Thermoelastic fracture problems using Extended Finite Element Method

By
Bharat Bhushan (153100048)

Under the guidance of
Prof. Salil S. Kulkarni

Department of Mechanical
Engineering, IIT Bombay

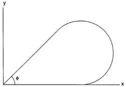
October 24, 2016

Outline

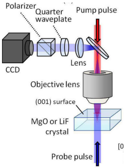
- Introduction
- Literature Survey
- Objectives and work done
- Computer implementation
- Validation of FEM program
- Example problems
- Conclusion

Introduction: Thermo-elastic Fracture Mechanics

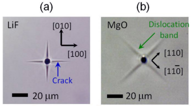
- Thermo-elastic Fracture mechanics: Study of propagation of crack in presence of temperature field.
- Atkinson's solution of Laplace's equation: $u(x, y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta$
- For $\phi = 2\pi$: $u \propto r^{\frac{1}{2}}$ and $u' \propto r^{-\frac{1}{2}}$



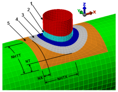
- Application areas:



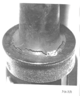
(a) Ultra fast laser



(b) Cracks generated in material



(c) The cylinder-nozzle intersection.

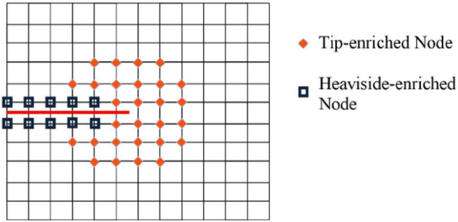


(d) Cracked baffle bolt of Belgian Nuclear Reactor

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

Extended Finite Element Method in Thermoelasticity

- In FEM we have to use very fine mesh to capture the behaviour of crack.
- In X-FEM, we enrich the polynomial approximation to include the effects of singular discontinuous field.
- Advantages over FEM:
  - Accurate solutions without the very fine mesh at the discontinuity
  - No need of remeshing



X-FEM enrichment strategy

Enrichment functions in X-FEM

- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

$$h(x,y) = \begin{cases} 1, & \textit{for} \quad y \geq 0 \\ -1, & \textit{for} \quad y \leq 0 \end{cases}$$

- The nodes of elements which contains cracktip are enriched by γ:

$$u^h = \sum_i N_i(x)u_i + \sum_{j\in J} N_j(x)h(x)a_j + \sum_{k\in K} N_k(x)\left(\sum_{l=1}^{4}\gamma_l(x)b_{kl}\right)$$

$$v^h = \sum_i N_i(x)v_i + \sum_{j\in J} N_j(x)h(x)c_j + \sum_{k\in K} N_k(x)\left(\sum_{l=1}^{4}\gamma_l(x)d_{kl}\right)$$

$$\textit{where}, \quad \gamma = \left[\sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)\right]$$

Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

Objectives and work Done

- Objectives:
  - To develop an Extended Finite Element Program for coupled thermoelastic fracture problems
  - To utilize the developed X-FEM Program in other fields which is governed by Laplace's equation i.e hydrogen diffusion
- Work Done:
  - Finite Element Formulation of coupled thermoelasticity
  - Development of a MATLAB program for semi-coupled thermoelastic problems.
  - Patch tests to validate the developed FEM program.
  - Solution of various thermoelastic fracture problems and comparison with the analytical solutions.
  - Comparison between the results of FEM and X-FEM programs.

Finite Element Formulation of Thermo-elasticity

- We have formulated the semi-coupled in which we neglected the effect of displacements on temperature field.
- In thermoelastic case the total strain is given as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

- It can be inverted to get following stress-strain relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

Governing Equations of Thermoelasticity

- The governing equations of the thermo-elasticity is derived as:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial x}-f_x=0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x}+c_{22}\frac{\partial v}{\partial y}\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial y}-f_y=0$$

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=q$$

- We can develop a week form of above equations by approximating u,v and T over a typical finite element Ω^e as:

$$u(x,y)=\sum_{i=1}^{n}N_i(x,y)u_i \quad ,v(x,y)=\sum_{i=1}^{n}N_i(x,y)v_i \quad ,T(x,y)=\sum_{i=1}^{n}N_i(x,y)T_i$$

