Thermoelastic fracture problems using Extended Finite Element Method

By
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Outline

- Introduction
- Motivation
- Literature Survey
- Objectives
- Computer implementation
- Example problems
- Conclusion and future work

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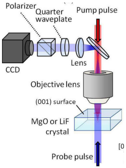

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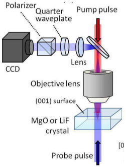

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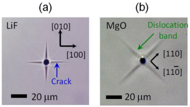

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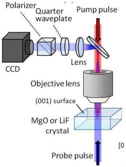

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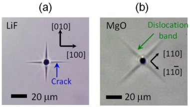

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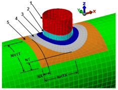

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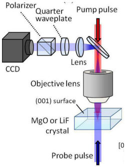

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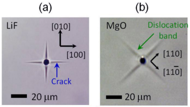

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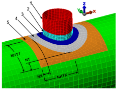

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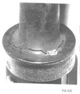

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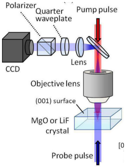

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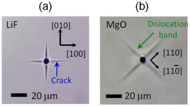

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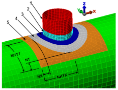

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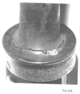

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source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

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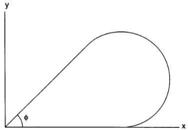

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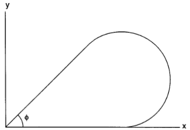

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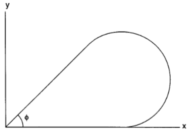

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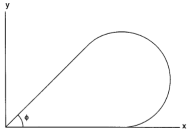

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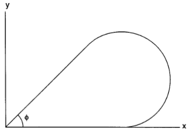

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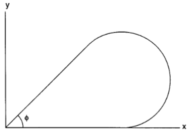

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- Thus, around the crack tip, thermal stresses also has square root singularity : our motivation for analyzing a thermo-elastic crack problem.

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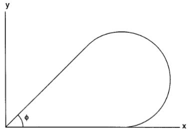

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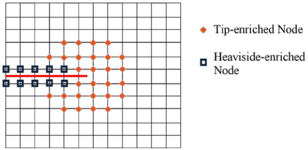

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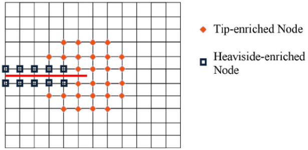

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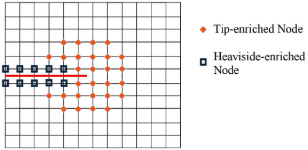

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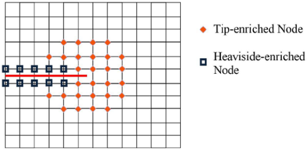

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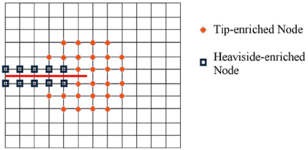

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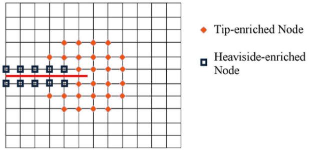

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Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

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$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

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- The governing equations of the thermoelasticity are derived as follows:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial x} - f_x = 0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}\right] - (c_{11}+c_{12})\alpha\frac{\partial T}{\partial y} - f_y = 0$$

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = q$$

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$$-\int_{\Omega}\left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_{\Omega}\left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$
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$$k\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_\Omega N_i N_j q dxdy = \int_\Gamma N_i \bar{Q} ds$$

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- Neglecting the body forces, above equations can be written in matrix form as:

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Where,

$$[K_{11}^e] = \int_\Omega [B]^T[C][B]dxdy \quad [K_{12}^e] = \int_\Omega [B]^T[\beta][N^\theta]dxdy \quad [K_{22}^e] = \int_\Omega [B^\theta]^T[K][B^\theta]dxdy$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \quad \{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds$$

Finite Element Model

- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

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- Computer implementation:
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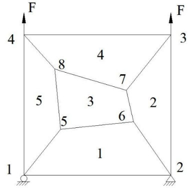
  - 2×2 Gauss quadrature rule is used for numerical integration

Patch Test 1: Validation of Elasticity code

Patch Test 1: Validation of Elasticity code

- A square plate is taken and meshed with 5 elements as shown in the figure below.



