3, 15.

3 **Theorem 22 :**

8 **Cayley - Hamilton Theorem**

Statement : Every square matrix satisfies its characteristic equation

Proof :

Let A be a n-rowed square matrix i.e., Let $A = [a_{ij}]_{n \times n}$

The characteristic equation of A is given by

$$|A - \lambda I| = f(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n]$ where a_i's $\in F$

Since all the elements of $(A - \lambda I)$ are at most of Ist degree in λ, all the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $(n - 1)$ or less and hence adj $(A - \lambda I)$ can be expressed as a matrix polynomial in λ.

Let adj $(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-2}\lambda^1 + B_{n-1}$

where $B_0, B_1, B_2, \dots B_{n-1}$ are n - rowed square matrices.

Now $(A - \lambda I)$ adj $(A - \lambda I) = |A - \lambda I|\, I$ $\quad$ ($\because$ $A(\text{adj } A) = |A| I$)

$\Rightarrow (A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-2}\lambda + B_{n-1})$

$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n] I$

Comparing like coefficients of like powers of λ, we obtain

$-B_0 = (-1)^n I$

$AB_0 - B_1 = (-1)^n a_1 I,$

$AB_1 - B_2 = (-1)^n a_2 I,$

- - - - - - - - - - - -

$AB_{n-1} = (-1)^n a_n I$

Premultiplying the above equations successively by A^n, A^{n-1}, ... I and adding, we obtain

$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \dots + (-1)^n a_n I$

$\Rightarrow (-1)^n [A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_n I] = 0$

$\Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_n I = 0$

$\Rightarrow$ A satisfies its characteristic equation

(0 is zero matrix)

Problem 23 :

If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ *verify Cayley - Hamilton theorem. Hence find* A^{-1}.

Sol :

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = 0$

i.e., $(2 - \lambda)[-6 - 3\lambda + 2\lambda + \lambda^2] - 5[-2-\lambda] - 1[3 - 6 + 2\lambda] = 0$

i.e., $(2 - \lambda)[-6 - \lambda + \lambda^2] + 10 + 5\lambda + 3 - 2\lambda = 0.$

i.e., $-12 - 2\lambda + 2\lambda^2 + 6\lambda + \lambda^2 - \lambda^3 + 13 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$

i.e., $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$

To prove that A satisfies $\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$, the characteristic equation of A

Now $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$

$A^3 = A^2.A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & 7 \end{bmatrix}$

Now $A^3 - 3A^2 - 7A - I$

$= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & 7 \end{bmatrix} + \begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix}$

$+ \begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A satisfies its characteristic equation.

$A^3 - 3A^2 - 7A - I = 0 \Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = 0$

(Here $|A| \neq 0$)

$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - 3A - 7I$

$\therefore A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ -15 & -9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$

$\therefore AA^{-1} = I$

Problem 24 :

Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ *by using Cayley - Hamilton theorem.*

Sol :

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$

The characteristic equation of A is

$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

i.e., $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$

i.e., $(1 - \lambda)[2 - \lambda - 2\lambda + \lambda^2 - 1] + 2(-1) = 0$

i.e., $(1 - \lambda)(1 - 3\lambda + \lambda^2) - 2 = 0.$

i.e., $\lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0$

By Cayley - Hamilton theorem, A satisfies its characteristic equation.

$\therefore A^3 - 4A^2 + 4A + I = 0$

$\Rightarrow A^{-1}(A^3 - 4A^2 + 4A + I) = 0$ $\quad$ ($\because$ $|A| = -1 \neq 0$)

$\Rightarrow A^2 - 4A + 4I + A^{-1} = 0$

$\Rightarrow A^{-1} = -4I + 4A - A^2$

$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$

$\therefore AA^{-1} = I.$

Theorem 27 :

A square matrix A of order n over a field F is a diagonalizable if and only if A has linearly independent characteristic vectors in $V_n(F)$.

Proof :

A is diagonalizable

$\Rightarrow$ A is similar to a diagonal matrix D (say)

$\Rightarrow$ there exists an invertible matrix P such that $P^{-1}AP = D \Rightarrow AP = PD$.

If $\lambda_1, \lambda_2, \dots \lambda_n$ are the diagonal elements of D, then $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic values of D. Since similar matrices have the same characteristic values, $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic values of A.

Let $X_1, X_2, \dots X_n$ be the column vectors of P. Now equating corresponding columns of (1) on either side, we get $AX_i = \lambda_i X_i$, $i = 1, 2, 3, \dots n$ implying that X_i is a characteristic vector of A corresponding to the characteristic value λ_i. Since P is invertible, its n column vectors $X_1, X_2, \dots X_n$ are L.I. and belong to $V_n(F)$. Thus A has n L.I. vectors in $V_n(F)$.

