VECTOR SPACES :

Definition : Let 'V' be a non-empty set whose elements are called vectors. Let 'F" be any set whose elements are called scalars whre (F, +, .) is a field.

The set 'V' is said to be a vector space if

1. There is defined an internal composition in 'V" called addition of vectors denoted by '+', for which (V, +) is an abelian group.
2. There is defined an external composition in 'V' over 'F', called the scalar multiplication in which $a\alpha \in V$ for all $a \in F$ and $\alpha \in V$.
3. The above two compositions satisfy the following postulates.
   i. $a(\alpha + \beta) = a\alpha + a\beta$
   ii. $(a + b)\alpha = a\alpha + b\alpha$
   iii. $(ab)\alpha = a(b\alpha)$
   iv. $1\alpha = \alpha$

$\forall\, a, b \in F$ and $\alpha, \beta \in V$ and 1 is the unity element of F.

Theorem 1 :

Let V(F) be a vector space. A non-empty set $W \subseteq V$.

The necessary and sufficient condition for W to be a subspace of V is $a, b \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in W$.

Proof :

Condition is necessary

W(F) is a subspace of V(F) $\Rightarrow$ W(F) is a vector space.

$\therefore$ Now $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$ and $b \in F$, $\beta \in W$

$\Rightarrow b\alpha \in W$

Now $a\alpha \in W$, $b\beta \in W \Rightarrow a\alpha + b\beta \in W$

Condition is Sufficient

Let W be the non-empty subset of V

Satisfying the given condition

i.e., $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ ... (1)

Taking $a = 1$, $b = -1$, and $\alpha, \beta \in W \Rightarrow 1\alpha + (-1)\beta \in W$

$\Rightarrow \alpha - \beta \in W$ [$\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

($H \subseteq G$ and $a, b \in H \Rightarrow a\, o\, b^{-1} \in H$ then (H, 0) is subgroup of (G, 0)).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +)

$\Rightarrow$ (W, +) is an abelian group. Again taking $b = 0$

$a, O \in F$ and $\alpha, \in W \Rightarrow a\alpha + O\beta \in W \Rightarrow a\alpha \in W \Rightarrow a \in F$ and $\alpha \in W \Rightarrow a\alpha \in W$.

$\therefore$ W is closed under scalar multiplication.

The remaining postulates of vector space hold in W as $W \subseteq V$.

$\therefore$ W(F) is a vector space of V(F).

Theorem 2 :

Let V(F) be a vector space and let $W \subseteq V$. The necessary and sufficient conditions for W to be a subspace of V are :

i. $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

ii. $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

Proof :

Conditions are necessary

i. W is a vector subspace of V

$\Rightarrow$ W is a subgroup of (V, +) $\Rightarrow$ (W, +) is a group.

$\Rightarrow$ If $a, \beta \in W$ then $a - \beta \in W$

(ii) W is a subspace of V

$\Rightarrow$ W is closed under scalar multiplication

$\Rightarrow$ for $a \in F$, $\alpha \in W$; $a\alpha \in W$

Conditions are sufficient

Let W be a non-empty subset of 'V" satisfying the two given conditions

$\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \bar{O} \in W$ (by (i))

$\therefore$ The zero vector of 'V' is also the zero vector of W.

$\bar{O} \in W, \in W \Rightarrow \bar{O} - \alpha \in W \Rightarrow (-\alpha) \in W$ (by(i))

$\Rightarrow$ additive inverse of each element of W is also in W.

Again $\alpha \in W$, $\beta \in W \Rightarrow \alpha \in W, (-\beta) \in W \Rightarrow \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$ (by (i))

i.e., W is closed under vector addition

AS $W \subseteq V$, all the elements of W are also the elements of V. Thereby vector addition in W will be associative and cimmutative. This implies that (W, +) is an abelian group.

Further by (ii), W is closed under scalar multiplication and other postulates of vector space hold in 'ω' as $W \subseteq V$

$\therefore$ W itself is a vector space under the operations of V.

