ECE64700: Homework I

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I. Convex Sets

A. Solutions to Exercise 1

$$C \subseteq \mathbb{R}^n,\ \lambda_1 > 0\ and\ \lambda_2 > 0$$

$$C\ is\ convex$$

$$A\ \triangleq\ \alpha S\ =\ \{\alpha x : x \in S\}$$

$$B\ \triangleq\ S_1 + S_2\ =\ \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$$

For any two points $x_1,\ x_2\ \in\ C$, $\theta x_1 + \bar{\theta} x_2\ \in\ C$, for any $0 \leq \theta \leq 1$.
In other words, the line segment between any two points in C lies in C.
In order to prove two sets A and B are equal, we need to prove that all the elements in A are in B and all the elements in B are in A.
Mathematically,

$$A \subseteq B,\ and$$

$$B \subseteq A$$

If $x_1\ =\ x_2$ in the set definition of B,

$$\lambda_1 x_1 + \lambda_2 x_1\ \in\ B,\ for\ any\ x_1\ \in\ C$$

Since, $x_1\ \in\ C$,

$$\lambda_1 x_1 + \lambda_2 x_1\ \in\ A$$

Similarly,

$$\lambda_1 x_2 + \lambda_2 x_2\ \in\ B,\ for\ any\ x_2\ \in\ C$$

Since, $x_2\ \in\ C$,

$$\lambda_1 x_2 + \lambda_2 x_2\ \in\ A$$

Now, if $x_1 \neq x_2$ in the set definition of B,

$$\lambda_1 x_1 + \lambda_2 x_2 \in B, \; for \; any \; x_1, \; x_2 \in C$$

If $x_1, \; x_2 \in C$, $\theta x_1 + \bar{\theta} x_2 \in C$, for any $0 \le \theta \le 1$.
So,

$$(\lambda_1 + \lambda_2)(\theta x_1 + \bar{\theta} x_2) \in A$$

$$(\lambda_1 + \lambda_2)\theta x_1 + (\lambda_1 + \lambda_2)\bar{\theta} x_2$$

Let,

$$\theta = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$\bar{\theta} = 1 - \theta = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

These substitutions are valid because,

$$0 \le \theta \le 1, \; since \; \lambda_1 > 0, \; and \; \lambda_2 > 0$$

Using these substitutions,

$$\lambda_1 x_1 + \lambda_2 x_2 \in A, \; for \; any \; x_1, \; x_2 \in C$$

This property need not be true when C is not convex because,

$$\theta x_1 + \bar{\theta} x_2 \notin C, \; for \; any \; x_1, \; x_2 \in C \; and \; for \; any \; 0 \le \theta \le 1$$

This means that for some $x_1, \; x_2 \in C$,

$$\lambda_1 x_1 + \lambda_2 x_2 \in B$$

$$\lambda_1 x_1 + \lambda_2 x_2 \notin A$$

Consider the following example,

$$Let, \; C = \{5, \; 6\}$$

C is not a convex set because it does not contain the line segment between 5 and 6.

$$Let, \; \lambda_1 = 1, \; \lambda_2 = 1$$

$$A = \{10, \; 12\}$$

$$B = \{10, \; 11, \; 12\}$$

11 in set B does not exist in set A.

However, now consider the case in which C is a convex set, i.e. C contains the line segment between 5 and 6.

Since, $0.5(5) + 0.5(6) \;=\; 5.5 \;\in\; C, \; for \; \theta \;=\; \bar{\theta} \;=\; 0.5$,

$$(\lambda_1 + \lambda_2)5.5 \;=\; 11 \;\in\; A$$

This can be done for all the elements in C (convex). In other words, for all the elements in C (convex), we can prove that $A \;\subseteq\; B$ and $B \;\subseteq\; A$.

B. Solutions to Exercise 2

1) $S_1 \;\equiv\; \mathcal{B}(0,\; r)$**:** A closed Euclidean Ball in $\mathbb{R}^2$ centered at 0 with radius r is defined as,

$$S_1 \;\equiv\; \mathcal{B}(0,\; r) \;=\; \{x \;\in\; \mathbb{R}^2 :\; ||x||_2 \leq r\}$$

For S_1 to be a convex set, $\theta x_1 + \bar{\theta} x_2 \;\in\; S_1$ for any two points $x_1,\; x_2 \;\in\; S_1$ and for any $0 \leq \theta \leq 1$.

In other words, we need to prove that,

$$||\theta x_1 + \bar{\theta} x_2||_2 \leq r$$

To prove this result, we know from the Triangle Inequality that,

$$||\theta x_1 + \bar{\theta} x_2||_2 \leq ||\theta x_1||_2 + ||\bar{\theta} x_2||_2$$

This can be written as,

$$||\theta x_1 + \bar{\theta} x_2||_2 \leq \theta ||x_1||_2 + \bar{\theta} ||x_2||_2$$

Since, $||x_1||_2 \leq r$, $||x_2||_2 \leq r$, and $0 \leq \theta \leq 1$,

$$\theta ||x_1||_2 + \bar{\theta} ||x_2||_2 \leq r$$

Which implies,

$$||\theta x_1 + \bar{\theta} x_2||_2 \leq r$$

Therefore, S_1 is a convex set.

In order to represent $S_1 \;\equiv\; \mathcal{B}(0,\; r)$ as an intersection of halfspaces, consider the points on the boundary of S_1. Then, treating tangents at all points on the boundary of S_1 as hyperplanes, we can write,

$$S_1 \;\equiv\; \mathcal{B}(0,\; r) \;=\; \bigcap_{y \;\in\; \mathbb{R}^2 :\; ||y||_2 \;=\; r} \{x \;\in\; \mathbb{R}^2 :\; x^T y \leq r\}$$

2) $S_2 \equiv \{x \in \mathbb{R}^n : x_i \in (-1, 1), \forall i = 1, 2, ..., n\}$**:** An open set in $\mathbb{R}^n$

For S_2 to be a convex set, $\theta x + \bar{\theta} y \in S_2$ for any two points $x, y \in S_2$ and for any $0 \leq \theta \leq 1$. To prove this, consider,

$$\theta x + \bar{\theta} y$$

Since $x_i \in (-1, 1), \forall i = 1, 2, ..., n$, $y_i \in (-1, 1), \forall i = 1, 2, ..., n$, and $0 \leq \theta \leq 1$,

$$\theta x_i + \bar{\theta} y_i \in (-1, 1), \forall i = 1, 2, ..., n$$

Which implies,

$$\theta x + \bar{\theta} y \in S_2$$

Therefore, S_2 is a convex set.

