Problem 57 :

If $f = (2x^2 y - x^4)\, i + (e^{xy} - y\sin x)\, j + (x^2 \cos y)\, k$. Find $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

Sol :

Given $f = (2x^2y - x^4)\, i + (e^{xy} - y \sin x)\, j + (x^2 \cos y)\, k$

$$\therefore \frac{\partial f}{\partial x} = (4xy - 4x^3)\, i + (ye^{xy} - y\cos x)\, j + (2x \cos y)\, k$$

$$\frac{\partial^2 f}{\partial x^2}\ (4y - 12x^2)\, i + (y^2 e^{xy} + y \sin x)\, j + (2 \cos y)\, k$$

with

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4x\, i + (e^{xy} + y^2 e^{xy} - \cos x)\, j - 2x \sin y\, k.$$

with

Problem 58 :

Find div f and curl f where $f = grad\,(x^3 + y^3 + z^3 - 3xyz)$

Sol :

$$f = grad\,(x^3 + y^3 + z^3 - 3xyz)$$

$$= \Sigma i \frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \Sigma i\,(3x^2 - 3yz)$$

$$i\,(3x^2 - 3yz) + j\,(3y^2 - 3zx) + k\,(3z^2 - 3xy)$$

$$\text{div } f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 6x + 6y + 6z = 6(x + y + z).$$

$$\text{curl } f = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= i(-3x + 3x) - j(-3y + 3y) + k(-3z + 3z)$$

$$= 0$$

Problem 59 :

Prove that $(f \times \nabla) \times r = -2f$

Sol :

$$(f \times \nabla) = (f \times i)\frac{\partial}{\partial x} + (f \times j)\frac{\partial}{\partial y} + (f \times k)\frac{\partial}{\partial z}$$

$$(f \times \nabla) \times r = (f \times i) \times \frac{\partial r}{\partial x} + (f \times j) \times \frac{\partial r}{\partial y} + (f \times k) \times \frac{\partial r}{\partial z}$$

$$= \Sigma\,(f \times i) \times i$$

$$= \Sigma\,\{(f \cdot i) \cdot i - f\}$$

$$= (f \cdot i)\, i + (f \cdot j)\, j + (f \cdot k)\, k - 3f$$

$$= f - 3f$$

$$= -2f.$$

Theorem 60 :

Prove that grad $(A \cdot B) = (B \cdot \nabla) A + (A \cdot \nabla) B$ + B X Curl A + A X Curl B.

Sol :

Now $A \times \text{Curl } B = A \times (\nabla \times B)$

curl B = ∇×B
∇×B = Σi × ∂B/∂x

$$= A \times \Sigma i \times \frac{\partial B}{\partial x} = \Sigma A \times \left(i \times \frac{\partial B}{\partial x}\right)$$

$$= \Sigma \left\{\left(A \cdot \frac{\partial B}{\partial x}\right) i - (A \cdot i)\frac{\partial B}{\partial x}\right\}$$

$$= S_i\left(A \cdot \frac{\partial B}{\partial x}\right) - \left(A - \times \Sigma i \frac{\partial}{\partial x}\right) B$$

$$\therefore A \times \text{Curl } B = \Sigma i \left(A \cdot \frac{\partial B}{\partial x}\right) - (A \cdot \nabla) B$$

$$\Rightarrow A \times \text{Curl } B + (A \cdot \nabla) B = \Sigma i\left(A \cdot \frac{\partial B}{\partial x}\right) \quad \dots (1)$$

Similarly $B \times \text{Curl } A = \Sigma i \left(B \cdot \frac{\partial A}{\partial x}\right) - (B \cdot \nabla) A$

$$\Rightarrow B \times \text{Curl } A + (B \cdot \nabla) A = \Sigma i \left(B \cdot \frac{\partial A}{\partial x}\right) \quad \dots (2)$$

Adding (1) and (2) we have

$$\therefore A \times \text{Curl } B + (A \cdot \nabla) B + B \times \text{Curl } A + (B \cdot \nabla) A$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x}\right) + \Sigma i \left(B \cdot \frac{\partial A}{\partial x}\right)$$

$$= \Sigma i \left(A \cdot \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x} \cdot B\right)$$

$$= \Sigma i \frac{\partial}{\partial x}(A \cdot B)$$

$$= \nabla (A \cdot B)$$

$$= \text{grad } (A \cdot B)$$

Theorem 61 :

Prove that curl $(A \times B) = A\, divB - B\, divA + (B \cdot \nabla) A - (A \cdot \nabla) B$.

