QUESTION BANK
UNIT - I

VECTOR SPACES :

Definition : *Let 'V' be a non-empty set whose elements are called vectors. Let 'F" be any set whose elements are called scalars whre (F, +, .) is a field.*

The set 'V' is said to be a vector space if

1. There is defined an internal composition in 'V" called addition of vectors denoted by '+', for which (V, +) is an abelian group.
2. There is defined an external composition in 'V' over 'F', called the scalar multiplication in which $a\alpha \in V$ for all $a \in F$ and $\alpha \in V$.
3. The above two compositions satisfy the following postulates.
   i. $a(\alpha + \beta) = a\alpha + a\beta$
   ii. $(a + b)\alpha = a\alpha + b\alpha$
   iii. $(ab)\alpha = a(b\alpha)$
   iv. $1\alpha = \alpha$

$\forall$ a, b $\in$ F and $\alpha, \beta \in V$ and 1 is the unity element of F.

Theorem 1 :

Let V(F) be a vector space. A non-empty set $W \subseteq V$.

The necessary and sufficient condition for W to be a subspace of V is a, b $\in$ F and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in W$.

Proof :

Condition is necessary

W(F) is a subspace of V(F) $\Rightarrow$ W(F) is a vector space.

$\therefore$ Now $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$ and $b \in F$, $\beta \in W$

$\Rightarrow b\alpha \in W$

Now $a\alpha \in W$, $b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Condition is Sufficient

Let W be the non-empty subset of V

Satisfying the given condition

i.e., a, b $\in$ F and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ ... (1)

Taking a = 1, b = –1, and $\alpha, \beta \in W \Rightarrow 1\alpha + (-1)\beta \in W$

$\Rightarrow \alpha - \beta \in W$ [$\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

($H \subseteq G$ and $a, b \in H \Rightarrow ab^{-1} \in H$ then (H, 0) is subgroup of (G, 0)).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +)

$\Rightarrow$ (W, +) is an abelian group. Again taking b = 0

a, O $\in$ F and α, $\in W \Rightarrow a\alpha + O\beta \in W \Rightarrow a\alpha \in W \Rightarrow a \in F$ and $\alpha \in W \Rightarrow a\alpha \in W$.

$\therefore$ W is closed under scalar multiplication.

The remaining postulates of vector space hold in W as $W \subseteq V$.

$\therefore$ W(F) is a vector space of V(F).

Theorem 2 :

Let V(F) be a vector space and let $W \subseteq V$. The necessary and sufficient conditions for W to be a subspace of V are :

i. $\alpha \in W$, $\beta \in W \Rightarrow \alpha - \beta \in W$

ii. $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$.

Proof :

Conditions are necessary

i. W is a vector subspace of V

$\Rightarrow$ W is a subgroup of (V, +) $\Rightarrow$ (W, +) is a group.

$\Rightarrow$ If $\alpha, \beta \in W$ then $\alpha - \beta \in W$

(ii) W is a subspace of V

$\Rightarrow$ W is closed under scalar multiplication

$\Rightarrow$ for $a \in F$, $\alpha \in W$; $a\alpha \in W$

Conditions are sufficient

Let W be a non-empty subset of 'V" satisfying the two given conditions

$\alpha \in W$, $\alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \bar{O} \in W$ (by (i))

$\therefore$ The zero vector of 'V' is also the zero vector of W.

$\bar{O} \in W$, $\alpha \in W \Rightarrow \bar{O} - \alpha \in W \Rightarrow (-\alpha) \in W$ (by(i))

$\Rightarrow$ additive inverse of each element of W is also in W.

Again $\alpha \in W$, $\beta \in W \Rightarrow \alpha \in W, (-\beta) \in W \Rightarrow \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$ (by (i))

i.e., W is closed under vector addition

AS $W \subseteq V$, all the elements of W are also the elements of V. Thereby vector addition in W will be associative and cimmutative. This implies that (W, +) is an abelian group.

Further by (ii), W is closed under scalar multiplication and other postulates of vector space hold in 'ω' as $W \subseteq V$

$\therefore$ W itself is a vector space under the operations of V.

Hence W(F) is a vector subspace of V(F).

Theorem 3 :

The union of two subspaces if and only if one is contained in the other.

Proof :

Let W_1 and W_2 be two subspaces of V(F)

Condition is Necessary

Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\therefore$ $W_1 \cup W_2 = W_2$ or W_1

$\Rightarrow$ $W_1 \cup W_2$ is a subspace of V(F)

Condition is Sufficient

Let $W_1 \cup W_2$ be a subspace

Let us suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

Now $W_1 \not\subset W_2 \Rightarrow$ there exists $x \in W_1$ and $x \notin W_2$ ... (1)

$W_2 \not\subset W_1 \Rightarrow$ there exists $y \in W_2$ and $y \notin W_1$ ... (2)

$\Rightarrow x + y \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow x + y \in W_1$ or $x + y \in W_2$.

Now x + y, $x \in W_1$ (subspace)

$\Rightarrow 1(x + y) + (-1)x \in W_1 \Rightarrow y \in W_1$ ... (3)

Similarly x + y, $y \in W_2$ (subspace)

$\Rightarrow 1(x + y) + (-1)y \in W_2 \Rightarrow x \in W_2$ ... (4)

Thus (3) and (4) contradict (2) and (1)

$\therefore$ Either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Linear Combination of Vectors

Definition

Let V(F) be a vector space. If $\alpha_1, \alpha_2, ..., a_n \in V$ then any vector.

$\gamma = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$ where $a_1, a_2, ..., a_n \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, ..., \alpha_n$.

Note (1)

γ is a vector belonging to V(F).