Hence W(F) is a vector subspace of V(F).

Theorem 3 :

The union of two subspaces if and only if one is contained in the other.

Proof :

Let W_1 and W_2 be two subspaces of V(F)

Hence α cannot be expr... $\alpha_1, \alpha_2, \alpha_3$.

Linear Dependence of Vectors

Definitiion

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly dependent (L.D.) set if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{O}$.

Linear Independence of Vectors

Definition

Let V(F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ of vectors of V is said to be a linearly independent (L.I.) if every relation of the form.

$$a_1\alpha_1 + a_2\alpha_2 + a_n\alpha_n = \bar{O},\ a_1^{-1} s \in F$$

$$\Rightarrow a_1 = 0,\ a_2 = 0,\ ...,\ a_n = 0$$

Problem 6 :

Show that the system of vectors (1, 3, 2), (1, −7, −8) (2, 1, −1) of $V_3(R)$ is linearly dependent.

Sol :

Let $a, b, c \in R$, then $a(1, 2, 3) + b(1, -7, -8) + c(2, 1, -1) = \bar{O}$

$\Rightarrow (a + b + 2c,\ 3a - 7b + c,\ 2a - 8b - c) = (0, 0, 0)$

$\Rightarrow (a + b + 2c = 0,\ 3a - 7b + c = 0,\ 2a - 8b - c = 0)$

$\Rightarrow a = 3,\ b = 1,\ c = -2$ [$\because$ The given vectors are linearly dependent]

Problem 7 :

Show that the system of vectors (1, 2, 0), (0, 3, 1), (−1, 0, 1) of $V_3(Q)$ is linearly independent where Q is the field of rational numbers.

Sol :

Let $x, y, z \in Q$ then $x(1, 2, 0) + y(0, 3, 1) + z(-1, 0, 1) = \bar{O}$

$\Rightarrow (x - z,\ 2x + 3y,\ y + z) = (0, 0, 0)$

$\Rightarrow x - z = 0,\ 2x + 3y = 0,\ y + z = 0$

$\Rightarrow x = 0,\ y = 0,\ z = 0.$

Hence the system is linearly independent.

independent.

Theorem 9 :

Let V(F) be a vector space and $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a finite subset of non-zero vectors of V(F). Then S is linearly dependent if and only if some vector $\alpha_k \in S$, $2 \le k \le n$, can be expressed as a linear combination of its preceeding vectors.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_k ... \alpha_n\}$ is linearly dependent. Then there exist $\alpha_1, \alpha_2, ... \alpha_n \in F$, not all zero, such that $a_1\alpha_1 + a_2\alpha_2 + ... a_n\alpha_n = \bar{O}$.

Let k be the greatest suffix of a for which $\alpha_k \neq 0$.

Then $a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k + 0\alpha_{k+1} + ... + 0\alpha_n = \bar{O}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + ... a_k\alpha_k = \bar{O}$

Now suppose $k = 1$, (1) $\Rightarrow a_1\alpha_1 = \bar{O}$

But $a_1 \neq 0$, so $\alpha_1 = 0$, which contradicts that each element of S is a non-zero vector.

Hence $k > 1$, i.e., $2 \le k \le n$.

Again from (1) we have,

$$a_k\alpha_k = -a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1}$$

$$\Rightarrow a_k^{-1}(a_k\alpha_k) = a_k^{-1}(-a_1\alpha_1 - a_2\alpha_2 ... - a_{k-1}\alpha_{k-1})$$

$$\Rightarrow \alpha_k = (-a_k^{-1} a_1)\alpha_1 + (-a_k^{-1} a_2)\alpha_2 + ... + (-a_k^{-1} \alpha_{k-1})\alpha_{k-1}$$

= (linear combination of its preceding vectors)

Conversely

Let some $\alpha_p \in S$ be expressible as a linear combination of its preceding vectors i.e., for $b_1, b_2, ... b_{p-1} \in F$.

$$\alpha_p = b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1}$$

$$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + ... + b_{p-1}\alpha_{p-1} + (-1)\alpha_p = \bar{O}$$