Proof :

$$\text{Curl } (A \times B) = \nabla \times (A \times B)$$

$$= \Sigma i \times \frac{\partial}{\partial x}(A \times B) = \Sigma i \times \left(\frac{\partial A}{\partial x} \times B + A \times \frac{\partial B}{\partial x}\right)$$

$$= \Sigma i \times \left(\frac{\partial A}{\partial x} \times B\right) + \Sigma i \times \left(A \times \frac{\partial B}{\partial x}\right)$$

$$= \Sigma \left\{(i \cdot B)\frac{\partial A}{\partial x} - \left(i \cdot \frac{\partial A}{\partial x}\right) B\right\} + \Sigma \left\{\left(i \cdot \frac{\partial B}{\partial x}\right) A - (i \cdot A)\frac{\partial B}{\partial x}\right\}$$

$$= \Sigma (B \cdot i)\frac{\partial A}{\partial x} - \Sigma\left(i \cdot \frac{\partial A}{\partial x}\right) B + \Sigma\left(i \cdot \frac{\partial B}{\partial x}\right) A - \Sigma (A \cdot i)\frac{\partial B}{\partial x}$$

$$= \left(B \cdot \Sigma i \frac{\partial}{\partial x}\right) A - \left(\Sigma i \frac{\partial A}{\partial x}\right) B + \left(\Sigma i \frac{\partial B}{\partial x}\right) A - \left(A \cdot \Sigma i \frac{\partial}{\partial x}\right) B$$

$$= (B \cdot \nabla) A - (\nabla \cdot A) B + (\nabla \cdot B) A - (A \cdot \nabla) B$$

$$= A\, divB - B\, divA + (B \cdot \nabla) A - (A \cdot \nabla) B.$$

Theorem 62 :

Prove that $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$

Proof :

$$\nabla \times (\nabla \times A) = \Sigma i \times \frac{\partial}{\partial x}(\nabla \times A)$$

$$\text{Now, } i \times \frac{\partial}{\partial x}(\nabla \times A) = i \times \frac{\partial}{\partial x}\left(i \times \frac{\partial A}{\partial x} + j \times \frac{\partial A}{\partial y} + k \times \frac{\partial A}{\partial z}\right)$$

$$= i \times \left(i \times \frac{\partial^2 A}{\partial x^2} + j \times \frac{\partial^2 A}{\partial x \partial y} + k \times \frac{\partial^2 A}{\partial x \partial z}\right)$$

$$= i \times \left(i \times \frac{\partial^2 A}{\partial x^2}\right) + i \times \left(j \times \frac{\partial^2 A}{\partial x \partial y}\right) + i \times \left(k \times \frac{\partial^2 A}{\partial x \partial z}\right)$$

$$= i \frac{\partial}{\partial x}\left(i \cdot \frac{\partial A}{\partial x}\right) + j \frac{\partial}{\partial y}\left(i \cdot \frac{\partial A}{\partial x}\right) + k \frac{\partial}{\partial z}\left(i \cdot \frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$= \nabla \left(i \cdot \frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$\therefore \Sigma i \times \frac{\partial}{\partial x}(\nabla \times A) = \nabla\, \Sigma i \cdot \frac{\partial A}{\partial x} - \Sigma \frac{\partial^2 A}{\partial x^2}$$

$$= \nabla (\nabla \cdot A) - \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}\right)$$

$$\therefore \nabla (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A.$$

Problem 63 :

If $F = (3x^2 + 6y)\, i - 14 yz\, j + 20 xz^2 k$. Calculate $\int_C F.dr$ along the lines from (0, 0, 0) to (1, 0, 0) then to (1, 1, 0) and then to (1, 1, 1).

Sol :

Given $F = (3x^2 + 6y)\, i - 14yz\, j + 20xz^2 k$

$r = xi + yj + zk$ $\quad dr = i\,dx + j\,dy + k\,dz$

$$\therefore \int_C F.dr = \int_C (3x^2 + 6y)\, dx - 14yz\, dy + 20xz^2\, dz$$

(i) Along the line from (0, 0, 0) to (1, 0, 0)

$y = 0,\ z = 0 \Rightarrow dy = 0,\ dz = 0$

x varies from 0 to 1.

∴ The integral over this part of the path.

$$\therefore \int_{C_1} F.dr = \int_{x=0}^{1} 3x^2\, dx = [x^3]_0^1 = 1.$$

(ii) Along the line from (1, 0, 0) to (1, 1, 0)

$x = 1,\ z = 0 \Rightarrow dx = 0,\ dz = 0$

y varies from 0 to 1.

∴ The integral over this part of the path.

$$\therefore \int_{C_2} F.dr = \int_{y=0}^{1} (0 - 0 + 0)\, dy = 0$$

(iii) Along the line from (1, 1, 0) to (1, 1, 1)

$x = 1,\ y = 1 \Rightarrow dx = 0,\ dy = 0$

z varies from 0 to 1.

