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Section : 0201

PROBLEM SET - 2

Problem 1 : Evaluate the Determinants

a) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{bmatrix} \rightarrow \det \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$

→ To calculate the determinant of the matrix, we consider the first row

→ Then we solve the determinants for the respective 3x3 matrices that are obtained subsequently.

$$\det = 1\begin{vmatrix} 1 & -2 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & -2 & 1 \\ 3 & 4 & -2 \\ -2 & -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -2 \\ -2 & 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 & -2 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{vmatrix}$$

↓ (1) ↓ (2) ↓ (3) ↓ (4)

(1) $\begin{vmatrix} 1 & -2 & 1 \\ -3 & 4 & -2 \\ 1 & -2 & 1 \end{vmatrix} = 1(4-4)+2(-3+2)+1(6-4) = 0-2+2 = 0$

(2) $\begin{vmatrix} 0 & -2 & 1 \\ 3 & 4 & -2 \\ -2 & -2 & 1 \end{vmatrix} = 0+2(3-4)+1(-6+6) = -2$

(3) $\begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -2 \\ -2 & 1 & 1 \end{vmatrix} = 0-1(3-4)+1(3-6) = -4$

(a) $\begin{vmatrix} 0 & 1 & -2 \\ 3 & -3 & 4 \\ -2 & 1 & -2 \end{vmatrix} = 0 - 1(-6+8) - 2(3-6) = -2+6 = 4$

$\det = 1\times 0 - 0\times -2 + 2\times 4 - 3\times 4 = \underline{\underline{-16}}$

b) $\det \begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$

$\det = gc \underbrace{\begin{vmatrix} b & b & b \\ e & e & b+e \\ b & b+f & b+d \end{vmatrix}}_{1} - ge \begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b+f & b+d \end{vmatrix} \to 2 + (a+ge) \underbrace{\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b & b+d \end{vmatrix}}_{3}$

$- (gb+ge) \begin{vmatrix} 0 & b & b \\ c & e & e \\ a & b & b+f \end{vmatrix} \to 4$

1 $\begin{vmatrix} b & b & b \\ e & e & b+e \\ b & b+f & b+d \end{vmatrix} = b(e(b+d) - (b+e)(b+f)) - b(e(b+d) - b(b+e))$
$+ b(e(b+f) - eb)$

$= b(\cancel{eb} + ed - b^2 - \cancel{eb} - bf - ef) - b(\cancel{eb} + ed - b^2 - \cancel{eb})$
$+ b(\cancel{eb} + ef - \cancel{eb})$

$= \cancel{bed} - \cancel{b^3} - b^2 f - \cancel{bef} - \cancel{bed} + \cancel{b^3} + \cancel{bef}$

$= \underline{\underline{-b^2 f}}$

② $\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b+f & b+d \end{vmatrix} = 0 - b(c(b+d) - a(b+e)) + b(c(b+f) - ea)$

$= -b(bc + cd - ab - ea) + b(bc + cf - ea)$

$= -\cancel{b^2c} - bcd + bab + \cancel{bea} + \cancel{b^2c} + bcf - \cancel{bea}$

$= b^2a - bcd + bcf$

③ $\begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & b & b+d \end{vmatrix} = 0 - b(c(b+d) - a(b+e)) + b(cb - ea)$

$= -b(cb + cd - ab - ae) + b(cb - ea)$

$= -\cancel{b^2c} - bcd + b^2a + \cancel{bae} + \cancel{b^2c} - \cancel{bae}$

$= b^2a - bcd$

④ $\begin{vmatrix} 0 & b & b \\ c & e & e \\ a & b & b+f \end{vmatrix} = 0 - b(c(b+f) - ea) + b(cb - ea)$

$= -b(cb + cf - ea) + b(cb - ea)$

$= -\cancel{b^2c} - bcf + \cancel{bea} + \cancel{b^2c} - \cancel{bea}$

$= -bcf$

$\det = gc(-b^2f) - ge(b^2a - bcd + bcf) + (a+ge)(b^2a - bcd)$
$\qquad - (gb + ge)(-bcf)$

$= -\cancel{b^2fgc} - \cancel{geb^2a} + \cancel{gebcd} - \cancel{gebcf} + a^2b^2 - abcd$
$\quad + \cancel{ageb^2} - \cancel{gebcd} + \cancel{gb^2cf} + \cancel{gebcf}$

$= a^2b^2 - abcd$

$\det = \underline{ab(ab - cd)}$

Problem 2 :

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

→ To solve the matrix with minimum calculations, convert the matrix into the Echelon form

Echelon Form:- The leading element in each row of the principal diagonal should be preceded by more zeros than the previous row.

