

EE201 - Signals and Systems

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Chapter 1

Analysis of Signals and Systems

1.1 Energy and Power

	Continuous Time Signal, $x(t)$	Discrete Time Signal, $x[n]$
Energy	$E_x = \int_{t_1}^{t_2} x(t) ^2 dt$	$E_x = \sum_{n=N_1}^{N_2} x[n] ^2$
Power	$P_x = \frac{E_x}{t_2 - t_1}$	$P_x = \frac{E_x}{N_2 - N_1 + 1}$

In infinite duration signals ($-\infty < t < +\infty, -\infty < n < +\infty$)

	Continuous Time Signal, $x(t)$	Discrete Time Signal, $x[n]$
Energy	$E_\infty = \lim_{T \rightarrow \infty} \int_{-T}^T x(t) ^2 dt$	$E_\infty = \lim_{N \rightarrow \infty} \sum_{n=-N}^N x[n] ^2$
Power	$P_\infty = \lim_{T \rightarrow \infty} \frac{E_\infty}{2T}$	$P_\infty = \lim_{N \rightarrow \infty} \frac{E_\infty}{2N + 1}$

1.2 Periodic Signals

A periodic signal repeats itself after sometime.

Continuous signals

If $x(t) = x(t + T) \forall t, T > 0$ then the signal is periodic. The minimum value of T that satisfies the condition is called the *fundamental period*.

Discrete signals

If $x[n] = x[n + N] \forall n, N > 0, N \in \mathbb{Z}$ then the signal is periodic. The minimum value of N that satisfies the condition is called the *fundamental period*.

1.3 Exponential Signals

1.3.1 Continuous Time Exponential Signals

A continuous time exponential signal can be represented as

$$x(t) = Ce^{at} \quad (1.1)$$

where $C, a \in \mathbb{C}$.

[1] Consider the general case where $C = |C|e^{j\theta}$ and $a = \sigma + j\omega$. The signal becomes

$$x(t) = |C|e^{\sigma t}e^{j(\omega t + \theta)}$$

- ω is angular frequency. If $\omega \neq 0$, the signal is sinusoidal, with $f = \frac{\omega}{2\pi}$.
- σ is the attenuation factor. If $\sigma \neq 0$, the signal decays/grows and is bounded by the envelope $|C|e^{\sigma t}$.
- θ is the initial phase.

1.3.2 Discrete Time Exponential Signals

A discrete time exponential signal can be represented as

$$x[n] = Ce^{an} \quad (1.2)$$

where $C, a \in \mathbb{C}$.

Consider the general case where $C = |C|e^{j\theta}$ and $a = \sigma + j\omega$. The signal becomes

$$x[n] = |C|e^{\sigma n}e^{j(\omega n + \theta)}$$

- ω is **related to** angular frequency. If $\omega \neq 2n\pi, n \in \mathbb{Z}$, the signal is periodic. Unlike continuous-time signals, the angular frequency cannot take all values. The range is $[0, 2\pi)$ or $[-\pi, \pi)$ because

$$e^{j(\omega+2\pi)n} = e^{j\omega n}e^{j2\pi n} = e^{j\omega n}$$

Rate of oscillation is small for $\omega \sim 2n\pi$, and high for $\omega \sim (2n+1)\pi, n \in \mathbb{Z}$.

- σ is the attenuation factor. If $\sigma \neq 0$, the signal decays/grows and is bounded by the envelope $|C|e^{\sigma n}$.
- θ is the initial phase.

