VECTOR SPACES :

Definition : *Let 'V' be a non-empty set whose elements are called vectors. Let 'F'' be any set whose elements are called scalars whre (F, +, .) is a field.*

The set 'V' is said to be a vector space if

1. There is defined an internal composition in 'V'' called addition of vectors denoted by '+', for which (V, +) is an abelian group.
2. There is defined an external composition in 'V' over 'F', called the scalar multiplication in which $a\alpha \in V$ for all $a \in F$ and $\alpha \in V$.
3. The above two compositions satisfy the following postulates.
   i. $a(\alpha + \beta) = a\alpha + a\beta$
   ii. $(a + b)\alpha = a\alpha + b\alpha$
   iii. $(ab)\alpha = a(b\alpha)$
   iv. $1\alpha = \alpha$

$\forall\, a, b \in F$ and $\alpha, \beta \in V$ and 1 is the unity element of F.

Theorem 1 :

Let V(F) be a vector space. A non-empty set $W \subseteq V$.

The necessary and sufficient condition for W to be a subspace of V is $a, b \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in W$.

Proof :

Condition is necessary

W(F) is a subspace of V(F) $\Rightarrow$ W(F) is a vector space.

$\therefore$ Now $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$ and $b \in F$, $\beta \in W \Rightarrow b\alpha \in W$

Now $a\alpha \in W$, $b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Condition is Sufficient

Let W be the non-empty subset of V

Satisfying the given condition

i.e., $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ ... (1)

Taking a= 1, b = –1, and $\alpha, \beta \in W \Rightarrow 1\alpha + (-1)\beta \in W$

$\Rightarrow \alpha - \beta \in W$ [$\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

($H \subseteq G$ and $a, b \in H \Rightarrow ab^{-1} \in H$ then (H, 0) is subgroup of (G, 0)).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +)

$\Rightarrow$ (W, +) is an abelian group. Again taking b = 0

$a, O \in F$ and $\alpha, \in W \Rightarrow a\alpha + O\beta \in W \Rightarrow a\alpha \in W \Rightarrow a \in F$ and $\alpha \in W \Rightarrow a\alpha \in W$.

$\therefore$ W is closed under scalar multiplication.

The remaining postulates of vector space hold in W as $W \subseteq V$.

$\therefore$ W(F) is a vector space of V(F).

Theorem 2 :

Let V(F) be a vector space and let $W \subseteq V$. The necessary and sufficient conditions for W to be a subspace of V are :

i. $\alpha \in W$, $\beta \in W \Rightarrow \alpha - \beta \in W$

ii. $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$.

Proof :

Conditions are necessary

i. W is a vector subspace of V

$\Rightarrow$ W is a subgroup of (V, +) $\Rightarrow$ (W, +) is a group.

$\Rightarrow$ If $a, \beta \in W$ then $a - \beta \in W$

(ii) W is a subspace of V

$\Rightarrow$ W is closed under scalar multiplication

$\Rightarrow$ for $a \in F$, $\alpha \in W$; $a\alpha \in W$

Conditions are sufficient

Let W be a non-empty subset of 'V'' satisfying the two given conditions

$\alpha \in W$, $\alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \overline{O} \in W$ (by (i))

$\therefore$ The zero vector of 'V' is also the zero vector of W.

$\overline{O} \in W$, $- \in W \Rightarrow \overline{O} - \alpha \in W \Rightarrow (-\alpha) \in W$ (by(i))

$\Rightarrow$ additive inverse of each element of W is also in W.

Again $\alpha \in W$, $\beta \in W \Rightarrow \alpha \in W, (-\beta) \in W \Rightarrow \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$ (by (i))

i.e., W is closed under vector addition

AS $W \subseteq V$, all the elements of W are also the elements of V. Thereby vector addition in W will be associative and cimmutative. This implies that (W, +) is an abelian group.

