UNIT - III

Problem 40 :

The sufficient conditions for the existence of the integral $\int_c f(x, y)\, dx$ to write is equal to $\int_\alpha^\beta f[\varphi(t), \psi(t)]$ $\varphi'(t)\, dt$ are that f, φ and ψ are continuous and φ prossesses a continuous derivative on [α, β].

Sol :

Since φ is derivative on [α, β], it is soon every sub-interval of [α, β].

∴ By Lagrange's mean value theorem, there exists

a $\eta_r \in [t_{r-1}, t_r]$ such that $\frac{\varphi(t_r) - \varphi(t_{r-1})}{t_r - t_{r-1}} = \varphi'(\eta_r)$

$\Rightarrow x_r - x_{r-1} = \varphi'(\eta_r)(t_r, t_{r-1})$

$= \varphi(\eta_r)\,\delta_r$, where $\delta_r = t_r - t_{r-1}$ ... (1)

Therefore, $S_n = \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]\, \varphi'(\xi_r)\, \delta_r + \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]\, [\varphi'(\eta_r) - \varphi'(\xi_r)]\, \delta_r$

If $S'_n = \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]\, \varphi'(\xi_r)\, \delta_r$ and

$S''_n = \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]\, [\varphi'(\eta_r) - \varphi'(\xi_r)]\, \delta_r$

Then $S_n = S'_n + S''_n$

Since f[φ(t), ψ(t)] and φ'(t) are continous on [α, β]

f[φ(t), ψ(t)] and φ'(t) is continous on [α, β].

∴ $\int_\alpha^\beta f[\varphi(t), \psi(t)]\, \varphi'(t)\, dt$ exists and

$\int_\alpha^\beta f[\varphi(t), \psi(t)]\, \varphi'(t)\, dt = \lim_{n \to \infty} \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]$

$\varphi'(\xi_r) \cdot \delta_r = \lim_{n \to \infty} S'_n$ ... (2)

Since f is continuous on [α, β], it is bounded

∴ There exists a k > 0 such that $|f[\varphi(t), \psi(t)]| \le k$, $\forall\, t \in [\alpha, \beta]$

φ', being bounded on [α, β], is bounded on $[t_{r-1}, t_r]$, $1 \le r \le n$.

If M_r and m_r be the supremum and infimum of φ' on $[t_{r-1}, t_r]$

$\varphi'(\eta_r) \le M_r$ [∵ $\eta_r \in [t_{r-1}, t_r]$]

$\varphi'(\xi_r) \ge m_r$ [∵ $\xi_r \in [t_{r-1}, t_r]$]

∴ $\varphi'(\eta_r) - \varphi'(\xi_r) \le M_r - m_r$ ... (3)

Also $\varphi'(\xi_r) \le M_r$ and $\varphi'(\eta_r) \ge m_r$

∴ $\varphi'(\xi_r) - \varphi'(\eta_r) \le M_r - m_r$ ... (4)

∴ $\pm(\varphi'(\eta_r) - \varphi'(\xi_r)) \le M_r - m_r$ [from (3) and (4)]

∴ $|\varphi'(\eta_r) - \varphi'(\xi_r)| \le M_r - m_r$

∴ $|\varphi'(\eta_r) - \varphi'(\xi_r)|\, \delta_r \le (M_r - m_r)\, \delta_r$ [∵ $\delta_r > 0$]

Since φ' is integrable on [α, β]; to $\frac{\varepsilon}{K} > 0$, there exists a δ > 0 such that for every division P of [α, β] whose norm is less than δ, $\sum_{r=1}^{n} (M_r - m_r)\, \delta_r < \frac{\varepsilon}{K}$ ... (5)

∴ $|S''_n - 0| = |S''_n| = \left| \sum_{r=1}^{n} f[\varphi(\xi_r), \psi(\xi_r)]\, [\varphi'(\xi_r) - \varphi'(\xi_r)]\, \delta_r \right|$

$\le K \sum_{r=1}^{n} |\varphi'(\eta_r) - \varphi'(\xi_r)|\, \delta_r$

$\le K \sum_{r=1}^{n} (M_r - m_r)\, \delta_r$

$< K \cdot \frac{\varepsilon}{K} = \varepsilon$

∴ $\lim_{n \to \infty} S''_n = 0$ ... (6)

∴ $\lim_{n \to \infty} = \lim_{n \to \infty} S'_n + S''_n$

$= \int_\alpha^\beta f[\varphi(t), \psi(t)]\, \varphi'(t)\, dt$ [from (2) and (6)]

∴ $\int_c f(x, y)\, dx = \int_\alpha^\beta f[\varphi(t), \psi(t)]\, \varphi'(t)\, dt$

Problem 41 :

Show that $\int_c [(x - y)^3\, dx + (x - y)^3\, dy] = 3\Pi a^4$, where C is a circle $x^2 + y^2 = a^2$ in the counter clockwise sense.

