3, 15.

3 **Theorem 22 :**

8 ***Cayley – Hamilton Theorem***

Statement :

Every square matrix satisfies its characteristic equation

Proof :

Let A be a n-rowed square matrix i.e., Let $A = [a_{ij}]_{n\times n}$

The characteristic equation of A is given by

$$|A-\lambda I| = f(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \Lambda & a_{1n} \\ a_{21} & a_{22}-\lambda & \Lambda & a_{2n} \\ M & M & M & M \\ a_{n1} & a_{n2} & \Lambda & a_{nn}-\lambda \end{vmatrix}$$

$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n]$ where a_i's $\in F$

Since all the elements of $(A-\lambda I)$ are at most of Ist degree in λ, all the elements of adj $(A-\lambda I)$ are polynomials in λ of degree $(n-1)$ or less and hence adj $(A-\lambda I)$ can be expressed as a matrix polynomial in λ.

Let adj $(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-2}\lambda^1 + B_{n-1}$

where $B_0, B_1, B_2, \dots B_{n-1}$ are n - rowed square matrices.

Now $(A-\lambda I)$ adj $(A-\lambda I) = |A-\lambda I|\, I$ (∵ A(adj A) = |A|I

$\Rightarrow (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-2}\lambda + B_{n-1})$

$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n]\, I$

Comparing like coefficients of like powers of λ,we obtain

$-B_0 = (-1)^n I$

$AB_0 - B_1 = (-1)^n a_1 I,$

$AB_1 - B_2 = (-1)^n a_2 I,$

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$AB_{n-1} = (-1)^n a_n I$

Premultiplying the above equations successively by A^n, A^{n-1}, ... I and adding, we obtain

$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \dots + (-1)^n a_n I$

$\Rightarrow (-1)^n [A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI] = 0$

$\Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI = 0$

$\Rightarrow$ A satisfies its characteristic equation

(0 is zero matrix)

4 **Problem 23 :**

If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ ***verify Cayley - Hamilton theorem.***

Hence find A^{-1}.

Sol :

The characteristic equation of A is $|A-\lambda I| = 0$

i.e., $\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$

i.e., $(2-\lambda)[-6-3\lambda+2\lambda+\lambda^2] - 5[-2-\lambda] - 1[3-6+2\lambda] = 0$

i.e., $(2-\lambda)[-6-\lambda+\lambda^2] + 10 + 5\lambda + 3 - 2\lambda = 0.$

i.e., $-12 - 2\lambda + 2\lambda^2 + 6\lambda + \lambda^2 - \lambda^3 + 13 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$

i.e., $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$

To prove that A satisfies $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$, the characteristic equation of A

Now $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$

$A^3 = A^2.A \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & 7 \end{bmatrix}$

Now $A^3 - 3A^2 - 7A - I$

$= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & 7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix}$

$+ \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

∴ A satisfies its characteristic equation.

∴ $A^3 - 3A^2 - 7A - I = 0 \Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = 0$

(Here $|A| \neq 0$)

$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - 3A - 7I$

∴ $A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ -15 & -9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$

∴ $AA^{-1} = I$

Problem 24 :

Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & \cancel{1}\,2 \end{bmatrix}$ ***by using Cayley - Hamilton theorem.***

Sol :

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

∴ $A^2 = \begin{bmatrix} -1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$

The characteristic equation of A is

$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

i.e., $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

i.e., $(1-\lambda)[2-\lambda-2\lambda+\lambda^2-1] + 2(-1) = 0$

i.e., $(1-\lambda)(1-3\lambda+\lambda^2) - 2 = 0.$

i.e., $\lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0$

By Cayley - Hamilton theorem, A satisfies its characteristic equation.