Converse :

Let $X_1, X_2, \dots X_n$ be n L.I. characteristic vectors of A corresponding to the characteristic values $\lambda_1, \lambda_2, \dots \lambda_n$ respectively.

Let $P = (X_1, X_2, \dots X_n)$. Since the n columns of P are L.I. $|P| \neq 0$. Hence P^{-1} exists.

Conside $AP = A[X_1, X_2, \dots X_n] = [AX_1, AX_2, \dots AX_n]$

$= [\lambda_1X_1, \lambda_2X_2, \dots \lambda_nX_n] = [X_1, X_2, \dots X_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$= PD$ where $D = \text{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$

$\Rightarrow P^{-1}(AP) = P^{-1}(PD) = (P^{-1}P)D = D$

$\Rightarrow P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$

$\Rightarrow$ A is diagonalizable.

Problem 28 :

Determine the modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *verify that* $P^{-1}AP$ *is a diagonal matrix.*

Sol :

The characteristic equation of A is $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda - 5)(\lambda + 3)^2 = 0$

Thus the eigen values are $\lambda = 5$, $\lambda = -3$ and $\lambda = -3$

When $\lambda = 5$ we have $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly for the values $\lambda = -3$ we can have two linearly independent eigen vectors.

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Observe that X_1, X_2 and X_3 are not orthogonal.

But they are linearly independent.

Consider $P = (X_1\ X_2\ X_3)$

$\therefore P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = Modal matrix of A

$\det P = 1(-1) - 2(2) + 3(0 - 1) = -1 - 4 - 3 = -8.$

$\therefore P^{-1} = \frac{1}{\det P}(\text{adj } P) = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

$\therefore P^{-1}A = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ +2 & 1 & 6 \\ -1 & -2 & -0 \end{bmatrix}$

$= \frac{-1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

$\therefore P^{-1}AP = \frac{-1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag}(5, -3, -3)$

Hence $P^{-1}AP$ is a diagonal matrix.

Norm or Length of a Vector

Definition

Let 'V' be an inner product space over the field F. The norm (length) of $\alpha \in V$ denoted by $\|\alpha\|$ is defined as the positive square root of $\langle \alpha, \alpha \rangle$

Norm or length of $\alpha \in V = \|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} \Rightarrow \|\alpha\|^2 = \langle \alpha, \alpha \rangle$

Unit Vector

Definition

Let V(F) be an inner produce space $\alpha \in V$ is called a unit vector if $\|\alpha\| = 1$.

If $\alpha \in V$ then $\frac{1}{\|\alpha\|}\alpha \in V$ is unit vector.

Theorem 29 6

In an inner product space V(F),

$|\langle \alpha, \beta \rangle| \leq \|\alpha\| . \|\beta\|$ *for all* $\alpha, \beta \in V$

Cauchy - Schwartz's inequality.

Sol :

Case (1)

Let $\alpha = \bar{0}$. Then $\langle \alpha, \beta \rangle = \langle \bar{0}, \beta \rangle = 0$ and

$\|\alpha\|^2 = \langle \alpha, \alpha \rangle = \langle \bar{0}, \bar{0} \rangle = 0$

$\therefore$ $|\langle \alpha, \beta \rangle| = 0$ and $\|\alpha\|\, \|\beta\| = 0$ $\quad$ ($\|\beta\| = 0$)

$\therefore$ $|\langle \alpha, \beta \rangle| = \|\alpha\|\, \|\beta\|$.

Case (2)

Let $\alpha \neq \bar{0}$

Then $\|\alpha\| > 0$ so that $\frac{1}{\|\alpha\|} > 0$

Take $\gamma \in V$ so that $\gamma = \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha$

Now $(\gamma, \gamma) = \left\langle \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha, \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha \right\rangle$

$= \langle \beta, \beta \rangle - \frac{\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}\langle \beta, \alpha \rangle - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\langle \alpha, \beta \rangle + \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\frac{\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}\langle \alpha, \alpha \rangle$

$= \|\beta\|^2 - \frac{\overline{\langle \beta, \alpha \rangle}\langle \beta, \alpha \rangle}{\|\alpha\|^2} - \frac{\langle \beta, \alpha \rangle\langle \alpha, \beta \rangle}{\|\alpha\|^2} + \frac{\langle \beta, \alpha \rangle\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}$

