Q) $\frac{d^2y}{dt^2}+y=0$ under conditions that $y=1$

Sol: $\frac{dy}{dt}=0$ when $t=0$

Taking the laplace transform of both sides of the given differential equation, we have

$$L\{y''\}+L\{y\}=0$$

or $p^2L\{y\}-py(0)-y'(0)+L\{y\}=0$

or $p^2L\{y\}-p.1L\{y\}=0$

or $L\{y\}=\frac{p}{p^2+1}$

∴ $y=L^{-1}\left\{\frac{p}{p^2+1}\right\}$

which is the required solution

80. Solve $(D+2)^2=4e^{-2t}$, $y(0)=-1$, $y'(0)=4$

Sol: The given equation can be written as

$$(D^2+4D+4)y=4e^{-2t}$$

∴ $L\{y''\}+4L\{y'\}+4L\{y\}=4L\{e^{-2t}\}$

or $p^2L\{y\}-py(0)-y'(0)+4[pL\{y\}-y(0)]+4\{L(y)\}$

$=4.\frac{1}{p+2}$

or $p^2L\{y\}+p-4+4pL\{y\}+4+4L(y)=\frac{1}{p+2}$

or $(p^2+4p+4)L\{y\}=\frac{4}{p+2}-p=\frac{4-2p-p^2}{p+2}$

or $L\{y\}=\frac{4-2p-p^2}{(p+2)^3}=\frac{4-(p+2)^2+2(p+2)}{(p+2)^3}$

∴ $y=L^{-1}\left\{\frac{4-(p+2)^2+2(p+2)}{(p+2)^3}\right\}=e^{-2t}L^{-1}\left\{\frac{4-p^2+2p}{p^3}\right\}$

$=e^{-2t}L^{-1}\left\{\frac{4}{p^3}+\frac{2}{p^2}-\frac{1}{p}\right\}$

$=ye^{-2t}\left(2t^2+2t-1\right)$

which is the required solution.

81. Solve $(D^2+9)y=\cos 2t$

y (0) = 1, $y(\pi/2)=-1$

Sol: Taking the laplace transform of both sides of the given equation, we have

$L\{y''\}+9L\{y\} = L\{\cos 2t\}$

or $p^2L\{y\}-py(0)-y'(0)+9L\{y\}=\frac{p}{p^2+4}$

or $(p^2+9)L\{y\}-p-A=\frac{p}{p^2+4}$ where $y'(0)=A$

or $L\{y\}=\frac{p+A}{p^2+9}+\frac{p}{(p^2+9)(p^2+4)}$

$=\frac{p}{p^2+9}+\frac{A}{p^2+9}+\frac{p}{5(p^2+4)}-\frac{p}{5(p^2+9)}$

∴ $y=L^{-1}\left\{\frac{p}{p^2+9}\right\}+AL^{-1}\left\{\frac{1}{p^2+9}\right\}+\frac{1}{5}L^{-1}\left\{\frac{p}{p^2+4}\right\}$

$-\frac{1}{3}L^{-1}\left\{\frac{p}{p^2+9}\right\}$

$=\cos 3t+\frac{A}{3}\sin 3t+\frac{1}{5}\cos 2t-\frac{1}{3}\cos 3t$

$=\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t+\frac{1}{5}\cos 2t$

Sine $y(\pi/2)=-1$

∴ $-1=\frac{4}{5}\cos\frac{3\pi}{2}+\frac{A}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\cos\pi$

or $-1=\frac{-A}{3}-\frac{1}{5}$

or ∴ $A=\frac{12}{5}$

Hence the required solution is

$y=\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t+\frac{1}{5}\cos 2t$

82. Solve $(D^2-D-2)y=20\sin 2t$

y=-1, Dy=2 When t=0

Sol: Taking the laplace transform of both sides of the given equation, we have.

$L\{y''\}-L\{y'\}-2L\{y\}=20L\{\sin 2t\}$

or $p^2L\{y\}-py(0)-y'(0)-[pL\{y\}-y(0)]-2L\{y\}=20\cdot\frac{2}{p^2+4}$

or $(p^2-p-2)L\{y\}+p-2-1=\frac{40}{p^2+4}$

or $(p-2)(p+1)L\{y\}=\frac{40}{p^2+4}+(-p+3)$

or $L\{y\}=\frac{40}{(p-2)(p+1)(p^2+4)}+\frac{-p+3}{(p-2)(p+1)}$

$=\frac{5}{3(p-2)}-\frac{8}{3(p+1)}-\frac{p-6}{(p^2+4)}+\frac{1}{3(p-2)}-\frac{4}{3(p+1)}$

$=\frac{2}{p-2}-\frac{4}{p+1}+\frac{p}{p^2+4}-\frac{6}{p^2+4}$

$y=2L^{-1}\left\{\frac{1}{p-2}\right\}-4L^{-1}\left\{\frac{1}{p+1}\right\}+L^{-1}\left\{\frac{p}{p^2+4}\right\}-6L^{-1}\left\{\frac{1}{p^2+4}\right\}$

