Thermoelastic fracture problems using Extended Finite Element Method

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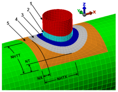
October 18, 2016

Outline

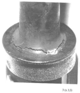
- Introduction
- Motivation
- Literature Survey
- Work done
- Problem definition
- Conclusion

Introduction: Thermo-elastic Fracture Problems

- Due to heat transfer, temperature field is set up in the material which induces thermal stresses in the body.
- These stresses becomes large in the vicinity of a discontinuity i.e. crack tip. If temperature variation is sufficiently large, it can lead to failure.
- Applications: Nuclear power plants, cylinder-nozzle intersection in pressure vessels, aerodynamic heating of high-speed aircraft, ultra fast pulse lasers etc.



(a) The cylinder-nozzle intersection. (www.knowledge.autodesk.com)



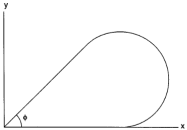
(b) Cracked head of baffle bolt of Belgian Nuclear Reactor.(www.miningawareness.wordpress.com)

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

Why thermal load on crack is important?

- Atkinson has solved the Dirichlet problem for Laplace's equation on a pie shaped region as $u(x,y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta,\ r > 0,\ 0 < \theta < \phi$
  - If $0 < \phi < \pi$: The first partial derivative of u with respect to x and y remains continuous as we approach towards the origin.
  - If $\pi < \phi < 2\pi$: The first derivative u with respect to x and y are not continuous as (x,y) approaches the origin.
- When $\phi = 2\pi$, the problem becomes a crack problem and displacement and derivative of displacement vary as $u \propto r^{\frac{1}{2}}$ and $u' \propto r^{-\frac{1}{2}}$ respectively.
- As thermo-elastic problems are also governed by Laplace's equation, temperature will vary as $r^{\frac{1}{2}}$ and heat fluxes will be unbounded at the crack tip.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



Pie-shaped region.

Methods of Solution

- Our main concern in case of thermally loaded fracture problems is to find the stress intensity factors. Methods for calculating stress intensity factors in FEM can be divided in two categories,
  - Substitution method
  - Energy method
- In substitution method, we can get the SIF's as:

$$u = \frac{1+\nu}{4E}\sqrt{\frac{2r}{\pi}}\left\{K_I\left[(2\kappa-1)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]+K_{II}\left[(2\kappa+3)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]\right\}$$

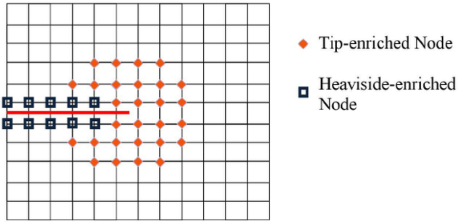
  A similar equation exist for v also.
- Using energy method, SIF's can be calculated as*

$$G = -\frac{d\Pi}{da}\ , \qquad \Pi = -\frac{1}{2}u^T[K]u + \frac{1}{2}\int \varepsilon_0{}^T[D]\varepsilon_0 dV$$

* **source**:Hellen, T. K., & Cesari, F. (1979). On the solution of the centre cracked plate with a quadratic thermal gradient. *Engineering Fracture Mechanics*, 12(4), 469-478.

Extended Finite Element Method in Thermoelasticity

- We have to use very fine mesh at capture the behaviour of crack.
- In XFEM, we enrich the polynomial approximation to include the effects of singular discontinuous field.
- Advantages:
  - Mesh is prepared without considering the existence of discontinuity.
  - No need of remeshing.



XFEM enrichment strategy

. . . Extended Finite Element Method

- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

$$h(x,y) = \begin{cases} 1, & \textit{for} \quad y \geq 0 \\ -1, & \textit{for} \quad y \leq 0 \end{cases}$$

- The nodes of elements which contains cracktip are enriched by γ:

$$u^h = \sum_i N_i(x)u_i + \sum_{j\in J} N_j(x)h(x)a_j + \sum_{k\in K} N_k(x)\left(\sum_{l=1}^{4}\gamma_l(x)b_{kl}\right)$$

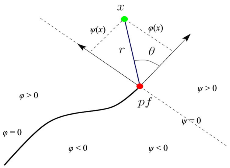
$$v^h = \sum_i N_i(x)v_i + \sum_{j\in J} N_j(x)h(x)c_j + \sum_{k\in K} N_k(x)\left(\sum_{l=1}^{4}\gamma_l(x)d_{kl}\right)$$

$$\textit{where}, \quad \gamma = \left[\sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)\right]$$

Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

Level Set Method

- The position of crack can be defined by level set methods. There are two level-set functions:
  - Normal level set, $\psi(x)$ = the signed distance from the crack surface.
  - Tangent level set, $\phi(x)$ = the signed distance to the plane including the crack front and perpendicular to the crack surface.
- To decide which enrichment should be used.
  - If $\phi < 0$ and $\psi_{min}\psi_{max} \leq 0$, nodes should be enriched with $h(x)$.
  - If $\phi_{min}\phi_{max} \leq 0$ and $\psi_{min}\psi_{max} \leq 0$, nodes should be enriched with γ.



Level Set Method

Source: Abdelaziz, Y., Bendahane, K., & Baraka, A. (2011). Extended Finite Element Modeling: Basic Review and Programming. Engineering, 3(07), 713.