$\left(\because \frac{\langle \alpha, \alpha \rangle}{\|\alpha\|^2} = 1\right)$

$= \|\beta\|^2 - \frac{\langle \beta, \alpha \rangle\langle \alpha, \beta \rangle}{\|\alpha\|^2} = \|\beta\|^2 - \frac{\overline{\langle \alpha, \beta \rangle}\langle \alpha, \beta \rangle}{\|\alpha\|^2}$

$\left(\because \langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}\right)$

But by the definition of the norm ; $(\gamma, \gamma) \geq 0$

$\therefore \|\beta\|^2 - \frac{\langle \alpha, \beta \rangle\overline{\langle \alpha, \beta \rangle}}{\|\alpha\|^2} \geq 0 \Rightarrow \|\beta\|^2 \geq \frac{|\langle \alpha, \beta \rangle|^2}{\|\alpha\|^2}$

$(\therefore Z\bar{Z} = |Z|^2)$

$\Rightarrow \|\beta\|^2\|\alpha\|^2 \geq |\langle \alpha, \beta \rangle|^2 \Rightarrow (\|\beta\|\ \|\alpha\|)^2 \geq |\langle \alpha, \beta \rangle|^2$

$(\because \|\alpha\|^2 > 0)$

$\therefore$ $(\|\beta\|\ \|\alpha\| \geq |\langle \alpha, \beta \rangle|$

as $\|\beta\|\ \|\alpha\|$ and $|\langle \alpha, \beta \rangle|$ are non-negative

nce. $|\langle \alpha, \beta \rangle| \leq \|\alpha\|\ \|\beta\|$.

Theorem 30 :

Triangle Inequality

Statement : In an inner product space V(F), $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ ***for all a, b*** $\in$ ***V.***

Proof :

By definition of norm,

$\|\alpha + \beta\|^2 = \langle \alpha + \beta, \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle$

$= \|\alpha\|^2 + \langle \alpha, \beta \rangle + \overline{\langle \alpha, \beta \rangle} + \|\beta\|^2$

$= \|\alpha\|^2 + 2\,\text{Re}\,(\alpha, \beta) + \|\beta\|^2$ $\quad$ ($\because$ $Z + \bar{Z} = 2\,\text{Re}\,Z$)

$\leq \|\alpha\|^2 + 2|\langle \alpha, \beta \rangle| + \|\beta\|^2$ $\quad$ ($\because$ $\text{Re}\,Z \leq |Z|$)

$\leq \|\alpha\|^2 + 2\|\alpha\|\ \|\beta\| + \|\beta\|^2$ $\quad$ ($\because$ $|\langle \alpha, \beta \rangle| \leq \|\alpha\|\ \|\beta\|$)

$\leq (\|\alpha\| + \|\beta\|)^2$ $\quad$ $\therefore \|\alpha + \beta\|^2 \leq (\|\alpha\| + \|\beta\|)^2$

As both $\|\alpha + \beta\|$ and $\|\alpha\| + \|\beta\|$ are non-negative we have

$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

Theorem 31 :

Parallelogram Law

If α, β are two vectors in an inner product space V(F) then

$$\|\alpha - \beta\|^2 + \|\alpha + \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$$

Proof :

$\|\alpha - \beta\|^2 = \langle \alpha - \beta, \alpha - \beta \rangle = \langle \alpha, \alpha \rangle - \langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle$

$= \|\alpha\|^2 - \langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle + \|\beta\|^2$

$\|\alpha + \beta\|^2 = \langle \alpha + \beta, \alpha + \beta \rangle = \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle$

$= \|\alpha\|^2 + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \|\beta\|^2$

$\therefore \|\alpha - \beta\|^2 + \|\alpha + \beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$

$\therefore \|\alpha - \beta\|^2 + \|\alpha + \beta\|^2 = 2(\|\alpha\|^2 + \|\beta\|^2)$

Problem 32 :

If α, β are two vectors in an inner product space, then α, β are linearly dependent if and only if $|\langle \alpha, \beta \rangle| = \|\alpha\|\ \|\beta\|$.

Sol :

Let α, β be linearly dependent

Then either $\alpha = \bar{0}$ or $\beta = \bar{0}$ or $\alpha = a\beta$ where 'a' is a scalar.

