Unit-I (Fourier Series:)

1. **Fourier Series:**

The Fourier Series for the function $f(x)$ in the interval $a < x < a+2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \operatorname{Sin} x.$$

where

$$a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\operatorname{Cos} nx\, dx$$

$$b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\operatorname{Sin} nx\, dx$$

These values a_0, a_n, b_n are known as Euler's formulae.

2. **Obtain the fourier Series for $f(x) = e^{-x}$ is the interval $0<x<2\pi$**

Sol: Let

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{Cos} nx + \sum_{n=1}^{\infty} b_n \operatorname{Sin} nx \quad \to(1)$$

Now Consider

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

III^{ly}

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\operatorname{Cos} nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\operatorname{Cos} nx\, dx$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\operatorname{Cos} nx + n \operatorname{Sin} nx)\right]_0^{2\pi}$$

$$\Rightarrow \left(\frac{1-e^{-2\pi}}{\pi}\right)\left(\frac{1}{(n^2+1)}\right)$$

Now Subs n = 1, 2, 3....

$$a_1 = \frac{1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right) \quad a_2 = \frac{1}{5}\left(\frac{1-e^{-2\pi}}{\pi}\right) \text{ and also on.}$$

III^{ly} Consider

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\operatorname{Sin} nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\operatorname{Sin} nx\, dx$$

$$\Rightarrow \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\operatorname{Sin} nx - n\operatorname{Cos} nx)\right]_0^{2\pi}$$

$$\Rightarrow \left(\frac{1-e^{-2\pi}}{\pi}\right)\left(\frac{n}{(n^2+1)}\right)$$

$$\therefore\ b_1 = \frac{1}{2}\left(\frac{1-e^{-2\pi}}{\pi}\right) \quad b_2 = \frac{1}{5}\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

So on

Substituting the vallues of a_0, a_n, b_n in (1) we get.

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left\{\frac{1}{2}\operatorname{Cos} x + \frac{1}{5}\operatorname{Cos} 2x + \frac{1}{10}\operatorname{Cos} 3x + \ldots\right\} + \left\{\frac{1}{2}\operatorname{Sin} x + \frac{2}{5}\operatorname{Sin} 2x + \frac{3}{10}\operatorname{Sin} 3x + \ldots\right\}\right]$$

3. **Proce that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \operatorname{Cos} nx}{n^2}$, $-\pi < x < \pi$**

Hence show that

(i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ (iv) $\sum \frac{1}{n^4} = \frac{\pi^4}{36}$

Sol:

Let $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{Cos} nx + \sum_{n=1}^{\infty} b_n \operatorname{Sin} nx \quad \to(1)$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3+\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$\therefore a_0 = \frac{2\pi^2}{3}$$

III^{ly}

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \operatorname{Cos} nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\operatorname{Sin} nx}{n}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\left(\frac{-\operatorname{Sin} nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi n}\left[x\left(\frac{\operatorname{Cos} nx}{n}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\operatorname{Cos} nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi n^2}\left[(x \operatorname{Cos} nx)_{-\pi}^{\pi} - \left(\frac{-\operatorname{Sin} nx}{n}\right)_{-\pi}^{\pi}\right]$$

$$= \frac{2}{n^2\pi}[\pi \operatorname{Cos} n\pi + \pi \operatorname{Cos} n(-\pi) - 0]$$

$$= \frac{2}{n^2\pi}[\pi \operatorname{Cos} n\pi + \pi \operatorname{Cos} n\pi]$$

$$= \frac{2}{n^2\cancel{\pi}} 2\cancel{\pi} \operatorname{Cos} n\pi$$

$$a_n = \frac{4(-1)^n}{n^2} \quad (\because \operatorname{Cos} n\pi = (-1)^n)$$

III^{ly} $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \operatorname{Sin} nx$

$\because$ Integral is an odd integral

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \operatorname{Sin} nx\, dx = 0$$

$$\therefore b_n = 0$$

Substituting a_0, a_n, b_n in (1)

$$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\operatorname{Cos} nx + 0$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \operatorname{Cos} nx}{n^2} \quad \to(2)$$

(i) Put in x = π (2)

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

(iii) Put x = 0 in (2)

