

# SDPB 2.7.0

April 20, 2024

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## 1 Introduction

SDPB is an arbitrary-precision semidefinite program solver, specialized for “polynomial matrix programs” (defined below). This document describes SDPB’s usage and input/output. Much more detail about its design is given in [1]. The reader is encouraged to look there for a better understanding of SDPB’s parameters and internal operation.

## 1.1 Installation and Requirements

See `Install.md` for up-to-date instructions on getting pre-made binaries or building from source.

## 2 Polynomial Matrix Programs

SDPB solves the following type of problem, which we call a *polynomial matrix program* (PMP). Consider a collection of symmetric polynomial matrices

$$M_j^n(x) = \begin{pmatrix} P_{j,11}^n(x) & \dots & P_{j,1m_j}^n(x) \\ \vdots & \ddots & \vdots \\ P_{j,m_j1}^n(x) & \dots & P_{j,m_jm_j}^n(x) \end{pmatrix} \quad (2.1)$$

labeled by  $0 \leq n \leq N$  and  $1 \leq j \leq J$ , where each element  $P_{j,rs}^n(x)$  is a polynomial in  $x$ . Given  $b \in \mathbb{R}^N$ , we would like to

$$\begin{aligned} & \text{maximize} && b_0 + b \cdot y && \text{over } y \in \mathbb{R}^N, \\ & \text{such that} && M_j^0(x) + \sum_{n=1}^N y_n M_j^n(x) \succeq 0 && \text{for all } x \geq 0 \text{ and } 1 \leq j \leq J. \end{aligned} \quad (2.2)$$

The notation  $M \succeq 0$  means “ $M$  is positive semidefinite.”

The constant  $b_0$  in formulation (2.2) is completely irrelevant to the solution algorithm, but is included for convenience.

## 3 Input to SDPB

You will normally start with a Polynomial Matrix Program (PMP) described in a file (or several files) in JSON, Mathematica, or XML format. PMP must first be converted, using `pmp2sdp` program, into an internal SDP format that SDPB can quickly load. SDP format is described in `docs/SDPB_input_format.md`.

`pmp2sdp` uses a PMP definition that is slightly different but equivalent to (2.2):

$$\begin{aligned} & \text{maximize} && a \cdot z && \text{over } z \in \mathbb{R}^{N+1}, \\ & \text{such that} && \sum_{n=0}^N z_n W_j^n(x) \succeq 0 && \text{for all } x \geq 0 \text{ and } 1 \leq j \leq J, \\ & && n \cdot z = 1. \end{aligned} \quad (3.1)$$

where  $W_j^n(x)$  are matrix polynomials. The normalization condition  $n \cdot z = 1$  can be used to solve for one of the components of  $z$  in terms of the others. For numerical stability, SDPB eliminates the component  $z_k$  that has the largest absolute value of  $|n_k|$ . Calling the remaining components  $y \in \mathbb{R}^N$ , we arrive at (2.2), where  $M_j^n(x)$  are linear combinations of  $W_j^n(x)$  and  $b_0, b_n$  are linear combinations of the  $a_n$ . This difference in convention is for convenient use in the conformal bootstrap. Note that conversion to (2.2) is trivial, if

you choose normalization vector  $n = (1, 0, \dots, 0)$ : in that case  $(b_0 \dots b_N) = (a_0 \dots a_N)$ ,  $(y_1 \dots y_N) = (z_1 \dots z_N)$ ,  $M_j^n(x) = W_j^n(x)$ .

To construct an SDP, `pmp2sdp` needs the following information:

- for each  $j = 1, \dots, J$ :
  - polynomial matrices  $W_j^0(x), \dots, W_j^N(x)$  of maximum degree  $d_j$ ,
  - bilinear bases  $q_m^{(j)}(x)$  ( $m = 0, \dots, \lfloor d_j/2 \rfloor$ ),
  - sample points  $x_k^{(j)}$  ( $k = 0, \dots, d_j$ ),
  - sample scalings  $s_k^{(j)}$  ( $k = 0, \dots, d_j$ ),
- an objective function  $a \in \mathbb{R}^{N+1}$ .
- a normalization vector  $n \in \mathbb{R}^{N+1}$ .

A bilinear basis is a collection of polynomials  $q_m^{(j)}(x)$  such that  $\deg q_m^{(j)} = m$ , for example monomials  $q_m^{(j)}(x) = x^m$ . (A better choice for numerical stability is usually orthogonal polynomials on the positive real line.) The sample points, sample scalings and bilinear bases determine how the PMP is represented internally as an SDP. In principle, they do not affect the solution of the PMP, but in practice they can affect numerical stability. See Section 3.3 in [1] for details.

`pmp2sdp` can parse input in **JSON**, **Mathematica** and **XML** formats, described below. We recommend using **JSON**; the two other formats are left for backward compatibility.

For convenience and/or for better efficiency when running `pmp2sdp` in parallel, you may split PMP into several files. In that case, you should provide an `.nsv` file with a null-separated list of PMP files. See details in `docs/Usage.md`.

### 3.1 JSON format

**Listing 1:** JSON format for PMP

```
PMP ≡ {
  "objective": ["a0", ..., "aN"],
  (optional) "normalization": ["n0", ..., "nN"],
  "PositiveMatrixWithPrefactorArray": [
    <positive matrix with prefactor 1>,
    ...
    <positive matrix with prefactor J>,
  ]
}

positive matrix with prefactor j ≡ {
  (optional) "DampedRational": <damped-rational prefactor>,
}
```

```

    "polynomials": [
      [⟨polynomial vector  $Q_{j,11}^n(x)$ ⟩, ... ⟨polynomial vector  $Q_{j,m_j1}^n(x)$ ⟩],
      ...
      [⟨polynomial vector  $Q_{j,1m_j}^n(x)$ ⟩, ..., ⟨polynomial vector  $Q_{j,m_jm_j}^n(x)$ ⟩]
    ],
    (optional) "samplePoints": [" $x_0^{(j)}$ ", ..., " $x_{d_j}^{(j)}$ "],
    (optional) "sampleScalings": [" $s_0^{(j)}$ ", ..., " $s_{d_j}^{(j)}$ "],
    (optional) "bilinearBasis": [
      ⟨polynomial  $q_0^{(j)}(x)$ ⟩,
      ...
      ⟨polynomial  $q_{\lfloor d_j/2 \rfloor}^{(j)}(x)$ ⟩
    ]
  }

polynomial vector  $Q_{j,rs}^n(x) \equiv [\langle \text{polynomial } Q_{j,rs}^0(x) \rangle, \dots, \langle \text{polynomial } Q_{j,rs}^N(x) \rangle]$ 

polynomial  $c_0 + c_1x + \dots + c_dx^d \equiv ["c_0", \dots, "c_d"]$ 

damped-rational prefactor  $c \frac{b^x}{\prod_{i=1}^k (x-p_i)} \equiv \{$ 
  "constant": " $c$ "
  "base": " $b$ "
  "poles": [" $p_1$ ", ..., " $p_k$ "]
}

```

Some fields in JSON format are optional.

If "normalization" is not specified, the default value  $n = (1, 0, \dots, 0)$  is assumed, effectively reducing formulation (3.1) to (2.2).

If "samplePoints" are not specified, pmp2sdp will make a choice of sample points automatically. For historical reasons, the choice is

$$x_k^{(j)} = \frac{\pi^2(-1 + 4k)^2}{-64(d_j + 1) \log(3 - 2\sqrt{2})}. \quad (3.2)$$

If "sampleScalings" and/or "bilinearBasis" are not specified, they are calculated automatically from the "DampedRational" prefactor. Specifically, if the prefactor is  $\chi(x)$ , the bilinear basis is a set of orthogonal polynomials with respect to measure  $\chi(x)dx$  on the positive real line, and sample scalings are  $\chi(x_k)$ , where the  $x_k$  are sample points. Only damped-rational prefactors are supported. Such prefactors are relevant to the conformal bootstrap problems. These stand for

$$\text{DampedRational}[c, \{p_1, \dots, p_k\}, b, x] \rightarrow c \frac{b^x}{\prod_{i=1}^k (x - p_i)}. \quad (3.3)$$

If "DampedRational" is not specified, default sample scalings and bilinear basis are calculated assuming the prefactor  $\chi(x) = e^{-x}$ .