When $\alpha = \bar{0}$: $\langle \alpha, \beta \rangle = \langle \bar{0}, \beta \rangle = 0$ and $\|\alpha\| = 0$

$\therefore$ $|\langle \alpha, \beta \rangle| = \|\alpha\|\ \|\beta\|$

When $\beta = \bar{0}$: $\langle \alpha, \beta \rangle = \langle \alpha, \bar{0} \rangle = \overline{\langle \bar{0}, \alpha \rangle} = 0$ and $\|\beta\| = 0$

$\therefore$ $|\langle \alpha, \beta \rangle| = \|\alpha\|\ \|\beta\|$

When $\alpha = a\beta$: $\langle \alpha, \beta \rangle = \langle a\beta, \beta \rangle = a\langle \beta, \beta \rangle$

$= a\|\beta\|^2$ and $\|\alpha\| = \|a\beta\| = |a|\ \|\beta\|$

$\therefore$ $|\langle \alpha, \beta \rangle| = |a|\ \|\beta\|^2 = (|a|\ \|\beta\|)(\|\beta\|) = \|\alpha\|\ \|\beta\|$

Conversely,

Let $|\langle \alpha, \beta \rangle| = \|a\|\ \|\beta\|$

When $\alpha = \bar{0}$ the vectors α, β are linearly dependent

When $\alpha \neq \bar{0}$; we have $\|\alpha\| > 0$.

Consider the vector $\gamma = \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha$

$\langle \gamma, \gamma \rangle = \left\langle \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha, \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha \right\rangle$

$= \langle \beta, \beta \rangle - \frac{\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}\langle \beta, \alpha \rangle - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\langle \alpha, \beta \rangle$

$+ \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\frac{\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}\langle \alpha, \alpha \rangle$

$= \|\beta\|^2 - \frac{\overline{\langle \beta, \alpha \rangle}}{\|\alpha\|^2}\langle \beta, \alpha \rangle - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\langle \alpha, \beta \rangle$

$+ \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\overline{\langle \beta, \alpha \rangle}$

$\left(\because \langle \alpha, \alpha \rangle = \|\alpha\|^2\right)$

$= \|\beta\|^2 - \frac{|\langle \alpha, \beta \rangle|^2}{\|\alpha\|^2}$

$= \|\beta\|^2 - \frac{\|\alpha\|^2\|\beta\|^2}{\|\alpha\|^2} = 0$

$\therefore$ $(\gamma, \gamma) = 0 \Rightarrow \gamma = \bar{0}$

$\Rightarrow \beta - \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha = \bar{0} \Rightarrow \beta = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}\alpha$

$\Rightarrow \beta = a\alpha$ where $a = \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2}$ is a scalar

$\therefore$ α, β are linearly dependent.

Problem 33 :

Two vectors α, β in an unitary space V(C) are such that $\langle \alpha, \beta \rangle = 0$ ***iff***

$\|a\alpha + b\beta\|^2 = |a|^2\ |\alpha|^2 + |b|^2\ |\beta|^2\ \forall$ ***a, b*** $\in$ ***c***

Sol :

Let $\langle \alpha, \beta \rangle = 0$. Then $\overline{\langle \alpha, \beta \rangle} = 0$

$\|a\alpha + b\beta\|^2 = \langle a\alpha + b\beta, a\alpha + b\beta \rangle$

$= a\langle \alpha, a\alpha + b\beta \rangle + b\langle \beta, a\alpha + b\beta \rangle$

$= a[\bar{a}\langle \alpha, \alpha \rangle + \bar{b}\langle \alpha, \beta \rangle] +$

$b[\bar{a}\langle \beta, \alpha \rangle + \bar{b}\langle \beta, \beta \rangle]$

$= a\bar{a}\langle \alpha, \alpha \rangle + a\bar{b}\langle \alpha, \beta \rangle + b\bar{a}\langle \beta, \alpha \rangle$

$+ b\bar{b}\langle \beta, \beta \rangle$

$= |a^2|\ \|\alpha\|^2 + a\bar{b}\langle \alpha, \beta \rangle + b\bar{a}\langle \beta, \alpha \rangle$

$+ |b^2|\ \|\beta\|^2$ $\quad$ ... (1)

$= |a|^2\|\alpha\|^2 + 0 + 0 + |b|^2\|\beta\|^2$

$= |a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2.$

Conversely,

Let $\|a\alpha + b\beta\|^2 = |a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2\ \forall\ a, b \in c$

Using (1) we have,

$|a|^2\|\alpha\|^2 + a\bar{b}\langle \alpha, \beta \rangle + b\bar{a}\overline{\langle \alpha, \beta \rangle} + |b|^2\|\beta\|^2$

$|a|^2\|\alpha\|^2 + |b|^2\|\beta\|^2$

$\Rightarrow a\bar{b}\langle \alpha, \beta \rangle + b\bar{a}\overline{\langle \alpha, \beta \rangle} = 0\ \forall\ a, b \in c$ $\quad$ ... (2)

Take a= 1, b = 1 so that $\bar{a} = 1$, $\bar{b} = 1$

Then (2) $\Rightarrow (1)(1)\langle \alpha, \beta \rangle + (1)(1)\overline{\langle \alpha, \beta \rangle} = 0$

