Mathematics Review (2)

CS-585

Natural Language Processing

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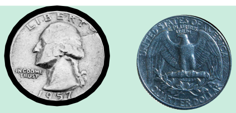
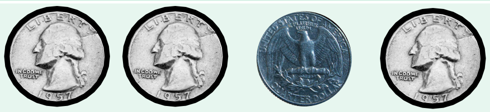
INFORMATION THEORY REVIEW

Information Theory

- Developed in the 1940s by Claude Shannon
- Concerned with the optimal compression of information for communication over a channel with limited capacity
- Basic measure of information is *bits*—the number of binary 1/0 indicators used to encode a value

Information content / Bits

		Bits
	$\begin{bmatrix}\mathbf{0}\\1\end{bmatrix}\begin{bmatrix}0\\\mathbf{1}\end{bmatrix}$	2
	$\begin{bmatrix}\mathbf{0}\\1\end{bmatrix}\begin{bmatrix}\mathbf{0}\\1\end{bmatrix}\begin{bmatrix}0\\\mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\1\end{bmatrix}$	4
	$\begin{bmatrix}\mathbf{1}\\2\\3\\4\\5\\6\end{bmatrix}\begin{bmatrix}1\\2\\\mathbf{3}\\4\\5\\6\end{bmatrix}$	$2\log_2 6$
	$\begin{bmatrix}1\\\vdots\\20\end{bmatrix}\begin{bmatrix}1\\\vdots\\20\end{bmatrix}\begin{bmatrix}1\\\vdots\\20\end{bmatrix}\begin{bmatrix}1\\\vdots\\20\end{bmatrix}\begin{bmatrix}1\\\vdots\\20\end{bmatrix}\begin{bmatrix}1\\\vdots\\20\end{bmatrix}$	$6\log_2 20$

Information content / Bits

- Generally, the information content or optimal code length of an event drawn from a distribution with N equiprobable outcomes is

$$-\log_2 \frac{1}{N} = \log_2 N \text{ bits}$$

- The information content of an event e drawn from a distribution P(X) over a discrete random variable X is

$$-\log_2 P(X = e) \text{ bits}$$

Bits and nats

- In information theory, we generally use base-2 logs because it makes information values interpretable as the number of 0/1 bits of information we use to encode data for computers

$$-\log_2 P(X = e) \text{ bits}$$

- But an alternative unit using the natural logarithm is *nats*:

$$-\ln P(X = e) \text{ nats}$$

- To convert from bits to nats, divide by $\log_2 e$:

$$-\log_2 P = -\log_2 e^{\ln P}$$

$$-\log_2 P = -\ln P \times \log_2 e$$

$$-\frac{\log_2 P}{\log_2 e} = -\ln P$$

Entropy

- Entropy (self-information) of a discrete random variable X is

$$H(X) = H(P(X)) = -E[\log_2 P(X)]$$
$$= -\sum_{x \in X} P(X = x) \log_2 P(X = x)$$

- Optimal code length for $X = x$:

$$-\log_2 P(X = x) \text{ (bits)}$$
$$-\ln P(X = x) \text{ (nats)}$$

Entropy example

$$\begin{aligned} H(\langle 0.5, 0.5\rangle) &= -E[\log_2 \langle 0.5, 0.5\rangle] \\ &= -\frac{1}{2}\log_2(0.5) - \frac{1}{2}\log_2(0.5) \\ &= 1 \end{aligned}$$

$$\begin{aligned} H\left(\left\langle \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right\rangle\right) &= -E\left[\log_2 \left\langle \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right\rangle\right] \\ &= -\sum_{1}^{6} \frac{1}{6}\log_2\left(\frac{1}{6}\right) \\ &= \log_2 6 \end{aligned}$$

Entropy example

$$H(\langle .1, .7, .15, .05\rangle) = -E[\log_2\langle .1, .7, .15, .05\rangle]$$

