MA201 - Differential Equations

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Chapter 1

Linear Differential Equations

A general n^{th} order differential equation can be written as

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0$$

An n^{th} order ODE is *linear*, if it can be written in the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = g(x)$$

where the functions $a_i(x)$ are called the coefficient functions.

A *non-linear* ODE is an ODE which is NOT linear.

1.1 Solution of an ODE

A solution of the n^{th} order ODE

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0$$

on an interval $I \subseteq \mathbb{R}$ is a function $y = \phi(x)$ which is defined on I, which is at least n times differentiable on I, and which satisfies the equation.

1.2 First Order Ordinary Differential Equations

Consider the first-order ODE of the form

$$\frac{dy}{dx} = f(x, y) \tag{1.1}$$

The eq. (1.1) can always be written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.2}$$

1.2.1 Solving First Order ODEs

We assume that the ODE of the form eq. (1.1) or eq. (1.2) has a solution.

1. **Separable Equations**
   If the eq. (1.1) or eq. (1.2) can be written in the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

or

$$f_1(x)\phi_1(y)dx + f_2(x)\phi_2(y)dy = 0$$

then the DE is called a separable equation and can be solved by separating variables and integrating.

2. **Homogeneous Equations**
If the eq. (1.2) can be written in the form

$$\frac{dy}{dx} = f(\frac{x}{y})$$

then the DE is called a homogeneous equation and can be solved by substituting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. After substitution, the equation changes to separable equation.

3. **Exact Equations**
If the eq. (1.2) can be written in the form

$$dF(x, y) = 0$$

without multiplying by any factor, then the DE is called a homogeneous equation and its general solution solution is

$$F(x, y) = c$$

where c is an arbitrary constant.

Note: Total differential := dF and it is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}dx + \frac{\partial F(x, y)}{\partial y}dy$$

1.2.2 Exact Differential Equations

How to check if a DE of form eq. (1.2) is exact or not.

Theorem 1.1 *Consider the differential equation*

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.3}$$

where $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives at all points (x, y) in rectangular domain D. Then, the differential equation 1.3 is exact ***iff***

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

1.2.3 Integrating Factors

Sometimes an equation **not exact** but can be **made exact** by multiplying it by some function of x and y. The function which when multiplied, makes the equation exact is called *integrating factor.*

1: If $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ be a function of x only

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = g(x)$$

then $e^{\int g(x)dx}$ is an IF of the equation.

2: If $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ be a function of y only

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = h(y)$$

then $e^{\int h(y)dy}$ is an IF of the equation.

3: If $Mx + Ny \neq 0$ and the equation is homogeneous, then $\frac{1}{Mx + Ny}$ is an IF of the equation.

4: If $Mx - Ny \neq 0$ and the equation can be written as

$$\{f_1(xy)\}\, ydx + \{f_2(xy)\}\, xdy = 0$$

then $\frac{1}{Mx - Ny}$ is an IF of the equation.

1.2.4 First Order Linear Differential Equation

The first order ODE is linear in the dependent varialbe y and independent variable x if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.4}$$

where P and Q are function of x only.

Solution

- If $P(x) = 0$, then the eq. (1.4) degenerates into a simple separable equation.
- If $P(x)! = 0$, then the eq. (1.4) is **not** exact. By rule 1 of section 1.2.3, integrating factor is $e^{\int P(x)dx}$. Therefore, the solution is

$$y(x) = \frac{\int Q(x)e^{\int P(x)dx}dx}{e^{\int P(x)dx}}$$

where c is a constant.

1.2.5 Bernoulli Equations

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1.5}$$

is called a Bernoulli DE, where P and Q are functions of X alone.

Note: When $n \in \{0, 1\}$, then eq. (1.5) is a linear DE.

Theorem 1.2 *Suppose $n \notin \{0, 1\}$, then the transformation $v = y^{1-n}$ reduces the eq. (1.5) to a linear DE in v.*

Bibliography