$\Rightarrow \langle \alpha, \beta \rangle + \overline{\langle \alpha, \beta \rangle} = 0$

$\Rightarrow 2\,\text{Re}\,\langle \alpha, \beta \rangle = 0 \Rightarrow \text{Re}\,\langle \alpha, \beta \rangle = 0$ $\quad$ ($\because$ $Z + \bar{Z} = 2\,\text{Re}\,Z$)

Take $a = i$, $b = 1$ so that $\bar{a} = -i$, $b = 1$

Then (2) $\Rightarrow i\langle \alpha, \beta \rangle - i\overline{\langle \alpha, \beta \rangle} = 0$

$\Rightarrow i[\langle \alpha, \beta \rangle - \overline{\langle \alpha, \beta \rangle}] = 0$

$\Rightarrow \langle\alpha,\beta\rangle - \overline{\langle\alpha,\beta\rangle} = 0$

$\Rightarrow 2\,\text{Im}\,\langle\alpha,\beta\rangle = 0 \quad (\because Z - \bar{Z} = 2\,\text{Im}\,\bar{Z})$

Thus we have Re $\langle\alpha,\beta\rangle = 0$ and Im $\langle\alpha,\beta\rangle = 0$

Hence $\langle\alpha,\beta\rangle = 0$.

Problem 34 :

If u, v are two vectors in a complex inner product space with standard inner product then prove that $4\langle u,v\rangle = \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$

Sol :

$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

$= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2 \quad \text{... (1)}$

$\|u-v\|^2 = \langle u-v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$

$= \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2 \quad \text{... (2)}$

$\|u+iv\|^2 = \langle u+iv, u+iv\rangle = \langle u,v\rangle + \bar{i}\langle u,v\rangle + i\langle v,u\rangle + i\bar{i}\langle v,v\rangle$

$= \|u\|^2 - i\langle u,v\rangle + i\langle v,u\rangle + \|v\|^2 \quad (\because i\bar{i} = i(-i) = 1)$

$i\|u-iv\|^2 = i\|u\|^2 + \langle u,v\rangle - \langle v,u\rangle + i\|v\|^2 \quad \text{... (3)}$

$\|u-iv\|^2 = \langle u-iv, u-iv\rangle = \langle u,v\rangle - \bar{i}\langle u,v\rangle - i\langle v,u\rangle + i\bar{i}\langle v,v\rangle$

$= \|u\|^2 + i\langle u,v\rangle - i\langle v,u\rangle + \|v\|^2$

$i\|u-iv\|^2 = i\|u\|^2 - \langle u,v\rangle + \langle v,u\rangle + i\|v\|^2 \quad \text{... (4)}$

From (1), (2), (3) and (4)

$\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$

$= \{2\langle u,v\rangle + 2\langle v,u\rangle\} + \{2\langle u,v\rangle - 2\langle v,u\rangle\} = 4\langle u,v\rangle$

Theorem 35 :

The vectors α, β of real inner product space V(F) are orthogonal if and only if, $\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$.

Proof :

$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$

$\Leftrightarrow \langle\alpha+\beta, \alpha+\beta\rangle = \|\alpha\|^2 + \|\beta\|^2$

$\Leftrightarrow \langle\alpha,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle + \langle\beta,\beta\rangle = \|\alpha\|^2 + \|\beta\|^2$

$\Leftrightarrow \|\alpha\|^2 + \langle\alpha,\beta\rangle + \overline{\langle\alpha,\beta\rangle} + \|\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$

$\Leftrightarrow \langle\alpha,\beta\rangle + \langle\alpha,\beta\rangle = 0 \quad (\because \langle\alpha,\beta\rangle \text{ is real } \overline{\langle\alpha,\beta\rangle} = \langle\alpha,\beta\rangle)$

$\Leftrightarrow \langle\alpha,\beta\rangle = 0$

$\Leftrightarrow$ α, β are orthogonal vectors.

Find the characteristic values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

:

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is

$|A - XI| \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$

$\Rightarrow 8 - x\,[(7-x)3 + 4(-4-x)]$

$+ 6[-18 + 8] + 2[24 + 6x - 14 + 2x] = 0$

$(8 - x)[21 - x - 16 - 4x] - 60 + 48 + 12x - 28 + 4x = 0$

$\Rightarrow 168 - 24x - 128 - 32x - 21x + 3x^2 + 16x$

$+ 4x^2 - 60 + 48 + 12x - 28 + 4x = 0$

$\Rightarrow 7x^2 - 45x = 0$

$7x^2 - 45x = 0$

$x = 0,\ 7x - 45 = 0$

$x = 0,\ x = \frac{45}{7}$

4. In an innerproduct space V(F), prove that (i) ...

11. (a) Determine the modal matrix P of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **verify that** $P^{-1}AP$ **is a giagonal matrix**

Sol :

The characteristic equation of A is $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda - 5)(\lambda + 3)^2 = 0$

Thus the eigen values are $\lambda = 5$, $\lambda = -3$ and $\lambda = -3$

When $\lambda = 5$ we have $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly for the values $\lambda = -3$ we can have two linearly independent eigen vectors.

