8. Piecewise Continous Function.

A function F(t) is said to be piecewise (or sectionally) continuous on a closed interval $a\le t\le b$, if it is defined on that interval and is such that the interval can be subdivided into a finite number of intervals, in each of which F(t) is continuous and has finite right and left hand limits.

9. EXISTENCE OF LAPLACE TRANSFORM THEOREM

If F(t) is a function which is piecewise continuous on every finite interval in the range $t\ge 0$ and satisfies $|F(t)|\le Me^{\alpha t}$ for all $t\ge 0$ and for some constants a and M, then the laplace transform of F(t) exists for all P>a.

PROOF: we have

$$L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

$$= \int_0^{t_0} e^{-pt}F(t)dt + \int_{t_0}^\infty e^{-pt}F(t)dt \to (1)$$

the integral $\int_0^{t_0} e^{-pt}F(t)dt$ at exists since F(t) is piecewise continuous on every finite interval $0\le t\le t_0$.

Now $\left|\int_{t_0}^\infty e^{-pt}F(t)dt\right| \le \int_{t_0}^\infty \left|e^{-pt}F(t)\right|dt$

$$\le \int_{t_0}^\infty e^{-pt}Me^{at} \quad \text{Since } |F(t)|\le Me^{at}$$

$$= \int_{t_0}^\infty e^{-(p-a)t}Me^{at}$$

$$= \frac{Me^{-(p-a)t_0}}{p-a}$$

$$\therefore \left|\int_{t_0}^\infty e^{-pt}F(t)dt\right| \le \frac{Me^{-(p-a)t_0}}{p-a}, p>a$$

But $\frac{Me^{-(p-a)t_0}}{p-a}$ can be made as small as we please by taking to sufficienly large.

thus from (1) we conclude that L{F(t)} exists for all p>a/

10. Functions of Exponential Order

A function F(t) is said to of exponential order α as $t\to\infty$ if there exists a positive constant (real) M, a number α and a finite number to such that

$$|F(t)|<Me^{\alpha t}$$

$$|e^{-\alpha t}F(t)|<M \text{ for all } t\ge t_0$$

If a function f(t) is of exponential order α, it is also of β, $\beta>\alpha$.

11. Function of Class A

A functions which is piecewise [(or) Sectionally] continuous on every finite interval in the range $t\ge 0$ and is of exponential order as $t\to\infty$ is known as "a function of class A"

12. find $L(\cosh^2 2t)$

Sol:

$$L(\cosh^2 2t) = L\left\{\frac12(1+\cosh 4t)\right\}.$$

$$= \frac12\{L(1)+L(\cosh 4t)\}$$

$$= \frac12\left[\frac1p+\frac{p}{p^2-16}\right] = \frac{p^2-8}{p(p^2-16)}, P>4$$

13. Find $L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$

Sol:

$$= L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$$

$$\Rightarrow 7L\{e^{2t}\}+9L\{e^{-2t}\}+5L\{Cost\}+7L\{t^3\}+5L\{Sin3t\}+2L\{1\}$$

$$\Rightarrow 7.\frac{1}{P-2}+9.\frac{1}{P+2}+5.\frac{p}{p^2+1^2}+5.\frac{3!}{p^4}+5.\frac{3}{p^2+3^2}+2\frac1p$$

$$\Rightarrow \frac{16p-4}{P^2-4}+\frac{5p}{P^2+1}+\frac{42+2P^3}{P^4}+\frac{15}{P^2+9}$$

14. Find laplace transform of the function F(t) where

$$F(t)=\begin{cases}Sint, & 0<t<\pi\\ 0, & t>\pi\end{cases}$$

Sol:

$$L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$$

$$= \int_0^\pi e^{-pt}Sint\,dt+\int_\pi^\infty e^{-pt}.0.dt$$

$$= \left[\frac{e^{-pt}}{p^2+1}(-PSint-Cost)\right]_0^\pi = \frac{e^{-p\pi}+1}{p^2+1}$$

15. find $L\{Sin\sqrt t\}$

Sol:

We have

$$L\{Sin\sqrt t\} = L\left\{\sqrt t-\frac{(\sqrt t)^3}{3!}+\frac{(\sqrt t)^5}{5!}....\right\}$$

$$= L\left\{t^{1/2}-\frac{t^{3/2}}{3!}+\frac{t^{5/2}}{5!}-\frac{t^{7/2}}{7!}+....\right\}$$

$$= L\{t^{1/2}\}-\frac1{3!}L\{t^{3/2}\}+\frac1{5!}L\{t^{5/2}\}\frac{-1}{7!}\{t^{7/2}\}+...$$

$$= \frac{\frac12\sqrt\pi}{p^{3/2}}-\frac16\frac{\frac32\cdot\frac12\sqrt\pi}{p^{5/2}}+\frac1{120}\frac{\frac52\cdot\frac32\cdot\frac12\sqrt\pi}{p^{7/2}}....$$

$$\Rightarrow \frac{\sqrt\pi}{2p^{3/2}}\left[1-\left(\frac1{4p}\right)+\frac1{2!}\left(\frac1{4p}\right)^2-\frac1{3!}\left(\frac1{4p}\right)^3+....\right]$$

$$\Rightarrow \frac{\sqrt\pi}{2p^{3/2}}e^{-1/4p}$$

16. Find the laplace transform of the function $F(t)=\frac{e^{at}-1}{a}$

Sol:

we have $L\{F(t)\} = L\left\{\frac{e^{at}-1}{a}\right\}$

$$= \frac1aL\{e^{at}\}-\frac1aL\{1\}$$

$$= \frac1a\left\{\frac1{p-a}-\frac1p\right\}, p>0$$

$$= \frac1{p(p-a)}, p>0$$

17. Evaluate $L\{F(t)\}$ if $F(t)=\begin{cases}(t-1)^2, & t>1\\0, & 0<t<1\end{cases}$

Sol:

We have $L\{F(t)\} = \int_0^\infty F(t)e^{-pt}dt$

$$= \int_0^1 0.e^{-pt}dt+\int_1^\infty (t-1)^2.e^{-pt}dt$$

$$\Rightarrow \left[-\frac{(t-1)^2}{p}e^{-pt}\right]_1^\infty+\frac2p\int_1^\infty(t-1).e^{-pt}dt$$

$$\Rightarrow \frac2p\left[-\frac{(t-1)}{p}e^{-pt}-\frac{e^{-pt}}{p^2}\right]_1^\infty=\frac{2e^{-p}}{p^3}$$

18. First Translation or Shifting Theorem.

If $L\{F(t)\}$ = f(p) when p> α,

then $L\{e^{at}F(t)\}=f(p-a), p>a+a;$

i.e if f(p) is laplace transform of F(t) then f(p-a) is the laplace transform of $e^{at}F(t)$.

Proof:

By definition, we have $f(p) = L\{F(t)\} = \int_0^\infty e^{-pt}F(t)dt$

$$f(p-a) = \int_0^\infty e^{-(p-a)t}F(t)dt = \int_\pi^\infty e^{-pt}.e^{at}F(t)dt$$

$$= L\{e^{at}F(t)\}$$

19. SECOND TRANSLATION OR SHIFTING THEOREM

If L{F(t)} = f(p) and

$G(t)=\begin{cases}F(t-a), & t>a\\0, & t<a\end{cases}$ then $L\{G(t)\}=e^{-ap}f(p)$.

Proof:

By definition, we have

$$L(G(t)) = \int_0^\infty e^{-pt}G(t)dt$$

$$= \int_0^a e^{-pt}0.dt+\int_a^\infty e^{-pt}F(t-a).dt$$

$$= \int_a^\infty e^{-pt}F(t-a).dt$$

$$= \int_0^\infty e^{-p(a+x)}F(x)dx$$

Putting t-a = x so that dt = dx

$$= e^{-pa}\int_0^\infty e^{-px}F(x)dx$$

$$= e^{-pa}\int_0^\infty e^{-pt}F(t)dt$$

By a property of definite integrals

$$= e^{-pa}L\{F(t)\}=e^{-pa}f(p)$$

20. CHANGE OF SCALE PROPERTY THEOREM

If $L\{F(t)\}$ = f(p)

then $L\{F(at)\} = \frac1af\left(\frac pa\right)$

Proof:

By definition, we have.

$$L\{F(at)\} = \int_0^\infty e^{-pt}F(at).dt$$

$$= \frac1a\int_0^\infty e^{-p(x/a)}F(x).dx$$

Putting at=x, so that dt=$\frac1adx$

$$\Rightarrow \frac1a\int_0^\infty e^{(p-a)t}F(t)dt \quad \left[\because \int_a^bf(x)dx=\int_a^bf(t)dt\right]$$

$$\Rightarrow \frac1af(p/a) \quad \left[\because f(p)=\int_0^\infty e^{-pt}F(t)dt\right]$$

21. Find $L\{e^{-2t}(3Cos6t=5Sin6t)\}$

Sol:

we have,

$\alpha\ \{3Cos6t-5Sin6t\}$

$$\Rightarrow 3.\frac{p}{p^2+6^2}-5\frac{6}{p^2+6^2}=\frac{3p-30}{p^2+36}=f(p) \text{ say}$$

$\therefore$ from first shifting theorem, we have

$$\alpha\{e^{-2t}(3Cos6t-5Sin6t)\}=f(p+2)$$

$$= \frac{3(p+2)-30}{(p+2)^2+36}=\frac{3p-24}{p^2+4p+40}$$

22. Find $L\{e^{-t}(3Sinh2t-5Cosh2t)\}$

Sol:

we have,

$L\{e^{-t}(3Sinh2t-5Cosh2t)\}$

$$= 3.\frac{2}{p^2-2^2}-5\frac{p}{p^2-2^2}=\frac{6-5p}{p^2-4}=f(p)$$

$\therefore$ from first shifting theorem

$$L\{e^{-t}(3Sinh2t-5Cosh2t)\}=f(p+1)$$

$$= \frac{6-5(p+1)}{(p+1)^2-4}=\frac{1-5p}{p^2+2p-3}$$

23. Find $L\{e^tSin^2t\}$

Sol:

we have $L\{Sin^2t\}=L\left\{\frac12(1-Cos2t)\right\}$

$$= \frac12\left[\frac1p-\frac{p}{p^2+2^2}\right]=\frac2{p(p^2+4)}=f(p)$$

$\therefore$ from first shifting theorem, we have,

$$L\{e^tSin^2t\}=f(p-1)=\frac2{(p-1)\{(p-1)^2+4\}}=\frac2{(p-1)\{p^2-2p+5\}}$$

24. given $L\{F(t)\}=\frac{p^2-p+1}{(2p+1)^2(p-1)}$ applying the change of scale property show that

$$L(F(2t))=\frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

Sol:

we have

$$L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2(p-1)}=f(p)$$

$$\therefore L\{F(2t)\} = \frac12f\left(\frac p2\right)=\frac12\frac{(p/2)^2-p/2+1}{\left(2(p/2)+1\right)^2(p/2-1)}$$

$$= \frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

25. find L{F(t)} where

$$F(t)=\begin{cases}Cos\left(t-\frac23\pi\right), & t>2\pi/3\\0, & t<2\pi/3\end{cases}$$