1.3.3 Periodicity

Continuous-time exponential signals

For a continuous-time exponential signal to be periodic

$$\begin{aligned} e^{j\omega(t+T)} &= e^{j\omega t} \\ \Rightarrow e^{j\omega T} &= 1 \\ \Rightarrow \omega T &= 2m\pi, m \in \mathbb{Z} \end{aligned}$$

Therefore, continuous-time exponential signal is periodic for all $\omega > 0$ and their *fundamental time period* is

$$T = \frac{2\pi}{\omega}$$

Discrete-time exponential signals

For a discrete-time exponential signal to be periodic

$$\begin{aligned} e^{j\omega(n+N)} &= e^{j\omega n} \\ \Rightarrow e^{j\omega N} &= 1 \\ \Rightarrow \omega N &= 2m\pi, m \in \mathbb{Z} \end{aligned}$$

Therefore, discrete-time exponential signal is periodic only if $\frac{\omega}{2\pi} = \frac{m}{N}$ i.e. $\frac{\omega}{2\pi}$ is a rational number. The *fundamental period* is

$$N = m \frac{2\pi}{\omega}$$

where m is smallest positive integer such that N evaluates to an integer.

1.3.4 Harmonics

Continuous-time exponential signals

$$\phi_k(t) = e^{jk[\frac{2\pi}{T}]t} = e^{jk\omega t}$$

Discrete-time exponential signals

$$\phi_k[n] = e^{jk[\frac{2\pi}{N}]n} = e^{jk\frac{\omega}{m}n}$$

1.4 Unit Impulse and Step Functions

1.4.1 Discrete-Time Case

Unit Impulse

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

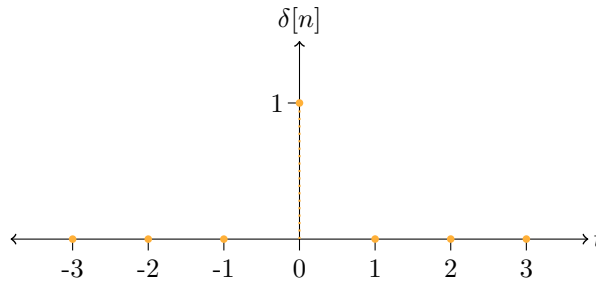


Figure 1.1: Unit impulse function in discrete-time

Unit Step

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

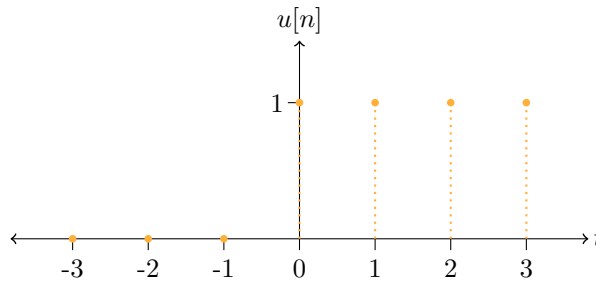


Figure 1.2: Unit step function in discrete-time

Properties

1. $\delta[n] = u[n] - u[n - 1]$
2. $u[n] = \sum_{m=-\infty}^n \delta[m] = \sum_{k=0}^{\infty} \delta[n - k]$

3. Sampling Property of unit impulse function

$$x[n] \cdot \delta[n] = x[0] \cdot \delta[n]$$

$$x[n] \cdot \delta[n - n_0] = x[n_0] \cdot \delta[n - n_0]$$

1.4.2 Continuous-Time Case**Unit Step**

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

This function is discontinuous at $t = 0$. It is modified a little bit to make it continuous.

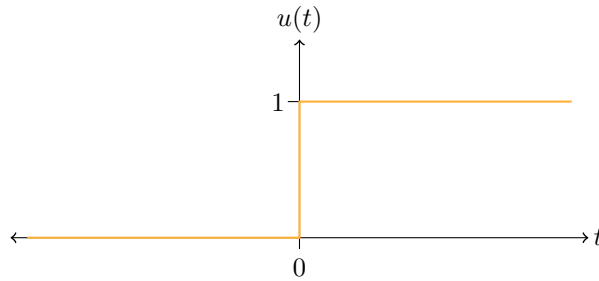


Figure 1.3: Unit step function in continuous-time

$$U_{\Delta}(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{\Delta} & 0 < t < \Delta \\ 1 & t \geq \Delta \end{cases}$$