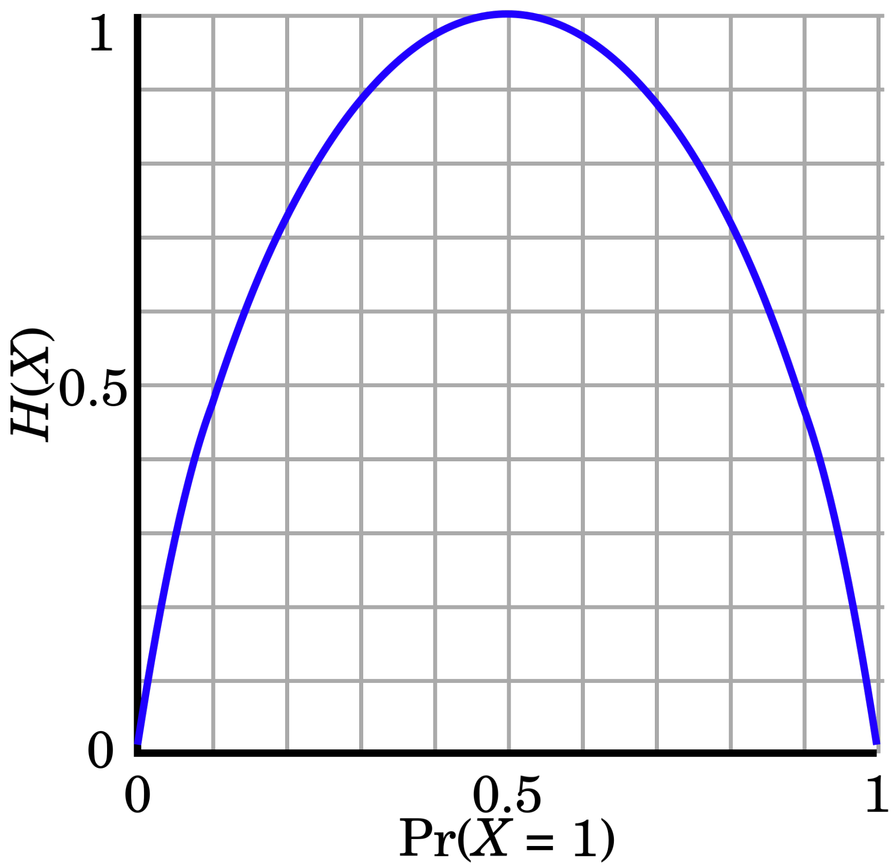
$$= \begin{matrix} -.1\log_2(.1) - .7\log_2(.7) \\ -.15\log_2(.15) - .05\log_2(.05) \end{matrix}$$

$$= .33 + .36 + .41 + .22 = 1.32$$

- Lower entropy than we would get for a uniform distribution $\langle 0.25, 0.25, 0.25, 0.25\rangle$ (which would be 2 bits)

Entropy of a weighted coin

- Think of entropy as uncertainty
- For a Bernoulli distribution, the uncertainty is maximized when both outcomes are equiprobable



https://en.wikipedia.org/wiki/Entropy_(information_theory)#/media/File:Binary_entropy_plot.svg

The Entropy of English

- We can think of a language as an orthographic symbol generation process governed by some unknown probability distribution $P_{lang}(X)$
- What is $H(P_{lang}(X))$?
- How uncertain/unpredictable is the next symbol in a text from a given language?

The Entropy of English: Upper Bounds

- We can calculate an upper bound on the entropy of a language like English by
  - Using a model to estimate the probabilities of symbols in the language
  - Calculating the average code length that an encoding based on this probability distribution would assign to symbols that actually occur in the language
- Model 1: unigram probabilities

$$P(X = x \in \{a - z, \_\}) = \frac{Count(x)}{Count(y \in \{a - z, \_\})}$$

- Unigram probabilities give us an estimate of $\hat{H}(X) = 4.03$ bits per letter for English

The Entropy of English: Upper Bounds

- N-gram probabilities

$$P(X = x \in \{a - z, \_\} | \mathcal{H}(X)) = \frac{Count(x, \mathcal{H})}{Count(y \in \{a - z, \_\}, \mathcal{H})}$$

- Bigram probabilities give us an estimate of $\hat{H}(X) = 2.8$ bits per letter for English
- Neural language models give us estimates of under 1.5 bits per letter.
- Experimental estimates put the "true" entropy of English at about 1.3 bits per character
  - https://www.princeton.edu/~wbialek/rome/refs/shannon_51.pdf