Sol:

Let $\phi(t)$ = Cos t

$$\therefore F(t)=\begin{cases}\phi(t-2\pi/3), & t>2\pi/3\\0, & t<2\pi/3\end{cases}$$

we have $L\{\phi(t)\}=L\{Cos t\}=\frac{p}{p^2+1}=f(p)$

$\therefore$ from second shifting theorem

$$L\{F(t)\} = e^{(-2\pi/3)p}f(p)=e^{-2\pi p/3}.\frac{p}{p^2+1}$$

26. Laplace Transform of the Derivatives of F(t) Theorem:

Let F(t) be continuous for all $t\ge0$ and be of exponential order a as $t\to\infty$ and if F'(t) is of class a, then laplace transform of the derivative F'(t) exists when p>a and L{F'(t)} = PL{F(t)}-F(0)

Proof:

CASE - 1 : If case F'(t) is continuous for all $t\ge0$. then

$$L\{F'(t)\}=\int_0^\infty e^{-pt}F'(t)dt \to (1)$$

$$\Rightarrow \left[e^{-pt}F(t)\right]_0^\infty+P\int_0^\infty e^{-pt}F(t)dt$$

$$= \lim_{t\to\infty}e^{-pt}F(t)-F(0)+PL\{F(t)\} \to (2)$$

Now $|F(t)|\le Me^{at}$ for all $t\ge0$ and for some constants a & M we have $|e^{-pt}F(t)|=e^{-pt}|F(t)|$

$$\le e^{-pt}Me^{at}$$

$$= Me^{-(p-a)t}\to0 \text{ as } t\to\infty \text{ if } p>a$$

$$\therefore \lim_{t\to\infty}e^{-pt}F(t)=0 \text{ for } p>a$$

therefore from (2) we conclude that $L\{F'(t)\}$ exists and L{F'(t)} =PL{F(t)}-F(0)

CASE -2: In case F'(t) is merely piecewise continuous? the integral (1) may be broken as the sum of integrals in different ranges from '0' to ∞ such that F'(t) is continuous in each of such parts.

Then proceeding as in case -I we shall have

$L\{F'(t)\}$ =PL{F(t)}-F(0)

27. Intial Value Theorem

Let F(t) be continuous forall $t\ge0$ and be of exponential order as $t\to\infty$ and if F'(t) is of class a then

$$\lim_{t\to\infty}F(t)=\lim_{t\to\infty}PL\{F(t)\}$$

Proof:

By the theorem of 1,12 we have

$$L\{F'(t)\}=\int_0^\infty e^{-pt}F'(t)dt=PL\{F(t)\}-F(0)\to(1)$$

Since F'(t) is sectionally continuous and of exponential order,

$$\therefore \lim_{p\to\infty}\int_0^\infty e^{-pt}F'(t)dt=0$$

Taking limit as $p\to\infty$ in (1) we have

$$0 = \lim_{p\to\infty}PL\{F(t)\}-F(0)$$

or $F(0)=\lim_{p\to\infty}PL\{F(t)\}$

or $\lim_{t\to0}\{F(t)\}=\lim_{p\to\infty}PL\{F(t)\}$

28. Final Value Theorem

Let F(t) be continuous for all $t\ge0$ and be of exponential order as $t\to\infty$ and if F'(t) is of class A, then

$$\lim_{p\to\infty}F(t) = \lim_{p\to0}PL\{F(t)\}$$

Proof:

By the theorem of 1.12 we have

$$L\{F'(t)\}=\int_0^\infty e^{-pt}F'(t)dt=PL\{F(t)\}-F(0)\to(1)$$

Taking limit as $p\to0$ in (1) we have

$$\lim_{p\to0}\int_0^\infty e^{-pt}F'(t)dt=\lim_{p\to0}PL\{F(t)\}-F(0)$$

or $\int_0^\infty F'(t)dt=\lim_{p\to0}PL\{F(t)\}-F(0)$

or $[F(t)]_0^\infty=\lim_{p\to0}PL\{F(t)\}-F(0)$

or $\lim_{t\to\infty}F(t)-F(0)=\lim_{p\to0}PL\{F(t)\}-F(0)$

or $\lim_{t\to\infty}F(t)=\lim_{p\to0}PL\{F(t)\}$

29. Multiplication By t^n Theorem

If F(t) is a function of class A and if L{F(t)}=f(p) then

$$L\{t^nF(t)\}=(-1)^n\frac{d^n}{dP^n}f(p)$$

Proof:

We shall prove this theorem by mathematical induction

we know that

$$L\{tF(t)\}=(-1)^1\frac{d}{dP}f(p)$$

or $\int_0^\infty e^{-pt}t^sF(t)dt=(-1)^s\frac{d^s}{dp^s}f(p)$

Now differentiating both sides w.r.t p we have

$$\frac{d}{dP}\int_0^\infty e^{-pt}t^sF(t)dt=(-1)^s\frac{d^{s+1}}{dP^{s+1}}f(p)$$

or $\int_0^\infty\frac{\partial}{\partial p}(e^{-pt}t^sF(t))dt=(-1)^s\frac{d^{s+1}}{dP^{s+1}}f(p)$