∴ $A^3 - 4A^2 + 4A + I = 0$

$\Rightarrow A^{-1}(A^3 - 4A^2 + 4A + I) = 0$ (∵ $|A|$ $-1 \neq 0$)

$\Rightarrow A^2 - 4A + 4I + A^{-1} = 0$

$\Rightarrow A^{-1} = -4I + 4A - A^2$

$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ 6 & -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$

∴ $AA^{-1} = I.$

Theorem 27 :

A square matrix A of order n over a field F is a diagonalizable if and only if A has linearly independent characteristic vectors in $V_n(F)$.

Proof :

A is diagonalizable

$\Rightarrow$ A is similar to a diagonal matrix D (say)

$\Rightarrow$ there exists an invertible matrix P such that $P^{-1}AP = D \Rightarrow AP = PD.$

If $\lambda_1, \lambda_2, \dots \lambda_n$ are the diagonal elements of D, then $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic values of D. Since similar matrices have the same characteristic values, $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic values of A.

Let $X_1, X_2, \dots X_n$ be the column vectors of P. Now equating corresponding columns of (1) on either side, we get $AX_i = \lambda_i X_i$, i = 1, 2, 3, ... n implying that X_i is a characteristic vector of A corresponding to the characteristic value λ_i. Since P is invertible, its n column vectors $X_1, X_2, \dots X_n$ are L.I. and belong to $V_n(F)$. Thus A has n L.I. vectors in $V_n(F)$.

Converse :

Let $X_1, X_2, \dots X_n$ be n L.I. characteristic vectors of A corresponding to the characteristic values $\lambda_1, \lambda_2, \dots \lambda_n$ respectively.

Let $P = (X_1, X_2, \dots X_n)$. Since the n columns of P are L.I. $|P| \neq 0$. Hence P^{-1} exists.

Consider $AP = A[X_1, X_2, \dots X_n] = [AX_1, AX_2, \dots AX_n]$

$= [\lambda_1X_1, \lambda_2X_2, \dots \lambda_nX_n] = [X_1, X_2, \dots X_n]\begin{bmatrix} \lambda_1 & 0 & L & 0 \\ 0 & \lambda_2 & L & 0 \\ L & L & L & L \\ L & L & L & L \\ 0 & 0 & L & \lambda_n \end{bmatrix}$

= PD where D = diag $(\lambda_1, \lambda_2, \dots \lambda_n)$

$\Rightarrow P^{-1}(AP) = P^{-1}(PD) = (P^{-1}P)D = D$

$\Rightarrow P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$

$\Rightarrow$ A is diagonalizable.

Problem 28 :

Determine the modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ ***verify that $P^{-1}AP$ is a diagonal matrix.***

Sol :

The characteristic equation of A is $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda = 5$, $\lambda = -3$ and $\lambda = -3$

When $\lambda = 5$ we have $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly for the values $\lambda = -3$ we can have two linearly independent eigen vectors.

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Observe that X_1, X_2 and X_3 are not orthogonal.

But they are linearly independent.

Consider $P = (X_1\ X_2\ X_3)$

∴ $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = Modal matrix of A

$\det P = 1(-1) - 2(2) + 3(0-1) = -1 - 4 - 3 = -8.$

∴ $P^{-1} = \dfrac{1}{\det P}(\text{adj } P) = \dfrac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

∴ $P^{-1}A = \dfrac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ +2 & 1 & 6 \\ -1 & -2 & -0 \end{bmatrix}$

$= \dfrac{-1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

∴ $P^{-1}AP = \dfrac{-1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ = diag (5, –3, –3)

Hence $P^{-1}AP$ is a diagonal matrix.

Norm or Length of a Vector

Definition

Let 'V' be an inner product space over the field F. The norm (length) of $\alpha \in V$ denoted by $\|\alpha\|$ is defined as the positive square root of $\langle\alpha, \alpha\rangle$

Norm or length of $\alpha \in V = \|\alpha\| = \sqrt{\langle\alpha,\alpha\rangle} \Rightarrow \|\alpha\|^2 = \langle\alpha, \alpha\rangle$

Unit Vector

Definition

Let V(F) be an inner produce space $\alpha \in V$ is called a unit vector if $\|\alpha\| = 1$.