$=2e^{2t}-4e^{-t}+\cos 2t-3\sin 2t$

which is the required solution

83. Solve $(D+1)^2y=t$

given that $y=-3$, **when t=0**

$y=-1$, **when t=1**

Sol: the given equation can be written as

$$(D^2+2D+1)y=t$$

∴ $L\{y''\}+2L\{y'\}+L\{y\}=L\{t\}$

or $P^2L\{y\}-py(0)-y'(0)+2[pL\{y\}-y(0)]+L\{y\}=\frac{1}{p^2}$

or $(p^2+2p+1)L\{y\}-p(-3)-A-2(-3)=\frac{1}{p^2}$ where $y'(0)=A$

or $(p+1)^2L\{y\}=\frac{1}{p^2}-3p-6+A$

or $L\{y\}=\frac{1}{p^2(p+1)^2}-\frac{3p+6}{(p+1)^2}+\frac{A}{(p+1)^2}$

$=\frac{-2}{p}+\frac{1}{p^2}+\frac{2}{p+1}+\frac{1}{(p+1)^2}-\frac{3}{p+1}-\frac{3}{(p+1)^2}+\frac{A}{(p+1)^2}$

$=\frac{-2}{p}+\frac{1}{p^2}-\frac{1}{(p+1)}+\frac{A-2}{(p+1)^2}$

∴ $y=-2L^{-1}\left\{\frac{1}{p}\right\}+L^{-1}\left\{\frac{1}{p^2}\right\}-L^{-1}\left\{\frac{1}{p+1}\right\}+(A-2)L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$

⇒ $-2.1+t-e^{-t}+(A-2)e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$

⇒ $-2+t-e^{-t}+(A-2)te^{-t}$

Now Since $y=-1$,, when $t=1$

∴ $-1=-2+1-e^{-1}+(A-2)e^{-1}$ or $A=3$

Hence the complete solution is

$y=-2+t-e^{-t}+te^{-t}$

84. Solve $(D^2+1)y=t\cos 2t$, **y=0,** $\frac{dy}{dt}=0$, **when t=0**

Sol: Taking the laplace transform of both sides of the given equation, we have

$L\{y''\}+L\{y\}-L\{t\cos 2t\}$

or $P^2L\{y\}-py(0)-y'(0)+L\{y\}=\frac{-d}{dp}(L\{\cos 2t\})$

or $(p^2+1)L\{y\}=\frac{-d}{dp}\left(\frac{p}{p^2+4}\right)$

$=\frac{-1}{p^2+4}+\frac{2p^2}{(p^2+4)^2}$

or $L\{y\}=\frac{p^2-4}{(p^2+1)(p^2+4)^2}$

$=\frac{-5}{9(p^2+1)}+\frac{5}{9(p^2+4)}+\frac{8}{3(p^2+4)^2}$

∴ $y=\frac{-5}{9}L^{-1}\left\{\frac{1}{p^2+1}\right\}+\frac{4}{9}L^{-1}\left\{\frac{1}{p^2+4}\right\}+\frac{8}{3}L^{-1}\left\{\frac{1}{(p^2+4)^2}\right\}$

$=\frac{-5}{9}\sin t+\frac{5}{18}\sin 2t+\frac{8}{3}\int_0^t\frac{1}{2}\sin 2x\,\frac{1}{2}\sin 2(t-x)dx$

By the convolution theorem since $L^{-1}\left\{\frac{1}{p^2+4}\right\}=\frac{1}{2}\sin 2t$

$=\frac{-5}{9}\sin t+\frac{5}{18}\sin 2t+\frac{1}{3}\int_0^t(\cos 2(t-2x)-\cos 2t)dx$

⇒ $\frac{-5}{9}\sin t+\frac{5}{18}\sin 2t+\frac{1}{3}\left[-\frac{1}{4}\sin 2(t-2x)-x\cos 2t\right]_0^t$

$=\frac{-5}{9}\sin t+\frac{5}{18}\sin 2t+\frac{1}{12}\sin 2t-\frac{t}{3}\cos 2t+\frac{1}{12}\sin 2t$

or $y=\frac{-5}{9}\sin t+\frac{4}{9}\sin 2t-\frac{t}{3}\cos 2t$

which is the required solution

85. Solve $(D^2-3D+2)y=1-e^{2t}$, $y=1$, $Dy=0$ **when t=0**

Sol: Taking the laplace transform of both sides of the given equation we have

$L\{y''\}-3L\{y'\}+2L\{y\}=L\{1\}-L\{e^{2t}\}$

or $p^2L\{y\}-py(0)-y'(0)-3[pL\{y\}-y(0)]+2L\{y\}=\frac{1}{p}-\frac{1}{p-2}$

or $(p^2-3p+2)L\{y\}-p+3=\frac{-2}{p(p-2)}$

or $(p-1)(p-2)L\{y\}=\frac{-2}{p(p-2)}+(p-3)$

or $L\{y\}=\frac{p^3-5p^2+6p-2}{p(p-1)(p-2)^2}=\frac{p^2-4p+2}{p(p-2)^2}$

$=\frac{1}{2p}+\frac{1}{2(p-2)}-\frac{1}{(p-2)^2}$

∴ $y=\frac{1}{2}L^{-1}\left\{\frac{1}{p}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{p-2}\right\}-L^{-1}\left\{\frac{1}{(p-2)^2}\right\}$