Weak Form Equations of Coupled Thermoelasticity

$$
\begin{aligned}
-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy \\
-\int_{\Omega}\frac{\partial N_i}{\partial x}\beta N_j T_j dxdy + \int_{\Omega}N_i f_x dxdy + \int_{\Gamma}N_i\vec{t}dx = 0
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}v_j + c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}u_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}v_j dxdy + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}u_j\right]dxdy \\
-\int_{\Omega}\frac{\partial N_i}{\partial y}\beta N_j T_j dxdy + \int_{\Omega}N_i f_y dxdy + \int_{\Gamma}N_i\vec{t}dy = 0
\end{aligned}
$$

$$
k\int_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_{\Omega}N_i N_j q dxdy = \int_{\Gamma}N_i\bar{Q}ds
$$

Finite Element Model

- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

Where,

$$[K_{11}^e] = \int_\Omega [B]^T [C][B] dxdy \quad [K_{12}^e] = \int_\Omega [B]^T [\beta][N^\theta] dxdy \quad [K_{22}^e] = \int_\Omega [B^\theta]^T [K][B^\theta] dxdy$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \quad \{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}, \quad [B^\theta] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}, \quad N^\theta = [N_1 \quad \dots \quad N_n]$$

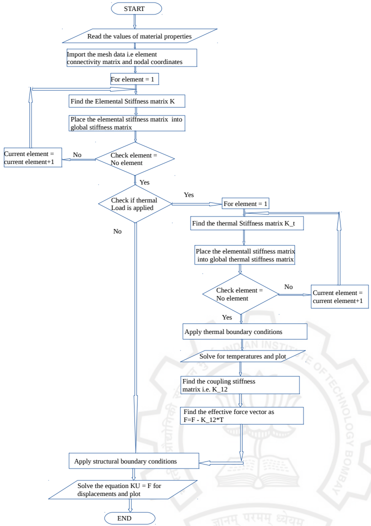
Computer implementation

A MATLAB program
is developed to solve the 2-dimensional
thermoelasticity problems.

Quadrilateral (Q4) elements
were used for meshing the body.

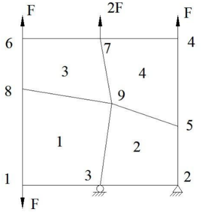
2×2 Gauss quadrature
rule is used for numerical integration.

A flow-chart showing the steps of
the programming is shown in the figure.



Patch Test 1

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.



Node (x,y)
1 (0,0)
2 (2,0)
3 (1,0)
4 (2,2)
5 (2,0.8)
6 (0,2)
7 (1,2)
8 (0,1.4)
9 (1.2,1.2)

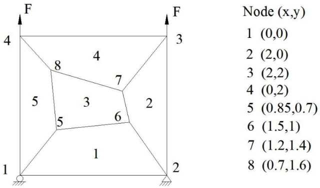
(a) Mesh Configuration

(b) Results of Patch Test 1

	Gauss Points	$\sigma_x/2F$	$\sigma_y/2F$	$\tau_{xy}/2F$
Element 1	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.2498×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	-0.222×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	0.111×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.1110×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 2

- Another patch test is performed on a plate with 5 element mesh and solved and results are tabulated as shown below
- The 2×2 quadrature rule is used for numerical integration



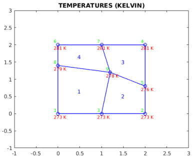
(a) Mesh Configuration

(b) Results of Patch Test 2

	Gauss Point	σ_x/F	σ_y/F	τ_{xy}/F
Element 1	1	0.166×10^{-15}	1	0.1665×10^{-15}
	2	-0.033×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	0.062×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	-0.222×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.2220×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0
Element 5	1	-0.422×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 3 : Thermal code

- Another patch was performed to validate the thermal code.
- Same problem as patch test was taken
- Temperature loads are applied to get the constant heat flux throughout the body

