Thermoelastic fracture problems using Extended Finite Element Method

By
Bharat Bhushan (153100048)

Under the guidance of
Prof. Salil S. Kulkarni

Department of Mechanical Engineering, IIT Bombay

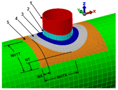
October 18, 2016

Outline

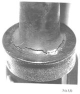
- Introduction
- Motivation
- Literature Survey
- Work done
- Problem definition
- Conclusion

Introduction: Thermo-elastic Fracture Problems

- Due to heat transfer, temperature field is set up in the material which induces thermal stresses in the body.
- These stresses becomes large in the vicinity of a discontinuity i.e. crack tip. If temperature variation is sufficiently large, it can lead to failure.
- Applications: Nuclear power plants, cylinder-nozzle intersection in pressure vessels, aerodynamic heating of high-speed aircraft, ultra fast pulse lasers etc.



(a) The cylinder-nozzle intersection. (www.knowledge.autodesk.com)



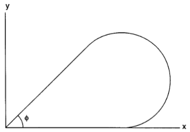
(b) Cracked head of baffle bolt of Belgian Nuclear Reactor.(www.miningawareness.wordpress.com)

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

Why thermal load on crack is important?

- Atkinson has solved the Dirichlet problem for Laplace's equation on a pie shaped region as $u(x,y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta, \; r > 0, \; 0 < \theta < \phi$
  - If $0 < \phi < \pi$: The first partial derivative of u with respect to x and y remains continuous as we approach towards the origin.
  - If $\pi < \phi < 2\pi$: The first derivative u with respect to x and y are not continuous as (x,y) approaches the origin.
- When $\phi = 2\pi$, the problem becomes a crack problem and displacement and derivative of displacement vary as $u \propto r^{\frac{1}{2}}$ and $u' \propto r^{-\frac{1}{2}}$ respectively.
- As thermo-elastic problems are also governed by Laplace's equation, temperature will vary as $r^{\frac{1}{2}}$ and heat fluxes will be unbounded at the crack tip.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



Pie-shaped region.

Methods of Solution

- Our main concern in case of thermally loaded fracture problems is to find the stress intensity factors. Methods for calculating stress intensity factors in FEM can be divided in two categories,
  - Substitution method
  - Energy method
- In substitution method, we can get the SIF's as:

$$u = \frac{1+\nu}{4E}\sqrt{\frac{2r}{\pi}}\left\{K_I\left[(2\kappa-1)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]+K_{II}\left[(2\kappa+3)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]\right\}$$

  A similar equation exist for v also.
- Using energy method, SIF's can be calculated as*

$$G = -\frac{d\Pi}{da}\ , \qquad \Pi = -\frac{1}{2}u^T[K]u+\frac{1}{2}\int \varepsilon_0{}^T[D]\varepsilon_0 dV$$

* **source**:Hellen, T. K., & Cesari, F. (1979). On the solution of the centre cracked plate with a quadratic thermal gradient. *Engineering Fracture Mechanics*, 12(4), 469-478.

Objectives

- Finite Element Formulation of thermo-elastic problems
- Computer implementation of the FEM model in MATLAB
- Solving the crack problems with thermal loading
- Application of the Extended Finite Element Method

Finite Element Formulation of Thermo-elasticity

- Since problems such as coupled structural-diffusion problems are governed by similar equations, we will be able to analyze all such coupled problems by only changing the field variable of the thermo-mechanical model.
- In semi-coupled analysis we neglect the effect of displacements on temperature field.
- The thermo-elastic constitutive relation for plane stress is given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

Constitutive relations in one dimensions

- we can get the total strain as

$$
\begin{aligned}
\varepsilon_{ij} &= \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} \\
&= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}
\end{aligned}
\tag{1}
$$

- For isotropic case equation 6 can be inverted to get the stresses

$$
\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij} \tag{2}
$$

where λ and μ are Lame constants and are given as

$$
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \qquad \mu = \frac{E}{2(1+\nu)} \tag{3}
$$

Constitutive relations in two dimensions

- Thus writing common constitutive relation for both plane strain and plane stress using equations

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha T \\ \varepsilon_y - \alpha T \\ \varepsilon_{xy} \end{Bmatrix} \tag{4}$$

where for plane strain

$$c_{11} = c_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \ , \ c_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)} \ , \ c_{66} = \frac{E}{1+\nu}$$

and for plane stress

$$c_{11} = c_{22} = \frac{E}{(1-\nu^2)} \ , \ c_{12} = \frac{E\nu}{(1-\nu^2)} \ , \ c_{66} = \frac{E}{1+\nu}$$

Finite element model

- The governing equations of the thermo-elasticity is given by:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial x}-f_x=0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x}+c_{22}\frac{\partial v}{\partial y}\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial y}-f_y=0$$

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=q$$

where k is the thermal conductivity, T is temperature and q is the heat source.

