Thermoelastic fracture problems using Extended Finite Element Method

By
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Outline

- Introduction
- Motivation
- Literature Survey
- Work done
- Problem definition
- Conclusion

Introduction: Thermo-elastic Fracture Mechanics

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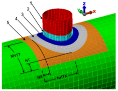
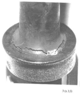
- Applications: Nuclear power plants, cylinder-nozzle intersection in pressure vessels, aerodynamic heating of high-speed aircraft, ultra fast pulse lasers etc.



(a) The cylinder-nozzle intersection. (www.knowledge.autodesk.com)



(b) Cracked head of baffle bolt of Belgian Nuclear Reactor.(www.miningawareness.wordpress.com)

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

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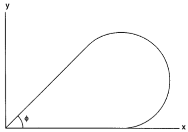

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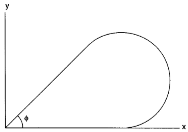

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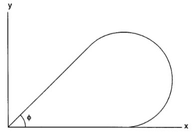

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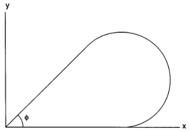

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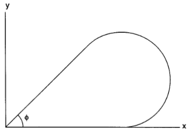
- As thermo-elastic problems are also governed by Laplace's equation, temperature will vary as $r^{\frac{1}{2}}$ and heat fluxes will be unbounded at the crack tip.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



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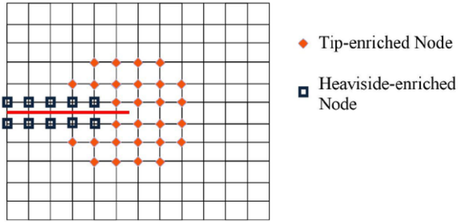

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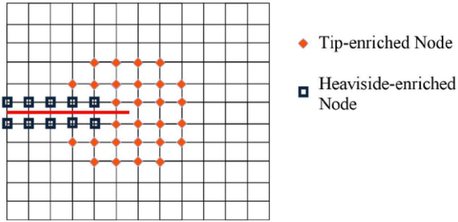

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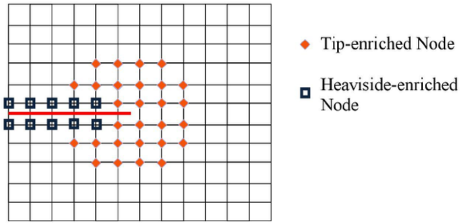

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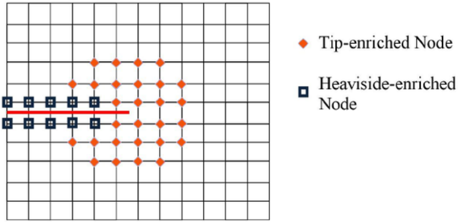

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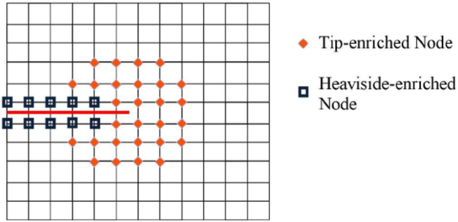
- Advantages:
  - Mesh is prepared without considering the existence of discontinuity
  - No need of remeshing
  - Accurate solution



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- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

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$$h(x,y) = \begin{cases} 1, & for \quad y \geq 0 \\ -1, & for \quad y \leq 0 \end{cases}$$

- The nodes of elements which contains cracktip are enriched by γ:

$$u^h = \sum_{i} N_i(x)u_i + \sum_{j \in J} N_j(x)h(x)a_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) b_{kl} \right)$$

$$v^h = \sum_{i} N_i(x)v_i + \sum_{j \in J} N_j(x)h(x)c_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) d_{kl} \right)$$

$$where, \quad \gamma = \left[\sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right]$$

Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

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- Comparison between the results of FEM and X-FEM programs.