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Observe that X_1, X_2 and X_3 are not orthogonal.

But they are linearly independent.

Consider $P = (X_1\ X_2\ X_3)$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = Modal matrix of A

$\det P = 1(-1) - 2(2) + 3(0 - 1) = -1 - 4 - 3 = -8.$

$\therefore P^{-1} = \frac{1}{\det P}(\text{adj } P) = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

$\therefore P^{-1}A = \frac{-1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ +2 & 1 & 6 \\ -1 & -2 & -0 \end{bmatrix}$

$= \frac{-1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

$\therefore P^{-1}AP = \frac{-1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag}(5, -3, -3)$

Hence $P^{-1}AP$ is a diagonal matrix.

Norm or Length of a Vector

Definition

Let 'V' be an inner product space over the field F. The norm (length) of $\alpha \in V$ denoted by $\|\alpha\|$ is defined as the positive square root of $\langle\alpha, \alpha\rangle$

Norm or length of $\alpha \in V = \|\alpha\| = \sqrt{\langle\alpha,\alpha\rangle} \Rightarrow \|\alpha\|^2 = \langle\alpha, \alpha\rangle$

Unit Vector

Definition

Let V(F) be an inner produce space $\alpha \in V$ is called a unit vector if $\|\alpha\| = 1$.

If $\alpha \in V$ then $\frac{1}{\|\alpha\|}\alpha \in V$ is unit vector.

(b) Find the eigen values and eigen vectors of A $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Sol :

The characteristic equation of A is

$|A - \lambda I| = 0$ i.e., $\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

i.e., $(\lambda + 2)(\lambda - 5) = 0$

Hence the eigen roots of A are −2, 5.

Case (i) :

Let $\lambda = -2$

Eigen vectors X corresponding to the eigen root −2 are given by

$(A - (-2)I)X = 0$

i.e., $\begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_{21}(-1)$ $\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 + 4x_2 = 0$

Let $x_2 = k\ (\neq 0)$

$\therefore x_1 = -\frac{4k}{3}$

$\therefore$ Eigen vectors corresponding to eigen root −2 are given by $k\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ where k is a non-zero parameter. Clearly the sub-space generated by $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ is a one-dimensional characteristic space of R^2.

Case (ii)

Let $\lambda = 5$

Eigen roots X corresponding to the eigen roots 5 are given by

$(A - 5I)X = 0$

i.e., $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_{21}\left(\frac{3}{4}\right)$ $\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + 4x_2 = 0$

Let $x_2 = k\ (\neq 0)$

$\therefore x_1 = k$

$\therefore$ Eigen vectors corresponding to eigen root 5 are given by $k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

where k is a non-zero parameter. Clearly the subspace generated by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a one-dimensional characteristic space of R^2.

12. (a) In a inner product space V(F), prove that $|\langle\alpha, \beta\rangle| \leq \|\alpha\|\,\|\beta\|\ \forall\ \alpha, \beta \in V$.

Sol :

Let $\alpha = \bar{0}$

then $\langle\alpha, \beta\rangle = \langle\bar{0}, \beta\rangle = 0$

and $\|\alpha\|^2 = \langle\alpha, \alpha\rangle = 0$

$\therefore |\langle\alpha, \beta\rangle| = \|\alpha\|\,\|\beta\|$

Case ii

Let $\alpha \neq \bar{0}$

then $\|\alpha\| > 0$ so that $\frac{1}{\|\alpha\|} > 0$

Now consider $\gamma \in v$

$\Rightarrow \gamma = \beta - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}.\alpha$

Now $\langle\gamma, \gamma\rangle = \left\langle\beta - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha, \beta - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\alpha\right\rangle$

$= \langle\beta, \beta\rangle - \frac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\beta, \alpha\rangle - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\langle\alpha, \beta\rangle + \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\frac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\alpha, \alpha\rangle$

$= \|\beta\|^2 - \frac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\langle\beta, \alpha\rangle - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\langle\alpha,\beta\rangle + \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2}\frac{\overline{\langle\beta,\alpha\rangle}}{\|\alpha\|^2}\|\alpha\|^2$