Linear span of a set

Definition

Let 'S' be a non-empty subset of a vector space V(F). The linear span of S is the set of all possible linear combinations of all possible finite subsets of S.

The linear span of S is denoted by L(S).

$\therefore$ $L(S) = \{\gamma . \gamma = \Sigma a_i \gamma_i, a_i \in F, \alpha_i \in S\}$

Problem 4 :

Express the vector $\alpha = (1, -2, 5)$ as a linear combination of the vectors $e_1 = (1, 1, 1)$, $e_2 = (1, 2, 3)$ and $e_3 = (2, -1, 1)$.

Sol :

Let $\alpha = (1, -2, 5) = x(1, 1, 1) + y(1, 2, 3) + z(2, -1, 1)$

$= (x + y + 2z, x + 2y - z, x + 3y + z)$

Hence x + y + 2z = 1, x + 2y – z = –2, x + 3y + z = 5

Reducing to echelon form, we get,

$\alpha + \beta + 2\gamma = 1$, $\beta - 2\gamma = -3$, $5\gamma = 10$

These equations are consistant and have a solution given by

$\alpha = -6$, $\beta = 3$, $\gamma = 2$.

Hence $\alpha = -6e_1 + 3e_2 + 2e_3$.

Problem 5 :

Show that the vector $\alpha = (2, -5, 3)$ in R^3 cannot be expressed as a linear combination of the vectors $e_1 = (1, -3, 2)$, $e_2 = (2, -4, -1)$, and $e_3 = (1, -5, 7)$.

Sol :

Let $\alpha = ae_1 + be_2 + ce_3$

i.e., (2, –5, 3) = a (1, –3, 2) + b (2, –4, –1) + c (1, –5, 7)

= (a + 2b + c, –3a – 4b – 5c, 2a – b + 7c)

$\Rightarrow$ a + 2b + c = 2, –3a –4b – 5c = –5, 2a – b + 7c = 3

Now reducing to echelan form, we get

a + 2b + c = 2, 2b – 2c = 1, O = 3

As the system is inconsistant, the equations have no solution. Hence α cannot be expressed as a *l*. c of e_1, e_2, e_3.

Linear Dependence of Vectors

Definitiion

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly dependent (L.D.) set if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{O}$.

Linear Independence of Vectors

Definition

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly independent (L.I.) if every relation of the form.

$$a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{O},\ a_1^{-1} s \in F$$

$$\Rightarrow a_1 = 0,\ a_2 = 0,\ ...,\ a_n = 0$$

Problem 6 :

Show that the system of vectors (1, 3, 2), (1, –7, –8) (2, 1, –1) of $V_3(R)$ is linearly dependent.

Sol :

Let a, b, c $\in$ R, then a (1, 2, 3) + b (1, –7, –8) + c (2, 1, –1) = $\bar{O}$

$\Rightarrow$ (a + b + 2c, 3a – 7b + c, 2a – 8b – c) = (0, 0, 0)

$\Rightarrow$ (a + b + 2c = 0, 3a – 7b + c = 0, 2a – 8b – c = 0)

$\Rightarrow$ a = 3, b = 1, c = –2 [$\because$ The given vectors are linearly dependent]

Problem 7 :

Show that the system of vectors (1, 2, 0), (0, 3, 1), (–1, 0, 1) of $V_3(Q)$ is linearly independent where Q is the field of rational numbers.

Sol :

Let x, y, z $\in$ Q then x (1, 2, 0) + y (0, 3, 1) + z (–1, 0, 1) = $\bar{O}$

$\Rightarrow$ (x – z, 2x + 3y, y + z) = (0, 0, 0)

$\Rightarrow$ x – z = 0, 2x + 3y = 0, y + z = 0

$\Rightarrow$ x = 0, y = 0, z = 0.

Hence the system is linearly independent.

Problem 8 :

Prove that the four vector $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$ $\gamma = (0, 0, 1)$, $\delta = (1, 1, 1)$ in $V_3(C)$ form linearly dependent set, but any three of them are linearly independent.

Sol :

a, b, c, d $\in$ C ($\because a\alpha + b\beta + c\chi + d\delta = \bar{O}$)

$\Rightarrow$ a (1, 0, 0) + b (0, 1, 0) + c (0, 0, 1) + d (1, 1, 1) = $\bar{O}$

$\Rightarrow$ (a + d, b + d, c + d) = (0, 0, 0)

$\Rightarrow$ a + d = 0, b + d = 0, c + d = 0

$\therefore$ a = –d, b = –d, c = –d.

Thus if d = –k then a = k, b = k, c = k.

Showing that $\alpha + \beta + \gamma - \delta = \bar{O}$

$\therefore$ The four vectors $\alpha, \beta, \gamma, \delta$ are linearly dependent.

(ii) But $a\alpha + b\beta + c\delta = \bar{O}$

$\Rightarrow$ a (1, 0, 0) + b (0, 1, 0) + c (1, 1, 1) = $\bar{O}$

$\Rightarrow$ (a + c, b + c, c) = (0, 0, 0)

$\Rightarrow$ a + c = 0, b + c = 0, c = 0

$\Rightarrow$ a = 0, b = 0, c = 0

$\Rightarrow$ the vectors α, β, δ are linearly independent.

Similarly we can show that any other three vectors are linearly independent.

Theorem 9 :

Let V(F) be a vector space and $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a finite subset of non-zero vectors of V(F). Then S is linearly dependent if and only if some vector $\alpha_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceeding vectors.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_k ... \alpha_n\}$ is linearly dependent. Then there exist $\alpha_1, \alpha_2, ... \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + ... a_n\alpha_n = \bar{O}$.