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- In thermoelastic case the total strain is given as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

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- It can be inverted to get following stress-strain relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

Governing Equations of Thermoelasticity

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- The governing equations of the thermo-elasticity is derived as:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial x}-f_x=0$$

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- We can develop a week form of above equations by approximating u,v and T over a typical finite element Ω^e as:

$$u(x,y)=\sum_{i=1}^{n}N_i(x,y)u_i \quad ,v(x,y)=\sum_{i=1}^{n}N_i(x,y)v_i \quad ,T(x,y)=\sum_{i=1}^{n}N_i(x,y)T_i$$

Weak Form Equations of Coupled Thermoelasticity

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$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$

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Weak Form Equations of Coupled Thermoelasticity

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$$k\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_\Omega N_i N_j q dxdy = \int_\Gamma N_i \bar{Q} ds$$

Finite Element Model

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- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

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$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}, \quad [B^\theta] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}, \quad N^\theta = [N_1 \ \dots \ N_n]$$

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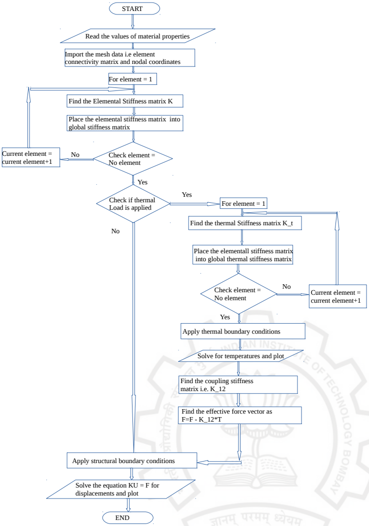

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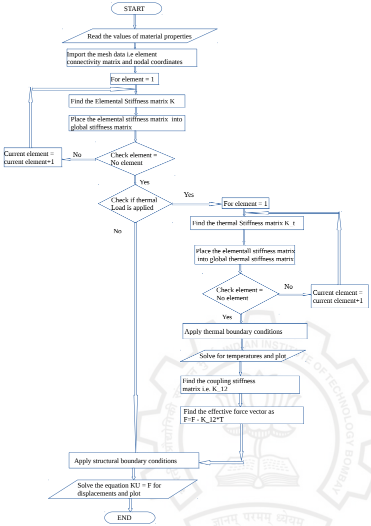

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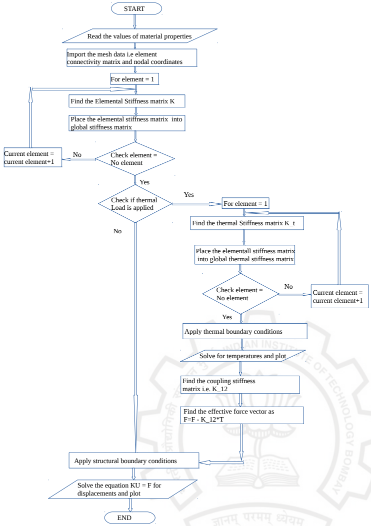

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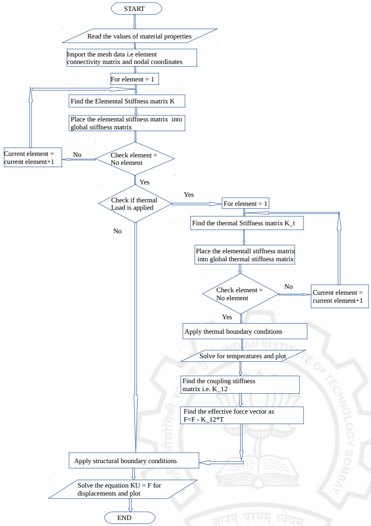
A flow-chart showing the steps of
the programming is shown in the figure.



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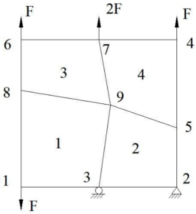

Node	(x,y)
1	(0,0)
2	(2,0)
3	(1,0)
4	(2,2)
5	(2,0.8)
6	(0,2)
7	(1,2)
8	(0,1.4)
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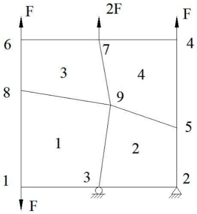

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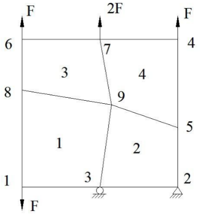
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body

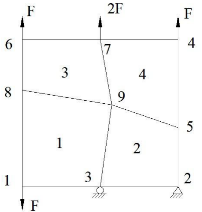


Node	(x,y)
1	(0,0)
2	(2,0)
3	(1,0)
4	(2,2)
5	(2,0.8)
6	(0,2)
7	(1,2)
8	(0,1.4)
9	(1.2,1.2)

(a) Mesh Configuration

Patch Test 1

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.