$= \|\beta\|^2 - \frac{\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle}{\|\alpha\|^2}$

$= \|\beta\|^2 - \frac{\langle\alpha,\beta\rangle\langle\alpha,\beta\rangle}{\|\alpha\|^2}$

By defination $\langle\gamma, \gamma\rangle \geq 0$

$\Rightarrow \|\beta\|^2 - \frac{|\langle\alpha,\beta\rangle|^2}{\|\alpha\|^2} \geq 0 \quad (\because Z.\bar{Z} = |z|^2)$

$\Rightarrow \|\beta\|^2\|\alpha\|^2 \geq |\langle\alpha, \beta\rangle|^2$

$\Rightarrow |\langle\alpha, \beta\rangle| \leq \|\alpha\|\,\|\beta\|$

(b) State and prove Gram – Schmidt orthogonalization process.

Sol :

Statement

Every finite dimensional inner product space has an orthonormal basis.

Proof

Let V(F) be an inner product space of dimension 'n' and $B = \{\beta_1, \beta_2 \ldots \beta_n\}$ be a basis of V

Now we shall obtain an orthonormal set $B' = \{\alpha_1 \alpha_2 \ldots \alpha_n\}$ in V(F) such that $L(B') = L(B) = V$

Let us prove by induction method

Since 'B' is a basis of V, B is linearly independent $\Rightarrow \beta_i \neq \bar{0}$ for $i = 1, 2, \ldots n$

Let $\alpha_1 = \frac{\beta_1}{\|\beta_1\|}$ then we have $(\alpha_1, \alpha_1) = 1$ and

α_1 is linear combination of β_1

Thus $\{\alpha_1\}$ is an orthonormal set which is a basis of v such that $L(\alpha_1) = L(\beta_1) = v$ when dim $v = n = 1$

Suppose that $\{\alpha_1, \alpha_2 \ldots \alpha_m\}$ where $1 \leq m < n$ has been constructed as on orthonormal basis of v

When dim V = m then $L\{\alpha_1, \alpha_2 \ldots \alpha_m\} = L(\beta_1, \beta_2 \ldots \beta_m) = V$ and each $\alpha_i = (1, 2 \ldots m)$ is a linear combination of $\beta_1, \beta_2, \ldots \beta_m$ consider the vector.

$\gamma_{m+1} = \beta_{m+1} - \langle\beta_{m+1}, \alpha_1\rangle\alpha_1 - \langle\beta_{m+1}, \alpha_2\rangle\alpha_2 \ldots - \langle\beta_{m+1}\alpha_m\rangle\alpha_m \quad \text{... (1)}$

$\Rightarrow \gamma_{m+1}$ is orthogonal to each of $\alpha_1 \alpha_2 \ldots \alpha_m$

Suppose that $\gamma_{m+1} = \hat{O}$ then $\beta_{m+1} = (\beta_{m+1}, \alpha_1)\alpha_1 + (\beta_{m+1}, \alpha_2)\alpha_2 + \ldots + (\beta_{m+1}, \alpha_m)\alpha_m$

$\Rightarrow \beta_{m+1}$ is a linear combination of $\alpha_1, \alpha_2 \ldots \alpha_m$

By induction hypothesis each $\alpha_i (i = 1, 2 \ldots m)$ is a linear combination of $\beta_1 \beta_2 \ldots \beta_n$

$\therefore \beta_{m+1}$ is a linear combination of $\beta_1 \beta_2 \ldots \beta_m$

This is a contradiction as $\{\beta_1 \beta_2 \ldots \beta_m, \beta_{m+1}\}$ is L.I

This is a contradiction

$\therefore \gamma_{m+1} \neq \bar{O}$

Take $\alpha_{m+1} = \frac{\gamma_{m+1}}{\|\gamma_{m+1}\|}$

Then we have $\|\alpha_{m+1}\| = 1$ and α_{m+1} is orthogonal to each of $\alpha_1\alpha_2 \ldots \alpha_m$

Also $\alpha_{m+1} \neq \alpha_i$ for $i = 1, 2, \ldots m$.

