## Homework 5

Due: April 13, 2022

- 1. Cost-constrained binary channels. Consider a cost-constrained binary input channel with cost function b(x) = x for  $x \in \{0,1\}$ . Find the capacity-cost function C(B) (a) for the binary symmetric channel and (b) for the binary erasure channel.
- 2. Parallel Gaussian channels. Consider 3 parallel Gaussian channels with noise variances given by

$$N_1 = 1, \qquad N_2 = 4, \qquad N_3 = 10.$$

Find the capacity as a function of total power P. Make sure your answer is in completely closed form (i.e., solve for  $\alpha^*$  as a function of P).

3. Parallel binary erasure channels. Consider k parallel binary erasure channels with an overall cost constraint. That is, the input is  $X = (X_1, X_2, \ldots, X_k)$  where  $X_j \in \{0, 1\}$ . The output is  $Y = (Y_1, Y_2, \ldots, Y_k)$ , where  $Y_j$  is the output of a binary erasure channel with  $X_j$  as the input and erasure probability  $p_j$ . There is a joint cost constraint on the input codeword  $x^n = (x_1^n, x_2^n, \ldots, x_k^n)$  given by

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} x_{ji} \le B.$$

Find the capacity of this channel. You may give your answer in parametric form (similar to the water filling solution, in which the capacity is written in terms of a Lagrange variable).

4. Problem 10.5 from Cover-Thomas: Rate distortion for uniform source with Hamming distortion. Consider a source X uniformly distributed on the set  $\{1, 2, ..., m\}$ . Find the rate distortion function for this source with Hamming distortion; that is,

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x \neq \hat{x}. \end{cases}$$

5. Problem 10.7 from Cover-Thomas: Erasure distortion. Consider  $X \sim \text{Bernoulli}(\frac{1}{2})$ , and let the distortion measure be given by a matrix (each row is a letter of  $\mathcal{X}$ , each columns is a letter of  $\hat{\mathcal{X}}$ )

$$d(x,\hat{x}) = \left[ \begin{array}{ccc} 0 & 1 & \infty \\ \infty & 1 & 0 \end{array} \right].$$

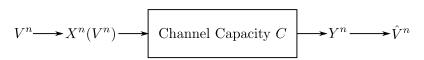
Calculate the rate distortion function for this source. Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

6. Rate-distortion for exponential random variable. Let X be an exponential random variable with expectation  $\mu$ ; i.e.,  $f_X(x) = \frac{1}{\mu}e^{-x/\mu}$  for  $x \ge 0$ . Define a distortion function for  $x, \hat{x} \in \mathbb{R}$  as

$$d(x, \hat{x}) = \begin{cases} x - \hat{x}, & \hat{x} \le x \\ \infty, & \hat{x} > x. \end{cases}$$

Find the rate-distortion function. *Hint:* Remember the results from problems 7 and 8 from Homework 4.

7. Problem 10.17 from Cover-Thomas: Source-channel separation theorem with distortion. Let  $V_1, V_2, \ldots, V_n$  be a finite alphabet i.i.d. source with is encoded as a sequence of n input symbols  $X^n$  of a discrete memoryless channel. The output of the channel  $Y^n$  is mapped onto the reconstruction alphabet  $\hat{V} = g(Y^n)$ . Let  $D = \mathbb{E}d(V^n, \hat{V}^n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}d(V_i, \hat{V}_i)$  be the average distortion achieved by this combined source and channel coding scheme.



- (a) Show that if C > R(D), where R(D) is the rate distortion function for V, it is possible to find encoders and decoders that achieve an average distortion arbitrarily close to D.
- (b) (Converse) Show that if the average distortion is equal to D, the capacity of the channel C must be greater than R(D).
- 8. Source-channel separation for a Gaussian source with a Gaussian channel. Consider a special case of the setup from problem 7 where  $V \sim \mathcal{N}(0, \sigma^2)$ , the channel is a Gaussian noise channel with power constraint P and noise variance N, and the distortion function is the squared error distortion; i.e.,  $d(v, \hat{v}) = (v \hat{v})^2$ .
  - (a) Using the result from problem 7, find the smallest possible expected distortion D given the other parameters.
  - (b) Now consider the following simple approach without coding. For each  $i=1,\ldots,n$ , the encoder sends  $X_i=\alpha V_i$ , and the decoder estimates the source using  $\hat{V}_i=\beta Y_i$ . Here,  $\alpha$  and  $\beta$  are constants that must be determined. What is the minimum expected distortion D achievable using this scheme? Be sure to choose  $\alpha$  to satisfy the power constraint of the channel, and  $\beta$  to minimize the expected distortion. Compare your to answer to that from part (a).