$\Rightarrow$ the set $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

[(−1) is a non-zero coefficient]

Hence the superset $S = \{-\alpha_1, \alpha_2, ..., \alpha_p - \alpha_n\}$ is linearly dependent

Note

If β is a linear combination of the set of vectors $\alpha_1, \alpha_2, ...$ then the set of vectors $\{b, \alpha_1, \alpha_2, ..., \alpha_n\}$ is linearly dependent

Basis of Vector Space

Definition

A subset S of a vector space V(F) is said to be the basis of V, if

i. S is linearly independent

ii. The linear span of S is V i.e., L(S) = V

Note

A vector space may have more than one basis.

Basis Extension

Theorem 10 :

Let V(F) be a finite dimensional vector space and $S = \{\alpha_1, \alpha_2, ... \alpha_m\}$ a linearly independent subset of V. Then either S itself a basis of V or S can be extended to form a basis of V.

Proof :

Given $S = \{\alpha_1, \alpha_2, ... \alpha_m\}$ is linearly independent subset of V

Since V(F) is finite dimensional it has a finite basis B.

Let $B = \{\beta_1, \beta_2, ... \beta_n\}$

Now consider the set $S_1 = \{\alpha_1, \alpha_2, ... \alpha_m, \beta_1, \beta_2, ... \beta_n\}$ written in this order.

Clearly $L(S_1) = V$.

Each α can be expressed as a *l.c.* of β's as B is the basis of V

$\Rightarrow S_1$ is linearly dependent.

Hence some vector in S_1 can be expressed as a *l.c.* of its preceding vectors. This vector cannot be any of α's, since S is linearly independent. So, this vector must be some β_i consider now the set.

$S_2 = \{\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_{i-1}, \beta_{i+1} ... \beta_n\} = S_1 - \{\beta_i\}$ obviously $L(S_2) = L(S_1) = V$

If S_2 is linearly independent, then S_2 forms a basis of V and it is the extended set.

If S_2 is linearly dependent then continue this procedure till we gt a set $S_k \subset S$ such that S_k is linearly independent.

$\therefore L(S_k) = L(S) = V$

S_k will be extended set of S forming a basis of V.

Theorem 11 :

V is a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ... \beta_n$ then any independent set of vectors in V is finite and contains no more than n elements.

Proof :

Let $\{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a linearly independent subset of vector space V(F)

Let $S = \{\beta_1, \beta_2, ..., \beta_n\}$ and L(S) = V.

Then any vector of V is *l.c.* of the elements of S.

Let $a_m \in V$ be a linear combination of its preceding vectors.

This vector must be one among β's.

Let it be β_i

$\therefore \beta_p$ = *l.c.* of $\alpha_m, \beta_1, \beta_2, ..., \beta_{i-1}$

If $S_1 = \{\alpha_m, \beta_1, \beta_2, ... \beta_{i-1}, \beta_{i+1}, ..., \beta_n\}$

Then $L(S_1) = L(S) = V$.

Again the vector $\alpha_{m-1} \in V$ is a *l.c.* of the elements of S_1.

∴ $\alpha_{m-1}, \alpha_m, \beta_1, \beta_2, \dots \beta_{i-1}, \beta_{i+1}, \dots, \beta_n$ are L.D.

∴ There exists a vector in this set which is a *l.c.* of its predecessors.

Such vector is one of β's as α_{m-1}, am are L.I.

Let it be β_k.

If $S_2 = \{\alpha_{m-1}, \alpha_m, \beta_1, \beta_{k-1}, \beta_{k+1}, \dots \beta_{i-1}, \beta_{i+1}, \dots, \beta_n\}$

Then β_k = *l.c.* of the elements of S_3.

∴ $L(S) = L(S_1) = L(S) = V$.

Continuing this process further for m − 3 times.

We get $S_{m-1} = \{\alpha_2, \alpha_3, \dots \alpha_m, \beta_{i1}, \beta_{i2}, \beta_{in-m+1}\}$.

