

# Thermoelastic fracture problems using Extended Finite Element Method



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# Outline

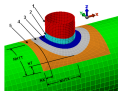
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- Introduction
- Motivation
- Literature Survey
- Work done
- Problem definition
- Conclusion



# Introduction: Thermo-elastic Fracture Mechanics

- Thermo-elastic Fracture mechanics is a field of mechanics where we study the propagation of crack in materials in presence of temperature field.
- Due to heat transfer, temperature field is set up in the material which induces thermal stresses in the body.
- These stresses becomes very large around a discontinuity i.e crack tip. If temperature variation is sufficiently large, it can lead to failure.
- Applications: Nuclear power plants, cylinder-nozzle intersection in pressure vessels, aerodynamic heating of high-speed aircraft, ultra fast pulse lasers etc.



(a) The cylinder-nozzle intersection.  
([www.knowledge.autodesk.com](http://www.knowledge.autodesk.com))



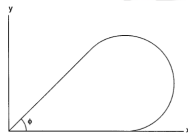
(b) Cracked head of baffle bolt of Belgian Nuclear Reactor. ([www.miningawareness.wordpress.com](http://www.miningawareness.wordpress.com))

**source:** Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

# Why thermal load on crack is important?

- Atkinson has solved the Dirichlet problem for Laplace's equation on a pie shaped region as  $u(x, y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta$ ,  $r > 0$ ,  $0 < \theta < \phi$ 
  - If  $0 < \phi < \pi$ : The first partial derivative of  $u$  with respect to  $x$  and  $y$  remains continuous as we approach towards the origin.
  - If  $\pi < \phi < 2\pi$ : The first derivative  $u$  with respect to  $x$  and  $y$  are not continuous as  $(x, y)$  approaches the origin.
- When  $\phi = 2\pi$ , the problem becomes a crack problem and displacement and derivative of displacement vary as  $u \propto r^{\frac{1}{2}}$  and  $u' \propto r^{-\frac{1}{2}}$  respectively.
- As thermo-elastic problems are also governed by Laplace's equation, temperature will vary as  $r^{\frac{1}{2}}$  and heat fluxes will be unbounded at the crack tip.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



Pie-shaped region.

# Solution of Thermoelastic Fracture Problems

- Methods for calculating stress intensity factors in FEM can be divided in two categories,
  - Substitution method
  - Energy method
- In substitution method, we can get the SIF's as:

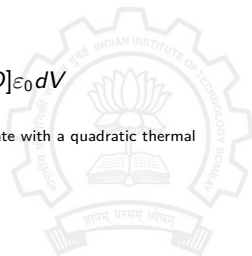
$$u = \frac{1+\nu}{4E} \sqrt{\frac{2r}{\pi}} \left\{ K_I \left[ (2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] + K_{II} \left[ (2\kappa + 3) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right] \right\}$$

A similar equation exist for  $v$  also.

- Using energy method, SIF's can be calculated as\*

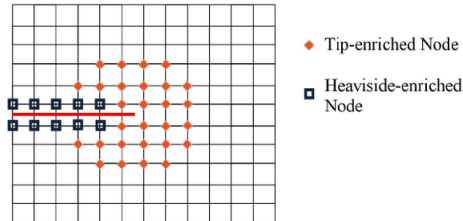
$$G = -\frac{d\Pi}{da}, \quad \Pi = -\frac{1}{2} u^T [K] u + \frac{1}{2} \int \epsilon_0^T [D] \epsilon_0 dV$$

\*source:Hellen, T. K., & Cesari, F. (1979). On the solution of the centre cracked plate with a quadratic thermal gradient. *Engineering Fracture Mechanics*, 12(4), 469-478.



