Thermoelastic fracture problems using Extended Finite Element Method

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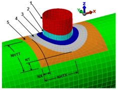
Outline

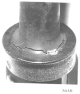
- Introduction
- Motivation
- Literature Survey
- Work done
- Problem definition
- Conclusion

Introduction: Thermo-elastic Fracture Problems

- Due to heat transfer, temperature field is set up in the material which induces thermal stresses in the body.
- These stresses becomes large in the vicinity of a discontinuity i.e. crack tip. If temperature variation is sufficiently large, it can lead to failure.
- Applications: Nuclear power plants, cylinder-nozzle intersection in pressure vessels, aerodynamic heating of high-speed aircraft, ultra fast pulse lasers etc.



(a) The cylinder-nozzle intersection. (www.knowledge.autodesk.com)



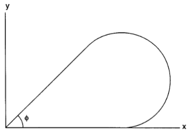
(b) Cracked head of baffle bolt of Belgian Nuclear Reactor.(www.miningawareness.wordpress.com)

source: Tian, X., Shen. (2006). A direct finite element method study of generalized thermoelastic problems. *International Journal of Solids and Structures*, 43(7), 2050-2063.

Why thermal load on crack is important?

- Atkinson has solved the Dirichlet problem for Laplace's equation on a pie shaped region as $u(x, y) = r^{\frac{\pi}{\phi}} \sin \alpha\theta,\ r > 0,\ 0 < \theta < \phi$
  - If $0 < \phi < \pi$: The first partial derivative of u with respect to x and y remains continuous as we approach towards the origin.
  - If $\pi < \phi < 2\pi$: The first derivative u with respect to x and y are not continuous as (x,y) approaches the origin.
- When $\phi = 2\pi$, the problem becomes a crack problem and displacement and derivative of displacement vary as $u \propto r^{\frac{1}{2}}$ and $u' \propto r^{-\frac{1}{2}}$ respectively.
- As thermo-elastic problems are also governed by Laplace's equation, temperature will vary as $r^{\frac{1}{2}}$ and heat fluxes will be unbounded at the crack tip.

source: Atkinson, K. E. (1997). *The numerical solution of integral equations of the second kind* (Vol. 4). Cambridge university press.



Pie-shaped region.

Finite Element Methods for Solving Thermoelastic Fracture Problems

- Methods for calculating stress intensity factors in FEM can be divided in two categories,
  - Substitution method
  - Energy method
- In substitution method, we can get the SIF's as:

$$u = \frac{1+\nu}{4E}\sqrt{\frac{2r}{\pi}}\left\{K_I\left[(2\kappa-1)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]+K_{II}\left[(2\kappa+3)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]\right\}$$

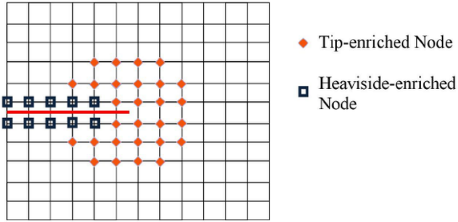
  A similar equation exist for v also.
- Using energy method, SIF's can be calculated as*

$$G = -\frac{d\Pi}{da}\ , \qquad \Pi = -\frac{1}{2}u^T[K]u + \frac{1}{2}\int {\varepsilon_0}^T[D]\varepsilon_0 dV$$

* **source**:Hellen, T. K., & Cesari, F. (1979). On the solution of the centre cracked plate with a quadratic thermal gradient. *Engineering Fracture Mechanics*, 12(4), 469-478.

Extended Finite Element Method in Thermoelasticity

- We have to use very fine mesh at capture the behaviour of crack.
- In XFEM, we enrich the polynomial approximation to include the effects of singular discontinuous field.
- Advantages:
  - Mesh is prepared without considering the existence of discontinuity.
  - No need of remeshing.