Further by (ii), W is closed under scalar multiplication and other postulates of vector space hold in 'ω' as $W \subseteq V$

$\therefore$ W itself is a vector space under the operations of V.

Hence W(F) is a vector subspace of V(F).

Theorem 3 :

The union of two subspaces if and only if one is contained in the other.

Proof :

Let W_1 and W_2 be two subspaces of V(F)

Hence α cannot be expressed ...

Linear Dependence of Vectors

Definitiion

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly dependent (L.D.) set if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + ... a_n\alpha_n = \overline{O}$.

Linear Independence of Vectors

Definition

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly independent (L.I.) if every relation of the form.

$$a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \overline{O},\ a_1^{-1}\ s \in F$$

$$\Rightarrow a_1 = 0,\ a_2 = 0,\ ...,\ a_n = 0$$

Problem 6 :

Show that the system of vectors (1, 3, 2), (1, –7, –8) (2, 1, –1) of $V_3(R)$ is linearly dependent.

Sol :

Let $a, b, c \in R$, then a (1, 2, 3) + b (1, –7, –8) + c (2, 1, –1) = $\overline{O}$

$\Rightarrow$ (a + b + 2c, 3a – 7b + c, 2a – 8b – c) = (0, 0, 0)

$\Rightarrow$ (a + b + 2c = 0, 3a – 7b + c = 0, 2a – 8b – c = 0)

$\Rightarrow$ a = 3, b = 1, c = –2 [$\because$ The given vectors are linearly dependent]

Problem 7 :

Show that the system of vectors (1, 2, 0), (0, 3, 1), (-1, 0, 1) of $V_3(Q)$ is linearly independent where Q is the field of rational numbers.

Sol :

Let $x, y, z \in Q$ then x (1, 2, 0) + y (0, 3, 1) + z (–1, 0, 1) = $\overline{O}$

$\Rightarrow$ (x – z, 2x + 3y, y + z) = (0, 0, 0)

$\Rightarrow$ x – z = 0, 2x + 3y = 0, y + z = 0

$\Rightarrow$ x = 0, y = 0, z = 0.

Hence the system is linearly independent.

independent.

Theorem 9 :

Let V(F) be a vector space and $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a finite subset of non-zero vectors of V(F). Then S is linearly dependent if and only if some vector $\alpha_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceeding vectors.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_k ... \alpha_n\}$ is linearly dependent. Then there exist $\alpha_1, \alpha_2, ... \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + ... a_n\alpha_n = \overline{O}$.

Let k be the greatest suffix of a for which $\alpha_k \ne 0$.

Then $a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k + 0\alpha_{k+1} + ... + 0\alpha_n = \overline{O}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k = \overline{O}$

Now suppose k = 1, (1) $\Rightarrow a_1\alpha_1 = \overline{O}$

But $a_1 \ne 0$, so $\alpha_1 = 0$, which contradicts that each element of S is a non-zero vector.

Hence k > 1, i.e., $2 \le k \le n$.

Again from (1) we have,

$a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1}$

$\Rightarrow a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1})$

$\Rightarrow \alpha_k = (-a_k^{-1}a_1)\alpha_1 + (-a_k^{-1}a_2)\alpha_2 + ... + (-a_k^{-1}\alpha_{k-1})\alpha_{k-1}$

= (linear combination of its preceding vectors)

Conversely

Let some $\alpha_p \in S$ be expressible as a linear combination of its preceding vectors i.e., for $b_1, b_2, ... b_{p-1} \in F$.

$\alpha_p = b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1} + (-1)\alpha_p = \overline{O}$

$\Rightarrow$ the set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

[(–1) is a non-zero coefficient]

Hence the superset $S = \{-\alpha_1, \alpha_2, ..., \alpha_p - \alpha_n\}$ is linearly dependent

Note

If β is a linear combination of the set of vectors $\alpha_1, \alpha_2, ..., \alpha_n$ then the set of vectors $\{b, \alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

Basis of Vector Space

Definition

A subset S of a vector space V(F) is said to be the basis of V if

i. S is linearly independent

ii. The linear span of S is V i.e., L(S) = V

Note

A vector space may have more than one basis.