The integral over this part of the path.

$$\therefore \int_{C_3} F.dr = 20z^2\, dz = \left[\frac{20z^3}{3}\right]_0^1 = \frac{20}{3}$$

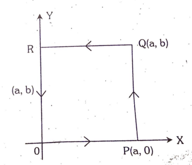
$$\therefore \text{Adding} \int_C F.dr = 1 + 0 + \frac{20}{3} = \frac{23}{3}.$$

Problem 64 :

If $F = (x^2 + y^2)\, i - 2xyj$ evaluate $\oint_C F.dr$ where the curve C is the rectangle in the xy plane bounded by $y = 0$, $y = b$, $x = 0$, $x = a$.

Sol :

Since the inegration takes place in xy place (z =)



$$\therefore \oint_C F.dr = \oint_C F_1 dx + F_2 dy$$

$$= \oint_C (x^2 + y^2)\, dx - 2xy dy$$

(i) Along OP $y = 0$, $dy = 0$ and x varies from 0 to a.

$$\int F.dr = \int_0^a x^2\, dx = \frac{a^3}{3}.$$

(ii) Along PQ $x = a$, $dx = 0$ and y changes from 0 to b.

$$\therefore \int F.dr = \int_0^b (-2ay)\, dy = -ab^2$$

(iii) Along QR, $y = b$, $dy = 0$ and x changes from a to 0.

$$\therefore \int_{QR} F.dr = \int_a^0 (x^2 + b^2)\, dx = \frac{-1}{3} a^3 - ab^2.$$

(iv) Along RO, $x = 0$, $dx = 0$ and y varies from 0 to b.

$$\therefore \int_{OR} F.dr = \int_b^0 0 = 0$$

$$\oint F.dr = \frac{a^3}{3} - ab^2 - \frac{1}{3} a^3 - ab^2$$

$$= -2ab^2$$

Problem 65 :

If $F = yi + zj + xk$. Find the circulation of F round the curve C where C is the circle $x^2 + y^2 = 1$, $z = 0$.

Sol :

Equation to the curve C

$x^2 + y^2 = 1,\ z = 0$

$\therefore dz = 0.$

In parametric form, $x = \cos\theta$, $y = \sin\theta$, $z = 0$

The circulation of $F = yi + zj + xk$, along C is

$$\oint_C F.dr = \oint F_1 dx + F_2 dy + F_3 dz$$

$$= \oint y\, dx + z\, dy + x\, dz$$

$$= \oint y\, dx$$

For the circle θ varies from 0 to 2Π

$$\therefore \oint_C F.dr = \int_0^{2\Pi} \sin\theta(-\sin\theta)\, d\theta$$

$$= -4 \int_0^{\Pi/2} \sin^2\theta\, d\theta = -4\left(\frac{1}{2}\right)\frac{\Pi}{2} = -\Pi.$$

Problem 66 :

Evaluate $\int_S F.N\, ds$, where $F = zi + xj - 3y^2 z\, k$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol :

Let $\phi = x^2 + y^2 = 16$

∴ The normal to the surface S is grand φ (i.e., ∇φ)

$$\therefore \text{Normal} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= 2xi + 2yj$$

$$\therefore \text{Unit normal } N = \frac{2xi + 2yj}{\sqrt{4x^2 + 4y^2}} = \frac{xi + yj}{\sqrt{x^2 + y^2}}$$

$$= \frac{xi + yj}{4} \quad (\because x^2 + y^2 = 16 \text{ on } S)$$

Let R be the projection of S on YZ plane, then $\iint_S F.N\, dS = \iint F.N \frac{dy\,dz}{|N.i|}$

Given $F = zi + xj - 3y^2 zk$

$$\therefore F.N = (zi + xj - 3y^2 zk) \cdot \frac{(xi + yj)}{4}$$

$$= \frac{1}{4}(xz + xy) \text{ and}$$

$$N.i = \frac{1}{4}(xi + yj) = \frac{x}{4}$$

For the surface $x^2 + y^2 = 16$ in yz plane $x = 0 \Rightarrow y = 4$. Hence in first octant y varies from 0 to 4

z varies from 0 to 5

Then the surface integral

$$\iint_R F.N \frac{dy\,dz}{|N.i|} = \iint_R \left(\frac{xz + xy}{4}\right)\left(\frac{4}{x}\right) dy\, dz$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (y + z)\, dy\, dz$$

$$= \int_{y=0}^{4}\int_{0}^{5} (y\,dy)\, dz + \int_{y=0}^{4}\int_{z=0}^{5} dy\,(z\,dz)$$

$$= \int_0^4 y dy \int_0^5 dz + \int_0^4 dy \int_0^5 z\, dz$$

$$= \left[\frac{y^2}{2}\right]_0^4 [z]_0^5 + [y]_0^4 \left[\frac{z^2}{2}\right]_0^5$$

$$= 8.5 + 4 \cdot \frac{25}{5} = 90.$$

Problem 67 :

Evaluate $\int_S F.N\, dS$ where $F = 18zi - 12j + 3yk$ and S is the part of the plane $2x + 3y + 6z = 12$ located in the first octant.