So that $L(S_{m-1}) = V$

This set consists of atleast one β_i.

otherwise $S_{m-1} = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$

So that α_1 is a *l.c.* of $\alpha_2, \alpha_3, \dots, \alpha_m$.

This cannot happen as the set $\alpha_2, \alpha_3, \dots, \alpha_m$ is L.I.

Hence S_{m-1} consists of atleast one β_i

⇒ $n - m + 1 \geq 1 \Rightarrow n \geq m$.

∴ The no. of elements of the L.I. set in V ≤ n.

12 **Theorem 12 :**

Let W_1 and W_2 be two subspaces of a finite dimensional vector space V(F). Then dim $(W_1 + W_2)$ = dim W_1 + dim W_2 − dim $(W_1 \cap W_2)$.

Proof :

Since W_1 and W_2 are subspaces of V, $W_1 + W_2$ and $W_1 \cap W_2$ are also subspace of V.

Let dim $(W_1 \cap W_2) = k$ and $S = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be a basis of $W_1 \cap W_2$.

Clearly $S \subseteq W_1$ and $S \subseteq W_2$ and S is L.I.

Since S is L.I and $S \subseteq W_1$, the set S can be extended to form a basis of W_1.

Let $B_1 = \{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_m\}$ be a basis of W_1

∴ dim $W_1 = k + m$.

Again since S is L.I. and $S \subseteq W_2$, the set S can be extended to form a basis of W_2.

Let $B_2 = \{\gamma_1, \gamma_2, \dots, \gamma_k, \beta_1, \beta_2, \dots, \beta_m\}$ be a basis of W_2

∴ dim $W_2 = k + t$

∴ dim W_1 + dim W_2 − dim $(W_1 \cap W_2)$

= (k + m) + (k + t) − k

= k + m + t.

Now we shall prove that the set

$S' = \{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_t\}$ is a basis of $W_1 + W_2$ and hence.

dim $(W_1 + W_2) = k + m + t$.

(i) To prove that S' is L.I.

Now $c_1\gamma_1 + c_2\gamma_2 + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \dots + b\beta = \bar{0}$ ... (1)

⇒ $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t = -c_1\gamma_1 - \dots - c_k\gamma_k - a_1\alpha_1 - \dots - a_m\alpha_m$

= *l.c.* of elements of B1 and (∵ ∈ W_1)

⇒ $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_1$

Again $0\gamma_1 + 0\gamma_2 + 0\gamma_k + b_1\beta_2 + b_2\beta_2 + \dots + b_t\beta_t$

= *l.c.* of elements of B_2 and (∵ ∈ W_2)

⇒ $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_2$

∴ by (1) and (2) ⇒ $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t \in W_1 \cap W_2$

Hence it can be expressed as a *l.c.* of the elements of the basis S of $W_1 \cap W_2$.

Let $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + \dots + d_k\gamma_k$.

∴ $b_1\beta_1 + b_2\beta_2 + \dots + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 \dots - d_k\gamma_k = \bar{O}$

⇒ (*l.c.* of elements of basis B_2) = $\bar{O}$

$b_1 = 0,\ b_2 = 0 \dots b_t = 0,\ d_1 = 0, \dots k_k = 0.$

Substituting in I we have,

$c_1\gamma_1 + c_2\gamma_2 + \dots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = \bar{O}$

⇒ (*l.c.* of elements of basis B_1) = $\bar{O}$

⇒ $c_1 = 0,\ c_2 = 0 \dots c_k = 0,\ a_1 = 0,\ a_2 = 0 \dots a_m = 0$

Thus relation I implies that,

$c_1 = c_2 = \dots = c_k = a_1 = a_2 = \dots \alpha_m = 0,\ b_1 = b_2 = \dots = b_t = 0$

∴ S' is L.I. set

(ii) To prove $L(S') = W_1 + W_2$

Every vector of S' is a vector of $W_1 + W_2$.