Otherwise from (1) we see that β_{m+1} will become a linear combination of $\beta_1\beta_2 \ldots \beta_m$

We see that α_{m+1} is a linear combination of $\beta_1\beta_2 \ldots \beta_m, \beta_{m+1}$

Thus $\{\alpha_1\ \alpha_2 \ldots \alpha_m, \alpha_{m+1}\}$ is an orthonormal basis of 'v' such that $L\{\alpha_1, \alpha_2 \ldots \alpha_{m+1}\} = L\{\beta_1\ \beta_2 \ldots \beta_{m+1}\} = V$ when dim v = m+1

$\therefore$ By induction $B' = \{\alpha_1, \alpha_2 \ldots \alpha_m\}$ is an orthonormal basis of v such that $L(B) = L(B') = V$ when dim v = n

MARCH / APRIL - 2016

find characteristic value and vectors

Question

The characteristic equation of A is $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)[(1 - \lambda)^2 - 1] - 1[1 - \lambda - 1] + 1[1 - 1 + \lambda] = 0$

$\Rightarrow (1 - \lambda)(\lambda^2 - 2\lambda + 1 - 1) + \lambda + \lambda = 0$

$\Rightarrow \lambda^2 - 2\lambda + 2\lambda - \lambda^3 + 2\lambda^2 = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 = 0 \Rightarrow \lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0$

$\Rightarrow \lambda = 0, 0, 3$

$\therefore$ Eigen values are $\boxed{\lambda = 0,0,3}$

Now the Eigen vectors for $\lambda = 0$ are

$|A - 0.\lambda I| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = B$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let $x_2 = s$, $x_3 = k$

$\Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -s - k$

The eigen vectors are $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

Eigen vectors for $\lambda = 3$ are

$[A - 3\lambda I]x = 0$

$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix}$

$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$

$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let $\boxed{x_3 = k} \Rightarrow \boxed{x_2 = k}$ & $\boxed{x_1 = k}$

$\therefore$ The eigen vectors for $\lambda = 3$ are $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

(b) Find the minimal polynomial for a real matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & 4 & 4 \end{bmatrix}$.

Ans :

The characteristic equation of 'A' is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & 4 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (7 - \lambda)[(7 - \lambda)(4 - \lambda) + 4] - 4[4(4 - \lambda) - 4] - 1$

$[16 + 4(7 - \lambda)] = 0$

$= (7 - \lambda)[28 - 7\lambda - 4\lambda + \lambda^2 + 4] - 4[16 - 4\lambda - 4]$

$- 1[16 + 28 - \lambda] = 0$

$\Rightarrow (7 - \lambda)[\lambda^2 - 11\lambda + 32] - 4[12 - 4\lambda] - 1[44 - \lambda] = 0$

$\Rightarrow (7 - \lambda)[\lambda^2 - 11\lambda + 32] - 48 + 16\lambda - 44 + \lambda = 0$

$\Rightarrow 224 - 77\lambda + 7\lambda^2 - 32\lambda + 11\lambda^2 - \lambda^3 - 48 + 16\lambda - 44 + \lambda = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 92\lambda + 176 = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 92\lambda - 176 = 0$

Now put $\lambda = x$

$\Rightarrow x^3 - 18x^2 + 92x - 176 = 0$

12. (a) Explain ...

4. Prove that $S = \left\{\left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right), \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)\right\}$ **is an orthonormal set in** R^3 **with standard inner product.**

Answer :

Let $\alpha = \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$, $\beta = \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$, $\gamma = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

then $S = \{\alpha, \beta, \gamma\}$

To show that S is an orthonormal set.

We have to prove $\|\alpha\| = \|\beta\| = \|\gamma\| = 1$

and $\langle\alpha, \beta\rangle = \langle\beta, \gamma\rangle = \langle\gamma, \alpha\rangle = 0$

$\therefore \|\alpha\|^2 = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right)$

$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{9}{9} = 1$

$\therefore \|\alpha\| = 1$

Similarly

$\|\beta\|^2 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{4}{9} + \frac{1}{9} + \frac{4}{9}$

$= \frac{9}{9} = 1$

$\gamma\| = 1$

Now

$\langle\alpha, \beta\rangle = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{-2}{3}\right)\left(\frac{2}{3}\right)$

$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$

$= 0$

Similarly

$\langle\beta, \gamma\rangle = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)$

$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$

$= 0$

Similarly

$\langle\gamma, \alpha\rangle = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right)$

$= \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$

$= 0$

$\therefore$ S is an orthonormal set.

... the arc of the parabol...

b) Find the characteristic equation of the matrix and verify that it is satisfied by A.

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Sol :

Given $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The charecteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(\lambda^2 - 4\lambda + 3) + (\lambda - 1) + (\lambda - 1)$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \text{... (1)}$

By caryley Hamilton theorem A satisfies equation (1)

$\Rightarrow A^3 - 6A^2 + 9A - 4I = 0$

$\Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 5 \end{bmatrix}$ $A^3 = \begin{bmatrix} 22 & -21 & 21 \\ 1 & 18 & -13 \\ 20 & -20 & 20 \end{bmatrix}$

$9A = \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$

$\Rightarrow A^2 - 6A^2 + 9A - 4I = 0$