Eg:- $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$ → principal diagonal

Now use Gaussian Elimination to achieve the required upper triangular matrix. Also use properties of determinants.

i) $R_4 \Rightarrow R_3 - R_4$

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ 0 & 0 & c-x & 0 \end{vmatrix} = 0$$

ii) $R_3 \Rightarrow R_3 - R_2$

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ 0 & b-x & x-b & 0 \\ 0 & 0 & c-x & 0 \end{vmatrix} = 0$$

iii) $R_2 \Rightarrow R_2 - \frac{a}{x} R_1$

$$\begin{vmatrix} x & a & a & 1 \\ 0 & x-\frac{a^2}{x} & b-\frac{a^2}{b} & 1-a/x \\ 0 & b-x & x-b & 0 \\ 0 & 0 & c-x & 0 \end{vmatrix}$$

Now if we look at the matrix upon swapping the columns we get the required matrix.

iv) Swapping $C_3 \leftrightarrow C_4$

$$(-1)\begin{vmatrix} x & a & 1 & a \\ 0 & x-\frac{a^2}{x} & 1-a/x & b-a^2/b \\ 0 & b-x & 0 & x-b \\ 0 & 0 & 0 & c-x \end{vmatrix} = 0$$

v) Swapping $C_2 \leftrightarrow C_3$

$$(-1)(-1)\begin{vmatrix} x & 1 & a & a \\ 0 & 1-a/x & x-a^2/x & b-a^2/b \\ 0 & 0 & b-x & x-b \\ 0 & 0 & 0 & c-x \end{vmatrix} = 0$$

~~Obs~~ Observe that the matrix is in Echelon form.

Property of Matrix :- The determinant of a matrix in the Echelon form is the product of all its diagonal elements.

$$\det = (x)\left(1-\frac{a}{x}\right)(b-x)(c-x)(-1)(-1) = 0$$

$$\Rightarrow \boxed{(x-a)(x-b)(x-c) = 0.}$$

✓

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not nec. calculat

Problem 3 :

$$A = \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Find the Eigen values and Eigen vectors of A.

To find Eigen values w.k.t $|A - \lambda I| = 0$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} 1-\lambda & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\det = (1-\lambda)\left((1-\lambda)(1-\lambda) - 0\right) - \gamma_1\left(\gamma_2(1-\lambda) - 0\right) = 0$$

$$= (1-\lambda)\left((1-\lambda)^2 - \gamma_1\gamma_2\right) = 0$$

$$\Rightarrow (1-\lambda)\left(1^2 + \lambda^2 - 2\lambda + \gamma_1\gamma_2\right) = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2 - 2\lambda + (1-\gamma_1\gamma_2)\right) = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 0$$

$$\frac{+2 \pm \sqrt{4 - 4(1)(1-\gamma_1\gamma_2)}}{2} = 0 \Rightarrow \frac{\cancel{2}(+1 \pm \sqrt{\gamma_1\gamma_2})}{\cancel{2}}$$

$$\Rightarrow 1 \pm \sqrt{\gamma_1\gamma_2} = 0$$

$$(\lambda-1)(\lambda-1-\sqrt{\gamma_1\gamma_2})(\lambda-1+\sqrt{\gamma_1\gamma_2}) = 0$$

The Eigen values that we get are.

$\lambda_1 = 1$

$\lambda_2 = 1+\sqrt{\gamma_1\gamma_2}$

$\lambda_3 = 1-\sqrt{\gamma_1\gamma_2}$

To find Eigen vectors $Ax^i = \lambda_i x^i$

$$\begin{bmatrix} 1 & \gamma_1 & 0 \\ \gamma_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 + \gamma_1 x_2 = \lambda_1 x_1$$
$$\gamma_2 x_1 + x_2 = \lambda_1 x_2 \qquad \boxed{\lambda_1 = 1}$$
$$x_3 = \lambda_1 x_3$$

$$\begin{bmatrix} 1-\lambda_1 & \gamma_1 & 0 \\ \gamma_2 & 1-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & \gamma_1 & 0 \\ \gamma_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\gamma_1 x_2 = 0$$
$$\gamma_2 x_1 = 0$$