Sol :

We know x = aCosθ, y = aSinθ are the parametric equations of $x^2 + y^2 = a^2$

Since the curve is traced in the counter - clockwise sense,

$0 \le \theta < 2\Pi$

∴ $\int_c [(x - y)^3\, dx + (x - y)^3\, dy]$

$= \int_0^{2\Pi} a^3 (\cos\theta - \sin\theta)^3 (-a\sin\theta + a\cos\theta)\, d\theta$

$= a^4 \int_0^{2\Pi} (\cos\theta - \sin\theta)^4 (\cos\theta - \sin\theta)\, d\theta$

$= a^4 \int_0^{2\Pi} (\cos\theta - \sin\theta)^4\, d\theta$

$= a^4 \int_0^{2\Pi} (1 - \sin\theta)^2\, d\theta$

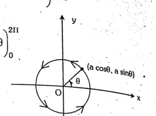
$= a^4 \int_0^{2\Pi} (1 + \sin^2 2\theta - 2\sin 2\theta)\, d\theta$

$= a^4 \int_0^{2\Pi} \left(1 + \frac{1 - \cos 4\theta}{2} - 2\sin 2\theta\right) d\theta$

$= a^4 \left(\frac{3}{2}\theta - \frac{\sin 4\theta}{8} + \cos 2\theta\right)_0^{2\Pi}$

$= a^4\, (3\Pi + 1) - (1)]$

$= 3\Pi\, a^4$



Problem 42 :

Find the value of $\int (x + y^2)\, dx + (x^2 - y)\, dy$, taken in the clockwise sense along the closed curve C formed by $y^2 = x$ and y = x between (0, 0) and (1, 1)

Sol :

$\int_c (x + y^2)\, dx + (x^2 - y)\, dy$

$= \int_{OPA} (x + y^2)\, dx\, (x^2 - y)\, dx + \int_{AQO} (x + y^2)\, dx + (x^2 - y)\, dy$

where $y^2 = x$, dx = 2y dy

∴ $\int_{OPA} (x + y^2)\, dx + (x^2 - y)\, dy$

$= \int_0^1 (y^2 + y^2)\, 2y\, dy + (y^4 - y)\, dy$

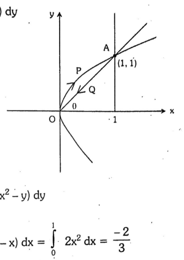
$= \int_0^1 (4y^3 + y^4 - y)\, dy$

$= \left[y^4 + \frac{y^5}{5} - \frac{y^2}{2}\right]_0^1$

$= 1 + \frac{1}{5} - \frac{1}{2}$

$= \frac{7}{10}$

when x = y, dy = dx



∴ $\int_{AQO} (x + y^2)\, dx + (x^2 - y)\, dy$

$= \int_1^0 (x + x^2 + x^2 - x)\, dx = \int_1^0 2x^2\, dx = \frac{-2}{3}$

∴ $\int_c (x + y^2)\, dx + (x^2 - y)\, dy = \frac{7}{10} - \frac{2}{3} = \frac{1}{30}$

Problem 43 :

The necessary and sufficient condition for the integrability of a bounded function f over a rectangle R is that to ∈ > 0, there exists a δ > 0 such that S(P, f) − s(P, f) < ∈ for every parition P of R whose norm is less than or equal to δ.

Proof :

Let f be integrable on R

∴ $\overline{\iint_R} f(x, y)\, dx\, dy = \iint_R f(x, y)\, dx\, dy$ ... (1)

To ε > 0, there exists a δ > 0 such that for every partition of R where norm is less than δ, $S(p, f) < \overline{\iint_R} f(x, y)\, dx\, dy + \frac{\varepsilon}{2}$ and

$S(p, f) > \underline{\iint_R} f(x, y)\, dx\, dy - \frac{\varepsilon}{2}$ [Darboux's theorem]

∴ $S(p, f) - S(p, f) < \left[\overline{\iint_R} f(x, y)\, dx\, dy + \frac{\varepsilon}{2}\right] - \left[\underline{\iint_R} f(x, y)\, dx\, dy - \frac{\varepsilon}{2}\right]$

⇒ S(p, f) − S(p, f) < ε

Conversely, suppose that S(p, f) − S(p, f) < ε, p being any partition of R whose norm is less than S we write S (p, f) − S(p, f)

$= \left[S(p, f) - \overline{\iint_R} f(x, y)\, dx\, dy\right] + \left[\overline{\iint_R} f(x, y)\, dx\, dy - \underline{\iint_R} f(x, y)\, dx\, dy\right] + \left[\underline{\iint_R} f(x, y)\, dx\, dy - S(p, f)\right] < \varepsilon$

Since the terms in each of the three brackets is non negative,

$0 \le \overline{\iint_R} f(x, y)\, dx\, dy - \underline{\iint_R} f(x, y)\, dx\, dy < \varepsilon$

Since ε > 0 is arbitrary,

$\overline{\iint_R} f(x, y)\, dx\, dy - \underline{\iint_R} f(x, y)\, dx\, dy = 0$

∴ $\overline{\iint_R} f(x, y)\, dx\, dy = \underline{\iint_R} f(x, y)\, dx\, dy$

∴ f is integrable on R.