# Extended Finite Element Method in Thermoelasticity

- In FEM we have to use very fine mesh to capture the behaviour of crack.
- In X-FEM, we enrich the polynomial approximation to include the effects of singular discontinuous field.
- Advantages:
  - Mesh is prepared without considering the existence of discontinuity
  - No need of remeshing
  - Accurate solution



X-FEM enrichment strategy



## Two types of enrichment functions in X-FEM

- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

$$h(x, y) = \begin{cases} 1, & \text{for } y \geq 0 \\ -1, & \text{for } y \leq 0 \end{cases}$$

- The nodes of elements which contains cracktip are enriched by  $\gamma$ :

$$u^h = \sum_i N_i(x) u_i + \sum_{j \in J} N_j(x) h(x) a_j + \sum_{k \in K} N_k(x) \left( \sum_{l=1}^4 \gamma_l(x) b_{kl} \right)$$

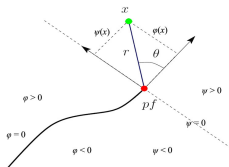
$$v^h = \sum_i N_i(x) v_i + \sum_{j \in J} N_j(x) h(x) c_j + \sum_{k \in K} N_k(x) \left( \sum_{l=1}^4 \gamma_l(x) d_{kl} \right)$$

$$\text{where, } \gamma = \left[ \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right]$$

**Source:** Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

# Which Nodes to Enrich? - Level Set Method

- There are two level-set functions defined as follows:
  - Normal level set,  $\psi(x)$  = the signed distance from the crack surface.
  - Tangent level set,  $\phi(x)$  = the signed distance to the plane including the crack front and perpendicular to the crack surface.
- To decide which enrichment should be used.
  - If  $\phi < 0$  and  $\psi_{min}\psi_{max} \leq 0$ , nodes should be enriched with  $h(x)$ .
  - If  $\phi_{min}\phi_{max} \leq 0$  and  $\psi_{min}\psi_{max} \leq 0$ , nodes should be enriched with  $\gamma$ .



Level Set Method

**Source:** Abdelaziz, Y., Bendahane, K., & Baraka, A. (2011). Extended Finite Element Modeling: Basic Review and Programming. Engineering, 3(07), 713.





# Work Done

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- Finite Element Formulation of coupled thermoelasticity
- Development of a MATLAB program for semi-coupled thermoelastic problems.
- Patch tests to validate the developed FEM program.
- Solution of various thermoelastic fracture problems and comparison with the analytical solutions.
- Comparison between the results of FEM and X-FEM programs.



# Finite Element Formulation of Thermo-elasticity

- We have formulated the semi-coupled in which we neglected the effect of displacements on temperature field.
- In thermoelastic case the total strain is given as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha (T - T_0) \delta_{ij}$$

- It can be inverted to get following stress-strain relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha \Delta T \\ \varepsilon_y - \alpha \Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$



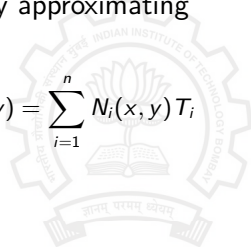
# Governing Equations of Thermoelasticity

- The governing equations of the thermo-elasticity is derived as:

$$\begin{aligned}\frac{\partial}{\partial x} \left[ c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial y} \left[ c_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] - (c_{11} + c_{12}) \alpha \frac{\partial T}{\partial x} - f_x &= 0 \\ \frac{\partial}{\partial x} \left[ c_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial y} \left[ c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y} \right] - (c_{11} + c_{12}) \alpha \frac{\partial T}{\partial y} - f_y &= 0 \\ k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) &= q\end{aligned}$$

- We can develop a weak form of above equations by approximating  $u, v$  and  $T$  over a typical finite element  $\Omega^e$  as:

$$u(x, y) = \sum_{i=1}^n N_i(x, y) u_i, \quad v(x, y) = \sum_{i=1}^n N_i(x, y) v_i, \quad T(x, y) = \sum_{i=1}^n N_i(x, y) T_i$$

