EEE 551 Information Theory (Spring 2022)

Homework 4

Due: March 30, 2022

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1. Problem 8.1 from Cover-Thomas: *Differential entropy.* Evaluate the differential entropy $h(X) = -\int f \log f$ for the following:
   (a) The exponential density, $f(x) = \lambda e^{-\lambda x},\ x \geq 0$.
   (b) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$.
   (c) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variances $\sigma_i^2, i = 1, 2$.

2. *Differential entropy.* Consider the continuous variables X, Y with joint PDF given by
$$f(x,y) = \begin{cases} 2, & 0 < x < 1,\ 0 < y < 1,\ x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Find $h(X)$, $h(Y)$, $h(X,Y)$, $h(X|Y)$, $h(Y|X)$, and $I(X;Y)$.

3. *Gaussian typical set.* Let $A_\epsilon^{(n)}$ be the typical set for the zero-mean Gaussian distribution $\mathcal{N}(0, \sigma^2)$. Find constants a and b such that $x^n \in A_\epsilon^{(n)}$ if and only if
$$a \leq \sum_{i=1}^{n} x_i^2 \leq b.$$

4. *Binary-input Gaussian noise channel.* Consider a channel where $X \in \{-\sqrt{P}, \sqrt{P}\}$, and $Y = X + Z$ where $Z \sim \mathcal{N}(0, N)$. This can be considered a model for binary phase-shift keying (BPSK) with power P used for a Gaussian noise channel. Assume that X is equally likely to be $\sqrt{P}$ or $-\sqrt{P}$.
   (a) Find a closed-form expression for $h(Y|X)$.
   (b) Find a formula for $h(Y)$. You do not need to evaluate the integral.
   (c) Use some numerical integration software (such as Matlab's function `integral`) to calculate $I(X;Y) = h(Y) - h(Y|X)$ for the following parameters: $P = 2$, $N = 3$. Compare your answer to the capacity of the standard Gaussian channel with the same power and noise variance.

5. *Mutual information with mixed random variable.* Let $X \sim \text{Bern}(1/2)$. Define a channel from X to Y as follows. Given $X = x$, Y is a mixed random variable with probability $1 - p$ of being equal to x, and with probability p of being drawn from a continuous uniform distribution on the interval $(0, 1)$. Find $I(X;Y)$.

6. Problem 9.2 from Cover-Thomas: *Two-look Gaussian channel.* Consider the ordinary Gaussian channel with two correlated looks at X, that is, $Y = (Y_1, Y_2)$, where
$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$
   with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$, where
$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$
   Find the capacity C for

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$

7. Let X be a nonnegative continuous random variable with pdf $g(x)$ and mean μ. Let Y be a random variable with exponential density and mean μ (i.e. $f(y) = \frac{1}{\mu}e^{-y/\mu}$, $y \geq 0$). Show that $h(X) \leq h(Y)$. (Hint: Evaluate the relative entropy $D(g\|f)$, where f is the pdf of Y.)

8. Problem 9.4 from Cover-Thomas: *Exponential noise channels.* $Y_i = X_i + Z_i$, where Z_i is i.i.d. exponentially distributed noise with mean μ. Assume that we have a mean constraint on the signal (i.e. $EX_i \leq \lambda$). Show that the capacity of such a channel is $C = \log(1 + \frac{\lambda}{\mu})$. (**Note**: This problem is misstated in the book. There is an additional constraint that the input to the channel is nonnegative, i.e. $X_i \geq 0$.)