By leibnitz rule for differentiating under the sign of integral

or $-\int_0^\infty e^{-pt}t^{s+1}F(t)dt=(-1)^s\frac{d^{s+1}}{dP^{s+1}}f(p)$

or $\int_0^\infty e^{-pt}\{t^{s+1}F(t)\}dt=(-1)^s\frac{d^{s+1}}{dP^{s+1}}f(p)$

or $L\{t^{s+1}F(t)\}=(-1)^s\frac{d^{s+1}f(p)}{dP^{s+1}}$

DIVISION BY t : THEOREM: If L{F(t)}=f(p), then

$L\left\{\frac1tF(t)\right\}=\int_p^\infty f(x)dx$ Provided $\lim_{t\to0}=\left\{\frac1tF(t)\right\}$ exists

Proof:

Let G(t) = $\frac1tF(t)$

F(t) = tG(t)

$$\therefore L\{F(t)\}=L\{tG(t)\}=\frac{-d}{dP}L\{G(t)\}$$

or $f(p)=\frac{-d}{dP}L\{G(t)\}$

Now Integrating both the sides w.r.t p from p to ∞ we have

$$-[L\{G(t)\}]_p^\infty=\int_p^\infty f(p)dp$$

or $-\lim_{p\to\infty}L\{G(t)\}+L\{G(t)\}=\int_p^\infty f(p)dp$

or $0+L\{G(t)\}=\int_p^\infty f(p)dp$

$$\left[\because \lim_{p\to\infty}L\{G(t)\}=\lim_{p\to\infty}\int_0^\infty e^{-pt}G(t)dt=0\right]$$

or $L\left\{\frac1tF(t)\right\}=\int_R^\infty f(x)dx$

31. L{-a sin at} = $-\frac{a^2}{p^2+a^2}$

Sol:

Hence Let F(t)=-a sin at

$F'(t)=-a^2$ cost and $F''(t)=-a^3$ Sin at

$F'(0)=-a^2$ and F(0)=0

$\therefore$ from $L\{F''(t)\}=-p^2L\{F(t)\}-PF(0)-F'(0)$

we have $L\{a^3Sinat\}=p^2L\{-aSinat\}-0-(a^2)$

or $(p^2+a^2)L\{-aSinat\}=-a^2$

$$\therefore L\{-aSinat\}=\frac{-a^2}{(p^2+a^2)}$$

$- dp\ (p\sqrt{p+4}) \quad p^2(p+4)^{3/2}$

THE INVERSE LAPLACE TRANSFORM

38. If f(p) is the laplace transform of a function F(t). i.e L{F(t)} = f(p) then F(t) is called the inverse laplace transform of the function f(p) and is written as $F(t)=L^{-1}\{f(p)\}$

L^{-1} is called the inverse laplace transformation operator.

39. Find $L^{-1}\left\{\frac6{2p-3}-\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\}$

Sol:

$$3L^{-1}\left\{\frac1{p-(3/2)}\right\}-3L^{-1}\left\{\frac1{9p^2-16}\right\}-4L^{-1}\left\{\frac{p}{9p^2-16}\right\}$$

$$+8L^{-1}\left\{\frac1{16p^2+9}\right\}-6L^{-1}\left\{\frac{p}{16p^2+9}\right\}$$

$$\Rightarrow 3L^{-1}\left\{\frac1{p-(3/2)}\right\}-\frac13L^{-1}\left\{\frac1{p^2-(4/3)^2}\right\}-\frac94L^{-1}\left\{\frac{p}{p^2-(4/3)^2}\right\}$$

$$+\frac12L^{-1}\left\{\frac1{p^2+(3/4)^2}\right\}-\frac38L^{-1}\left\{\frac{p}{p^2+(3/4)^2}\right\}$$

$$\Rightarrow 3e^{(3/2)t}-\frac13\cdot\frac1{(4/3)}Sinh\frac43t-\frac49Cosh\frac{4t}3$$

$$+\frac12\frac1{3/4}Sin\frac{3t}4-\frac38Cos\frac34t$$

$$\Rightarrow 3e^{(3/2)t}-\frac14Sinh\frac43-\frac49Cosh\frac{4t}3+\frac23Sin\frac{3t}4-\frac38Cos\frac34t$$

40. Find $L^{-1}\left\{\frac3{p^2-3}+\frac{3p+2}{p^3}-\frac{3p-27}{p^2+9}+\frac{6-30\sqrt p}{p^4}\right\}$

Sol:

$$\Rightarrow L^{-1}\left\{\frac3{p^2-3}+\frac{3p+2}{p^3}-\frac{3p-27}{p^2+9}+\frac{6-30\sqrt p}{p^4}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac3{p^2-3}+\frac3{p^2}+\frac2{p^3}-\frac{3p}{p^2+9}+\frac{27}{p^2+9}+\frac6{p^4}-\frac{30}{p^{7/2}}\right\}$$

$$\Rightarrow 3L^{-1}\left\{\frac1{p^2-(\sqrt3)^2}\right\}+3L^{-1}\left\{\frac1{p^2}\right\}+2L^{-1}\left\{\frac1{p^3}\right\}-3L^{-1}\left\{\frac{p}{p^2+3^2}\right\}$$

