PU occupancy behavior estimation

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1 System Model

1.1 Assumptions

1. There's only one Primary User (PU) in the wideband spectrum of interest.
2. There's only one Secondary User (SU) making observations of the PU occupancy in the wideband spectrum of interest.
3. If $B = \{b_1,\ b_2,\ b_3,\ .....,\ b_K\}$ represents the set of all sub-bands in the wideband spectrum of interest, then it's assumed that considering energy detection, for any band $b_k \in B$, $E[|X_k(i)|^2] = 1$ if it is occupied by the PU, else $E[|X_k(i)|^2] = 0$.
4. The noise samples $V_k(i)$ are i.i.d Complex Gaussian with zero mean and variance σ_V^2 independent of PU occupancy state in the wideband spectrum of interest. Furthermore, the noise samples are i.i.d across frequency and across observation rounds.
5. Furthermore, the PU occupancy behavior is assumed to be static during the estimation period of our algorithm.
6. The Hidden Markov Model parameters are assumed to be known for now in order to come up with an optimal algorithm for state estimation.

1.2 Model

$$y(n) = \sum_{m=0}^{M-1} h(m)x(n-m) + v(n) \tag{1}$$

Here, $y(n)$ is the wideband signal observed at the SU receiver expressed as a convolution of the PU signal $x(n)$ with the channel impulse response $h(n)$ added with a noise term v(n). Equation (1) can be written in the frequency domain by taking a K-point DFT which decomposes the observed wideband signal into K discrete narrow-band components as shown below,

$$Y_k(i) = H_k X_k(i) + V_k(i) \tag{2}$$

where,
$i \in \{1,\ 2,\ 3,\ ....,\ N\}$ represents the index of the observation
$k \in \{1,\ 2,\ 3,\ ....,\ K\}$ represents the index of the sub-band
$V_k(i) \sim \mathcal{N}(0,\ \sigma_V^2)$ represents the zero-mean additive Gaussian noise sample
$H_k \sim \mathcal{N}(0,\ \sigma_H^2)$ represents the k^{th} DFT coefficient of the impulse response $h(n)$ of the channel in between the PU and the SU receiver
The PU occupancy behavior in each sub-band $b_k \in B$ is modelled as X_k taking two possible values 0 *and* 1. Therefore, the PU occupancy behavior in the entire wideband spectrum of interest discretized into narrow-band frequency components can be modelled as a vector of size $|B| = K$ such that,

$$\vec{X} = [X_1, X_2, X_3, ..., X_K]^T \in \{0,1\}^K \tag{3}$$

The true states encapsulate the actual behavior of the PU which is an unobserved Markov process and the measurements at the SU are noisy observations of the true states which are modelled to be the observed states of the Hidden Markov Model. For some sub-band $j \in \{1,\ 2,\ 3,\ .....,\ K-1\}$, the system is assumed to satisfy the Markov property as shown below,

$$P(X_j(i)|X_{j-1}(i),\ X_{j-2}(i),\ .....,\ X_1(i)) \ = \ P(X_j(i)|X_{j-1}(i)),\ for\ j > 1,$$
$$And,\ we\ will\ use\ P(X_1(i))\ for\ j = 1.$$

Since **the PU is assumed to be static in the period of our estimation**, we can write the above assumption as,

$$P(X_j|X_{j-1},\ X_{j-2},\ .....,\ X_1) \ = \ P(X_j|X_{j-1}),\ for\ j > 1,$$
$$And,\ we\ will\ use\ P(X_1)\ for\ j = 1.$$

Now, we know that,

$$\vec{X} \ = \ [X_1,\ X_2,\ X_3,\ ......,\ X_K]^T$$

which realizes as,

$$\vec{x} \ = \ [x_1,\ x_2,\ x_3,\ ......,\ x_K]^T$$

So,

$$P(\vec{X} = \vec{x}) = P(X_1 = x_1) \prod_{k=2}^{K} P(X_k = x_k|X_{k-1} = x_{k-1}) \tag{4}$$

Now, let's expand on the observation model. With a realization $x_k\ in\ \{0,1\}$ of X_k, we have from assumption 3 that,

$$E[|X_k(i)|^2] \ = \ x_k,\ given\ X_k\ during\ observation\ cycle\ i\ has\ realized\ as\ x_k$$