S_2 can be written as the intersection of an infinite number of halfspaces as follows,

$$S_2 = \bigcap_{|r|\geq 1,\ a \in \mathbb{R}^n:\ ||a||\geq 1} \{x \in \mathbb{R}^n : -r < a^T x < r\}$$

Writing this halfspace representation in its minimal form,

$$S_2 = \bigcap_{a \in \mathbb{R}^n:\ ||a||=1} \{x \in \mathbb{R}^n : -1 < a^T x < 1\}$$

For instance, for $\mathbb{R}^2$,

$$S_2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < 1\} \bigcap \{x = (x_1, x_2) \in \mathbb{R}^2 : -1 < x_2 < 1\}$$

This intersection of 4 halfspaces in $\mathbb{R}^2$ represents the set S_2 which is an open square in $\mathbb{R}^2$.

C. Solutions to Exercise 3

The convex hull of a set C is the smallest convex set that contains C.

Consider a set A consisting of a finite number of elements in $\mathbb{R}^n$.

$$A \equiv \{x_1, x_2, x_3, ..., x_k\}$$

The set S_2 in Exercise 2 is an open set in $\mathbb{R}^n$ which proves to be a convex set.

But, since the set S_2 is an open set, it cannot be the smallest convex set containing A.

Mathematically,

If $A \subseteq S_2$, $conv(A) \neq S_2$

If A is the set of all points on the boundary of S_2,

$$conv(A) = cl(S_2)$$

But, $cl(S_2) \neq S_2$

Therefore, the set S_2 cannot be written as the convex hull of the set A.

The closure of a set C denoted by $cl(C)$ is the smallest closed set containing all the elements of C.

Therefore,

$$cl(S_2) = \{x \in \mathbb{R}^n : x_i \in [-1, 1], \forall i = 1, 2, ..., n\}$$

D. Solutions to Exercise 4

The image of a convex set under an affine transformation is convex.

Claim: The image of a convex set $S \subseteq \mathbb{R}^n$ under an affine transformation is convex.

Proof: Consider an affine function (sum of a linear function and a constant) $f : \mathbb{R}^n \to \mathbb{R}^m$ such that the image of a convex set $S \subseteq \mathbb{R}^n$ under this affine transformation is given by,

$$f(S) = \{Ax + b : x \in S\} \subseteq \mathbb{R}^m$$

where,

$$A \in \mathbb{R}^{m \times n}$$

$$b \in \mathbb{R}^m$$

$$x \in \mathbb{R}^n$$

Since S is a convex set, for any two points $x_1, x_2 \in S$ and for any $0 \leq \theta \leq 1$,

$$\theta x_1 + \bar{\theta} x_2 \in S$$

Now,

$$x_1 \in S \implies Ax_1 + b \in f(S)$$

$$x_2 \in S \implies Ax_2 + b \in f(S)$$

$$\theta x_1 + \bar{\theta} x_2 \in S \implies A(\theta x_1 + \bar{\theta} x_2) + b \in f(S)$$

To prove that $f(S)$ is a convex set, consider the points $Ax_1 + b, Ax_2 + b \in f(S)$ for any $x_1, x_2 \in S$.

The convex combination of these two points in $f(S)$ should exist in $f(S)$ for $f(S)$ to be a convex set.

Since $\theta + \bar{\theta} \;=\; 1$ and $A(\theta x_1 + \bar{\theta} x_2) + b \;\in\; f(S)$ from before, we can now see that,

$$\theta(Ax_1{+}b){+}\bar{\theta}(Ax_2{+}b) \;=\; A\theta x_1{+}b\theta{+}A\bar{\theta}x_2{+}b\bar{\theta} \;=\; A\theta x_1{+}A\bar{\theta}x_2{+}b \;=\; A(\theta x_1{+}\bar{\theta}x_2){+}b \;\in\; f(S)$$

This proves that the convex combination of any two points in $f(S)$ are contained in $f(S)$. Therefore, the image $f(S) \;\subseteq\; \mathbb{R}^m$ of a convex set $S \subseteq \mathbb{R}^n$ under an affine transformation $f:\; \mathbb{R}^n \to \mathbb{R}^m$ is a convex set.

E. Solutions to Exercise 5

1) [Boyd] 2.11: Hyperbolic sets

Claim: The Hyperbolic set $C \;\equiv\; \{x \;\in\; \mathbb{R}^2_{++} :\; x_1 x_2 \geq 1\}$ is convex.

Proof: We know from Jensen's inequality that for a convex function $f(x)$ and for any $0 \leq \theta \leq 1$,

$$f(\sum_i \theta_i x_i) \;\leq\; \sum_i (\theta_i f(x_i))$$

For $a,\; b \;\in\; dom(f)$,

$$f(\theta a + \bar{\theta} b) \;\leq\; \theta f(a) + \bar{\theta} f(b)$$

We know that $-log(x)$ is a convex function. So, for any $a,\; b \;\in\; dom(-log(x)) \;\equiv\; \mathbb{R}_{++}$,

$$-log(\theta a + \bar{\theta} b) \;\leq\; -\theta log(a) - \bar{\theta} log(b)$$

$$log(\theta a + \bar{\theta} b) \;\geq\; \theta log(a) + \bar{\theta} log(b)$$

Applying the properties of logarithms we get,

$$log(\theta a + \bar{\theta} b) \;\geq\; log(a^{\theta} b^{\bar{\theta}})$$

Rearranging,

$$a^{\theta} b^{\bar{\theta}} \;\leq\; \theta a + \bar{\theta} b$$

Now, in order to prove that the set C is a convex set, we need to prove that for any two points $x,\; y \;\in\; C$ and for any $0 \leq \theta \leq 1$, the convex combination of these two points $\theta x + \bar{\theta} y \;\in\; C$. Therefore, we need to prove that,

$$(\theta x_1 + \bar{\theta} y_1)(\theta x_2 + \bar{\theta} y_2) \;\geq\; 1$$

From the set definition, we know that,

$$x_1,\; x_2,\; y_1,\; y_2 \;\in\; \mathbb{R}_{++}$$

Using the inequality $a^\theta b^{\bar{\theta}} \;\leq\; \theta a + \bar{\theta} b$ we derived earlier,

$$x_1^\theta y_1^{\bar{\theta}} \;\leq\; \theta x_1 + \bar{\theta} y_1$$

$$x_2^\theta y_2^{\bar{\theta}} \;\leq\; \theta x_2 + \bar{\theta} y_2$$

Since all the quantities involved in the inequalities are positive, we can multiply the corresponding members of the two,

$$(x_1^\theta y_1^{\bar{\theta}})(x_2^\theta y_2^{\bar{\theta}}) \leq (\theta x_1 + \bar{\theta} y_1)(\theta x_2 + \bar{\theta} y_2)$$

$$(x_1 x_2)^\theta (y_1 y_2)^{\bar{\theta}} \leq (\theta x_1 + \bar{\theta} y_1)(\theta x_2 + \bar{\theta} y_2)$$

