Problem Set # 5

Due Date: Thursday Dec 16 in ELMS

Instructions:

- Please write clearly and show all your work. Credit will be given for all correct steps. Points will be deducted for any missing steps.
- Please write your name clearly on the assignment.
- Problems 1 and 2 are from a previous final exam.
- No extensions will be provided for this problem set.

Problem 1: Let $\mathbf{A} \in \Re^{n \times n}$ and $\mathbf{B} \in \Re^{m \times m}$. Show that $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0)e^{\mathbf{B}t}$ is the solution to the following equation

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{X}(t)\mathbf{B}$$

Problem 2: A (Euclidean) ball, $\mathbf{B}(x_c, r)$, in $\Re^n$ is given by

$$\mathbf{B}(x_c, r) = \left\{ x \in \Re^n \;\middle|\; \|x - x_c\| \leq r \right\}$$

where $r > 0$ and $\|.\|$ denotes the Euclidean norm, $\|x\| = \sqrt{x^T x}$. The vector $x_c \in \Re^n$ is the center of the ball and r is the radius of the ball. **Prove that $\mathbf{B}(x_c, r)$ is a Convex set.**

Gaussian Random Processes:

A random process $X(t)$ is a Gaussian random process if the samples $X_1 = X(t_1), X_2 = X(t_2), \ldots, X_k = X(t_k)$ are jointly Gaussian random variables for all k, and all choices of $t_1, t_2, \ldots, t_k$. This definition holds for both discrete-time and continuous-time processes. The joint probability density function (pdf) of jointly Gaussian random variables is determined by the vector of means $\mathbf{m}$ and by the covariance matrix Σ:

$$f_{X_1,X_2,\ldots,X_k}(x_1, x_2, \ldots, x_k) \;=\; \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} \exp\left(\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\Sigma^{-1}(\mathbf{x}-\mathbf{m})\right)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}; \quad \mathbf{m} = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_k) \end{bmatrix}; \quad \Sigma = \begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & \cdots & C_X(t_1,t_k) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & \cdots & C_X(t_2,t_k) \\ \vdots & \vdots & \ddots & \vdots \\ C_X(t_k,t_1) & C_X(t_k,t_2) & \cdots & C_X(t_k,t_k) \end{bmatrix}$$

and

$$C_X(t_i, t_j) \;=\; E\left[X(t_i)X(t_j)\right] \;-\; m_X(t_i)m_X(t_j).$$

Gaussian random processes have the special property that their joint pdf's are completely specified by the mean of the process $m_X(t)$ and by the covariance function $C_X(t_1, t_2)$. Gaussian random processes also have the property that the linear operation on a Gaussian process results in another Gaussian random process.

Problem 3: Let $X(t)$ be a zero-mean Gaussian random process with covariance function given by

$$C_X(t_1, t_2) \;=\; \sigma^2 e^{-|t_1 - t_2|}$$

Find the joint pdf of $X(t)$ and $X(t+s)$.

Problem 4: Consider the following state-space equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Consider the following cost function:

$$J = \int_0^\infty \left(\mathbf{x}^{\mathrm{T}} \mathbf{Q} \mathbf{x} + u^2 \right) dt$$

where $\mathbf{Q}$ is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}, \quad \gamma > 0$$

Design an LQR controller for this system and provide the state feedback $u = \mathbf{K}\mathbf{x}$. You need to do all these calculations by hand and show all your steps. No credit will be given for simulation based results. Points will be deducted for any missing steps.

Problem 5: Consider the following state-space representation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t)$$

- Write this system in the standard form of Uncontrollable Systems using the Similarity Transformation discussed in class.
- Write the controllable part of this system as a state equation.

Show all your work. No credit will be given for simulation based results. Points will be deducted for any missing steps.

Problem 6: Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \mathbf{x}$$

Investigate the stability of this system using the following Lyapunov equation:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show all your work. No credit will be given for simulation based results.