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

(ii) we have

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

$$\Rightarrow \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] + \left[\frac{1}{2^2} + \frac{1}{4^2} + \ldots\right] = \frac{\pi^2}{6}$$

$$\Rightarrow \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] = \frac{\pi^2}{6} - \left[\frac{1}{2^2} + \frac{1}{4^2} + \ldots\right]$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} \left(\because \sum \frac{1}{(2n)^2} = \frac{\pi^2}{24}\right)$$

$$= \frac{4\pi^2 - \pi^2}{24}$$

$$= \frac{3\pi^2}{24}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(iv) we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Squaring on both sides

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)^2 = \left(\frac{\pi^2}{6}\right)^2$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{36}$$

4. **Find a fouries Series to represent x^2 in the interval $(-l, l)$.**

Sol:

Since $f(x) = x^2$ is an even function in $(-l, l)$. we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{Cos}\frac{n\pi x}{l} \quad \to(1) x^2$$

where $a_0 = \frac{2}{l}\int_0^l x^2\, dx$

$$a_0 = \frac{2l^2}{3}$$

III^{ly} $a_n = \int_0^l x^2 \operatorname{Cos}\frac{n\pi x}{l} dx$

$$\Rightarrow \frac{2}{l}\left[x^2\left(\frac{\operatorname{Sin}\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{\operatorname{Cos}\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\operatorname{Sin}\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$\Rightarrow \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$\therefore a_1 = \frac{-4l^2}{\pi^2}, \quad a_2 = \frac{4l^2}{2^2\pi^2}, \quad a_3 = \frac{-4l^2}{3^2\pi^2},$$

Substituting these values in (1) we get.

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\operatorname{Cos}\frac{\pi x}{l}}{1^2} - \frac{\operatorname{Cos}\frac{2\pi x}{l}}{2^2} + \frac{\operatorname{Cos}\frac{3\pi x}{l}}{3^2} - \frac{\operatorname{Cos}\frac{4\pi x}{l}}{4^2} + \ldots\right]$$

which is the required Fourier series.

5. Obtain Fourier Series for the function f(x) given by

$f(x) = 1 + 2x/\pi, \quad -\pi \le x \le 0$

$f(x) = 1 - 2x/\pi, \quad 0 \le x \le \pi$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Sol: we have

$$f(-x) = 1 - \frac{2x}{\pi} \text{ in } (-\pi, 0) = f(x) \text{ in } (0, \pi)$$

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\Rightarrow f(x)$ is an even function in $(\pi, -\pi)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \operatorname{Cos} nx \quad \to(1)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$\Rightarrow \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi}$$

$$\Rightarrow \frac{2}{\pi}\left(\pi - \frac{\pi^2}{\pi}\right)$$

$$\Rightarrow \frac{2}{\pi}(0) = 0$$

III^{ly} Consider

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{Cos} nx\, dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\operatorname{Cos} nx\, dx$$

$$\Rightarrow \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{-\operatorname{Sin} nx}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{-2}{\pi}\left(\frac{-\operatorname{Sin} nx}{n}\right)dx\right]dx$$

$$\Rightarrow \frac{4}{n\pi^2}\left[\frac{\operatorname{Cos} nx}{n}\right]_0^{\pi}$$

$$\Rightarrow \frac{-4}{n^2\pi^2}[\operatorname{Cos} n\pi - 1]$$

$$\Rightarrow \frac{-4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n \Rightarrow \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$\therefore a_1 = \frac{8}{\pi^2}, \quad a_3 = \frac{8}{3^2\pi^2}, \quad a_5 = \frac{8}{5^2\pi^2} \ldots$$

and $a_2 = a_4 = a_6 = 0\ldots$

Substituting these values in (1) we get.

$$f(x) = \frac{8}{\pi^2}\left[\frac{\operatorname{Cos} x}{1^2} + \frac{\operatorname{Cos} 3x}{3^2} + \frac{\operatorname{Cos} 5x}{5^2} + \ldots\right]$$

Now put x = 0

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\because f(0) = 1$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

6. **Express f(x)=x as a half range cosine series in 0<x<2.**

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\operatorname{Cos} n\pi x}{2} \quad \to(1)$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)dx$

$$= \int_0^2 x\, dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} = 2$$

$$\therefore a_0 = 2$$

III^{ly} $a_n = \frac{2}{2}\int_0^2 f(x)\frac{\operatorname{Cos} n\pi x}{2}dx$

$$= \int_0^2 x\frac{\operatorname{Cos} n\pi x}{2}dx$$

$$= \left[\frac{2x}{n\pi}\operatorname{Sin}\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\operatorname{Cos}\frac{n\pi x}{2}\right]_0^2$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Substituting these values in then we get.