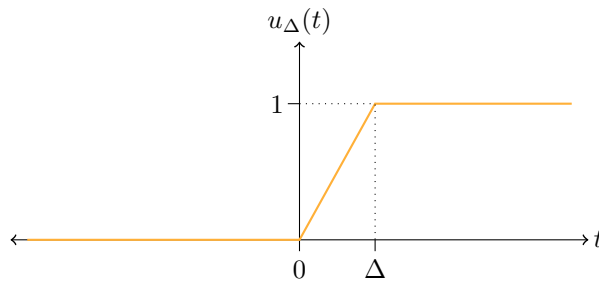


Figure 1.4: Continuous unit step function in continuous-time

Unit Impulse

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

$$\delta(t) = \frac{du(t)}{dt}$$

Since, $u(t)$ is not differentiable.

$$\delta(t) = \lim_{\Delta \rightarrow 0} \frac{dU_{\Delta}(t)}{dt}$$

$$\delta(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & t \geq \Delta \end{cases}$$

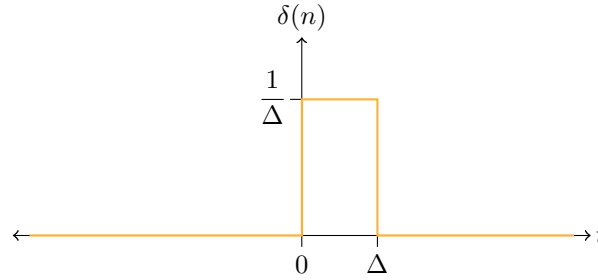


Figure 1.5: Unit impulse function in continuous-time

Properties

$$1. \int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$$

2. Sampling Property of unit impulse function

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$

$$x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$$

1.5 Systems

System is the interconnection of components/devices/subsystems. A *continuous-time system* is represented by *differential equation* and a *discrete-time system* is represented by *difference equation*.

1.5.1 Interconnection of Systems

1. Series (Cascade) Connection

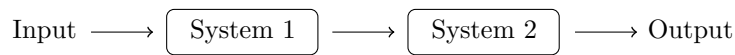


Figure 1.6: Example of series interconnection

2. Parallel Connection

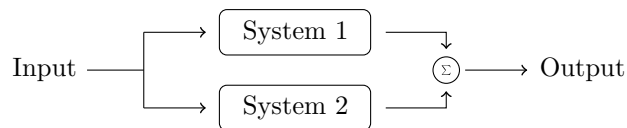


Figure 1.7: Example of parallel interconnection

3. Series-Parallel Connection

It is combination of series and parallel connections.

4. Feedback Interconnection

In this kind of interconnection, some part of output is again fed as input to the system.

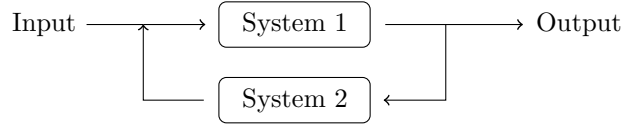


Figure 1.8: Example of feedback interconnection

1.6 Basic Properties of a System

Some basic properties of a system are

1. Memory
2. Invertibility
3. Causality
4. Stability
5. Time-invariance
6. Linearity

In all examples below, $x(t)$ (or $x[n]$) is the input to the system and the output is $y(t)$ (or $y[n]$).

1.6.1 Memory

Systems can be divided into two classes on basis of memory:

1. **Memoryless:** System whose output depends only on present input.

EXAMPLES

- $y[n] = (2x[n] - x^2[n])^2$
- $y(t) = Rx(t)$

2. **With memory:** System whose output depends on past/future value also.

EXAMPLES

- $y[n] = x[n - 1]$
- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y(t) = \frac{1}{c} \int_{-\infty}^{\infty} x(\tau) d\tau$

1.6.2 Invertibility

Systems can be divided into two classes on basis of invertibility:

1. **Invertible:** Systems for which distinct input leads to distinct outputs (i.e. they are one-one and onto).

EXAMPLES

- $y(t) = 2x(t)$

When a system and its inverse system are cascaded then output is same as input and such group of system is called *identity systems* ($x(t) = t$).