$=\frac{1}{2}+\frac{1}{2}e^{2t}-te^{2t}$

which is the required solution

86. Solve $(D^2-D)y=2\cos t$, **y=3, Dy=2,** $D^2y=1$

Sol: Taking the laplace transform of both sides of the given equation we have

$L\{y''\}-L\{y'\}=2L\{\cos t\}$

or $p^2L\{y\}-p^2y(0)-py'(0)-[pL\{y\}-y(0)]=\frac{2p}{p^2+1}$

or $(p^2-p)L\{y\}-3p^2-2p-1+3=\frac{2p}{p^2+1}$

or $p(p^2-1)L\{y\}=\frac{2p}{p^2+1}+3p^2+2p-2$

or $L\{y\}=\frac{2}{(p^2-1)(p^2+1)}+\frac{3p^2+2p-2}{p(p-1)(p+1)}$

$=\frac{1}{p^2-1}-\frac{1}{p^2+1}+\frac{2}{p}+\frac{3}{2(p-1)}-\frac{1}{2(p+1)}$

∴ $y=L^{-1}\left\{\frac{1}{p^2-1}\right\}-L^{-1}\left\{\frac{1}{p^2+1}\right\}+2L^{-1}\left\{\frac{1}{p}\right\}+\frac{3}{2}L^{-1}\left\{\frac{1}{p-1}\right\}$

$-\frac{1}{2}L^{-1}\left\{\frac{1}{p+1}\right\}$

$=\sinh t-\sin t+2+\frac{3}{2}e^t-\frac{1}{2}e^{-t}$

$=\sinh t-\sin t+2+\cosh t+2\sinh t$

$=3\sinh t-\sin t+\cosh t+2$

87. Solve $(D^3-2D^2+5D)y=0$

if y(0)=0, $y'(0)=1$, $y(\pi/8)=1$

Sol: Taking the laplace transform of both sides of the given equation, we have

$L\{y'''\}-2L\{y''\}+5L\{y'\}=0$

or $p^3L\{y\}-p^2y(0)-py'(0)-y''(0)-2[p^2L\{y\}-py(0)-y'(0)]$

$+5[pL\{y\}-y(0)]=0$

or $(p^3-2p^2+5p)L\{y\}-p-A-2[-1]+5.0=0$

where $y''(0)=A$

or $L\{y\}=\frac{A-2+p}{p^3-2p^2+5p}$

$=\frac{A-2}{p(p^2-2p+5)}+\frac{1}{p^2-2p+5}$

⇒ $\frac{A-2}{5p}-\frac{A-2}{5}\frac{p-2}{p^2-2p-5}+\frac{1}{p^2-2p+5}$

⇒ $\frac{A-2}{5p}-\frac{A-2}{5}\frac{(p-1)-1}{(p-1)^2+4}+\frac{1}{(p-1)^2+4}$

⇒ $\frac{A-2}{5p}-\frac{A-2}{5}\frac{(p-1)}{(p-1)^2+4}+\frac{A+3}{10}\frac{2}{(p-1)^2+4}$

∴ $y=\frac{A-2}{5}L^{-1}\left\{\frac{1}{p}\right\}-\frac{A-2}{5}L^{-1}\left\{\frac{p-1}{(p-1)^2+4}\right\}$

$+\frac{A+3}{10}L^{-1}\left\{\frac{2}{(p-1)^2+4}\right\}$

⇒ $\frac{A-2}{5}-\frac{A-2}{5}e^t\cos 2t+\frac{A+3}{10}e^t\sin 2t$

Since $y(\pi/8)-1$

∴ $1=\frac{A-2}{5}-\frac{A-2}{5}e^{\pi/8}\frac{1}{\sqrt{2}}+\frac{A+3}{10}e^{\pi/8}\frac{1}{\sqrt{2}}$

or $\frac{7-A}{5}=\frac{e^{\pi/8}}{10\sqrt{2}}(-2A+4+A+3)$

or $\left(\frac{7-A}{5}\right)\left[1-\frac{e^{\pi/8}}{2\sqrt{2}}\right]=0$

or A=-7

Hence the required solution is

$y=1+e^t(\sin 2t-\cos 2t)$

88. Solve $ty''+y'+4ty=0$

$y(0)=3, y'(0)=0$

Sol: Taking the laplace transform of both sides of the given equation we have

$L\{ty''\}+L\{y'\}+4L\{ty\}=0$

or $\frac{-d}{dp}L\{y''\}+L\{y'\}+4(-1)\frac{d}{dp}L\{y\}=0$

or $\frac{-d}{dp}[p^2L\{y\}-py(0)-y'(0)]+[pL\{y\}-y(0)]$

$-4\frac{d}{dp}L\{y\}=0$

or $-\frac{d}{dp}(p^2z-3p)+(pz-3)-4\frac{dz}{dp}=0$ where $z=L\{y\}$

or $-(p^2+4)\frac{dz}{dp}-pz=0$

or $\frac{dz}{z}+\frac{p}{p^2+4}dp=0$

integreating, $\log z-\frac{1}{2}\log(p^2+4)=\log C_1$

or $Z=\frac{C_1}{\sqrt{p^2+4}}$

or $L\{y\}=\frac{C_1}{\sqrt{p^2+4}}$

∴ $y=C_1L^{-1}\left\{\frac{1}{\sqrt{(p^2+4)}}\right\}$

or $y=C_1J_0(2t)$

Since $y(0)=3$, ∴ $3=C_1J_0(0)$ [∵ $J_0(0)=1$]

Hence $y=3J_0(2t)$

which is the required solution

89. Solve $y''-ty'+y=1$

$y(0)=1$, $y'(0)=2$

Sol: Taking the laplace transform of both sides of the given equation, we have

$L\{y''\}-L\{ty'\}+L\{y\}=L\{1\}$

or $p^2L\{y\}-py(0)-y'(0)+\frac{d}{dp}L\{y'\}+L\{y\}=\frac{1}{p}$

or $p^2L\{y\}-p-2+\frac{d}{dp}[pL\{y\}-y(0)]+L\{y\}=\frac{1}{p}$

or $p^2z-p-2+\frac{d}{dp}(pz-1)+Z=\frac{1}{p}$ where $Z=L\{y\}$

or $p\frac{dz}{dp}+(p^2+2)Z=p+2+\frac{1}{p}$

or $\frac{dz}{dp}+\left(p+\frac{2}{p}\right)Z=1+\frac{2}{p}+\frac{1}{p^2}$

which is linear.