The JSON format is also described by the schema in `docs/json_schema/pmp_schema.json`.

## 3.2 Mathematica interface

JSON can be generated with a `WritePmpJson` function in the included Mathematica package `mathematica/SDPB.m`. Here `file` is the JSON file to be written to, and `sdp` has a form described in Listing 2. You can also provide a custom function `samplePointsFn[d]` generating sample points  $x_0, \dots, x_d$  for a polynomial of degree  $d$ .

As an example bootstrap application, the included notebook `Bootstrap2dExample.m` computes a single-correlator dimension bound for 2d CFTs with a  $\mathbb{Z}_2$  symmetry, as in [2].

`SDPB.m` also defines `WritePmpXml[file, sdp]` function which writes PMP in XML format described in Section 3.3. In addition, `pmp2sdp` can also directly read `.m` files containing SDP definition.

**Listing 2:** Usage of `WritePmpJson` in `SDPB.m`

```
function call ≡
WritePmpJson[file, ⟨sdp⟩]
or
WritePmpJson[file, ⟨sdp⟩, ⟨samplePointsFn⟩]

sdp ≡ SDP[⟨objective⟩, ⟨normalization⟩, ⟨positive matrices with prefactors⟩]

objective ≡ {a0, ..., aN}

normalization ≡ {n0, ..., nN}

positive matrices with prefactors ≡ {
    ⟨positive matrix with prefactor 1⟩,
    ...,
    ⟨positive matrix with prefactor J⟩,
}

positive matrix with prefactor j ≡
PositiveMatrixWithPrefactor[⟨prefactor⟩,
{
    {
        {Qj,110(x), ..., Qj,11N(x)}, ..., {Qj,mj10(x), ..., Qj,mj1N(x)}
    },
    ...,
    {
        {Qj,1mj0(x), ..., Qj,1mjN(x)}, ..., {Qj,mjmj0(x), ..., Qj,mjmjN(x)}
    },
}
]
```

```

prefactor  $\equiv$ 
DampedRational[c, {p1, ..., pk}, b, x]
or
const

samplePointsFn[d_]  $\equiv$  function returning a list {x0, ..., xd}

```

### 3.3 XML format

pmp2sdp can read XML files describing PMP in form (2.2). Note that we recommend using JSON as more compact and efficient.

**Listing 3:** XML format for PMP

```

PMP  $\equiv$ 
<sdp>
  <xml for objective>
  <xml for polynomial vector matrices>
</sdp>

xml for objective  $\equiv$ 
<objective>
  <elt>b0</elt>
  ...
  <elt>bN</elt>
</objective>

xml for polynomial vector matrices  $\equiv$ 
<polynomialVectorMatrices>
  <xml for polynomial vector matrix M1n(x)>
  ...
  <xml for polynomial vector matrix Mjn(x)>
</polynomialVectorMatrices>

xml for polynomial vector matrix Mjn(x)  $\equiv$ 
<polynomialVectorMatrix>
  <rows>mj</rows>
  <cols>mj</cols>
  <elements>
    <xml for polynomial vector Pj,11n(x)>
    ...
    <xml for polynomial vector Pj,mj1n(x)>
    ...
    <xml for polynomial vector Pj,1mjn(x)>
    ...
    <xml for polynomial vector Pj,mjmjn(x)>
  </elements>

```

```

<samplePoints>
<elt> $x_0^{(j)}$ </elt>
...
<elt> $x_{d_j}^{(j)}$ </elt>
</samplePoints>
<sampleScalings>
<elt> $s_0^{(j)}$ </elt>
...
<elt> $s_{d_j}^{(j)}$ </elt>
</sampleScalings>
<bilinearBasis>
<xml for polynomial  $q_0^{(j)}(x)$ >
...
<xml for polynomial  $q_{\lfloor d_j/2 \rfloor}^{(j)}(x)$ >
</bilinearBasis>
</polynomialVectorMatrix>

xml for polynomial vector  $P_{j,rs}^n(x) \equiv$ 
<polynomialVector>
<xml for polynomial  $P_{j,rs}^0(x)$ >
...
<xml for polynomial  $P_{j,rs}^N(x)$ >
</polynomialVector>

xml for polynomial  $a_0 + a_1x + \dots a_dx^d \equiv$ 
<polynomial>
<coeff> $a_0$ </coeff>
...
<coeff> $a_d$ </coeff>
</polynomial>

```

### 3.4 An Example

Let's look at an example. Consider the following problem: maximize  $-y$  such that

$$1 + x^4 + y \left( \frac{x^4}{12} + x^2 \right) \geq 0 \quad \text{for all } x \geq 0 \quad (3.4)$$

This is an PMP with  $1 \times 1$  positive-semidefiniteness constraints. We will arbitrarily choose a prefactor of  $e^{-x} = \text{DampedRational}[1, \{\}, 1/\text{E}, x]$ , so that the bilinear basis consists of Laguerre polynomials. The `Mathematica` code for this example (see `mathematica/Tests.m`) is

---

**Listing 4:** Mathematica input for the example (3.4)

---

```

Module[
  {
    polys = {
      PositiveMatrixWithPrefactor[
        DampedRational[1,{}, 1/E,x],
        {{{1 + x^4, x^4/12 + x^2}}}
      ]
    },
    norm = {1, 0},
    obj = {0, -1}
  },
  WritePmpJson["pmp.json", SDP[obj, norm, polys]];
];

```

It produces the following JSON file

**Listing 5:** JSON file `pmp.json` produced by listing 4. Decimals are truncated at 12 digits.

```

{
  "objective": [
    "0",
    "-1"
  ],
  "normalization": [
    "1",
    "0"
  ],
  "PositiveMatrixWithPrefactorArray": [
    {
      "DampedRational": {
        "base": "0.3678794411",
        "constant": "1",
        "poles": []
      },
      "polynomials": [
        [
          [
            "1",
            "0",
            "0",
            "0",
            "1"
          ],
          [
            "0",
            "0",
            "1",
            "0",
            "0.083333333333"
          ]
        ]
      ]
    }
  ]
}

```



## 4 Internal SDP

$$\begin{aligned} \mathcal{D}: \quad & \text{maximize} \quad \text{Tr}(CY) + b_0 + b \cdot y \quad \text{over} \quad y \in \mathbb{R}^N, Y \in \mathcal{S}^K, \\ & \text{such that} \quad \text{Tr}(A_*Y) + By = c, \text{ and} \\ & Y \succeq 0. \end{aligned} \tag{4.1}$$

$$\begin{aligned} c &\in \mathbb{R}^P, \\ B &\in \mathbb{R}^{P \times N}, \\ A_1, \dots, A_P, C &\in \mathcal{S}^K. \end{aligned} \quad (4.3)$$

$$\begin{aligned} P &\equiv \sum_i A_i x_i - X - C, \\ p &\equiv b - B^T x, \\ d &\equiv c - \text{Tr}(A_* Y) - B y, \end{aligned} \tag{4.4}$$

9

An optimal point should be both primal and dual feasible, and have (nearly) equal primal and dual objective values. Specifically, let us define `dualityGap` as the normalized difference between the primal and dual objective functions

$$\begin{aligned} \text{dualityGap} &\equiv \frac{|\text{primalObjective} - \text{dualObjective}|}{\max\{1, |\text{primalObjective} + \text{dualObjective}|\}}, \\ \text{primalObjective} &\equiv b_0 + c \cdot x, \\ \text{dualObjective} &\equiv \text{Tr}(CY) + b_0 + b \cdot y. \end{aligned} \tag{4.5}$$

A point is considered “optimal” if

$$\text{dualityGap} < \text{dualityGapThreshold}, \tag{4.6}$$

where `dualityGapThreshold`  $\ll 1$  is chosen by the user.

