62. Fourier Transform: Let f(x) be a function defined on $(-\infty,\infty)$ and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty,\infty)$ then.

$$F\{f(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$ is called the fourier trousform of f(x) and is denoted by F{f(x)} or f(p).

63. Inversion theorem for complex fourier transform

If $\widetilde{f(p)}$ is the fourier transform of f(x) and if f(x) satisfies the Dirichlets conditions in every finite in terval $(-l,l)$ and further if $\int_{-\infty}^{\infty}|f(x)|dx$ is conuergent , then at every point of continuity of f(x)

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tilde{p})e^{-ipx}dp$$

Proof: From Fourier integral formulae we have.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{iw(x-v)}\,dw\right]dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\,dw\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv$$

Put w=-p, dw=-dp

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}dp.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ipx}dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\widetilde{f(p)}dp$$

64. Fourier Sine transform:

The infinite fourier Sine transform of f(x), 0<x<∞ is defined by F_s {f(x)} is $\widetilde{f_s(p)}$ Such that

$$F_s\{f(x)\} = f_s(\tilde{p}) = \sqrt{\frac{2}{\pi}} = \int_0^{\infty} f(x)\,Sin\,px\,dx$$

Inversion formula for fourier sine transform:

If $f_s(\tilde{p})$ is the fourier sine transform of the function f(x) which satifies the D.C in every finite interval $(0,l)$ such that $\int_0^{\infty}|f(x)|dx$ exists then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f_s p}$ Sin px dp at every point of continuity of f(x)

65. Fourier Cosine Transform:

The infinite fourier cosine transform of f(x), $0<x<\infty$ is defined by $F_c\{f(x)\}$ or $\widetilde{f_c(p)}$ such that

$$F_c\{f(x)\} = \widetilde{f_c(p)} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} f(x)\,Cos\,px\,dx$$

Inversion formula for fourier Cosine formula:

If $\widetilde{f_c p}$ is the fourier Cosine transform of the function f(x) which Satisfies the D.C in every finite interval $(0,l)$ and is such that $\int_0^{\infty}|f(x)|dx$ exists then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f_c(p)}\,Cos\,px\,dp$ Cos px dp at every point of continuity of f(x).

66. Change of Scale property for fourier Transform:

If $\widetilde{f(p)}$ is the Complex fourier transform of f(x). thus the complex fourier transform of f(ax) is $\frac{1}{a}\widetilde{f(p/a)}$

Proof: we have

$$\widetilde{f(p)} = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx \quad \to (1)$$

$$\text{Now } F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

Put ax=t $\quad dx = \frac{1}{a}dt$

$$\Rightarrow \frac{1}{a}.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p/a)t} f(t)\,dt$$

$$F\{f(ax)\} = \frac{1}{a}\widetilde{f(p/a)}$$

67. Shifting property

If $f(\tilde{p})$ is the complex fourier transform of f(x), then complex fourier transform of f(x–a) is $e^{ipa}f(\tilde{p})$.

Proof: we have

$$f(\tilde{p}) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

$$\Rightarrow F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x-a)dx$$

Put x–a=t $\quad \Rightarrow x=t+a \quad$ dx=dt

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ip(t+a)} f(t)\,dt$$

$$= e^{ipa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f(t)\,dt$$

$$F\{f(x-a)\} = e^{ipa}\widetilde{f(p)}$$

68. Modulation theorem:

If $\widetilde{f(p)}$ is the complex fourier transform of f(x), then the fourier transform of f(x) cos ax is $\frac{1}{2}[\tilde{f}(p-a)+\tilde{f}(p+a)]$

Proof: we have

$$\widetilde{f(p)} = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx \to (1)$$

$$\text{Now } F\{f(x)Cos\,ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\frac{e^{iax}+e^{-iax}}{2}dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p-a)x} f(x)dx\right]$$

$$\Rightarrow \frac{1}{2}\left[\widetilde{f(p+a)}+\widetilde{f(p-a)}\right]$$

69. Find the fourier transform of F(x) defined by $F(x) = \begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ and hence evaluate

a) $\int_{-\infty}^{\infty}\frac{Sin\,pa\,Cos\,px}{p}dp \quad$ b) $\int_0^{\infty}\frac{Sin\,p}{p}dp$

