Coin Change Problem

- Finding the number of ways of making changes for a particular amount of cents, *n*, using a given set of denominations $C=\{c_1 \ldots c_d\}$ (e.g, the US coin system: {1, 5, 10, 25, 50, 100})
  - An example: n = 4,C = {1,2,3}, solutions: {1,1,1,1}, {1,1,2},{2,2},{1,3}.
- Minimizing the number of coins returned for a particular quantity of change **(available coins {1, 5, 10, 25})**
  - 30 Cents (solution: 25 + 2, two coins)
  - 67 Cents ?
- 17 cents given denominations = {1, 2, 3, 4}?

Find the Fewest Coins: Casher's algorithm

- Given 30 cents, and coins {1, 5, 10, 25}
- Here is what a casher will do: always go with coins of highest value first
  - Choose the coin with highest value 25
    - 1 quarter
  - Now we have 5 cents left
    - 1 nickel

The solution is: 2 (one quarter + one nickel)

Greedy Algorithm Does not Always Give Optimal Solution to Coin Change Problem

- Coins = {1, 3, 4, 5}
- 7 cents = ?

- Greedy solution:
  - 3 coins: one 5 + two 1

- Optimal solution:
  - 2 coins: one 3 + one 4

Find the Fewest Coins: Divide and Conquer

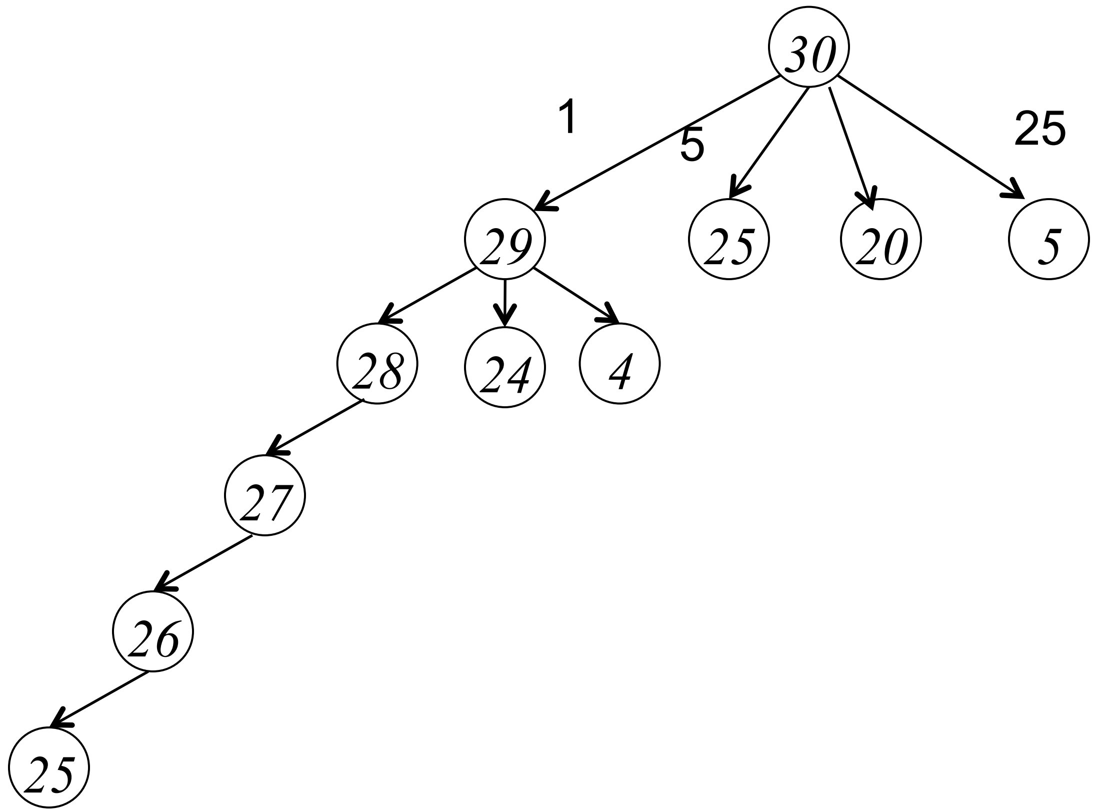
- 30 cents, given coins {1, 5, 10, 25, 50}, we need to calculate MinChange(30)
- Choose the smallest of the following:
  - 1 + MinChange(29) #give a penny
  - 1 + MinChange(25) #give a nickel
  - 1 + MinChange(10) #give a dime
  - 1 + MinChange(5) #give a quarter

- Do not know MinChange(29), MinChange(25), MinChange(10), MinChange(5)?

Coin Change Problem: A Recursive Algorithm

```
MinChange(M)
  if M = 0
    return 0
  v ← ∞
  for c in denominations ≤ M
      v ← min {MinChange(M−c) + 1, v}
  return v
```

Recursive Algorithm Is Not Efficient



- It recalculates the optimal coin combination for a given amount of money repeatedly

How to avoid computing the same function multiple times

- We're re-computing values in our algorithm more than once
- Save results of each computation for 0 to ***M***
- This way, we can do a reference call to find an already computed value, instead of re-computing each time
- Running time ***M********d***, where ***M*** is the value of money and ***d*** is the number of denominations

Coin Change Problem: Save the Intermediate Results

```
MinChange(M)
if minChange[M] not empty
    return minChange[M]
    if M = 0
        return 0
    for c in denominations ≤ M
        v ← min {MinChange(M−c) + 1, v}
    minChange[M] = v
    return v
```

The Change Problem: Dynamic Programming

```
MinChangeDP(M)
   minCoin[0] ← 0
   for m ← 1 to M
      minCoin[m] ← infinity
      for c in denominations ≤ M
         if minCoin[m−c] + 1 < minCoin[m]
            minCoin[m] ← minCoin[m−c] + 1
   return minCoin[M]
```

Greedy algorithm outputs optimal solutions for coin values 10, 5, 1

Proof:

Let N be the amount to be paid. Let the optimal solution be $P=A*10 + B*5 + C$. Clearly $B \le 1$ (otherwise we can decrease B by 2 and increase A by 1, improving the solution). Similarly, $C \le 4$.

Let the solution given by GreedyCoinChange be $P=a*10 + b*5 + c$. Clearly $b \le 1$ (otherwise the algorithm would output 10 instead of 5). Similarly $c \le 4$.

From $0 \le C \le 4$ and $P=(2A+B)*5+C$ we have $C=P \bmod 5$.
Similarly $c=P \bmod 5$, and hence $c=C$. Let $Q=(P-C)/5$.
From $0 \le B \le 1$ and $Q = 2A + B$ we have $B=Q \bmod 2$.
Similarly $b=Q \bmod 2$, and hence $b=B$.
Thus $a=A$, $b=B$, $c=C$, i.e., the solution given by GreedyCoinChange is optimal.