Since $x_1 x_2 \geq 1$, $y_1 y_2 \geq 1$, and $0 \leq \theta \leq 1$,

$$(\theta x_1 + \bar{\theta} y_1)(\theta x_2 + \bar{\theta} y_2) \;\geq\; 1$$

Therefore,

$$\theta x + \bar{\theta} y \;\in\; C$$

Hence,

$$C \;\equiv\; \{x \;\in\; \mathbb{R}^2_{++} :\; x_1 x_2 \geq 1\} \;is\; a\; convex\; set.$$

In order to generalize the above result, we need to prove that,

$$D \;\equiv\; \{x \;\in\; \mathbb{R}^n_{++} :\; \prod_{i=1}^{n} \; x_i = 1\}$$

From the set definition, we know that, for any two points $x, y \;\in\; D$,

$$x_i,\; y_i \;\in\; \mathbb{R}_{++},\; \forall i \;=\; 1,\; 2,\; ....,\; n$$

Using the inequality $a^\theta b^{\bar{\theta}} \;\leq\; \theta a + \bar{\theta} b$ we derived earlier, we can write,

$$(\prod_{i=1}^{n} \; x_i)^\theta (\prod_{i=1}^{n} \; y_i)^{\bar{\theta}} \leq \prod_{i=1}^{n} \; (\theta x_i + \bar{\theta} y_i)$$

Since,

$$\prod_{i=1}^{n} \; x_i \geq 1,$$

$$\prod_{i=1}^{n} \; y_i \geq 1,\; and$$

$$0 \leq \theta \leq 1$$

$$(\prod_{i=1}^{n} \; x_i)^\theta (\prod_{i=1}^{n} \; y_i)^{\bar{\theta}} \geq 1$$

Which implies,

$$\prod_{i=1}^{n} (\theta x_i + \bar{\theta} y_i) \geq 1$$

Therefore,

$$\theta x + \bar{\theta} y \ \in \ D$$

Hence,

$$D \ \equiv \ \{x \ \in \ \mathbb{R}^n_{++} : \ \prod_{i=1}^{n} x_i \geq 1\} \ is \ a \ convex \ set.$$

2) [Boyd] 2.12: Identifying convex sets

1) A slab, $S \ \equiv \ \{x \ \in \ \mathbb{R}^n : \ \alpha \leq a^T x \leq \beta\}$ is a convex set because it is an intersection of two halfspaces. Halfspaces are convex sets and the intersection of convex sets is another convex set.
2) A rectangle, $S \ \equiv \ \{x \ \in \ \mathbb{R}^n : \ \alpha_i \leq x_i \leq \beta_i, \ i \ = \ 1, \ 2, \ ...., \ n\}$ is a convex set because it is an intersection of a finite number of halfspaces. Halfspaces are convex sets and the intersection of convex sets is another convex set.
3) A wedge, $S \ \equiv \ \{x \ \in \ \mathbb{R}^n : \ a_1^T x \leq b_1, \ a_2^T x \leq b_2\}$ is a convex set because it is an intersection of two halfspaces. Halfspaces are convex sets and the intersection of convex sets is another convex set.
4) The set of points closer to a given point than to a given set,
$A \ \equiv \ \{x : ||x - x_0||_2 \leq ||x - y||_2, \ \forall y \ \in \ S\}$ is a convex set because it is an intersection of a finite number of half-spaces.
For a fixed $y \ \in \ S$, consider the set,

$$A_y \ \equiv \ \{x : \ ||x - x_0||_2 \leq ||x - y||_2\}$$

Square on both sides and expand the Euclidean norm,

$$= \ \{x : \ (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)\}$$

$$= \ \{x : \ x^T x - x^T x_0 - x_0^T x + x_0^T x_0 \leq x^T x - x^T y - y^T x + y^T y\}$$

Rearranging the terms,

$$= \ \{x : \ x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2y^T x + y^T y\}$$

Simplifying,

$$= \ \{x : \ 2y^T x - 2x_0^T x \leq y^T y - x_0^T x_0\}$$

Rearranging into the standard form of a halfspace,

$$= \{x : \ 2(y - x_0)^T x \le y^T y - x_0^T x_0\}$$

Which is a halfspace of the form $\{x : \ a^T x \le b\}$.

For any $y \in S$, the set A can be written as follows,

$$A = \bigcap_{y \, \in \, S} A_y$$

Therefore, the set S is a convex set because it is an intersection of a finite number of halfspaces.

5) The set of points closer to one set than another, $A \equiv \{x : dist(x, \ S) \le dist(x, \ T)\}$, where $S, \ T \subseteq \mathbb{R}^n$, and $dist(x, \ S) = inf\{||x - z||_2 \mid z \in S\}$ is not a convex set.
Consider the following counter-example.
Let, $S \equiv \{x \in \mathbb{R}^2 : \ x_1 < -3\} \cup \{x \in \mathbb{R}^2 : \ x_1 > 3\}$.
Let $T \equiv \{x \in \mathbb{R}^2 : \ x_1 = 0\}$ be a hyperplane in $\mathbb{R}^2$
In this example, the set of all points closer to set S than to set T is not a convex set.

6) The set $C \equiv \{x : \ x + S_2 \subseteq S_1\}$, where $S_1, \ S_2 \subseteq \mathbb{R}^n$ with S_1 being convex is a convex set because it is an intersection of convex sets. Consider the set,

$$C_y \equiv \{x : \ x + y \in S_1\} = \{z - y : \ z \in S_1\}$$

C_y is a convex set because it is the image of the convex set S_1 under an affine transformation.

$$C = \bigcap_{y \, \in \, S_2} C_y$$

Therefore, the set C is a convex set because it is an intersection of convex sets.

7) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, $S \equiv \{x : \ ||x - a||_2 \le \theta ||x - b||_2\}$, where $a \ne b$ and $0 \le \theta \le 1$ is a convex set for reasons detailed below.
If $\theta = 1$, the set S reduces to $\{x : \ ||x - a||_2 \le ||x - b||_2\}$ which as we showed in item 4 [The set of all points closer to a point $x_0 = a$ than to another point $y = b$] is a convex set.
For $0 \le \theta < 1$, let's square on both sides,

$$\{x : \ ||x - a||_2^2 \le \theta^2 ||x - b||_2^2\}$$

Expand the norms,

$$\{x : \ (x - a)^T (x - a) \le \theta^2 (x - b)^T (x - b)\}$$

$$\{x: \ x^T x - x^T a - a^T x + a^T a \leq \theta^2 x^T x - \theta^2 x^T b - \theta^2 b^T x + \theta^2 b^T b\}$$

Rearranging the terms,

$$\{x: \ (1-\theta^2)x^T x - 2a^T x + 2\theta^2 b^T x \leq \theta^2 b^T b - a^T a\}$$