## 5 Output of SDPB

### 5.1 Terminal Output

The output from running SDPB on the example problem in section 3.4 is in listing 6. The input, output, and checkpoint files are listed first, followed by various parameters. After each iteration, SDPB prints the following:

**time:** The current solver runtime in seconds.

**mu:** The value of the complementarity  $\text{Tr}(XY)/K$ .

**P-obj:** The primal objective value  $b_0 + c \cdot x$ .

**D-obj:** The dual objective value  $\text{Tr}(CY) + b_0 + b \cdot y$ .

**gap:** The value of `dualityGap`.

**P-err:** The primal error  $\max_{i,j}\{|P_{ij}|\}$ .

**p-err:** The primal error  $\max_i\{|p_i|\}$ .

**D-err:** The dual error  $\max_i\{|d_i|\}$ .

**P-step:** The primal step length  $\alpha_{\mathcal{P}}$  described in [1].

**D-step:** The dual step length  $\alpha_{\mathcal{D}}$  described in [1].

**beta:** The corrector centering parameter  $\beta_c$  described in [1].

These values are also written to `iterations.json` in the output folder (see Section 5.3).

If an optimal solution exists, the primal and dual error will decrease until the problem becomes primal and dual feasible. Then the primal and dual objective functions start to converge, and the complementarity  $\mu$  decreases until the duality gap becomes smaller than `dualityGapThreshold`.

The terminal output ends with the final values of the primal/dual objectives, primal/dual errors and duality gap.

**Listing 6:** Output of SDPB for the input file in listing 5

```
$ ./build/sdpb --noFinalCheckpoint -s test/data/sdp.zip -o test/out/sdpb
2024-Jan-04 20:04:14 Start SDPB
SDPB version: 2.6.1-14-ga7479df1
MPI processes: 1, nodes: 1
SDP directory   : "test/data/sdp.zip"
out directory   : "test/out/sdpb"

Parameters:
maxIterations      = 500
maxRuntime         = 9223372036854775807
checkpointInterval = 3600
findPrimalFeasible = false
findDualFeasible   = false
detectPrimalFeasibleJump = false
detectDualFeasibleJump = false
precision(actual)  = 400(448)
dualityGapThreshold = 1e-30
primalErrorThreshold = 1e-30
dualErrorThreshold = 1e-30
initialMatrixScalePrimal = 1e+20
initialMatrixScaleDual   = 1e+20
feasibleCenteringParameter = 0.1
infeasibleCenteringParameter = 0.3
stepLengthReduction      = 0.7
maxComplementarity       = 1e+100
initialCheckpointDir      = "test/data/sdp.zip.ck"
checkpointDir             = "test/data/sdp.zip.ck"
noFinalCheckpoint         = true
writeSolution             = x,y
procGranularity           = 1
verbosity                = 1

Initialize SDP solver
  primal dimension: 5
  dual dimension: 1
  SDP blocks: 1
2024-Jan-04 20:04:14 Start solver iterations
```

|    | time | mu      | P-obj     | D-obj     | gap  | P-err     | p-err     | D-err     | P-step | D-step | beta  |
|----|------|---------|-----------|-----------|------|-----------|-----------|-----------|--------|--------|-------|
| 1  | 0    | 1.0e+40 | +0.00     | +0.00     | 0.00 | +1.00e+20 | +1.00     | +2.88e+20 | 0.631  | 0.647  | 0.300 |
| 2  | 0    | 5.0e+39 | +9.49e+19 | -1.64e+20 | 1.00 | +3.69e+19 | +0.369    | +1.02e+20 | 0.653  | 0.639  | 0.300 |
| 3  | 0    | 2.5e+39 | +1.04e+20 | -2.92e+20 | 1.00 | +1.28e+19 | +0.128    | +3.68e+19 | 0.660  | 0.639  | 0.300 |
| 4  | 0    | 1.2e+39 | +1.43e+20 | -4.30e+20 | 1.00 | +4.35e+18 | +0.0435   | +1.33e+19 | 0.652  | 0.638  | 0.300 |
| 5  | 0    | 5.8e+38 | +1.91e+20 | -6.13e+20 | 1.00 | +1.51e+18 | +0.0151   | +4.80e+18 | 0.645  | 0.636  | 0.300 |
| 6  | 0    | 2.8e+38 | +2.52e+20 | -8.65e+20 | 1.00 | +5.38e+17 | +0.00538  | +1.75e+18 | 0.640  | 0.634  | 0.300 |
| 7  | 0    | 1.4e+38 | +3.36e+20 | -1.21e+21 | 1.00 | +1.94e+17 | +0.00194  | +6.38e+17 | 0.636  | 0.633  | 0.300 |
| 8  | 0    | 7.2e+37 | +4.52e+20 | -1.69e+21 | 1.00 | +7.05e+16 | +0.000705 | +2.34e+17 | 0.635  | 0.633  | 0.300 |
| 9  | 0    | 3.6e+37 | +6.16e+20 | -2.34e+21 | 1.00 | +2.57e+16 | +0.000257 | +8.58e+16 | 0.634  | 0.633  | 0.300 |
| 10 | 0    | 1.8e+37 | +8.43e+20 | -3.23e+21 | 1.00 | +9.43e+15 | +9.43e-05 | +3.15e+16 | 0.633  | 0.633  | 0.300 |
| 11 | 0    | 9.3e+36 | +1.16e+21 | -4.46e+21 | 1.00 | +3.46e+15 | +3.46e-05 | +1.16e+16 | 0.633  | 0.633  | 0.300 |
| 12 | 0    | 4.7e+36 | +1.59e+21 | -6.16e+21 | 1.00 | +1.27e+15 | +1.27e-05 | +4.24e+15 | 0.633  | 0.633  | 0.300 |
| 13 | 0    | 2.4e+36 | +2.20e+21 | -8.49e+21 | 1.00 | +4.65e+14 | +4.65e-06 | +1.56e+15 | 0.633  | 0.633  | 0.300 |
| 14 | 0    | 1.2e+36 | +3.03e+21 | -1.17e+22 | 1.00 | +1.71e+14 | +1.71e-06 | +5.71e+14 | 0.633  | 0.633  | 0.300 |