Problem 44 :

Evaluate $\int_{y=-1}^{y=0} \left[\int_{x=-1}^{x=1} (x + y + 1)\, dx\right]$ and $\int_{x=-1}^{x=1} \left[\int_{y=-1}^{y=0} (x + y + 1)\, dy\right] dx$

Sol :

$\int_{x=-1}^{x=1} (x + y + 1)\, dx = \left[\frac{x^2}{2} + xy + x\right]_{x=-1}^{x=1}$

$= \left(\frac{1}{2} + y + 1\right) - \left(\frac{1}{2} - y - 1\right) = 2y + 2$

∴ $\int_{y=-1}^{y=0} \left[\int_{x=-1}^{x=1} (x + y + 1)\, dx\right] dy$

$= \int_{y=-1}^{y=0} (2y + 2)\, dy$

$= [y^2 + 2y]_{y=-1}^{y=0}$

$= 0 - (1 - 2) = 1$

Also $\int_{y=-1}^{y=0} (x + y + 1)\, dy = \left[xy + \frac{y^2}{2} + y\right]_{y=-1}^{y=0}$

$= 0 - \left[-x + \frac{1}{2} - 1\right]$

$= x + \frac{1}{2}$

∴ $\int_{x=-1}^{x=1} \left[\int_{y=-1}^{y=0} (x + y + 1)\, dy\right] dx = \int_{x=-1}^{x=1} \left(x + \frac{1}{2}\right) dx$

$= \left[\frac{x^2}{2} + \frac{1}{2}x\right]_{x=-1}^{x=1}$

$= \left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right)$

$= 1$

∴ The repeated integrals exists and are equal

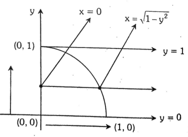
Problem 45 :

Write the repeated ingegral $\int_{x=0}^{x=1} \left\{\int_{y=0}^{y=\sqrt{1-x^2}} \frac{dy}{(1 + e^y)\sqrt{1 - x^2 - y^2}}\right\} dx$ as a double integral. Evaluate it after reversing the order.

Sol :

From the limits given in the repeated integral, $0 \le x \le 1$ and $0 \le y \le \sqrt{1 - x^2}$

If E be the region of integration, then



$E = \{x, y) / 0 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\}$. ... (1).

which is the sector of the circle $x^2 + y^2 = 1$ bounded by positive axes (or) quadrant of the circle in the I quadrant.

The double integral corresponding to the given repeated, integral is

$\iint_E \frac{1}{(1 + e^y)\sqrt{1 + x^2 - y^2}}\, dx\, dy$ where E is as given in (1)

$\iint_E \frac{1}{(1 + e^y)\sqrt{1 + x^2 - y^2}}\, dx\, dy = \int_{y=0}^{y=1} \left\{\int_{x=0}^{x=\sqrt{1-y^2}} \frac{1}{(1 + e^y)\sqrt{1 - x^2 - y^2}}\, dx\right\} dy$

$= \int_{y=0}^{y=1} \left\{\frac{1}{1 + e^y} \int_{x=0}^{x=\sqrt{1-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2}}\, dx\right\} dy$

$= \int_{y=0}^{y=1} \left\{\frac{1}{1 + e^y} \left[\sin^{-1}\left(\frac{x}{\sqrt{1 - y^2}}\right)\right]_{x=0}^{x=\sqrt{1-y^2}}\right\} dy$

$= \int_{y=0}^{y=1} \frac{1}{1 + e^y} \{\sin^{-1}(1) - \sin^{-1}(0)\}\, dy$

$= \frac{\Pi}{2} \int_{y=0}^{y=1} \frac{e^{-y}}{e^y + 1}\, dy = \frac{-\Pi}{2} [\log(e^{-y} + 1)]_{y=0}^{y=1}$

$= \frac{\Pi}{2} [\log(e^{-1} + 1) - \log(e^0 + 1)]$

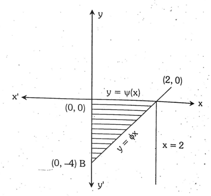
$= \frac{-\Pi}{2} [\log(1 + e) - \log e - \log 2]$

$= \frac{\Pi}{2} [1 + \log 2 - \log(1 + e)]$

Problem 46 :

Evaluate $\iint_E f(x, y)\, dx\, dy$ where $f(x, y) = \frac{2y + 1}{x + 1}$ and E is the region bounded by x = 0, y = 0, y = 2x − 4.