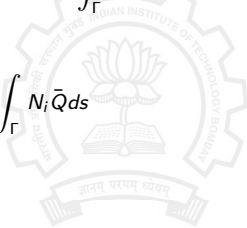


# Weak Form Equations of Coupled Thermoelasticity

$$\begin{aligned}
 - \int_{\Omega} \left[ c_{11} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} u_j + c_{12} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} v_j \right] dx dy + \int_{\Omega} \left[ c_{66} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} u_j dx dy + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} v_j \right] dx dy \\
 - \int_{\Omega} \frac{\partial N_i}{\partial x} \beta N_j T_j dx dy + \int_{\Omega} N_i f_x dx dy + \int_{\Gamma} N_i \bar{t} dx = 0
 \end{aligned}$$

$$\begin{aligned}
 - \int_{\Omega} \left[ c_{11} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} v_j + c_{12} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} u_j \right] dx dy + \int_{\Omega} \left[ c_{66} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} v_j dx dy + \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} u_j \right] dx dy \\
 - \int_{\Omega} \frac{\partial N_i}{\partial y} \beta N_j T_j dx dy + \int_{\Omega} N_i f_y dx dy + \int_{\Gamma} N_i \bar{t} dy = 0
 \end{aligned}$$

$$k \int_{\Omega} \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) T_j dx dy - \int_{\Omega} N_i N_j q dx dy = \int_{\Gamma} N_i \bar{Q} ds$$



# Finite Element Model

- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

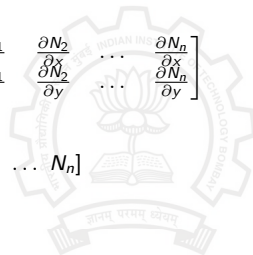
Where,

$$[K_{11}^e] = \int_{\Omega} [B]^T [C] [B] dx dy \quad [K_{12}^e] = \int_{\Omega} [B]^T [\beta] [N^{\theta}] dx dy \quad [K_{22}^e] = \int_{\Omega} [B^{\theta}]^T [K] [B^{\theta}] dx dy$$

$$\{F\} = \int_{\Gamma} [N]^T \bar{t} ds \quad \{Q\} = \int_{\Gamma} [N^{\theta}]^T \bar{Q} ds$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}, \quad [B^{\theta}] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}, \quad N^{\theta} = [N_1 \quad \dots \quad N_n]$$



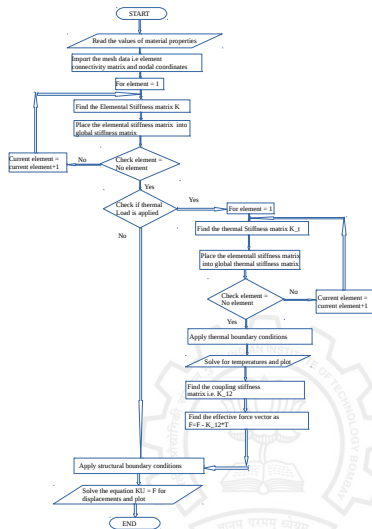
# Computer implementation

A MATLAB program is developed to solve the 2-dimensional thermoelasticity problems.

Quadrilateral (Q4) elements were used for meshing the body.

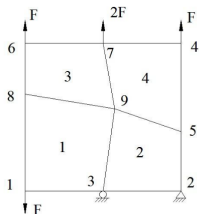
$2 \times 2$  Gauss quadrature rule is used for numerical integration.

A flow-chart showing the steps of the programming is shown in the figure.



# Patch Test 1

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using  $2 \times 2$  Gauss quadrature rule.



(a) Mesh Configuration

Node (x,y)

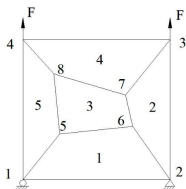
- 1 (0,0)
- 2 (2,0)
- 3 (1,0)
- 4 (2,2)
- 5 (2,0.8)
- 6 (0,2)
- 7 (1,2)
- 8 (0,1.4)
- 9 (1.2,1.2)

(b) Results of Patch Test 1

	Gauss Points	$\sigma_x/2F$	$\sigma_y/2F$	$\tau_{xy}/2F$
Element 1	1	$-0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.2498 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	3	$0.3058 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	4	0	1	0
Element 2	1	$-0.222 \times 10^{-15}$	1	0
	2	$-0.41633 \times 10^{-15}$	1	$0.111 \times 10^{-15}$
	3	$0.02775 \times 10^{-15}$	1	$0.138 \times 10^{-15}$
	4	$-0.1110 \times 10^{-15}$	1	0
Element 3	1	$0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.0555 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	3	$-0.0555 \times 10^{-15}$	1	0
	4	$0.22204 \times 10^{-15}$	1	0
Element 4	1	$-0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	3	$0.3058 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	4	$0.222 \times 10^{-15}$	1	0