|    |   |         |           |           |      |            |            |            |       |       |       |
|----|---|---------|-----------|-----------|------|------------|------------|------------|-------|-------|-------|
| 15 | 0 | 6.1e+35 | +4.18e+21 | -1.62e+22 | 1.00 | +6.26e+13  | +6.26e-07  | +2.09e+14  | 0.633 | 0.633 | 0.300 |
| 16 | 0 | 3.1e+35 | +5.77e+21 | -2.23e+22 | 1.00 | +2.30e+13  | +2.30e-07  | +7.68e+13  | 0.633 | 0.633 | 0.300 |
| 17 | 0 | 1.6e+35 | +7.96e+21 | -3.08e+22 | 1.00 | +8.42e+12  | +8.42e-08  | +2.82e+13  | 0.633 | 0.633 | 0.300 |
| 18 | 0 | 7.9e+34 | +1.10e+22 | -4.25e+22 | 1.00 | +3.09e+12  | +3.09e-08  | +1.03e+13  | 0.633 | 0.633 | 0.300 |
| 19 | 0 | 4.0e+34 | +1.52e+22 | -5.86e+22 | 1.00 | +1.13e+12  | +1.13e-08  | +3.79e+12  | 0.633 | 0.633 | 0.300 |
| 20 | 0 | 2.0e+34 | +2.09e+22 | -8.09e+22 | 1.00 | +4.15e+11  | +4.15e-09  | +1.39e+12  | 0.633 | 0.633 | 0.300 |
| 21 | 0 | 1.0e+34 | +2.89e+22 | -1.12e+23 | 1.00 | +1.52e+11  | +1.52e-09  | +5.09e+11  | 0.633 | 0.633 | 0.300 |
| 22 | 0 | 5.2e+33 | +3.98e+22 | -1.54e+23 | 1.00 | +5.58e+10  | +5.58e-10  | +1.87e+11  | 0.633 | 0.633 | 0.300 |
| 23 | 0 | 2.6e+33 | +5.50e+22 | -2.13e+23 | 1.00 | +2.05e+10  | +2.05e-10  | +6.85e+10  | 0.633 | 0.633 | 0.300 |
| 24 | 0 | 1.3e+33 | +7.59e+22 | -2.93e+23 | 1.00 | +7.51e+09  | +7.51e-11  | +2.51e+10  | 0.633 | 0.633 | 0.300 |
| 25 | 0 | 6.7e+32 | +1.05e+23 | -4.05e+23 | 1.00 | +2.75e+09  | +2.75e-11  | +9.21e+09  | 0.633 | 0.633 | 0.300 |
| 26 | 0 | 3.4e+32 | +1.44e+23 | -5.59e+23 | 1.00 | +1.01e+09  | +1.01e-11  | +3.38e+09  | 0.633 | 0.633 | 0.300 |
| 27 | 0 | 1.7e+32 | +1.99e+23 | -7.71e+23 | 1.00 | +3.70e+08  | +3.70e-12  | +1.24e+09  | 0.633 | 0.633 | 0.300 |
| 28 | 0 | 8.7e+31 | +2.75e+23 | -1.06e+24 | 1.00 | +1.36e+08  | +1.36e-12  | +4.54e+08  | 0.633 | 0.633 | 0.300 |
| 29 | 0 | 4.4e+31 | +3.80e+23 | -1.47e+24 | 1.00 | +4.98e+07  | +4.98e-13  | +1.67e+08  | 0.633 | 0.633 | 0.300 |
| 30 | 0 | 2.2e+31 | +5.24e+23 | -2.03e+24 | 1.00 | +1.83e+07  | +1.83e-13  | +6.11e+07  | 0.633 | 0.633 | 0.300 |
| 31 | 0 | 1.1e+31 | +7.23e+23 | -2.80e+24 | 1.00 | +6.70e+06  | +6.70e-14  | +2.24e+07  | 0.633 | 0.633 | 0.300 |
| 32 | 0 | 5.7e+30 | +9.98e+23 | -3.86e+24 | 1.00 | +2.46e+06  | +2.46e-14  | +8.22e+06  | 0.633 | 0.633 | 0.300 |
| 33 | 0 | 2.9e+30 | +1.38e+24 | -5.32e+24 | 1.00 | +9.01e+05  | +9.01e-15  | +3.01e+06  | 0.633 | 0.633 | 0.300 |
| 34 | 0 | 1.5e+30 | +1.90e+24 | -7.35e+24 | 1.00 | +3.30e+05  | +3.30e-15  | +1.11e+06  | 0.633 | 0.633 | 0.300 |
| 35 | 0 | 7.4e+29 | +2.62e+24 | -1.01e+25 | 1.00 | +1.21e+05  | +1.21e-15  | +4.05e+05  | 0.633 | 0.633 | 0.300 |
| 36 | 0 | 3.7e+29 | +3.62e+24 | -1.40e+25 | 1.00 | +4.44e+04  | +4.44e-16  | +1.49e+05  | 0.633 | 0.633 | 0.300 |
| 37 | 0 | 1.9e+29 | +4.99e+24 | -1.93e+25 | 1.00 | +1.63e+04  | +1.63e-16  | +5.45e+04  | 0.633 | 0.633 | 0.300 |
| 38 | 0 | 9.6e+28 | +6.89e+24 | -2.66e+25 | 1.00 | +5.98e+03  | +5.98e-17  | +2.00e+04  | 0.633 | 0.633 | 0.300 |
| 39 | 0 | 4.8e+28 | +9.51e+24 | -3.68e+25 | 1.00 | +2.19e+03  | +2.19e-17  | +7.33e+03  | 0.633 | 0.633 | 0.300 |
| 40 | 0 | 2.4e+28 | +1.31e+25 | -5.07e+25 | 1.00 | +803.      | +8.03e-18  | +2.69e+03  | 0.634 | 0.634 | 0.300 |
| 41 | 0 | 1.2e+28 | +1.81e+25 | -7.00e+25 | 1.00 | +294.      | +2.94e-18  | +985.      | 0.634 | 0.634 | 0.300 |
| 42 | 0 | 6.3e+27 | +2.50e+25 | -9.64e+25 | 1.00 | +108.      | +1.08e-18  | +360.      | 0.636 | 0.636 | 0.300 |
| 43 | 0 | 3.2e+27 | +3.44e+25 | -1.32e+26 | 1.00 | +39.1      | +3.91e-19  | +131.      | 0.642 | 0.642 | 0.300 |
| 44 | 0 | 1.6e+27 | +4.72e+25 | -1.81e+26 | 1.00 | +14.0      | +1.40e-19  | +47.5      | 0.657 | 0.657 | 0.300 |
| 45 | 0 | 7.8e+26 | +6.42e+25 | -2.41e+26 | 1.00 | +4.80      | +4.80e-20  | +16.7      | 0.699 | 0.699 | 0.300 |
| 46 | 0 | 3.6e+26 | +8.51e+25 | -3.04e+26 | 1.00 | +1.44      | +1.44e-20  | +5.40      | 0.785 | 0.785 | 0.300 |
| 47 | 0 | 1.6e+26 | +1.02e+26 | -3.19e+26 | 1.00 | +0.310     | +3.10e-21  | +1.25      | 1.00  | 1.00  | 0.300 |
| 48 | 0 | 5.3e+25 | +8.01e+25 | -1.83e+26 | 1.00 | +7.61e-116 | +2.88e-118 | +5.79e-90  | 0.734 | 0.734 | 0.100 |
| 49 | 0 | 1.8e+25 | +2.73e+25 | -6.20e+25 | 1.00 | +9.24e-116 | +2.94e-117 | +1.08e-85  | 0.736 | 0.736 | 0.100 |
| 50 | 0 | 6.0e+24 | +9.21e+24 | -2.09e+25 | 1.00 | +1.56e-115 | +4.24e-117 | +4.71e-86  | 0.736 | 0.736 | 0.100 |
| 51 | 0 | 2.0e+24 | +3.10e+24 | -7.08e+24 | 1.00 | +1.09e-115 | +1.74e-117 | +1.41e-86  | 0.736 | 0.736 | 0.100 |
| 52 | 0 | 6.9e+23 | +1.04e+24 | -2.39e+24 | 1.00 | +1.07e-115 | +6.29e-117 | +3.94e-87  | 0.737 | 0.737 | 0.100 |
| 53 | 0 | 2.3e+23 | +3.51e+23 | -8.08e+23 | 1.00 | +1.33e-115 | +4.65e-118 | +1.05e-87  | 0.737 | 0.737 | 0.100 |
| 54 | 0 | 7.8e+22 | +1.18e+23 | -2.72e+23 | 1.00 | +1.02e-115 | +9.43e-117 | +2.79e-88  | 0.738 | 0.738 | 0.100 |
| 55 | 0 | 2.6e+22 | +3.95e+22 | -9.14e+22 | 1.00 | +7.61e-116 | +9.56e-117 | +7.36e-89  | 0.738 | 0.738 | 0.100 |
| 56 | 0 | 8.8e+21 | +1.32e+22 | -3.07e+22 | 1.00 | +1.02e-115 | +5.74e-117 | +1.93e-89  | 0.739 | 0.739 | 0.100 |
| 57 | 0 | 2.9e+21 | +4.43e+21 | -1.03e+22 | 1.00 | +1.15e-115 | +1.19e-116 | +5.03e-90  | 0.739 | 0.739 | 0.100 |
| 58 | 0 | 9.8e+20 | +1.48e+21 | -3.44e+21 | 1.00 | +8.15e-116 | +1.77e-116 | +1.31e-90  | 0.739 | 0.739 | 0.100 |
| 59 | 0 | 3.3e+20 | +4.95e+20 | -1.15e+21 | 1.00 | +7.61e-116 | +3.24e-117 | +3.42e-91  | 0.740 | 0.740 | 0.100 |
| 60 | 0 | 1.1e+20 | +1.66e+20 | -3.85e+20 | 1.00 | +1.27e-115 | +8.39e-118 | +8.91e-92  | 0.740 | 0.740 | 0.100 |
| 61 | 0 | 3.7e+19 | +5.54e+19 | -1.29e+20 | 1.00 | +1.20e-115 | +3.30e-118 | +2.32e-92  | 0.740 | 0.740 | 0.100 |
| 62 | 0 | 1.2e+19 | +1.85e+19 | -4.30e+19 | 1.00 | +1.02e-115 | +7.83e-119 | +6.03e-93  | 0.740 | 0.740 | 0.100 |
| 63 | 0 | 4.1e+18 | +6.18e+18 | -1.44e+19 | 1.00 | +6.16e-116 | +1.44e-119 | +1.57e-93  | 0.740 | 0.740 | 0.100 |
| 64 | 0 | 1.4e+18 | +2.06e+18 | -4.80e+18 | 1.00 | +1.60e-116 | +3.76e-120 | +4.08e-94  | 0.740 | 0.740 | 0.100 |
| 65 | 0 | 4.6e+17 | +6.89e+17 | -1.60e+18 | 1.00 | +4.17e-117 | +9.76e-121 | +1.06e-94  | 0.740 | 0.740 | 0.100 |
| 66 | 0 | 1.5e+17 | +2.30e+17 | -5.36e+17 | 1.00 | +1.08e-117 | +2.54e-121 | +2.75e-95  | 0.740 | 0.740 | 0.100 |
| 67 | 0 | 5.1e+16 | +7.68e+16 | -1.79e+17 | 1.00 | +2.81e-118 | +6.60e-122 | +7.16e-96  | 0.740 | 0.740 | 0.100 |
| 68 | 0 | 1.7e+16 | +2.56e+16 | -5.97e+16 | 1.00 | +7.31e-119 | +1.72e-122 | +1.86e-96  | 0.740 | 0.740 | 0.100 |
| 69 | 0 | 5.7e+15 | +8.56e+15 | -1.99e+16 | 1.00 | +1.90e-119 | +4.58e-123 | +4.83e-97  | 0.740 | 0.740 | 0.100 |
| 70 | 0 | 1.9e+15 | +2.86e+15 | -6.65e+15 | 1.00 | +4.93e-120 | +1.38e-123 | +1.25e-97  | 0.740 | 0.740 | 0.100 |
| 71 | 0 | 6.3e+14 | +9.53e+14 | -2.22e+15 | 1.00 | +1.28e-120 | +7.26e-124 | +3.26e-98  | 0.740 | 0.740 | 0.100 |
| 72 | 0 | 2.1e+14 | +3.18e+14 | -7.41e+14 | 1.00 | +3.33e-121 | +4.71e-124 | +8.47e-99  | 0.740 | 0.740 | 0.100 |
| 73 | 0 | 7.1e+13 | +1.06e+14 | -2.47e+14 | 1.00 | +8.64e-122 | +1.22e-124 | +2.20e-99  | 0.740 | 0.740 | 0.100 |
| 74 | 0 | 2.4e+13 | +3.54e+13 | -8.26e+13 | 1.00 | +2.24e-122 | +3.18e-125 | +5.71e-100 | 0.740 | 0.740 | 0.100 |
| 75 | 0 | 7.9e+12 | +1.18e+13 | -2.76e+13 | 1.00 | +5.83e-123 | +8.26e-126 | +1.48e-100 | 0.740 | 0.740 | 0.100 |
| 76 | 0 | 2.6e+12 | +3.94e+12 | -9.20e+12 | 1.00 | +1.51e-123 | +2.14e-126 | +3.85e-101 | 0.740 | 0.740 | 0.100 |
| 77 | 0 | 8.8e+11 | +1.32e+12 | -3.07e+12 | 1.00 | +3.93e-124 | +5.57e-127 | +1.00e-101 | 0.740 | 0.740 | 0.100 |
| 78 | 0 | 2.9e+11 | +4.39e+11 | -1.02e+12 | 1.00 | +1.02e-124 | +1.45e-127 | +2.60e-102 | 0.740 | 0.740 | 0.100 |
| 79 | 0 | 9.8e+10 | +1.47e+11 | -3.42e+11 | 1.00 | +2.65e-125 | +3.75e-128 | +6.74e-103 | 0.740 | 0.740 | 0.100 |
| 80 | 0 | 3.3e+10 | +4.89e+10 | -1.14e+11 | 1.00 | +6.88e-126 | +9.74e-129 | +1.75e-103 | 0.740 | 0.740 | 0.100 |
| 81 | 0 | 1.1e+10 | +1.63e+10 | -3.80e+10 | 1.00 | +1.79e-126 | +2.53e-129 | +4.54e-104 | 0.740 | 0.740 | 0.100 |
| 82 | 0 | 3.6e+09 | +5.44e+09 | -1.27e+10 | 1.00 | +4.64e-127 | +6.57e-130 | +1.18e-104 | 0.740 | 0.740 | 0.100 |