2. **Non-invertible:** Systems which are not invertible.

EXAMPLES

- $y(t) = c$

1.6.3 Causality

Systems can be divided into two classes on basis of causality:

1. **Causal/Non-anticipative:** Systems whose output is dependent only on present and past (but not future) values of the input.

EXAMPLES

- All memoryless systems(they use only present input)

All practical systems are causal, unless they use recorded signals as future values.

Note: *Causal signals* are signals which start after $t = 0$, *non-causal signals* are signals that start before $t = 0$ and *anti-causal signals* are signals that end after $t = 0$.

2. **Non-causal system:** Systems which are not causal.

EXAMPLES

- $y(t) = x(t + 1)$

1.6.4 Stability

Systems which produce bounded output for bounded input are called *stable system*. Such systems are called BIBO(bounded input, bounded output) stable.

EXAMPLES

- Charging/discharging capacitor

Note: A signal is called bounded if $\exists B > 0$ such that the signal magnitude never exceeds B . [2]

1.6.5 Time Invariance

Systems can be divided into two classes on basis of time-invariance:

1. **Time invariant:** System whose output does not depend on the instant or time of applying the input. Delay in input produces same delay in output.

EXAMPLES

- $y(t) = x(t)$

2. **Time variant:** Systems whose output depend on the time of application of input.

EXAMPLES

- $y(t) = tx(t)$,

1.6.6 Linearity

Systems can be divided into two classes on basis of linearity:

1. **Linear:** Systems that can be superimposed i.e. they are additive and homogeneous.

EXAMPLES

- $y(t) = x(t_0)$

Additivity: If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, then

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

Homogeneity: If $x(t) \rightarrow y(t)$, then

$$ax(t) \rightarrow ay(t) \quad (a \in \mathbb{C})$$

2. **Non-linear:** Systems which do not follow additivity or homogeneity or both.

EXAMPLES

- $y(t) = x^2(t) + c$

1.7 Linear Time Invariant(LTI) Systems

Unit impulse signal will be used as a basis for creating other signals. In other words, we will be writing signals as a linear combination of shifted unit-impulse signals.

1.7.1 Discrete-Time LTI Systems

Consider an arbitrary signal $x[n]$, it can be represented as linear combination of shifted unit-impulse signals.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad (1.3)$$

This is called *sifting property* of impulse in discrete-time case.

Let $h_k[n]$ be the impulse response (system response to the unit-impulse $\delta[n-k]$). ($h_0 = h$)

\therefore System is linear

\therefore From eq. (1.3), combining the system response to different unit impulses will give the output

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \quad (1.4)$$

\therefore System is time-invariant

$\therefore h_k[n] = h[n-k]$

The eq. (1.4), reduces to

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad (1.5)$$

Using convolution sum[3], eq. (1.5) reduces to

$$y[n] = x[n] * h[n]$$

1.7.2 Continuous-Time LTI Systems

Like discrete-time case, we can write any arbitrary signal $x(t)$ as a linear combination of shifted unit-impulse signals.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad (1.6)$$

This is called *sifting property* of impulse in continuous-time case.

Let $h_\tau(t)$ be the impulse response (system response to the unit-impulse $\delta(t - \tau)$). ($h_0 = h$)

\therefore System is linear

\therefore From eq. (1.6), combining the system response to different unit impulses will give the output

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau \quad (1.7)$$

\therefore System is time-invariant

$\therefore h_\tau(t) = h(t - \tau)$

The eq. (1.7), reduces to

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \quad (1.8)$$

Using convolution integral[3], eq. (1.8) reduces to

$$y(t) = x(t) * h(t)$$

Corollary: Impulse response is complete characterization of LTI system.

1.7.3 Properties of LTI Systems

The properties given below are for LTI systems only.