Sol :

Let $\phi = 2x + 3y + 6z - 12$

∴ Normal to the plane φ is ∇φ

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = 2i + 3j + 6k$$

$$\therefore \text{Unit normal, } N = \frac{2i + 3j + 6k}{\sqrt{4 + 9 + 36}} = \frac{1}{7}(2i + 3j + 6k)$$

Let R be the projection of S on XY plane, then

$$\int_S F.N\, ds = \iint_R F.N \frac{dxdy}{|N.k|}$$

Given $F = 18\,zi - 12\,j + 3\,yk$

$$\therefore F.N = (18\,zi - 12\,j + 3\,yk) \cdot \frac{1}{7}(2i + 3j + 6k)$$

$$= \frac{1}{7}(36z + 36 + 18y) = \frac{6}{7}(6z - 6 + 3y)$$

$$N.k = \frac{1}{7}(2i + 3j + 6k) \cdot k = \frac{6}{7}$$

Given surface S is $2x + 3y + 6z = 12$

∴ R of xy plane is $2x + 3y = 12$.

$$\Rightarrow y = \frac{12 - 2x}{3} \qquad \therefore y = 0 \Rightarrow x = 6$$

∴ x varies from 0 to 6 any y varies from 0 to $\frac{12 - 2x}{3}$

The surface integral $\iint_R F.N \frac{dxdy}{|N.k|}$

$$= \iint_R \frac{6}{7}(6z - 6 + 3y) \cdot \frac{7}{6}\, dx\, dy$$

$$= \iint_R (12 - 2x - 3y - 6 + 3y)\, dx\, dy$$

$$(\because 6z = 12 - 2x - 3y)$$

$$= \iint_R (6 - 2x)\, dx\, dy$$

$$= 2\int_0^6 (3 - x)\left[\int_0^{\frac{1}{3}(12 - 2x)} dy\right] dx$$

$$= 2\int_0^6 (3 - x)\, [y]_0^{\frac{1}{3}(12-2x)}\, dx$$

$$= 2\int_0^6 \left((3 - x)\frac{1}{3}(12 - 2x)\right) dx$$

$$= \frac{4}{3}\int_0^6 (18 - 9x + x^2)\, dx$$

$$= \frac{4}{3}\left[18x - \frac{9}{2}x^2 + \frac{1}{3}x^3\right]_0^6$$

$$= 24$$

Problem 68 :

If $F = 4xzi - y^2 j + yzk$, evalute $\int F.NdS$ where S is the surface of the cube bounded by $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.

Sol :

Consider the cube surrounded by the following faces.

(i) For the face PQRS, i is the outward normal

∴ $N = i$, $x = a$, $dS = dydz$

$$\therefore \int F.NdS$$

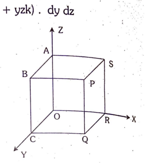
$$= \int_{y=0}^{a}\int_{z=0}^{a} (4x\,zi - y^2 j + yzk) \cdot dy\, dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} 4xz\, dy\, dz$$

$$= 4a \int_{y=0}^{a} dy \int_0^a z\, dz$$

$\because$ x = a on this face

$$= 4a\, [y]_0^a \left[\frac{z^2}{2}\right]_0^a = 2a^4.$$



(ii) For the face OABC, $-i$ is the outward normal

∴ $N = -i$, $x = 0$, and $dS = dy\, dz$

$$\therefore \int F.N\, dS = \iint_{R_2} (4x\,zi - y^2 j + yzk) \cdot (-i)\, dy\, dz$$

$$= -\iint_{R_2} 4x\, zdy\, dz = 0$$

$\because$ x = 0 on this face.

(iii) For the face BPQC, j is the outward normal.

∴ $N = j$, $y = a$ and $dS = dx\, dz$

$$\therefore \int_{R_3} F.N\, dS = \iint_{R_3} (4x\,zi - y^2 j + yzk) \cdot j\, dx\, dz$$

$$= -\iint_{R_3} y^2\, dx\, dz$$

$$= -a^2 \int_{x=0}^{a}\int_{z=0}^{a} dy\, dz$$

$$(\because y = a, \text{ on this face } \dots -a^2 [x]_0^a [z]_0^a = -a^4)$$

(iv) For the face ORSA, $-j$ is the outward normal.