Node	(x,y)
1	(0,0)
2	(2,0)
3	(2,2)
4	(0,2)
5	(0.85,0.7)
6	(1.5,1)
7	(1.2,1.4)
8	(0.7,1.6)

(a) Mesh Configuration

Patch Test 1: Validation of Elasticity code

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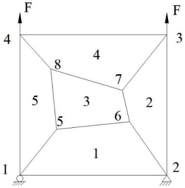
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions



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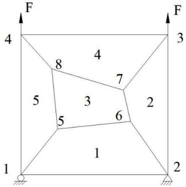

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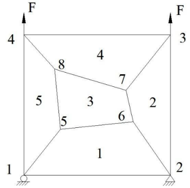

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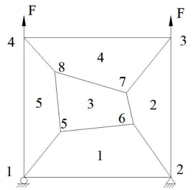

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(a) Mesh Configuration

(b) Results of Patch Test

	Gauss Point	σ_x/F	σ_y/F	τ_{xy}/F
Element 1	1	0.166×10^{-15}	1	0.1665×10^{-15}
	2	-0.033×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	0.062×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	-0.222×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.2220×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0
Element 5	1	-0.422×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 2: Validation of Thermoelasticity code

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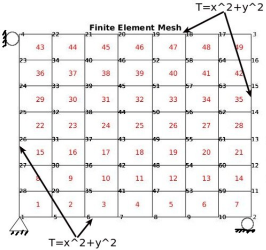
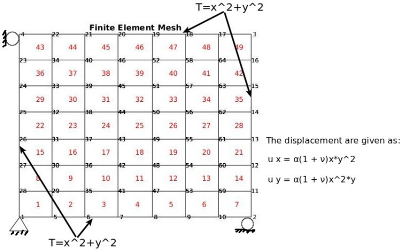


The displacement are given as:

$u_x = \alpha(1+\nu)x*y^2$

$u_y = \alpha(1+\nu)x^2*y$

Patch Test 2: Validation of Thermoelasticity code



- A temperature distribution of $T = x^2 + y^2$ is applied on 4 boundaries of plate which satisfies the Poisson's equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 4$.

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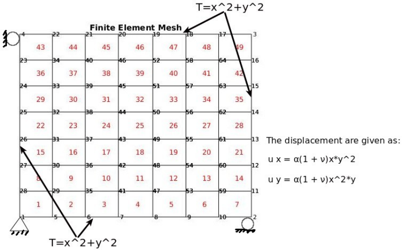

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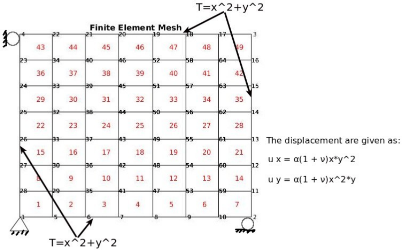

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- Heat source $q = 4$ is applied throughout the body.
- Results are compared with the analytical solutions in the following table:

Nodes	% Error in Temperature T	% Error in displacement u_y
30	$-0.0523342237 \times 10^{-12}$	0.00034894013
33	$0.1046684475 \times 10^{-12}$	0.00082764871
63	$0.0916486147 \times 10^{-12}$	0.00015321553

Mesh Refinements

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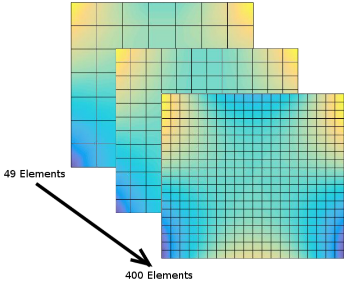

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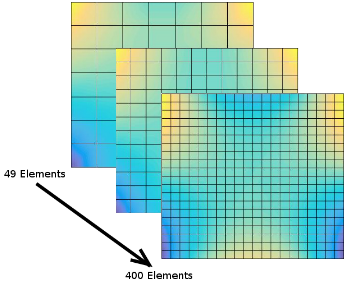

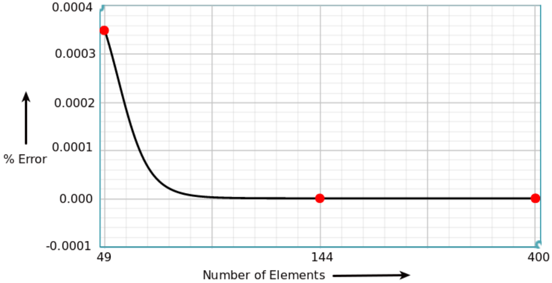

Errors obtained on node 30 after mesh refinements

Number of Elements	% Error in displacement u_x	% Error in displacement u_y
7×7	0.00034894013	0.00034894013
12×12	1.764697×10^{-7}	1.764697×10^{-7}
20×20	1.265804×10^{-8}	1.265804×10^{-8}

Plate with edge crack

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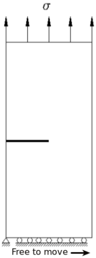
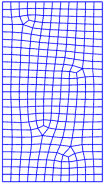


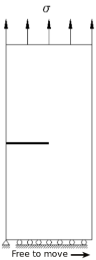
(a) Edge crack



(b) Mesh configuration

- An edge crack problem is solved applying stress $\sigma_y = 100$ Mpa and taking E=200 Gpa and ν=0.3.