∴ $L(S') \leq W_1 + W_2$

Let $S \in W_1 + W_2$

∴ $\delta = \alpha + \beta$ where $\alpha \in W$, $\beta \in W_2$

δ = (*l.c.* of elements of B_1) + (*l.c.* of elements of B_2)

= (*l.c.* of γ's and α's) + (*l.c.* of γ's and β's)

= (*l.c.* of γ's, α's and β's.

= (l.c. of elements of S'

∴ δ ∈ L(S')

∴ $W_1 + W_2 \leq L(S')$

∴ $L(S') = W_1 + W_2$

Hence S' is the basis of $W_1 + W_2$

∴ dim $(W_1 + W_2) = k + m + t$.

Hence the theorem.

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14. Describe explicitly the linear transformation $T : R^2 \to R$ such that T(2, 3) = (4, 5) and T(1, 0) = (0, 0)

Sol :

First of all we have to show that the vectors (2, 3) and (1, 0) are L.I.

Let $a(2, 3) + b(1, 0) = \bar{O}$

⇒ (2a + b, 3a + 0) = (0, 0)

⇒ 2a + b = 0, 3a = 0

⇒ 2a = 0, b = 0

∴ S = {(2, 3), (1, 0)} is L.I.

Let us prove that $L(S) = R^2$

Let $(x, y) \in R^2$ and (x, y) = a (2, 3) + b(1, 0) − (2a + b, 3a)

⇒ 2a + b = x, 3a = y

⇒ $a = \frac{y}{3}$; $b = \frac{3x - 2y}{3}$

Hence S spans R^2.

Now $T(x, y) = T\left[\frac{y}{3}(2, 3) + \frac{3x - 2y}{3}(1, 0)\right]$

$= \frac{y}{3} T(2, 3) + \frac{3x - 2y}{3} T(1, 0)$

$= \frac{y}{3}(4, 5) + \frac{3x - 2y}{3}(0, 0)$

$= \left(\frac{4y}{3}, \frac{5y}{3}\right)$

∴ This is the required transformation.

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Theorem 16 :

Rank - Nullity theorem

Statement

Let U(F) and V(F) be two vector spaces and T : U → V be a linear transformation. Let U be finite dimensional

Then $P(T) + \nu(T) = \dim U$

i.e., rank (T) + nullity (T) = dim U

Proof :

The null space N(T) is a subspace of finite dimensional space U(F).

⇒ N(T) is finite dimensional

Let $S = \{\alpha_1, \alpha_2, \dots \alpha_k\}$ be a basis set of N(T)

∴ dim $N(T) = \nu(T) = k$

∴ $T(\alpha_1) = \hat{\bar{O}}, T(\alpha_2) = \hat{\bar{O}} \dots T(\alpha_k) = \hat{\bar{O}}$ ... (1)

($\hat{\bar{O}} \in V$)

As S is L.I. it can be extended to form a basis of U(F)

∴ dim U = k + m.

Now we show that the set of images of additional vectors.

S2 = $\{T(\theta_1), T(\theta_2), \dots T(\theta_m)\}$ is a basis of R(T) clearly $S_2 \subseteq R(T)$

(i) To prove S_2 is L.I

Let $a_1, a_2, \dots a_m \in F$ be such that

$a_1T(\theta_1) + a_2T(\theta_2) + \dots + a_mT(\theta_m) = \hat{\bar{O}}$

⇒ $T[a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m] = \hat{\bar{O}}$

⇒ $a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m \in N(T)$, null space of T.

But each vector is N(T) is a *l.c.* of elements of basis S.