## Patch Test 2

- Another patch test is performed on a plate with 5 element mesh and solved and results are tabulated as shown below
- The  $2 \times 2$  quadrature rule is used for numerical integration



(a) Mesh Configuration

Node (x,y)
1 (0,0)
2 (2,0)
3 (2,2)
4 (0,2)
5 (0.85,0.7)
6 (1.5,1)
7 (1.2,1.4)
8 (0.7,1.6)

(b) Results of Patch Test 2

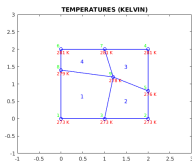
	Gauss Point	$\sigma_x/F$	$\sigma_y/F$	$\tau_{xy}/F$
Element 1	1	$0.166 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$-0.033 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	3	$-0.063 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	4	0	1	0
Element 2	1	$0.062 \times 10^{-15}$	1	0
	2	$-0.41633 \times 10^{-15}$	1	$-0.222 \times 10^{-15}$
	3	$0.02775 \times 10^{-15}$	1	$0.138 \times 10^{-15}$
	4	$-0.2220 \times 10^{-15}$	1	0
Element 3	1	$0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.0555 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	3	$-0.0555 \times 10^{-15}$	1	0
	4	$0.22204 \times 10^{-15}$	1	0
Element 4	1	$-0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	3	$-0.063 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	4	$0.222 \times 10^{-15}$	1	0
Element 5	1	$-0.422 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	2	$0.222 \times 10^{-15}$	1	$0.1665 \times 10^{-15}$
	3	$-0.063 \times 10^{-15}$	1	$0.222 \times 10^{-15}$
	4	$0.222 \times 10^{-15}$	1	0



## Patch Test 3 : Thermal code

- Another patch was performed to validate the thermal code.
- Same problem as patch test was taken
- Temperature loads are applied to get the constant heat flux throughout the body

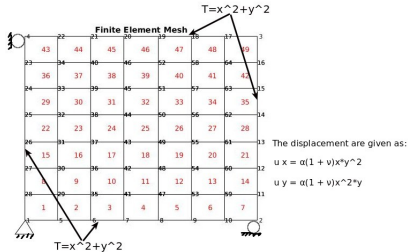
(b) Results of Patch test 3



(a) Temperature distribution

Nodes	Temperatures	$Q_y (W)$	$Q_x (W)$
1	273	200	$0.125 \times 10^{-10}$
2	273	200	$0.155 \times 10^{-10}$
3	273	200	$0.222 \times 10^{-10}$
4	281	200	0
5	276	200	0
6	281	200	0
7	281	200	$0.111 \times 10^{-10}$
8	279	200	$0.138 \times 10^{-10}$
9	278	200	0

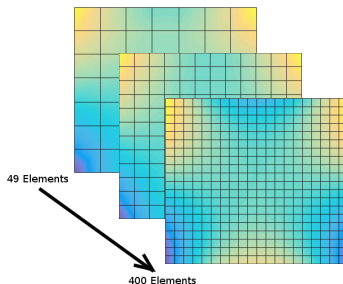
## Patch Test 4: Thermo-elastic code



- A square plate of dimension  $2 \times 2$  is taken and meshed as shown
- Temperature distribution of  $T = x^2 + y^2$  is applied on 4 boundaries of plate
- a heat source  $q = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 4$  is applied throughout the body
- Mesh is refined and to get the more accurate results



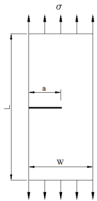
# Mesh Refinements



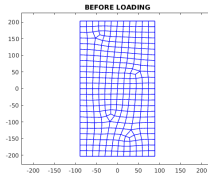
Errors obtained after mesh refinements

Number of Elements	Error in $u_x$	Error in $u_y$	Error in $\sigma_x$	Error in $\sigma_y$	Error in $\tau_{xy}$
$7 \times 7$	$5.783469 \times 10^{-10}$	$5.783469 \times 10^{-10}$	$6.668255 \times 10^2$	$6.668255 \times 10^2$	$3.615391 \times 10^2$
$12 \times 12$	$1.764697 \times 10^{-11}$	$1.764697 \times 10^{-11}$	1.000597121	1.000597121	0.031722
$20 \times 20$	$1.265804 \times 10^{-12}$	$1.265804 \times 10^{-12}$	0.994299467	0.994299467	0.005781