∴ $N = -j$, $y = 0$ and $dS = dx\, dz$

$$F.N = (4x\,zi - y^2 j + yzk) \cdot (-j) = y^2$$

$$\therefore \int_{R_4} F.N\, dS = \iint_{R_4} y^2\, dx\, dz = 0$$

$\because$ y = 0, on this face.

(v) For the face ABPS, k is the outward normal

∴ $N = k$, $z = a$ and $dS = dx\, dy$

$$F.N = (4x\,zi - y^2 j + yzk) \cdot k = yz$$

$$\therefore \int_{R_5} F.N\, dS = \iint_{R_5} yz\, dx\, dy$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} ay\, dx\, dy$$

$$= a\, [x]_0^a \cdot \left[\frac{y^2}{2}\right]_0^a$$

$$= \frac{a^4}{4}$$

(vi) For the face OCQR, $-k$ is the outward normal.

∴ $N = -k$, $z = 0$ and $dS = dx\, dy$

$$F.N = (4x\,zi - y^2 j + yzk) \cdot (-k) = -yz$$

$$\therefore \int_{R_6} F.N\, dS = -\iint_{R_6} yz\, dx\, dy = 0$$

$\because$ z = 0, on this face.

Hence adding all we get,

$$\int_S F.N\, dS = 2a^4 + 0 - a^4 + 0 + \frac{1}{2}a^4 + 0 = \frac{3}{2}a^4.$$

Problem 69 :

If $F = (2x^2 - 3z)\, i - 2xyj - 4xk$ evaluate (a) $\int_V \nabla . F\, dV$ and (b) $\int_V \nabla \times F\, dV$ where V is the closed region bounded by $x = 0$, $y = 0$, $z = 0$, $2x + y + z = 4$.

Sol :

$$\text{(a)} \quad \nabla . F = \left(i \cdot \frac{\partial F}{\partial x} + j \cdot \frac{\partial F}{\partial y} + k \cdot \frac{\partial F}{\partial z}\right) = 4x - 2x = 2x$$

The limits are $z = 0$ to $z = 4 - 2x - 2y$

$y = 0$ to $\frac{4 - 2x}{2}$ i.e., $(2 - x)$ $x = 0$ to $\frac{4}{2}$ i.e., 2

$$\therefore \int_V \nabla . \bar{F}\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x\,[z]_0^{4-2x-2y}\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x\,(4-2x-2y)\,dx\,dy$$

$$= 4\int_{x=0}^{2}\int_{y=0}^{2-x} (2x - x^2 - xy)\,dx\,dy$$

$$= 4\int_0^2 \left[2x - x^2 - x\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= 4\int_0 \left[(2x-x^2)(2-x) - \frac{x}{2}(2-x^2)\right] dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x)\,dx$$

$$= \left[\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2\right] = \frac{8}{3}$$

(b) $$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = j - 2yk.$$

$$\therefore \int_V \nabla \times F\,dV = \iiint_V (j - 2yk)\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \int (j-2yk)\,[z]_0^{4-2x-2y}\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (j-2yk)\,(4-2x-2y)\,dx\,dy$$

$$= \int_{x=0}^{2} j\left[(4-2x)y - y^2\right]_0^{2-x} dx$$

$$-k\int_{x=0}^{2}\left[(4-2x)y^2 - \frac{4y^3}{3}\right] dx$$

$$= j\int_0^2 (2-x)^2\,dx - k\int_0^2 \frac{2}{3}(2-x)^3\,dx$$

$$= j\left[\frac{(2-x)^3}{3}\right]_0^2 - \frac{2k}{3}\left[\frac{(2-x)^4}{4}\right]_0^2$$

$$= \frac{8}{3}(j - k)$$

Theorem 70 :

GAUSS'S DIVERGENCE THEOREM.

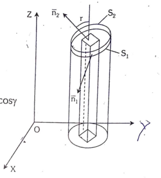
Let S be a closed surface enclosed a volume V.

If F is a continuously differentiable vector point function, then

$$\int_V \operatorname{div} F\,dv = \int_S F\,.\,N\,dS$$

where N is the outward drawn unit normal vector at any point of S.

Cartesian Form :



Let $F = F_1 i + F_2 j + F_3 k$ and

$N = i\cos\alpha + j\cos\beta + k\cos\gamma$

where $\cos\alpha$, $j\cos\beta$, $\cos\gamma$ are the direction cosines of N

$F.N = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$

Also $\operatorname{div}F = \nabla\,.\,F$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Hence the divergence theorem can be written as

$$\iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

$$= \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS$$

$$= \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

Proof :

Let S be a closed surface. Let us choose the coordinate axes so that any line parallel to the coordinate axes cuts S in at most two points. Let R be the projection of S on xy-plane.

S_1 and S_2 be the lower and upper parts of S.

