Thermoelastic fracture problems using Extended Finite Element Method

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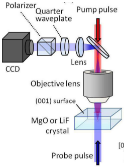
Outline

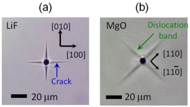
- Introduction
- Motivation
- Literature Survey
- Objectives
- Computer implementation
- Example problems
- Conclusion and future work

Introduction: Thermo-elastic Fracture Mechanics

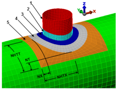
- Thermo-elastic Fracture mechanics is a field in which we Study the propagation of the crack in presence of temperature field.
- Some application areas are as follows:



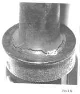
(a) Ultrafast laser



(b) Cracks generated in material



(c) The cylinder-nozzle intersection.



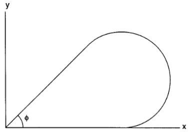
(d) Cracked baffle bolt of Belgian Nuclear Reactor

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

Motivation

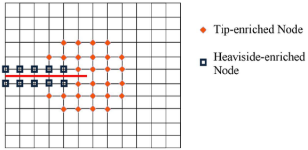
- Atkinson has solved a Dirichlet problem for Laplace's equation on a pie shaped region as:
$$u(x,y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta, \;\; r > 0, \;\; 0 < \theta < \phi$$
  - Case I: $0 < \phi < \pi$: u' is continuous as we approach towards the origin
  - Case II: $\pi < \phi < 2\pi$: u' is not continuous as (x,y) approaches the origin
- When $\phi = 2\pi$, the problem becomes a crack problem:
$$u \propto r^{\frac{1}{2}} \text{ and } u' \propto r^{-\frac{1}{2}}$$
- Substituting T in place of u: $T \propto r^{\frac{1}{2}}$ and $T' \propto r^{-\frac{1}{2}}$
- Thus, around the crack tip, thermal stresses also has square root singularity : our motivation for analyzing a thermo-elastic crack problem.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



Pie-shaped region.

Introduction to Extended Finite Element Method



X-FEM enrichment strategy

$$\textit{Heavyside enrichment functions}: \qquad h(x,y) = \begin{cases} 1, & \textit{for} \quad y \geq 0 \\ -1, & \textit{for} \quad y \leq 0 \end{cases}$$

$$u^h = \sum_i N_i(x)u_i + \sum_{j \in J} N_j(x)h(x)a_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) b_{kl} \right)$$

$$v^h = \sum_i N_i(x)v_i + \sum_{j \in J} N_j(x)h(x)c_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) d_{kl} \right)$$

$$\text{where,}\gamma = \left[\sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right]$$

Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

Objectives:

- Finite element formulation of semi-coupled thermoelasticity and implementation in MATLAB
- Validation of FEM programs by performing patch tests.
- Development of Extended Finite Element Program for coupled thermoelastic fracture problems
- Utilization of the developed X-FEM Program in other fields which is governed by Laplace's equation

Finite Element Formulation of Thermoelasticity

- A semi-coupled thermoelasticity problem is formulated.
- In semi-coupled problems we neglected the effect of displacements on temperature field.
- In presence of temperature field, the Hooke's law can be given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha \Delta T \\ \varepsilon_y - \alpha \Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

- The governing equations of the thermoelasticity are derived as follows:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial x} - f_x = 0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial y} - f_y = 0$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = q$$

Weak Form Equations of Coupled Thermoelasticity

$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$
$$-\int_{\Omega}\frac{\partial N_i}{\partial x}\beta N_j T_j dxdy + \int_{\Omega} N_i f_x dxdy + \int_{\Gamma} N_i \vec{t} dx = 0$$

$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}v_j + c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}u_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}v_j dxdy + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}u_j\right]dxdy$$
$$-\int_{\Omega}\frac{\partial N_i}{\partial y}\beta N_j T_j dxdy + \int_{\Omega} N_i f_y dxdy + \int_{\Gamma} N_i \vec{t} dy = 0$$

$$k\int_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_{\Omega} N_i N_j q dxdy = \int_{\Gamma} N_i \bar{Q} ds$$

Finite Element Model

- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

Where,

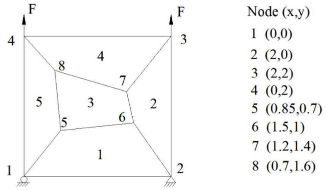
$$[K_{11}^e] = \int_\Omega [B]^T[C][B]dxdy \quad [K_{12}^e] = \int_\Omega [B]^T[\beta][N^\theta]dxdy \quad [K_{22}^e] = \int_\Omega [B^\theta]^T[K][B^\theta]dxdy$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \quad \{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds$$

- Computer implementation:
  - A MATLAB program is developed to solve the 2-dimensional thermoelasticity problems based on above Finite Element model
  - 4-noded quadrilateral (Q4) elements and 3-degrees of freedom per node
  - 2×2 Gauss quadrature rule is used for numerical integration

Patch Test 1: Validation of Elasticity code

- A square plate is taken and meshed with 5 elements as shown in the figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.

