EEE 551 Information Theory (Spring 2022)

Homework 5

Due: April 13, 2022

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1. *Cost-constrained binary channels.* Consider a cost-constrained binary input channel with cost function $b(x) = x$ for $x \in \{0, 1\}$. Find the capacity-cost function $C(B)$ (a) for the binary symmetric channel and (b) for the binary erasure channel.

2. *Parallel Gaussian channels.* Consider 3 parallel Gaussian channels with noise variances given by
$$N_1 = 1, \qquad N_2 = 4, \qquad N_3 = 10.$$
Find the capacity as a function of total power P. Make sure your answer is in completely closed form (i.e., solve for $\alpha^\star$ as a function of P).

3. *Parallel binary erasure channels.* Consider k parallel binary erasure channels with an overall cost constraint. That is, the input is $X = (X_1, X_2, \ldots, X_k)$ where $X_j \in \{0, 1\}$. The output is $Y = (Y_1, Y_2, \ldots, Y_k)$, where Y_j is the output of a binary erasure channel with X_j as the input and erasure probability p_j. There is a joint cost constraint on the input codeword $x^n = (x_1^n, x_2^n, \ldots, x_k^n)$ given by
$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ji} \le B.$$
Find the capacity of this channel. You may give your answer in parametric form (similar to the water filling solution, in which the capacity is written in terms of a Lagrange variable).

4. Problem 10.5 from Cover-Thomas: *Rate distortion for uniform source with Hamming distortion.* Consider a source X uniformly distributed on the set $\{1, 2, \ldots, m\}$. Find the rate distortion function for this source with Hamming distortion; that is,
$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x \ne \hat{x}. \end{cases}$$

5. Problem 10.7 from Cover-Thomas: *Erasure distortion.* Consider $X \sim \text{Bernoulli}(\frac{1}{2})$, and let the distortion measure be given by a matrix (each row is a letter of $\mathcal{X}$, each columns is a letter of $\hat{\mathcal{X}}$)
$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix}.$$
Calculate the rate distortion function for this source. Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

6. *Rate-distortion for exponential random variable.* Let X be an exponential random variable with expectation μ; i.e., $f_X(x) = \frac{1}{\mu}e^{-x/\mu}$ for $x \ge 0$. Define a distortion function for $x, \hat{x} \in \mathbb{R}$ as
$$d(x, \hat{x}) = \begin{cases} x - \hat{x}, & \hat{x} \le x \\ \infty, & \hat{x} > x. \end{cases}$$
Find the rate-distortion function. *Hint:* Remember the results from problems 7 and 8 from Homework 4.

7. Problem 10.17 from Cover-Thomas: *Source-channel separation theorem with distortion.* Let $V_1, V_2, \ldots, V_n$ be a finite alphabet i.i.d. source with is encoded as a sequence of n input symbols X^n of a discrete memoryless channel. The output of the channel Y^n is mapped onto the reconstruction alphabet $\hat{V} = g(Y^n)$. Let $D = \mathbb{E}d(V^n, \hat{V}^n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}d(V_i, \hat{V}_i)$ be the average distortion achieved by this combined source and channel coding scheme.

$$V^n \longrightarrow X^n(V^n) \longrightarrow \boxed{\text{Channel Capacity } C} \longrightarrow Y^n \longrightarrow \hat{V}^n$$

   (a) Show that if $C > R(D)$, where $R(D)$ is the rate distortion function for V, it is possible to find encoders and decoders that achieve an average distortion arbitrarily close to D.

   (b) (Converse) Show that if the average distortion is equal to D, the capacity of the channel C must be greater than $R(D)$.

8. *Source-channel separation for a Gaussian source with a Gaussian channel.* Consider a special case of the setup from problem 7 where $V \sim \mathcal{N}(0, \sigma^2)$, the channel is a Gaussian noise channel with power constraint P and noise variance N, and the distortion function is the squared error distortion; i.e., $d(v, \hat{v}) = (v - \hat{v})^2$.

   (a) Using the result from problem 7, find the smallest possible expected distortion D given the other parameters.

   (b) Now consider the following simple approach without coding. For each $i = 1, \ldots, n$, the encoder sends $X_i = \alpha V_i$, and the decoder estimates the source using $\hat{V}_i = \beta Y_i$. Here, α and β are constants that must be determined. What is the minimum expected distortion D achievable using this scheme? Be sure to choose α to satisfy the power constraint of the channel, and β to minimize the expected distortion. Compare your to answer to that from part (a).