|     |   |         |           |           |          |            |            |            |       |       |       |
|-----|---|---------|-----------|-----------|----------|------------|------------|------------|-------|-------|-------|
| 83  | 0 | 1.2e+09 | +1.82e+09 | -4.24e+09 | 1.00     | +1.20e-127 | +1.71e-130 | +3.06e-105 | 0.740 | 0.740 | 0.100 |
| 84  | 0 | 4.0e+08 | +6.06e+08 | -1.41e+09 | 1.00     | +3.13e-128 | +4.43e-131 | +7.95e-106 | 0.740 | 0.740 | 0.100 |
| 85  | 0 | 1.3e+08 | +2.02e+08 | -4.72e+08 | 1.00     | +8.12e-129 | +1.15e-131 | +2.06e-106 | 0.740 | 0.740 | 0.100 |
| 86  | 0 | 4.5e+07 | +6.75e+07 | -1.57e+08 | 1.00     | +2.11e-129 | +2.99e-132 | +5.36e-107 | 0.740 | 0.740 | 0.100 |
| 87  | 0 | 1.5e+07 | +2.25e+07 | -5.25e+07 | 1.00     | +5.47e-130 | +7.74e-133 | +1.39e-107 | 0.740 | 0.740 | 0.100 |
| 88  | 0 | 5.0e+06 | +7.51e+06 | -1.75e+07 | 1.00     | +1.42e-130 | +2.02e-133 | +3.61e-108 | 0.740 | 0.740 | 0.100 |
| 89  | 0 | 1.7e+06 | +2.51e+06 | -5.85e+06 | 1.00     | +3.69e-131 | +5.17e-134 | +9.38e-109 | 0.740 | 0.740 | 0.100 |
| 90  | 0 | 5.6e+05 | +8.36e+05 | -1.95e+06 | 1.00     | +9.57e-132 | +1.32e-134 | +2.44e-109 | 0.740 | 0.740 | 0.100 |
| 91  | 0 | 1.9e+05 | +2.79e+05 | -6.51e+05 | 1.00     | +2.48e-132 | +4.07e-135 | +6.32e-110 | 0.740 | 0.740 | 0.100 |
| 92  | 0 | 6.2e+04 | +9.31e+04 | -2.17e+05 | 1.00     | +6.41e-133 | +3.42e-136 | +1.64e-110 | 0.740 | 0.740 | 0.100 |
| 93  | 0 | 2.1e+04 | +3.11e+04 | -7.24e+04 | 1.00     | +1.65e-133 | +3.73e-136 | +4.26e-111 | 0.740 | 0.740 | 0.100 |
| 94  | 0 | 6.9e+03 | +1.04e+04 | -2.41e+04 | 1.00     | +4.13e-134 | +7.06e-137 | +1.11e-111 | 0.741 | 0.741 | 0.100 |
| 95  | 0 | 2.3e+03 | +3.47e+03 | -8.04e+03 | 1.00     | +5.50e-135 | +1.24e-135 | +2.87e-112 | 0.741 | 0.741 | 0.100 |
| 96  | 0 | 7.7e+02 | +1.17e+03 | -2.67e+03 | 1.00     | +5.50e-135 | +2.24e-136 | +7.45e-113 | 0.741 | 0.741 | 0.100 |
| 97  | 0 | 2.6e+02 | +394.     | -886.     | 1.00     | +5.50e-135 | +1.33e-135 | +1.93e-113 | 0.741 | 0.741 | 0.100 |
| 98  | 0 | 85.     | +136.     | -291.     | 1.00     | +6.88e-135 | +1.38e-135 | +5.01e-114 | 0.738 | 0.738 | 0.100 |
| 99  | 0 | 29.     | +49.8     | -93.6     | 1.00     | +8.25e-135 | +0.00      | +1.31e-114 | 0.722 | 0.722 | 0.100 |
| 100 | 0 | 10.     | +21.0     | -29.2     | 1.00     | +8.25e-135 | +0.00      | +3.65e-115 | 0.694 | 0.694 | 0.100 |
| 101 | 0 | 3.8     | +10.5     | -8.33     | 1.00     | +6.88e-135 | +0.00      | +1.11e-115 | 0.707 | 0.707 | 0.100 |
| 102 | 0 | 1.4     | +5.45     | -1.41     | 1.00     | +8.73e-135 | +7.64e-136 | +3.27e-116 | 0.765 | 0.765 | 0.100 |
| 103 | 0 | 0.43    | +2.99     | +0.854    | 0.555    | +2.75e-135 | +6.29e-136 | +7.67e-117 | 0.768 | 0.768 | 0.100 |
| 104 | 0 | 0.13    | +2.20     | +1.54     | 0.176    | +2.02e-135 | +1.55e-136 | +1.78e-117 | 0.764 | 0.764 | 0.100 |
| 105 | 0 | 0.041   | +1.95     | +1.75     | 0.0557   | +3.92e-136 | +0.00      | +4.21e-118 | 0.764 | 0.764 | 0.100 |
| 106 | 0 | 0.013   | +1.88     | +1.81     | 0.0175   | +3.57e-136 | +0.00      | +9.94e-119 | 0.766 | 0.766 | 0.100 |
| 107 | 0 | 0.0040  | +1.85     | +1.83     | 0.00543  | +1.70e-136 | +1.38e-135 | +2.33e-119 | 0.768 | 0.768 | 0.100 |
| 108 | 0 | 0.0012  | +1.84     | +1.84     | 0.00168  | +5.13e-136 | +0.00      | +5.39e-120 | 0.770 | 0.770 | 0.100 |
| 109 | 0 | 0.00038 | +1.84     | +1.84     | 0.000514 | +6.37e-136 | +0.00      | +1.24e-120 | 0.772 | 0.772 | 0.100 |
| 110 | 0 | 0.00012 | +1.84     | +1.84     | 0.000157 | +9.57e-136 | +1.86e-136 | +2.82e-121 | 0.773 | 0.773 | 0.100 |
| 111 | 0 | 3.5e-05 | +1.84     | +1.84     | 4.77e-05 | +9.95e-137 | +1.08e-136 | +6.39e-122 | 0.774 | 0.774 | 0.100 |
| 112 | 0 | 1.1e-05 | +1.84     | +1.84     | 1.44e-05 | +5.12e-136 | +2.90e-136 | +1.44e-122 | 0.775 | 0.775 | 0.100 |
| 113 | 0 | 3.2e-06 | +1.84     | +1.84     | 4.37e-06 | +8.62e-136 | +3.56e-136 | +3.24e-123 | 0.776 | 0.776 | 0.100 |
| 114 | 0 | 9.7e-07 | +1.84     | +1.84     | 1.32e-06 | +7.51e-136 | +4.08e-136 | +7.26e-124 | 0.776 | 0.776 | 0.100 |
| 115 | 0 | 2.9e-07 | +1.84     | +1.84     | 3.97e-07 | +5.08e-136 | +2.73e-136 | +1.62e-124 | 0.777 | 0.777 | 0.100 |
| 116 | 0 | 8.8e-08 | +1.84     | +1.84     | 1.19e-07 | +5.77e-136 | +3.84e-136 | +3.62e-125 | 0.777 | 0.777 | 0.100 |
| 117 | 0 | 2.6e-08 | +1.84     | +1.84     | 3.59e-08 | +9.94e-136 | +4.95e-136 | +8.08e-126 | 0.777 | 0.777 | 0.100 |
| 118 | 0 | 7.9e-09 | +1.84     | +1.84     | 1.08e-08 | +5.35e-136 | +2.62e-136 | +1.80e-126 | 0.777 | 0.777 | 0.100 |
| 119 | 0 | 2.4e-09 | +1.84     | +1.84     | 3.24e-09 | +2.19e-136 | +1.90e-136 | +4.01e-127 | 0.777 | 0.777 | 0.100 |