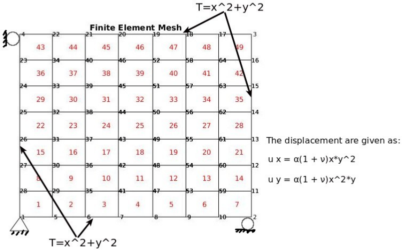


(a) Mesh Configuration

(b) Results of Patch Test

	Gauss Point	σ_x/F	σ_y/F	τ_{xy}/F
Element 1	1	0.166×10^{-15}	1	0.1665×10^{-15}
	2	-0.033×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	0.062×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	-0.222×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.2220×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0
Element 5	1	-0.422×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

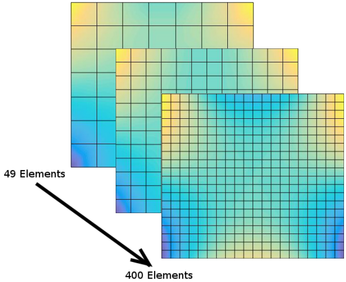
Patch Test 2: Validation of Thermoelasticity code

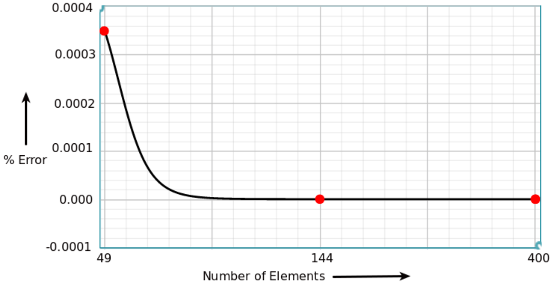


- A temperature distribution of $T = x^2 + y^2$ is applied on 4 boundaries of plate which satisfies the Poisson's equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 4$.
- Heat source $q = 4$ is applied throughout the body.
- Results are compared with the analytical solutions in the following table:

Nodes	% Error in Temperature T	% Error in displacement u_y
30	$-0.0523342237 \times 10^{-12}$	0.00034894013
33	$0.1046684475 \times 10^{-12}$	0.00082764871
63	$0.0916486147 \times 10^{-12}$	0.00015321553

Mesh Refinements

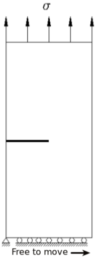




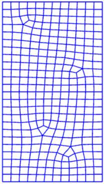
Errors obtained on node 30 after mesh refinements

Number of Elements	% Error in displacement u_x	% Error in displacement u_y
7×7	0.00034894013	0.00034894013
12×12	1.764697×10^{-7}	1.764697×10^{-7}
20×20	1.265804×10^{-8}	1.265804×10^{-8}

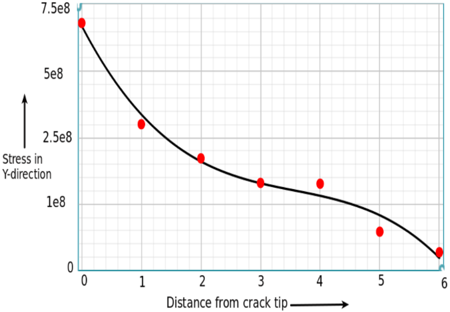
Plate with edge crack



(a) Edge crack



(b) Mesh configuration

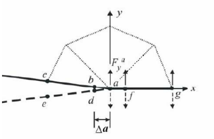


(c) σ_y ahead of the crack tip

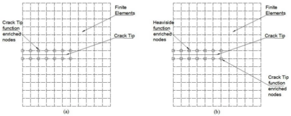
- An edge crack problem is solved applying stress σ_y = 100 Mpa and taking E=200 Gpa and ν=0.3.
- Stress intensity factor is calculated using the crack closure integral technique.
- Same problem is solved in X-FEM program developed by Parnaik and compared by our solution

source: Koustubh Parnaik. (2013). Implementation of Extended Finite Element Method (X-FEM) for static fracture problems. *Department of Mechanical Engineering, IIT Bombay*.

Comparison of FEM and X-FEM methods



(a) Crack closure technique



(b) X-FEM enrichment of nodes

Crack closure integral: $G_I = \dfrac{W}{B\Delta a} = \dfrac{F_y^a u_y^b}{2B\Delta a}$ $\quad Thus, \quad K_I = \sqrt{\dfrac{G_I}{E}} = 5.24586910 \times 10^9 \quad Pa\sqrt{m}$

Analytical: $K_I = C * \sigma\sqrt{\pi a} = 4.726065 \times 10^9 \quad Pa\sqrt{m}$

Method used	$\%\ Error = \{K_{theoretical} - K_{numerical} / K_{theoretical}\} \times 100$
Finite Element Method	11.1%
Extended Finite Element Method	4%

source:Kumar, P., & Prashant, K. (2009). Elements of fracture mechanics. Tata McGraw-Hill Education.

Conclusions and Future Work

- Conclusions:
  - Finite Element Formulation of semi-coupled thermoelasticity is performed.
  - MATLAB program is developed based on the semi-coupled formulation.
  - Patch tests were performed to validate the FEM program.
  - Results of FEM and X-FEM programs were compared and it is shown that solution improves when X-FEM is used.
- Future work:
  - Finite Element Formulation of Fully-coupled problems
  - Application of the Extended Finite Element enrichments to both displacement and temperature fields
  - Utilization of the X-FEM program in other fields which is governed by the Laplace's equation.

Thank You!