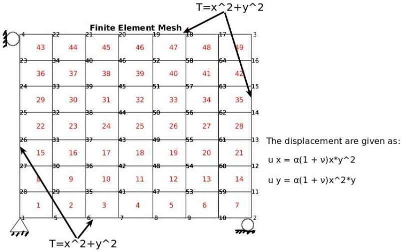


(a) Temperature distribution

(b) Results of Patch test 3

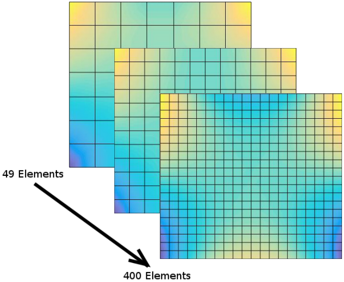
Nodes	Temperatures	$Q_y(W)$	$Q_x(W)$
1	273	200	0.125×10^{-10}
2	273	200	0.155×10^{-10}
3	273	200	0.222×10^{-10}
4	281	200	0
5	276	200	0
6	281	200	0
7	281	200	0.111×10^{-10}
8	279	200	0.138×10^{-10}
9	278	200	0

Patch Test 4: Thermo-elastic code



- A square plate of dimension 2×2 is taken and meshed as shown
- Temperature distribution of $T = x^2 + y^2$ is applied on 4 boundaries of plate
- a heat source $q = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 4$ is applied throughout the body
- Mesh is refined and to get the more accurate results

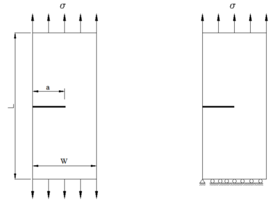
Mesh Refinements



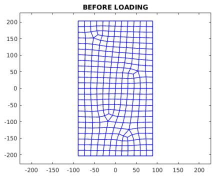
Errors obtained after mesh refinements

Number of Elements	Error in u_x	Error in u_y	Error in σ_x	Error in σ_y	Error in τ_{xy}
7×7	5.783469×10^{-10}	5.783469×10^{-10}	6.668255×10^{2}	6.668255×10^{2}	3.615391×10^{2}
12×12	1.764697×10^{-11}	1.764697×10^{-11}	1.000597121	1.000597121	0.031722
20×20	1.265804×10^{-12}	1.265804×10^{-12}	0.994299467	0.994299467	0.005781

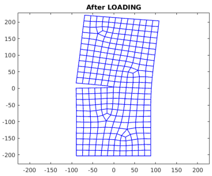
Plate with edge crack



(a) Plate with edge crack



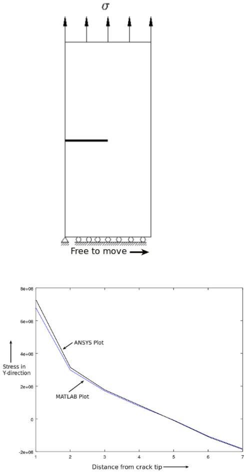
(b) Before loading



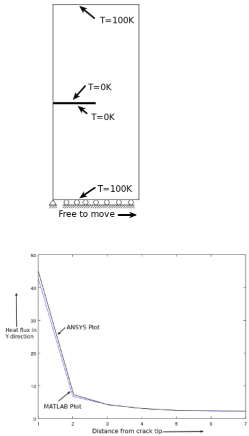
(c) After loading

- A plate with edge crack is meshed in ANSYS and imported to MATLAB
- Boundary conditions are applied and stresses ahead of cracktip is plotted
- The results obtained by MATLAB code is compared with the ANSYS solution

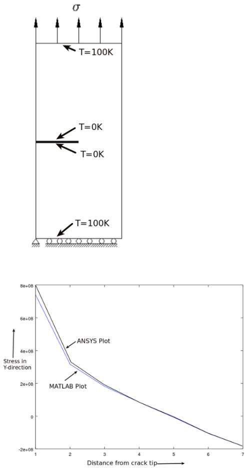
Stresses Ahead of Crack-tip: Comparison with ANSYS



(a) σ_y ahead of the crack tip

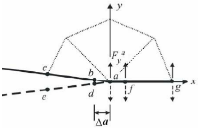


(b) Heat flux ahead of crack tip

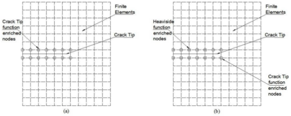


(c) σ_y ahead of the crack tip

Comparison of FEM and X-FEM methods



(a) Crack closure technique



(b) X-FEM enrichment of nodes

Crack closure integral: $G_I = \dfrac{W}{B\Delta a} = \dfrac{F_y^a u_y^b}{2B\Delta a}$ $\quad Thus, \quad K_I = \sqrt{\dfrac{G_I}{E}} = 5.24586910 \times 10^9 \quad Pa\sqrt{m}$