Sol :



for y = 2x − 4

The triangle OAB is the region of integration we use Fubine's theorem to evaluate the integral. If we take y = φ(x) = 2x − 4 and y = χ(x) = 0 φ(x) and Ψ(x) are continuous on (0, 2)

Also φ(x) ≤ Ψ(x)

f(x, y) is continuous and bounded on E.

∴ $\iint_E f(x, y)\, dx\, dy = \int_{x=0}^{x=2} \left\{\int_{y=\phi(x)}^{y=\psi(x)} \frac{2y - 1}{x + 1}\, dy\right\} dx$

$= \int_{x=0}^{x=2} \left\{\int_{y=2x+4}^{y=0} \frac{2y - 1}{x + 1}\, dy\right\} dx$

$= \int_{x=0}^{x=2} \left\{\left[\frac{y^2 - y}{x + 1}\right]_{y=2x-4}^{y=0}\right\} dx$

$= \int_{x=0}^{x=2} -\left\{\frac{(2x - 4)^2 - (2x - 4)}{x + 1}\right\} dx$

$= -\int_{x=0}^{x=2} \left(\frac{4x^2 - 18x + 20}{x + 1}\right) dx$

$= -2 \int_{x=0}^{x=2} \left(\frac{2x^2 - 9x + 10}{x + 1}\right) dx$

$= -2 \int_{x=0}^{x=2} \left(\frac{(2x + 11)(x + 1)}{x + 1} + \frac{21}{x + 1}\right) dx$

$= -2 \int_{x=0}^{x=2} \left(2x - 11 + \frac{21}{x + 1}\right) dx$

$= -2\, [x^2 - 11x + 21 \log(x + 1)]_{x=0}^{x=2}$

$= -2\, [4 - 22 + 21 \log 3]$

$= -2\, [-18 + 21 \log 3]$

$= 6\, [6 - 7 \log 3]$

Problem 47 :

By changing the order of integration, evaluate $\int_0^1 dy \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} \frac{dx}{(x^2 - 2x + y - 3)^2}$.

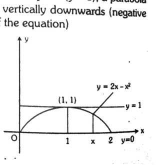
Sol :

Here the region of integration is bounded by $0 \le y \le 1$ and $1 - \sqrt{1 - y} \le x \le 1 + \sqrt{1 - y}$

The curve bounding is $(x - 1)^2 = 1 - y = -(y - 1)$, a parabola with vertex at (1, 1) and its axis is vertically downwards (negative sign being on the right hand side of the equation)

For $(x - 1)^2 = -(y - 1)$

x	0	1	2
y	0	1	0



The each x ∈ [0, 2]

$y - 1 = -(x - 1)^2$, $y = 1 - (x - 1)^2$

$= 2x - x^2$

∴ y-limits are : y = 0, $y = 2x - x^2$

x-limits are x = 0, x = 2.

Here the region has only two boundaries, the parabola and the x–axis in [0, 2].

No division of the region is needed.

∴ The given repeated integral after reversal becomes.

$\int_{x=0}^{x=2} \left\{\int_{y=0}^{y=2x-x^2} \frac{dy}{[(x^2 - 2x - 3) + y]^2}\right\} dx$

$= -\int_{x=0}^{x=2} \left\{\left[\frac{1}{(x^2 - 2x - 3) + y}\right]_{y=0}^{y=2x-x^2}\right\} dx$

$= -\int_{x=0}^{x=2} \left(-\frac{1}{3} - \frac{1}{x^2 - 2x - 3}\right) dx$

$= -\left[\frac{1}{3}x + \frac{1}{4} \log\left|\frac{x - 3}{x + 1}\right|_{x=0}^{x=2}\right]$

$= \frac{2}{3} + \frac{1}{4} \log \frac{1}{3} - \frac{1}{4} \log(3)$

$= \frac{2}{3} + \frac{1}{4} \left[\log\left(\frac{1}{3}\right) + \log\left(\frac{1}{3}\right)\right]$

$= \frac{2}{3} + \frac{1}{2} \log\left(\frac{1}{3}\right)$

Problem 48 :

In the integral $\int_2^4 \int_{\frac{4}{x}}^{\frac{20-8x}{8-x}} (4 - y)\, dy\, dx$, change the order ...tion, and evaluate the integral.