XFEM enrichment strategy

Two types of enrichment functions in XFEM

- The nodes which belongs to the elements totally cut by the crack, are enriched by and Heaviside function.

$$h(x,y) = \begin{cases} 1, & \textit{for} \quad y \geq 0 \\ -1, & \textit{for} \quad y \leq 0 \end{cases}$$

- The nodes of elements which contains cracktip are enriched by γ:

$$u^h = \sum_i N_i(x) u_i + \sum_{j \in J} N_j(x) h(x) a_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) b_{kl} \right)$$

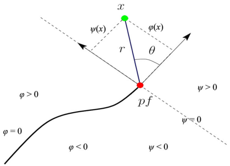
$$v^h = \sum_i N_i(x) v_i + \sum_{j \in J} N_j(x) h(x) c_j + \sum_{k \in K} N_k(x) \left(\sum_{l=1}^{4} \gamma_l(x) d_{kl} \right)$$

$$\textit{where}, \quad \gamma = \left[\sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right]$$

Source: Belytschko, T., & Black, T. (1999). Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5), 601-620.

Which Nodes to Enrich? - Level Set Method

- There are two level-set functions defined as follows:
  - Normal level set, $\psi(x)$ = the signed distance from the crack surface.
  - Tangent level set, $\phi(x)$ = the signed distance to the plane including the crack front and perpendicular to the crack surface.
- To decide which enrichment should be used.
  - If $\phi < 0$ and $\psi_{min}\psi_{max} \leq 0$, nodes should be enriched with $h(x)$.
  - If $\phi_{min}\phi_{max} \leq 0$ and $\psi_{min}\psi_{max} \leq 0$, nodes should be enriched with γ.



Level Set Method

Source: Abdelaziz, Y., Bendahane, K., & Baraka, A. (2011). Extended Finite Element Modeling: Basic Review and Programming. Engineering, 3(07), 713.

Work Done

- Finite Element Formulation of coupled thermoelasticity
- Development of a MATLAB program for semi-coupled thermoelastic problems.
- Patch tests to validate the developed FEM program.
- Solution of various thermoelastic fracture problems and comparison with the analytical solutions.
- Comparison between the solutions of FEM and XFEM programs.

Finite Element Formulation of Thermo-elasticity

- We will formulate the semi-coupled in which we neglect the effect of displacements on temperature field.
- In thermoelastic case the total strain is given as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(M)} + \varepsilon_{ij}^{(T)} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

- Which can be inverted to get following stress-strain relationship:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha\Delta T \\ \varepsilon_y - \alpha\Delta T \\ \varepsilon_{xy} \end{Bmatrix}$$

Governing Equations of Thermoelasticity

- The governing equations of the thermo-elasticity is derived as:

$$\frac{\partial}{\partial x}\left[c_{11}\frac{\partial u}{\partial x}+c_{12}\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial y}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial x}-f_x=0$$

$$\frac{\partial}{\partial x}\left[c_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[c_{12}\frac{\partial u}{\partial x}+c_{22}\frac{\partial v}{\partial y}\right]-(c_{11}+c_{12})\alpha\frac{\partial T}{\partial y}-f_y=0$$

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=q$$

- We can develop a week form of above equations by approximating u,v and T over a typical finite element Ω^e as:

$$u(x,y)=\sum_{i=1}^{n}N_i(x,y)u_i \quad ,v(x,y)=\sum_{i=1}^{n}N_i(x,y)v_i \quad ,T(x,y)=\sum_{i=1}^{n}N_i(x,y)T_i$$