$$+27L^{-1}\left\{\frac1{p^2+3^2}\right\}+6L^{-1}\left\{\frac1{p^4}\right\}-30L^{-1}\left\{\frac1{p^{7/2}}\right\}$$

$$\Rightarrow 3.\frac1{\sqrt3}Sinh\sqrt3t+\frac{3.t^{2-1}}{(2-1)!}+\frac{2t^{3-1}}{(3-1)!}-3Cos3t+\frac{27}3Sin3t$$

$$+\frac{6t^{4-1}}{(4-1)!}-30\frac{t^{7/2-1}}{\Gamma(7/2)}$$

$$\Rightarrow \sqrt{3}\,Sinh\sqrt{3}t+3t+t^2-3Cos3t+9Sin3t+t^3-16t^2\sqrt{1/\pi}$$

41. Prove that

$$L^{-1}\left\{\frac{5}{p^2}+\left(\frac{\sqrt{p}-1}{p}\right)^2-\frac{7}{3p+2}\right\}=1+6t-4\sqrt{t/\pi}-\frac{7}{3}e^{-2t/3}$$

Sol:

$$\Rightarrow L^{-1}\left\{\frac{5}{p^2}+\left(\frac{\sqrt{p}-1}{p}\right)^2-\frac{7}{3p+2}\right\}\Rightarrow L^{-1}\left\{\frac{5}{p^2}+\frac{p-2\sqrt{p}+1}{p^2}-\frac{7}{3}\frac{1}{p+2/3}\right\}$$

$$\Rightarrow 6L^{-1}\left\{\frac{1}{p^2}\right\}+L^{-1}\left\{\frac{1}{p}\right\}-2L^{-1}\left\{\frac{1}{p^{3/2}}\right\}-\frac{7}{3}L^{-1}\left\{\frac{1}{P+2/3}\right\}$$

$$\Rightarrow 6.\frac{t^{2-1}}{1!}+1-2\frac{t^{3/2-1}}{\Gamma(3/2)}-\frac{7}{3}e^{-2t/3}$$

$$\Rightarrow 6t+1-4\sqrt{(t/\pi)}-\frac{7}{3}e^{-2t/3}$$

42. First Translation or Shifting Theorem

If $L^{-1}\{f(p)\}=F(t)$ then $L^{-1}\{f(p-a)\}=e^{at}F(t)=e^{at}L^{-1}\{f(p)\}$

Proof:

we have $f(p)=\int_0^\infty e^{-pt}F(t)dt$

$$\therefore f(p-a)=\int_0^\infty e^{-(p-a)t}F(t)dt$$

$$=\int_0^\infty e^{-pt}\{e^{at}F(t)\}dt$$

$$=L\{e^{at}F(t)\}$$

$$\therefore L^{-1}\{f(p-a)\}=e^{at}F(t)=e^{-at}L^{-1}\{f(p)\}$$

43. Second Translation or Shifting Theorem

If $L^{-1}\{f(p)\}=F(t)$ then $L^{-1}\{e^{-ap}f(p)\}=G(t)$

where G(t)=F(t–a) H (t–a)

i.e $G(t)=\begin{cases}F(t-a), & t>a\\ 0, & t<a\end{cases}$

Proof:

we have $f(p)=\int_0^\infty e^{-pt}F(t)dt$

$$e^{-ap}f(p)=\int_0^\infty e^{-p(t+a)}F(t)dt$$

$$=\int_a^\infty e^{-px}F(x-a)dx \quad \text{Putting t+a=x, so that dt=dx.}$$

$$\Rightarrow \int_0^a e^{-px}0.dx+\int_a^\infty e^{-px}F(x-a).dx$$

$$\Rightarrow \int_0^a e^{-pt}0.dt+\int_a^\infty e^{-pt}F(t-a).dt$$

$$\Rightarrow \int_0^\infty e^{-pt}G(t)=L\{G(t)\}$$

where $G(t)=\begin{cases}F(t-a), & t>a\\ 0, & t<a\end{cases}$

44. Change of Scale Property Theorem :

If $L^{-1}f(p)=F(t)$, then

$$L^{-1}f(ap)=\frac{1}{a}F\left(\frac{t}{a}\right)$$

Proof:

we have $f(p)=\int_0^\infty e^{-pt}F(t)dt$

$$\therefore f(ap)=\int_0^\infty e^{-apt}F(t)dt$$

$$\Rightarrow \frac{1}{a}\int_0^\infty e^{-px}F\left(\frac{x}{a}\right)dx \quad \text{Putting at=x, so that dt=(1/a)dx}$$

$$\Rightarrow \frac{1}{a}\int_0^\infty e^{-pt}F\left(\frac{t}{a}\right)dt=\frac{1}{a}L\left\{F\left(\frac{t}{a}\right)\right\}=L\left\{\frac{1}{a}F\left(\frac{t}{a}\right)\right\}$$