Let k be the greatest suffix of α for which $\alpha_k \neq 0$.

Then $a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k + 0\alpha_{k+1} + ... + 0\alpha_n = 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k = 0$

Now suppose k = 1, (1) $\Rightarrow a_1\alpha_1 = 0$

But $a_1 \neq 0$, so $\alpha_1 = 0$, which contradicts that each element of S is a non-zero vector.

Hence k > 1, i.e., $2 \le k \le n$.

Again from (1) we have,

$a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1}$

$\Rightarrow a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1})$

$\Rightarrow \alpha_k = (-a_k^{-1}a_1)\alpha_1 + (-a_k^{-1}a_2)\alpha_2 + ... + (-a_k^{-1}a_{k-1})\alpha_{k-1}$

= (linear combination of its preceding vectors)

Conversely

Let some $\alpha_p \in S$ be expressible as a linear combination of its preceding vectors i.e., for $b_1, b_2, ... b_{p-1} \in F$.

$\alpha_p = b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1} + (-1)\alpha_p = \bar{O}$

$\Rightarrow$ the set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

[(–1) is a non-zero coefficient]

Hence the superset $S = \{-\alpha_1, \alpha_2, ..., \alpha_p - \alpha_n\}$ is linearly dependent

Note

If β is a linear combination of the set of vectors $\alpha_1, \alpha_2, ..., \alpha_n$ then the set of vectors $\{b, \alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

Basis of Vector Space

Definition

A subset S of a vector space V(F) is said to be the basis of V, if

i. S is linearly independent

ii. The linear span of S is V i.e., L(S) = V

Note

A vector space may have more than one basis.

Basis Extension

Theorem 10 :

Let V(F) be a finite dimensional vector space and $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_m\}$ is linearly independent subset of V

Since V(F) is finite dimensional it has a finite basis B.

Let $B = \{\beta_1, \beta_2, ... \beta_n\}$

Now consider the set $S_1 = \{\alpha_1, \alpha_2, ... \alpha_m, \beta_1, \beta_2, ... \beta_n\}$ writ in this order.

Clearly L(S) = V.

Each α can be expressed as a *l*.c. of β's as B is the basis of

$\Rightarrow S_1$ is linearly dependent.

Hence some vector in S_1 can be expressed as a *l*.c. of its preceding vectors. This vector cannot be any of α's, since S is linearly independent. So, this vector must be some β_i consider now the set.

$S_2 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_{i-1}, \beta_{i+1} ... \beta_n\} = S_1 - \{\beta_i\}$ obviously $L(S_2) = L(S_1) = V$

If S_2 is linearly independent, then S_2 forms a basis of V and it is the extended set.

If S_2 is linearly dependent then continue this procedure till we get a set $S_k \subset S$ such that S_k is linearly independent.

$\therefore$ $L(S_k) = L(S) = V$

S_k will be extended set of S forming a basis of V.

Theorem 11 :

V is a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ... \beta_n$ then any independent set of vectors in V is finite and contains no more than n elements.

Proof :

Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a linearly independent subset of vector space V(F)

Let $S = \{\beta_1, \beta_2, ..., \beta_n\}$ and L(S) = V.

Then any vector of V is *l*.c. of the elements of S.

Let $a_m \in V$ be a linear combination of its preceding vectors.

This vector must be one among β's.

Let it be β_i

$\therefore$ β_p = *l*.c. of $\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}$

If $S_1 = \{\alpha_m, \beta_1, \beta_2, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$

Then $L(S_1) = L(S) = V$.

Again the vector $\alpha_{m-1} \in V$ is a *l*.c. of the elements of S_1.

$\therefore$ $\alpha_{m-1}, \alpha_m, \beta_1, \beta_2, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n$ are L.D.

There exists a vector in this set which is a *l*.c. of its predecessors.

Such vector is one of β's as α_{m-1}, am are L.I.

Let it be β_k.

If $S_2 = \{\alpha_{m-1}, \alpha_m, \beta_1, \beta_2, ..., \beta_{k-1}, \beta_{k+1}, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$

Then β_k = *l*.c. of the elements of S_3.

$\therefore$ $L(S_2) = L(S_1) = L(S) = V$.

Continuing this process further for m – 3 times.

We get $S_{m-1} = \{\alpha_2, \alpha_3, ... \alpha_m, \beta_{i1}, \beta_{i2}, ... \beta_{in-m+1}\}$.

So that $L(S_{m-1}) = V$

This set consists of atleast one β_i.

otherwise $S_{m-1} = \{\alpha_1, \alpha_2, ..., \alpha_m\}$

So that α_1 is a *l*.c. of $\alpha_2, \alpha_3, ..., \alpha_m$.

This cannot happen as the set $\alpha_2, \alpha_3, ..., \alpha_m$ is L.I.

Hence S_{m-1} consists of atleast one β_i

$\Rightarrow n - m + 1 \ge 1 \Rightarrow n \ge m$.

$\therefore$ The no. of elements of the L.I. set in $V \le n$.

Theorem 12 :

Let W_1 and W_2 be two subspaces of a finite dimensional vector space V(F). Then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Proof :

Since W_1 and W_2 are subspaces of V, $W_1 + W_2$ and $W_1 \cap W_2$ are also subspace of V.

Let $\dim(W_1 \cap W_2) = k$ and $S = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ be a basis of $W_1 \cap W_2$.

Clearly $S \subseteq W_1$ and $S \subseteq W_2$ and S is L.I.

Since S is L.I and $S \subseteq W_1$, the set S can be extended to form a basis of W_1.

Let $B_1 = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1

$\therefore$ $\dim W_1 = k + m$.

Again since S is L.I. and $S \subseteq W_2$, the set S can be extended to form a basis of W_2.