Sol: we have

$$\tilde{F}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} F(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ipx}}{ip}\right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ipa}}{ip}-\frac{e^{-ipa}}{ip}\right)$$

$$\Rightarrow \frac{2i\,Sin\,pa}{ip\sqrt{2\pi}}$$

$$= \frac{2\,Sin\,pa}{p\sqrt{2\pi}},\ p\neq 0$$

For p=0, $F(\tilde{p}) = \frac{2a}{\sqrt{2\pi}}$

a) we know that if $F(\tilde{p}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{ipx}dx$

then $F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\tilde{p})e^{-ipx}dp$

$$\therefore \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2\,Sin\,pa}{p\sqrt{2\pi}}e^{-ipx}dp = \begin{cases}1, & |x|<a\\ 0 & |x|>a\end{cases}$$

$$\text{but LHS} \Rightarrow \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Sin\,pa\,Cos\,px}{p}dp - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{Sin\,pa\,Sin\,px}{p}dp$$

$$\Rightarrow \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Sin\,pa\,Cos\,px}{p}dp$$

(∵ 2nd integral is an odd integral)

$$\therefore \int_{-\infty}^{\infty}\frac{Sin\,pa\,Cos\,px}{p}dp = \begin{cases}\pi, & |x|<a\\ 0 & |x|>a\end{cases}$$

b) If x=0 and a=1 in (a) then we get

$$\int_{-\infty}^{\infty}\frac{Sin\,p}{p}dp = \pi$$

$$\Rightarrow 2\int_0^{\infty}\frac{Sin\,p}{p}dp = \pi$$

$$\Rightarrow \int_0^{\infty}\frac{Sin\,p}{p}dp = \pi/2$$

70. Find the fourier transform of $F(x)= \begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$ **and hence evaluate** $\int_0^{\infty}\left(\frac{x\,Cosx - Sinx}{x^3}\right)Cos\frac{x}{2}dx$

Sol: we have

$$F(\tilde{p}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}F(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{ipx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{ipx}}{ip}\right]_{-1}^{1} + \frac{2}{\sqrt{2\pi}}\int_{-1}^{1}x.\frac{e^{ipx}}{ip}dx$$

$$= 0+\frac{\sqrt{2}}{ip\sqrt{\pi}}\int_{-1}^{1}x.e^{ipx}dx$$

$$\Rightarrow \frac{\sqrt{2}}{ip\sqrt{\pi}}\left[\left[\frac{xe^{ipx}}{ip}\right]_{-1}^{1} - \int_{-1}^{1}1.\frac{e^{ipx}}{ip}dx\right]$$

$$\Rightarrow \frac{\sqrt{2}}{ip\sqrt{\pi}}\left[\left(\frac{e^{ip}+e^{-ip}}{ip}-\left[\frac{e^{ipx}}{(ip)^2}\right]_{-1}^{1}\right)\right]$$

$$\Rightarrow \frac{\sqrt{2}}{ip\sqrt{\pi}}\left(\frac{2Cos\,p}{ip}+\frac{e^{ip}-e^{-ip}}{p^2}\right)$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{-2Cos\,p}{p^2}+\frac{2i\left(\frac{e^{ip}-e^{-ip}}{2}\right)}{ip^3}\right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\left[\frac{-2Cos\,p}{p^2}+\frac{2iSin\,p}{ip^3}\right]$$

$$\Rightarrow -2\sqrt{\frac{2}{\pi}}\left[\frac{p\,Cos\,p - Sin\,p}{p^3}\right]$$

we know that if

$$F(\tilde{p}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{ipx}dx \quad \text{then}$$

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\tilde{p})e^{-ipx}dp$$

$$\therefore -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2\left(\sqrt{2/\pi}\right)(p\,Cos\,p - Sin\,p)}{p^3}e^{-ipx}dp = \begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Cos\,px\,dp + i\int_{-\infty}^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Sin\,px\,dp = \begin{cases}\frac{\pi}{2}(1-x^2), & |x|<1\\ 0, & |x|>1\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Cos\,px\,dp = \begin{cases}\frac{\pi}{2}(1-x^2), & |x|<1\\ 0, & |x|>1\end{cases}$$

Now put $x=\frac{1}{2}$ on both sides

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Cos\frac{p}{2}dp = \frac{\pi}{2}\left(1-\frac{1}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_0^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Cos\frac{p}{2}dp = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{p\,Cos\,p - Sin\,p}{p^3}\right)Cos\frac{p}{2}dp = \frac{3\pi}{16}$$

Replace p by 'x'

$$\Rightarrow \int_0^{\infty}\left(\frac{x\,Cos\,x - Sin\,x}{x^3}\right)Cos\frac{x}{2}dx = \frac{3\pi}{16}$$

71. Find the fourier Complex transform of f(x), if $f(x) = \begin{cases}e^{iwx}, & a<x<b\\ 0, & x<a,\ x<b\end{cases}$

Sol: we have

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{a}^{b}e^{ipx}e^{iwx}\,dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left\{\frac{e^{i(p+w)x}}{i(p+w)}\right\}_a^b$$

$$\Rightarrow \frac{i}{\sqrt{2\pi}}\left\{\frac{e^{-i(p+w)b}+e^{i(p+w)a}}{p+w}\right\}_a^b$$