Sol :

For $y = \frac{4}{x}$

x	1	2	3	4
y	4	2	4/3	1

For $y = \frac{20 - 8x}{8 - x}$

x	1	2	3	4
y	16/7	2	8/5	1

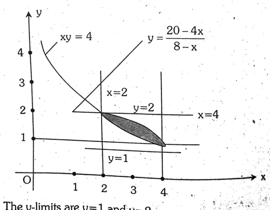
The two curves intersect at (2, 2) and (4, 1)

For y ∈ [1, 2], the x-limits are $x = \frac{4}{y}$ and the order is given by

$8y - xy = 20 - 4x$

$\Rightarrow x(4 - y) = 20 - 8y$

$\Rightarrow x = \frac{20 - 8y}{4 - y}$



The y-limits are y=1 and y=2

Therefore, the integral after change in the order, becomes,

$\int_{y=1}^{y=2} \left\{\int_{x=\frac{4}{y}}^{x=\frac{20-8y}{4-y}} (4 - y)\, dx\right\} dy$

$= \int_{y=1}^{y=2} \left\{[(4 - y)x]_{x=\frac{4}{y}}^{x=\frac{20-8y}{4-y}}\right\} dy$

$= \int_{y=1}^{y=2} \left\{(4 - y)\left[\frac{20 - 8y}{4 - y} - \frac{4}{y}\right]\right\} dy$

$= \int_{y=1}^{y=2} (4 - y) \left[\frac{20y - 8y^2 - 16 + 4y}{(4 - y)y}\right] dy$

$= \int_{y=1}^{y=2} \left(\frac{24y - 8y^2 - 16}{y}\right) dy$

$= \int_{y=1}^{y=2} \left(24y - 8y - \frac{16}{y}\right) dy$

$= [24y - 4y^2 - 16 \log y]_{y=1}^{y=2}$

$= [48 - 16 - 16 \log 2] - [24 - 4 - 0]$

$= 32 - 16 \log 2 - 20$

$12 - 16 \log 2.$

Problem 49 :

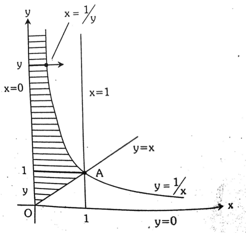
Show in a diagram the field of integration of the integral

$$\int_0^1 dx \int_x^{1/x} \frac{y^2 dy}{(x+y)^2\sqrt{1+y^2}}$$ ***and by changing the order of integration, show that the value of the integral is*** $\sqrt{2}-\frac{1}{2}$.

Sol :

The shaded portion is the region of integration. We now change the order of integration as required.

The region of integration is divided into two regions OAB and BAC.



After changing the order, the given one can be written as

$$\int_{y=0}^{y=1}\left\{\int_{x=0}^{x=y}\frac{y^2}{\sqrt{1+y^2}\,(x+y)^2}dx\right\}dy + \int_{y=1}^{y=\infty}\left\{\int_{x=0}^{x=1/y}\frac{y^2}{\sqrt{1+y^2}\,(x+y)^2}dx\right\}dy$$

$$= \int_{y=0}^{y=1}\left\{\left[\frac{-y^2}{\sqrt{1+y^2}}\cdot\frac{1}{(x+y)}\right]_{x=0}^{x=y}\right\}dy + \int_{y=1}^{y=\infty}\left\{\left[\frac{-y^2}{\sqrt{1+y^2}}\cdot\frac{1}{(x+y)}\right]_{x=0}^{x=1/y}\right\}dy$$

$$= \frac{1}{2}\int_{y=0}^{y=1}\frac{y}{\sqrt{1+y^2}}dy + \int_{y=1}^{y=\infty}(1+y^2)^{-3/2}dy$$

$$= \frac{1}{2}\left[\sqrt{1+y^2}\right]_{y=0}^{y=1} - \left[\frac{1}{\sqrt{1+y^2}}\right]_{y=1}^{y=\infty}$$

$$= \frac{1}{2}[\sqrt{2}-1] - \left[0-\frac{1}{\sqrt{2}}\right]$$

$$= \left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right]-\frac{1}{2}$$

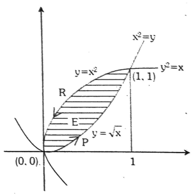
$$= \sqrt{2}-\frac{1}{2}.$$

Problem 50 :

Verify Green's theorem in the plane for $\int_c (2xy-x^2)\,dx + (x^2+y^2)\,dy$, ***where C is the boundary of the region enclosed by*** $y = x^2$ ***and*** $y^2 = x$.

Sol :

We know the two curves intersect at the origin and (1, 1).



The curve C consists of two parts OPQ, QRO.

OPQ along $x^2 = y$ and QRO along $y^2 = x$.