Divide by $(1-\theta^2)$ on both sides of the inequality,

$$\{x: \ x^T x - \frac{2(a^T - \theta^2 b^T)x}{(1-\theta^2)} \leq \frac{\theta^2 b^T b - a^T a}{(1-\theta^2)}\}$$

$$\{x: \ x^T x - \frac{2(a - \theta^2 b)^T x}{(1-\theta^2)} \leq \frac{\theta^2 b^T b - a^T a}{(1-\theta^2)}\}$$

Add $\frac{(a-\theta^2 b)^T}{(1-\theta^2)}\frac{(a-\theta^2 b)}{(1-\theta^2)}$ to both sides of the inequality,

$$\{x: \ x^T x - \frac{2(a - \theta^2 b)^T x}{(1-\theta^2)} + \frac{(a-\theta^2 b)^T}{(1-\theta^2)}\frac{(a-\theta^2 b)}{(1-\theta^2)} \leq \frac{\theta^2 b^T b - a^T a}{(1-\theta^2)} + \frac{(a-\theta^2 b)^T}{(1-\theta^2)}\frac{(a-\theta^2 b)}{(1-\theta^2)}\}$$

Treating $\frac{(a-\theta^2 b)}{(1-\theta^2)}$ as x_c and $\frac{\theta^2 b^T b - a^T a}{(1-\theta^2)} + \frac{(a-\theta^2 b)^T}{(1-\theta^2)}\frac{(a-\theta^2 b)}{(1-\theta^2)}$ as r^2,

$$\{x: \ x^T x - 2x_c^T x + x_c^T x_c \leq r^2\}$$

$$\{x: \ ||x - x_c||_2^2 \leq r^2\}$$

$$\{x: \ ||x - x_c||_2 \leq r\}$$

This is a Euclidean Ball $\mathcal{B}(x_c, \ r)$ which is a convex set.

Therefore, the set S is a convex set.

II. Convex Functions

A. *Solutions to Exercise 1*

1) Non-negative, Non-zero weighted sum of strictly convex functions:

$$g(x) \ = \ \sum_{i=1}^{N} \ \alpha_i f_i(x)$$

Here $f_i : \chi \rightarrow \mathbb{R}$, for $i \ = \ 1, \ 2, \ ...., \ N$ constitute N strictly convex functions with domain $\chi \subseteq \mathbb{R}^n$ (convex).

Claim: For N strictly convex functions $f_i(x), \ i \ = \ 1, \ 2, \ ...., \ N$, if $\alpha_i > 0$ for $i \ = \ 1, \ 2, \ ...., \ N$ then, $g(x) \ = \ \sum_{i=1}^{N} \ \alpha_i f_i(x)$ is a strictly convex function.

Proof: The domain of $g(x)$, i.e. $dom(g) \ = \ \bigcap_{i=1}^{N} \ dom(f_i) \ = \ \chi$ is a convex set because the intersection of convex sets is another convex set.

For $g(x)$ to be a strictly convex function, the following inequality must be satisfied.
For any $x,\ y\ \in\ dom(g)$ and for any $0 \leq \theta \leq 1$,

$$g(\theta x + \bar{\theta} y) < \theta g(x) + \bar{\theta} g(y)$$

$$\sum_{i=1}^{N}\ \alpha_i f_i(\theta x + \bar{\theta} y) < \theta \sum_{i=1}^{N}\ \alpha_i f_i(x) + \bar{\theta} \sum_{i=1}^{N}\ \alpha_i f_i(y)$$

$$\sum_{i=1}^{N}\ \alpha_i f_i(\theta x + \bar{\theta} y) < \sum_{i=1}^{N}\ \alpha_i(\theta f_i(x) + \bar{\theta} f_i(y))$$

Now, since $f_i(x)$ is a strictly convex function, i.e.,

$$\theta f_i(x) + \bar{\theta} f_i(y) > f_i(\theta x + \bar{\theta} y)$$

The inequality

$$\sum_{i=1}^{N}\ \alpha_i f_i(\theta x + \bar{\theta} y) < \sum_{i=1}^{N}\ \alpha_i(\theta f_i(x) + \bar{\theta} f_i(y))$$

will hold true only if $\alpha_i > 0,\ for\ i\ =\ 1,\ 2,\ ....,\ N$.
Therefore, a non-negative, non-zero weighted sum of strictly convex functions is strictly convex.
No, the above mentioned condition for the weights, i.e. $\alpha_i > 0$ is not a necessary condition.
Let us show this using a counter-example.
Consider the following scenario,

$$f_1(x)\ =\ -log(x)\ is\ a\ strictly\ convex\ function$$

$$f_2(x)\ =\ -log(x)\ is\ a\ strictly\ convex\ function$$

$$dom(f_1)\ =\ dom(f_2)\ =\ \mathbb{R}_{++}\ is\ a\ convex\ set$$

$$g(x)\ =\ \alpha_1 f_1(x) + \alpha_2 f_2(x)$$

Let, $\alpha_1\ =\ 0\ and\ \alpha_2\ =\ 1$.

$$g(x)\ =\ -log(x)$$

So, g(x) is a strictly convex function despite the weights not satisfying the result derived in the previous segment, i.e. $\alpha_i > 0$.
Yes, the set of $\alpha\ =\ \{\alpha_1,\ \alpha_2,\ ....,\ \alpha_N\}$ is a convex set because the set $\alpha\ =\ \{a : a\ \in\ \mathbb{R}_{++}\}$ is a convex set.

2) Maximum of strictly convex functions:

$$g(x) \;=\; \max_i \; f_i(x)$$

Claim: For N strictly convex functions $f_i(x),\; i \;=\; 1,\; 2,\; ....,\; N$, $\max_{i=1}^{N} \; f_i(x)$ is a convex function.

Proof: The $dom(g) \;=\; \bigcap_{i=1}^{N} \; dom(f_i) \;=\; \chi$ is a convex set.

Consider the epigraph of g(x).

$$epi\;\, g \;=\; \{(x,\; t):\; t \geq \max_{i=1}^{N} \; f_i(x)\}$$

$$epi\;\, g \;=\; \bigcap_{i=1}^{N} \{(x,\; t):\; t \geq f_i(x)\}$$

$$epi\;\, g \;=\; \bigcap_{i=1}^{N} epi\;\, f_i$$

Since f_i is a convex function, $epi\;\, f_i$ is a convex set.

The intersection of convex sets is another convex set.

Therefore, $epi\;\, g$ is a convex set.

Hence, $g(x)$ is a convex function.