| 120 | 0 | 7.2e-10 | +1.84     | +1.84     | 9.73e-10 | +7.67e-136 | +4.85e-136 | +8.92e-128 | 0.778 | 0.778 | 0.100 |
| 121 | 0 | 2.2e-10 | +1.84     | +1.84     | 2.92e-10 | +8.89e-136 | +5.16e-136 | +1.99e-128 | 0.778 | 0.778 | 0.100 |
| 122 | 0 | 6.5e-11 | +1.84     | +1.84     | 8.77e-11 | +5.98e-136 | +3.68e-136 | +4.41e-129 | 0.778 | 0.778 | 0.100 |
| 123 | 0 | 1.9e-11 | +1.84     | +1.84     | 2.63e-11 | +7.13e-136 | +2.89e-136 | +9.82e-130 | 0.778 | 0.778 | 0.100 |
| 124 | 0 | 5.8e-12 | +1.84     | +1.84     | 7.90e-12 | +3.06e-136 | +7.54e-137 | +2.18e-130 | 0.778 | 0.778 | 0.100 |
| 125 | 0 | 1.7e-12 | +1.84     | +1.84     | 2.37e-12 | +9.92e-136 | +4.60e-136 | +4.85e-131 | 0.778 | 0.778 | 0.100 |
| 126 | 0 | 5.2e-13 | +1.84     | +1.84     | 7.11e-13 | +5.20e-136 | +3.37e-136 | +1.08e-131 | 0.778 | 0.778 | 0.100 |
| 127 | 0 | 1.6e-13 | +1.84     | +1.84     | 2.13e-13 | +2.32e-136 | +3.97e-136 | +2.40e-132 | 0.778 | 0.778 | 0.100 |
| 128 | 0 | 4.7e-14 | +1.84     | +1.84     | 6.40e-14 | +2.24e-136 | +2.86e-136 | +5.43e-133 | 0.778 | 0.778 | 0.100 |
| 129 | 0 | 1.4e-14 | +1.84     | +1.84     | 1.92e-14 | +2.54e-136 | +9.40e-137 | +1.17e-133 | 0.778 | 0.778 | 0.100 |
| 130 | 0 | 4.2e-15 | +1.84     | +1.84     | 5.76e-15 | +4.15e-136 | +6.62e-137 | +3.99e-134 | 0.778 | 0.778 | 0.100 |
| 131 | 0 | 1.3e-15 | +1.84     | +1.84     | 1.73e-15 | +6.32e-136 | +2.25e-136 | +1.79e-134 | 0.778 | 0.778 | 0.100 |
| 132 | 0 | 3.8e-16 | +1.84     | +1.84     | 5.19e-16 | +2.14e-136 | +7.38e-137 | +8.25e-134 | 0.778 | 0.778 | 0.100 |
| 133 | 0 | 1.1e-16 | +1.84     | +1.84     | 1.56e-16 | +3.37e-136 | +2.46e-136 | +2.20e-134 | 0.778 | 0.778 | 0.100 |
| 134 | 0 | 3.4e-17 | +1.84     | +1.84     | 4.67e-17 | +1.12e-136 | +6.53e-137 | +3.85e-134 | 0.778 | 0.778 | 0.100 |
| 135 | 0 | 1.0e-17 | +1.84     | +1.84     | 1.40e-17 | +4.30e-136 | +6.07e-137 | +8.25e-134 | 0.778 | 0.778 | 0.100 |
| 136 | 0 | 3.1e-18 | +1.84     | +1.84     | 4.20e-18 | +5.18e-136 | +3.61e-136 | +1.99e-133 | 0.778 | 0.778 | 0.100 |
| 137 | 0 | 9.3e-19 | +1.84     | +1.84     | 1.26e-18 | +2.71e-136 | +1.17e-136 | +1.35e-133 | 0.778 | 0.778 | 0.100 |
| 138 | 0 | 2.8e-19 | +1.84     | +1.84     | 3.78e-19 | +2.81e-136 | +1.63e-136 | +6.49e-133 | 0.778 | 0.778 | 0.100 |
| 139 | 0 | 8.4e-20 | +1.84     | +1.84     | 1.13e-19 | +3.12e-136 | +1.75e-136 | +2.82e-133 | 0.778 | 0.778 | 0.100 |
| 140 | 0 | 2.5e-20 | +1.84     | +1.84     | 3.40e-20 | +7.02e-136 | +5.05e-136 | +8.94e-133 | 0.778 | 0.778 | 0.100 |
| 141 | 0 | 7.5e-21 | +1.84     | +1.84     | 1.02e-20 | +7.15e-136 | +5.85e-137 | +1.22e-132 | 0.778 | 0.778 | 0.100 |
| 142 | 0 | 2.3e-21 | +1.84     | +1.84     | 3.06e-21 | +8.79e-136 | +4.70e-137 | +8.71e-133 | 0.778 | 0.778 | 0.100 |
| 143 | 0 | 6.8e-22 | +1.84     | +1.84     | 9.19e-22 | +1.15e-135 | +2.78e-136 | +2.74e-133 | 0.778 | 0.778 | 0.100 |
| 144 | 0 | 2.0e-22 | +1.84     | +1.84     | 2.76e-22 | +5.67e-136 | +4.45e-136 | +5.94e-133 | 0.778 | 0.778 | 0.100 |
| 145 | 0 | 6.1e-23 | +1.84     | +1.84     | 8.27e-23 | +6.16e-136 | +3.91e-136 | +3.77e-132 | 0.778 | 0.778 | 0.100 |
| 146 | 0 | 1.8e-23 | +1.84     | +1.84     | 2.48e-23 | +1.14e-135 | +4.40e-136 | +1.11e-132 | 0.778 | 0.778 | 0.100 |
| 147 | 0 | 5.5e-24 | +1.84     | +1.84     | 7.44e-24 | +6.65e-136 | +4.59e-136 | +1.32e-132 | 0.778 | 0.778 | 0.100 |
| 148 | 0 | 1.6e-24 | +1.84     | +1.84     | 2.23e-24 | +3.93e-136 | +2.56e-136 | +7.97e-133 | 0.778 | 0.778 | 0.100 |
| 149 | 0 | 4.9e-25 | +1.84     | +1.84     | 6.70e-25 | +7.85e-136 | +3.06e-136 | +1.77e-132 | 0.778 | 0.778 | 0.100 |
| 150 | 0 | 1.5e-25 | +1.84     | +1.84     | 2.01e-25 | +4.17e-136 | +2.62e-136 | +4.13e-133 | 0.778 | 0.778 | 0.100 |

```

151      0 4.4e-26 +1.84      +1.84      6.03e-26 +5.77e-136 +3.80e-136 +3.89e-132 0.778 0.778 0.100
152      0 1.3e-26 +1.84      +1.84      1.81e-26 +4.39e-136 +1.70e-136 +6.24e-132 0.778 0.778 0.100
153      0 4.0e-27 +1.84      +1.84      5.43e-27 +7.21e-136 +4.33e-136 +2.04e-132 0.778 0.778 0.100
154      0 1.2e-27 +1.84      +1.84      1.63e-27 +5.54e-136 +3.51e-136 +4.22e-132 0.778 0.778 0.100
155      0 3.6e-28 +1.84      +1.84      4.88e-28 +5.55e-136 +3.99e-136 +7.05e-132 0.778 0.778 0.100
156      0 1.1e-28 +1.84      +1.84      1.47e-28 +2.16e-136 +2.71e-136 +1.73e-131 0.778 0.778 0.100
157      0 3.2e-29 +1.84      +1.84      4.40e-29 +8.40e-136 +3.49e-136 +1.20e-131 0.778 0.778 0.100
158      0 9.7e-30 +1.84      +1.84      1.32e-29 +2.15e-136 +4.09e-136 +3.41e-131 0.778 0.778 0.100
159      0 2.9e-30 +1.84      +1.84      3.96e-30 +6.19e-136 +4.88e-136 +3.44e-131 0.778 0.778 0.100
160      0 8.7e-31 +1.84      +1.84      1.19e-30 +1.88e-136 +1.63e-136 +1.42e-131 0.778 0.778 0.100