If $\alpha \in V$ then $\dfrac{1}{\|\alpha\|}\alpha \in V$ is unit vector.

Theorem 29 : 6

In an inner product space V(F),

$|\langle\alpha, \beta\rangle| \leq \|\alpha\| . \|\beta\|$ for all $\alpha, \beta \in V$

Cauchy - Schwartz's inequality.

Sol :

Case (1)

Let $\alpha = \bar{0}$. Then $\langle\alpha, \beta\rangle = \langle\bar{0}, \beta\rangle = 0$ and

$\|\alpha\|^2 = \langle\alpha, \alpha\rangle = \langle\bar{0}, \bar{0}\rangle = 0$

∴ $|\langle\alpha, \beta\rangle| = 0$ and $\|\alpha\|\ \|\beta\| = 0$ $(\|\beta\| = 0)$

∴ $|\langle\alpha, \beta\rangle| = \|\alpha\|\ \|\beta\|.$

Case (2)

Let $\alpha = \bar{0}$

Then $\|\alpha\| > 0$ so that $\dfrac{1}{\|\alpha\|} > 0$

Take $\gamma \in V$ so that $\gamma = \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha$

Now $(\gamma, \gamma) = \left\langle \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha,\ \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha \right\rangle$

$= \langle\beta,\beta\rangle - \dfrac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\beta,\alpha\rangle - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\langle\alpha,\beta\rangle + \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\dfrac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\alpha,\alpha\rangle$

$= \|\beta\|^2 - \dfrac{\langle\beta,\alpha\rangle\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2} - \dfrac{\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle}{\|\alpha\|^2} + \dfrac{\langle\beta,\alpha\rangle\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}$

$\left(∵ \dfrac{\langle\alpha,\alpha\rangle}{\|\alpha\|^2} = 1\right)$

$= \|\beta\|^2 - \dfrac{\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle}{\|\alpha\|^2} = \|\beta\|^2 - \dfrac{\overline{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle}{\|\alpha\|^2}$

$(∵ \langle\beta,\alpha\rangle = \overline{\langle\alpha,\beta\rangle})$

But by the definition of the norm ; $(\gamma, \gamma) \geq 0$

∴ $\|\beta\|^2 - \dfrac{\langle\alpha,\beta\rangle\overline{\langle\alpha,\beta\rangle}}{\|\alpha\|^2} \geq 0 \Rightarrow \|\beta\|^2 \geq \dfrac{|\langle\alpha,\beta\rangle|^2}{\|\alpha\|^2}$

$(∴ Z\bar{Z} = |Z|^2)$

$\Rightarrow \|\beta\|^2\|\alpha\|^2 \geq |\langle\alpha,\beta\rangle|^2 \Rightarrow (\|\beta\|\ \|\alpha\|)^2 \geq |\langle\alpha,\beta\rangle|^2$

$(∵ \|\alpha\|^2 > 0)$

∴ $(\|\beta\|\ \|\alpha\| \geq |\langle\alpha, \beta\rangle|$

as $\|\beta\|\ \|\alpha\|$ and $|\langle\alpha, \beta\rangle|$ are non-negative

nce. $|\langle\alpha, \beta\rangle| \leq \|\alpha\|\ \|\beta\|$.

Theorem 30 :

Triangle Inequality

Statement : In an inner product space V(F), $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$ for all $\alpha, b \in V$.