Taking the expectation operator on both sides of equation (2) given X_k has realized as x_k, we have,

$$E[Y_k(i)\ |\ X_k(i) = x_k] \ = \ E[H_k x_k] \ + \ E[V_k(i)]$$

$$E[Y_k(i) \mid X_k(i) = x_k] \ = \ E[H_k]E[x_k] \ + \ E[V_k(i)]$$
$$E[Y_k(i) \mid X_k(i) = x_k] \ = \ 0 \ + \ 0$$
$$E[Y_k(i)|X_k(i) = x_k] = 0 \tag{5}$$

because, as already discussed, $V_k(i) \ \sim \ \mathcal{N}(0, \ \sigma_V^2)$ and $H_k \ \sim \ \mathcal{N}(0, \ \sigma_H^2)$. Furthermore, the variance of $Y_k(i)$ given X_k at observation cycle i has realized as x_k, is calculated to be,

$$Var[Y_k(i)|X_k(i) = x_k] = E[(Y_k(i)|X_k(i) = x_k)^2] - [E[Y_k(i)|X_k(i) = x_k]]^2$$
$$Var[Y_k(i)|X_k(i) = x_k] = E[|H_kX_k(i)|^2 + |V_k(i)|^2 + 2H_kX_k(i)V_k(i)] - (0)^2$$
$$Var[Y_k(i)|X_k(i) = x_k] = \sigma_H^2 E[|X_k(i)|^2] + \sigma_V^2 + 2E[H_k]E[X_k(i)]E[V_k(i)]$$
$$Var[Y_k(i)|X_k(i) = x_k] = \sigma_H^2 x_k + \sigma_V^2 \tag{6}$$

2 The Estimator

Given: The observations of the K frequency sub-bands in the wideband spectrum of interest, i.e. $Y_1, \ Y_2, \ Y_3, \ ....., \ Y_K$
Assuming the state transition probability matrix A and the array of initial probabilities Π are known.
From the observation model, we already know that the emission probabilities are given by,

$$P(Y_k|X_k = x_k) \ \sim \mathcal{N}(0, \ \sigma_H^2 x_k + \sigma_V^2)$$

Now, the problem of estimating a sequence of states across the frequency bands in a Hidden Markov Model can be solved using Dynamic Programming to give us the most likely sequence of hidden states called the **Viterbi Path** based on the sequence of noisy observations of the true states of the frequency sub-bands. From the above statements we can write,

$$P(\vec{X} = \vec{x}) = P(X_1 = x_1) \prod_{k=2}^{K} P(X_k = x_k|X_{k-1} = x_{k-1})$$

Now, the optimization problem can be written as follows,

$$\vec{x}^* = argmax_{\vec{x}} P(\vec{X}|\vec{Y}) \tag{7}$$

Here, $\vec{Y}$ represents the observation vector consisting of the observations of the K sub-bands given by equation (2), as shown below,

$$\vec{Y} \ = \ [Y_1, \ Y_2, \ ...., \ Y_K]^T$$

In other words,

$$\vec{x}^* \ represents \ the \ Viterbi \ path \ across \ frequency \ sub-bands$$

$\vec{Y}$ *represents the sequence of observations across frequency sub − bands*

This argmax problem can be re-written as a maximization problem of the joint distribution due to the proportional relation between the joint and the conditional. Therefore, Equation (7) can be written as,

$$V_i^{(j)} = max_{x_1,x_2,...,x_{i-1}} P(y_1, y_2, ..., y_{i-1}, x_1, x_2, ..., x_{i-1}, y_i, x_i = j) \tag{8}$$

Here, $V_i^{(j)}$ represents **a value function in our optimization problem tracking the sequence of states of sub-bands that maximize the joint distribution of states and observations as detailed in Equation (8)**.

Now, for the $(i+1)^{th}$ sub-band in state l, repeating the same step, we have,

$$V_{i+1}^{(l)} = max_{x_1,x_2,...,x_i} P(y_1, y_2, ..., y_i, x_1, x_2, ..., x_i, y_{i+1}, x_{i+1} = l) \tag{9}$$

Using the definition of conditional probability, we have,

$$V_{i+1}^{(l)} = max_{x_1,x_2,...,x_i} P(y_{i+1}, x_{i+1} = l | y_1, y_2, ..., y_i, x_1, x_2, ..., x_i) P(y_1, y_2, ..., y_i, x_1, x_2, ..., x_i) \tag{10}$$

Now, from the Markov Property, we have,

$$V_{i+1}^{(l)} = max_{x_1,x_2,...,x_i} P(y_{i+1}, x_{i+1} = l | x_i) P(y_1, y_2, ..., y_i, x_1, x_2, ..., x_i) \tag{11}$$

Pushing the maximization operator in,

$$V_{i+1}^{(l)} = max_j [P(y_{i+1}, x_{i+1} = l | x_i = j) max_{x_1,x_2,...,x_{i-1}} [P(y_1, y_2, ..., y_{i-1}, x_1, x_2, ..., x_{i-1}, y_i, x_i = j)]] \tag{12}$$