```

```
-----found primal-dual optimal solution-----
```

```

primalObjective = 1.84026576313204924668804017173055420056358532030282556465761906133430166726537336826049865612094019
0211160188629478172102847851400971545
dualObjective   = 1.84026576313204924668804017172924388084784907020307957926406455972756967820389551729116356865203683
7213248476950467408148728442213127677
dualityGap      = 3.56013718775636270149999059635335050723442743109168831293885607041894974620853522385695676694435617
2630595608074240236952679818626742854e-31
primalError     = 8.00472462794652854216192415154691251952338782833660449388851026694624386819579233719251965820695076
6138417240341523554115275581567613244e-136
dualError       = 3.67963333625613903189297919337418392740579208646582684485331852877210732905668816990827126132539445
5152562009003409010941981912150248860e-131

```

```
Saving solution to      : "test/out/sdpb"
```

For larger, parallel runs, the exact solution that SDPB computes can vary between different machines, or even different configurations on the same run. For example, the step size is computed by adding up many other numbers. The order in which these numbers are added can change the final sum in the lowest significant bits. This does not make the solver unstable, but it does introduce a minor amount of variation. The solver still converges to the same answer up to the error thresholds and duality gap, but the actual value of the residual errors and gap will be different.

## 5.2 Termination

The possible termination reasons for SDPB are as follows

**found primal-dual optimal solution**

Found a solution for  $x, X, y, Y$  that is simultaneously primal feasible, dual feasible, and optimal.

**found primal feasible solution**

Found a solution for  $x, X$  that is primal feasible. SDPB will only terminate with this result if the option `--findPrimalFeasible` is specified.

**found dual feasible solution**

Found a solution for  $y, Y$  that is dual feasible. SDPB will only terminate with this result if the option `--findDualFeasible` is specified.

**primal feasible jump detected**

A Newton step with primal step length  $\alpha_P$  just occurred, without resulting in a primal feasible solution. (Usually this means one should increase `precision`.)

**dual feasible jump detected**

A Newton step with dual step length  $\alpha_{\mathcal{D}}$  just occurred, without resulting in a dual feasible solution. (Usually this means one should increase **precision**.)

**maxIterations exceeded**

SDPB has run for more iterations than specified by the option `--maxIterations`.

**maxRuntime exceeded**

SDPB has run for longer than specified by the option `--maxRuntime`.