Node	(x,y)
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2	(2,0)
3	(1,0)
4	(2,2)
5	(2,0.8)
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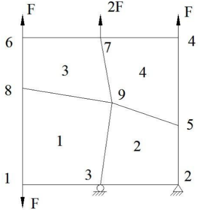


Node	(x,y)
1	(0,0)
2	(2,0)
3	(1,0)
4	(2,2)
5	(2,0.8)
6	(0,2)
7	(1,2)
8	(0,1.4)
9	(1.2,1.2)

(a) Mesh Configuration

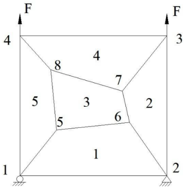
(b) Results of Patch Test 1

	Gauss Points	$\sigma_x/2F$	$\sigma_y/2F$	$\tau_{xy}/2F$
Element 1	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.2498×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	-0.222×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	0.111×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.1110×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 2

Patch Test 2

- Another patch test is performed on a plate with 5 element mesh and solved and results are tabulated as shown below

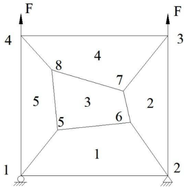


Node	(x,y)
1	(0,0)
2	(2,0)
3	(2,2)
4	(0,2)
5	(0.85,0.7)
6	(1.5,1)
7	(1.2,1.4)
8	(0.7,1.6)

(a) Mesh Configuration

Patch Test 2

- Another patch test is performed on a plate with 5 element mesh and solved and results are tabulated as shown below
- The 2×2 quadrature rule is used for numerical integration



Node	(x,y)
1	(0,0)
2	(2,0)
3	(2,2)
4	(0,2)
5	(0.85,0.7)
6	(1.5,1)
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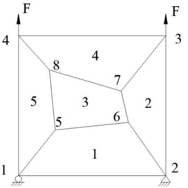

Node (x,y)
1 (0,0)
2 (2,0)
3 (2,2)
4 (0,2)
5 (0.85,0.7)
6 (1.5,1)
7 (1.2,1.4)
8 (0.7,1.6)

(a) Mesh Configuration

(b) Results of Patch Test 2

	Gauss Point	σ_x/F	σ_y/F	τ_{xy}/F
Element 1	1	0.166×10^{-15}	1	0.1665×10^{-15}
	2	-0.033×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	0.062×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	-0.222×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.2220×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
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	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0
Element 5	1	-0.422×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 3 : Thermal code

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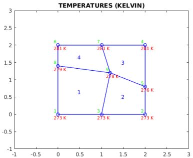
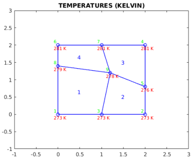
- Another patch was performed to validate the thermal code.



(a) Temperature distribution

Patch Test 3 : Thermal code

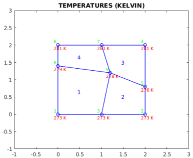
- Another patch was performed to validate the thermal code.
- Same problem as patch test was taken



(a) Temperature distribution

Patch Test 3 : Thermal code

- Another patch was performed to validate the thermal code.
- Same problem as patch test was taken
- Temperature loads are applied to get the constant heat flux throughout the body



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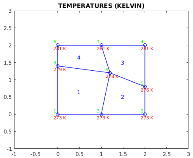

(a) Temperature distribution

(b) Results of Patch test 3

Nodes	Temperatures	$Q_y(W)$	$Q_x(W)$
1	273	200	0.125×10^{-10}
2	273	200	0.155×10^{-10}
3	273	200	0.222×10^{-10}
4	281	200	0
5	276	200	0
6	281	200	0
7	281	200	0.111×10^{-10}
8	279	200	0.138×10^{-10}
9	278	200	0

Patch Test 4: Thermo-elastic code

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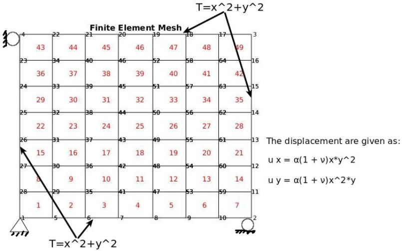
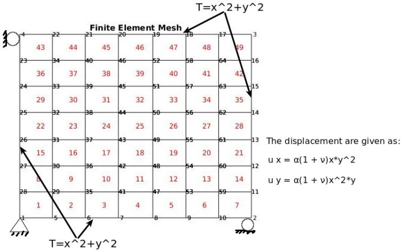