∴ $IF=e^{\int(p+2/p)dp}=e^{p^2/2+2\log p}=p^2e^{p^2/2}$

∴ $p^2e^{p^2/2}.Z=C_1t-^2\int\left(1+2/p+2/p^2\right)p^2e^{p^2/2}dp$

⇒ $C_1+\int(p^2+2p+1)e^{p^2/2}dp$

⇒ $C_1+\int(p^2+1)e^{p^2/2}dp+2\int p.e^{p^2/2}dp$

⇒ $C_1+\int(2y+1)e^y\frac{dy}{\sqrt{2y}}+2\int\sqrt{2\upsilon}\,e^{\upsilon}\frac{d\upsilon}{\sqrt{(2\upsilon)}}$ putting $\frac{p^2}{2}=\upsilon$

so that pdp=dv or dp=$\frac{d\upsilon}{\sqrt{(2\upsilon)}}$

⇒ $C_1+\int\sqrt{2\upsilon}\,e^{\upsilon}d\upsilon+\int\sqrt{\frac{e^{\upsilon}}{2\upsilon}}d\upsilon+2\int\sqrt{e^{\upsilon}}d\upsilon$

⇒ $C_1+\int(\sqrt{2\upsilon})e^y+2e^y=1+pe^{p^2/2}+2e^{p^2/2}$

or $Z=L\{y\}=\frac{C_1}{p^2}e^{-p^2/2}+\frac{1}{p}+\frac{2}{p^2}$

⇒ $\frac{C_1}{p^2}\left(1-\frac{p^2}{2}+\frac{p^4}{4.2!}.....\right)+\frac{1}{p}+\frac{2}{p^2}$

⇒ $\frac{(2+C_1)}{p^2}-\frac{C_1}{2}+\frac{C_1}{8}p^2.....+\frac{1}{p}$

∴ $y=(2+C_1)L^{-1}\left\{\frac{1}{p^2}\right\}-\frac{C_1}{2}L^{-1}\{1\}+\frac{C_1}{8}L^{-1}\{p^2\}$

$+....+L^{-1}\left\{\frac{1}{p}\right\}$

$=(2+C_1)t+1$ ∵ $L^{-1}\{p^n\}=0$ n=0, 1, 2...

Hence y = 2t+1

which is the required solution

90. Solve $(D^2-3)x-4y=0$

$x+(D^2+1)y=0, t>0$

if x=y=Dy=0, Dx=2, when t=0

Sol: Taking the laplace transform of both sides of the two equations, we have

$L\{x''\}-3L\{x\}-4L\{y\}=0$

and $L\{x\}+L\{y''\}+L\{y\}=0$

or $p^2\bar{x}-px(0)-x'(0)-3\bar{x}-\bar{y}=0$ where $\bar{x}=L\{x\}$

and $\bar{y}=L\{y\}$

and $\bar{x}+p^2\bar{y}-py(0)-y'(0)+\bar{y}=0$

or $(p^2-3)\bar{x}-4\bar{y}=2$

and $\bar{x}+(p^2+1)\bar{y}=0$

Solving for $\bar{x}$ and $\bar{y}$ we have

$\bar{x}=\frac{2(p^2+1)}{(p^2-1)^2}=\frac{1}{(p-1)^2}+\frac{1}{(p+1)^2}$

and $\bar{y}=\frac{-2}{(p+1)^2(p-1)^2}=\frac{1}{2}\left[\frac{-1}{p+1}+\frac{1}{p-1}-\frac{1}{(p+1)^2}-\frac{1}{(p-1)^2}\right]$

∴ $x=L^{-1}\left[\frac{1}{(p-1)^2}\right]+L^{-1}\left[\frac{1}{(p+1)^2}\right]$

$=e^tL^{-1}\left\{\frac{1}{p^2}+\right\}+e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}=\left(e^t+e^{-t}\right)t$

and $y=\frac{1}{2}\left[-L^{-1}\left\{\frac{1}{p+1}\right\}+L^{-1}\left\{\frac{1}{p-1}\right\}-L^{-1}\left\{\frac{1}{(p+1)^2}\right\}-L^{-1}\left\{\frac{1}{(p-1)^2}\right\}\right]$

$=\frac{1}{2}\left[-e^{-t}+e^t-te^{-t}-te^t\right]$

$=\frac{1}{2}(1-t)e^t-\frac{1}{2}(1-t)e^{-t}$

91. Solve $Dx=Dy=t$

$D^2x-y=e^{-t}$

x(0)=3, $x'(0)=2$, **y(0)=0**

Sol: Taking the laplace transform of both sides of the two equations, we have

$L\{x'\}+L\{y'\}=L\{t\}$

& $L\{x''\}-L\{y\}=L\{e^{-t}\}$

or $p\bar{x}-x(0)+p\bar{y}-y(0)=\frac{1}{p^2}$

and $p^2\bar{x}-px(0)-x'(0)-\bar{y}=\frac{1}{p+1}$

or $p\bar{x}+p\bar{y}=3+\frac{1}{p^2}$

& $p^2\bar{x}-\bar{y}=3p-2+\frac{1}{p+1}$

Solving for $\bar{x}$ and $\bar{y}$, we have

$\bar{x}=\frac{3p^2+1}{p^3(1+p^2)}+\frac{3p}{1+p^2}-\frac{2}{1+p^2}-\frac{1}{(p+1)(p^2+1)}$