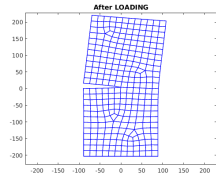
# Plate with edge crack



(a) Plate with edge crack



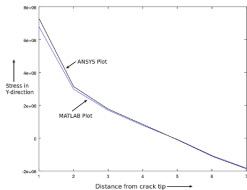
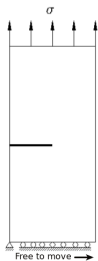
(b) Before loading



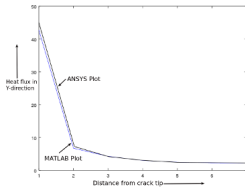
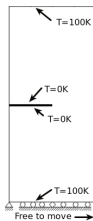
(c) After loading

- A plate with edge crack is meshed in ANSYS and imported to MATLAB
- Boundary conditions are applied and stresses ahead of cracktip is plotted
- The results obtained by MATLAB code is compared with the ANSYS solution

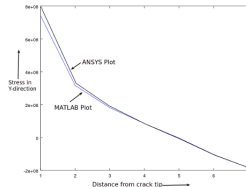
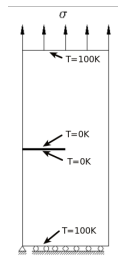
# Stresses Ahead of Crack-tip: Comparison with ANSYS



(a)  $\sigma_y$  ahead of the crack tip

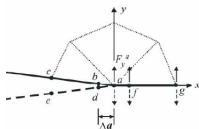


(b) Heat flux ahead of crack tip

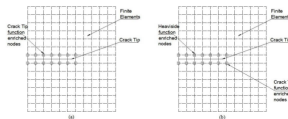


(c)  $\sigma_y$  ahead of the crack tip

# Comparison of FEM and X-FEM methods



(a) Crack closure technique



(b) X-FEM enrichment of nodes

Crack closure integral:  $G_I = \frac{W}{B\Delta a} = \frac{F_y^a u_y^b}{2B\Delta a}$  Thus,  $K_I = \sqrt{\frac{G_I}{E}} = 5.24586910 \times 10^9 \text{ Pa}\sqrt{\text{m}}$

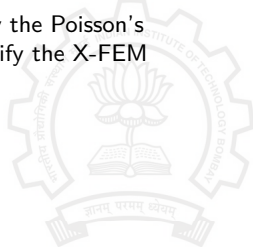
Analytical:  $K_I = C * \sigma \sqrt{\pi a} = 4.726065 \times 10^9 \text{ Pa}\sqrt{\text{m}}$

Method used	% Error = $\{K_{theoretical} - K_{numerical} / K_{theoretical}\} \times 100$
Finite Element Method	11.1%
Extended Finite Element Method	4%

# Conclusions

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- A semi-coupled thermoelastic problem is formulated, FEM program is developed in MATLAB and validated by performing different patch tests.
- It is shown that when solving fracture problems with traditional FEM program, solutions are not accurate around the crack tip
- Same problem is solved in an XFEM program developed by parnaik to show that the solution improves drastically when using XFEM
- The Extended Finite Element enrichments will be applied to both displacement and temperature fields of thermoelasticity program in stage 2 of the project
- Q8 elements will be applied in our program to solve curved boundary problems.
- As the hydrogen diffusion in cracked bodies are governed by the Poisson's equation similar to the thermoelastic problems, we will modify the X-FEM program to solve problems of hydrogen-diffusion



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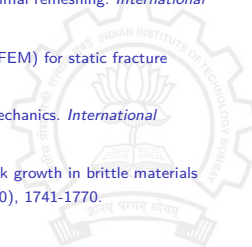
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