Proof :

By definition of norm,

$\|\alpha+\beta\|^2 = \langle\alpha+\beta, \alpha+\beta\rangle = \langle\alpha,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle$

$= \|\alpha\|^2 + \langle\alpha,\beta\rangle + \overline{\langle\alpha,\beta\rangle} + \|\beta\|^2$

$= \|\alpha\|^2 + 2\,\text{Re}\,\langle\alpha,\beta\rangle + \|\beta\|^2$ (∵ $Z + \bar{Z} = 2\,\text{Re}\,Z$)

$\leq \|\alpha\|^2 + 2|\langle\alpha,\beta\rangle| + \|\beta\|^2$ (∵ Re Z ≤ |Z|)

$\leq \|\alpha\|^2 + 2\|\alpha\|\ \|\beta\| + \|\beta\|^2$ (∵ $|\langle\alpha,\beta\rangle| \leq \|\alpha\|\ \|\beta\|$)

$\leq (\|\alpha\| + \|\beta\|)^2$ ∴ $\|\alpha+\beta\|^2 \leq (\|\alpha\| + \|\beta\|)^2$

As both $\|\alpha+\beta\|$ and $\|\alpha\| + \|\beta\|$ are non-negative we have

$\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$

Theorem 31 :

Parallelogram Law

If α, β are two vectors in an inner product space V(F) then

$$\|\alpha-\beta\|^2 + \|\alpha+\beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$$

Proof :

$\|\alpha-\beta\|^2 = \langle\alpha-\beta, \alpha-\beta\rangle = \langle\alpha,\alpha\rangle - \langle\alpha,\beta\rangle - \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle$

$= \|\alpha\|^2 - \langle\alpha,\beta\rangle - \langle\beta,\alpha\rangle + \|\beta\|^2$

$\|\alpha+\beta\|^2 = \langle\alpha+\beta, \alpha+\beta\rangle = \langle\alpha,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle$

$= \|\alpha\|^2 + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \|\beta\|^2$

∴ $\|\alpha-\beta\|^2 + \|\alpha+\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$

∴ $\|\alpha-\beta\|^2 + \|\alpha+\beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$

Problem 32 :

If α, β are two vectors in an inner product space, then α, β are linearly dependent if and only if $|\langle\alpha,\beta\rangle| = \|\alpha\|\ \|\beta\|$.

Sol :

Let α, β be linearly dependent

Then either $\alpha = \bar{0}$ or $\beta = \bar{0}$ or $\alpha = a\beta$ where 'a' is a scalar.

When $\alpha = \bar{0}$: $\langle\alpha,\beta\rangle = \langle\bar{0},\beta\rangle = 0$ and $\|\alpha\| = 0$

∴ $|\langle\alpha,\beta\rangle| = \|\alpha\|\ \|\beta\|$

When $\beta = \bar{0}$: $\langle\alpha,\beta\rangle = \langle\alpha,\bar{0}\rangle = \overline{\langle\bar{0},\alpha\rangle} = 0$ and $\|\beta\| = 0$

∴ $|\langle\alpha,\beta\rangle| = \|\alpha\|\ \|\beta\|$

When $\alpha = a\beta$: $\langle\alpha,\beta\rangle = \langle a\beta,\beta\rangle = a\langle\beta,\beta\rangle$

$= a\|\beta\|^2$ and $\|\alpha\| = \|a\beta\| = |a|\ \|\beta\|$

∴ $|\langle\alpha,\beta\rangle| = \|\alpha\|\ \|\beta\|^2 = (|a|\ \|\beta\|)(\|\beta\|) = \|\alpha\|\ \|\beta\|$

Conversely,

Let $|\langle\alpha,\beta\rangle| = \|a\|\ \|\beta\|$

When $\alpha = \bar{0}$ the vectors α, β are linearly dependent

When $\alpha \neq \bar{0}$; we have $\|\alpha\| > 0$.

