

# MA201 - Differential Equations

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# Chapter 1

## Linear Differential Equations

A general  $n^{th}$  order differential equation can be written as

$$F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$$

An  $n^{th}$  order ODE is *linear*, if it can be written in the form

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x) y = g(x)$$

where the functions  $a_i(x)$  are called the coefficient functions.

A *non-linear* ODE is an ODE which is NOT linear.

### 1.1 Solution of an ODE

A solution of the  $n^{th}$  order ODE

$$F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$$

on an interval  $I \subseteq \mathbb{R}$  is a function  $y = \phi(x)$  which is defined on  $I$ , which is at least  $n$  times differentiable on  $I$ , and which satisfies the equation.

### 1.2 First Order Ordinary Differential Equations

Consider the first-order ODE of the form

$$\frac{dy}{dx} = f(x, y) \tag{1.1}$$

The eq. (1.1) can always be written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.2}$$

#### 1.2.1 Solving First Order ODEs

We assume that the ODE of the form eq. (1.1) or eq. (1.2) has a solution.

##### 1. Separable Equations

If the eq. (1.1) or eq. (1.2) can be written in the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

or

$$f_1(x)\phi_1(y)dx + f_2(x)\phi_2(y)dy = 0$$

then the DE is called a separable equation and can be solved by separating variables and integrating.

## 2. Homogeneous Equations

If the eq. (1.2) can be written in the form

$$\frac{dy}{dx} = f\left(\frac{x}{y}\right)$$

then the DE is called a homogeneous equation and can be solved by substituting  $y = vx$  and  $\frac{dy}{dx} = v + x\frac{dv}{dx}$ . After substitution, the equation changes to separable equation.

## 3. Exact Equations

If the eq. (1.2) can be written in the form

$$dF(x, y) = 0$$

without multiplying by any factor, then the DE is called a homogeneous equation and its general solution is

$$F(x, y) = c$$

where  $c$  is an arbitrary constant.

**Note:** Total differential  $:= dF$  and it is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}dx + \frac{\partial F(x, y)}{\partial y}dy$$

### 1.2.2 Exact Differential Equations

How to check if a DE of form eq. (1.2) is exact or not.

**Theorem 1.1** Consider the differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.3}$$

where  $M(x, y)$  and  $N(x, y)$  have continuous first partial derivatives at all points  $(x, y)$  in rectangular domain  $D$ . Then, the differential equation 1.3 is exact **iff**

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

### 1.2.3 Integrating Factors

Sometimes an equation **not exact** but can be **made exact** by multiplying it by some function of  $x$  and  $y$ . The function which when multiplied, makes the equation exact is called *integrating factor*.

**1:** If  $\frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$  be a function of  $x$  only

$$\frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = g(x)$$

then  $e^{\int g(x)dx}$  is an IF of the equation.

**2:** If  $\frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right]$  be a function of  $y$  only

$$\frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = h(y)$$

then  $e^{\int h(y)dy}$  is an IF of the equation.

**3:** If  $Mx + Ny \neq 0$  and the equation is homogeneous, then  $\frac{1}{Mx + Ny}$  is an IF of the equation.

**4:** If  $Mx - Ny \neq 0$  and the equation can be written as

$$\{f_1(xy)\} ydx + \{f_2(xy)\} xdy = 0$$

then  $\frac{1}{Mx - Ny}$  is an IF of the equation.

### 1.2.4 First Order Linear Differential Equation

The first order ODE is linear in the dependent variable  $y$  and independent variable  $x$  if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (1.4)$$

where  $P$  and  $Q$  are function of  $x$  only.

#### Solution

- If  $P(x) = 0$ , then the eq. (1.4) degenerates into a simple separable equation.
- If  $P(x) \neq 0$ , then the eq. (1.4) is **not** exact. By rule 1 of section 1.2.3, integrating factor is  $e^{\int P(x)dx}$ . Therefore, the solution is

$$y(x) = \frac{\int Q(x)e^{\int P(x)dx} dx}{e^{\int P(x)dx}}$$

where  $c$  is a constant.

### 1.2.5 Bernoulli Equations

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (1.5)$$

is called a Bernoulli DE, where  $P$  and  $Q$  are functions of  $X$  alone.

**Note:** When  $n \in \{0, 1\}$ , then eq. (1.5) is a linear DE.

**Theorem 1.2** Suppose  $n \notin \{0, 1\}$ , then the transformation  $v = y^{1-n}$  reduces the eq. (1.5) to a linear DE in  $v$ .

## 1.3 Initial Value Problems

Consider the first order DE of the form

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

where  $f$  is a real-valued function defined in some domain  $D$  in  $xy$ -plane.

$$D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$$

where  $a, b, c$  and  $d$  are real finite constants.

This type of DE is called *initial value problem* (IVP). and  $y(x_0) = y_0$  is called an *initial condition*. A IVP may have zero, one, finite or infinite solutions.

### 1.3.1 Peano Existence Theorem

Let  $D$  be a rectangular domain that contains point  $x_0, y_0$  in its interior. If  $f(x, y)$  is a **continuous** function in the domain  $D$ , then there exists a solution to the initial value problem on some interval  $I = (x_0 - h, x_0 + h)$  where  $h > 0$ , sufficiently small.

### 1.3.2 Picard Existence and Uniqueness Theorem

Let  $D$  be a rectangular domain that contains point  $x_0, y_0$  in its interior. If  $f(x, y)$  and  $\frac{\partial f}{\partial y}(x, y)$  are **continuous** function in the domain  $D$ , then there exists some interval  $I = (x_0 - h, x_0 + h)$ ,  $h > 0$  contained in  $[a, b]$  and a **unique** function  $\phi(x)$  defined on  $I$  that is a solution of the initial value problem.

**Note:** If  $f(x, y)$  does not satisfy the hypotheses of *Picard Existence and Uniqueness Theorem*, then we **cannot** conclude about the solution.

### 1.3.3 Lipschitz Condition

The function  $f : D \rightarrow \mathbb{R}$  is said to be LIPSCHITZ w.r.t.  $y$  in  $D$  if  $\exists k > 0$  such that

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

for all  $(x, y_1), (x, y_2) \in D$ .

$k$  is called *Lipschitz constant*.

### 1.3.4 Picard Existence and Uniqueness Theorem - 2

Let  $D$  be a rectangular domain that contains point  $(x_0, y_0)$  in its interior. Let  $f$  satisfies the following two conditions:

- 1:  $f(x, y)$  is continuous in  $D$
- 2:  $f(x, y)$  satisfies a Lipschitz condition w.r.t. in  $y$  in  $D$

Consider the rectangle  $R : |x - x_0| \leq \alpha, |y - y_0| \leq \beta$  in  $D$ . Then, there exists a **unique** solution to the IVP in the interval  $I = [x_0 - h, x_0 + h]$ , where  $h = \min\{\alpha, \frac{\beta}{M}\}$ ,  $M = \max_{(x,y) \in R} |f(x, y)|$ .

# Bibliography