Basis Extension

Theorem 10 :

Let V(F) be a finite dimensional vector space and $S = \{\alpha_1, \alpha_2, ... \alpha_m\}$ a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_m\}$ is linearly independent subset of V

Since V(F) is finite dimensional it has a finite basis B.

Let $B = \{\beta_1, \beta_2, ... \beta_n\}$

Now consider the set $S_1 = \{\alpha_1, \alpha_2, ... \alpha_m, \beta_1, \beta_2, ... \beta_n\}$ written in this order.

Clearly L(S) = V.

Each α can be expressed as a *l*.c. of β's as B is the basis of V

$\Rightarrow S_1$ is linearly dependent.

Hence some vector in S_1 can be expressed as a *l*.c. of its preceding vectors. This vector cannot be any of α's, since S is linearly independent. So, this vector must be some β_i consider now the set.

$S_2 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_{i-1}, \beta_{i+1} ... \beta_n\} = S_1 - \{\beta_i\}$ obviously $L(S_2) = L(S_1) = V$

If S_2 is linearly independent, then S_2 forms a basis of V and it is the extended set.

If S_2 is linearly dependent then continue this procedure till we gt a set $S_k \subset S$ such that S_k is linearly independent.

$\therefore L(S_k) = L(S) = V$

S_k will be extended set of S forming a basis of V.

Theorem 11 :

V is a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ... \beta_n$ then any independent set of vectors in V is finite and contains no more than n elements.

Proof :

Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a linearly independent subset of vector space V(F)

Let $S = \{\beta_1, \beta_2, ..., \beta_n\}$ and L(S) = V.

Then any vector of V is *l*.c. of the elements of S.

Let $\alpha_m \in V$ be a linear combination of its preceding vectors.

This vector must be one among β's.

Let it be β_i

$\therefore \beta_p$ = *l*.c. of $\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}$

If $S_1 = \{\alpha_m, \beta_1, \beta_2, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$

Then L(S1) = L(S) = V.

Again the vector $\alpha_{m-1} \in V$ is a *l*.c. of the elements of S_1.

$\therefore \alpha_{m-1}, \alpha_m, \beta_1, \beta_2, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n$ are L.D.

$\therefore$ There exists a vector in this set which is a *l*.c. of its predecessors.

Such vector is one of β's as α_{m-1}, am are L.I.

Let it be β_k.

If $S_2 = \{\alpha_{m-1}, \alpha_m, \beta_1, \beta_{k-1}, \beta_{k+1}, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$

Then β_k = *l*.c. of the elements of S_3.

$\therefore L(S) = L(S_1) = L(S) = V$.

Continuing this process further for m – 3 times.

We get $S_{m-1} = \{\alpha_2, \alpha_3, ... \alpha_m, \beta_{i1}, \beta_{i2}, \beta_{in-m+1}\}$.

So that $L(S_{m-1}) = V$

This set consists of atleast one β_i.

otherwise $S_{m-1} = \{\alpha_1, \alpha_2, ..., \alpha_m\}$

So that α_1 is a *l*.c. of $\alpha_2, \alpha_3, ..., \alpha_m$.

This cannot happen as the set $\alpha_2, \alpha_3, ..., \alpha_m$ is L.I.

Hence S_{m-1} consists of atleast one β_i

$\Rightarrow n - m + 1 \ge 1 \Rightarrow n \ge m$.

$\therefore$ The no. of elements of the L.I. set in $V \le n$.

Theorem 12 :

Let W_1 and W_2 be two subspaces of a finite dimensional vector space V(F). Then dim $(W_1 + W_2)$ = dim W_1 + dim W_2 – dim $(W_1 \cap W_2)$.