If C_1 and C_2 respectively denote the curves OPQ and QRO in the anti-clockwise direction, then,

$$\int_{c_1}(2xy-x^2)\,dx + (x^2+y^2)\,dy$$

$$= \int_0^1 (2x\,.\,x^2 - x^2)\,dx + (x^2+x^4)\,2x\,dx$$

$$= \int_0^1 (2x^3-x^2+2x^3+2x^5)\,dx$$

$$= \int_0^1 (4x^3-x^2+2x^5)\,dx$$

$$= \left[x^4-\frac{x^3}{3}+\frac{x^6}{3}\right]_0^1 = 1-\frac{1}{3}+\frac{1}{3} = 1.$$

$$\int_{c_2}(2xy-x^2)\,dx + (x^2+y^2)\,dy$$

$$= \int_1^0 (2\,.\,y^2\,.\,y - y^4)\,2y\,dy + (y^2+y^2)\,dy$$

$$= \int_1^0 (2xy-x^2)\,dx + (x^2+y^2)\,dy$$

$$= \int_1^0 (5y^4-2y^5+y^2)\,dy$$

$$= \left[y^5-\frac{y^6}{3}+\frac{y^3}{3}\right]_1^0$$

$$= 0-\left(1-\frac{1}{3}+\frac{1}{3}\right) = -1.$$

$$\therefore \oint_C (2xy-x^2)\,dx + (x^2+y^2)\,dy$$

$$= \oint_{C_1}(2xy-x^2)\,dx + (x^2+y^2)\,dy + \oint_{C_2}(2xy-x^2)\,dx + (x^2+y^2)\,dy$$

$= 1 - 1 = 0.$

Applying Green's theorem : $M = 2xy - x^2$, $N = x^2 + y^2$.

$$\frac{\partial N}{\partial x} = 2x \text{ and } \frac{\partial M}{\partial y} = 2x$$

$$\therefore \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} = 0 \qquad \text{... (1)}$$

$$\iint_E\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \int_{x=0}^{x=1}\left\{\int_{y=\sqrt{x}}^{y=x^2}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy\right\}dx \text{ [Fubini's theorem]}$$

$= 0$ (∵ from (1))

Green's theorem is verified.

Problem 51 :

Apply Green's theorem to evaluate $\oint_C$ ***{(y − sinx) dx + cosx dy}, where C is the triangle enclosed by the lines y = 0,*** $x = \frac{\Pi}{2}$, $\Pi y = 2x$.

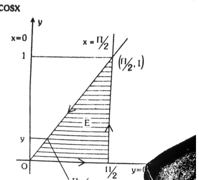
Sol :

Here $M = y - \sin x$, $N = \cos x$

$$\frac{\partial N}{\partial x} = -\sin x,\ \frac{\partial M}{\partial y} = 1$$

$$\therefore \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} = -\sin x - 1$$

By Green's theorem



$$\oint_C (y-\sin x)\,dx + \cos x\,dy$$

$$= \iint_E\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \int_{y=0}^{y=1}\left\{\int_{x=\Pi y/2}^{x=\Pi/2}(-\sin x-1)\,dx\right\}dy$$

$$= \int_{y=0}^{y=1}\left\{[\cos x - x]_{x=\Pi y/2}^{x=\Pi/2}\right\}dy$$

$$= \int_{y=0}^{y=1}\left\{\left(0-\frac{\Pi}{2}\right)-\left(\cos\frac{\Pi y}{2}-\frac{\Pi y}{2}\right)\right\}dy$$

$$= \left[\frac{\Pi y^2}{4}-\frac{\Pi}{2}y-\frac{2}{\Pi}\sin\left(\frac{\Pi y}{2}\right)\right]_{y=0}^{y=1}$$

$$= \frac{\Pi}{4}-\frac{\Pi}{2}-\frac{2}{\Pi}$$

$$= \frac{-\Pi}{4}-\frac{2}{\Pi}.$$

Problem 52 :

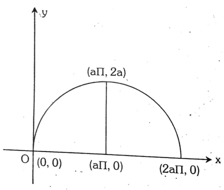
Find the area bounded by one arch of the cycloid $x = a(\theta-\sin\theta)$, $y = a(1-\cos\theta)$, $a > 0$ ***and the x-axis.***

Sol :

One arch of the cycloid is in between $\theta = 0$, $\theta = 2\Pi$.

Since $a > 0$

We know $\frac{1}{2}\left|\int_C (x\,dy - y\,dx)\right|$ represents the area of the region bounded by 'C' where C is given by



$x = a(\theta-\sin\theta)$, $y = a(1-\cos\theta)$

where $0 \le \theta \le 2\Pi$

$$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta.$$

$$\text{Area} = \frac{1}{2}\left|\int_0^{2\Pi}[a(1-\cos\theta)\,a\sin\theta - a(1-\cos\theta)\,a(1-\cos\theta)]\,d\theta\right|$$

$$= \frac{a^2}{2}\left|\int_0^{2\Pi}[\theta\sin\theta-\sin^2\theta-(1+\cos^2\theta-2\cos\theta)]\,d\theta\right|$$

$$= \frac{a^2}{2}\left|\int_0^{2\Pi}(\theta\sin\theta-2+2\cos\theta)\,d\theta\right|$$

$$= \frac{a^2}{2}\left|[\theta(-\cos\theta)+\sin\theta-2\theta+2\sin\theta]_0^{2\Pi}\right|$$

$$= \frac{a^2}{2}|-2\Pi-4\Pi|$$

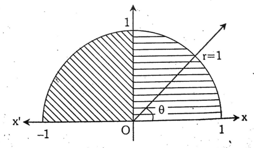
$= 3\Pi a^2$ sq. units.