Consider the vector $\gamma = \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha$

$\langle\gamma,\gamma\rangle = \left\langle \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha,\ \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha \right\rangle$

$= \langle\beta,\beta\rangle - \dfrac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\beta,\alpha\rangle - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\langle\alpha,\beta\rangle$

$+ \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\dfrac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\alpha,\alpha\rangle$

$= \|\beta\|^2 - \dfrac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\beta,\alpha\rangle - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\langle\alpha,\beta\rangle$

$+ \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\overline{\langle\beta,\alpha\rangle}$

$(∵ \langle\alpha,\alpha\rangle = \|\alpha\|^2)$

$= \|\beta\|^2 - \dfrac{|\langle\alpha,\beta\rangle|^2}{\|\alpha\|^2}$

$= \|\beta\|^2 - \dfrac{\|\alpha\|^2\|\beta\|^2}{\|\alpha\|^2} = 0$

∴ $(\gamma, \gamma) = 0 \Rightarrow \gamma = \bar{0}$

$\Rightarrow \beta - \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha = \bar{0} \Rightarrow \beta = \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha$

$\Rightarrow \beta = a\alpha$ where $a = \dfrac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}$ is a scalar

∴ α, β are linearly dependent.

Problem 33 :

Two vectors α, β in an unitary space V(C) are such that $\langle\alpha,\beta\rangle = 0$ iff

$\|a\alpha + b\beta\|^2 = |a|^2\,|\alpha|^2 + |b|^2\,|\beta|^2\ \forall$ ***a, b*** $\in$ ***c***

Sol :

Let $\langle\alpha,\beta\rangle = 0$. Then $\overline{\langle\alpha,\beta\rangle} = 0$

$\|a\alpha + b\beta\|^2 = \langle a\alpha + b\beta, a\alpha + b\beta\rangle$

$= a\langle\alpha, a\alpha + b\beta\rangle + b\langle\beta, a\alpha + b\beta\rangle$

$= a[\bar{a}\langle\alpha,\alpha\rangle + \bar{b}\langle\alpha,\beta\rangle] +$

$b[\bar{a}\langle\beta,\alpha\rangle + \bar{b}\langle\beta,\beta\rangle]$

$= a\bar{a}\langle\alpha,\alpha\rangle + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\langle\beta,\alpha\rangle$

$+ b\bar{b}\langle\beta,\beta\rangle$

$= |a^2|\,\|\alpha\|^2 + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\langle\beta,\alpha\rangle$

$+ |b^2|\,\|\beta\|^2$ ... (1)

$= |a|^2\|\alpha\|^2 + 0 + 0 + |b|^2\|\beta\|^2$

$= |a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2.$

Conversely,

Let $\|a\alpha + b\beta\|^2 = |a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2,\ \forall\ a, b \in c$

Using (1) we have,

$|a|^2\|\alpha\|^2 + a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\overline{\langle\alpha,\beta\rangle} + |b|^2\|\beta\|^2$

$|a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2$

$\Rightarrow a\bar{b}\langle\alpha,\beta\rangle + b\bar{a}\overline{\langle\alpha,\beta\rangle} = 0\ \forall\ a, b \in c$ ... (2)

Take a= 1, b = 1 so that $\bar{a} = 1$, $\bar{b} = 1$

Then (2) $\Rightarrow (1)(1)\langle\alpha,\beta\rangle + (1)(1)\overline{\langle\alpha,\beta\rangle} = 0$

$\Rightarrow \langle\alpha,\beta\rangle + \overline{\langle\alpha,\beta\rangle} = 0$

$\Rightarrow 2\,\text{Re}\,\langle\alpha,\beta\rangle = 0 \Rightarrow \text{Re}\,\langle\alpha,\beta\rangle = 0$ (∵ $Z + \bar{Z} = 2\,\text{Re}\,Z$)

Take a = i, b = 1 so that $\bar{a} = -i$, b = 1

Then (2) $\Rightarrow i\langle\alpha,\beta\rangle - i\overline{\langle\alpha,\beta\rangle} = 0$

$\Rightarrow i[\langle\alpha,\beta\rangle - \overline{\langle\alpha,\beta\rangle}] = 0$

$$\Rightarrow \langle \alpha, \beta \rangle - \overline{\langle \alpha, \beta \rangle} = 0$$

$$\Rightarrow 2\,\text{Im}\,\langle \alpha, \beta \rangle = 0 \qquad (\because Z - \bar{Z} = 2\,\text{Im}\,\bar{Z})$$

Thus we have Re $\langle \alpha, \beta \rangle = 0$ and Im $\langle \alpha, \beta \rangle = 0$

Hence $\langle \alpha, \beta \rangle = 0$.