Let $B_2 = \{\gamma_1, \gamma_2, ..., \gamma_k, \beta_1, \beta_2, ..., \beta_m\}$ be a basis of W_2

$\therefore$ $\dim W_2 = k + t$

$\therefore$ $\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= (k + m) + (k + t) - k$

$= k + m + t$.

Now we shall prove that the set

$S' = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_t\}$ is a basis of $W_1 + W_2$ and hence.

$\dim(W_1 + W_2) = k + m + t$.

(i) To prove that S' is L.I.

Now $c_1\gamma_1 + c_2\gamma_2 + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + ... + b\beta = 0$ ... (1)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = -c_1\gamma_1 - ... - c_k\gamma_k - a_1\alpha_1 - ... - a_m\alpha_m$

= *l*.c. of elements of B1 and ($\because \in W_1$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1$

Again $0\gamma_1 + 0\gamma_2 + 0\gamma_k + b_1\beta_2 + b_2\beta_2 + ... + b_t\beta_t$.

= *l*.c. of elements of B_2 and ($\because \in W_2$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_2$

$\therefore$ by (1) and (2) $\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1 \cap W_2$

Hence it can be expressed as a *l*.c. of the elements of the basis S of $W_1 \cap W_2$.

Let $b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + ... + d_k\gamma_k$.

$\therefore$ $b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 ... - d_k\gamma_k = \bar{O}$

$\Rightarrow$ (*l*.c. of elements of basis B_2) = $\bar{O}$

$b_1 = 0$, $b_2 = 0 ... b_t = 0$, $d_1 = 0, ... k_k = 0$.

Substituting in I we have,

$c_1\gamma_1 + c_2\gamma_2 + ... + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m$

$\Rightarrow$ (*l*.c. of elements of basis B_1) = $\bar{O}$

$\Rightarrow c_1 = 0$, $c_2 = 0 ... c_k = 0$, $a_1 = 0$, $a_2 = 0 ... a_m = 0$

Thus relation I implies that,

$c_1 = c_2 = ... = c_k = a_1 = a_2 = ... \alpha_m = 0$, $b_1 = b_2 = ... = b_t = 0$

$\therefore$ S' is L.I. set

(ii) To prove $L(S') = W_1 + W_2$

Every vector of S' is a vector of $W_1 + W_2$.

$\therefore$ $L(S') \le W_1 + W_2$

Let $S \in W_1 + W_2$

$\therefore$ $\delta = \alpha + \beta$ where $\alpha \in W$, $\beta \in W_2$

δ = (*l*.c. of elements of B_1) + (*l*.c. of elements of B_2)

= (*l*.c. of γ's and α's) + (*l*.c. of γ's and β's)

= (*l*.c. of γ's, α's and β's.

= (*l*.c. of elements of S'

$\therefore$ $\delta \in L(S')$

$\therefore$ $W_1 + W_2 \le L(S')$

$\therefore$ $L(S') = W_1 + W_2$

Hence S' is the basis of $W_1 + W_2$

$\therefore$ $\dim(W_1 + W_2) = k + m + t$.

Hence the theorem.

Problem 13 :

V is the vector space of polynomials over R. W_1 and W_2 are the subspaces generated by $\{x^3 + x^2 - 1, x^3 + 2x^2 + 3x, 2x^3 + 3x^2 + 3x - 1\}$ and $\{x^3 + 2x^2 + 2x - 2, 2x^3 + 3x^2 + 2x - 3, x^3 + 3x^2 + 4x - 3\}$ respectively. Find

(i) $\dim(W_1 + W_2)$ (ii) $\dim(W_1 \cap W_2)$

Sol :

The coordinates of the polynomials w.r.to the basis $\{x^3, x^2, x, 1\}$ are respectively.

{(1, 1, 0, –1), (1, 2, 3, 0), (2, 3, 3, –1)} and

{(1, 2, 2, –2), (2, 3, 2, –3), (1, 3, 4, –3)}

(i) Arranging the vectors generating W_1 as rows of a matrix and reducing to echelon form

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ The basis of W_1 contains 2 vectors.

$\Rightarrow$ $\dim W_1 = 2$ similarly we get $\dim W_2 = 2$.

(ii) Now the subspace $W_1 + W_2$ is by all the six vectors. Hence arranging them in rows of matrix and reducing to echelon form.

$$P = \begin{bmatrix} 1 & 1 & 0 & -1 \\ & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & \\ 0 & 0 & -1 & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Q$$

$\Rightarrow$ The basis of $W_1 + W_2$ consists of the three non-zero rows vectors in Q.

$\Rightarrow$ $\dim(W_1 + W_2) = 3$.

Now we have the formula,

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$

$= 2 + 2 - 3 = 1$

14. ***Describe explicitly the linear transformation*** $T: R^2 \to R$ ***such that*** $T(2, 3) = (4, 5)$ ***and*** $T(1, 0) = (0, 0)$

Sol :

First of all we have to show that the vectors (2, 3) and (1, 0) are L.I.

Let $a(2, 3) + b(1, 0) = \bar{O}$

$\Rightarrow (2a + b, 3a + 0) = (0, 0)$

$\Rightarrow 2a + b = 0,\ 3a = 0$

$\Rightarrow 2a = 0,\ b = 0$

$\therefore S = \{(2, 3), (1, 0)\}$ is L.I.

Let us prove that $L(S) = R^2$

Let $(x, y) \in R^2$ and $(x, y) = a(2, 3) + b(1, 0) - (2a + b, 3a)$

$\Rightarrow 2a + b = x,\ 3a = y$

$\Rightarrow a = \dfrac{y}{3};\ b = \dfrac{3x - 2y}{3}$

Hence S spans R^2.