∴ For some $b_1, b_2, \dots b_m \in F$,

Let $a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \dots + b_k\alpha_k$

⇒ $a_1\theta_1 + \dots + a_m\theta_m - b_1\alpha_1 - \dots - b_k\alpha_k = \hat{\bar{O}}$

⇒ $a_1 = 0, a_2 = 0 \dots a_m = 0, b_1 = 0, \dots b_k = 0$

(∵ S_1 is L.I

⇒ S_2 is L.I. set

(ii) To prove $L(S_2) = R(T)$

Let β ∈ range space R(T), then there exists α ∈ U suc that T(α) = β

Now α ∈ U ⇒ there exists c's, d's ∈ F such that

$\alpha = c_1\alpha_1 + c_2\alpha_2 - \dots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \dots + d_m\theta_m$

⇒ $T(\alpha) = T[c_1\alpha_1 + \dots + c_k\alpha_k + d_1\theta_1 + \dots + d_m\theta_m]$

$= c_1T(\alpha_1) + \dots + c_kT(\alpha_k) + d_1T(\theta_1) + \dots + d_mT(\theta_m)$

⇒ $\beta = d_1T(\theta_1) + d_2T(\theta_2) + \dots + d_mT(\theta_m)$ (by (1))

⇒ $\beta \in L(S_2)$

∴ S_2 is a basis of R(T) and dim R(T) = m

Thus dim R(T) + dim N(T) = m + k = dim U

i.e., $\rho(T) + \nu(T) = \dim U$.

Problem 17 :

Let $T : V_2 \to V_3$ be defined by T(x, y) = (x + y, 2x − y, 7y) Find [T : B_1, B_2] where B_1 and B_2 are the standard bases of V_2 and V_3.

Sol :

B_1 is the standard basis of V_2 and B_2 that of V_3

∴ $B_1 = \{(1, 0), (0, 1)\}$

$B_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

∴ T(1, 0) = (1, 2, 0) = 1 (1, 0, 0) + 2 (0, 1, 0) + 0 (0, 0, 1)

T(0, 1) = (1, −1, 7) = 1(1, 0, 0) −1(0, 1, 0) + 7 (0, 0, 1)

∴ The matrix of T relative of B_1 and B_2 is

$$[T : B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}$$

15 ***Problem 18 :***

Let $T : R^3 \to R^2$ be the linear transformations defined by T(x, y, z) = (3x + 2y − 4z, x − 5y + 3x). Find the matrix of T relative to the bases B_1={(1, 1, 1), (1, 1, 0), (1, 0, 0)}, B_2 = {(1, 3), (2, 5)}

Sol :

Let $(a, b) \in R^2$ and let (a, b) = p(1, 3) + q (2, 5)

= (p + 2q, 3p + 5q)

⇒ p + 2q = q, 3p + 5q = b

Solving p = −5a + 2b, q = 3a − b

∴ (a, b) = (−5a + 2b) (1, 3) + (3a − b) (2, 5)

Now T(1, 1, 1) = (1, −1) = −7 (1, 3) + 4 (2, 5)

T(1, 1, 0) = (5, −4) = −33 (1, 3) + 19 (2, 5)

T(1, 0, 0) = (3, 1) = −13 (1, 3) + 8 (2, 5)

∴ The matrix of L.T. relative to B_1 and B_2

$$[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

Problem 19 :

Let T be the linear operator r on R^2 defined by T(x, y) = (4x − 2y, 2x + y). Find the matrix of T w.r.to the basis T{(1, 1), (−1, 0)}, Also verify $[T]_B\ [\alpha]_B = [T]\ (\alpha)_B\ \forall \alpha \in R^2$.

Sol :

Let $(a, b) \in R^2$. Then;

(a, b) = p(1, 1) + q(−1, 0) = (p, −q, p)

⇒ a = p − q and b = p

⇒ p = b and q = b − a

∴ (a, b) = b(1, 1) + (b − a) (−1, 0) ... (1)

(i) Given transformation is T(x, y) = (4x − 2y, 2x + y)

∴ T(1, 1) = (2, 3) = 3(1, 1) + 1 (−1, 0) ... (by (1))

T(−1, 0) = (−4, 2) = −2 (1, 1) + 2 (−1, 0)

$$\therefore\ [T : B] = [T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

(ii) Let $\alpha \in R^2$ where α = (a, b)