Weak Form Equations of Coupled Thermoelasticity

$$-\int_\Omega \left[c_{11}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}u_j + c_{12}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}v_j\right]dxdy + \int_\Omega \left[c_{66}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}u_j dxdy + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}v_j\right]dxdy$$
$$-\int_\Omega \frac{\partial N_i}{\partial x}\beta N_j T_j dxdy + \int_\Omega N_i f_x dxdy + \int_\Gamma N_i \vec{t} dx = 0$$

$$-\int_\Omega \left[c_{11}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}v_j + c_{12}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}u_j\right]dxdy + \int_\Omega \left[c_{66}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}v_j dxdy + \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}u_j\right]dxdy$$
$$-\int_\Omega \frac{\partial N_i}{\partial y}\beta N_j T_j dxdy + \int_\Omega N_i f_y dxdy + \int_\Gamma N_i \vec{t} dy = 0$$

$$k\int_\Omega \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j dxdy - \int_\Omega N_i N_j q dxdy = \int_\Gamma N_i \bar{Q} ds$$

Finite Element Model

- Neglecting the body forces, above equations can be written in matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} U^e \\ T^e \end{Bmatrix} = \begin{Bmatrix} F \\ Q \end{Bmatrix}$$

Where,

$$[K_{11}^e] = \int_\Omega [B]^T [C][B] dxdy \quad [K_{12}^e] = \int_\Omega [B]^T [\beta][N^\theta] dxdy \quad [K_{22}^e] = \int_\Omega [B^\theta]^T [K][B^\theta] dxdy$$

$$\{F\} = \int_\Gamma [N]^T \bar{t} ds \quad \{Q\} = \int_\Gamma [N^\theta]^T \bar{Q} ds$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}, \quad [B^\theta] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \dots & 0 & N_n \end{bmatrix}, \quad N^\theta = [N_1 \ \dots \ N_n]$$

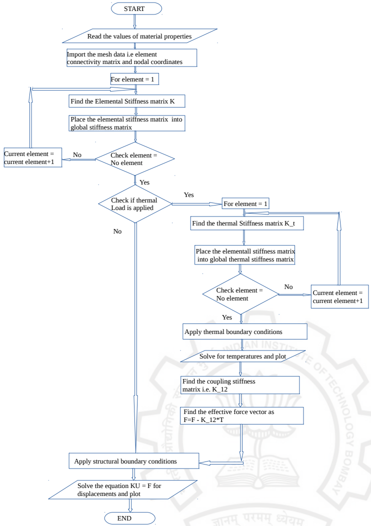
Computer implementation

A MATLAB program
is developed to solve the 2-dimensional thermoelasticity problems.

Quadrilateral (Q4) elements
were used for meshing the body.

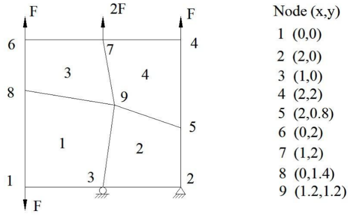
2×2 Gauss quadrature
rule is used for numerical integration.

A flow-chart showing the steps of the programming is shown in the figure.



Patch Test 1

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.



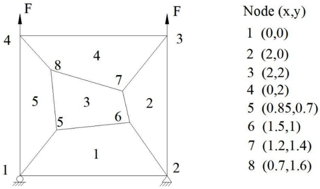
(a) Mesh Configuration

(b) Results of Patch Test 1

	Gauss Points	$\sigma_x/2F$	$\sigma_y/2F$	$\tau_{xy}/2F$
Element 1	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.2498×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	-0.222×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	0.111×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.1110×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	0.3058×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0

Patch Test 2

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.



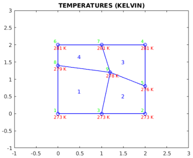
(a) Mesh Configuration

(b) Results of Patch Test 2

	Gauss Point	σ_x/F	σ_y/F	τ_{xy}/F
Element 1	1	0.166×10^{-15}	1	0.1665×10^{-15}
	2	-0.033×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0	1	0
Element 2	1	0.062×10^{-15}	1	0
	2	$-.41633 \times 10^{-15}$	1	-0.222×10^{-15}
	3	0.02775×10^{-15}	1	0.138×10^{-15}
	4	-0.2220×10^{-15}	1	0
Element 3	1	0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.0555×10^{-15}	1	0.222×10^{-15}
	3	-0.0555×10^{-15}	1	0
	4	0.22204×10^{-15}	1	0
Element 4	1	-0.222×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}
	4	0.222×10^{-15}	1	0
Element 5	1	-0.422×10^{-15}	1	0.1665×10^{-15}
	2	0.222×10^{-15}	1	0.1665×10^{-15}
	3	-0.063×10^{-15}	1	0.222×10^{-15}