$=\frac{1}{p^3}\left(1+2p^2-\frac{2p^4}{1+p^2}\right)+\frac{3p}{1+p^2}-\frac{2}{1+p^2}+\frac{1}{2(p+1)}$

$\frac{-p}{2(p^2+1)}+\frac{1}{2(p^2+1)}$

$=\frac{2}{p}+\frac{1}{p^3}+\frac{1}{2(p+1)}+\frac{p}{2(1+p^3)}-\frac{3}{2(p^2+1)}$

and $\bar y = \frac{1}{p(p+1)(p^2+1)} + \frac{2}{p^2+1}$

$= \frac{1}{p} - \frac{1}{2(p+1)} - \frac{p}{2(p^2+1)} + \frac{3}{2(p^2+1)}$

$\therefore x = 2L^{-1}\left\{\frac1p\right\} + L^{-1}\left\{\frac{1}{p^2}\right\} + \frac12 L^{-1}\left\{\frac{1}{p+1}\right\} + \frac12 L^{-1}\left\{\frac{p}{p^2+1}\right\}$

$-\frac32 L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$\Rightarrow 2 + \frac12 t^2 + \frac12 e^{-t} + \frac12 \cos t - \frac32 \sin t$

and $y = L^{-1}\left\{\frac1p\right\} - \frac12 L^{-1}\left\{\frac{1}{p+1}\right\} - \frac12 L^{-1}\left\{\frac{p}{p^2+1}\right\}$

$+\frac32 L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$= 1 - \frac12 e^{-t} - \frac12 \cos t + \frac32 \sin t$

92. Solve $(D-2)x-(D+1)y=6e^{3t}$

$(2D-3)x+(D-3)y=6e^{3t}$

$x=3,\ y=0$ when $t=0$

Sol: Taking the laplace transform of both sides of the two equations, we have

$L\{x'\}-2L\{x\}-L\{y'\}-L\{y\}=6L\{e^{3t}\}$

and $2L\{x'\}-3L\{x\}+L\{y'\}-3L\{y\}=6L\{e^{3t}\}$

or $p\bar x - x(0) - 2\bar x - \{p\bar y - y(0)\} - \bar y = \frac{6}{p-3}$

and $2\{p\bar x - x(0)\} - 3\bar x + p\bar y - y(0) - 3\bar y = \frac{6}{p-3}$

or $(p-2)\bar x - (p+1)\bar y = 3 + \frac{6}{p-3} = \frac{3p-3}{p-3}$

and $(2p-3)\bar x + (p-3)\bar y = 6 + \frac{6}{p-3} = \frac{6p-12}{p-3}$

Solving for $\bar x$ and $\bar y$, we have

$\bar x = \frac{3p^2-6p-1}{(p-3)(p-1)^2} = \frac{1}{p-1} + \frac{2}{(p-1)^2} - \frac{1}{p-3}$

and $\bar y = \frac{-3p+5}{(p-3)(p-1)^2} = \frac{1}{p-1} - \frac{1}{(p-1)^2} - \frac{1}{p-3}$

$\therefore x = L^{-1}\left\{\frac{1}{p-1}\right\} + 2L^{-1}\left\{\frac{1}{(p-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{p-3}\right\}$

$= e^t + 2te^t + 2e^{3t}$

and $y = L^{-1}\left\{\frac{1}{p-1}\right\} - L^{-1}\left\{\frac{1}{(p-1)^2}\right\} - L^{-1}\left\{\frac{1}{p-3}\right\}$

$= e^t - te^t - e^{3t}$

93. Solve $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y,\ y(x,0) = 6e^{-3x}$

which is bounded for x>0, t>0

Sol: Taking the laplace transform of both sides of the given equation we have

$L\left\{\frac{\partial y}{\partial x}\right\} = 2L\left\{\frac{\partial y}{\partial t}\right\} + L\{y\}$

or $\frac{d\bar y}{dx} = 2[p\bar y - y(x,0) + \bar y]$

or $\frac{d\bar y}{dx} - (2p+1)\bar y = 12e^{-3x} \to (1)$

which is a linear Equation.

$I.F = e^{-\int(2p+1)dx} = e^{-(2p+1)x}$

$\therefore$ the solution of (i) is

$e^{-(2p+1)x}\bar y = C - 12\int e^{-3x}.e^{-(2p+1)x}dx$

$= C - 12\int e^{-(2p+4)x}dx$

$= C + \frac{6}{p+2}e^{-(2p+4)x}$

$\bar y = L\{y\} = Ce^{(2p+1)x} + \frac{6}{p+2}e^{-3x}$

Since y(x,t) is bounded as $x\to\infty$ therefore, we must have $\bar y(x,p)$ also bounded as $x\to\infty$ $\therefore$ C=0