$z = f(x, y)$ and $z = g(x, y)$ be the equations of S_1 and S_2.

The relation can be put in the form of $f(x, y) \le z \le g(x, y)$

$$\therefore \int_V \frac{\partial F_3}{\partial z}\,dV = \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$$

$$= \iint_R \left[\frac{\partial F_3}{\partial z}\,dz\right]_{z=f(x,y)}^{z=g(x,y)} dx\,dy$$

$$= \iint_R [F_3(x,y,z)]_f^g\,dx\,dy$$

$$= \iint_R [F_3(x,y,g) - F_3(x,y,F)]\,dx\,dy$$

$$= \iint_R F_3(x,y,g)\,dx\,dy - \iint_R [F_3(x,y,g)$$

$$- F_3(x,y,F)]_{dy}^{dx} \quad \text{... (1)}$$

For the upper part S_2 and dx dy = dS. $\cos\gamma$ = N . kdS.

Since the normal to S_2 makes an acute γ with k.

$$\therefore \iint_R F_3(x,y,g)\,dx\,dy = \int_{S_1} F_3 N\,.\,k\,dS$$

For the lower portion S_1 dx dy = $-\cos\gamma$ dS = $-$N . kdS

Since the normal to S_1 makes an dotuse angle γ with k.

$$\therefore \iint_R F_3(x,y,g)\,dx\,dy = -\int_{S_1} F_3 N\,.\,k\,dS$$

Hence from (1)

$$\therefore \int_V \frac{\partial F_3}{\partial z}\,dV = \int_{S_2} F_3 k\,.\,N\,dS + \int_{S_1} F_3 k\,.\,N\,dS$$

$$= \int_S F_3 k\,.\,N\,dS \quad \text{... (1)}$$

Similarly, by projecting S on the other coordinate planes

$$\int_V \frac{\partial F_2}{\partial y}\,dV = \int_S F_2 j\,.\,N\,dS \quad \text{... (2)}$$

$$\int_V \frac{\partial F_1}{\partial x}\,dV = \int_S F_1 i\,.\,N\,dS \quad \text{... (3)}$$

Adding we have,

$$\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV = \int (F_1 i + F_2 j + F_3 k)\,.\,N\,dS$$

$$\Rightarrow \int \nabla\,.\,F\,dV = \int F\,.\,N\,dS.$$

Problem 71 :

Show that $\int_S (axi + byj + czk)\,.\,N\,dS = 4\frac{\Pi}{3}(a+b+c)$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Sol :

Here $\Gamma = axi + byj + czk$

$$\operatorname{div}\dot{F} = \nabla\,.\,F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

Gauss's theorem,

$$\int_S F\,.\,N\,dS = \int_V \nabla\,.\,F\,dV$$

$$= \int_V (a+b+c)\,dV$$

$$= (a+b+c)\,V$$

$$= \frac{4\Pi}{3}(a+b+c)$$

$V = \dfrac{4\Pi}{3}$, fo the given sphere.

Problem 72 :

Verify Gauss's divergence theorem to evaluate $\int_S (x^3 - yz)i - 2x^2yj + zk)\,.\,N\,dS$ ***over the surface of a cube bounded by the coordinate planes*** $x = y = z = a$.

Sol :

Gauss's theorem states that

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

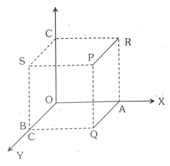
From the problem $F_1 = x^3 - yz$, $F_2 = 2x^2y$, $F_3 = z$

$$\frac{\partial F_1}{\partial x} = 3x^2,\quad \frac{\partial F_2}{\partial y} = -2x^2,\quad \frac{\partial F_3}{\partial z} = 1$$

$$\therefore \text{RHS} = \iiint_V (3x^2 - 2x^2 + 1)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a\int_0^a \left[\frac{x^3}{3} + x\right]_0^a dy\,dz$$

$$= \frac{1}{3}a^5 + a^3$$



Verification

Let us calculate directly the value of $\int$ F . N dS over the six faces of the cube

(i) For the face PQAR N = i, dS = dz dy and x = a

$$\therefore \int F\,.\,N\,dS = \iint_{S_1} [(x^3 - yz)\,i - 2x^2yj + zk],\ i\,dz\,dy$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} (x^3 - yz)\,dz\,dy = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)\,dz\,dy$$

$$= a^3 [y]_0^a [z]_0^a - \left[\frac{y^2}{2}\right]_0^a \left[\frac{z^2}{2}\right]_0^a$$

$$= a^5 - \frac{1}{4}a^4$$

(ii) For the face OBSC N = $-$i, dS = dy dz and x = 0.