72. Find the Cosine transform of a function of x which is unity for 0<x<a. and Zero for $x\ge a$. What is the function whose Cosine transform is $\sqrt{\frac{2}{\pi}}\left(\frac{Sin\,ap}{p}\right)$

Sol: Given that $f(x)=\begin{cases}1, & 0<x<a\\ 0, & x\ge a\end{cases}$

$$\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\,Cos\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{a}Cos\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{Sin\,pa}{p}\right)$$

Again we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f_c(p)}\,Cos\,px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{Sin\,pa}{p}Cos\,px\,dp$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{Sin(a+x)p+Sin(a-x)p}{p}dp$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{Sin(a+x)p}{p}dp+\frac{1}{\pi}\int_0^{\infty}\frac{Sin(a-x)p}{p}dp$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1,\ \text{if } x<a$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = 0,\ \text{if } x>a$$

73. Find fourier sine and cosine transform of e^{-x} and using the inversion formulae recover the original functions in both the cases.

Sol :

Let $f(x) = e^{-x}$

$$\widetilde{f_s(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\,Sin\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\,Sin\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+p^2}(-Sin\,px - P\,Cos\,px)\right]_0^{\infty}$$

$$\widetilde{f_s(p)} = \left[\frac{p}{1+p^2}\right]\sqrt{\frac{2}{\pi}}$$

$$\text{and} \quad \widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\,Cos\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}Cos\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+p^2}(-Cos\,px + P\,Sin\,px)\right]_0^{\infty}$$

$$= \left[\frac{p}{1+p^2}\right]\sqrt{\frac{2}{\pi}}$$

Applying inversion to the sine transform, we have.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f_s(p)}.Sin\,px\,dp$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{P\,Sin\,px}{1+p^2}dp \quad \to (1)$$

and applying inversion to the cosine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\widetilde{f_c(p)}.Cos\,px\,dp$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{Cos\,px}{1+p^2}dp \to (2)$$

Now from fourier integral theorem, we have

$$f(x) = \frac{1}{\pi}\int_0^{\infty}dp\int_{-\infty}^{\infty}f(v)\,Cos\,p(x-v)\,dv$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}Cos\,px\,dp+\int_{-\infty}^{\infty}f(v)\,Cos\,pv\,dv + \frac{1}{\pi}\int_0^{\infty}Sin\,px\,dp+\int_{-\infty}^{\infty}f(v)\,Sin\,pv\,dv \quad \to (3)$$

Case (i) Defining f(x) in $(-\infty,0)$ such that f(x) is an even function of x, from (3) we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty}Cos\,px\,dp\int_0^{\infty}f(v)\,Cos\,pv\,dv$$

Taking $f(x) = e^{-x}$ we have.

$$e^{-x} = \frac{2}{\pi}\int_0^{\infty}Cos\,px\,dp\int_0^{\infty}e^{-v}\,Cos\,pv\,dv$$

$$= \frac{2}{\pi}\int_0^{\infty}Cos\,px\left[\frac{e^{-v}}{1+p^2}(-Cos\,pv+P\,Sin\,pv)\right]_0^{\infty}dp$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{Cos\,px}{1+p^2}dp$$

$$= \int_0^{\infty}\frac{Cos\,px}{1+p^2}dp$$

$$= \frac{\pi}{2}e^{-x}$$

∴ from (2) we have

$$f(x) = \frac{2}{\pi}.\frac{\pi}{2}.e^{-x}$$

$$f(x) = e^{-x}$$

Case (ii): Again defining f(x) in $(-\infty,0)$ such that f(x) is an odd function of x from (2) we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty}Sin\,px\,dp\int_0^{\infty}f(v)\,Sin\,pv\,dv$$

Taking f(x) = e^{-x} and simplifying, we have

$$= \int_0^{\infty}\frac{P\,Sin\,px}{1+p^2}dp = \frac{\pi}{2}e^{-x}$$

∴ from (1) we have

$$f(x) = \frac{2}{\pi}.\frac{\pi}{2}.e^{-x}$$

$$\Rightarrow f(x) = e^{-x}$$

74. Find Fourier cosine transform of $f(x)=\frac{1}{1+x^2}$ and hence find fourier sine transform of $F(x)=\frac{x}{1+x^2}$

Sol: we have

$$\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\,Cos\,px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Cos\,px}{1+x^2}dx$$