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\frac{\operatorname{Cos} n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{\operatorname{Cos}\frac{\pi x}{2}}{1^2} + \frac{\operatorname{Cos}\frac{3\pi x}{2}}{3^2} + \frac{\operatorname{Cos}\frac{5\pi x}{2}}{5^2} \ldots\right]$$

Solutions to Model Paper - 1

SECTION - A

UNIT - I

(6 × 6 = ...)

1. **Find the fourier series of f(x) = x sinx is (-Π, Π).**

Ans :

Clearly f(x) is an even function and hence $b_n = 0\ \forall n$

Now $a_0 = \frac{2}{\Pi}\int_0^{\Pi} x \sin x\, dx$

$$= \frac{2}{\Pi}[-x\cos x + \sin x]_0^{\Pi}$$

$$= \frac{2}{\Pi}[\Pi\cos\Pi + \sin\Pi]$$

$$= \frac{2}{\Pi}[\Pi]$$

$$a_0 = \frac{2}{\Pi}$$

Similarly,

$$a_n = \frac{2}{\Pi}\int_0^{\Pi} x \sin x \cos nx\, dx$$

$$= \frac{1}{\Pi}\int_0^{\Pi}[\sin(n+1)x - \sin(n-1)x]\, dx$$

Now

$$a_1 = \frac{1}{\Pi}\int_0^{\Pi} x \sin 2x\, dx$$

$$= \frac{1}{\Pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\Pi}$$

$$= \frac{1}{\Pi}\left(\frac{-\Pi}{2}\right)$$

$$a_1 = \frac{-1}{2}$$

when n > 1

$$a_n = \frac{1}{\Pi}\int_0^{\Pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\Pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2} - x\left(\frac{-\cos(n-1)x}{(n-1)}\right) - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\Pi}$$

$$= \frac{1}{\Pi}\left[-\Pi\frac{(-1)^{n+1}}{n+1} + \frac{\Pi(-1)^{n-1}}{n-1}\right]$$

$$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$\therefore a_n = \frac{2(-1)^{n+1}}{(n+1)(n-1)}$$

$$\therefore f(x) = 1 - \frac{\cos x}{2} + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{(n+1)(n-1)}\cos nx$$

$$= 1 - \frac{\cos x}{2} - 2\left[\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \ldots\right]$$

2. **Find Half range sine series of f(x) = x in 0 < x < 2.**

Ans :

f(x) = x is an odd function

$\therefore a_0$ and $a_n = 0$

$\therefore$ Half range contains only b_n coefficients in the interval (-2, 2)

$\therefore f(x) = \Sigma\, b_n \sin nx$

where $b_n = \frac{2}{2}\int_0^2 f(x) \sin nx\, dx$

$$\therefore b_n = \int_0^2 x \sin nx\, dx$$

$$\Rightarrow x\left[\left(\frac{-\cos x}{n}\right)^2 - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^2$$

$$\therefore b_n = \frac{-2\cos 2n}{n} + \frac{\sin 2n}{n^2}$$

$$\therefore f(x) = \Sigma\left(\frac{-2\cos 2n}{n} + \frac{\sin 2n}{n^2}\right)\sin nx$$

$\ldots - y_1(x, 0)$

$$\therefore L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = P^2\bar{y}(x, p) - Py(x, 0) - y_1(x, 0)$$

SECTION : B

UNIT - I

(4 × 16 = 64)

9. (a) **Find fourier series for e^x is (-Π, Π)**

Ans :

Let $f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore a_0 = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\, dx = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} e^x\, dx = \frac{1}{\Pi}\left(\frac{e^{\Pi} - e^{-\Pi}}{2}\right)$$

$$\therefore a_0 = \frac{e^{\Pi} - e^{-\Pi}}{2\Pi}$$

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\Pi}\int_{-\Pi}^{\Pi} e^x \cos nx\, dx$$

$$= \frac{1}{\Pi}\left[\frac{e^x}{1+n^2}[\cos nx + n\sin nx]\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{(1+n^2)\Pi}[e^{\Pi}\cos n\Pi - e^{-\Pi}\cos n\Pi]$$

$$a_n = \frac{\cos n\Pi(e^{\Pi} - e^{-\Pi})}{\Pi(1+n^2)} = \frac{2(-1)^n \sin n\Pi}{\Pi(1+n^2)}$$