$\therefore \bar y = L\{y\} = \frac{6}{p+2}e^{-3x}$

$\therefore y = L^{-1}\left\{\frac{6-e^{-3x}}{p+2}\right\}$

$= 6e^{-2t}.e^{-3x}$

or $y(x,t) = 6e^{-2t-3x}$

which is the required solution

94. Solve $\frac{\partial y}{\partial t} = 2\frac{\partial^2 y}{\partial x^2}$

where y(0,t) = 0= y(5,t) and y(x,0) =10 Sin $4\pi x$

Sol:

Taking the laplace transform of both sides of given equations, we have

$L\left\{\frac{\partial y}{\partial t}\right\} = 2L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$

or $p\bar y - y(x,0) = 2\frac{d^2\bar y}{dx^2}$

or $\frac{d^2\bar y}{dx^2} - \frac{p}{2}.y = -5\sin 4\pi x$

Whose general solution is

$\bar y = C_1 e^{\sqrt{(p/2)}x} + C_2 e^{\sqrt{(p/2)}x} - \frac{5\sin 4\pi x}{-(4\pi)^2 - \frac{p}{2}}$

or $\bar y = C_1 e^{\sqrt{(p/2)}x} + C_2 e^{\sqrt{(p/2)}x} + \frac{10}{32\pi^2+p}\sin 4\pi x \to (1)$

But y (0,t) = 0 $\Rightarrow$ y (5,t)

$\therefore \bar y(0,p) = 0,\ \bar y(5,p) = 0$

$\therefore$ from (1) we have $0 = C_1 + C_2$

and $0 = C_1 e^{5\sqrt{(p/2)}} + C_2 e^{-5\sqrt{(p/2)}} + \frac{10}{32\pi^2+p}\sin 20\pi$

$= C_1 e^{5\sqrt{(p/2)}} + C_2 e^{-5\sqrt{(p/2)}} + 0$

Solving $C_1 = 0 = C_2$

$\therefore$ from (1) we have

$\bar y = \frac{10}{32\pi^2+p}\sin 4\pi x$

$\therefore y = L^{-1}\left\{\frac{10}{32\pi^2+p}\sin 4\pi x\right\}$

or $y = 10^{-32\pi^2 t}\sin 4\pi x$

which is the required solution

95.Solve $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = xt$

where y=0= $\frac{\partial y}{\partial t}$ **at t=0 and y(0,t) =0**

Sol: Taking the laplace transform of both the sides of the given equation, we have

$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} - l\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\{xt\}$

or $\frac{d^2\bar y}{dx^2} - [p^2\bar y(x,p) - py(x,0) - y_t(x,0)] = xL\{t\}$

or $\frac{d^2\bar y}{dx^2} - [p^2\bar y(x,p) - py(x,0) - y_t(x,0)] = xL\{t\}$

or $\frac{d^2\bar y}{dx^2} - p^2\bar y = \frac{x}{p^2}$

whose general solution is

$\bar y = C_1e^{px} + C_2e^{-px} - \frac{x}{p^4}$

Since $\bar y$ = 0 for all values of x, $\therefore C_1 = 0$, Otherwise

$\bar y = \infty$ as $x\to\infty$

$\therefore \bar y = C_2e^{-px} - \frac{x}{p^4}$

Again $\bar y = 0$ when x=0 $\therefore C_2 = 0$

$\therefore \bar y = -\frac{x}{p^4}$

or $y = L^{-1}\left\{\frac{x}{p^4}\right\}$

or $y = \frac{xt^3}{6}$

which is the required solution

96. Solve the boundary value problem

$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}(t>0, x>0)$

where u(x,0) = 0, $x\geq 0$, $u_t(x,0)=0$, **x>0**

u(0,t) = t, $\lim_{x\to\infty} u(x,t)=0$,

$t\geq 0$

Sol: Taking the laplace transform of both sides of the given equation and using boundary conditions we have

$L\left\{\frac{\partial^2u}{\partial t^2}\right\} = a^2 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$

$\Rightarrow p^2\bar u(x,p) - pu(x,0) - u_t(x,0) = a^2\frac{d^2u}{dx^2}$

$\Rightarrow \frac{d^2\bar u}{dx^2} - \frac{p^2}{a^2}\bar u = 0 \to (1)$

Also $\bar u(0,p) = L\{t\} = \frac{1}{p^2}$ and $\bar u(x,p) = 0$ as $x\to\infty$.

The solution of (1) is given by

$\bar u(x,p) = Ae^{(p/a)x} + Be^{-(p/a)x}$

Since $\bar u(x,p) = 0$ as $x\to\infty$

$\Rightarrow A = 0$

and $\bar u(0,p) = \frac{1}{p^2} = B$

Hence $\bar u(x,p) = \frac{1}{p^2}e^{-(p/a)x}$

$\therefore u(x,t) = L^{-1}\left\{\frac{1}{p^2}e^{-(p/a)x}\right\}$

$u(x,t) = \begin{cases} t-\frac{x}{a}, & t>\frac{x}{a} \\ 0, & t<\frac{x}{a}\end{cases} \qquad \because L^{-1}\left\{\frac{1}{p^2}\right\} = t$

$u(x,t) = \left(t-\frac{x}{a}\right).H\left(t-\frac{x}{a}\right)$

in terms of Heavisides unit step functions

$\therefore u(x,t) = \sum_{n=0}^\infty L^{-1}\{\bar F(p).e^{-\sqrt{p}(m-x)}\}$

$-\sum_{n=0}^\infty L^{-1}\{\bar F(p).e^{-\sqrt{p}(m+x)}\}$

By convolution theorem we get

$= \sum_{n=0}^\infty \left[\int_0^t \{F(t-\lambda)\}.\frac{(m-x)}{2\sqrt{\pi}.\lambda^{3/2}}e^{-\frac{1}{4\lambda}(m-x)^2}d\lambda - \int_0^t\{F(t-\lambda)\}.\frac{(m+x)}{2\sqrt{\pi}.\lambda^{3/2}}e^{-\frac{1}{4\lambda}(m+x)^2}d\lambda\right] \to (3)$