For $g(x)$ to be a strictly convex function, we need to prove that, for any $x,\; y \;\in\; dom(g)$ and for any $0 \leq \theta \leq 1$,

$$g(\theta x + \bar{\theta} y) < \theta g(x) + \bar{\theta} g(y)$$

Since $f_i(x)$ for $i \;=\; 1,\; 2,\; ....,\; N$ are strictly convex functions,

$$f_i(\theta x + \bar{\theta} y) < \theta f_i(x) + \bar{\theta} f_i(y)$$

Taking max on both sides of the inequality,

$$\max_{i=1}^{N} \; f_i(\theta x + \bar{\theta} y) < \max_{i=1}^{N} \; (\theta f_i(x) + \bar{\theta} f_i(y))$$

This can be written as,

$$\max_{i=1}^{N} \; f_i(\theta x + \bar{\theta} y) < \max_{i=1}^{N} \; (\theta f_i(x)) + \max_{i=1}^{N}(\bar{\theta} f_i(y))$$
$$\max_{i=1}^{N} \; f_i(\theta x + \bar{\theta} y) < \theta \max_{i=1}^{N} \; (f_i(x)) + \bar{\theta} \max_{i=1}^{N}(f_i(y))$$

Now, from the definition of $g(x)$,

$$g(\theta x + \bar{\theta} y) \;\, < \theta g(x) + \bar{\theta} g(y)$$

Therefore, $g(x)$ is a strictly convex function.

B. Solutions to Exercise 2

1) M/M/1 Queueing System:

1) The set of λ values such that the expected delay in the M/M/1 Queueing system is less than a given d is a convex set due to the reasons detailed below.

$$A \; \equiv \; \{\lambda \; \in \; \mathbb{R}_{++} : \; \mathbb{E}[W] < d\}$$

For an M/M/1 Queueing system,

$$\mathbb{E}[W] \; = \; \frac{1}{\mu(1-\rho)}$$

where,

$\mu \; \triangleq Service\ Rate\ of\ the\ Server$

and,

$\rho \; \triangleq \; System\ Load\ or\ System\ Utilization \; = \; \frac{\lambda}{\mu}$

Therefore,

$$\begin{aligned} A \; &\equiv \; \{\lambda \; \in \; \mathbb{R}_{++} : \; \frac{1}{\mu(1-\rho)} < d\} \\ &= \; \{\lambda \; \in \; \mathbb{R}_{++} : \; \frac{1}{\mu - \lambda} < d\} \\ &= \; \{\lambda \; \in \; \mathbb{R}_{++} : \; 1 < d\mu - d\lambda\} \\ &= \; \{\lambda \; \in \; \mathbb{R}_{++} : \; d(\lambda - \mu) + 1 < 0\} \end{aligned}$$

This can be treated as the sub-level set of $f(\lambda) \; = \; d(\lambda - \mu) + 1$.

We know that, if $f(\lambda)$ is a convex function, then all its sub-level sets are convex sets.

So, in order to prove that the set A is a convex set, we need to prove that $f(\lambda)$ is a convex function.

After which, it is trivial to prove that $C_0 \; = \; \{\lambda \; \in \; \mathbb{R}_{++} : \; f(\lambda) < 0\} \; = \; A$ is a convex set.

The function $f(\lambda) \; = \; d(\lambda - \mu) + 1$ is an affine function because,

$$f^{''}(\lambda) \; = \; 0$$

and, for any $\lambda, \; \nu \; \in \; dom(f)$ and for any $0 \leq \theta \leq 1$,

$$f(\theta\lambda + \bar{\theta}\nu) \; = \; \theta f(\lambda) + \bar{\theta} f(\nu)$$

Therefore, $f(\lambda)$ is a convex function and as a result $\{\lambda \in \mathbb{R}_{++} : \mathbb{E}[W] < d\}$ is a convex set.

2) The set of $(\lambda,\ d)$ pairs such that the expected delay in the M/M/1 queueing system is less than a given d is a convex set for reasons detailed below.

$$\begin{aligned} B &\equiv \{(\lambda,\ d):\ \mathbb{E}[W] < d\} \\ &= \{(\lambda,\ d):\ \frac{1}{\mu(1-\rho)} < d\} \\ &= \{(\lambda,\ d):\ \frac{1}{\mu-\lambda} < d\} \\ &= \{(\lambda,\ d):\ d(\lambda-\mu)+1 < 0\} \end{aligned}$$

This can be treated as the epigraph of the convex function $f(\lambda)$.
Since a function is convex if and only if its epigraph is a convex set and knowing that $f(\lambda)$ is a convex function from the previous item,
The set $B \equiv \{(\lambda,\ d):\ \mathbb{E}[W] < d\}$ is a convex set.

C. Multi-User Wireless Communication System

1) The set of vectors $\vec{P} = (P_0,\ P_1,\ ....,\ P_I)$ such that the achievable rate of user 0 is greater than a given value r is a convex set for reasons detailed below.

$$S \equiv \{\vec{P}:\ C_0(\vec{P}) > r\}$$

$$S \equiv \{\vec{P}:\ r - C_0(\vec{P}) < 0\}$$

$$C_0(\vec{P}) = W log_2[1 + \frac{P_0}{(\sum_{i=1}^{I} P_i) + N_0}]$$

We can prove that $C_0(P)$ is a concave function using the following approximations.

$$\frac{W}{log_e 2} log_e[\frac{P_0}{(\sum_{i=1}^{I} P_i) + N_0}] \geq C_0(\vec{P}) \geq \frac{W}{log_e 2}[\frac{P_0}{(\sum_{i=1}^{I} P_i) + N_0}]$$

Using substitutions $P_i = e^{x_i}$,

$$\frac{W}{log_e 2} log_e[\frac{e^{x_0}}{(\sum_{i=1}^{I} e^{x_i}) + N_0}] \geq C_0(\vec{P}) \geq \frac{W}{log_e 2}[\frac{e^{x_0}}{(\sum_{i=1}^{I} e^{x_i}) + N_0}]$$

$$\frac{W}{log_e 2}[x_0 - log_e(\sum_{i=1}^{I} e^{x_i} + N_0)] \geq C_0(\vec{P}) \geq log_e[\frac{W}{log_e 2}] + x_0 - log_e(\sum_{i=1}^{I} e^{x_i} + N_0)]$$

Both approximations (Very Low SINR regime and Very High SINR regime) of the function $C_0(\vec{P})$ are concave because log is a concave function, x_0 is affine (both convex and concave),

$-log_e(\sum_{i=1}^{I}\ e^{x_i}+N_0)$ is concave, and sum of concave functions is concave.
To prove the concavity of $-log_e(\sum_{i=1}^{I}\ e^{x_i}+N_0)$, let's prove the convexity of

$$g(\vec{x})\ =\ log_e(\sum_{i=1}^{I}\ e^{x_i}+N_0)$$

For any $\vec{x},\ \vec{y}\ \in\ dom(g)$ where, $dom(g)\ \subseteq\mathbb{R}^I\ (convex)$ and for any $0\leq\theta\leq 1$, we need to prove that,

$$g(\theta\vec{x}+\bar{\theta}\vec{y})\leq\theta g(\vec{x})+\bar{\theta}g(\vec{y})$$

We can write the left hand side of the above inequality as follows,

$$g(\theta\vec{x}+\bar{\theta}\vec{y})=log_e(\sum_{i=1}^{I}\ e^{\theta x_i+\bar{\theta}y_i}+N_0)$$

