The following problems are based on connecting **probability distributions** with **physical intuition**, in particular symmetry arguments, velocity moments, and thermal estimates, covered in the class in the last two weeks.

1: Even–Odd Symmetry in Probabilities

Consider a particle velocity distribution that depends only on the square of the velocity:

$$P(v) \propto e^{-av^2}, \quad v \in (-\infty, \infty).$$

(a) Show that $P(v)$ is an even function of v.

(b) Argue which of the following moments vanish:

$$\langle v \rangle, \quad \langle v^2 \rangle, \quad \langle v^3 \rangle.$$

(c) Explain the physical meaning of your answer in terms of left-moving vs. right-moving particles.

2: First and Second Moments

Take the normalized one-dimensional velocity distribution

$$P(v) = \sqrt{\frac{a}{\pi}}\, e^{-av^2}.$$

(a) Verify the normalization: $\displaystyle\int_{-\infty}^{\infty} P(v)\, dv = 1$.

(b) Compute the first and second moments $\langle v \rangle$ and $\langle v^2 \rangle$.

(c) By physical reasoning: Why is $\langle v \rangle = 0$ while $\langle v^2 \rangle \neq 0$? Relate this to kinetic energy.

3: Order of Magnitude of $\langle v^2 \rangle$

In kinetic theory there are two relevant scales in the problem: the mass of the particle m, which sets the inertia, and the thermal energy $k_B T$, which sets the "kick" provided by random collisions. When combined, these two scales must determine the typical size of velocity fluctuations.

(a) By dimensional reasoning, argue why the average squared velocity must take the form

$$\langle v^2 \rangle \sim \frac{k_B T}{m}.$$

(Hint: there are only two parameters, m and $k_B T$, available to set the units.)

(b) Compare this thermal estimate with the exact result for $\langle v^2 \rangle$ you computed in Problem 2, after identifying the parameter a with $m/(2k_B T)$.

(c) Discuss the physical implication: at the same temperature, lighter molecules (e.g. H_2) have larger $\sqrt{\langle v^2 \rangle}$ than heavier molecules (e.g. O_2). Give one consequence of this fact (examples: diffusion speed in air, likelihood of atmospheric escape from a planet).