Constitutive relations in two dimensions

- Approximating u,v and T over a typical finite element Ω^e by the expression [?].

$$u(x, y) = \sum_{i=1}^{n} N_i(x, y) u_i \tag{5}$$

$$v(x, y) = \sum_{i=1}^{n} N_i(x, y) v_i \tag{6}$$

$$T(x, y) = \sum_{i=1}^{n} N_i(x, y) T_i \tag{7}$$

weak form

-

$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$
$$-\int_{\Omega}\frac{\partial N_i}{\partial x}\beta N_j T_j dxdy + \int_{\Omega} N_i f_x dxdy + \int_{\Gamma} N_i \vec{t} dx = 0 \quad (8)$$

$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}v_j + c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}u_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}v_j dxdy + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}u_j\right]dxdy$$
$$-\int_{\Omega}\frac{\partial N_i}{\partial y}\beta N_j T_j dxdy + \int_{\Omega} N_i f_y dxdy + \int_{\Gamma} N_i \vec{t} dy = 0 \quad (9)$$

$$k\int_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_{\Omega} N_i N_j q dxdy = \int_{\Gamma} N_i \bar{Q} ds \quad (10)$$

Neglecting the body forces, above equations can be written in matrix form as follows

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix} \quad (11)$$

weak form

- Where $U^e = [u \quad v]^T$ and T^e are the unknown variables to be found and K_{11}, K_{12} and K_{22} are the stiffness matrices which are defined as below:

$$K_{11} = \int_\Omega \begin{bmatrix} c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} & c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} + c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} \\ c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} & c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} \end{bmatrix} dxdy$$

$$K_{12} = -\beta \int_\Omega \begin{bmatrix} \frac{\partial N_i}{\partial x} N_j \\ \frac{\partial N_i}{\partial y} N_j \end{bmatrix} dxdy$$

$$K_{22} = \int_\Omega K \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

$$F = \int_\Omega N_i \begin{bmatrix} f_x \\ f_y \end{bmatrix} dxdy \qquad Q = \int_\Gamma N_i \bar{Q} ds \quad \text{and} \quad U = [u \quad v]^T$$

where the constants c_{11}, c_{12}, c_{66} for plane strain is given by:

$$c_{11} = c_{22} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad , \quad c_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad c_{66} = \frac{E}{1+\nu}$$

and for plane

weak form

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix} \text{ and } N^\theta = [N_1 \ \dots \ N_n]$$

Here N and N^θ are the shape functions for displacement and temperature fields respectively. So the approximation of fields withing one element in matrix form can be written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = N \ U^e \quad \text{and} \quad T = N^\theta \ T^e$$

If we define the matrices $[B]$ and $[B^\theta]$ as follows:

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix} \text{ and } [B^\theta] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix} \tag{12}$$

The strains $\{\varepsilon\}$ and temperature gradients $\{\theta'\}$ can be written as follows:

$$\{\varepsilon\} = [B] \{U^{(e)}\} \ , \qquad \{\theta'\} = \left[B^\theta\right] \{T^{(e)}\} \tag{13}$$

weak form

- These expressions can also be written in the matrix as described by Tian [**?**]:

$$[K^e_{11}] = \int_\Omega [B]^T [C][B] dxdy \tag{14}$$

$$[K^e_{12}] = \int_\Omega [B]^T [\beta][N^\theta] dxdy \tag{15}$$

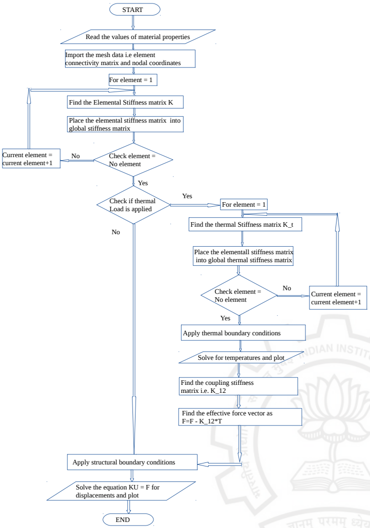
$$[K^e_{22}] = \int_\Omega [B^\theta]^T [K][B^\theta] dxdy \tag{16}$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \tag{17}$$