Proof :

Since W_1 and W_2 are subspaces of V, $W_1 + W_2$ and $W_1 \cap W_2$ are also subspace of V.

Let dim $(W_1 \cap W_2)$ = k and $S = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ be a basis of $W_1 \cap W_2$.

Clearly $S \subseteq W_1$ and $S \subseteq W_2$ and S is L.I.

Since S is L.I and $S \subseteq W_1$, the set S can be extended to form a basis of W_1.

Let $B_1 = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m\}$ be a basis of W_1

$\therefore$ dim W_1 = k + m.

Again since S is L.I. and $S \subseteq W_2$, the set S can be extended to form a basis of W_2.

Let $B_2 = \{\gamma_1, \gamma_2, ..., \gamma_k, \beta_1, \beta_2, ..., \beta_m\}$ be a basis of W_2

$\therefore$ dim W_2 = k + t

$\therefore$ dim W_1 + dim W_2 – dim $(W_1 \cap W_2)$

= (k + m) + (k + t) – k

= k + m + t.

Now we shall prove that the set

$S' = \{\gamma_1, \gamma_2, ..., \gamma_k, \alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_t\}$ is a basis of $W_1 + W_2$ and hence.

dim $(W_1 + W_2)$ = k + m + t.

(i) To prove that S' is L.I.

Now $c_1\gamma_1 + c_2\gamma_2 + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + ... + b\beta = 0$ ... (1)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = -c_1\gamma_1 - ... - c_k\gamma_k - a_1\alpha_1 - ... - a_m\alpha_m$

= *l*.c. of elements of B1 and ($\because \in W_1$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1$

Again $0\gamma_1 + 0\gamma_2 + 0\gamma_k + b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t$

= *l*.c. of elements of B_2 and ($\because \in W_2$)

$\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_2$

$\therefore$ by (1) and (2) $\Rightarrow b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t \in W_1 \cap W_2$

Hence it can be expressed as a *l*.c. of the elements of the basis S of $W_1 \cap W_2$.

Let $b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + ... + d_k\gamma_k$.

$\therefore b_1\beta_1 + b_2\beta_2 + ... + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 - ... - d_k\gamma_k = \overline{O}$

$\Rightarrow$ (*l*.c. of elements of basis B_2) = $\overline{O}$

$b_1 = 0, b_2 = 0 ... b_t = 0, d_1 = 0, ... k_t = 0$.

Substituting in I we have,

$c_1\gamma_1 + c_2\gamma_2 + ... + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + ... + a_m\alpha_m = \overline{O}$

$\Rightarrow$ (*l*.c. of elements of basis B_1) = $\overline{O}$

$\Rightarrow c_1 = 0, c_2 = 0 ... c_k = 0, a_1 = 0, a_2 = 0 ... a_m = 0$

Thus relation I implies that,

$c_1 = c_2 = ... = c_k = a_1 = a_2 = ... \alpha_m = 0, b_1 = b_2 = ... = b_t = 0$

$\therefore$ S' is L.I. set

(ii) To prove $L(S') = W_1 + W_2$

Every vector of S' is a vector of $W_1 + W_2$.

$\therefore L(S') \le W_1 + W_2$

Let $S \in W_1 + W_2$

$\therefore \delta = \alpha + \beta$ where $\alpha \in W$, $\beta \in W_2$

δ = (*l*.c. of elements of B_1) + (*l*.c. of elements of B_2)

= (*l*.c. of γ's and α's) + (*l*.c. of γ's and β's)

= (*l*.c. of γ's, α's and β's.

= (l.c. of elements of S'

$\therefore \delta \in L(S')$

$\therefore W_1 + W_2 \le L(S')$

$\therefore L(S') = W_1 + W_2$

Hence S' is the basis of $W_1 + W_2$

$\therefore$ dim $(W_1 + W_2)$ = k + m + t.