Analytical: $K_I = C * \sigma\sqrt{\pi a} = 4.726065 \times 10^9 \quad Pa\sqrt{m}$

Method used	$\%\ Error = \{K_{theoretical} - K_{numerical} / K_{theoretical}\} \times 100$
Finite Element Method	11.1%
Extended Finite Element Method	4%

Conclusions and Future Work

- A **FEM program** is developed for Thermo-elastci Fracture problems based on **semi-coupled formulation** and **patch tests** were performed to validate the program.
- It is shown that when solving fracture problems with traditional FEM program, solutions are **not accurate** around the crack tip
- Same problem is solved in an X-FEM program developed by parnaik to show that the **solution improves** drastically when using X-FEM
- The Extended Finite Element enrichments will be applied to both displacement and temperature fields of thermoelasticity program in stage 2 of the project
- **Q8 elements** will be applied in our program to solve curved boundary problems.
- As the **hydrogen diffusion** problems governed by Laplace's equation same as thermoelastic problems, we will modify the X-FEM program to solve these problems.

References

Kumar, P., & Prashant, K. (2009). Elements of fracture mechanics. Tata McGraw-Hill Education.

Fillery, B. P., Hu, X., & Fisher, G. (2006). Thermo-elastic fracture of edge cracked plate under surface âĂŸshockâĂŹloading. In *Fracture of Nano and Engineering Materials and Structures* (pp. 385-386). Springer Netherlands.

Tian, X., Shen, Y., Chen, C., & He, T. (2006). A direct finite element method study of generalized thermoelastic problems. International Journal of Solids and Structures, 43(7), 2050-2063.

Sih, G. C. (1962). On the singular character of thermal stresses near a crack tip. Journal of Applied Mechanics, 29(3), 587-589.

Hellen, T. K., & Cesari, F. (1979). On the solution of the centre cracked plate with a quadratic thermal gradient. *Engineering Fracture Mechanics*, 12(4), 469-478.

Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

Koustubh Parnaik. (2013). Implementation of Extended Finite Element Method (X-FEM) for static fracture problems. *Department of Mechanical Engineering, IIT Bombay*.

Duflot, M. (2008). The extended finite element method in thermoelastic fracture mechanics. *International Journal for Numerical Methods in Engineering*, 74(5), 827-847.

Budyn, E., Zi, G., MoÃńs, N., & Belytschko, T. (2004). A method for multiple crack growth in brittle materials without remeshing. *International journal for numerical methods in engineering*, 61(10), 1741-1770.

. . . References

Stazi, F. L., Budyn, E., Chessa, J., & Belytschko, T. (2003). An extended finite element method with higher-order elements for curved cracks. *Computational Mechanics*, 31(1-2), 38-48.

Khoei, A. R., & Nikbakht, M. (2007). An enriched finite element algorithm for numerical computation of contact friction problems. *International Journal of Mechanical Sciences*, 49(2), 183-199.

Liang, J., Huang, R., Prevost, J. H., & Suo, Z. (2003). Evolving crack patterns in thin films with the extended finite element method. *International Journal of Solids and Structures*, 40(10), 2343-2354.

Xu, Y., & Yuan, H. (2009). On damage accumulations in the cyclic cohesive zone model for X-FEM analysis of mixed-mode fatigue crack growth.*Computational Materials Science*, 46(3), 579-585.

Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.

Sad, M. H. (2004). Elasticity: Theory. *Applications and Numerics*.

Reddy, J. N. (1989). An introduction to the finite element method.

Fries, T. P., & Belytschko, T. (2010). The extended/generalized finite element method: an overview of the method and its applications. *International Journal for Numerical Methods in Engineering*, 84(3), 253-304.

Liu, X. Y., Xiao, Q. Z., & Karihaloo, B. L. (2004). X-FEM for direct evaluation of mixed mode SIFs in homogeneous and biâĂŘmaterials. *International Journal for Numerical Methods in Engineering*, 59(8), 1103-1118.