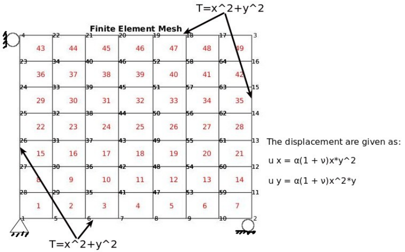
- A square plate of dimension 2×2 is taken and meshed as shown

Patch Test 4: Thermo-elastic code



- A square plate of dimension 2×2 is taken and meshed as shown
- Temperature distribution of $T = x^2 + y^2$ is applied on 4 boundaries of plate

Patch Test 4: Thermo-elastic code



- A square plate of dimension 2×2 is taken and meshed as shown
- Temperature distribution of $T = x^2 + y^2$ is applied on 4 boundaries of plate
- a heat source $q = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 4$ is applied throughout the body
- Mesh is refined and to get the more accurate results

Mesh Refinements

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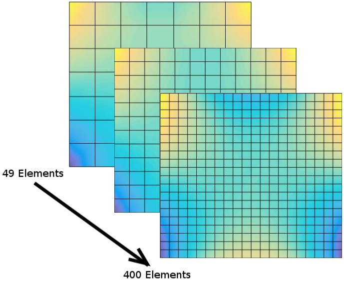

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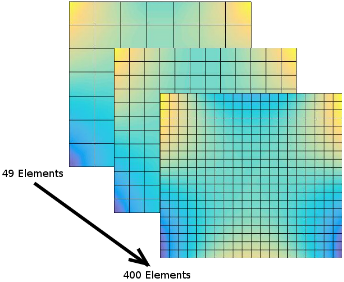
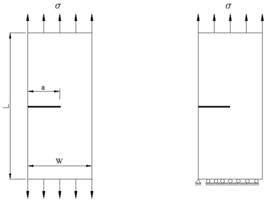


Errors obtained after mesh refinements

Number of Elements	Error in u_x	Error in u_y	Error in σ_x	Error in σ_y	Error in τ_{xy}
7×7	5.783469×10^{-10}	5.783469×10^{-10}	6.668255×10^{2}	6.668255×10^{2}	3.615391×10^{2}
12×12	1.764697×10^{-11}	1.764697×10^{-11}	1.000597121	1.000597121	0.031722
20×20	1.265804×10^{-12}	1.265804×10^{-12}	0.994299467	0.994299467	0.005781

Plate with edge crack

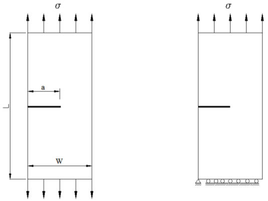
Plate with edge crack



(a) Plate with edge crack

- A plate with edge crack is meshed in ANSYS and imported to MATLAB

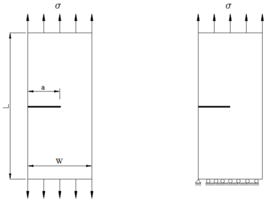
Plate with edge crack



(a) Plate with edge crack

- A plate with edge crack is meshed in ANSYS and imported to MATLAB
- Boundary conditions are applied and stresses ahead of cracktip is plotted

Plate with edge crack



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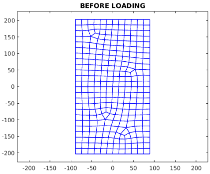
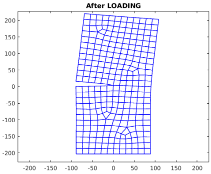


(b) Before loading



(c) After loading

- A plate with edge crack is meshed in ANSYS and imported to MATLAB
- Boundary conditions are applied and stresses ahead of cracktip is plotted
- The results obtained by MATLAB code is compared with the ANSYS solution