**maxComplementarity exceeded**

$\mu = \text{Tr}(XY)/\dim(X)$  exceeded the value specified by `--maxComplementarity`. This might indicate that the problem is unbounded and no optimal solution will be found.

When using SDPB to determine primal or dual feasibility, one can specify the options `--findPrimalFeasible` or `--findDualFeasible`. This will cause the solver to terminate immediately once the primal or dual errors are sufficiently small. This often occurs immediately after the primal or dual step lengths become equal to 1. A step length of 1 means that the solver has found a Newton step that exactly solves the primal or dual constraints, while preserving positive-semidefiniteness of  $X, Y$ . Sometimes a step length of 1 does not result in sufficiently small primal/dual errors. This is indicative of numerical instabilities and usually means **precision** should be increased. The options `--detectPrimalFeasibleJump` and `--detectDualFeasibleJump` cause SDPB to terminate if a step length of 1 occurs without resulting in primal/dual feasibility. If desired, one can then restart the solver with a higher value of **precision**.

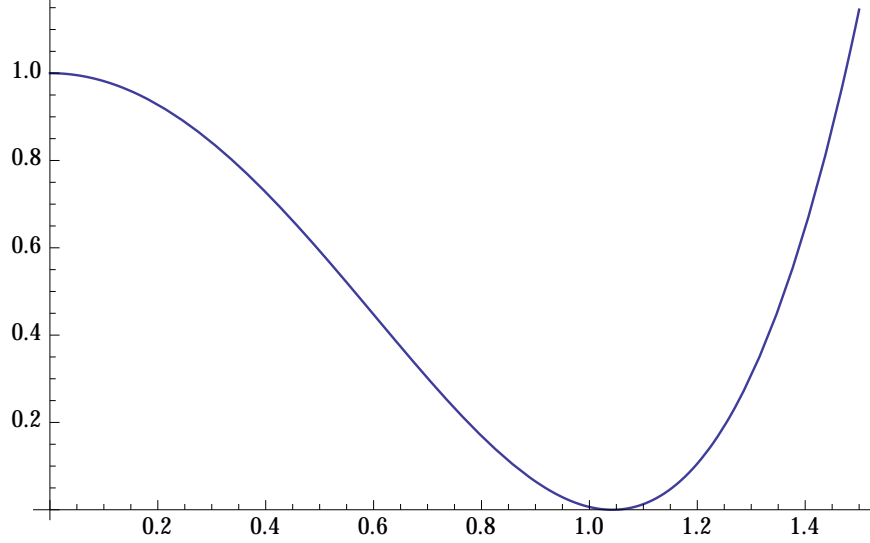
## 5.3 Output File

**Listing 7:** Contents of the output file `test/out/sdpb/out.txt` corresponding to listing 5. Decimal expansions have been truncated for brevity. *Mathematica* uses `*` instead of the character `e` for scientific notation. Thus, the output format is not quite suitable for import into *Mathematica* without modification. This could be changed in future versions.

```
terminateReason = "found primal-dual optimal solution";
primalObjective = 1.8402657631320492466880401717305542005635853203028255646...;
dualObjective   = 1.8402657631320492466880401717292438808478490702030795792...;
dualityGap      = 3.5601371877563627014999905963533505072344274310916883129...e-31;
primalError     = 8.0047246279465285421619241515469125195233878283366044938...e-136;
dualError       = 3.6796333362561390318929791933741839274057920864658268448...e-131;
Solver runtime  = 0;
```

The output file `test/out/sdpb/out.txt` corresponding to listing 5 is shown in listing 7. It includes the reason for termination, the final primal/dual objective values, the final duality gap, the final primal/dual errors, and the total runtime.

The output file `test/out/sdpb/iterations.json` contains information about each iteration in machine-readable JSON format. It includes all fields printed to terminal (see Section



**Figure 1:** A plot of  $1 + x^4 + y \left( \frac{x^4}{12} + x^2 \right)$  with  $y = -1.840265763084$  equal to its optimal value. The zero near  $x = 1$  shows that  $-y$  cannot be further increased without violating the positivity constraint.

5.1), as well as the quantity  $\mathbf{R}\text{-err} = \|\mu I - XY\|_{\max}$ , condition number for matrix  $Q$ , the worst condition number across all SDP blocks and the name of the corresponding block.

The vector  $y$  is saved in `test/out/sdpb/y.txt`, and the two blocks of the  $x$  vector are saved in `test/out/sdpb/x.0.txt` and `test/out/sdpb/x.1.txt`. For JSON and Mathematica, you can also restore the vector  $z$  from  $y$  using normalization condition  $n \cdot z = 1$  (3.1). If you call SDPB with `--writeSolution="x,y,z"` option,  $z$  is saved in `test/out/sdpb/z.txt`. To write additional files for the matrices  $X$  and  $Y$ , add the option `--writeSolution="x,y,z,X,Y"`.

The value of  $y$  gives the solution to our optimization problem. The function

$$1 + x^4 + (-1.840265763084) \left( \frac{x^4}{12} + x^2 \right) \quad (5.1)$$

is plotted in figure 1. The zero near  $x = 1$  shows that  $y$  is optimal.

## 5.4 Checkpoints

Every `checkpointInterval`, SDPB saves a new checkpoint in a directory with the `.ck` extension. SDPB also saves a checkpoint after termination, provided the option `--noFinalCheckpoint` is not specified.

A checkpoint file encodes the values of  $x, X, y, Y$ . If SDPB detects an existing checkpoint file on startup, it will use those values of  $x, X, y, Y$  as initial conditions in the solver. Thus, SDPB can be stopped and started at will without losing progress.



A typical workflow for long-running computations on shared machines is to specify a moderate `checkpointInterval` (e.g. one hour) and a somewhat larger `maxRuntime` (e.g. 12 hours). SDPB will terminate after 12 hours and can then be restarted without losing progress. If SDPB is killed prematurely, then at most 1 hour of progress will be lost. This pattern of restarting gives other users chances to run their processes. It can be sustained indefinitely, allowing extremely long computations.

Checkpoints are written in binary format to conserve space and speed up loading and unloading. If you specify the `--writeSolution="x,y,X,Y"` option, the output directory can also be used to restart a computation with the `-i` option. It will not be bitwise identical to restarting from a binary checkpoint, but it should be very, very, very close.

Text checkpoints can be useful if you want to solve a different system by starting closer to previously solved system. You can also use it to continue a calculation with a different number of cores, or even on a different machine. Using the previous input as an example,

```
$ ./build/sdpb -s test/data/sdp.zip -o test/test/out/sdpb --noFinalCheckpoint --writeSolution="x,y,X,Y"
$ ./build/sdpb -s test/data/sdp.zip -o test/test/out/sdpb --noFinalCheckpoint -i test/test/out/sdpb
```

the second calculation will start from the end of the first calculation.

## 6 Attribution

If you use SDPB in work that results in publication, please cite [1]. Depending on how SDPB is used, the following sources might also be relevant:

- The first use of semidefinite programming in the bootstrap [3].
- The generalization of semidefinite programming methods to arbitrary spacetime dimension [4].
- The generalization of semidefinite programming methods to arbitrary systems of correlation functions [5].

## 7 Acknowledgements

SDPB makes extensive use of the parallel linear algebra library `Elemental` [11], the Boost C++ libraries [9], the `libxml2` library [10], and the multiprecision libraries `GMP` [12], and `MPFR` [13].

SDPB was partially based on the solvers `SDPA` and `SDPA-GMP` [6–8], which were essential sources of inspiration and examples.

Thanks to Filip Kos, David Poland, and Alessandro Vichi for collaboration in developing semidefinite programming methods for the conformal bootstrap and assistance testing SDPB.

Thanks to Amir Ali Ahmadi, Hande Benson, Pablo Parrilo, and Robert Vanderbei for advice and discussions about semidefinite programming.

I am supported by DOE grant number DE-SC0009988 and a William D. Loughlin Membership at the Institute for Advanced Study.

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