Similarly

$$b_n = \frac{2n(-1)^n \sin n\Pi}{\Pi(1+n^2)}$$

$$\therefore e^x = \frac{\sin n\Pi}{\Pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n \sin n\Pi}{\Pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n \sin n\Pi}{\Pi(1+n^2)}\sin nx$$

(b) **Find the fourier half range series for x^2 in the range of 0 < x < Π.**

Ans :

$f(x) = x^2\ \forall\ 0 < x < \Pi$

By Fourier series we have

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$\therefore a_0 = \frac{1}{\Pi}\int_0^{\Pi} x^2\, dx$ here $b_n = 0$

$= \frac{1}{\Pi}\left[\frac{x^3}{3}\right]_0^{\Pi}$ $\therefore$ f(x) is an even function

$$= \frac{\Pi^3}{3\Pi}$$

$$a_0 = \frac{\Pi^3}{3}$$

Similarly

$$a_0 = \frac{1}{\Pi}\int_0^{\Pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\Pi}\int_0^{\Pi} x^2\cos nx\,dx$$

$$= \frac{1}{\Pi}\left[x^2\frac{\sin nx}{n}\Big|_0^{\Pi} - \int_0^{\Pi} 2x\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\Pi}\left[\Pi^2\frac{\sin n\Pi}{n} - 0 - \frac{2}{n}\int_0^{\Pi} x\sin nx dx\right]$$

$$= \frac{1}{\Pi}\left[\frac{-2}{n}\right]\left[\frac{x \in \cos nx}{n}\Big|_0^{\Pi} - \int_0^{\Pi} -\frac{\cos nx}{n}dx\right]$$

$$\Rightarrow \frac{-2}{n\Pi}\left[\frac{-x\cos nx}{n}\Big|_0^{\Pi} + \frac{\sin nx}{n^2}\Big|_0^{\Pi}\right]$$

$$\therefore \quad a_0 = \frac{-2}{n\Pi}\left[\frac{-\Pi\cos n\Pi + 0}{n} + \frac{\sin n\Pi - \sin(0)}{n^2}\right]$$

$$\Rightarrow a_n = \frac{2\cos n\Pi}{n^2} = \frac{2(-1)^n}{n^2}$$

$$\therefore \quad f(x) = \frac{\Pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx$$

10. (a) *Derive Euler's formula in Fourier Series expansion of a function.*

Ans :

The fourier series for the function f(x) is the interval $\alpha < x < \alpha + 2\Pi$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{\Pi}\int_{\alpha+2\Pi}^{\alpha+2\Pi} f(x)\,dx$

$a_n = \frac{1}{\Pi}\int_{\alpha+2\Pi}^{\alpha+2\Pi} f(x)\cos nx\,dx$ & b_n

$= \frac{1}{\Pi}\int_{\alpha+2\Pi}^{\alpha+2\Pi} f(x)\sin nx\,dx$

These values of a_n, b_n are known as Euler's formulae.

Proof :

Let f(x) is represented in the interval $(\alpha, \alpha + 2\Pi)$ by the fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ... (1)

To find the coefficients a_0, a_n, b_n we assume that the series (1) can be integrated term by term from $x = \alpha$ to $\alpha + 2\Pi$.

To find a_0 integrate both sides of (1) from $x = \alpha$ to $\alpha + 2\Pi$ then we get

$$\int_{\alpha}^{\alpha+2\Pi} f(x)\,dx = \frac{1}{2}a_0 = \int_{\alpha}^{\alpha+2\Pi} dx + \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)dx + \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)dx$$

$$= \frac{1}{2}a_0(\alpha + 2\Pi - \alpha) + 0 + 0$$

$$= \Pi a_0$$

$$\therefore \quad a_n = \frac{1}{\Pi}\int_{\alpha}^{\alpha+2\Pi} f(x)\,dx$$

Similarly

To find a_n multiply each side of (1) by cosnx and integrate from $x = \alpha$ to $\alpha + 2\Pi$ then

$$\int_{\alpha}^{\alpha+2\Pi} f(x)\cos nx\,dx$$

$$\Rightarrow \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\Pi}\cos nx\,dx + \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\cos nx\,dx + \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\cos nx\,dx$$

$$= 0 + \Pi a_n + 0$$

$$\therefore \quad a_n = \frac{1}{\Pi}\int_{\alpha}^{\alpha+2\Pi} f(x)\cos nx\,dx$$