Problem 53 :

Evaluate $\iint_E e^{x^2+y^2}\,dy\,dx$, ***where E is the semi-circular region bounded by the x-axis and the curve*** $y = \sqrt{1-x^2}$.

Sol :

There is no direct procedure to integrate $e^{x^2+y^2}$ either w.r.t 'x' or w.r.t. 'y' in the cartesian plane (xy plane). If we apply polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, x^2+y^2 becomes r^2 and from the Jacobian we get r; so that e^{r^2} r can be integrated w.r.t 'r'.



Since $r = 1$ for every point (r, θ) on the semi-circle being the boundary of the region E, the limits of r are $r = 0$ and $r = 1$.

The limits of θ are; $\theta = 0$ to $\theta = \Pi$ (rotation starts from $\overline{OX}$.

Since $J = r$, we have,

$$\iint_E e^{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\theta=\Pi}\left\{\int_{r=0}^{r=1}e^{r^2}\,r\,dr\right\}d\theta$$

$$= \int_{\theta=0}^{\theta=\Pi}\left\{\left[\frac{1}{2}e^{r^2}\right]_{r=0}^{r=1}\right\}d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=\Pi}(e-1)\,d\theta$$

$$= \frac{1}{2}[(e-1)\theta]_{\theta=0}^{\theta=\Pi}$$

$$= \frac{1}{2}[(e-1)\Pi] = \frac{\Pi}{2}(e-1)$$

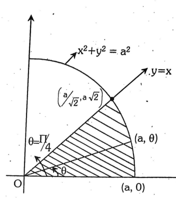
Problem 54 :

Evaluate $\iint_E \frac{xy(x+y)^2}{x^2+y^2}\,dx\,dy$, ***where E is bounded by y = 0, y = x,*** $x^2+y^2 = a^2$ ***in the first quadrant.***

Sol :

Since the slope of the line $y = x$ is '1'

∴ The bounding line $y = x$ makes the angle $\frac{\Pi}{4}$ with $\overline{Ox}$



The limits of θ are ; $\theta = 0$ and $\theta = \frac{\Pi}{4}$

The limits of r are ; $r = 0$ and $r = a$

$$\therefore \iint_E \frac{xy(x+y)^2}{x^2+y^2}\,dx\,dy$$

$$= \int_{\theta=0}^{\theta=\frac{\Pi}{4}}\left\{\int_{r=0}^{r=a}\frac{r^2\sin\theta\cos\theta\,r^2(\sin\theta+\cos\theta)^2}{r^2}\,r\,dr\right\}d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\Pi}{4}}\left\{\int_{r=0}^{r=a}\sin\theta\cos\theta(1+\sin 2\theta)\,r^3\,dr\right\}d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\Pi}{4}}\left\{\sin\theta\cos\theta(1+\sin 2\theta)\left[\frac{r^4}{4}\right]_{r=0}^{r=a}\right\}d\theta$$

$$= \frac{a^4}{4}\int_{\theta=0}^{\theta=\frac{\Pi}{4}}\frac{1}{2}(\sin 2\theta+\sin^2 2\theta)\,d\theta$$

$$= \frac{a^4}{8}\int_{\theta=0}^{\theta=\frac{\Pi}{4}}(\sin\alpha+\sin^2\alpha)\left(\frac{1}{2}d\alpha\right)\left[2\theta=\alpha, d\theta=\frac{d\alpha}{2}\right]$$

$$= \frac{a^4}{16}\left[[1-\cos\alpha]_0^{\Pi/2}+\frac{1}{2}\cdot\frac{\Pi}{2}\right]$$

$$= \frac{a^4}{16}\left[1+\frac{\Pi}{4}\right]$$

$$= \frac{a^4}{64}(4+\Pi)$$

Problem 55 :

Find the area of the surface cut from the cylinder $x^2+z^2 = a^2$ ***by the cyclinder*** $x^2+y^2 = a^2$.