Problem 34 :

If u, v are two vectors in a complex inner product space with standard inner product then prove that 4 $\langle u, v \rangle$ $= \|u+v\|^2 - \|u-v\|^2 + i\,\|u+iv\|^2 - i\,\|u-iv\|^2$

Sol :

$$\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u+u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$$

$$= \|u\|^2 + \langle u,v \rangle + \langle v,u \rangle + \|v\|^2 \qquad \text{... (1)}$$

$$\|u-v\|^2 = \langle u-v, u-v \rangle = \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle$$

$$= \|u\|^2 - \langle u,v \rangle - \langle v,u \rangle + \|v\|^2 \qquad \text{... (2)}$$

$$\|u+iv\|^2 = \langle u+iv, u+iv \rangle = \langle u,v \rangle + \bar{i}\,\langle u,v \rangle + i\langle v,u \rangle + i\bar{i}\,\langle v,v \rangle$$

$$= \|u\|^2 - i\langle u,v \rangle + i\langle v,u \rangle + \|v\|^2 \quad (\because i\bar{i} = i(-i) = 1)$$

$$i\|u-iv\|^2 = i\|u\|^2 + \langle u,v \rangle - \langle v,u \rangle + i\,\|v\|^2 \qquad \text{... (3)}$$

$$\|u-iv\|^2 = \langle u-iv, u-iv \rangle = \langle u,v \rangle - \bar{i}\,\langle u,v \rangle - i\,\langle v,u \rangle + i\bar{i}\,\langle v,v \rangle$$

$$= \|u\|^2 + i\langle u,v \rangle - i\,\langle v,u \rangle + \|v\|^2$$

$$i\|u-iv\|^2 = i\|u\|^2 - \langle u,v \rangle + \langle v,u \rangle + i\|v\|^2 \qquad \text{... (4)}$$

From (1), (2), (3) and (4)

$$\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$$

$$= \{2(u,v) + 2\,\langle v,u \rangle\} + \{2\langle u,v \rangle - 2\langle v,u \rangle\} = 4\langle u,v \rangle$$

Theorem 35 :

The vectors α, β ***of real inner product space V(F) are orthogonal if and only if,*** $\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$.

Proof :

$$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$$

$$\Leftrightarrow \langle \alpha+\beta, \alpha+\beta \rangle = \|\alpha\|^2 + \|\beta\|^2$$

$$\Leftrightarrow \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle = \|\alpha\|^2 + \|\beta\|^2$$

$$\Leftrightarrow \|\alpha\|^2 + \langle \alpha, \beta \rangle + \overline{\langle \alpha, \beta \rangle} + \|\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$$

$$\Leftrightarrow \langle \alpha, \beta \rangle + \langle \alpha, \beta \rangle = 0 \quad (\because \langle \alpha, \beta \rangle \text{ is real } \overline{\langle \alpha, \beta \rangle} = \langle \alpha, \beta \rangle)$$

$$\Leftrightarrow \langle \alpha, \beta \rangle = 0$$

$\Leftrightarrow$ α, β are orthogonal vectors.

Theorem 36 :

GRAM-SCHIMDT ORTHOGONALISATION PROCESS

Statement : Every finite dimensional inner product space has an orthogonal basis.

Proof :

Let V(F) be an inner product space of dimension n and B = $\{\beta_1, \beta_2, \dots \beta_n\}$ be a basis for V.

Now we shall obtain an orthonormal set B' = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ in V(F) with the help of the vectors in B by means of a construction known as Gram-Schmidt orthogonalisation process.

The main idea behind the construction by using induction on the dimension n of V.