Diff: both sides w.r.t 'p' we have

$$\frac{d}{dp}\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x\,Sin\,px}{1+x^2}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{(x^2+1-1)Sin\,px}{x(1+x^2)}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{x}dx+\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{x(1+x^2)}dx$$

$$= -\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)+\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{x(1+x^2)}dx$$

$$\frac{d}{dp}\widetilde{f_c(p)} = -\sqrt{\frac{2}{\pi}}+\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{x(1+x^2)}dx$$

Differentiating w.r.t 'p' on both sides.

$$\frac{d^2}{dp^2}\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Cos\,px}{1+x^2}dx = \widetilde{f_c(p)}$$

$$\Rightarrow (D^2-1)\widetilde{f_c(p)} = 0$$

Whose general solution is $\widetilde{f_c(p)} = Ae^{p}+Be^{-p} \quad \to (1)$

wehn p = 0,

$$\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{dx}{1+x^2}$$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}x\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\left(\pi/2\right)$$

$$\widetilde{f_c(p)} = \sqrt{\pi/2}$$

Also when p = 0

$$\frac{d}{dp}\widetilde{f_c(p)} = -\sqrt{\frac{\pi}{2}}$$

∴ from (1) we have

$$A+B = \sqrt{\frac{\pi}{2}}$$

$$\text{and} \quad A-B = -\sqrt{\frac{\pi}{2}}$$

Solving A = 0,

$$B = \sqrt{\pi/2}$$

∴ from (1) we have $\widetilde{f_c(p)} = \sqrt{\frac{\pi}{2}}e^{-p}$ Now we have

$$\widetilde{f_c(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Cos\,px}{1+x^2}dx = \sqrt{\pi/2}e^{-p}$$

Now differentiating both sides w.r.t 'p' we have

$$\Rightarrow -\int_0^{\infty}\frac{x\,Sin\,px}{1+x^2}dx = -\frac{\pi}{2}e^{-p}$$

$$\Rightarrow \widetilde{F_s(p)} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x}{1+x^2}Sin\,px\,dx$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}e^{-p}$$

75. Show that the fourier transform of $f(x)=e^{-x^2/2}$ is $e^{-p^2/2}$

Sol: we have

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{ipx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}e^{ipx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x-ip)^2}e^{-p^2/2}dx$$

Substituting $\frac{x-ip}{\sqrt{2}} = y$,

$$\Rightarrow dx = \sqrt{2}\,dy$$

$$= \frac{e^{-p^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y^2}dy\left(\sqrt{2}\right)$$

$$= \frac{e^{-p^2/2}}{\sqrt{\pi}}.\sqrt{\pi} \quad \left(\because \int_{-\infty}^{\infty}e^{-y^2}dy = \sqrt{\pi}\right)$$

$$F\{f(x)\} = e^{-p^2/2}$$

76. Use the sine inversion formula to obtain $f(x)$ if $\widetilde{f_s(p)} = \frac{p}{1+p^2}$.

Sol: using sine inversion formula we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{p}{1+p^2}Sin\,px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{(p^2+1-1)}{p(1+p^2)}Sin\,px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{p}dp-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{p(1+p^2)}dp$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{p(1+p^2)}dp \to (1)$$

$$f(x) = \sqrt{\frac{\pi}{2}}-\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Sin\,px}{p(1+p^2)}dp$$

$$\therefore \frac{df}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{Cos\,px}{1+p^2}dp \to (2)$$

$$\text{and} \quad \frac{d^2f}{dx^2} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{p\,Sin\,px}{1+p^2}dp$$

$$\Rightarrow \frac{d^2f}{dx^2}-f = 0$$

whose solution is

$$f = Ae^{x} + Be^{-x} \rightarrow (3)$$

$$\therefore \quad \frac{df}{dx} = Ae^{x} - Be^{-x}$$

when $x = 0$,

$$f = \sqrt{\frac{\pi}{2}} \text{ from (1)}$$

and

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{dp}{1+p^2} = -\sqrt{\frac{\pi}{2}}$$

$\therefore$ from (3)

$$\sqrt{\frac{\pi}{2}} = A+B,$$

and

$$-\sqrt{\frac{\pi}{2}} = A-B$$

on solving we get

$$A = 0, \quad B = \sqrt{\pi/2}$$

$$\therefore \quad f(x) = \sqrt{\frac{\pi}{2}}e^{-x}$$

77. Find the finite fourier sine and Cosine transforms of the function $f(x) = 2x$, $0<x<4$.

Sol: we have

$$\overline{f_s}(p) = \int_0^{\ell} f(x)\,\mathrm{Sin}\frac{p\pi x}{\ell}\,dx$$

$$= \int_0^4 2x\,\mathrm{Sin}\frac{p\pi x}{4}\,dx \quad \text{when } \ell = 4$$

$$= \left(\frac{-2x\,\mathrm{Cos}\frac{p\pi x}{4}}{p\pi/4}\right)_0^4 + 2\int_0^4 \frac{\mathrm{Cos}\frac{p\pi x}{4}}{p\pi/4}\,dx$$