Hence the theorem.

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14. ***Describe explicitly the linear transformation $T : R^2 \to R$ such that T(2, 3) = (4, 5) and T(1, 0) = (0, 0)***

Sol :

First of all we have to show that the vectors (2, 3) and (1, 0) are L.I.

Let $a(2, 3) + b(1, 0) = \overline{O}$

$\Rightarrow$ (2a + b, 3a + 0) = (0, 0)

$\Rightarrow$ 2a + b = 0, 3a = 0

$\Rightarrow$ 2a = 0, b = 0

$\therefore$ S = {(2, 3), (1, 0)} is L.I.

Let us prove that $L(S) = R^2$

Let $(x, y) \in R^2$ and (x, y) = a (2, 3) + b(1, 0) – (2a + b, 3a)

$\Rightarrow$ 2a + b = x, 3a = y

$\Rightarrow a = \frac{y}{3}$; $b = \frac{3x - 2y}{3}$

Hence S spans R^2.

$$\text{Now}\quad T(x, y) = T\left[\frac{y}{3}(2, 3) + \frac{3x - 2y}{3}(1, 0)\right]$$

$$= \frac{y}{3}T(2, 3) + \frac{3x - 2y}{3}T(1, 0)$$

$$= \frac{y}{3}(4, 5) + \frac{3x - 2y}{3}(0, 0)$$

$$= \left(\frac{4y}{3}, \frac{5y}{3}\right)$$

$\therefore$ This is the required transformation.

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Theorem 16 :

Rank - Nullity theorem

Statement

Let U(F) and V(F) be two vector spaces and $T : U \to V$ be a linear transformation. Let U be finite dimensional

Then P(T) + ν(T) = dim U

i.e., rank (T) + nullity (T) = dim U

Proof :

The null space N(T) is a subspace of finite dimensional space U(F).

$\Rightarrow$ N(T) is finite dimensional

Let $S = \{\alpha_1, \alpha_2, ... \alpha_k\}$ be a basis set of N(T)

$\therefore$ dim N(T) = ν(T) = k

$\therefore T(\alpha_1) = \hat{\overline{O}}, T(\alpha_2) = \hat{\overline{O}} ... T(\alpha_k) = \hat{\overline{O}}$ ... (1)

($\hat{\overline{O}} \in V$)

As S is L.I. it can be extended to form a basis of U(F)

$\therefore$ dim U = k + m.

Now we show that the set of images of additional vectors.

S2 = $\{T(\theta_1), T(\theta_2), ... T(\theta_m)\}$ is a basis of R(T) clearly $S_2 \subseteq R(T)$

(i) To prove S_2 is L.I

Let $a_1, a_2, ... a_m \in F$ be such that

$a_1T(\theta_1) + a_2T(\theta_2) + ... + a_mT(\theta_m) = \hat{\overline{O}}$

$\Rightarrow T[a_1\theta_1 + a_2\theta_2 + ... + a_m\theta_m] = \hat{\overline{O}}$

$\Rightarrow a_1\theta_1 + a_2\theta_2 + ... + a_m\theta_m \in N(T)$, null space of T.

But each vector is N(T) is a *l*.c. of elements of basis S.

$\therefore$ For some $b_1, b_2, ... b_m \in F$,

Let $a_1\theta_1 + a_2\theta_2 + ... + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + ... + b_k\alpha_k$

$\Rightarrow a_1\theta_1 + ... + a_m\theta_m - b_1\alpha_1 - ... - b_k\alpha_k = \hat{\overline{O}}$

$\Rightarrow a_1 = 0, a_2 = 0 ... a_m = 0, b_1 = 0, ... b_k = 0$

($\because S_1$ is L.I)

$\Rightarrow S_2$ is L.I. set

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space R(T), then there exists $\alpha \in U$ such that $T(\alpha) = \beta$