Using Equation (8),

$$V_{i+1}^{(l)} = max_j [P(y_{i+1}, x_{i+1} = l | x_i = j) V_i^{(j)}] \tag{13}$$

We know that, for three random variables R, U, and W,

$$P(R, U|W) = P(U|R, W) P(R|W)$$

Using this, we have,

$$V_{i+1}^{(l)} = max_j [P(y_{i+1} | x_{i+1} = l, x_i = j) P(x_{i+1} = l | x_i = j) V_i^{(j)}] \tag{14}$$

$$V_{i+1}^{(l)} = max_j [P(y_{i+1} | x_{i+1} = l) P(x_{i+1} = l | x_i = j) V_i^{(j)}] \tag{15}$$

Let, $m_l(y_{i+1})$ be the emission probability, i.e. the probability of emission of observation y_{i+1} in state l.

Let, a_{jl} be the state transition probability. Then,

$$V_{i+1}^{(l)} = m_l(y_{i+1}) \; max_j [a_{jl} V_i^{(j)}] \tag{16}$$

Here, from the observation model,

$$m_l(y_{i+1}) \sim \mathcal{N}(0, \; \sigma_H^2 l + \sigma_V^2)$$

And, from the system's Markov model,

$$a_{jl} \ \in \ A, \ : \ a_{jl} \ = \ P(x_{i+1} = l | x_i = j)$$

Equation (16) constitutes the **Forward Recursion aspect of the Viterbi algorithm**.
Now, we analytically derive the **Backtrack feature of the Viterbi algorithm** below.
The state of the K^{th} sub-band, i.e the last state in the Viterbi path is given by,

$$k^* \ = \ argmax_k \ V_K^{(k)} \tag{17}$$

This can be written as follows,

$$k^* \ = \ argmax_k \ max_{x_1,x_2,...,x_{K-1}} \ P(x_1, x_2, ..., x_{K-1}, x_K = k, y_1, y_2, ...y_K) \tag{18}$$

Essentially, the idea here is to prove the an earlier sub-band in the sequence is in a certain state given that a later sub-band in the sequence is in a certain state.
So,
Given: $x_{i+1} = l^*$ is the state of the $(i+1)^{th}$ sub-band in the most likely state sequence.
To find an analytical solution for the state of the i^{th} sub-band in the most likely state-sequence.
Consider the pointer,

$$Ptr_{i+1} \ = \ argmax_j \ (a_{jl} V_i^{(j)})$$

Now, substituting in the definitions of the state transition probabilities and the value function,

$$Ptr_{i+1} = argmax_j \ P(x_{i+1} = l^* | x_i = j) \ max_{x_1,x_2,...,x_{i-1}} P(y_1, y_2, ..., y_{i-1}, x_1, x_2, ..., x_{i-1}, y_i, x_i = j) \tag{19}$$

Moving the constant in or taking max operator outside,

$$Ptr_{i+1} = argmax_j \ max_{x_1,x_2,...,x_{i-1}} \ P(x_{i+1} = l^* | x_i = j) P(y_1, y_2, ..., y_{i-1}, x_1, x_2, ..., x_{i-1}, y_i, x_i = j) \tag{20}$$

We can write Equation (20) as,

$$Ptr_{i+1} = argmax_j \ max_{x_1,x_2,...,x_{i-1}} \ P(x_{i+1} = l^* | x_1, x_2, ..., x_{i-1} x_i = j, y_1, y_2, ..., y_{i-1}, y_i) \\ P(y_1, y_2, ..., y_{i-1}, x_1, x_2, ..., x_{i-1}, y_i, x_i = j) \tag{21}$$

Using Chain Rule, this product becomes the joint distribution,

$$Ptr_{i+1} = argmax_j \ max_{x_1,x_2,...,x_{i-1}} \ P(x_{i+1} = l^*, x_1, x_2, ..., x_{i-1}, x_i = j, y_1, y_2, ..., y_{i-1}, y_i) \tag{22}$$

Adding a constant to the argmax operation, i.e. j should not feature in this constant, we have,

$$Ptr_{i+1} = argmax_j \ (max_{x_{i+1},xi+2,...,x_K} \ P(x_{i+2}, x_{i+3}, ..., x_K, y_{i+1}, y_{i+2}, ..., y_K | x_{i+1} = l^*)) \\ max_{x_1,x_2,...,x_{i-1}} \ P(x_{i+1} = l, x_1, x_2, ..., x_{i-1}, x_i = j, y_1, y_2, ..., y_{i-1}, y_i)) \tag{23}$$

$$Ptr_{i+1} = argmax_j \ max_{x_{i+1},xi+2,...,x_K} \ max_{x_1,x_2,...,x_{i-1}} \ P(x_{i+2}, x_{i+3}, ..., x_K, y_{i+1}, y_{i+2}, ..., y_K | x_{i+1} = l^*) \\ P(x_{i+1} = l, x_1, x_2, ..., x_{i-1}, x_i = j, y_1, y_2, ..., y_{i-1}, y_i) \tag{24}$$