Patch Test for thermal code

- A square plate is taken and meshed with 4 elements as shown in figure below.
- Minimum number of essential boundary conditions is fixed to eliminate the rigid body motions
- Loads are applied such that there is constant state of stress in the body
- Numerical integration is performed using 2×2 Gauss quadrature rule.

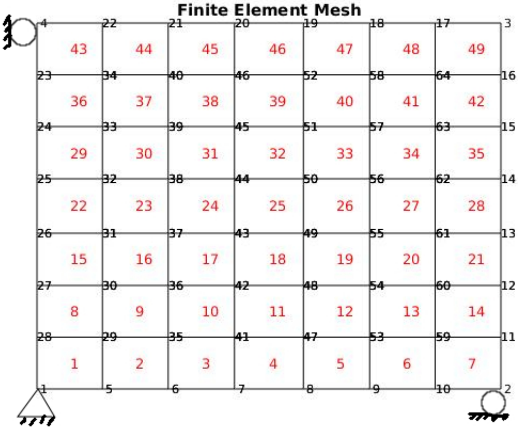


(a) Temperature distribution

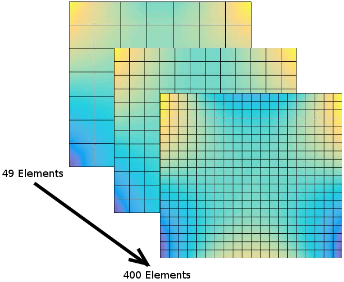
(b) Results of Patch test 3

Nodes	Temperatures	$Q_y(W)$	$Q_x(W)$
1	273	200	0.125×10^{-10}
2	273	200	0.155×10^{-10}
3	273	200	0.222×10^{-10}
4	281	200	0
5	276	200	0
6	281	200	0
7	281	200	0.111×10^{-10}
8	279	200	0.138×10^{-10}
9	278	200	0

Patch Test for Thermo-elastic code: Poisson's equation



Mesh Refinements



Errors obtained after mesh refinements

Number of Elements	Error in u_x	Error in u_y	Error in σ_x	Error in σ_y	Error in τ_{xy}
7×7	5.783469×10^{-10}	5.783469×10^{-10}	6.668255×10^{2}	6.668255×10^{2}	3.615391×10^{2}
12×12	1.764697×10^{-11}	1.764697×10^{-11}	1.000597121	1.000597121	0.031722
20×20	1.265804×10^{-12}	1.265804×10^{-12}	0.994299467	0.994299467	0.005781

Plate with edge crack

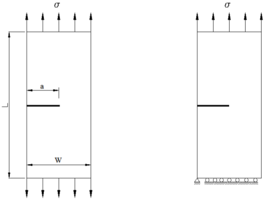
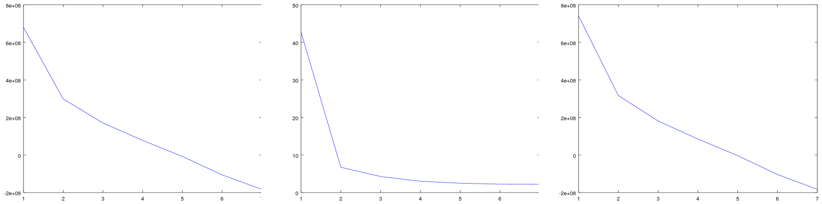


Plate with edge crack



(a) σ_y ahead of the crack tip (b) Heat flux ahead of crack tip (c) σ_y ahead of the crack tip