Again to find b_n multiply each side of (1) by sinnx and integrate from $x = \alpha$ to $\alpha + 2\Pi$ then

$$\int_{\alpha}^{\alpha+2\Pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\Pi}\sin nx\,dx + \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+ \int_{\alpha}^{\alpha+2\Pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \Pi b_n$$

$$\therefore \quad b_n = \frac{1}{\Pi}\int_{\alpha}^{\alpha+2\Pi} f(x)\sin nx\,dx$$

(b) *Expand* $f(x) = \begin{cases}\left(\frac{1}{4} - x\right), & \text{if } 0 < x < \frac{1}{2}\\ \left(\frac{x-1}{4}\right), & \text{if } \frac{1}{2} < x < 1\end{cases}$

Ans :

Fourier series of sine terms consists only bn coefficients

$\therefore$ fourier series of sine terms is given by $F(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{1}\int_0^1 f(x)\sin nx\,dx$

$$\Rightarrow 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin nx dx + \int_0^{1/2}\left(x - \frac{1}{4}\right)\sin nx dx\right]$$

$$\Rightarrow 2\left[\frac{1}{4}\int_0^{1/2}\sin nx dx - \int_0^{1/2} x\sin nx dx + \int_0^{1/2} x\sin nx dx - \frac{1}{4}\int_0^{1/2}\sin nx dx\right]$$

$$\Rightarrow \frac{2}{4}\left[\frac{-\cos nx}{n}\right]_0^{1/2} - 2\left\{\left[\frac{x\cos nx}{n}\right]_0^{1/2} - (1)\left[\frac{-\sin nx}{n^2}\right]_0^{1/2}\right\}$$

$$+ 2\left[x\left(\frac{-\cos nx}{n}\right)_{1/2}^{1} - (1)\left(\frac{-\sin nx}{n^2}\right)_{1/2}^{1}\right] - \frac{1}{4}\left(\frac{-\cos nx}{n}\right)_{1/2}^{1}$$

$$\Rightarrow \frac{1}{2n}\left[\frac{-\cos n}{2} + 1\right] - 2\left[\frac{1}{2n}\left(\frac{-\cos n}{2}\right) + \frac{\sin\frac{n}{2}}{n^2}\right]$$

$$+ \frac{2}{n}\left[-\cos n + \frac{1}{2}\cos\frac{n}{2} + \frac{\sin n}{n^2} - \frac{\sin\frac{n}{2}}{n^2}\right]$$

$$+ \frac{1}{4}\left[\frac{\cos n}{n} - \frac{\cos\frac{n}{2}}{n}\right]$$

$$\Rightarrow \cos\frac{n}{2}\left[\frac{-1}{2n} + \frac{1}{n} - \frac{1}{n}\right] + \sin\frac{n}{2}\left[\frac{-2}{n^2} - \frac{1}{n^2}\right] + \frac{1}{2n} - \frac{2}{n}\cos n$$

$$\therefore \quad b_n = \left(\frac{-1}{2n} + \frac{2}{n}\right)\cos\frac{n}{2} - \frac{3}{n^2}\sin\frac{n}{2} + \frac{1}{2n} - \frac{2}{n}\cos n$$

$$b_n = \frac{3}{2n}\cos\frac{n}{2} - \frac{3}{n^2}\sin\frac{n}{2} + \frac{1}{2n} - \frac{2}{n}\cos n$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty}\left(\frac{3}{2n}\cos\frac{n}{2} - \frac{3}{n^2}\sin\frac{n}{2} + \frac{1}{2n} - \frac{2}{n}\cos n\right)\sin nx$$

10. (a) *Obtain the Fourier series to represent the function* $f(x) = |x|$ ***in*** $-\Pi < x < \Pi$ ***and deduce*** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Ans :