Since B is a basis of V, B is linearly independent.

$$\Rightarrow \beta_i \neq \bar{0} \quad \text{for} \quad i = 1, 2, \dots n$$

$$\text{Let} \quad \alpha_1 = \frac{\beta_1}{\|\beta_1\|}$$

Then we have $(\alpha_1, \alpha_1) = 1$ and α_1 is a linear combination of β_1

Thus $\{\alpha_1\}$ is an orthonormal set which is a basis of V such that $L(\{\alpha_1\}) = L(\{\beta_1\}) = V$ when $\dim V = n = 1$.

Suppose that $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ where $1 \le m < n$ has been constructed as an ortho-normal basis of V when dim V = m.

Then $L(\{\alpha_1, \alpha_2, \dots \alpha_m\}) = L(\{\beta_1, \beta_2, \dots \beta_m\}) = V$

and each α_i (i = 1, 2, ... m) is a linear combination of $\beta_1, \beta_2, \dots \beta_m$

Consider the vector

$$\gamma_{m+1} = \beta_{m+1} - \langle \beta_{m+1}, \alpha_1 \rangle\, \alpha_1 - \langle \beta_{m+1}, \alpha_2 \rangle\, \alpha_2 - \dots - \langle \beta_{m+1}, \alpha_m \rangle\, \alpha_m \qquad \text{... (1)}$$

Now we know that

γ_{m+1} is orthogonal to each of $\alpha_1, \alpha_2, \dots, \alpha_m$.

Suppose that $\gamma_{m+1} = \bar{0}$. Then

$$\beta_{m+1} = (\beta_{m+1}, \alpha_1)\,\alpha_1 + (\beta_{m+1}, \alpha_2)\,\alpha_2 + \dots + (\beta_{m+1}, \alpha_m)\,\alpha_m$$

So that β_{m+1} is a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_m$

By our induction hypothesis, each

α_i (i = 1, 2, ... n) is a linear combination of $\beta_1, \beta_2, \dots, \beta_m$.

$\therefore$ β_{m+1} is a linear combination of $\beta_1, \beta_2, \dots, \beta_m$.

This is a contradiction, as $\{\beta_1, \beta_2, \dots, \beta_m\, \beta_{m+1}\}$ is linearly independent.

$$\therefore \quad \gamma_{m+1} \neq \bar{0}, \text{ Take } \alpha_{m+1} = \frac{\gamma_{m+1}}{\|\gamma_{m+1}\|}$$

Then we have $\|\ _{m+1}\| = 1$ and α_{m+1} is orthogo[illegible] to each of $\alpha_1, \alpha_2, \dots \alpha_m$.

Also $\alpha_{m+1} \neq \alpha_i$; for i = 1, 2, ... m.

Otherwise, from (1) we see that β_{m+1} will become a linear combination of $\beta_1, \beta_2, \dots, \beta_m$.

Further from (1) we see that α_{m+1} is a linear combination of $\beta_1, \beta_2, \dots, \beta_m, \beta_{m+1}$.

Thus $\{\alpha_1, \alpha_2, \dots, \alpha_m, \alpha_{m+1}\}$ is an orthonormal basis of V such that $L\{\alpha_1, \alpha_2, \dots, \alpha_{m+1}\} = L(\{\beta_1, \beta_2, \dots, \beta_{m+1}\}) = V$ when dim V = m + 1.

$\therefore$ by induction, B' = $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is an orthonormal basis of V such that

$$L(B) = L(B') = V \text{ when } \dim V = n.$$

$$\left[\left(\sqrt{14} \ \dots\right)\left(\dots \sqrt{378}\right)\left(\sqrt{3}\ \sqrt{3}\ \sqrt{3}\right)\right]$$

Theorem 38 :

Bessel's Inequality and Parseval's Identity

Statement :

(Bessel's inequality) If $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ ***is any orthonormal subset in an inner product space V(F) and*** β ***is any vector in V, then*** $\sum_{i=1}^{n} |\langle \beta, \alpha_i \rangle|^2 \le \|\beta\|^2$.