$$\therefore \int_{S_2} F\,.\,N\,dS = -\int_{y=0}^{a}\int_{z=0}^{a} (x^3 - yz)\,dy\,dz$$

$$= \int_0^a\int_0^a yz\,dy\,dz = \left[\frac{y^2}{2}\right]_0^a \left[\frac{z^2}{2}\right]_0^a = \frac{1}{4}a^4.$$

$\because$ x = 0 for this face

(iii) For the face PQBS N = j, dS = dz dx and y = a

$$\therefore \int_{S_3} F\,.\,N\,dS = -\int_{x=0}^{a}\int_{z=0}^{a} (2x^2 y)\,dx\,dz$$

$$= -2a\int_0^a\int_0^a x^2\,dx\,dz \quad \because y = a \text{ for this face}$$

$$= -2a\left[\frac{x^3}{3}\right]_0^a [z]_0^a = \frac{-2}{3}a^5.$$

(iv) For the face OARC N = $-$j, dS = dz dx and y = 0

$$\int_{S_4} F\,.\,N\,dS = \iint_{S_4} 2x^2 y\,dx\,dz = 0$$

$\because$ on this face y = 0

(v) For the face PRCS N = k, dS = dx dy and z = a

$$\therefore \int_{S_5} F\,.\,N\,dS = -\int_{x=0}^{a}\int_{y=0}^{a} z\,dx\,dy$$

$$= a\int_0^a\int_0^a dx\,dz \quad \because z = a \text{ on this face}$$

$$= a\,[x]_0^a\,[y]_0^a = a^3.$$

(vi) For the face OBQA N = $-$k, dS = dx dy and z = 0

$$\therefore \int_{S_6} F\,.\,N\,dS = -\int_{x=0}^{a}\int_{y=0}^{a} z\,dx\,dy = 0$$

$\because$ on this face z = 0

Hence for the total faces

$$\int_S F\,.\,N\,dS = a^5 - \frac{1}{4}a^4 + \frac{1}{4}a^4 - \frac{2}{3}a^5 + 0 + a^3$$

$$= \frac{1}{3}a^5 + a^3$$

Theorem 73 : 12

GREEN'S THEOREM IN A PLANE.

Let S be a closed region in xy-plane enclosed by a curve C. Let P and Q be continuous and differentiable scalar functions of x and y in S. Then

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

The line integral being taken round C such that S is on left as one advances along C.

Proof :

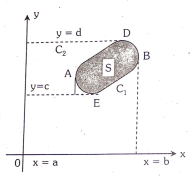
Let any line parallel to either coordinate axes cut C in atmost two points. Let S lies between the lines x = a, x = b, and y = c, y = d.

Let y = f(x) be the curve C_1 (i.e., AEB)

and y = g(x) be the curve C_2 (i.e., ADB)

where $f(x) \le g(x)$

$$\text{Consider } \iint_S \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} \frac{\partial P}{\partial y}\,dx\,dy$$



$$= \int_a^b [P(x,y)]_{y=f(x)}^{g(x)}\,dx$$

$$= \int_a^b [P(x,g) - P(x,f)]\,dx$$

$$= \int_a^b P(x,g)\,dx - \int_a^b P(x,f)\,dx$$

$$= -\int_b^a P(x,y)\,dx - \int_a^b P(x,f)\,dx$$

$$= -\int_{C_2} P(x,y)\,dx - \int_{C_1} P(x,y)\,dx$$

$$\therefore \iint_S \frac{\partial P}{\partial y}\,dx\,dy = -\oint_C P\,dx \quad \text{... (1)}$$

Similarly we can prove that $\iint_S \frac{\partial Q}{\partial x}\,dx\,dy = \oint_C Q\,dy$ ... (2)

Hence adding (1) and (2),

We get $\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

Problem 74 :

Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - ydx$ ***and hence find the area of the ellipse*** $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\Pi$.

Sol :

By Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Put P = $-$y and Q = +x $\quad \frac{\partial P}{\partial y} = -1$, $\partial Q = +1$

$$\therefore \oint_C x\,dy - ydx = 2\int_S dx\,dy = 2A$$

where A is the are of surface S.

$$\therefore \frac{1}{2}\int_C x\,dy - ydx = A$$

Now for the ellipse $x = a\cos\theta$, $y = b\sin\theta$

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - ydx$$

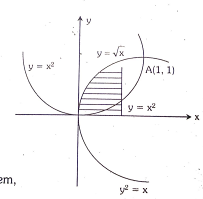
$$= \frac{1}{2}\int_0^{2\Pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\Pi} d\theta = \Pi ab$$

Problem 75 :

Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ ***where C is the region bounded by*** $y = \sqrt{x}$ ***and*** $y = x^2$.

