A^2B is $0_{4\times1}$, A^3B is bound to be $0_{4\times1}$

$$A^3B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{evidently.}$$

Now $C([B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B]) = \begin{bmatrix} 0 & 1/J_1 & 0 & 0 \\ 1/J_1 & 0 & 0 & 0 \\ 0 & 1/J_2 & 0 & 0 \\ 1/J_2 & 0 & 0 & 0 \end{bmatrix}$

We observe that matrix C has 2 columns which are zeros, so the rank(C) < n=4

$\therefore$ The given system is UNCONTROLLABLE.

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* In order to make a system controllable, a possible way would be to perform an invertible state transform of the form $y = Tx$ such that the new 'A' & 'B' matrices of the system make the controllability matrix have full rank.

Problem 4: Given the linear second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u$$

find a linear state feedback control $u = K_1 x_1 + K_2 x_2$ so that the closed loop system has poles at $s = -2, 2$.

Answer:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{B} u$$

linear state feedback $u = K_1 x_1 + K_2 x_2$

Such that closed loop poles are at $s = -2, 2$

Checking for controllability:

$$B = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

controllability matric $C = \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$

There is a way to check for linear independence of a matrix.

If the det of a matrix is not zero then the matrix has full rank and is therefore controllable.

$\det(C) = 5 - (-14) = 19, \quad \because \det(C) \neq 0$

$\Rightarrow$ Matrix C has full rank of 2.

$\therefore$ The system is controllable.

With the state feedback $u = kx$,
The closed-loop system matrix A_c is

$$A_c = A + BK$$

$$= \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 1+k_1 & -3+k_2 \\ 1-2k_1 & -2-2k_2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(\lambda I - A_c) = 0$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1+k_1 & -3+k_2 \\ 1-2k_1 & -2-2k_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda-1-k_1 & -(-3+k_2) \\ -(1-2k_1) & \lambda+2+2k_2 \end{bmatrix} = 0$$

$$\det(\lambda I - A_c) = (\lambda-1-k_1)(\lambda+2+2k_2) - (-3+k_2)(1-2k_1) -$$

$$\lambda^2 + 2\lambda + 2k_2\lambda - \lambda - 2 - 2k_2 - k_1\lambda - 2k_1 - 2k_1k_2 - k_2 + 3 + 2k_1k_2 \ldots$$
$$\ldots - 6k_1 = 0$$

$$\lambda^2 + 2\lambda + k_2\lambda - \lambda - 2 - 2k_2 - k_1\lambda - 2k_1 - k_2 + 3 - 6k_1 = 0$$

$$\lambda^2 + (1 - k_1 + 2k_2)\lambda + (1 - 8k_1 - 3k_2) = 0 \quad \rightarrow (1).$$

we know that poles are -2 & 2.

substitute pole values in λ of (1).

we get $-(1-k_1+2k_2)=0$

$1-8k_1-3k_2=-4$

Solving for k_1 and k_2 we get

$k_1 = \frac{13}{9}$, $k_2 = \frac{-3}{19}$

Now the state feedback control is

$u = k_1 x_1 + k_2 x_2$

$$\boxed{u = \frac{13}{19}x_1 - \frac{3}{19}x_2}$$

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Problem 5 :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Repeating the process for problem 4 on above system.

Answer :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

Check for contrallability

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad AB = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

here, $\det(C) = 0 \Rightarrow$ UNCONTRALLABLE

Closed loop system matrix:

$$A_c = A + BK = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \quad K_2]$$

$$A_c = \begin{bmatrix} -1 & 0 \\ K_1 & K_2+2 \end{bmatrix}$$

Characteristic Equation :- $\det(\lambda I - A_c) = 0$.

$$\Rightarrow \begin{vmatrix} \lambda+1 & 0 \\ -K_1 & \lambda-2-K_2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)(\lambda-2-K_2) = 0$$

$\Rightarrow$ The closed loop poles are -1 & K_2+2

* can the closed loop poles be placed at -2?

consider the value of $K_2 = -4$ so one pole will be placed at -2.

closed loop pole values are $-1, -2$.

$\therefore$ for any value of K_1 and K_2

$\hookrightarrow$ it is impossible to place closed loop poles at $(-2, -2)$

Since, one of the poles, is always on the Left hand plane (LHP) and the other pole can be placed on the LHP by appropriately choosing the value of K_2 $(K_2 < -2)$

$\therefore$ The System is STABILIZABLE.

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Problem 6:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Repeat the process of problem 5

Answer:

Check for controllability:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$C = [B \;\; AB] = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\det C = \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} \neq 0$$

$\therefore$ Not controllable

closed loop system matrix $A_c = A + BK$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \;\; K_2]$$

$$A_c = \begin{bmatrix} 1 & 0 \\ K_1 & K_2+2 \end{bmatrix}$$

characteristic eqn $\det(\lambda I - A_c) = 0$

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Rightarrow \det \begin{bmatrix} \lambda-1 & 0 \\ -K_1 & \lambda-K_2-2 \end{bmatrix} = 0$$

$$\begin{vmatrix} \lambda-1 & 0 \\ -K_1 & \lambda-K_2-2 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-K_2-2) = 0$$

$\Rightarrow$ the closed loop poles are :- 1 and K_2+2

* can the closed loop poles be placed at -2

consider the value of $K_2 = -4$, so one pole will be placed at -2.

closed loop pole values are 1 and -2.

for any value of k_1 and k_2, it is impossible to place the closed loop poles at $(-2,-2)$.