N_0 can be written as $e^{log_e N_0}$,

$$g(\theta\vec{x}+\bar{\theta}\vec{y})=log_e(\sum_{i=1}^{I}\ e^{\theta x_i+\bar{\theta}y_i}+e^{log_e N_0})$$

This can further be simplified into,

$$g(\theta\vec{x}+\bar{\theta}\vec{y})=log_e(\sum_{i=1}^{I+1}\ e^{\theta x_i+\bar{\theta}y_i})$$

Let $a_i\ =\ e^{x_i}$ and $b_i\ =\ e^{y_i}$.

$$g(\theta\vec{x}+\bar{\theta}\vec{y})=log_e(\sum_{i=1}^{I+1}\ a_i^{\theta}b_i^{\bar{\theta}})$$

We know from Hölder's inequality that,

$$\sum_{i=1}^{N}\ x_i y_i\ \leq\ (\sum_{i=1}^{N}\ |x_i|^p)^{\frac{1}{p}}(\sum_{i=1}^{N}\ |y_i|^q)^{\frac{1}{q}}$$

Using $x_i\ =\ a_i^{\theta}$, $y_i\ =\ b_i^{\bar{\theta}}$, $p\ =\ \frac{1}{\theta}$, $q\ =\ \frac{1}{\bar{\theta}}$, and $N\ =\ I+1$,

$$\sum_{i=1}^{I+1}\ a_i^{\theta}b_i^{\bar{\theta}}\ \leq\ (\sum_{i=1}^{I+1}\ a_i^{\theta\frac{1}{\theta}})^{\theta}(\sum_{i=1}^{I+1}\ b_i^{\bar{\theta}\frac{1}{\bar{\theta}}})^{\bar{\theta}}$$

Taking the log on both sides of the inequality,

$$log_e[\sum_{i=1}^{I+1}\ a_i^{\theta}b_i^{\bar{\theta}}]\ \leq\ log_e[(\sum_{i=1}^{I+1}\ a_i^{\theta\frac{1}{\theta}})^{\theta}]+log_e[(\sum_{i=1}^{I+1}\ b_i^{\bar{\theta}\frac{1}{\bar{\theta}}})^{\bar{\theta}}]$$

$$log_e[\sum_{i=1}^{I+1}\ a_i^{\theta}b_i^{\bar{\theta}}]\ \leq\theta log_e[\sum_{i=1}^{I+1}\ a_i]+\bar{\theta}log_e[\sum_{i=1}^{I+1}\ b_i]$$

Reverting the substitutions,

$$log_e[\sum_{i=1}^{I+1} e^{x_i\theta} e^{y_i\bar{\theta}}] \leq \theta log_e[\sum_{i=1}^{I+1} e^{x_i}] + \bar{\theta} log_e[\sum_{i=1}^{I+1} e^{y_i}]$$

$$log_e[\sum_{i=1}^{I+1} e^{\theta x_i + \bar{\theta} y_i}] \leq \theta log_e[\sum_{i=1}^{I+1} e^{x_i}] + \bar{\theta} log_e[\sum_{i=1}^{I+1} e^{y_i}]$$

Hence,

$$g(\theta\vec{x} + \bar{\theta}\vec{y}) \leq \theta g(\vec{x}) + \bar{\theta}\vec{y}$$

Therefore, $g(\vec{x}) = log_e(\sum_{i=1}^{I} e^{x_i} + N_0)$ is a convex function due to which $-g(\vec{x}) = -log_e(\sum_{i=1}^{I} e^{x_i} + N_0)$ is a concave function.

Since $C_0(\vec{P})$ is concave, the function $r - C_0(\vec{P})$ is convex.

If a function is convex, its sub-level sets are convex.

Therefore, $\{\vec{P} : \; r - C_0(\vec{P}) < 0\}$ is a convex set.

2) The set of vectors $\vec{x} = (r, \; P_0, \; P_1, \; ...., \; P_I)$ such that the achievable rate of user 0 is greater than r is a convex set because the hypograph of a concave function is a convex set. This is shown below.

$$S \; \equiv \; \{(\vec{P}, \; r) : \; C_0(\vec{P}) > r\}$$

This is of the form,

$$\{(x, \; t) : \; f(x) > t\}$$

which represents the hypograph of a concave function $f(x)$.

From the previous item, we know that, $C_0(\vec{P})$ is a concave function.

Therefore, the set $S \; \equiv \; \{(\vec{P}, \; r) : \; C_0(\vec{P}) > r\}$ is a convex set.

D. Solutions to Exercise 3

The capacity region for two user MAC is defined as,

$$\mathcal{R} \; \equiv \; \{(R_1, \; R_2) : \; R_1 < log_2(1+\frac{g_1 P_1}{N}), \; R_2 < log_2(1+\frac{g_2 P_2}{N}), \; R_1+R_2 < log_2(1+\frac{(g_1 P_1 + g_2 P_2)}{N})\}$$

The set of all $(g_1, \; g_2)$ such that $(R_1, \; R_2)$ lies in $\mathcal{R}$ is a convex set because the intersection of convex sets is also a convex set. This is shown below.

Let,

$$S \; \equiv \{(g_1, \; g_2) : \; R_1 < log_2(1+\frac{g_1 P_1}{N}), \; R_2 < log_2(1+\frac{g_2 P_2}{N}), \; R_1+R_2 < log_2(1+\frac{(g_1 P_1 + g_2 P_2)}{N})\}$$

We know that, $log_2(1+\frac{g_iP_i}{N})$ is a concave function (for $i = 1, 2$ in this problem).
The Hessian of $f(\vec{g}) = log_2(1+\frac{g_1P_1+g_2P_2}{N})$ is negative semidefinite.

$$\nabla^2 f(\vec{g}) \preccurlyeq 0$$

Therefore, $log_2(1+\frac{g_1P_1+g_2P_2}{N})$ is a concave function.
Now, the set S can be treated as the intersection of super-level sets which are convex sets because the associated functions are concave functions.
Hence, S is a convex set because the intersection of convex sets is also a convex set.