Now $\alpha \in U \Rightarrow$ there exists c's, d's $\in$ F such that

$\alpha = c_1\alpha_1 + c_2\alpha_2 - ... + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + ... + d_m\theta_m$

$\Rightarrow T(\alpha) = T[c_1\alpha_1 + ... + c_k\alpha_k + d_1\theta_1 + ... + d_m\theta_m]$

$= c_1T(\alpha_1) + ... + c_kT(\alpha_k) + d_1T(\theta_1) + ... + d_mT(\theta_m)$

$\Rightarrow \beta = d_1T(\theta_1) + d_2T(\theta_2) + ... + d_mT(\theta_m)$ (by (1))

$\Rightarrow \beta \in L(S_2)$

$\therefore S_2$ is a basis of R(T) and dim R(T) = m

Thus dim R(T) + dim N(T) = m + k = dim U

i.e., ρ(T) + ν(T) = dim U.

Problem 17 :

Let $T : V_2 \to V_3$ be defined by $T(x, y) = (x + y, 2x - y, 7y)$ Find $[T : B_1, B_2]$ where B_1 and B_2 are the standard bases of V_2 and V_3.

Sol :

B_1 is the standard basis of V_2 and B_2 that of V_3

$\therefore \quad B_1 = \{(1, 0), (0, 1)\}$

$B_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$\therefore \quad T(1, 0) = (1, 2, 0) = 1(1, 0, 0) + 2(0, 1, 0) + 0(0, 0, 1)$

$T(0, 1) = (1, -1, 7) = 1(1, 0, 0) - 1(0, 1, 0) + 7(0, 0, 1)$

$\therefore$ The matrix of T relative of B_1 and B_2 is

$$[T : B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

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Problem 18 :

Let $T : R^3 \to R^2$ be the linear transformations defined by $T(x, y, z) = (3x + 2y - 4x, x - 5y + 3x)$. Find the matrix of T relative to the bases $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$, $B_2 = \{(1, 3), (2, 5)\}$

Sol :

Let $(a, b) \in R^2$ and let $(a, b) = p(1, 3) + q(2, 5)$

$= (p + 2q, 3p + 5q)$

$\Rightarrow p + 2q = q, \; 3p + 5q = b$

Solving $p = -5a + 2b, \; q = 3a - b$

$\therefore \quad (a, b) = (-5a + 2b)(1, 3) + (3a - b)(2, 5)$

Now $T(1, 1, 1) = (1, -1) = -7(1, 3) + 4(2, 5)$

$T(1, 1, 0) = (5, -4) = -33(1, 3) + 19(2, 5)$

$T(1, 0, 0) = (3, 1) = -13(1, 3) + 8(2, 5)$

$\therefore$ The matrix of L.T. relative to B_1 and B_2

$$[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

Problem 19 :

Let T be the linear operator r on R^2 defined by $T(x, y) = (4x - 2y, 2x + y)$. Find the matrix of T w.r.to the basis $T\{(1, 1), (-1, 0)\}$, Also verify $[T]_B\,[\alpha]_B = [T]\,(\alpha)_B \;\forall \alpha \in R^2$.

Sol :

Let $(a, b) \in R^2$. Then;

$(a, b) = p(1, 1) + q(-1, 0) = (p, -q, p)$

$\Rightarrow a = p - q$ and $b = p$

$\Rightarrow p = b$ and $q = b - a$

$\therefore \quad (a, b) = b(1, 1) + (b - a)(-1, 0)$ ... (1)

(i) Given transformation is $T(x, y) = (4x - 2y, 2x + y)$

$\therefore \quad T(1, 1) = (2, 3) = 3(1, 1) + 1(-1, 0)$ ... (by (1))