Now

$$a_0 = \frac{2}{\Pi}\int_0^{\Pi} f(x)\,dx$$

$$= \frac{2}{\Pi}\int_0^{\Pi} |x|\,dx$$

$$= \frac{2}{\Pi}\int_0^{\Pi} x\,dx$$

$$= \frac{2}{\Pi}\left[\frac{x^2}{2}\right]_0^{\Pi}$$

$$\therefore \quad a_0 = \Pi$$

Similarly

$$a_n = \frac{2}{\Pi}\int_0^{\Pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\Pi}\int_0^{\Pi} x\cos nx\,dx$$

$$= \frac{2}{\Pi}\left\{\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\Pi}$$

$$= \frac{2}{\Pi}\left[\frac{\cos n\Pi}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\Pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$a_n = \frac{2}{\Pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\therefore \quad a_n = \begin{cases}0, & \text{if n is even}\\ \frac{-4}{n\Pi^2} & \text{if n is odd}\end{cases}$$

$$\therefore \quad |x| = \frac{\Pi}{2} - \frac{4}{\Pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\Pi^2}{8}$$

(b) *Define Fourier series of a function f(x) in the interval (–p, p). Find the Fourier series expansion of* $f(x) = x^2$ ***in the interval*** $(-\Pi, \Pi)$.

Ans :

The Fourier series of the function F(x) in the interval (–P, P) is given by $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ... (1)

where $a_0 = \frac{1}{\Pi}\int_{-P}^{P} f(x)\,dx$, $an = \frac{1}{\Pi}\int_{-P}^{P} f(x)\cos nx\,dx$

and $b_n = \frac{1}{\Pi}\int_{-P}^{P} f(x)\sin nx\,dx$

Given that $F(x) = x^2$ in the interval $(-\Pi, \Pi)$.

$$a_0 = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} x^2\,dx$$

$\therefore$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}$ $a_n\cos nx + b_n\sin nx$

$$= \frac{2}{\Pi}\int_0^{\Pi} x^2\,dx$$

$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$

$$= \frac{2}{\Pi}\left[\frac{x^3}{3}\right]_0^{\Pi}$$

$x^2 = \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \ldots\right]$

$$\therefore \quad a_0 = \frac{2\Pi^2}{3}$$

Similarly

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} x^2\cos nx\,dx = \frac{2}{\Pi}\int_0^{\Pi} x^2\cos nx\,dx$$

$$= \frac{2}{\Pi}\left[\frac{x^2\sin nx}{n} - \int\frac{2x\sin nx}{n}dx\right]_0^{\Pi}$$

$$a_n = \frac{2}{\Pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]\right]_0^{\Pi}$$

$$= \frac{2}{\Pi}\left[\frac{2}{n^2}\Pi(-1)^n\right]$$

$$\therefore \quad a_n = \frac{4}{n^2}(-1)^n \quad \because \sin n\Pi = 0 \text{ and } \cos n\Pi = (-1)^n$$

Given F(x) is even function then b_n function tends zero

$\therefore$ $b_n = 0$

Hence $\tilde{U}_s = \sqrt{\frac{2}{\Pi}}\left(\frac{p}{1 + p^2}\right)e^{-2p^2t}$.

Now applying the inversion Fourier sine transform, we have

$$U(x, t) = \frac{2}{\Pi}\int_0^{\infty}\frac{pe^{-2p^2t}\sin px}{1 + p^2}dp$$

SECTION - B (4 × 16 = 64

UNIT - I

9. (a) *Find the Fourier series expansion for f(x), if* $f(x) = \begin{cases}-\Pi, & -\Pi < x < 0\\ x, & 0 < x < \Pi\end{cases}$. ***Hence deduce that*** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{5^2} + \ldots\ldots + = \frac{\Pi^2}{8}$

Ans :