We can write Equation (24) as follows due to the independence relation exhibited by the Markov Model,

$$Ptr_{i+1} = argmax_j \ max_{x_{i+1},xi+2,...,x_K} \ max_{x_1,x_2,...,x_{i-1}} P(x_{i+2}, x_{i+3}, ..., x_K, y_{i+1}, y_{i+2}, ..., y_K | x_{i+1} = l^*, \\ x_i = j, x_{i-1}, ..., x_1, y_i, y_{i-1}, ..., y_1) P(x_{i+1} = l, x_1, x_2, ..., x_{i-1}, x_i = j, y_1, y_2, ..., y_{i-1}, y_i) \tag{25}$$

Using Chain Rule again and consolidating the max operator,

$$Ptr_{i+1} = argmax_j \ max_{x_1,x_2,...,x_{i-1},x_{i+1},x_{i+2},...,x_K} \ P(x_{i+2}, x_{i+3}, ..., x_K, x_{i+1} = l^*, x_i = j, x_{i-1}, ..., x_1, \\ y_{i+1}, y_{i+2}, ..., y_K, y_i, y_{i-1}, ..., y_1) \tag{26}$$

Now, the right-hand side of Equation (26) corresponds to the state of the i^{th} sub-band in most-likely state sequence.
Therefore,

$$Ptr_{i+1} \ = \ x_i* \ = \ j* \tag{27}$$

This constitutes an overlapping sub-problems solution which can be solved using Dynamic Programming. The idea is to recursively traverse through the Trellis diagram to find the next state which maximizes the probability of the traversed path. Using the analytical results obtained above, we can now write the algorithm.

3 The Algorithm

Initialization: The array of initial probabilities Π is known.
Forward Recursion: $V_j^{(r)} \ = \ m_r(y_j) \ max_l[a_{lr} V_{j-1}^{(l)}]$ and $Ptr_j \ = \ argmax_l \ (a_{lr} \ V_{j-1}^{(l)})$
Backtrack: $k^* \ = \ argmax_k(V_K^{(k)})$ and $x_{i-1}^* \ = \ Ptr_i$
Termination: $P(y, \vec{x}^*) \ = \ max_k \ (V_K^{(k)})$
This will be implemented in Python and the results such as P_{FA} : False Alarm Probability, P_D : Detection Probability, and P_{MD} : Missed Detection Probability will be reported.

4 Changes in this version of the document (v3.3.0) over (v3.2.0)

1. Removed the part which said $\vec{X}$ is a Bernoulli vector consisting of K Bernoulli random variables because this would imply i.i.d among sub-bands which would be counter-intuitive to our System Model which assumes Markov across the sub-bands
2. Refined the Markov property in the System Model for $j = 1$
3. Defined the parameters $m_l(y_{i+1})$ and a_{jl} mathematically in the Forward Recursion analysis segment
4. Included the Backtrack analysis to derive the Viterbi algorithm in complete detail
5. Added extensions to the To-Do List based on 11-07-2018 meeting.

5 Extensions based on 11-07-2018 Meeting

1. Implement this Viterbi algorithm assuming initial probabilities and check the functionality by tuning the probabilities in the state transition matrix using the initial probabilities assumption
2. Compare the results of this implementation with the results from the algorithm detailed in reference [1]
3. Extend it to Markovian across time (Dynamic PU behavior)
4. Extend it to a case where we only sense a few bands (as directed by our Bandit) instead of sensing all the bands
5. Bandits and Deep Reinforcement Learning
6. Applications of Coupled Hidden Markov Models (CHMMs) $\longrightarrow$ References [2], [3], and [4]

5.1 External References

1. Generalized correlation model using correlation coefficients:
https://ieeexplore.ieee.org/document/#
2. Factorial HMMs:
http://www.ee.columbia.edu/ sfchang/course/svia-F03/papers/factorial-HMM-97.pdf
3. Coupled HMMs:
http://www.ee.columbia.edu/ sfchang/course/svia-F03/papers/brand96coupled-hmm.pdf

4. Modelling twitter user activity:
https://arxiv.org/pdf/1305.1980.pdf