Hence $L^{-1}\{f(ap)\}=\frac{1}{a}F\left(\frac{t}{a}\right)$

45. Find $L^{-1}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\}$

Sol:

$$L^{-1}\left\{-\frac{4}{5(p+3)}+\frac{4p+1}{5(p^2+2p+2)}\right\}$$

$$\Rightarrow \frac{-4}{5}L^{-1}\left\{\frac{1}{p+3}\right\}+\frac{1}{5}L^{-1}\left\{\frac{4(p+1)-3}{(p+1)^2+1}\right\}$$

$$\Rightarrow \frac{-4}{5}e^{-t}+\frac{e^{-t}}{5}\left[L^{-1}\left\{\frac{4p}{p^2+1}-\frac{3}{p^2+1}\right\}\right]$$

$$\Rightarrow \frac{-4}{5}e^{-3t}+\frac{e^{-t}}{5}(4Cost-3Sint)$$

46. $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\}$

$$\Rightarrow L^{-1}\left\{\frac{3(p-1)+10}{(p-1)^2-4}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{3(p-1)}{(p-1)^2-4}+\frac{10}{(p-1)^2-4}\right\}$$

$$\Rightarrow 3L^{-1}\left\{\frac{(p-1)}{(p-1)^2-4}\right\}+10L^{-1}\left\{\frac{1}{(p-1)^2-4}\right\}$$

$$\Rightarrow 3e^tL^{-1}\left\{\frac{p}{p-2^2}\right\}+10e^tL^{-1}\left\{\frac{1}{p-2^2}\right\}$$

$\Rightarrow 3e^t$ Cos h 2t+ $5e^t$ Sin h 2t = $4e^{3t}-e^t$

47. Evaluate $L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\}$

Sol:

$$\Rightarrow L^{-1}\left\{\frac{(p+1)-1}{(p+1)^{5/2}}\right\}=e^{-t}L^{-1}\left\{\frac{p-1}{p^{5/2}}\right\}$$

$$\Rightarrow e^{-t}L^{-1}\left\{\frac{1}{p^{3/2}}-\frac{1}{p^{5/2}}\right\}$$

$$\Rightarrow e^{-t}L^{-1}\left\{\frac{1}{p^{3/2}}\right\}-e^{-t}L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$$

$$\Rightarrow e^{-t}\frac{t^{3/2-1}}{\Gamma(3/2)}-e^{-t}\frac{t^{5/2-1}}{\Gamma(5/2)}$$

$$\Rightarrow 2e^{-t}\sqrt{(t/\pi)}-\frac{4}{3}e^{-t}t\sqrt{(t/\pi)}$$

$$\Rightarrow \frac{2}{3}e^{-t}\sqrt{(t/\pi)}(3-2t)$$

48. If $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\}=\frac{Cos2\sqrt{t}}{\sqrt{(\pi t)}}$, **find** $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$

Sol:

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\}=\frac{Cos2\sqrt{t}}{\sqrt{(\pi t)}}$$

$$\therefore L^{-1}\left\{\frac{e^{-1/pk}}{(pk)^{1/2}}\right\}=\frac{1}{k}\frac{Cos2\sqrt{t/k}}{\sqrt{(\pi t/k)}}$$

or $L^{-1}\left\{\frac{e^{-1/pk}}{p^{1/2}}\right\}=\frac{Cos2\sqrt{t/k}}{\sqrt{(\pi t/k)}}$

taking $k=\frac{1}{a}$ we have $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}=\frac{Cos2\sqrt{(at)}}{\sqrt{(\pi t)}}$.

52. CONVOLUTION (Definition)

Let F(t) and G(t) be two Function of class a, then the convolution of the two junctions F(t) & G(t) denoted by F*G is defined by the relation

$$F*G=\int_0^t F(x)\,G(t-x)dx$$

this relation F*G is also called the resultant or falting of Fand G.

(i) F*G is commutative i.e F*G=G*F

Proof: $F*G=\int_0^t F(x)\,G(t-x)dx$

$$=-\int_t^0 F(t-y)\,G(y)dy \quad \text{Putting t–x=y so that dx=–dy}$$

$$=\int_0^t G(y)F(t-y)dy=G*F$$

(ii) F*G is associative i.e (F*G)*H =F*(G*H)

(iii) F*G id distributive wrt addition

i.e F*(G+H)=F*G+F*H

Proof: $F*(G+H)=\int_0^t F(x)[G(t-x)+H(t-x)]dx$

$$=\int_0^t F(x)G(t-x)dx+\int_0^t F(x)H(t-x)dx$$

$$=F*G+F*H$$

53. CONVOLUTION THEOREM

Let F(t) and G(t) be two functions of class A and Let $L^{-1}\{f(p)\}=F(t)$ and $L^{-1}\{g(p)\}=G(t)$ then

$$L^{-1}\{f(p)g(p)\}=\int_0^t F(x)G(t-x)dx=F*G$$

Proof: Here we shall prove that

$$L\left[\int_0^t F(x)G(t-x)dx\right]=f(p)g(p)$$

from which the required result follows immediately we have

$$L\left[\int_0^t F(x)G(t-x)dx\right]$$

$$\Rightarrow \int_{t=0}^\infty e^{-pt}\left\{\int_0^t F(x)G(t-x)dx\right\}dt$$

$$\Rightarrow \int_{t=0}^\infty\int_{x=0}^t e^{-pt}F(x)\,G(t-x)dx\,dt$$

$$\Rightarrow \underset{T\to\infty}{Lt}\int_{t=0}^T\int_{x=0}^t e^{-pt}F(x)\,G(t-x)dx\,dt$$

$$\Rightarrow \underset{T\to\infty}{Lt}\,I_T$$

where $I_T=\int_{t=0}^T\int_{x=0}^t e^{-pt}F(x)\,G(t-x)dx\,dt$

Taking t–x= y or t=x+y we see that the shaded area A_{tx} of the tx–plane is transformed into Area A_{xy} of the xy-plane as shown in the figure.