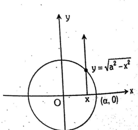
Sol :

The surface where area is to be found out can be written as

$z^2 = a^2 - x^2$ ... (1)

$$\therefore 2z\frac{\partial z}{\partial x} = -2x \text{ and}$$

$$2z\frac{\partial z}{\partial x} = 0$$



$$\therefore \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

$$= \sqrt{1+\frac{x^2}{z^2}+0}$$

$$= \sqrt{\frac{z^2+x^2}{z}} = \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2}} \text{ (from (1))}$$

Since (1) is symmetric about $z = 0$, the required area

$$\|S\| = 2\iint_E \frac{a}{\sqrt{a^2-x^2}}$$

where $E : x^2+y^2 = a^2$

$$= 8\int_0^{x=a}\left[\int_{y=0}^{y=\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}dy\right]dx$$

$$= 8a\int_{x=0}^{x=a}\frac{1}{\sqrt{a^2-x^2}}\left[\sqrt{a^2-x^2}\right]dx$$

$= 8a\,[x]_0^a$

$= 8a^2.$

Problem 56 :

Find the area of the surface cut from the bottom of the paraboloid $x^2+y^2+z^2 = 0$ ***by the plane z = 4.***

Sol :

We write the surface at

$$z = x^2+y^2, \text{ so that } \frac{\partial z}{\partial x} = 2x \text{ and } \frac{\partial z}{\partial y} = 2y.$$

$$\therefore \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+4x^2+4y^2}$$

The surface is cut by the plane $z = 4$ in a circle $x^2+y^2 = 4$.

$$\therefore \|S\| = \iint_E \sqrt{1+4x^2+4y^2}\,dx\,dy,\ E : x^2+y^2 \le 4$$

For the polar coordinates,

$$\|S\| = \int_{\theta=0}^{\theta=2\Pi}\left\{\int_{r=0}^{r=2}\sqrt{1+4r^2}\,r\,dr\right\}d\theta$$

$$= \frac{1}{12}\int_{\theta=0}^{\theta=2\Pi}\left\{(1+4r^2)^{3/2}\right\}_{r=0}^{r=a}d\theta$$

$$= \frac{1}{12}\int_{\theta=0}^{\theta=2\Pi}\left(17^{3/2}-1\right)d\theta$$

$$= \frac{1}{12}\left(17^{3/2}-1\right).2\Pi$$

$$= \frac{\Pi}{6}(17\sqrt{17}-1)$$

14. (a) Find the length of the cardiod r = a(1 − cos θ).

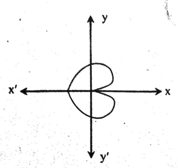
Sol :

$$\text{Length of the curve} = \int_0^{2\pi}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \quad ∵ f(\theta) = r = a(1-\cos\theta)$$

$$\Rightarrow \int_0^{2\pi}\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$\Rightarrow a\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$



$$\Rightarrow 2a\int_0^{2\pi}\sin\theta/2\,d\theta$$

$$\Rightarrow -2a.2(\cos\theta/2)_0^{2\pi}$$

$$\Rightarrow -4a(-1-1) = 8a$$

5. Evaluate $\int_C \frac{dx}{x+y}$ **where C is the curve** $x = at^2$, $y = 2at$ **where** $0 \le t \le 2$.

Sol :

Given $x = at^2$ $Y = 2at$

$dx = 2at\,dt$ $dY = 2a$

$$\text{Now sub in } \int_C\frac{dx}{x+y} = \int_0^2\frac{2at}{at^2+2at}$$

$$= \int_0^2\frac{2at\,dt}{at(t+2)}$$

$$= \int_0^2\frac{2}{t+2}dt$$

$= [2\log(t+2)]_0^2 \Rightarrow 2\log 4 - 2\log 2 = 2\log 2$

$= \log 4$

6. Evaluate $\iint_R xy\,dx\,dy$ **where R = [2, 5] × [1, 2].**

Sol :

Given $\iint_R xy$ x varies from 2 to 5

y varies form 1 to 2

$$\Rightarrow \int_2^5\int_1^2 xy\,dx\,dy = \int_2^5 x\left(\frac{y^2}{2}\right)_1^2 dx$$

$$= \int_2^5 x\left[\frac{4}{2}-\frac{1}{2}\right]dx = \frac{3}{2}\int_2^5 x\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_2^5 = \frac{3}{2}\left[\frac{25}{2}-\frac{4}{2}\right] = \frac{3}{2}\left(\frac{21}{2}\right) = \frac{63}{4}$$

14. a) Change the order of integration in the double integral $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ **and hence find the value.**

Answer :

The limits of integration are given by the straight line

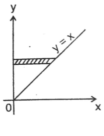
$y = x$, $y = \infty$, $x=0$ and $x = \infty$.

i.e., the region of integration is bounded by $x=0$, $y=x$ and an infinite boundary.

Hence taking the strips parallel to x-axis, the limits for y are from 0 to ∞.

Hence changing the order of integration, we have

$$\int_0^\infty\int_x^\infty\frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty\int_0^y\frac{e^{-y}}{y}\,dy\,dx$$



$$\Rightarrow \int_0^\infty\frac{e^{-y}}{y}[x]_0^y\,dy$$

$$\Rightarrow \int_0^\infty\frac{e^{-y}}{y}(y)\,dy$$

$$\Rightarrow \int_0^\infty e^{-y}\,dy = \frac{e^{-y}}{-1}\Big]_0^\infty$$

$\Rightarrow 1.$