$$\{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds \tag{18}$$

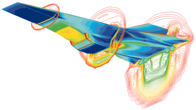
Computer implementation

We have developed the 2-dimensional Finite Element Program and the elements used is quadrilateral elements i.e. Q4 elements. We transformed the quadrilateral element of a mesh to the master element $\hat{\Omega}$ (fig. and used 2×2 Gauss quadrature rule for numerical integration.

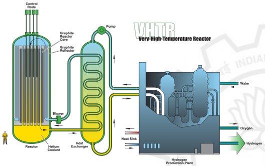


Coupled Thermo-elasticity Problems

- Coupled thermoelasticity problems have become very important in recent years because of its use in various industries.
- Some application areas include
  - aerodynamic heating of high speed air crafts as shown in figure (a).
  - the nuclear reactors where very high-temperatures and temperature gradients are developed as shown in figure (b).
  - the ultra fast pulse lasers which is used for micro-machining.
  - non destructive detection.
  - natural characterisation etc



(a) Hyper-X vehicle at Mach 7.Source: www.dfrc.nasa.gov



(b) High temperature reactor. Source: https://commons.wikimedia.org

Extended Finite Element Method

- Present work describes the application of the eXtended Finite Element Method (XFEM) in coupled thermoelastic problems.
- In classical finite element method, mesh should conform to the boundaries of discontinuity for accurate modeling of the problem. So remeshing has to be done every time the crack grows.
- An alternate method is to enrich the polynomial approximation with functions which can model the discontinuities.

$$u^h = \sum_i N_i(x) u_i + \sum_{j \in J} N_j(x) h(x) a_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) b_{kl} \right)$$

$$v^h = \sum_i N_i(x) v_i + \sum_{j \in J} N_j(x) h(x) c_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) d_{kl} \right)$$

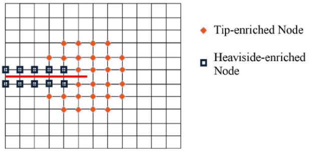
. . . Extended Finite Element Method

- $$\gamma = \left[\sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)\right]$$
- As can be seen above the third singular function γ_3 is the only enrichment function which is discontinuous across the crack. Thus the discontinuity of the displacement field at $\theta = \pm\pi$ in the singular zone is only modeled by γ_3 on the elements containing the crack tip.
- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

$$h(x,y) = \begin{cases} 1, & \textit{for} \quad y \geq 0 \\ -1, & \textit{for} \quad y \leq 0 \end{cases}$$

...Extended Finite Element Method

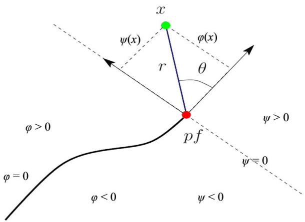
- In this work we have studied a recent method of modeling crack growth without re-meshing. The main advantage of this method is that the mesh is prepared without considering the existence of discontinuity.
- XFEM is based on the partition of unity method in which we enrich the classical finite element approximation to include the effects of singular discontinuous field around the crack.



XFEM enrichment strategy. Source: Abdelaziz, Y., Bendahane, K., & Baraka, A. (2011). Extended Finite Element Modeling: Basic Review and Programm

Level Set Method

- The description of crack in extended finite element method is often described by the level-set method. A crack is described by two level-set methods as shown in figure below.
  - A normal level set, $\psi(x)$ which is the signed distance from the crack surface.
  - A tangent level set, $\phi(x)$ which is the signed distance to the plane including the crack front and perpendicular to the crack surface.
- To know which element should be enriched by which function we see the following
  - If $\phi < 0$ and $\psi_{min}\psi_{max} \leq 0$, then the crack cuts through the element and the nodes of the element are to be enriched with h(x).
  - If in the element $\phi_{min}\phi max \leq 0$ and $\psi_{min}\psi max \leq 0$, then the tip lies within that element and its nodes are to be enriched with γ.



Level Set Method