Now $T(x, y) = T\left[\dfrac{y}{3}(2, 3) + \dfrac{3x - 2y}{3}(1, 0)\right]$

$= \dfrac{y}{3} T(2, 3) + \dfrac{3x - 2y}{3} T(1, 0)$

$= \dfrac{y}{3}(4, 5) + \dfrac{3x - 2y}{3}(0, 0)$

$= \left(\dfrac{4y}{3}, \dfrac{5y}{3}\right)$

$\therefore$ This is the required transformation.

Problem 15 :

Find $T(x, y, z)$ ***where*** $T: R^3 \to R$ ***is defined by*** $T(1, 1, 1) = 3,\ T(0, 1, -2) = 1,\ T(0, 0, 1) = -2$.

Sol :

Let $S = \{(1, 1, 1), (0, 1, -2), (0, 0, 1)\}$

(i) Let $a(1, 1, 1) + b(0, 1, -2) + c(0, 0, 1) = \bar{O}$

$\Rightarrow (a, a + b, a - 2b + c) = (0, 0, 0)$ ($\because \bar{O} \in R^3$)

$\Rightarrow a = 0,\ a + b = 0,\ a - 2b + c = 0$

$\Rightarrow a = 0,\ b = 0,\ c = 0.$ ($\because$ S is L.I. set)

(ii) Let $(x, y, z) \in R^3$

$(x, y, z) = a(1, 1, 1) + b(0, 1, -2) + c(0, 0, 1)$

$= (a, a + b, a - 2b + c)$

$\Rightarrow a = x,\ a + b = y,\ a - 2b + c = z$

$\Rightarrow a = x,\ b = y - x,\ c = z + 2y - 3x$

$\therefore$ S spans R^3

Hence $T(x, y, z) = T[x(1, 1, 1) + (y - x)(0, 1, -2) + (z + 2y - 3x)(0, 0, 1)]$

$= xT(1, 1, 1) + (y - x)T(0, 1, -2) + (z + 2y - 3x)T(0, 0, 1)$

$= x(3) + (y - x)(1) + (z + 2y - 3x)(-2)$

$= 8x - 3y - 2z$ which is the required.

Linear functional.

Theorem 16 :

Rank - Nullity theorem

Statement

Let U(F) and V(F) be two vector spaces and $T: U \to V$ be a linear transformation. Let U be finite dimensional

Then $P(T) + \nu(T) = \dim U$

i.e., rank (T) + nullity (T) = dim U

Proof :

The null space N(T) is a subspace of finite dimensional space U(F).

$\Rightarrow$ N(T) is finite dimensional

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_k\}$ be a basis set of N(T)

$\therefore \dim N(T) = \nu(T) = k$

$\therefore T(\alpha_1) = \bar{O},\ T(\alpha_2) = \bar{O} \ldots T(\alpha_k) = \bar{O}$

$(\bar{O} \in V)$

As S is L.I. it can be extended to form a basis of U(F).

$\therefore \dim U = k + m.$

Now we show that the set of images of additional vectors.

$S2 = \{T(\theta_1), T(\theta_2), \ldots T(\theta_m)\}$ is a basis of R(T) clearly $S_2 \subseteq R(T)$

(i) To prove S_2 is L.I

Let $a_1, a_2, \ldots a_m \in F$ be such that

$a_1T(\theta_1) + a_2T(\theta_2) + \ldots + a_mT(\theta_m) = \bar{O}$

$\Rightarrow T[a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m] = \bar{O}$

$\Rightarrow a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m \in N(T)$, null space of T.

But each vector is N(T) is a l.c. of elements of basis S.

$\therefore$ For some $b_1, b_2, \ldots b_m \in F$,

Let $a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_k\alpha_k$

$\Rightarrow a_1\theta_1 + \ldots + a_m\theta_m - b_1\alpha_1 - \ldots - b_k\alpha_k = \bar{O}$

$\Rightarrow a_1 = 0,\ a_2 = 0 \ldots a_m = 0,\ b_1 = 0, \ldots b_k = 0$

($\because S_1$ is L.I)

$\Rightarrow S_2$ is L.I. set

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space R(T), then there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

Now $\alpha \in U \Rightarrow$ there exists c's, d's $\in F$ such that

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \ldots + d_m\theta_m$

$\Rightarrow T(\alpha) = T[c_1\alpha_1 + \ldots + c_k\alpha_k + d_1\theta_1 + \ldots + d_m\theta_m]$

$= c_1T(\alpha_1) + \ldots + c_kT(\alpha_k) + d_1T(\theta_1) + \ldots + d_mT(\theta_m)$

$\Rightarrow \beta = d_1T(\theta_1) + d_2T(\theta_2) + \ldots + d_mT(\theta_m)$ (by (1))

$\Rightarrow \beta \in L(S_2)$

$\therefore S_2$ is a basis of R(T) and dim R(T) = m

Thus dim R(T) + dim N(T) = m + k = dim U

i.e., $\rho(T) + \nu(T) = \dim U$.

Problem 17 :

Let $T: V_2 \to V_3$ ***be defined by*** $T(x, y) = (x + y, 2x - y, 7y)$ ***Find*** $[T: B_1, B_2]$ ***where*** B_1 ***and*** B_2 ***are the standard bases of*** V_2 ***and*** V_3.