E. Solutions to Exercise 3

1) [Boyd] 3.5 - Running average of a convex function:

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

$dom(F) = \mathbb{R}_{++}$ is a convex set
$f(t)$ is a convex function
To prove the convexity of $F(x)$, let's prove that the second order derivative is greater than or equal to 0.
Mathematically, we need to prove that,

$$\frac{d^2}{dx^2}F(x) \geq 0$$

Using Leibnitz's rule of differentiation under the integral sign,

$$\frac{d}{dx}F(x) = \frac{1}{x}f(x) - \frac{1}{x^2}\int_0^x f(t)dt$$

$$\frac{d^2}{dx^2}F(x) = \frac{-1}{x^2}f(x) + \frac{1}{x}f^{'}(x) - \frac{1}{x^2}f(x) + \frac{2}{x^3}\int_0^x f(t)dt$$

$$\frac{d^2}{dx^2}F(x) = \frac{-2}{x^2}f(x) + \frac{1}{x}f^{'}(x) + \frac{2}{x^3}\int_0^x f(t)dt$$

$$\frac{d^2}{dx^2}F(x) = \frac{2}{x^3}(\int_0^x f(t)dt + \frac{x^2}{2}f^{'}(x) - xf(x))$$

Now, $xf(x)$ can be written as $\int_0^x f(x)dt$.
Furthermore, $\frac{x^2f^{'}(x)}{2}$ can be written as $\int_0^x xf^{'}(x)dt - \int_0^x tf^{'}(x)dt$.
Performing these substitutions, we get,

$$\frac{d^2}{dx^2}F(x) = \frac{2}{x^3}(\int_0^x f(t)dt + \int_0^x xf^{'}(x)dt - \int_0^x tf^{'}(x)dt - \int_0^x f(x)dt)$$

Taking the integral outside,

$$\frac{d^2}{dx^2}F(x) \;=\; \frac{2}{x^3}\int_0^x \;(f(t)+xf^{'}(x)-tf^{'}(x)-f(x))dt$$

Let's convert this to the First Order Condition form,

$$\frac{d^2}{dx^2}F(x) \;=\; \frac{2}{x^3}\int_0^x \;(f(t)-f(x)-f^{'}(x)(t-x))dt$$

Since, $f(x)$ is a convex function, we know it satisfies the First Order Condition, i.e.,

$$f(t) \geq f(x) + (t-x)f^{'}(x)$$

Therefore,

$$\frac{d^2}{dx^2}F(x) \geq 0$$

Hence, the running average of a convex function $F(x)$ is also a convex function.

2) [Boyd] 3.15(b) - Modelling effect of satiation using economic utility functions:

$$u_\alpha \;=\; \frac{x^\alpha - 1}{\alpha}$$

For any $0 < \alpha \leq 1$ and for any $x,\ y\ \in\ dom(u_\alpha)$ such that $x \leq y$, we have,

$$u_\alpha(x) \leq u_\alpha(y)$$

Therefore, $u_\alpha(x)$ is a monotonically increasing function.

By inspection, for any $0 < \alpha \leq 1$,

$$u_\alpha(1) \;=\; 0$$

To prove the concavity of u_α, let us prove the second order condition,

$$\frac{d}{dx}\;u_\alpha(x) \;=\; x^{\alpha-1}$$

$$\frac{d^2}{dx^2}\;u_\alpha(x) \;=\; (\alpha-1)x^{\alpha-2}$$

Since $0 < \alpha \leq 1$,

$$\frac{d^2}{dx^2}\;u_\alpha(x) \leq 0$$

Therefore, $u_\alpha(x)$ is a concave function.

3) [Boyd] 3.16 - Identifying convexity and/or concavity:

1) $f(x) = e^x - 1$

$dom(f) = \mathbb{R}$ is a convex set

$$f^{'}(x) = e^x$$

$$f^{''}(x) = e^x$$

$$f^{''}(x) \geq 0, \ \forall x \in dom(f)$$

Therefore, $f(x) = e^x - 1$ is a convex function.

2) $f(x) = x_1 x_2$

$dom(f) = \mathbb{R}^2_{++}$ is a convex set

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\nabla^2 f(x)$ is neither positive semidefinite nor negative semidefinite.

Therefore, $f(x)$ is neither convex nor concave.

3) $f(x) = \frac{1}{x_1 x_2}$

$dom(f) = \mathbb{R}^2_{++}$ is a convex set

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_2^3 x_1} \end{bmatrix}$$

Since $x_1, \ x_2 > 0$, the Hessian matrix above is symmetric and has positive eigenvalues.

Hence, $\nabla^2 f(x)$ is positive definite.

Therefore, $f(x)$ is a convex function.

4) $f(x) = \frac{x_1}{x_2}$

$dom(f) = \mathbb{R}^2_{++}$ is a convex set

$$\nabla^2 f(x) = \begin{bmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

$\nabla^2 f(x)$ is neither positive semidefinite nor negative semidefinite.

Therefore, $f(x)$ is neither convex nor concave.

5) $f(x) = \frac{x_1^2}{x_2}$

$dom(f) = \mathbb{R} \times \mathbb{R}_{++}$ is a convex set

$$\nabla^2 f(x) = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

$$\nabla^2 f(x) \;=\; \frac{2}{x_2}\begin{bmatrix} 1 & \frac{-x_1}{x_2} \\ \frac{-x_1}{x_2} & \frac{x_1^2}{x_2^2} \end{bmatrix}$$

$\nabla^2 f(x)$ is positive semidefinite because the Hessian matrix is symmetric and has non-negative eigenvalues.

Therefore, $f(x)$ is a convex function.

6) $f(x) \;=\; x_1^{\alpha} x_2^{1-\alpha}$
   $dom(f) \;=\; \mathbb{R}^2_{++}$ is a convex set
   $0 \leq \alpha \leq 1$

$$\nabla^2 f(x) \;=\; \begin{bmatrix} \alpha(\alpha-1)x_2^{1-\alpha}x_1^{\alpha-2} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix}$$

$$\nabla^2 f(x) \;=\; \alpha(1-\alpha)x_1^{\alpha}x_2^{1-\alpha}\begin{bmatrix} \frac{-1}{x_1^2} & \frac{1}{x_1x_2} \\ \frac{1}{x_1x_2} & \frac{-1}{x_2^2} \end{bmatrix}$$

The Hessian above is a symmetric matrix with non-positive eigenvalues.

Therefore, $f(x)$ is a concave function.

4) [Boyd] 3.21 - Point-wise maximum and supremum:

1) The function $f(x) \;=\; \max_{i\;=\;1,\;2,\;....,\;k}\; ||A^{(i)}x - b^{(i)}||$ is a convex function due to the reasons detailed below.

   We know that the norm is a convex function because it satisfies Jensen's inequality, i.e.,

   For any $x,\; y \;\in\; dom(f)\;(convex)$ and for any $0 \leq \theta \leq 1$, we know from the Triangle inequality that,

$$||\theta x + \bar{\theta} y|| \leq ||\theta x|| + ||\bar{\theta} y||$$

$$||\theta x + \bar{\theta} y|| \leq \theta ||x|| + \bar{\theta}||y||$$

   Hence, $||x||$ is a convex function.

   The affine mapping of a convex function preserves convexity.

$$A^{(i)} \;\in\; \mathbb{R}^{m\times n},\; b^{(i)} \;\in\; \mathbb{R}^m,\; and\; ||.||\; is\; the\; norm\; on\; \mathbb{R}^m$$

   Therefore, $g_i(x) \;=\; ||A^{(i)}x - b^{(i)}||$ is a convex function.

