MA201 - Differential Equations

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Contents

Chapter 1

Linear Differential Equations

A general n^{th} order differential equation can be written as

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0$$

An n^{th} order ODE is *linear*, if it can be written in the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n(x)y = g(x)$$

where the functions $a_i(x)$ are called the coefficient functions.

A *non-linear* ODE is an ODE which is NOT linear.

1.1 Solution of an ODE

A solution of the n^{th} order ODE

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0$$

on an interval $I \subseteq \mathbb{R}$ is a function $y = \phi(x)$ which is defined on I, which is at least n times differentiable on I, and which satisfies the equation.

1.2 First Order Ordinary Differential Equations

Consider the first-order ODE of the form

$$\frac{dy}{dx} = f(x, y) \tag{1.1}$$

The eq. (1.1) can always be written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.2}$$

1.2.1 Solving First Order ODEs

We assume that the ODE of the form eq. (1.1) or eq. (1.2) has a solution.

1. **Separable Equations**
   If the eq. (1.1) or eq. (1.2) can be written in the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

or

$$f_1(x)\phi_1(y)dx + f_2(x)\phi_2(y)dy = 0$$

then the DE is called a separable equation and can be solved by separating variables and integrating.

2. **Homogeneous Equations**
If the eq. (1.2) can be written in the form

$$\frac{dy}{dx} = f(\frac{x}{y})$$

then the DE is called a homogeneous equation and can be solved by substituting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. After substitution, the equation changes to separable equation.

3. **Exact Equations**
If the eq. (1.2) can be written in the form

$$dF(x, y) = 0$$

without multiplying by any factor, then the DE is called a homogeneous equation and its general solution solution is

$$F(x, y) = c$$

where c is an arbitrary constant.

Note: Total differential $:= dF$ and it is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}dx + \frac{\partial F(x, y)}{\partial y}dy$$

1.2.2 Exact Differential Equations

How to check if a DE of form eq. (1.2) is exact or not.

Theorem 1.1 *Consider the differential equation*

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.3}$$

*where $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives at all points (x, y) in rectangular domain D. Then, the differential equation 1.3 is exact **iff***

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

1.2.3 Integrating Factors

Sometimes an equation **not exact** but can be **made exact** by multiplying it by some function of x and y. The function which when multiplied, makes the equation exact is called *integrating factor.*

1: If $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ be a function of x only

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = g(x)$$

then $e^{\int g(x)dx}$ is an IF of the equation.

2: If $\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]$ be a function of y only

$$\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=h(y)$$

then $e^{\int h(y)dy}$ is an IF of the equation.

3: If $Mx+Ny\neq 0$ and the equation is homogeneous, then $\frac{1}{Mx+Ny}$ is an IF of the equation.

4: If $Mx-Ny\neq 0$ and the equation can be written as

$$\{f_1(xy)\}\,ydx+\{f_2(xy)\}\,xdy=0$$

then $\frac{1}{Mx-Ny}$ is an IF of the equation.

1.2.4 First Order Linear Differential Equation

The first order ODE is linear in the dependent varialbe y and independent variable x if it can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{1.4}$$

where P and Q are function of x only.

Solution

- If $P(x)=0$, then the eq. (1.4) degenerates into a simple separable equation.
- If $P(x)!=0$, then the eq. (1.4) is **not** exact. By rule 1 of section 1.2.3, integrating factor is $e^{\int P(x)dx}$. Therefore, the solution is

$$y(x)=\frac{\int Q(x)e^{\int P(x)dx}dx}{e^{\int P(x)dx}}$$

where c is a constant.

1.2.5 Bernoulli Equations

An equation of the form

$$\frac{dy}{dx}+P(x)y=Q(x)y^n \tag{1.5}$$

is called a Bernoulli DE, where P and Q are functions of X alone.

Note: When $n\in\{0,1\}$, then eq. (1.5) is a linear DE.

Theorem 1.2 *Suppose $n\notin\{0,1\}$, then the transformation $v=y^{1-n}$ reduces the eq. (1.5) to a linear DE in v.*

1.3 Initial Value Problems

Consider the first order DE of the form

$$\begin{cases}\frac{dy}{dx}=f(x,y)\\ y(x_0)=y_0\end{cases}$$

where f is a real-valued function defined in some domain D in xy-plane.

$$D=\{(x,y):a\leq x\leq b, c\leq y\leq d\}$$

where a,b,c and d re real finite constants.

This type of DE is called *initial value problem*(IVP). and $y(x_0)=y_0$ is called an *initial condition.* A IVP may have zero, one, finite or infinite solutions.

1.3.1 Peano Existence Theorem

Let D be a rectangular domain that contains point x_0, y_0 in its interior. If $f(x, y)$ is a **continuous** function in the domain D, then there exists a solution to the initial value problem on some interval $I = (x_0 - h, x_0 + h)$ where $h > 0$, sufficiently small.

1.3.2 Picard's Existence and Uniqueness Theorem

Let D be a rectangular domain that contains point x_0, y_0 in its interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are **continuous** function in the domain D, then there exists some interval $I = (x_0 - h, x_0 + h)$, $h > 0$ contained in $[a, b]$ and a **unique** function $\phi(x)$ defined on I that is a solution of the initial value problem.

Note: If $f(x, y)$ does not satisfy the hypotheses of *Picard Existence and Uniqueness Theorem*, then we **cannot** conclude about the solution.

1.3.3 Lipschitz Condition

The function $f : D \to \mathbb{R}$ is said to be LIPSCHITZ w.r.t. y in D if $\exists\ k > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le k|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

k is called *Lipschitz constant.*

1.3.4 Picard's Existence and Uniqueness Theorem - 2

Let D be a rectangular domain that contains point (x_0, y_0) in its interior. Let f satisfies the following two conditions:

1: $f(x, y)$ is continuous in D

2: $f(x, y)$ satisfies a Lipschitz condition w.r.t. in y in D

Consider the rectangle $R : |x - x_0| \le \alpha, |y - y_0| \le \beta$ in D. Then, there exists a **unique** solution to the IVP in the interval $I = [x_0 - h, x_0 + h]$, where $h = \min\{\alpha, \frac{\beta}{M}\}, M = \max\limits_{(x,y)\in R} |f(x, y)|$.

1.3.5 Picard's Method of Successive Approximations

In the IVP

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Approximations are as follows

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)dt \qquad \text{First approximation} \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))dt \qquad \text{Second approximation} \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt \qquad n^{th} \text{ approximation} \end{aligned}$$

These approximations converge to the solution of the IVP

$$\boxed{\text{Solution} = \lim_{n\to\infty} y_n(x)}$$

Bibliography