Further equality holds if and only if β ***is in the subspace generated by*** $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ ***or if and only if*** $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ ***is basis for V.***

Sol :

$S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is an orthonormal set

$$\Rightarrow (\alpha_i, \alpha_k) = 1 \quad \text{if} \quad i \neq j$$

$$\text{and} \quad (\alpha_i, \alpha_j) = 0 \quad \text{if} \quad i \neq j$$

Consider the vector $\gamma = \beta - \sum_{i=1}^{n} (\beta, \alpha_i)\,\alpha_i$

$$\|\gamma\|^2 = \langle \gamma, \gamma \rangle = \left\langle \beta - \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle \alpha_i, \beta - \sum_{j=1}^{n} \langle \beta, \alpha_j \rangle \alpha_j \right\rangle$$

$$= \langle \beta, \beta \rangle - \sum_{j=1}^{n} \overline{\langle \beta, \alpha_j \rangle}\, \langle \beta, \alpha_j \rangle - \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \langle \alpha_i, \beta \rangle + \left(\sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\right)\left(\sum_{j=1}^{n} \overline{\langle \beta, \alpha_j \rangle}\right) \langle \alpha_i, \alpha_j \rangle$$

$$= \langle \beta, \beta \rangle - \sum_{j=1}^{n} \overline{\langle \beta, \alpha_j \rangle}\, \langle \beta, \alpha_j \rangle - \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle} + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_j \rangle}\, \langle \alpha_i, \alpha_j \rangle$$

$$= \langle \beta, \beta \rangle - 2\sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle} + \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle}\, \langle \alpha_i, \alpha_i \rangle$$

(On summing with respect to j and using $(\alpha_i, \alpha_j) = 0$ for $i \neq j$)

$$= \|\beta\|^2 - 2\sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle} + \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle}$$

$$= \|\beta\|^2 - \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \overline{\langle \beta, \alpha_i \rangle}$$

$$\therefore \quad \|\gamma\|^2 = \|\beta\|^2 - \sum_{i=1}^{n} |\langle \beta, \alpha_i \rangle|^2 \qquad (\because Z\bar{Z} = |Z|^2)$$

Since $\|\gamma\|^2 \ge 0$ we have,

$$\|\beta\|^2 - \sum_{i=1}^{n} |\langle \beta, \alpha_i \rangle|^2 \ge 0$$

$$\Rightarrow \sum_{i=1}^{n} |\langle \beta, \alpha_i \rangle|^2 \le \|\beta\|^2$$

If the equality holds, then

$$\|\gamma\|^2 = 0 \Leftrightarrow \langle \gamma, \gamma \rangle = 0$$

$$\Leftrightarrow \gamma = \bar{0}$$

$$\Leftrightarrow \beta = \sum_{i=1}^{n} \langle \beta, \alpha_i \rangle\, \alpha_i$$

$\Leftrightarrow$ β is a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$

$\Leftrightarrow$ β is a vector in the subspace spanned by

$S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ i.e., $\beta \in L(S)$

$\Leftrightarrow$ $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V.

By $R_{21}(-1)$ $\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The characteric eqn of A is

$|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$

$(\lambda+2)(\lambda-2) = 0$

$\Rightarrow 3x_1 + 4x_2 = 0$

Let $x_2 = k\ (\neq 0)$

$\therefore \quad x1 = -4k/3$

$\therefore$ Eigen vectors corresponding to eigen root –2 are given by $k\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ where k is a non-zero parameter. Clearly the sub-space generated by $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ is a one-dimensional characteristic space of R^2.

roots are –2, 5

case 1 $\lambda = -2$

$\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Case (ii) :

Let $\lambda = 5$

Eigen roots X corresponding to the eigen roots 5 are given by

$$(A - 5I)\,X = 0$$

i.e., $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_{21}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + 4x_2 = 0$

Let $x_2 = k\ (\neq 0)$

$x_1 = k$

$\therefore$ Eigen vector to the root 5 are given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$