$$\Rightarrow \frac{-32}{p\pi}\mathrm{Cos}\,p\pi + \frac{8}{p\pi}\left(\frac{\mathrm{Sin}\frac{p\pi x}{4}}{p\pi/4}\right)_0^4$$

$$\Rightarrow \frac{-32}{p\pi}\mathrm{Cos}\,p\pi$$

and

$$\overline{f_c}(p) = \int_0^{l} f(x)\,\mathrm{Cos}\frac{p\pi x}{l}\,dx$$

$$= \int_0^4 2x\,\frac{\mathrm{Cos}\,p\pi x}{4}\,dx \quad (\because \ell = 4)$$

$$= \left(\frac{2x\,\mathrm{Sin}\frac{p\pi x}{4}}{p\pi/4}\right)_0^4 - 2\int_0^4 \frac{\mathrm{Sin}\left(\frac{p\pi x}{4}\right)}{p\pi/4}\,dx$$

$$= -\frac{8}{p\pi}\left(\frac{-\mathrm{Cos}\frac{\pi x}{4}}{p\pi/4}\right)_0^4$$

$$= \frac{32}{p^2\pi^2}(\mathrm{Cos}\,p\pi - 1) \quad \text{if } p > 0$$

nd if p=0 then $\overline{f_c}(p) = \int_0^4 2x.1\,dx = 16$

78. Find the finite cosine transform of $\left(1-\frac{x}{\pi}\right)^2$

Sol: we have

$$\overline{f_c}(p) = \int_0^\pi \left(1-\frac{x}{\pi}\right)^2 \mathrm{Cos}\,px\,dx$$

$$= \left[\left(1-\frac{x}{\pi}\right)^2\frac{\mathrm{Sin}\,px}{p}\right]_0^\pi + \frac{2}{\pi p}\int_0^\pi\left(1-\frac{x}{\pi}\right)\mathrm{Sin}\,px\,dx$$

$$= \frac{2}{\pi p}\left[-\left(1-\frac{x}{\pi}\right)\frac{\mathrm{Cos}\,px}{p}\right]_0^\pi - \frac{2}{p\pi}\cdot\frac{1}{p\pi}\int_0^\pi \mathrm{Cos}\,px\,dx$$

$$= \frac{2}{\pi p^2} - \frac{2}{p^2\pi^2}\left(\frac{\mathrm{Sin}\,px}{p}\right)_0^\pi$$

$$\overline{f_c}(p) = \frac{2}{\pi p^2} \quad \text{if } p>0$$

and if p = 0 then $\overline{f_c}(p) = \int_0^\pi\left(1-\frac{x}{\pi}\right)^2 dx = \pi/3$

UNIT-4

(b) Find fourier sine and cosine transform of e^{-x} and using the inversion formulae recover the original functions in both the cases.

Ans :

Let . $f(x) = e^{-x}$

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\Pi}}\int_0^\infty f(x)\sin px\,dx$$

$$= \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x}\sin px\,dx$$

$$= \sqrt{\frac{2}{\Pi}}\left[\frac{e^{-x}}{1+P^2}(-\sin px - p\cos px)\right]_0^\infty$$

$$\tilde{f}_s(p) = \frac{P}{1+P^2}\sqrt{\frac{2}{\Pi}}$$

Similarly

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\Pi}}\int_0^\infty f(x)\cos px\,dx$$

$$= \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x}\cos px\,dx$$

$$= \sqrt{\frac{2}{\Pi}}\left[\frac{e^{-x}}{1+P^2}(-\cos px + p\sin px)\right]_0^\infty$$

$$\tilde{f}_c(p) = \frac{1}{1+P^2}\sqrt{\frac{2}{\Pi}}$$

Applying inversion to the sine transform, we have

$$f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \tilde{f}_c(p)\sin px\,dp$$

$$= \frac{2}{\Pi}\int_0^\infty \frac{p\sin px}{1+P^2}\,dp \quad \ldots (1)$$

and applying inversion to the cosine transform, we have

$$f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \tilde{f}_c(p)\cos px\,dp$$

$$= \frac{2}{\Pi}\int_0^\infty \frac{\cos px}{1+P^2}\,dp \quad \ldots (2)$$

Now from fourier integral theorem we have

$$f(x) = \frac{1}{\Pi}\int_0^\infty dp\int_{-\infty}^\infty f(v) - \mathrm{cosp}\,(x-v)\,dv$$

$$\Rightarrow f(x) = \frac{1}{\Pi}\int_0^\infty \mathrm{cospx}\,dp\int_{-\infty}^\infty f(v)\,\mathrm{cospv}\,dv$$

$$+ \frac{1}{\Pi}\int_0^\infty \mathrm{sinpx}\,dp\int_{-\infty}^\infty f(v)\,\mathrm{sinpv}\,dv \quad \ldots (3)$$