From Fourier series we have

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

here $a_0 = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} F(x)\,dx$

$$= \frac{1}{\Pi}\left[\int_{-\Pi}^{0} -\Pi dx + \int_0^{\Pi} x dx\right]$$

$$= \frac{1}{\Pi}\left[\{-\Pi x\}_{-\Pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\Pi}\right]$$

$$= \frac{1}{\Pi}\left[\frac{-\Pi^2}{2}\right]$$

$$\therefore \quad a_0 = \frac{-\Pi}{2}$$

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\Pi}\left[\int_{-\Pi}^{0} -\Pi\cos nx dx + \int_0^{\Pi} x\cos nx dx\right]$$

$$= \frac{1}{\Pi}\left[-\Pi\left(\frac{\sin nx}{n}\right)_{-\Pi}^{0} + \left[x\frac{\sin nx}{n}\right|_0^{\Pi} - \int_0^{\Pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\Pi}\left[0 + 0 + \frac{1}{n^2}\cos nx\Big|_0^{\Pi}\right]$$

$$= \frac{1}{n^2\Pi}(\cos n\Pi - \cos 0)$$

$$a_n = \frac{(-1)^n - 1}{n^2\Pi}$$

Similarly

$$b_n = \frac{1}{\Pi}\int_{-\Pi}^{\Pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\Pi}\left[\int_{-\Pi}^{0} -\Pi\sin nx dx + \int_0^{\Pi} x\sin nx dx\right]$$

$$= \frac{1}{\Pi}\left[\frac{+\Pi\cos nx}{n}\Big|_{-\Pi}^{0} + \left(\frac{-x\cos nx}{n}\right)_0^{\Pi} + \int\frac{\cos nx}{n}dx\right]$$

$$= \frac{1 - (-1)^n}{n} - \frac{\Pi(-1)^n}{\Pi n} + 0$$

$$= \frac{1 - (-1)^n - (-1)^n}{n}$$

$$b_n = \frac{1 - 2(-1)^n}{n}$$

$$\therefore \quad F(x) = \frac{-\Pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\Pi}\cos nx + \sum_{n=1}^{\infty}\frac{1 - 2(-1)^n}{n}\sin nx$$

$$F(x) = \frac{-\Pi}{4} - \frac{2}{\Pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$$

$$+ 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots$$

Now

Put $x = 0$

$$\therefore \quad F(0) = \frac{-\Pi}{4} - \frac{2}{\Pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\infty\right)$$

Now f(x) is discontinuous at $x = 0$

$f(0 - 0) = -\Pi$, $f(0 + 0) = 0$

$$\therefore \quad f(0) = \frac{1}{2}[f(0 - 0) + f(0 + 0)] = -\Pi/2$$

$$\Rightarrow \frac{-\Pi}{2} = \frac{-\Pi}{4}\ \frac{-2}{\Pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$$

$$\Rightarrow \frac{-\Pi}{2} + \frac{-\Pi}{4} = \frac{-2}{\Pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\Rightarrow \frac{-2\Pi + \Pi}{4} = \frac{-2}{\Pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\Rightarrow \frac{-\Pi}{4} = \frac{-2}{\Pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

$$\Rightarrow \frac{\Pi^2}{8} = \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right]$$

(b) *Express* $f(x) = \frac{1}{2}(\Pi - x)$ ***as fourier series with period*** 2Π ***to be valid in the interval 0 to*** 2Π.

Ans :

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \ldots (1)$$

where $a_0 = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\,dx$

$$= \frac{1}{\Pi}\int_0^{2\Pi}\frac{1}{2}(\Pi - x)\,dx$$

$$= \frac{1}{2\Pi}\left[\Pi x - \frac{x^2}{2}\right]_0^{2\Pi}$$

$$= \frac{1}{2\Pi}\left[2\Pi^2 - \frac{4\Pi^2}{2}\right]$$

$$= \frac{1}{2\Pi}(0)$$

$$\therefore \quad a_0 = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\,dx$$

$$a_n = \frac{1}{\Pi}\int_0^{2\Pi}\frac{1}{2}(\Pi - x)\cos nx\,dx$$

$$= \frac{1}{2\Pi}\left[(\Pi - x)\frac{a\sin nx}{n}\Big|_0^{2\Pi} - \int_0^{2\Pi}(-1)\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{2\Pi}\left[(\Pi - x)\frac{\sin nx}{n} - \frac{1}{n^2}\cos nx\right]_0^{2\Pi}$$

$$= \frac{1}{2\Pi}\left[\left\{0 - \frac{1}{n^2}(1)\right\} - \left\{0 - \frac{1}{n^2}\right\}\right] = 0$$

$$\therefore \quad a_n = 0$$

$$b_n = \frac{1}{\Pi}\int_0^{2\Pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\Pi}\int_0^{2\Pi}\frac{1}{2}(\Pi - x)\sin nx\,dx$$