Optimal Coding

- We know that the optimal code length for message m drawn from distribution X is $\log P(X = \mathrm{m})$, but how to construct code that approximates this bound?
- Multiple algorithms:
  - Huffman coding
  - Arithmetic coding
  - Hu-Tucker coding

Huffman Coding

1. Initialize queue **Q** with pairs $\phi_i = [s_i, p_i]$, where *s* is a symbol from the vocabulary, and *p* is its associated probability
2. While $\|Q\| > 1$
   1. Remove the two lowest-probability elements ϕ_j and ϕ_k from Q, and create a new pair $\phi_{\|Q\|+1} = [\langle s_j, s_k \rangle, p_j + p_k]$
   2. Add the new node to **Q**
3. The remaining pair in the queue contains a binary tree that can be used to assign codes

- [Notebook]

Cross-entropy and perplexity

- If *entropy* is the information in bits required to represent a message using an optimal encoding derived from the **true** distribution…
- *Cross-entropy* is the information in bits required to represent a message using an optimal encoding derived from a **different** distribution
- We encountered this already when bounding the entropy of English
- Cross-entropy is always an upper bound on the entropy

Cross-entropy and perplexity

- The cross-entropy between two distributions *P* and *Q* (where *Q* is often a model of the true distribution *P*) is

$$H\big(P(X), Q(X)\big) = -E_{P(X)}[\log_2 Q(X)]$$
$$= -\sum_{x \in X} P(X = x) \log_2 Q(X = x)$$

- This is the expected number of bits required to encode messages from P using an encoding system from Q, and is <u>not</u> symmetric

$$H(P, Q) \neq H(Q, P)$$
$$H(P, Q) \geq H(P)$$

Cross-entropy and perplexity

- Many speech recognition and language modeling tasks use *perplexity*, rather than cross-entropy as an evaluation measure

$$Perplexity\big(P(X),Q(X)\big) = 2^{H(P(X),Q(X))}$$

- For a sequence of observations (words, characters), the perplexity is just

$$\prod_{i=1..n} Q(X_i = x_i)^{-1}$$

Where *n* is the length of the sequence

- Perplexity is the inverse of the probability of the sequence under the model

Conditional Entropy

- How related are two random variables X and Y to one another?
- In information-theoretic terms, how efficiently can you encode X given the value of Y?
- This is the conditional entropy:

$$H(X|Y) = \sum_{y \in Y} P(Y = y) H(X|Y = y)$$

Mutual Information

- The difference between the entropy *H(X)* and the conditional entropy *H(X|Y)* is called the mutual information between the two random variables:

$$I(X;Y) = H(X) - H(X|Y)$$

- When *Y* provides no information about *X*, $I(X;Y) = 0$
- When *Y* provides complete information about *X*, $I(X;Y) = H(X)$
- Mutual information is symmetric:

$$I(X;Y) = I(Y;X)$$

Distributional Similarity Measures

- How different are two distributions $P(X)$ and $Q(X)$?
  - We looked at cross-entropy, which tells us how efficient a coding system designed for one distribution is for encoding a different distribution
  - But cross-entropy depends on the entropy of the distribution to be encoded:

$$H(P,Q) \geq H(P)$$

Distributional Similarity Measures: KL Divergence

- Solution: measure the incremental encoding length, rather then the encoding length directly.
- This measure is the Kullback-Leibler (KL) Divergence
- It is defined as the cross-entropy minus the entropy of the distribution to be encoded:

$$D_{KL}\big(P(X) \parallel Q(X)\big) = H(P(X), Q(X)) - H\big(P(X)\big)$$

$$= -\textstyle\sum_{x \in X} P(X = x) \log_2 Q(X = x) + \sum_{x \in X} P(X = x) \log_2 P(X = x)$$

$$= -\sum_{x \in X} P(X = x)(\log_2 Q(X = x) - \log_2 P(X = x))$$

$$= -\sum_{x \in X} P(X = x)\left(\log_2 \frac{Q(X = x)}{P(X = x)}\right)$$

KL Divergence example

$$P(X) = \langle .1, .5, .4 \rangle$$

$$Q(X) = \langle .2, .2, .6 \rangle$$

$$\begin{aligned}
D_{KL}(P \parallel Q) &= -\sum_{x \in X} P(X = x)\left(\log_2 \frac{Q(X = x)}{P(X = x)}\right) \\
&= -.1 \log_2(\frac{.2}{.1}) - .5 \log_2(\frac{.2}{.5}) - .4 \log_2(\frac{.6}{.4}) \\
&= -0.1 + 0.66 - .23 = \mathbf{0.33}
\end{aligned}$$