Case I :

Defining f(x) in $(-\infty, 0)$ such that f(x) is an even function of 'x' from (3) we have

$$f(x) = \frac{2}{\Pi}\int_0^\infty \mathrm{cospx}\,dp\int_0^\infty f(v)\,\mathrm{cospv}\,dv$$

Taking $f(x) = e^{-x}$ we have

$$e^{-x} = \int_0^\infty \mathrm{cospx}\,dp\int_0^\infty e^{-P}\,\mathrm{cospv}\,dv$$

$$= \frac{2}{\Pi}\int_0^\infty \mathrm{cospx}\left[\frac{e^{-P}}{1+P^2}(-\cos pv + p\sin pv)\right]_0^\infty dp$$

$$= \frac{2}{\Pi}\int_0^\infty \frac{\cos px}{1+P^2}\,dp$$

$$\therefore \int_0^\infty \frac{\cos px}{1+P^2}\,dp = \frac{\Pi}{2}e^{-P}$$

$\therefore$ from (2) we ahve $f(x) = \frac{2}{\Pi}\cdot\frac{\Pi}{2}e^{-x}$

$\therefore$ $f(x) = e^{-x}$

Case II :

Again defining f(x) in $(-\infty, 0)$ such that f(x) is an odd function of x, from (ii) we have

$$\therefore \quad f(x) = \frac{2}{\Pi}\int_0^\infty \sin px\,dp\int_0^\infty f(v)\sin pv\,dv$$

Taking $f(x) = e^{-x}$ and simplifying, we have

$$\int_0^\infty \frac{p\sin px}{1+P^2}\,dp = \frac{\Pi}{2}e^{-x}$$

$\therefore$ from (1) $f(x) = \frac{2}{\Pi}\cdot\frac{\Pi}{2}e^{-x}$

$\therefore$ $f(x) = e^{-x}$

14. (a) Find f(x) is $\tilde{f}_s(p) = \frac{e^{-ap}}{p}$. Hence deduce $F_s^{-1}\left\{\frac{1}{p}\right\}$.

Ans :

Using fourier sine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \frac{e^{-ap}}{p}\sin px\,dp$$

$$\therefore \quad \frac{df}{dx} = \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-ap}\cos px\,dp$$

$$= \sqrt{\frac{2}{\Pi}}\left(\frac{a}{a^2+x^2}\right)$$

$$\Rightarrow \quad f = \sqrt{\frac{2}{\Pi}}\int\frac{adx}{a^2+x^2} + A$$

$$\Rightarrow \quad f = \sqrt{\frac{2}{\Pi}}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) + A$$

But when $x = 0$, $f = 0$

$\therefore$ $A = 0$

Hence $f(x) = F_s^{-1}\left\{\frac{e^{-ap}}{p}\right\}$

$$= \sqrt{\frac{2}{\Pi}}\tan^{-1}\left(\frac{x}{a}\right) \quad \ldots (1)$$

Now put a= 0 in (1) we have

$$F_s^{-1}\left\{\frac{1}{p}\right\} = \sqrt{\frac{2}{\Pi}}\tan^{-1}(\infty) = \sqrt{\frac{\Pi}{2}}$$

(b) Find the fourier cosine transform of e^{-x^2}.

Ans :

We have

$$F_c\{e^{-x^2}\} = \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x^2}\,\mathrm{cospx}\,dx = I \quad \ldots (1)$$

Different w.r.t 'p' on both sides we have

$$\frac{dI}{dp} = \frac{1}{2}\sqrt{\frac{2}{\Pi}}\int_0^\infty (2xe^{-x^2})\,\mathrm{sinpx}\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\Pi}}\left[\left(e^{-x^2}\sin px\right)_0^\infty - p\int_0^\infty e^{-x^2}\cos pxdx\right]$$

$$= \frac{-p}{2}\sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x^2}\cos px\,dx$$

$$\frac{dI}{dp} = \frac{-p}{2}I \text{ from (1)}$$

$$\Rightarrow \quad \frac{dI}{dp} = \frac{-p}{2}\,dp$$

Integrating on both sides we get

$$\int\frac{dI}{I} = -\frac{1}{2}\int p\,dp$$

$$\Rightarrow \log I = \frac{-p^2}{4} + \log A$$

$$\Rightarrow I = Ae^{-p^2/4}$$

But when $p = 0$ from (1)

$$I = \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x^2}dx = \frac{1}{\sqrt{2}}$$