$(\because L^{-1}\{\overline{F(p)}\} = F(t))$

$L^{-1}\{e^{-\sqrt{p}(m-x)}\} = \frac{(m-x)}{2\sqrt{\pi}.t^{3/2}}e^{-\frac{1}{4t}(m-x)^2}$

and $L^{-1}\{e^{-\sqrt{p}(m+x)}\} = \frac{(m+x)}{2\sqrt{\pi}.t^{3/2}}e^{-\frac{1}{4t}(m+x)^2}$

Now Subs : $\frac{1}{4\lambda}(m-x)^2 = \mu^2$ in the first integral then we get.

$d\lambda = \frac{-(m-x)^2}{2\mu^2}d\mu$

$= \frac{4\lambda^{3/2}}{(m-x)}d\mu$

III^{ly} Put $\frac{1}{4\lambda}(m+x)^2 = V^2$ in the second integral we get

$u(x,t) = \sum_{n=0}^\infty \frac{2}{\sqrt{\pi}}\left[\int_{\frac{(m-x)}{2\sqrt{t}}}^\infty F\left(t - \frac{(m-x)^2}{4\mu^2}\right)e^{-\mu^2}d\mu - \int_{\frac{(m+x)}{2\sqrt{t}}}^\infty F\left(t+\frac{(m+x)^2}{4\mu^2}\right)e^{-v^2}dv\right]$

where m=2n+1

97. Solve the boundary Value problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

(0<x<1, t>0) subject to the conditions

U(x,0) = 0, 0<x<1

U(0,t) = 0 U(1,t) = F(t) (t>0)

Sol: Taking the laplace transform of both sides of the given equation and using boundary conditions we have

$p\bar u(x,p) - u(x,0) = \frac{d^2\bar u}{dx^2}$

$\frac{d^2\bar u}{dx^2} - p\bar u = 0 \to (1)$

Now $\bar u(0,p)$ and $\bar u(1,p) = L\{F(t)\} = \overline{F(p)}$

The solution of (1) is given by

$\bar u(x,p) = A\cosh(\sqrt{p})x + B\sinh(\sqrt{p})x$

$\therefore \bar u(0,p) = 0 = A$

and $\bar u(1,p) = \overline{F(p)} = B\sinh\sqrt{p}$

$\Rightarrow B = \frac{\overline{F(p)}}{\sinh\sqrt{p}}$

Hence $\bar u(x,p) = \overline{F(p)}\frac{\sinh(\sqrt{p})x}{\sinh(\sqrt{p})}$

$= \overline{F(p)}.\frac{e^{x\sqrt{p}} - e^{-x\sqrt{p}}}{2}.\frac{2}{e^{\sqrt{p}} - e^{-\sqrt{p}}}$

$= \overline{F(p)}.\{e^{-\sqrt{p}(1-x)} - e^{-\sqrt{p}(1+x)}\}\sum_{n=0}^\infty e^{-2x\sqrt{p}}$

$= \sum_{n=0}^\infty \overline{F(p)}e^{-\sqrt{p}(m-x)} - \sum_{n=0}^\infty\overline{F(p)}e^{-\sqrt{p}(m+x)}$

where m=2n+1

98. Find the solution of the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ **which tends to zero as** $x\to\infty$ **and which satisfies the conditions u=f(t) where x=0, t>0, and u=0 when x>0, t=0**

Sol: Taking the place transform of both sides of the given equation we have

$L\left\{\frac{\partial u}{\partial t}\right\} = k\left\{\frac{\partial^2u}{\partial x^2}\right\}$

$\Rightarrow p\bar u(x,p) - u(x,0) = k\frac{d^2\bar u}{dx^2}$

$\frac{d^2\bar u}{dx^2} - \frac{p}{k}\bar u = 0 \to (1)$

Solution of equation (1) is

$\bar u = A.e^{(\sqrt{p/k})x} + B.e^{-(\sqrt{p/k})x}$

$\because u\to 0$ as $x\to\infty$

$\therefore \bar u\to 0$ as $x\to\infty$

$\Rightarrow A = 0$

$\therefore \bar u = B.e^{-(\sqrt{p/k})x} \to (2)$.

III^{ly} when x = 0 $\bar u = \int_0^\infty f(t).e^{-pt}dt = \overline{f(p)}$

$\therefore$ form (1) we have

$\overline{f(p)} = B$

Hence $\bar u = \overline{f(p)}e^{-(\sqrt{p/k})x}$

$\therefore$ by laplace inversion theorem we have

$u(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\overline{f(p)}.e^{-(\sqrt{p/k})x}dp$

99.A String is Stretched between two fixed point (0,0) and (c,0). If it is displaced into the Curve y=b sin $\left(\frac{\pi x}{c}\right)$ **and released from rest in that position at time t>0 find its displacement at any line t>0 and any point 0<x<c.**

Sol: Here the displacement y(x,t) of the string is governed by the differential equation.