$x_2 = 0 \quad x_1 = 0 \quad x_3 = x_3$

$$\therefore x^1 = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \Rightarrow x^1 = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

consider x_3 to be of unit's value

$$x^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = (+)\sqrt{\gamma_1\gamma_2}$

$$x^2 = \begin{bmatrix} -\sqrt{\gamma_1\gamma_2} & \gamma_1 & 0 \\ \gamma_2 & -\sqrt{\gamma_1\gamma_2} & 0 \\ 0 & 0 & -\sqrt{\gamma_1\gamma_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-\sqrt{\gamma_1\gamma_2}\, x_1 + \gamma_1 x_2 = 0 \Rightarrow x_1 = \sqrt{\frac{\gamma_1}{\gamma_2}}\, x_2$$

$$-\sqrt{\gamma_1\gamma_2}\, x_2 + x_1\gamma_2 = 0$$

$$\Rightarrow x^2 = x_2 \begin{bmatrix} \sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$$

consider x_2 to be unit's value

$$x^2 = \begin{bmatrix} \sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$$

$-\sqrt{\gamma_1\gamma_2}\, x_3 = 0$

$\lambda_3 = 1 - \sqrt{\gamma_1\gamma_2} \Rightarrow \begin{bmatrix} \sqrt{\gamma_1\gamma_2} & \gamma_1 & 0 \\ \gamma_2 & \sqrt{\gamma_1\gamma_2} & 0 \\ 0 & 0 & \sqrt{\gamma_1\gamma_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\sqrt{\gamma_1\gamma_2}\, x_1 + \gamma_1 x_2 = 0 \Rightarrow x_1 = -\sqrt{\frac{\gamma_1}{\gamma_2}}\, x_2$

$\gamma_2 x_1 + \sqrt{\gamma_1\gamma_2}\, x_2 = 0.$

$\sqrt{\gamma_1\gamma_2}\, x_3 = 0 \rightarrow x_3 = 0$

$x^3 = x_2 \begin{bmatrix} -\sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$ consider x_2 to be unit's ~~length~~ value.

$x^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $x^2 = \begin{bmatrix} \sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$ $x^3 = \begin{bmatrix} -\sqrt{\frac{\gamma_1}{\gamma_2}} \\ 1 \\ 0 \end{bmatrix}$

Conditions:-

→ Given that γ_1, γ_2 are non-zero complex numbers. Hence for the Eigen values to be real, $\gamma_1\gamma_2$ should be real and non-negative

→ This is possible when γ_1 and γ_2 are complex conjugates

Given that the matrix satisfies $A^TA = AA^T$, then Eigen vectors are orthogonal.

$A^TA = \begin{bmatrix} 1+\gamma_1^2 & \gamma_1+\gamma_2 & 0 \\ \gamma_1+\gamma_2 & \gamma_2^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+\gamma_2^2 & \gamma_1+\gamma_2 & 0 \\ \gamma_1+\gamma_2 & \gamma_1^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^T$

comparing $1+\gamma_1^2 = 1+\gamma_2^2 \rightarrow |\gamma_1| = |\gamma_2|$

Problem 4: Make an LU Decomposition matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix}$$

Hence solve for $Ax = b$

i) $b = \begin{bmatrix} -4 & 1 & 8 & -5 \end{bmatrix}^T = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$

LU Decomposition :- making the matrix decompose into two different matrices, upper triangular matrix and a lower triangular matrix.

w.k.t $\quad A = LU \qquad\qquad Ax = b$

considering $x = [x_1 \; x_2 \; x_3 \; x_4]^T$

$(LU)x = b$

$L(Ux) = b$ $\quad$ considering $Ux = y$

$\boxed{Ly = b}$ $\qquad$ $\boxed{Ux = y}$

$$A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \square & 1 & 0 & 0 \\ \square & \square & 1 & 0 \\ \square & \square & \square & 1 \end{bmatrix}}_{L} \begin{bmatrix} \\ & U & \\ \\ \end{bmatrix}$$

$A = LU$

$\Rightarrow$ Transformation using Gaussian Elimination

① $\rightarrow R_2 \Rightarrow R_2 - R_1/2$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L$$

② $\rightarrow R_3 \Rightarrow R_3 - \frac{5R_1}{2}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 21/2 & -7/2 & -17/2 \\ 3 & -6 & -3 & 1 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L$$

③ $\rightarrow R_4 \Rightarrow R_4 - \frac{3R_1}{2}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 21/2 & -7/2 & -17/2 \\ 0 & -3/2 & -9/2 & -7/2 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 0 & 1 & 0 \\ 3/2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L$$

④ $\rightarrow R_3 \Rightarrow R_3 - \frac{21R_2}{11}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & -3/2 & -9/2 & -7/2 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L$$

(5) $R_4 \Rightarrow R_4 + \frac{3R_2}{11}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & -60/11 & -52/11 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & 0 & 1 \end{bmatrix} \Rightarrow L$$

(6) $R_4 \Rightarrow R_4 + \frac{12R_3}{7}$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \Rightarrow U \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix}}_{U}$$

The order of L and U is very important.