Plate with edge crack



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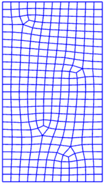

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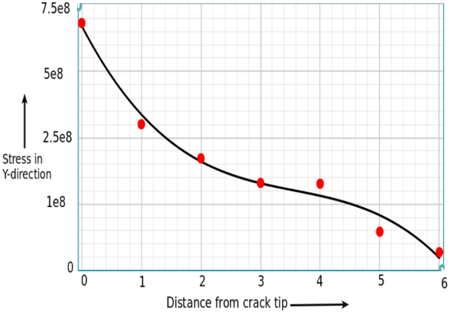

(c) σ_y ahead of the crack tip

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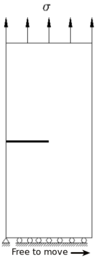

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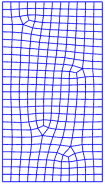

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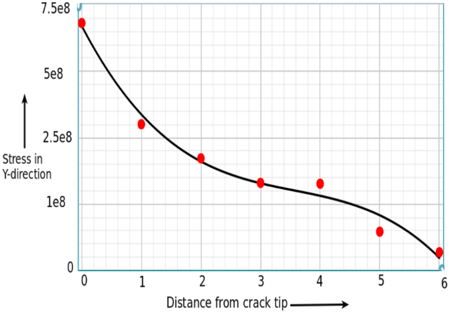

(c) σ_y ahead of the crack tip

- An edge crack problem is solved applying stress $\sigma_y = 100$ Mpa and taking E=200 Gpa and ν=0.3.
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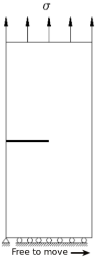
Plate with edge crack



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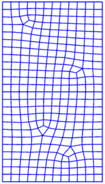

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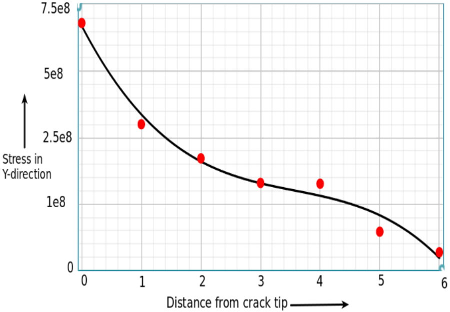

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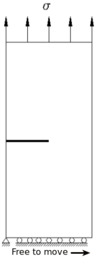

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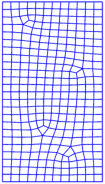

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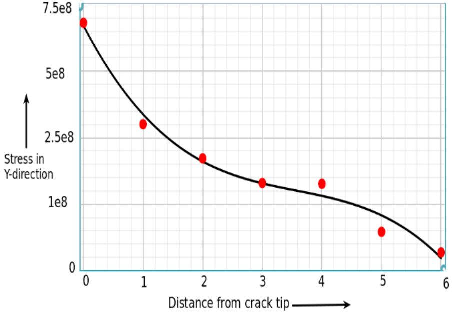

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source: Koustubh Parnaik. (2013). Implementation of Extended Finite Element Method (X-FEM) for static fracture problems. *Department of Mechanical Engineering, IIT Bombay*.

Comparison of FEM and X-FEM methods

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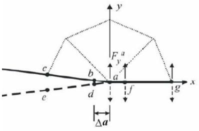

(a) Crack closure technique

Crack closure integral: $$G_I = \frac{W}{B\Delta a} = \frac{F_y^a u_y^b}{2B\Delta a}$$

source:Kumar, P., & Prashant, K. (2009). Elements of fracture mechanics. Tata McGraw-Hill Education.

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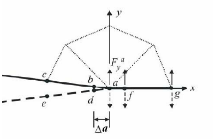

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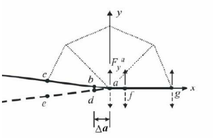

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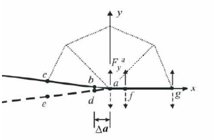

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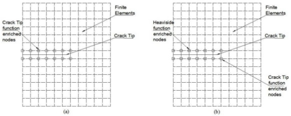

(b) X-FEM enrichment of nodes

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Method used	$\%\ Error = \{K_{theoretical} - K_{numerical} / K_{theoretical}\} \times 100$
Finite Element Method	11.1%
Extended Finite Element Method	4%

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  - Finite Element Formulation of Fully-coupled problems
  - Application of the Extended Finite Element enrichments to both displacement and temperature fields
  - Utilization of the X-FEM program in other fields which is governed by the Laplace's equation.

Thank You!