$T(-1, 0) = (-4, 2) = -2(1, 1) + 2(-1, 0)$

$$\therefore \quad [T : B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

(ii) Let $a \in R^2$ where $\alpha = (a, b)$

$\therefore \quad \alpha = (a, b) = b(1, 1) + (b - a)(-1, 0)$

$$\therefore \quad [\alpha]_B = \begin{bmatrix} b \\ b - a \end{bmatrix}$$

$$\therefore \quad [T]_B\,[\alpha]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} b \\ b - a \end{bmatrix} = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix} \quad \text{... (1)}$$

Again $T(\alpha) + T(a, b) = (4a - 2b, 2a + b)$

Let $(4a - 2b, 2a + b) = x(1, 1) + y(-1, 0) = (x, -y, x)$

$\Rightarrow 4a - 2b = x - y$ and $2a + b = x$

$\Rightarrow x = 2a + b$ and $y = -2a + 3b$.

Hence $T(\alpha) = T(a, b) = (2a + b)(1, 1) + (-2a + 3b)(-1, 0)$

$\therefore$ The matrix of $T(\alpha)$ w.r.to the base B is

$$[T(\alpha)]B = \begin{bmatrix} 2a + b \\ -2a + 3b \end{bmatrix} \quad \text{... (2)}$$

$\therefore$ from (1) and (2) $[T]_B\,[\alpha]_B = [T(\alpha)]_B$.

1. *Define a subspace. Prove that the intersection of two subspaces is again a subspace.*

Ans :

Subspace

Let V(F) be a vector space and $W \subseteq V$. Then W is said to be a subspace of v if w itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.

Proof :

W_1 and W_2 are subspace V(F)

$\Rightarrow \bar{0} \in W_1$ and $\bar{0} \in W_2$ $\quad \therefore W_1 \cap W_2 \neq \emptyset$

$\Rightarrow \bar{0} \in W_1 \cap W_2$ $\quad \therefore \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$

Now $a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

$\therefore \quad a\alpha + b\beta \in W_1 \cap W_2$

$\therefore \quad W_1 \cap W_2$ is a subspace of V(F)

$$\Rightarrow i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - j\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + k\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$\Rightarrow 0.$

SECTION - B (4 × 16 = 64)

UNIT - I

9. a) Let V(F) be a finite dimensional vector space. Then show that any two bases of V have the same number of elements.

Sol. :

Basis of a vector space

A subset S of a vector space V(F) is said to be the basis of V, if

(i) S is linearly independent

(ii) the linear span of S is V i.e., L(S) = V

Proof :

Let S_m and S_n be the two bases of V(F) where $S_m = \{\alpha_1, \alpha_2, \dots \alpha_m\}$ $S_n = \{\beta_1, \beta_2, \dots \beta_m\}$ obviously both S_m and S_n are L.I subsets of V.

(i) Consider S_m as the basis of V and S_n and L.I set

$\Rightarrow L(S_m) = V$ and $n(S_m) = m$

$\therefore \quad S_n$ can be extended to be a basis of V

$\Rightarrow n \leq m$

(ii) Consider S_n as the basis of V and S_m as L.I set

$\Rightarrow L(S_n) = V$ and $n(S_n) = n$

$\therefore \quad S_m$ can be extended to be a basis of V

$\Rightarrow m \leq n$

But both S_m and S_n are bases of V

$\therefore \quad n \leq m$ and $m \leq n \Rightarrow m = n$

Thus any two bases of V have the same number of elements.

b) Show that every non empty subset of a linearly independent set of vectors is linearly independent.

Sol. :

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_m\}$ be a linearly independent set of vectors.

Let us consider the subset $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_k\}$ where $1 \leq k \leq m$.

Now for some scalars consider the linear combination

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k = \bar{0}$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_k\alpha_k + 0_{k+1}\alpha_{k+1} + \dots + 0_m\alpha_m = \bar{0}.$$

Q $S_1 \subset S$ which is a linearly independent set.

$\therefore \quad a_2 = a_2 = \dots = a_k = 0$

$\therefore \quad S_1$ is a linearly independent set.