Since, one pole is always on RHP, the system is UNSTABILIZABLE.

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Problem 7 :

Given: $y(t)$ is the output of an LTI system for input $u(t)$. initial state, $x(t) = 0$

To show: Output of the system for $\dot{u}(t)$ is $\dot{y}(t)$

Answer : Let the LTI system be written as follows for an input $u(t)$

$$\dot{X}(t) = AX(t) + Bu(t) \quad - (1)$$
$$y(t) = CX(t) + Du(t) \quad - (2)$$

$X(t) \rightarrow$ State of the system
$y(t) \rightarrow$ output of the system
$u(t) \rightarrow$ input of the system

Taking Laplace Transform of (1) & (2)

$$\Rightarrow \quad sX(s) = AX(s) + BU(s) \quad - (3)$$
$$Y(s) = CX(s) + DU(s) \quad - (4)$$

Eqn (3) Equating $X(s)$ terms

$$(sI - A)X(s) = BU(s)$$
$$\rightarrow X(s) = (sI - A)^{-1} BU(s)$$

Substituting (5) in (4)

$$\Rightarrow Y(s) = [C(sI - A)^{-1} + D]U(s) \quad - (6)$$

Now, let the input be $\hat{u}(t) = \dot{u}(t)$ and output be $\hat{y}(t)$

$$\dot{x}(t) = Ax(t) + B\dot{u}(t) \quad — (7)$$
$$\hat{y}(t) = Cx(t) + D\dot{u}(t) \quad — (8)$$

Taking Laplace Transform of (7) assuming zero initial conditions.

$$sX(s) = AX(s) + BsU(s) \qquad [\because \mathcal{L}\{\dot{u}(t)\} = sU(s)]$$

$$\Rightarrow X(s) = (sI - A)^{-1} B\, sU(s) \quad — (9)$$

Taking Laplace transform of (8) and substituting (9) for $X(s)$

$$\hat{Y}(s) = CX(s) + DsU(s)$$
$$= C(sI-A)^{-1}BsU(s) + DsU(s)$$
$$= [C(sI-A)^{-1}B + D]\, sU(s)$$
$$= s[C(sI-A)^{-1}B + D]\, U(s)$$

$$\hat{Y}(s) = sY(s) \quad — (10) \qquad [\text{from } (6)]$$

Taking Inverse Laplace Transform of (10) $\Rightarrow \hat{y}(t) = \dot{y}(t)$.

For an input $\dot{u}(t)$, the output is $\dot{y}(t)$ for an LTI system with zero initial conditions.

Problem 8 :-

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} x(t) \qquad x(t_0) = x_0$$

find the solution of the linear state eqn:-

Answer:-

Obtaining $x(t)$ for the given linear, unforced system

$$\dot{X} = AX$$

Taking Laplace transform on both sides.

$$sX(s) - x(t_0) = AX(s)$$

$$sX(s) - A(X(s)) = x(t_0)$$

$$\Rightarrow (sI - A)X(s) = x(t_0) \qquad (x(t_0) = x_0)$$

$$X(s) = (sI - A)^{-1} x_0$$

Taking Inverse Laplace transform

$$x(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\} x_0 \quad \text{--- (1)}$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{bmatrix}$$

$$\det(sI - A) = (s+1)(s^2+4s-4) = (s+1)(s+2)^2$$

$$[(sI-A)]^{-1} = \frac{Adj\,[sI-A]}{|sI-A|}$$

$$Adj\,[sI-A] = \begin{bmatrix} (s+2)^2 & 0 & 0 \\ 0 & s(s+1) & 4(s+1) \\ 0 & -1(s+1) & (s+4)(s+1) \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{(s+1)(s+2)^2} \begin{bmatrix} (s+2)^2 & 0 & 0 \\ 0 & s(s+1) & 4(s+1) \\ 0 & -1(s+1) & (s+4)(s+1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix}$$

Now taking inverse laplace.

$$\mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] & 0 & 0 \\ 0 & \mathcal{L}^{-1}\left[\frac{s}{(s+2)^2}\right] & \mathcal{L}^{-1}\left[\frac{4}{(s+2)^2}\right] \\ 0 & \mathcal{L}^{-1}\left[\frac{-1}{(s+2)^2}\right] & \mathcal{L}^{-1}\left[\frac{s+4}{(s+2)^2}\right] \end{bmatrix}$$

$$\Rightarrow \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & -4te^{-2t} \\ 0 & -e^{-2t}t & (1+2t)e^{-2t} \end{bmatrix}$$

$\therefore$ from eqn (1)

$$x(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}(1-2t) & -4te^{-2t} \\ 0 & -te^{-2t} & (1+2t)e^{-2t} \end{bmatrix} x_0$$

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