$\Rightarrow \quad Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}$$

$y_1 = 4 \quad , \quad \frac{1}{2}y_1 + y_2 = 1$ - (1)

$\frac{5}{2}y_1 + \frac{21}{11}y_2 + y_3 = 8$ - (3) $\qquad \frac{3}{2}y_1 - \frac{3}{11}y_2 - \frac{12}{7}y_3 + y_4 = -5$ (4)

Upon solving the eqns we get values of y_1, y_2 and y_3, y_4

$y_1 = -4$, $y_2 = 3$, $y_3 = \frac{135}{11}$ $y_4 = \frac{160}{7}$

$Ux = y$.

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 135/11 \\ 160/7 \end{bmatrix}$$

$-\frac{32}{7} x_4 = 160/7$ — ①

$35/11\, x_3 + 1/11\, x_4 = 135/11$ — ②

$11/2\, x_2 - 7/2\, x_3 - 9/2\, x_4 = 3$ — ③

$2x_1 - 3x_2 + x_3 + 3x_4 = -4$ — ④

$x_4 = -5$

x_3 upon solving we get $x_3 \Rightarrow 4$

upon solving $x_2 = -1$

and upon solving $x_1 = 2$.

** We solve the eqns by basic substitution method

$x = [2 \ -1 \ 4 \ -5]^T$ for $b = [-4 \ 1 \ 8 \ -5]^T$

ii) $b = [-10 \quad 0 \quad -3 \quad -24]^T$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix}}_{U}$$

$Ax = b \qquad (LU)x = b$

$L(Ux) = b \qquad Ux = y.$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 5/2 & 21/11 & 1 & 0 \\ 3/2 & -3/11 & -12/7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -3 \\ -24 \end{bmatrix}$$

$\boxed{y_1 = -10}$

$\frac{1}{2}y_1 + y_2 = 0 \Rightarrow \boxed{y_2 = 5}$

$\frac{5}{2}y_1 + \frac{21}{11}y_2 + y_3 = -3 \Rightarrow \boxed{y_3 = 137/11}$

$\frac{3}{2}y_1 - \frac{3}{11}y_2 - \frac{12}{7}y_3 + y_4 = -24 \Rightarrow \boxed{y_4 = 96/7}$

$Ux = y$

$$\begin{bmatrix} 2 & -3 & 1 & 3 \\ 0 & 11/2 & -7/2 & -9/2 \\ 0 & 0 & 35/11 & 1/11 \\ 0 & 0 & 0 & -32/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ 137/11 \\ 96/7 \end{bmatrix}$$

$-\frac{32}{7}x_4 = \frac{96}{7} \Rightarrow x_4 = -3$

$\frac{35}{11}x_3 + \frac{1}{11}x_4 = \frac{137}{11} \Rightarrow x_3 = 4$

$\frac{11}{2}x_2 - \frac{7}{2}x_3 - \frac{9}{2}x_4 = 5 \Rightarrow x_2 = 1$

$2x_1 - 3x_2 + x_3 + x_4 \Rightarrow x_1 = -4$

$x = [-4 \quad 1 \quad 4 \quad -3]^T$ for $b = [-10 \quad 0 \quad -3 \quad -24]^T$

Finding the det of the matrix.

$$\begin{vmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{vmatrix}$$

prove det $= -160$

$$\det = 2\underbrace{\begin{vmatrix} 4 & -3 & -3 \\ 3 & -1 & -1 \\ -6 & -3 & 1 \end{vmatrix}}_{1} + 3\underbrace{\begin{vmatrix} 1 & -3 & -3 \\ 5 & -1 & -1 \\ 3 & -3 & 1 \end{vmatrix}}_{2} + 1\underbrace{\begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & 1 \end{vmatrix}}_{3} - 3\underbrace{\begin{vmatrix} 1 & 4 & -3 \\ 5 & 3 & -1 \\ 3 & -6 & -3 \end{vmatrix}}_{4}$$

1 $4[(-1\times1) - (-1\times-3)] - (-3)[(3\times1) - (-6\times-1)] + (-3)[(3\times-3) + (-6\times-1)]$

$= 20$

2 $1[(-1\times1) - (-1\times-3)] - (-3)[(5\times1) - (-1\times3)] + (-3)[(5\times-3) - (-1\times3)]$

$= 56$

3 $1[(3\times1) - (-6\times-1)] - 4[(5\times1) - (-1\times3)] + (-3)[(5\times-6) - (-1\times3)]$

$= 82$

4 $1[(3\times-3) - (-6\times-1)] - 4[(5\times-3) - (-1\times3)] + (-3)[(5\times-6) - (-1\times3)]$

$= 150$

$\det = 2(20) + 3(56) + 1(82) - 3(150) = \underline{-160}$

Hence proved.