Sol :

B_1 is the standard basis of V_2 and B_2 that of V_3

$\therefore B_1 = \{(1, 0), (0, 1)\}$

$B_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$T(1, 0) = (1, 2, 0) = 1(1, 0, 0) + 2(0, 1, 0) + 0(0, 0, 1)$

$T(0, 1) = (1, -1, 7) = 1(1, 0, 0) - 1(0, 1, 0) + 7(0, 0, 1)$

The matrix of T relative of B_1 and B_2 is

$$[T: B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

Problem 18 :

Let $T: R^3 \to R^2$ ***be the linear transformations defined by*** $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$. ***Find the matrix of T relative to the bases*** $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$, $B_2 = \{(1, 3), (2, 5)\}$

Sol :

Let $(a, b) \in R^2$ and let $(a, b) = p(1, 3) + q(2, 5)$

$= (p + 2q, 3p + 5q)$

$\Rightarrow p + 2q = a,\ 3p + 5q = b$

Solving $p = -5a + 2b,\ q = 3a - b$

$\therefore (a, b) = (-5a + 2b)(1, 3) + (3a - b)(2, 5)$

Now $T(1, 1, 1) = (1, -1) = -7(1, 3) + 4(2, 5)$

$T(1, 1, 0) = (5, -4) = -33(1, 3) + 19(2, 5)$

$T(1, 0, 0) = (3, 1) = -13(1, 3) + 8(2, 5)$

$\therefore$ The matrix of L.T. relative to B_1 and B_2

$$[T: B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

Problem 19 :

Let T be the linear operator r on R^2 ***defined by*** $T(x, y) = (4x - 2y, 2x + y)$. ***Find the matrix of T w.r.to the basis*** $T\{(1, 1), (-1, 0)\}$, ***Also verify*** $[T]_B [\alpha]_B = [T(\alpha)]_B\ \forall \alpha \in R^2$.

Sol :

Let $(a, b) \in R^2$. Then;

$(a, b) = p(1, 1) + q(-1, 0) = (p, -q, p)$

$\Rightarrow a = p - q$ and $b = p$

$\Rightarrow p = b$ and $q = b - a$

$\therefore (a, b) = b(1, 1) + (b - a)(-1, 0)$ ... (1)

(i) Given transformation is $T(x, y) = (4x - 2y, 2x + y)$

$\therefore T(1, 1) = (2, 3) = 3(1, 1) + 1(-1, 0)$ ... (by (1))

$T(-1, 0) = (-4, 2) = -2(1, 1) + 2(-1, 0)$

$$\therefore [T: B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

(ii) Let $a \in R^2$ where $\alpha = (a, b)$

$\therefore \alpha = (a, b) = b(1, 1) + (b - a)(-1, 0)$

$$\therefore [\alpha]_B = \begin{bmatrix} b \\ b - a \end{bmatrix}$$

$$\therefore [T]_B [\alpha]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} b \\ b - a \end{bmatrix} = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix} \quad \ldots (1)$$

Again $T(\alpha) + T(a, b) = (4a - 2b, 2a + b)$

Let $(4a - 2b, 2a + b) = x(1, 1) + y(-1, 0) = (x - y, x)$

$\Rightarrow 4a - 2b = x - y$ and $2a + b = x$

$\Rightarrow x = 2a + b$ and $y = -2a + 3b$.

Hence $T(\alpha) = T(a, b) = (2a + b)(1, 1) + (-2a + 3b)(-1, 0)$

$\therefore$ The matrix of $T(\alpha)$ w.r.to the base B is

$$[T(\alpha)]B = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix} \quad \ldots (2)$$

$\therefore$ from (1) and (2) $[T]_B [\alpha]_B = [T(\alpha)]_B$.

UNIT - I

9. a) **Let V(F) be a finite dimensional vector space. Then show that any two bases of V have the same number of elements.**

Sol. :

Basis of a vector space

A subset S of a vector space V(F) is said to be the basis of V, if

(i) S is linearly independent

(ii) the linear span of S is V i.e., L(S) = V

Proof :

Let S_m and S_n be the two bases of V(F) where $S_m = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ $S_n = \{\beta_1, \beta_2, \ldots \beta_n\}$ obviously both S_m and S_n are L.I subsets of V.

(i) Consider S_m as the basis of V and S_n and L.I set

$\Rightarrow L(S_m) = V$ and $n(S_m) = m$

$\therefore S_n$ can be extended to be a basis of V

$\Rightarrow n \leq m$

(ii) Consider S_n as the basis of V and S_m as L.I set

$\Rightarrow L(S_n) = V$ and $n(S_n) = n$

$\therefore S_m$ can be extended to be a basis of V

$\Rightarrow m \leq n$

But both S_m and S_n are bases of V

$\therefore n \leq m$ and $m \leq n \Rightarrow m = n$

Thus any two bases of V have the same number of elements.

b) **Show that every non empty subset of a linearly independent set of vectors is linearly independent.**

Sol. :

Let $S = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ be a linearly independent set of vectors.

Let us consider the subset $S_1 = \{\alpha_1, \alpha_2 \ldots \alpha_k\}$ where $1 \leq k \leq m$.

Now for some scalars consider the linear combination

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k = \bar{0}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_k\alpha_k + 0\alpha_{k+1} + \ldots + 0_m\alpha_m = \bar{0}.$

$\because S_1 \subset S$ which is a linearly independent set.

$\therefore a_2 = a_2 = \ldots = a_k = 0$

$\therefore S_1$ is a linearly independent set.

10. a) **Define Range and Null space of a linear transform.**

Sol. :

a) **Range of Linear Transformation**

Let U(F) and V(F) be two vector spaces and Let $T: U \to V$ be a linear transformation. The range of T is defined to be the set Range (T)

$R(T) = \{T(\alpha) / \alpha \in u\}$.

Obviously the range of T is a subset of V i.e., $R(T) \subseteq V$.