$\therefore$ from (2) $A = \frac{1}{\sqrt{2}}$

Substitutes 'A' values in (2) we get

$$I = F_C\{e^{-x^2}\} = \frac{1}{\sqrt{2}}e^{-p^2/4}$$

6. Find the fourier transform of f(x) if $f(x) = \begin{cases}\frac{\sqrt{(2\Pi)}}{2\in}, & |x| \le \in\\ 0, & |x| \le \in\end{cases}$

Ans :

We have

$$F\{f(x)\} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty f(x)\,e^{ipx}\,dx$$

$$= \int_{-\in}^{\in}\frac{1}{2\in}e^{ipx}\,dx$$

$$= \frac{1}{2ip\in}\left(e^{ipx}\right)_{-\in}^{\in}$$

$$= \frac{e^{ip\in} - e^{-ip\in}}{2ip\in}$$

$$F\{f(x)\} = \frac{\sin p\in}{p\in}$$

UNIT - IV

Q prove inversion formula of fourier sin transform

The function f(x) is called the inverse fourier sine transform of $\tilde{f}_s(p)$.

Statement

If $\tilde{f}_s(p)$ is the fourier sine transform of the function f(x) which satisfies the Dirichlet conditions in every finite interval (0, l) and is such that $\int_0^\infty |f(x)|\,dx$ exists. Then $f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \tilde{f}_s(p)\sin px\,dp$ at every point of continuity of f(x).

Proof :

From Fourier integral formula we have

$$f(x) = \frac{1}{\Pi}\int_0^\infty dw\int_{-\infty}^\infty f(v)\,\mathrm{cosw}\,(x-v)\,dv$$

$$= \frac{1}{\Pi}\left[\int_0^\infty dp\int_{-\infty}^\infty [f(v)\cos px\cos pv + f(v)\sin px\sin pv]\,dv\right.$$

where $w = p$

$$= \frac{1}{\Pi}\int_0^\infty \cos px\,dp\int_{-\infty}^\infty f(v)\cos pv\,dv + \frac{1}{\Pi}\int_0^\infty \sin px\,dp\int_{-\infty}^\infty f(v)\sin pv\,dv$$

$$f(x) = \frac{1}{\Pi}\int_0^\infty \cos px\,dp\int_{-\infty}^\infty f(x)\cos px\,dx + \frac{1}{\Pi}\int_0^\infty \sin px\,dp\int_{-\infty}^\infty f(x)\sin px\,dx \quad \ldots (1)$$

Now defining f(x) in $(-\infty, 0)$, such that f(x) is an odd function of x, f(x) cos px is an odd function of x and f(x) sin px is an even function of x.

$\therefore$ from (1) we have

$$f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \sin px\,dp\,\sqrt{\frac{2}{\Pi}}\int_0^\infty f(x)\sin px\,dp$$

$$f(x) = \sqrt{\frac{2}{\Pi}}\int_0^\infty \tilde{f}_s(p)\sin px\,dp$$

(b) Find the fourier transform of F(x) defined by $F(x) = \begin{cases}1, & |x| < a\\ 0, & |x| > a\end{cases}$ and hence evaluate (a) $\int_{-\infty}^\infty \frac{\sin pa\cos px}{p}\,dp$ and (b) $\int_0^\infty \frac{\sin p}{p}\,dp$.

Ans :

We have

$$\tilde{F}(p) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}F(x)\,dx$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-a}^a e^{ipx}\,dx$$

$$= \frac{1}{\sqrt{2\Pi}}\left(\frac{e^{ipx}}{ip}\right)_{-a}^a$$

$$= \frac{1}{\sqrt{2\Pi}}\left(\frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}\right)$$

$$= \frac{2i\sin pa}{ip\sqrt{2\Pi}}$$

$$\tilde{F}(p) = \frac{2\sin pa}{p\sqrt{2\Pi}}, \quad p \ne 0$$

For $p = 0$, $\tilde{F}(p) = \frac{2a}{\sqrt{2\Pi}}$

(a) We know that if

$$\tilde{F}(p) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty F(x)\,e^{ipx}\,dx$$

Then

$$F(x) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty \tilde{F}(p)\,e^{-ipx}\,dp$$

$$\therefore \quad \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty \frac{2\sin pa}{p\sqrt{2\Pi}}\cdot e^{-ipx}\,dp = \begin{cases}1, & |x| < a\\ 0, & |x| > a\end{cases}$$