$$= \frac{1}{2\Pi}\int_0^{2\Pi}(\Pi - x)\sin nx\,dx$$

$$= \frac{1}{2\Pi}\left[(\Pi - x)\left(\frac{-\cos nx}{n}\right)\Big|_0^{2\Pi} - \int_0^{2\Pi}\frac{\cos nx}{n}dx\right]$$

$$= \frac{1}{2\Pi}\left[(\Pi - x)\left(\frac{-\cos nx}{n}\right)\Big|_0^{2\Pi} - \left(\frac{1}{n^2}\sin nx\right)_0^{2\Pi}\right]$$

$$= \frac{1}{2\Pi}\left[\left(\frac{\Pi}{n} - 0\right) - \left(\frac{-\Pi}{n}\right)\right]$$

$$= \frac{1}{2\Pi}\left(\frac{2\Pi}{n}\right)$$

$$\therefore \quad b_n = \frac{1}{n}$$

$$\therefore \quad F(x) = 0 + 0 + \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\Rightarrow \frac{1}{2}(\Pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$$

10. (a) *Find the fourier series for f(x) in the interval* $(-\Pi, \Pi)$

(b) *Obtain a Half range cosine series for*

$$f(x) = \begin{cases}Kx, & \text{for } 0 \le x \le l/2\\ K(l - x), & \text{for } l/2 \le x \le l\end{cases}$$

Ans :

Half range cosine series is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

where $a_0 = \frac{1}{\Pi}\left[\int_0^{l/2} kx dx + \int_{l/2}^{l} + klx - \frac{kx^2}{2}\Big|_{l/2}^{l}\right]$

$$= \frac{1}{\Pi}\left[\frac{kx^2}{2}\Big|_0^{l/2} + klx - \frac{kx^2}{2}\Big|_{l/2}^{l}\right]$$

$$= \frac{1}{\Pi}\left[\frac{kl^2}{8} + kl^2 - \frac{kl^2}{2} - \frac{kl^2}{2} + \frac{kl^2}{8}\right]$$

$$= \frac{1}{\Pi}\left[\frac{2kl^2}{8} + kl^2 - kl^2\right]$$

$$a_0 = \frac{kl^2}{4\Pi}$$

Similarly

$$a_n = \frac{1}{\Pi}\left[\int_0^{l/2} kx\cos nx dx\right] + \frac{1}{\Pi}\left[\int_{l/2}^{l} k(l - x)\cos nx dx\right]$$

$$= \frac{k}{\Pi}\left[\frac{x\sin nx}{n}\Big|_0^{l/2} - \int\frac{\sin nx}{n}dx\right]$$

$$+ \frac{k}{\Pi}\left[(l - x)\frac{\sin nx}{n}\Big|_{l/2}^{l} + \int\frac{\sin nx}{n}dx\right]$$

$$= \frac{k}{\Pi}\left[\frac{\left(\frac{l}{2}\right)\frac{\sin nl}{2}}{n} + \frac{\cos nx}{n^2}\Big|_0^{l/2}\right]$$

$$+ \frac{k}{\Pi}\left[\frac{-\left(l - \frac{l}{2}\right)\frac{\sin nl}{2}}{n} - \frac{\cos nx}{n^2}\Big|_{l/2}^{l}\right]$$

$$= \frac{k}{\Pi}\left[\frac{\frac{l\sin nl}{2}}{2n} + \frac{\frac{\cos nl}{2} - 1}{n^2} - \left(\frac{l}{2}\right)\frac{\frac{\sin nl}{2}}{n} - \frac{\cos nl + \frac{\cos nl}{2}}{n^2}\right]$$

$$= \frac{k}{\Pi}\left[\frac{\frac{l\sin nl}{2}}{2n} + \frac{\frac{\cos nl}{2} - 1}{n^2} - \frac{\frac{l\sin nl}{2}}{2n} - \frac{\cos nl}{n^2} - \frac{\frac{\cos nl}{2}}{n^2}\right]$$

$a_n = \frac{-k(\cos nl + 1)}{n^2\pi}$

$\therefore$ F(x)

$= \frac{kl^2}{8\pi} - \sum_{n=l}^{\infty}\frac{k(\cos nl + l)}{n^2\pi}$

$\cos nx$