Problem 5: Consider the following 3x3 matrix

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

i) is this matrix diagonalizable

first we need to find the det $(A - \lambda I)$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix}$$

$$\det = \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0$$

Now we will perform some column and row matrices transformations to simplify the determinant.

$$R \to R_1 + R_2 + R_3 \quad \begin{vmatrix} -(1+\lambda)+4 & -(1+\lambda)+4 & -(1+\lambda)+4 \\ 2 & -(1+\lambda) & 2 \\ 2 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$[-(1+\lambda)+4] \begin{vmatrix} 1 & 1 & 1 \\ 2 & -(1+\lambda) & 2 \\ 2 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$(-\lambda+3) \begin{vmatrix} 1 & 1 & 1 \\ 2 & -(1+\lambda) & 2 \\ 2 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$C_1 \to C_1 - C_3$ $\begin{vmatrix} 0 & 1 & 1 \\ 0 & -(1+\lambda) & 2 \\ 2+(1+\lambda) & 2 & -(1+\lambda) \end{vmatrix}$ $(-\lambda+3)$

Now finding the det along column 1:

$$\det \Rightarrow (-\lambda+3)\left[(\lambda+3)(-2 \mp (1+\lambda))\right] = 0$$

$$= (3-\lambda)(\lambda+3)(3+\lambda) = 0$$

$\lambda = -3, -3, 3$ ← Eigen values.

w.r.t symmetric matrices might have distinct Eigen vectors even when the eigen values are repeated, (by the property of orthogonality).

To find Eigen vectors:-

① $Ax' = 3x'$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

System of Eqns:-

$$-x_1 + 2x_2 + 2x_3 = 3x_1$$
$$2x_1 - x_2 + 2x_3 = 3x_2$$
$$2x_1 + 2x_2 - x_3 = 3x_3$$

when solving, it is only possible that this set of eqns can be solved when $\boxed{x_1 = x_2 = x_3 = K}$

so we get $x^1 = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(2) $Ax^2 = -3x^2$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_1 \\ -3x_2 \\ -3x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0. \Rightarrow$

Now we can have more than one solutions.

consider

$x_1 = 0 \Rightarrow x_2 = -x_3$

then we get $x^2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$x_2 = 0 \Rightarrow x_1 = -3$

then we get $x^3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now we have 3 distinct Eigen vectors

To check if the matrix is diagonalizable and do the diagonalization
→ Yes the matrix is diagonalizable because it has 3 distinct Eigenvectors.

$$S = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$ ← Eigen vector matrix.

$$S^{-1} = \frac{1}{|S|} S^{T}$$ $|S| = 1(-1) - 1(1+1) = -1 - 2 = -3$

$$S^{-1} = \frac{1}{-3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Now the Diagonalization D can be found by

$\boxed{A = SDS^{-1}}$

$$D = S^{-1} A S$$

$$= \frac{1}{-3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 6:- Let $A \in R^{n\times n}$, $U \in R^{n\times k}$ and $V \in R^{n\times k}$ be given matrices. Suppose that A, $A + UV^T$ and $I + V^T A^{-1} U$ are non-singular matrices.

Prove that :-

$$(A+UV^T)^{-1} = A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$

Answers:-

$$(A+UV^T)X = Y \Rightarrow \frac{X}{Y} = \frac{1}{(A+UV^T)} \quad X$$

$$\Rightarrow XY^{-1} = (A+UV^T)^{-1} \quad -2$$

$$AX + UV^TX = Y \rightarrow AX = Y - UV^TX$$

matrix multiplication does not commute

$$X = A^{-1}Y - A^{-1}UV^TX$$

$$V^TX = V^T[A^{-1}Y - A^{-1}UV^TX]$$

$$V^TX = V^TA^{-1}Y - V^TA^{-1}UV^TX$$

$$V^TX + V^TA^{-1}UV^TX = V^TA^{-1}Y$$

$$V^TX[I + V^TA^{-1}U] = V^TA^{-1}Y$$

$$V^TX = \frac{V^TA^{-1}Y}{I+V^TA^{-1}U}$$

$$\Rightarrow X = A^{-1}Y - \frac{A^{-1}UV^TA^{-1}Y}{I+V^TA^{-1}U} \; X = \left(A^{-1} - \frac{A^{-1}UV^TA^{-1}}{I+V^TA^{-1}U}\right)Y$$

$$\boxed{XY^{-1} = (A+UV^T)^{-1} = A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}} \quad ✓$$