Stresses Ahead of Crack-tip: Comparison with ANSYS

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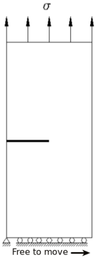
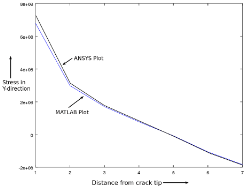
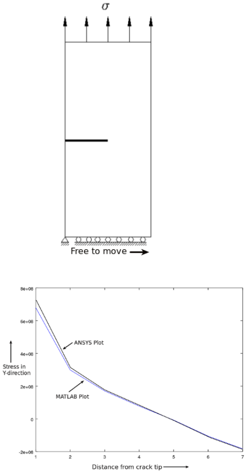




(a) σ_y ahead of the crack tip

Stresses Ahead of Crack-tip: Comparison with ANSYS



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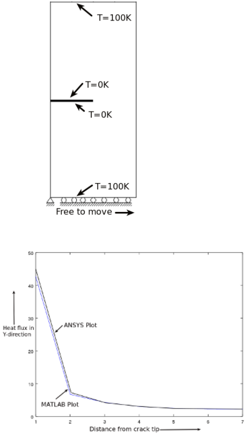
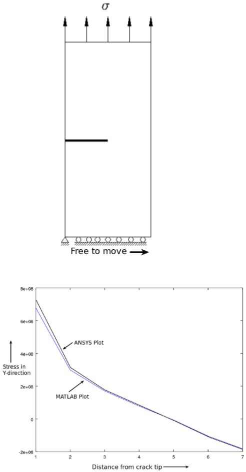


(b) Heat flux ahead of crack tip

Stresses Ahead of Crack-tip: Comparison with ANSYS



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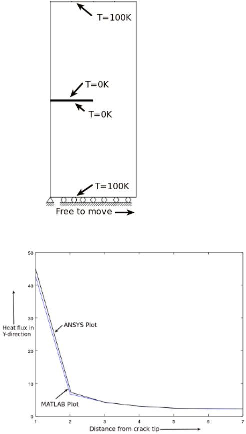
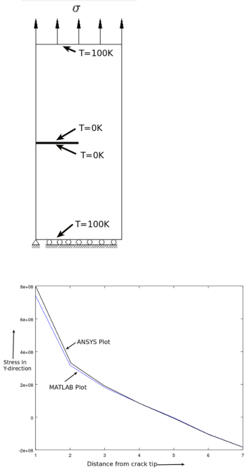


(b) Heat flux ahead of crack tip



(c) σ_y ahead of the crack tip

Comparison of FEM and X-FEM methods

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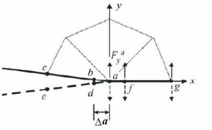
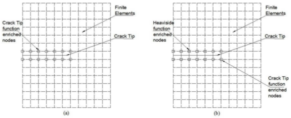
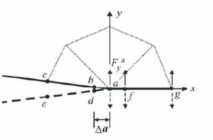


(a) Crack closure technique



(b) X-FEM enrichment of nodes

Comparison of FEM and X-FEM methods



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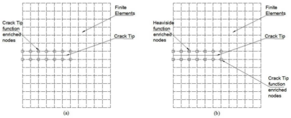

(b) X-FEM enrichment of nodes

Crack closure integral: $G_I = \frac{W}{B\Delta a} = \frac{F_y^a u_y^b}{2B\Delta a}$ $Thus,$ $K_I = \sqrt{\frac{G_I}{E}} = 5.24586910 \times 10^9 \quad Pa\sqrt{m}$

Analytical: $K_I = C * \sigma\sqrt{\pi a} = 4.726065 \times 10^9 \quad Pa\sqrt{m}$

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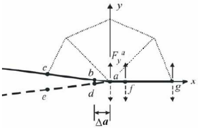

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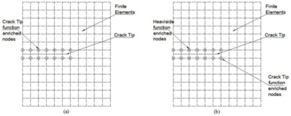

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Analytical: $K_I = C * \sigma\sqrt{\pi a} = 4.726065 \times 10^9 \quad Pa\sqrt{m}$

Method used	$\%\ Error = \{K_{theoretical} - K_{numerical}/K_{theoretical}\} \times 100$
Finite Element Method	11.1%
Extended Finite Element Method	4%

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- Q8 elements will be applied in our program to solve curved boundary problems.
- As the hydrogen diffusion in cracked bodies are governed by the Poisson's equation similar to the thermoelastic problems, we will modify the X-FEM program to solve problems of hydrogen-diffusion

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