Therefore by the theorem on transformation of multiple integrats, we have.

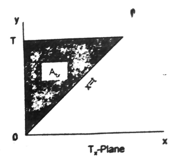
$$I_T=\int\int_{A_{tx}}e^{-pt}F(x)\,G(t-x)dx\,dt$$

$$=\int\int_{A_{xy}}e^{-p(x+y)}F(x)\,G(y)\left|\frac{\partial(x,t)}{\partial(x,y)}\right|dx\,dy$$

where jacobian of the transformation

$$J=\left|\frac{\partial(x,t)}{\partial(x,y)}\right|=\begin{vmatrix}\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y}\\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y}\end{vmatrix}=\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}=1$$

$$\therefore I_T=\int_{y=0}^T\int_{x=0}^{T-y}e^{-p(x+y)}F(x)G(y)dx\,dy$$

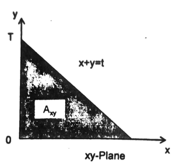


Now let us define a function

$$k(x,y)=\begin{cases}e^{-p(x+y)}F(x)G(y), & x+y\le T\\ 0, & x+y>T\end{cases}$$

Here the function K(x,y) is defined over the square of side such that t the integral IT over the unshaded area shown in the fi ure is zero.

$$\therefore I_T=\int_{y=0}^T\int_{x=0}^T K(x,y)dx\,dy$$



thus $L\left\{\int_0^t F(x)G(t-x)dx\right\}$

$$=\underset{T\to\infty}{Lt}\,I_t$$

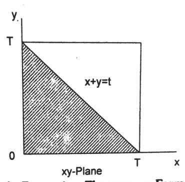
$$=\int_{y=0}^\infty\int_{x=0}^\infty K(x,y)dx\,dy$$

$$=\int_{y=0}^\infty\int_{x=0}^\infty e^{-p(x+y)}F(x)\,G(y)dx\,dy$$

$$=\left[\int_0^\infty e^{-px}F(x)\,dx\right]\left[\int_0^\infty e^{-py}G(y)\,dy\right]$$

= L {F(t)} L{G(t)} =f(p) g(p)

Hence $\int_0^t F(x)G(t-x)dx=L^{-1}\{f(p)g(p)\}=F*G$



54. Heaviside's Expansion Theorem or Formula

Let F(p) and G(p) be two polynomials in p where F(p) has degree less than that of G(p). If G(p) has n distinct zero's x_r, r=1, 2...n i.e G(p) = $(p-\alpha_1)(p-\alpha_1).....(p-\alpha_n)$ then

$$L^{-1}\left[\frac{F(p)}{G(p)}\right]=\sum_{r=1}^n\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}$$

Proof: Since F(p) is a polynomial of degree less than that of G(p) and G(p) has n distinct zero's α_r = r = 1,2....n

$$\therefore \frac{F(p)}{G(p)}=\frac{F(p)}{(p-\alpha_1)(p-\alpha_1).....(p-\alpha_n)}$$

$$\Rightarrow \frac{A_1}{(p-\alpha_1)}+\frac{A_2}{(p-\alpha_2)}+.....+\frac{A_r}{(p-\alpha_r)}+....\frac{A_n}{(p-\alpha_n)}$$

Multiplying both sides by $(p-\alpha_r)$ and taking limits as $p\to\alpha_r$, we have

$$A_r=\underset{p\to\alpha_r}{Lt}\frac{F(p).p-\alpha_r}{G(p)}$$

$$=F(\alpha_r)\underset{p\to\alpha_r}{Lt}\frac{p-\alpha_r}{G(p)}$$

$$=F(\alpha_r)\underset{p\to\alpha_r}{Lt}\frac{1}{G'(p)}$$

$$=F(\alpha_r)\underset{p\to\alpha_r}{Lt}\frac{1}{G'(\alpha_r)}$$

$$\therefore \frac{F(p)}{G(p)}=\frac{F(\alpha_1)}{G'(\alpha_1)}\frac{1}{(p-\alpha_1)}+\frac{F(\alpha_2)}{G'(\alpha_2)}\frac{1}{(p-\alpha_2)}+.....+$$

$$\frac{F(\alpha_r)}{G'(\alpha_r)}\cdot\frac{1}{(p-\alpha_r)}+.....+\frac{F(\alpha_n)}{G'(\alpha_n)}\cdot\frac{1}{(p-\alpha_n)}$$

Hence $L^{-1}\frac{F(p)}{G(p)}=\frac{F(\alpha_1)}{G'(\alpha_1)}L^{-1}\left[\frac{1}{p-\alpha_1}\right]+\frac{F(\alpha_2)}{G'(\alpha_2)}L^{-1}\left[\frac{1}{p-\alpha_2}\right]+....$

$$+\frac{F(\alpha_r)}{G'(\alpha_r)}L^{-1}\left[\frac{1}{p-\alpha_r}\right]+...+\frac{F(\alpha_n)}{G'(\alpha_n)}L^{-1}\left[\frac{1}{p-\alpha_n}\right]$$

$$\Rightarrow \frac{F(\alpha_1)}{G'(\alpha_1)}e^{\alpha_1 t}+.....+\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}+...+\frac{F(\alpha_n)}{G'(\alpha_n)}e^{\alpha_n t}$$

$$\Rightarrow \sum_{r=1}^n\frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t}$$