Sol :



Here $P = 3x^2 - 8y^2$

$$\frac{\partial P}{\partial y} = -16y$$

$Q = 4y - 6xy$

$$\frac{\partial Q}{\partial x} = -6y$$

Hence by Green's theorem,

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dx$$

$$= \iint_S (-6y + 16y)\,dx\,dy = 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} y\,dy\,dx$$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\,dx = \frac{3}{2}$$

(ii) Verification

The line integral along C

= line integral along $y = x^2$ (from 0 to A)

+ line integral along $y^2 = x$ (from A to 0)

$= I_1 + I_2$

$$\therefore I_1 = \int_{x=0}^{1} [3x^2 - 8(x^2)^2]\,dx + [4x^2 - 6x(x^2)]\,2x\,dx$$

$$= \int_0^1 (3x^2 + 8x^2 - 20x^4)\,dx = -1$$

$$I_2 = \int_0^1 (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x^{3/2})\frac{1}{2\sqrt{x}}\,dx$$

$\because y = \sqrt{x}$

$$= \int_0^1 (3x2 - 11x + 2)\,dx = \frac{5}{2}$$

$$\therefore I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$$

Hence the theorem is verified.

12 **Stokes Theorem**

Let S be a surface bounded by a closed, noninteresting curve C. If F is any differentiable vector point function, the

$$\oint_C F\,.\,dr = \int_S \operatorname{Curl} F\,.\,N\,dS$$

where C is traversed in the positive direction. The direction of C is called positive if a person, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal N to S, has the surface on his left.

Problem 76 :

If $F = yi + (x - 2xz)\,j - xyk$ ***evaluate*** $\int_S (\nabla \times F)\,.\,N\,dS$ ***where S is the surface of the sphere*** $x^2 + y^2 + z^2 = a^2$ ***above the xy - plane.***

Sol :

Given $\quad F = yi + (x - 2xz)\,j - xyk$

By stokes theorem

$$\int_S \nabla \times F\,.\,N\,dS = \int_C F\,.\,dr = \int_C F_1dx + F_2dy + F_3dz$$

$$= \int_C ydx + (x - 2xz)\,dy - xy\,dz$$

Above the xy - plane the sphere is z = 0,

$x^2 + y^2 = a^2$.

$$\therefore \int_C F\,.\,dr = \int_C ydx + x\,dy$$

Put $x = a\cos\theta$, $y = a\sin\theta$

$dx = -a\sin\theta$, $dy = a\cos\theta\,d\theta$ and

θ varies from 0 to 2Π.

$$\therefore \int_C F\,.\,dr = \int_0^{2\Pi} [a\sin\theta + (a\cos\theta)(a\cos\theta)]\,d\theta$$

$$= a^2\int_0^{2\Pi} \cos 2\theta\,.\,d\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{2\Pi}$$

$$= 0$$

Problem 77 :

Verify stokes theorem for $A = (2x - y)\,i - yz^2j - y^2zk$, ***where S is the upper half surface of the sphere*** $x^2 + y^2 + z^2 = 1$ ***and C is its boundary.***

Sol :

The boundary C of S is a circle in xy-plane

i.e., $x^2 + y^2 = 1$, $z = 0$

The parametric equation is

$x = \cos t$, $y = \sin t$, $z = 0$ for $0 \le t \le 2\Pi$

$$\int_C A\,.\,dr = \int_C F_1dx + F_2dy + F_3dz$$

$$= \int_C (2x - y)\,dx - yz^2\,dy - yz^2dz$$

$$= \int_C (2x - y)\,dx$$

$\therefore$ z = 0, dz = 0

$$= -\int_0^{2\Pi} (2\cos t\ \ \sin t)\,\sin t\,dt$$

$$= \int_0^{2\Pi} \sin^2 t\,dt - \int_0^{2\Pi} \sin 2t\,dt$$

$$= 4\int_0^{\Pi/2} \sin^2 t\,.\,dt + \left[\frac{\cos 2t}{2}\right]_0^{2\Pi}$$

$$= 4\,.\,\frac{1}{2}\,.\,\frac{\Pi}{2} + \frac{1}{2}(1 - 1) = \Pi.$$

$$\text{Also } \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = k$$

$$\therefore \int_z (\nabla \times A)\,.\,N\,dS = \int_z k\,.\,N\,dS = \iint_R dx\,dy$$

Since k . N dS dx dy and R is the projection of S on xy - plane.

$$\text{Now } \iint_R dx\,dy$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx$$

$$= 4\int_0^1 \sqrt{(1-x^2)}\,dx, \text{ put } x = \sin\theta,\ dx = \cos\theta\,d\theta$$

$$= 4\int_0^{\Pi/2} \cos^2\theta\,.\,d\theta$$

$$= 4\,.\,\frac{1}{2}\,\frac{\Pi}{2} = \Pi$$

Thus stokes theorem is verified.