   Now, putting it all together,

$$f(x) \;=\; h(g_i(x)) \;=\; \max_{i\;=\;1,\;2,\;....,\;k}\; g(x_i)$$

Now, $\max_i \; g_i(x)$ is a convex function because the epigraph of $f(x)$ is a convex set.

Mathematically,

$$epi \; f \;=\; \{(x,\; t):\; t \geq \max_i \; g_i(x)\}$$

$$epi \; f \;=\; \bigcap_i \{(x,\; t):\; t \geq g_i(x)\}$$

$g_i(x)$ is a convex function.

So, the epigraph of $g_i(x)$ is a convex set and the intersection of convex sets is another convex set.

Therefore, $f(x) \;=\; \max_{i\;=\;1,\;2,\;....,\;k} \; ||A^{(i)}x - b^{(i)}||$ is a convex function.

2) $f(x) \;=\; \sum_{i\;=\;1}^{r} \; |x|_{[i]}$ is a convex function due to reasons detailed below.

$dom(f) \;=\; \mathbb{R}^n$ is a convex set.

We know that, $|x|_i$ is a convex function.

Now, $f(x)$ can be written as follows.

Let, $x \;\in\; \mathbb{R}^n$ be defined as $(x_1,\; x_2,\; ....,\; x_n)$

For any $r \leq n$,

$$f(x) \;=\; \max_i \; |x|_i + \max_{i\;\neq\;i_1} \; |x|_i + ... + \max_{i\;\neq\;i_{r-1}} \; |x|_i$$

We know from the previous item that the $\max$ of a convex function is convex and the sum of convex functions is convex.

Therefore, $f(x) \;=\; \sum_{i\;=\;1}^{r} \; |x|_{[i]}$ is a convex function.

5) [Boyd] 3.22 - Composition Rules:

1) $f(x) \;=\; -log(-log(\sum_{i=1}^{m} \; e^{a_i^T x + b_i}))$ is a convex function. The reasoning is provided below.

$dom(f) \;=\; \{x \;:\; \sum_{i=1}^{m} \; e^{a_i^T x + b_i} < 1\}$ is a convex set because e^x is a convex function, the affine mapping of e^x is a convex function, the sum of convex functions is a convex function, and the sub-level sets of a convex function are convex sets.

Let,

$$g(x) \;=\; log(\sum_{i=1}^{m} \; e^{x_i})$$

$g(x)$ is a convex function. Refer to the proof in subsection C of section II.

$g(a_i^T x + b_i),\; for \; i \;=\; 1,\; 2,\; ....,\; m$, is a convex function. because affine mappings of a convex function is also a convex function.

$l(x) \;=\; \sum_{i=1}^{m} \; g(a_i^T x + b_i)$ is a convex function because the sum of convex functions is also a convex function.

$-l(x)$ is a concave function.

Let,

$$f(x) \ = \ -log(-log(\sum_{i=1}^{m} \ e^{a_i^T x + b_i})) \ = \ h(-l(x))$$

Now, using composition rules,

$-l(x)$ is a concave function.

$h(x)$ is a convex function and it's decreasing.

Therefore, $f(x)$ is a convex function.

2) $f(x, \ u, \ v) \ = \ -\sqrt{uv - x^T x}$ is a convex function using the following rationale.

$$f(x, \ u, \ v) \ = \ -\sqrt{u(v - \frac{x^T x}{u})}$$

$dom(f) \ = \ \{(x, \ u, \ v) : \ uv > x^T x, \ u, \ v > 0\}$ is a convex set.

Using the given fact that $\frac{x^T x}{u}$ is a convex function,

$v - \frac{x^T x}{u}$ is a concave function

Expressing $f(x, \ u, \ v)$ to show the composition operation,

$$f(x, \ u, \ v) \ = \ h(g(x, \ u, \ v))$$

Using the given fact that $-\sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2_{++}$,

h is convex and decreasing.

g is concave.

Therefore, $f(x, \ u, \ v)$ is a convex function.

3) $f(x, \ u, \ v) \ = \ -log(uv - x^T x)$ is a convex function due to the following reasoning.

$$f(x, \ u, \ v) \ = \ -log(uv - x^T x) \ = \ -log(u(v - \frac{x^T x}{u})) \ = \ -log(u) - log(v - \frac{x^T x}{u})$$

$dom(f) \ = \ \{(x, \ u, \ v) : \ uv > x^T x, \ u, \ v > 0\}$ is a convex set.

$v - \frac{x^T x}{u}$ is a concave function from the previous item.

$-log(v - \frac{x^T x}{u})$ is a convex function because for $f(x, \ u, \ v) \ = \ h(g(x, \ u, \ v))$, h is convex and decreasing, and g is concave.

$-log(u)$ is a convex function.

The sum of convex functions is also a convex function.

Therefore, $f(x, \ u, \ v) \ = \ -log(uv - x^T x)$ is a convex function.

4) $f(x,\ t)\ =\ -(t^p - ||x||_p^p)^{\frac{1}{p}}$

$$dom(f)\ =\ \{(x,\ t):\ t \geq ||x||_p\}$$

The $dom(f)$ is a convex set because $||x||_p$ is a convex function and the epigraph of a convex function is a convex set.

$$f(x,\ t)\ =\ -t^{1-\frac{1}{p}}(t - \frac{||x||_p^p}{t^{p-1}})^{\frac{1}{p}}$$

Using the fact that $\frac{||x||p^p}{u^{p-1}}$ is convex in $(x,\ u)$ for $u > 0$,
$t - \frac{||x||_p^p}{t^{p-1}}$ is a concave function.
Using the given fact that $-x^{\frac{1}{p}}y^{1-\frac{1}{p}}$ is a convex function on $\mathbb{R}_+^2$, we can finally say that,
$f(x,\ t)\ =\ -(t^p - ||x||_p^p)^{\frac{1}{p}}$ is a convex function.

5) $f(x,\ t)\ =\ -log(t^p - ||x||_p^p)$ is a convex function based on the following rationale.

$$dom(f)\ =\ \{(x,\ t):\ t > ||x||_p\}$$

$dom(f)$ is a convex set because the epigraph of a convex function, i.e. $||x||_p$ is a convex set.

$$f(x,\ t)\ =\ -log(t^{p-1}(t - \frac{||x||_p^p}{t^{p-1}}))$$

Using the fact that $\frac{||x||_p^p}{t^{p-1}}$ is a convex function, $t - \frac{||x||_p^p}{t^{p-1}}$ is a concave function.

$$f(x,\ t)\ =\ -log(t^{p-1}) - log(t - \frac{||x||_p^p}{t^{p-1}})$$

$$f(x,\ t)\ =\ -(p-1)log(t) - log(t - \frac{||x||_p^p}{t^{p-1}})$$

Therefore, since $p > 1$, $f(x,\ t)\ =\ -log(t^p - ||x||_p^p)$ is a convex function.