∴ α = (a, b) = b(1, 1) + (b − a) (−1, 0)

$$\therefore\ [\alpha]_B = \begin{bmatrix} b \\ b - a \end{bmatrix}$$

$\therefore \quad [T]_B\,[\alpha]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ b-a \end{bmatrix} = \begin{bmatrix} 2a+b \\ -2a+3b \end{bmatrix}$ ... (1)

Again $T(\alpha) + T(a, b) = (4a - 2b, 2a + b)$

Let $(4a - 2b, 2a + b) = x(1, 1) + y(-1, 0) = (x, -y, x)$

$\Rightarrow \; 4a - 2b = x - y$ and $2a + b = x$

$\Rightarrow \; x = 2a + b$ and $y = -2a + 3b$.

Hence $T(\alpha) = T(a, b) = (2a + b)(1, 1) + (-2a + 3b)(-1, 0)$

$\therefore$ The matrix of $T(\alpha)$ w.r.to the base B is

$$[T(\alpha)]B = \begin{bmatrix} 2a+b \\ -2a+3b \end{bmatrix} \quad \text{... (2)}$$

$\therefore$ from (1) and (2) $[T]_B\,[\alpha]_B = [T(\alpha)]_B$.

1. Define a subspace. Prove that the intersection of two subspaces is again a subspace.

Ans :

Subspace

Let V(F) be a vector space and $W \subseteq V$. Then W is said to be a subspace of v if w itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.

Proof :

W_1 and W_2 are subspace V(F)

$\Rightarrow \; \bar{0} \in W_1$ and $\bar{0} \in W_2$ $\qquad \therefore \; W_1 \cap W_2 \neq \varnothing$

$\Rightarrow \; \bar{0} \in W_1 \cap W_2$ $\qquad \therefore \; \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$

Now $a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$

$\therefore \; a\alpha + b\beta \in W_1 \cap W_2$

$\therefore \; W_1 \cap W_2$ is a subspace of V(F)

$$\Rightarrow i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - j\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + k\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$\Rightarrow 0.$

SECTION - B (4 × 16 = 64)

UNIT - I

9. a) Let V(F) be a finite dimensional vector space. Then show that any two bases of V have the same number of elements.

Sol. :

Basis of a vector space

A subset S of a vector space V(F) is said to be the basis of V, if

(i) S is linearly independent

(ii) the linear span of S is V i.e., L(S) = V

Proof :

Let S_m and S_n be the two bases of V(F) where $S_m = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ $S_n = \{\beta_1, \beta_2, \ldots \beta_m\}$ obviously both S_m and S_n are L.I subsets of V.

(i) Consider S_m as the basis of V and S_n and L.I set

$\Rightarrow \; L(S_m) = V$ and $n(S_m) = m$

$\therefore \; S_n$ can be extended to be a basis of V

$\Rightarrow \; n \le m$

(ii) Consider S_n as the basis of V and S_m as L.I set

$\Rightarrow \; L(S_n) = V$ and $n(S_n) = n$

$\therefore \; S_m$ can be extended to be a basis of V

$\Rightarrow \; m \le n$

But both S_m and S_n are bases of V

$\therefore \; n \le m$ and $m \le n \Rightarrow m = n$

Thus any two bases of V have the same number of elements.

b) Show that every non empty subset of a linearly independent set of vectors is linearly independent.

Sol. :

Let $S = \{\alpha_1, \alpha_2 \ldots. \alpha_m\}$ be a linearly independent set of vectors.

Let us consider the subset $S_1 = \{\alpha_1, \alpha_2 \ldots. \alpha_k\}$ where $1 \le k \le m$.

Now for some scalars consider the linear combination

$$a_1\alpha_1 + a_2\alpha_2 + \ldots.. + a_k\alpha_k = \bar{0}$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots. + a_k\alpha_k + 0_{k+1}\alpha_{k+1} + \ldots + 0_m\alpha_m = \bar{0}.$$

Q $S_1 \subset S$ which is a linearly independent set.

$\therefore \; a_2 = a_2 = \ldots = a_k = 0$

$\therefore \; S_1$ is a linearly independent set.