Nullspace of a Linear Transformation

Let U(F) and V(F) be two vector spaces and $T: U \to V$ be a linear transformation. The null space denoted by N(T) is the set of all vectors $\alpha \in U$ such that $T(\alpha) = \bar{0}$ (zero vector of V). The null space of N(T) is also called the kernel of T. i.e., $N(T) = \{\alpha \in U / T(\alpha) = \bar{0} \in V\}$.

Obviously the null space is a subset of U i.e., $N(T) \subseteq U$.

b) **Let T be a linear transformation from a vector space U(F) into a vector space V(F). Then T is non-singular if and only if T is one-one.**

Sol. :

Suppose that a Linear transformation $T: u \to v$ is non singular.

Let $\alpha_1, \alpha_2 \in U$ and $T(\alpha_1) = T(\alpha_2)$

Consider

$T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = 0$

$\Rightarrow T(\alpha_1 - \alpha_2) = 0$

$\Rightarrow \alpha_1 - \alpha_2 \in N(T)$

$\Rightarrow \alpha_1 - \alpha_2 = 0$ by definition

$\Rightarrow \alpha_1 = \alpha_2$

$\Rightarrow T: U \to V$ is one - one

Conversely suppose that $T: U \to V$ is one-one.

Let $\alpha \in N(T)$ = The null space of T.

$\Rightarrow T(\alpha) = 0$

$\Rightarrow T(\alpha) = T(0)$

$\Rightarrow \alpha = 0$

$\therefore N(T) = \{0\}$

$\Rightarrow$ T is non singular.

Solutions to Issued by O.U.

SECTION - A $(6 \times 6 = 36)$

UNIT - I

1. ***Define a subspace. Prove that the intersection of two subspaces is again a subspace.***

Ans :

Subspace

Let V(F) be a vector space and $W \subseteq V$. Then W is said to be a subspace of v if w itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.

Proof :

W_1 and W_2 are subspace V(F)

$\Rightarrow \bar{0} \in W_1$ and $\bar{0} \in W_2$ $\quad \therefore W_1 \cap W_2 \neq \emptyset$

$\Rightarrow \bar{0} \in W_1 \cap W_2$ $\quad \therefore \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$

Now $a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

$\therefore a\alpha + b\beta \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a subspace of V(F)

2. ***Define linear transformation. Show that the mapping*** $T: V_3(R) \to V_2(R)$ ***defined as*** $T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$ ***is a linear transformation from*** $V_3(R)$ ***into*** $V_2(R)$.

Ans :

Definition

Let U(F) and V(F) be two vector spaces.

Then the function $T: U \to V$ is called a linear transformation of U into V if $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)\ \forall a, b \in F;\ \alpha, \beta \in U$

Let $\alpha = (a, b, c)$ and $\beta = (a_2\ b_2\ c_2)$ be two vectors of $V_3(R)$ for $a, b \in F$

Consider

$T(a\alpha + b\beta) = T[a(a_1\ b_1\ c_1) + b(a_2\ b_2\ c_2)]$

$= T[aa_1 + ba_2,\ ab_1 + bb_2,\ ac_1 + bc_2]$

$= [3(aa_1 + ba_2) - 2(ab_1 + bb_2) + (ac_1 + bc_2),\ (aa_1 + ba_2) - 3(ab_1 + bb_2) - 2(ac_1 + bc_2)]$

$= [(3a_1 - 2b_1 + c_1)a + (3a_2 - 2b_2 + c_2)b,\ (a_1 - 3b_1 - 2c_1)a + (a_2 - 3b_2 - 2c_2)b]$

$= a[3a_1 - 2b_1 + c_1,\ a_1 - 3b_1 - 2c_1] + b[3a_2 - 2b_2 + c_2,\ a_2 - 3b_2 - 2c_2]$

$T(a\alpha + b\beta) = aT(a_1\ b_1\ c_1) + bT(a_2\ b_2\ c_2)$

$= aT(\alpha) + bT(\beta)$

$\therefore$ T is a linear transformation from $V_3(R)$ to $V_2(R)$

(b) **If α, β, γ are linearly independent vectors of V(R) then show that $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also linearly independent.**

Sol :

Let $a, b, c \in R$

$\Rightarrow a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$

$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0$

$\because \alpha, \beta, \gamma$ are linearly independent.

$\Rightarrow a + c = 0,\ a + b = 0,\ b + c = 0$

$\Rightarrow a = -c$

$\Rightarrow -c + b = 0 \Rightarrow b = c$

$\therefore 2c = 0$

$\Rightarrow c = 0 \Rightarrow b = 0 \Rightarrow a = 0$

$\therefore \alpha + \beta, \beta + \gamma$ and $\gamma + \alpha$ are LI.

10. (a) **Let W be a subspace of a finite dimensional vector space V(F) then dim(V/W) = dim V − dim W.**

Sol :

Since V is finite dimensional

$\Rightarrow$ W is also finite dimensional.

Let the set $\{\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_m\}$ be the basis of w.

$\therefore \dim w = m$

$\because$ The set B is LI, it can be extended to form a basis of V.

Let the set $S = \{\alpha_1\ \alpha_2 \ldots \alpha_m\ \beta_1\ \beta_2 \ldots \beta_l\}$ be the basis of V.

$\therefore \dim V = m + l.$

$\Rightarrow \dim V - \dim w = (m + l) - m = l$

Now we shall prove that the set $s' = \{w + \beta_1, w + \beta_2 \ldots w + \beta_l\}$ is a basis of v/w and hence dim v/w = l.

(i) To prove s' is LI

We know that the zero vector of v/w is w.