55. Find $L^{-1}\left\{\frac{1}{p^2-a^2}\right\}$

Sol: Since $\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)=-\frac{2p}{(p^2-a^2)^2}$

$$\therefore \frac{p}{(p^2-a^2)^2}=-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)$$

Hence $L^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\}=-\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)\right\}$

$$=-\frac{1}{2}(-1)^1tL^{-1}\left\{\frac{1}{p^2-a^2}\right\}$$

$$=-\frac{1}{2}t-\frac{1}{a}Sinhat=\frac{1}{2a}Sinhat$$

56. Find $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$

Sol: Since $L^{-1}\left\{\frac{1}{p^2+1}\right\}$ = Sin t

$$L^{-1}\left\{\frac{1}{p(p^2+1)}\right\}=\int_0^t Sinx\,dx=1-cost$$

$$L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\}=\int_0^t(1-Cosx)dx=1-Sint$$

and $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}=\int_0^t(x-\sin x)dx=\frac{t^2}{2}+Cost-1$

57. Find $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}$

Sol: $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}=L^{-1}\left\{\frac{1}{(p+1-1)(p+1)^3}\right\}$

$$=e^{-t}L^{-1}\left\{\frac{1}{(p-1)p^3}\right\}\quad\to(1)$$

Since $L^{-1}\left\{\frac{1}{(p-1)}\right\}=e^t$

$$\therefore L^{-1}\left\{\frac{1}{p(p-1)}\right\}=\int_0^t e^x dx=(e^t-1)$$

$$L^{-1}\left\{\frac{1}{p^2(p-1)}\right\}=\int_0^t(e^x-1)dx=e^t-t-1$$

and $L^{-1}\left\{\frac{1}{p^3(p-1)}\right\}=\int_0^t(e^x-x-1)dx=e^t-\frac{t^2}{2}-t-1$

∴ from (1) we have

$$L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}=e^{-t}\left\{e^t-\frac{t^2}{2}-t-1\right\}$$

$$=1-e^{-t}\left(1+t+\frac{t^2}{2}\right)$$

58. Use the Convolution Theoremto Find $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

Sol: Since $L^{-1}\left\{\frac{p}{(p^2+a^2)}\right\}$ = Cos at

$$L^{-1}\left\{\frac{1}{(p^2+a^2)}\right\}=\frac{1}{a}\text{Sin at}$$

∴ By the convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}=\int_0^t\cos ax\frac{1}{a}Sina(t-x)dx$$

$$\Rightarrow \frac{1}{a}\int_0^t\cos ax\,(\text{Sin at Cosax-Cosat sinax})dx$$

$$\Rightarrow \frac{1}{a}Sinat\int_0^t\cos^2ax\,dx-\frac{1}{a}Cosat\int_0^t Sinax\,Cos\,ax\,dx$$

$$\Rightarrow \frac{1}{2a}Sinat\int_0^t(1+\cos 2ax)dx-\frac{1}{2a}Cosat\int_0^t Sin\,2ax\,dx$$

$$\Rightarrow \frac{1}{2a}Sinat\left[\frac{1+Sin2at}{2a}\right]-\frac{1}{2a}Cosat\left[\frac{1-Cos2at}{2a}\right]$$

$$\Rightarrow \frac{tSinat}{2a}+\frac{Sinat\,2Sinat\,Cosat}{4a^2}-\frac{Cosat.2Sin^2at}{4a^2}$$

$$\Rightarrow \frac{tSinat}{2a}$$

UNIT - II

3. Given $L\{F(t)\}=\frac{P^2-P+1}{(2P+1)^2(P-1)}$ **applying the change of scale property show that** $L\{F(2t)\}=\frac{P^2-2P+4}{4(P+1)^2(P-2)}$

Ans :

We have

$$L\{F(t)\}=\frac{p^2-p+1}{(2p+1)^2(p-1)}=f(p)$$

$$\therefore L\{F(2t)\}=\frac{1}{2}f(p/2)$$

$$=\frac{1}{2}\left[\frac{(p/2)^2-(p/2)+1}{\left(\frac{2p}{2}+1\right)^2\left(\frac{p}{2}-1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{p^2/4-p/2+1}{(p+1)^2\left(\frac{p-2}{2}\right)}\right]$$

$$=\frac{1}{2}\cdot\frac{p^2-2p+4}{4(p+1)^2(p-2)}\times\frac{2}{1}$$

$$L\{F(2t)\}=\frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

MODEL PAPER - II

4. Evaluate $L^{-1}\left\{\frac{3P-2}{P^2-4P+20}\right\}$

Ans :

$$L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{3p-2}{(p-2)^2+16}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{3p-6+4}{(p-2)^2+16}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{3(p-2)}{(p-2)^2+16}\right\}+L^{-1}\left\{\frac{4}{(p-2)^2+16}\right\}$$

$$\Rightarrow e^{2t}L^{-1}\left\{\frac{3p}{p^2+16}\right\}+e^{2t}L^{-1}\left\{\frac{4}{p^2+16}\right\}$$

$$\Rightarrow 3e^{2t}L^{-1}\left\{\frac{p}{p^2+16}\right\}+4e^{2t}L^{-1}\left\{\frac{1}{p^2+16}\right\}$$

$$\Rightarrow 3e^{2t}\cos 4t+4e^{2t}\sin 4t$$

$$\Rightarrow e^{2t}(3\cos 4t+4\sin 4t)$$

UNIT - III