$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2y}{\partial x^2} \to (1)$

With boundary conditions y(0,t)=0, y(c,t)=0 and initial conditions

y(x,0)=bSin $\frac{\pi x}{c}$ and $y_t(x,0)=0$

Taking the laplace transforms of (1) and the boundary conditions, we have

$L\left\{\frac{\partial^2y}{\partial t^2}\right\} = a^2L\left\{\frac{\partial^2y}{\partial x^2}\right\}$

$\Rightarrow p^2\bar y(x,p) - py(x,0) - y_t(x,0) = a^2\frac{d^2\bar y}{dx^2}$

$\Rightarrow \frac{d^2\bar y}{dx^2} - \frac{p^2\bar y}{a^2} = \frac{-bp}{a^2}\sin\frac{\pi x}{c} \to (2)$

Also $\bar y(0,p)=0$ and $\bar y(c,p)=0$

Now sol. of 2 is given by

$\bar y(x,p) = A.e^{(p/a)x} + B.e^{-(p/a)x} + \frac{bp\sin\frac{\pi x}{c}}{p^2+\frac{\pi^2a^2}{c^2}}$

$\therefore \bar y(0,p) = 0 = A+B \to (3)$

$\therefore \bar y(c,p) = 0 = Ae^{pc/a} + Be^{-pc/a}$

Solving (3) and (4) we get

A=0, B= 0

Hence $\bar y(x,p) = \frac{bp\sin\frac{\pi x}{c}}{p^2+\frac{\pi^2a^2}{c^2}}$

$\therefore y(x,t) = bL^{-1}\left\{\frac{p}{p^2+\left(\frac{\pi a}{c}\right)^2}\right\}.\sin\frac{\pi x}{c}$

$y(x,t) = b\cos\frac{\pi at}{c}.\sin\frac{\pi x}{c}$

16. (a) Solve $\frac{\partial u}{\partial t} = \frac{2\partial^2u}{\partial x^2}$ *if u(0, t) = 0, u(x, 0) =* e^{-x}*, x > 0, u(x, t) is bounded where x > 0, t > 0.*

Ans :

Given equation is $\frac{\partial u}{\partial t} = \frac{2\partial u^2}{\partial x^2}$

Taking the fourier sine transform on both the sides then we get

$\sqrt{\frac{2}{\Pi}}\int_0^\infty \frac{\partial u}{\partial t}\sin px\,dx = 2\sqrt{\frac{2}{\Pi}}\int_0^\infty\frac{\partial^2u}{\partial x^2}\sin px\,dx$

$\Rightarrow \frac{d}{dt}\sqrt{\frac{2}{\Pi}}\int_0^\infty U\sin px\,dx = 2\sqrt{\frac{2}{\Pi}}\left(\frac{\partial u}{\partial x}\sin px\right)$

$-2p\sqrt{\frac{2}{\Pi}}\int_0^\infty\frac{\partial u}{\partial x}\cos px\,dx$

If $\frac{\partial u}{\partial x}\to 0$ as $x\to\infty$

$\Rightarrow \frac{d}{dt}\bar U_s = -2p\sqrt{\frac{2}{\Pi}}(u\cos px)_0^\infty - 2p^2\sqrt{\frac{2}{\Pi}}\int_0^\infty u\sin px\,dx$

If $U\to 0$ as $x\to\infty$

$= 2p\sqrt{\frac{2}{\Pi}}U(0,t) - 2p^2\bar U_s$

$\Rightarrow \frac{d\bar U_s}{dt} + 2p^2\bar U_s = 2p\sqrt{\frac{2}{\Pi}} = Ae^{-2p^2t}$ ... (1)

But u(x, 0) = e^{-x}

$\therefore$ when t = 0,

$\bar U_s = \sqrt{\frac{2}{\Pi}}\int_0^\infty e^{-x}\sin px\,dx$

$= \sqrt{\frac{2}{\Pi}}\left[\frac{e^{-x}}{1+p^2}(-\sin pxdx - p\cos px)\right]_0^\infty$

$= \frac{p}{1+p^2}\sqrt{\frac{2}{\Pi}}$

$\therefore$ from (1) when t = 0

$\bar U_s = \frac{p}{1+p^2}\sqrt{\frac{2}{\Pi}} = A$

$\therefore \bar U_s = \sqrt{\frac{2}{\Pi}}\left(\frac{p}{1+p^2}\right)e^{-2p^2t}$

Now applying the inversion Fourier sine transform, we have

$U(x,t) = \frac{2}{\Pi}\int_0^\infty\frac{pe^{-2p^2t}\sin px}{1+p^2}dp$

8. Solve $\frac{d^2x}{dt^2} + x = f(t)$ *if x = x' = 0 for t = 0.*

Ans :

Given equation is $\frac{d^2x}{dt^2} + x = f(t)$

Taking Laplace transform on both sides

$\Rightarrow L\{x''(t)\} + L\{x(t)\} + L\{f(t)\}$

$\Rightarrow P^2L\{x(t)\} - Px(0) - x'(0) + L\{x(t)\} = L\{(t)\}$

$\Rightarrow (P^2 + 1)L\{x(t)\} = f(p)$

$\Rightarrow L\{x(t)\} = \frac{f(p)}{P^2+1}$

$\Rightarrow x(t) = L^{-1}\left\{\frac{1}{P^2+1}.f(p)\right\}$

$\Rightarrow x(t) = \sin t * f(p)$

$\therefore x(t) = \int_0^t \sin(t-x)\,f(x)\,dx$ (by convolution theorem)

SECTION - B $(4 \times 16 = 64)$