KL Divergence example

$$P(X) = \langle .1, .5, .4 \rangle$$

$$Q(X) = \langle .2, .2, .6 \rangle$$

$$
\begin{aligned}
D_{KL}(Q \parallel P) &= -\sum_{x \in X} Q(X=x) \left(\log_2 \frac{P(X=x)}{Q(X=x)} \right) \\
&= -.2 \log_2 \left(\frac{.1}{.2}\right) - .2 \log_2 \left(\frac{.5}{.2}\right) - .6 \log_2 \left(\frac{.4}{.6}\right) \\
&= 0.2 - 0.26 + 0.35 = \mathbf{0.29}
\end{aligned}
$$

Distributional Similarity Measures: JS Divergence

- KL Divergence is not symmetric:

$$D_{KL}\big(P(X) \parallel Q(X)\big) \neq D_{KL}\big(Q(X) \parallel P(X)\big)$$

- A commonly-used symmetric measure of distributional distance is the Jensen-Shannon (JS) Divergence:

$$M(X) \stackrel{\text{def}}{=} \frac{(P(X) + Q(X))}{2}$$

$$D_{JS}(P(X) \parallel Q(X)) = \frac{D_{KL}\big(P(X) \parallel M(X)\big) + D_{KL}(Q(X) \parallel M(X))}{2}$$

- Why not $\frac{D_{KL}(P\|Q)+D_{KL}(Q\|P)}{2}$?
  - Eliminate case where $\log Q(X = x) = 0$, infinite values

JS Divergence example

$$P(X) = \langle .1, .5, .4 \rangle$$

$$Q(X) = \langle .2, .2, .6 \rangle$$

$$M(X) = \frac{P(X) + Q(X)}{2} = \langle .15, .35, .5 \rangle$$

$$\begin{aligned}
D_{KL}(P \parallel M) &= -\sum_{x \in X} P(X = x) \left(\log_2 \frac{M(X = x)}{P(X = x)} \right) \\
&= -.1 \log_2 \left(\frac{.15}{.1}\right) - .5 \log_2 \left(\frac{.35}{.5}\right) - .4 \log_2 \left(\frac{.5}{.4}\right) \\
&= -0.058 + 0.257 - 0.129 = 0.070
\end{aligned}$$

JS Divergence example

$$P(X) = \langle .1, .5, .4 \rangle$$

$$Q(X) = \langle .2, .2, .6 \rangle$$

$$M(X) = \frac{P(X) + Q(X)}{2} = \langle .15, .35, .5 \rangle$$

$$\begin{aligned}
D_{KL}(Q \parallel M) &= -\sum_{x \in X} Q(X = x) \left(\log_2 \frac{M(X = x)}{Q(X = x)} \right) \\
&= -.2 \log_2 (\frac{.15}{.2}) - .2 \log_2 (\frac{.35}{.2}) - .6 \log_2 (\frac{.5}{.6}) \\
&= 0.083 - 0.161 + 0.158 = 0.079
\end{aligned}$$

JS Divergence example

$$
\begin{aligned}
D_{JS}(P \parallel Q) &= \frac{D_{KL}(P \parallel M) + D_{KL}(Q \parallel M)}{2} \\
&= \frac{0.70 + 0.79}{2} \\
&\approx 0.75
\end{aligned}
$$

$$
\begin{aligned}
D_{JS}(Q \parallel P) &= \frac{D_{KL}(Q \parallel M) + D_{KL}(P \parallel M)}{2} \\
&= \frac{0.79 + 0.70}{2} \\
&\approx 0.75
\end{aligned}
$$