E2 212: Compressive Sensing as an Example in Linear Algebra

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1 Compressive Sensing

Consider the problem of finding a unique solution to the following linear equation

$$y = Ax, \tag{1}$$

where $A \in \mathcal{R}^{m\times n}$ and $x \in \mathcal{R}^n$. From elementary linear algebra, we know that a unique solution to the above problem exits if A is full rank. However, if (1) is under determined, i.e., A has more columns than rows, then there are infinitely many solutions. In this case, we can hope to find a unique solution if we impose additional constraints on the solution that we seek to find. One such constraint is the sparsity assumption on x, i.e., most of the entries of the vector x are zero. Although, we can impose other constraints on x, sparsity is the most simple yet effective way of capturing many real world scenarios that (1) models. Thus, we restrict the problem to one of finding a unique solution to (1) when x is $k-$*sparse*. While we know that there are k non-zero entries in x, we do not know exactly where x is sparse. Absence of this support information is one of the key challenges faced while solving this problem. Just imposing a sparsity constraint on x may not guarantee a unique solution (for example, when $m < k$). One may hope to find a solution if $m \geq k$. However, the following proposition asserts that we need $m \geq 2k$.[1]

Proposition 1 *If every $2k < n$ columns of A are linearly independent (i.e., $m \geq 2k$), then a k sparse vector can be uniquely recovered from* (1) *.*

Proof: Suppose if there exists two k sparse vectors x_1 and x_2, $x_1 \neq x_2$ such that $Ax_1 = Ax_2$, then $A(x_1 - x_2) = 0$. Note that $x_1 - x_2$ is at most $2k$ sparse. Now, it is easy to see from $A(x_1 - x_2) = 0$ that there exits a $2k$ linearly dependent columns of A leading to a contradiction. $\square$

[1] The proof of the following proposition is taken from:
`http://terrytao.files.wordpress.com/2009/08/compressed-sensing1.pdf`.