1. Commutative Property

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

2. Distributive Property

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

3. Associative Property

$$x[n] * [h_1[n] * h_2[n]] = [x[n] * h_1[n]] * h_2[n]$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

4. When LTI system is memoryless

$$h[n] = 0 \text{ for } n \neq 0 \text{ and } h(t) = 0 \text{ for } t \neq 0$$

$$h(t) = k\delta(t)$$

$$h[n] = k\delta[n]$$

$$y(t) = kx(t)$$

$$y[n] = kx[n]$$

Note: $x(t) * \delta(t) = x(t)$ and $x[n] * \delta[n] = x[n]$

5. LTI system $h_1(t)$ (or $h_1[n]$) is invertible if $\exists h_2(t)$ (or $h_2[n]$), such that

$$h_1(t) * h_2(t) = \delta(t)$$

$$\text{or } h_1[n] * h_2[n] = \delta[n]$$

and $h_2(t)$ (or $h_2[n]$) is its inverse.

6. When LTI system is causal

$$\begin{aligned}
 y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
 &= \sum_{k=-\infty}^n x[k]h[n-k] \quad (\because x[k] = 0, k > n) \\
 &= \sum_{k=0}^{\infty} x[n-k]h[k]
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
 &= \int_{-\infty}^t x(\tau)h(t-\tau)d\tau \quad (\because x(\tau) = 0, \tau > t) \\
 &= \int_0^{\infty} x(t-\tau)h(\tau)d\tau
 \end{aligned}$$

So, we can also say that for a causal **system**

$$\begin{aligned}
 h(t) &= 0 \quad \forall t < 0 \\
 h[n] &= 0 \quad \forall n < 0
 \end{aligned}$$

Similarly, for a causal **signal**

$$\begin{aligned}
 x(t) &= 0 \quad \forall t < 0 \\
 x[n] &= 0 \quad \forall n < 0
 \end{aligned}$$

Initial test: A causal system gives no output unless an input is given.

7. When LTI system is stable

$$\begin{aligned}
 |x[n]| &\leq B \quad \forall n \\
 |y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[n]h[n-k] \right| \\
 |y[n]| &\leq \sum_{k=-\infty}^{\infty} |x[n]| \cdot |h[n-k]| \\
 |y[n]| &\leq B \sum_{k=-\infty}^{\infty} |h[n-k]|
 \end{aligned}$$

Therefore, for a system to be stable the ‘impulse response must be absolutely integrable’.

$$\begin{aligned}
 \sum_{k=-\infty}^{\infty} |h[k]| &< \infty \\
 \int_{-\infty}^{\infty} |h(\tau)|d\tau &< \infty
 \end{aligned}$$

1.7.4 Unit Step Response

Unit step response(response to a unit step signal) is denoted by $s[n]$ or $s(t)$. It can be calculated using impulse response.

$$\begin{aligned} s[n] &= u[n] * h[n] \\ &= \sum_{k=-\infty}^n h[k] \\ s(t) &= u(t) * h(t) \\ &= \int_{-\infty}^t h(\tau) d\tau \end{aligned}$$

Similarly, impulse response can be calculated from step response.

$$\begin{aligned} h[n] &= s[n] - s[n-1] \\ h(t) &= \frac{ds(t)}{dt} = s'(t) \end{aligned}$$

1.7.5 Differential Equation

An LTI system, in continuous-time case, can be represented using **linear** differential equation with **constant coefficients**. A general N^{th} order linear constant coefficient differential equation is

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

When $N = 0$, we get an explicit relation. In other cases, the relation between input and output is implicit.

General solution

$$y(t) = \underbrace{y_p(t)}_{\text{Particular solution}} + \underbrace{y_h(t)}_{\text{Homogeneous solution}}$$

1.7.6 Difference Equation

An LTI system, in discrete-time case, can be represented using **linear** difference equation with **constant coefficients**. A general N^{th} order linear constant coefficient difference equation is

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

When $N = 0$, we get an explicit relation and the system has finite impulse responses(FIR). In other cases, the relation between input and output is implicit/recursive and the system has infinite impulse responses (IIR).

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