But

$$LHS = \frac{1}{\Pi}\int_{-\infty}^\infty \frac{\sin pa\cos px}{p}\,dp - \frac{i}{\Pi}\int_{-\infty}^\infty \frac{\sin pa\sin px}{p}\,dp$$

$$= \frac{1}{\Pi}\int_{-\infty}^\infty \frac{\sin pa\cos px}{p}\,dp$$

$\because$ Integrand is the other integral is an odd function

$$\therefore \quad \int_{-\infty}^\infty \frac{\sin pa\cos px}{p}\,dp = \begin{cases}\Pi, & |x| < a\\ 0, & |x| > a\end{cases}$$

(b) If x = 0 and a = 1 in (a), then

$$\int_{-\infty}^\infty \frac{\sin p}{p}\,dp = \Pi$$

$$\Rightarrow 2\int_0^\infty \frac{\sin p}{p}\,dp = \Pi$$

$$\Rightarrow \int_0^\infty \frac{\sin p}{p}\,dp = \Pi/2$$

14. (a) Find the finite cosine transform of $\left(1-\frac{x}{\Pi}\right)^2$.

Ans :

We have

$$\tilde{f}_c(p) = \int_0^\Pi \left(1-\frac{x}{\Pi}\right)^2\cos px\,dx$$

$$= \left[\left(1-\frac{x}{\Pi}\right)^2\frac{\sin px}{p}\right]_0^\Pi + \frac{2}{\Pi p}\int_0^\Pi\left(1-\frac{x}{\Pi}\right)\sin px\,dx$$

$$= \frac{2}{\Pi p}\left[-\left(1-\frac{x}{\Pi}\right)\frac{\cos px}{p}\right]_0^\Pi - \frac{2}{p\Pi}\cdot\frac{1}{p\Pi}\int_0^\Pi \cos px\,dx$$

$$= \frac{2}{\Pi p^2} - \frac{2}{p^2\Pi^2}\left(\frac{\sin px}{p}\right)_0^\Pi = \frac{2}{\Pi p^2} \text{ if } p > 0$$

and if $p = 0$, then

$$\tilde{f}_c(p) = \int_0^\Pi\left(1-\frac{x}{\Pi}\right)^2 dx = \left[\frac{-\Pi}{3}\left(1-\frac{x}{\Pi}\right)^3\right]_0^\Pi = \frac{\Pi}{3}.$$

(b) State and prove parseval's identity for the fourier transforms.

Ans :

Statement

If $\tilde{f}(p)$ is the fourier transform of f(x), then $\int_{-\infty}^\infty |f(x)|^2\,dx = \int_{-\infty}^\infty |\tilde{f}(p)|^2\,dp$.

Proof :

Let f*(x) be the complex conjugate of the function f(x). If $\tilde{f}^*(p)$ is the fourier transform of f*(x), then we hvae

$$\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty |f(x)|^2\,dx = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty f(x)\,f^*(x)\,dx$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty f(x)\,f^*(x)\,e^{ip'x}\,dp$$

where $p' = 0$

$$\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty |f(x)|^2\,dx = \tilde{f}(p') * \tilde{f}^*(-p')$$

Since the fourier transform of the product is the convolution of the fourier transforms

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty \tilde{f}(p)\cdot\tilde{f}^*(p-p')\,dp, \quad p' = 0$$

$$= \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty \tilde{f}(p)\cdot\tilde{f}^*(p)\,dp = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty |\tilde{f}(p)|^2\,dp$$

Hence $\int_{-\infty}^\infty |f(x)|^2\,dx = \int_{-\infty}^\infty |\tilde{f}(p)|^2\,dp$

UNIT - III

State and prove linearity property of fourier transform.

Ans :

Statement

If $\tilde{f}(p)$ and $\tilde{g}(p)$ are fourier transforms of f(x) and g(x) respectively then $F\{a\,f(x)+b\,g(x)\} = a\,\tilde{f}(p)+b\,\tilde{g}(p)$ where a and b are constants

Proof :

We have

$$F\{f(x)\} = \tilde{f}(p) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}f(x)\,dx$$

Similarly

$$F\{g(x)\} = \tilde{g}(p) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}g(x)\,dx$$

$$\therefore \quad F\{a\,f(x) + b\,g(x)\} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}\{a\,f(x) + b\,g(x)\}\,dx$$

$$= \frac{a}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}f(x)\,dx + \frac{b}{\sqrt{2\Pi}}\int_{-\infty}^\infty e^{ipx}g(x)\,dx$$

$$\therefore \quad F\{af(x) + bg(x)\} = a\,\tilde{f}(p) + b\,\tilde{g}(p)$$