Now

$b_1(w + \beta_1) + b_2(w + \beta_2) + \ldots + b_l(w + \beta_l) = w$ ... (1)

$\Rightarrow (w + b_1\beta_1) + (w + b_2\beta_2) + \ldots + (w + b_l\beta_l) = w$

$\Rightarrow w + (b_1\beta_1 + b_2\beta_2 + \ldots + b_l\beta_l) = w$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_l\beta_l \in W.$

But any vector of w is L.C of elts. of B

Let $b_1\beta_1 + b_2\beta_2 + \ldots + b_l\beta_l = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_l\beta_l - a_1\alpha_1 - a_2\alpha_2 - \ldots - a_m\alpha_m = 0$

$\Rightarrow$ L.C of elts. of LI set = 0

$\Rightarrow b_1 = b_2 = \ldots = b_l = 0,\ a_1 = a_2 = \ldots = a_m = 0$

$\Rightarrow$ The set s' is LI.

(ii) To prove L(s') = v/w

wdf Since s is the basis of V, $\alpha \in V$ can be expressed

as $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_m\alpha_m + d_1\beta_1 + d_2\beta_2 + \ldots + d_l\beta_l$

$\Rightarrow r + d_1\beta_1 + d_2\beta_2 + \ldots + d_l\beta_l$ where c's d's $\in F$

and $r = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_m\alpha_m$

$\Rightarrow$ (l.c of elts. of B)

$\Rightarrow r \in w$

for $r \in V$, $w + r \in v/w$.

$\Rightarrow w + d_1\beta_1 + d_2\beta_2 + \ldots + d_l\beta_l$ ($r \in w \Rightarrow w + r \in w$)

$\Rightarrow (w + d_1\beta_1) + (w + d_2\beta_2) + \ldots + (w + d_l\beta_l)$

$\Rightarrow d_1(w + \beta_1) + d_2(w + \beta_2) + \ldots + d_l(w + \beta_l)$

$\Rightarrow$ L.C of elts. of s'

$\Rightarrow w + \alpha \in L(s')$

$\therefore L(s') = v/w$

$\therefore$ s' is a basis of v/w

$\Rightarrow \dim v/w = l$

$\Rightarrow \dim v/w = \dim v - \dim w$

UNIT - I

1. **Let V(F) be a vector space then if $a, b \in F$ and α is a non-zero vector of V, we have $a\alpha = b\alpha \Rightarrow a = b$. And if $\alpha, \beta \in V$ and a is a non-zero element of F, we have $a\alpha = a\beta \Rightarrow \alpha = \beta$.**

Sol :

Given V(F) is a vector space & $a, b \in F$

(i) Let $a\alpha = b\alpha\ \forall \alpha \in V$

$\Rightarrow a\alpha + (-b\alpha) = \bar{0}$

$\Rightarrow a\alpha - b\alpha = \bar{0}$

$\Rightarrow (a - b)\alpha = \bar{0}$

$\because a \neq \bar{0} \Rightarrow a - b = \bar{0} \Rightarrow a = b$

$\Rightarrow \boxed{a\alpha = b\beta \Rightarrow a = b}$

(ii) Let $a\alpha = a\beta \Rightarrow a\alpha + (-a\beta) = \bar{0}$

$\Rightarrow a\alpha - (a\beta) = \bar{0}$

$a\alpha - a\beta = \bar{0}$

$\Rightarrow a(\alpha - \beta) = \bar{0}$

$\because a \neq \bar{0} \Rightarrow \alpha - \beta = \bar{0} \Rightarrow \alpha = \beta$

$\boxed{a\alpha = a\beta \Rightarrow \alpha = \beta}$

PART - B (4×16 = 64 Marks)

UNIT - I

9. a) **Prove that each subspace W of a finite dimensional vector space V(F) of dimension n is a finite dimensional space with dim W ≤ n. Also V = W iff dim V = dim W.**

Answer :

Dim V = n $\Rightarrow$ each (n + 1) or more vectors of V form an LD set

Given W is a subspace of V(F)

$\Rightarrow$ Each set of (n + 1) vectors in W is subset of V and hence LD

Thus any LI set of vectors in W can contain at the most n vectors.

Let $S = \{\alpha, \alpha_2 \ldots \alpha_m\}$ be the largest LI subset of W where $m \leq n$.

Now we will prove that S is the basis of W for any $\beta \in W$

Consider

$S_1 = \{\alpha, \alpha_2 \ldots \alpha_m, \beta\}$

$\because$ S is the largest set of LI vectors, S_1 is LD.

$\therefore \exists$ scalars $a_1, a_2, a_3 \ldots a_m, b \in F$ not all zero such that $a_1\alpha_1 + a_2\alpha_2 + \ldots a_m\alpha_m + b\beta = 0$.

Let b = 0 them we have $a_1\alpha_1 + a_2\alpha_2 + \ldots a_m\alpha_m = 0$

$\Rightarrow a_1 = 0,\ a_2 = 0, \ldots a_m = 0$ as S is LI.

$\Rightarrow S_1$ is LI

which is a contradiction

$\therefore b \neq 0 \Rightarrow \exists$ an let, $b^{-1} \in F \ni bb^{-1} = 1$

$\therefore a_1\alpha_1 + a_2\alpha_2 + a_m\alpha_m + b\beta = 0$

$\Rightarrow b\beta = -a_1\alpha_1 - a_2\alpha_2 \ldots a_m\alpha_m$

$\Rightarrow \beta = (-b^{-1}a_1)\alpha_1 + (-b^{-1}a_2)\alpha_2 + \ldots + (-b^{-1}a_m)\alpha_m.$

$\Rightarrow \beta$ = a Linear combination of elts. of S $\Rightarrow \beta \in L(S)$

$\therefore$ W is a finite dimensional vector space