Work done

- Finite Element Formulation of coupled thermo-elastic problems
- MATLAB program is developed using the FEM formulation
- Patch test for the different modules of program is performed
- Various thermoelastic fracture problems are solved using the developed code
- Need for using XFEM is shown by solving same problems and getting more accurate results

Finite Element Formulation of Thermo-elasticity

- We will formulate the semi-coupled in which we neglect the effect of displacements on temperature field.
- In thermoelastic case the total strain is given as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

- Which can be inverted to get following stress-strain relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

Finite element model

- The governing equations of the thermo-elasticity is given by:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial x} - f_x = 0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial y} - f_y = 0$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = q$$

  where k is the thermal conductivity, T is temperature and q is the heat source.
- We can develop a week form of above equations by approximating u,v and T over a typical finite element Ω^e as:

$$u(x,y) = \sum_{i=1}^{n} N_i(x,y)u_i \quad , v(x,y) = \sum_{i=1}^{n} N_i(x,y)v_i \quad , T(x,y) = \sum_{i=1}^{n} N_i(x,y)T_i$$

Weak form of Thermoelastic problem

$$-\int_\Omega \left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_\Omega \left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$
$$-\int_\Omega \frac{\partial N_i}{\partial x}\beta N_j T_j dxdy + \int_\Omega N_i f_x dxdy + \int_\Gamma N_i \vec{t} dx = 0$$

$$-\int_\Omega \left[c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}v_j + c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}u_j\right]dxdy + \int_\Omega \left[c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}v_j dxdy + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}u_j\right]dxdy$$
$$-\int_\Omega \frac{\partial N_i}{\partial y}\beta N_j T_j dxdy + \int_\Omega N_i f_y dxdy + \int_\Gamma N_i \vec{t} dy = 0$$
$$k\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_\Omega N_i N_j q dxdy = \int_\Gamma N_i \bar{Q} ds$$

weak form

Neglecting the body forces, above equations can be written in matrix form as follows

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

- Where $U^e = [u \;\; v]^T$ and T^e are the unknown variables to be found and K_{11}, K_{12} and K_{22} are the stiffness matrices which are defined as below:

$$K_{11} = \int_\Omega \begin{bmatrix} c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} & c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} \\ c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} & c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} \end{bmatrix} dxdy$$

$$K_{12} = -\beta \int_\Omega \begin{bmatrix} \frac{\partial N_i}{\partial x} N_j \\ \frac{\partial N_i}{\partial y} N_j \end{bmatrix} dxdy$$

$$K_{22} = \int K \left(\frac{\partial N_i}{\partial} \frac{\partial N_j}{\partial} + \frac{\partial N_i}{\partial} \frac{\partial N_j}{\partial} \right) dxdy$$

weak form

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix} \; and \; N^{\theta} = [N_1 \;\; \dots \;\; N_n]$$

Here N and N^{θ} are the shape functions for displacement and temperature fields respectively. So the approximation of fields withing one element in matrix form can be written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = N \; U^e \quad and \quad T = N^{\theta} \; T^e$$

If we define the matrices $[B]$ and $[B^{\theta}]$ as follows:

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix} \; and \; [B^{\theta}] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix} \tag{1}$$

The strains $\{\varepsilon\}$ and temperature gradients $\{\theta'\}$ can be written as follows:

$$\{\varepsilon\} = [B]\,\{U^{(e)}\}\;, \qquad \{\theta'\} = \left[B^{\theta}\right]\{T^{(e)}\} \tag{2}$$

weak form

- These expressions can also be written in the matrix as described by Tian [?]:

$$[K_{11}^e] = \int_\Omega [B]^T [C][B] dxdy \tag{3}$$

$$[K_{12}^e] = \int_\Omega [B]^T [\beta][N^\theta] dxdy \tag{4}$$

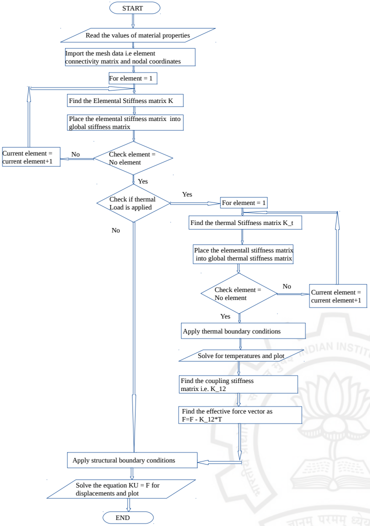
$$[K_{22}^e] = \int_\Omega [B^\theta]^T [K][B^\theta] dxdy \tag{5}$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \tag{6}$$

$$\{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds \tag{7}$$

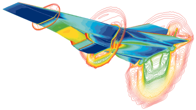
Computer implementation

We have developed the 2-dimensional Finite Element Program and the elements used is quadrilateral elements i.e. Q4 elements. We transformed the quadrilateral element of a mesh to the master element $\hat{\Omega}$ (fig. and used 2×2 Gauss quadrature rule for numerical integration.

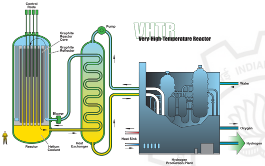


Coupled Thermo-elasticity Problems

- Coupled thermoelasticity problems have become very important in recent years because of its use in various industries.
- Some application areas include
  - aerodynamic heating of high speed air crafts as shown in figure (a).
  - the nuclear reactors where very high-temperatures and temperature gradients are developed as shown in figure (b).
  - the ultra fast pulse lasers which is used for micro-machining.
  - non destructive detection.
  - natural characterisation etc



(a) Hyper-X vehicle at Mach 7.Source: www.dfrc.nasa.gov



(b) High temperature reactor. Source: https://commons.wikimedia.org