EE201 - Signals and Systems

Indian Institute of Technology Ropar

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Contents

Chapter 1

Analysis of Signals and Systems

1.1 Energy and Power

	Continuous Time Signal, $x(t)$	**Discrete Time Signal**, $x[n]$
Energy	$E_x = \int_{t_1}^{t_2} \lvert x(t)\rvert^2 dt$	$E_x = \sum_{n=N_1}^{N_2} \lvert x[n]\rvert^2$
Power	$E_x = \frac{E_x}{t_2 - t_1}$	$P_x = \frac{E_x}{N_2 - N_1 + 1}$

In infinite duration signals $(-\infty < t < +\infty, -\infty < n < +\infty)$

	Continuous Time Signal, $x(t)$	**Discrete Time Signal**, $x[n]$
Energy	$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$	$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} \lvert x[n]\rvert^2$
Power	$E_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T}$	$P_\infty = \lim_{N\to\infty} \frac{E_\infty}{2N+1}$

1.2 Periodic Signals

A periodic signal repeats itself after sometime.

Continuous signals

If $x(t) = x(t+T)$ $\forall t$, $T > 0$ then the signal is periodic. The minimum value of T that satisfies the condition is calle the *fundamental period*.

Discrete signals

If $x[n] = x[n+N]$ $\forall n$, $N > 0, N \in \mathbb{Z}$ then the signal is periodic. The minimum value of N that satisfies the condition is calle the *fundamental period*.

1.3 Exponential Signals

1.3.1 Continuous Time Exponential Signals

A continuous time exponential signal can be represented as

$$x(t) = Ce^{at} \tag{1.1}$$

where $C, a \in \mathbb{C}$.

[1]Consider the general case where $C = |C|e^{j\theta}$ and $a = \sigma + \mathrm{j}\omega$. The signal becomes

$$x(t) = |C|e^{\sigma t}e^{\mathrm{j}(wt+\theta)}$$

- ω is angular frequency. If $\omega \neq 0$, the signal is sinusoidal, with $f = \frac{\omega}{2\pi}$.
- σ is the attenuation factor. If $\sigma \neq 0$, the signal decays/grows and is bounded by the envelope $|C|e^{\sigma t}$.
- θ is the initial phase.

1.3.2 Discrete Time Exponential Signals

A discrete time exponential signal can be represented as

$$x[n] = Ce^{an} \tag{1.2}$$

where $C, a \in \mathbb{C}$.

Consider the general case where $C = |C|e^{j\theta}$ and $a = \sigma + \mathrm{j}\omega$. The signal becomes

$$x[n] = |C|e^{\sigma n}e^{\mathrm{j}(wn+\theta)}$$

- ω is **related to** angular frequency. If $\omega \neq 2n\pi, n \in \mathbb{Z}$, the signal is periodic.
  Unlike continuous-time signals, the angular frequency cannot take all values. The range is $[0, 2\pi)$ or $[-\pi, \pi)$ because
  $$e^{\mathrm{j}(\omega+2\pi)n} = e^{\mathrm{j}\omega n}e^{\mathrm{j}2n\pi} = e^{\mathrm{j}\omega n}$$
  Rate of oscillation is small for $\omega \sim 2n\pi$, and high for $\omega \sim (2n+1)\pi, n \in \mathbb{Z}$.
- σ is the attenuation factor. If $\sigma \neq 0$, the signal decays/grows and is bounded by the envelope $|C|e^{\sigma n}$.
- θ is the initial phase.

1.3.3 Periodicity

Continuous-time exponential signals

For a continuous-time exponential signal to be periodic

$$\begin{aligned} e^{\mathrm{j}\omega(t+T)} &= e^{\mathrm{j}\omega t} \\ \Rightarrow e^{\mathrm{j}\omega T} &= 1 \\ \Rightarrow \omega T &= 2m\pi, m \in \mathbb{Z} \end{aligned}$$

Therefore, continuous-time exponential signal is periodic for all $\omega > 0$ and their *fundamental time period* is

$$T = \frac{2\pi}{\omega}$$

Discrete-time exponential signals

For a discrete-time exponential signal to be periodic

$$\begin{aligned} e^{\mathrm{j}\omega(n+N)} &= e^{\mathrm{j}\omega n} \\ \Rightarrow e^{\mathrm{j}\omega N} &= 1 \\ \Rightarrow \omega N &= 2m\pi, m \in \mathbb{Z} \end{aligned}$$

Therefore, discrete-time exponential signal is periodic only if $\frac{\omega}{2\pi} = \frac{m}{N}$ i.e. $\frac{w}{2\pi}$ is a rational number. The *fundamental period* is

$$N = m\frac{2\pi}{\omega}$$

where m is smallest positive integer such that N evaluates to an integer.

1.3.4 Harmonics

Continuous-time exponential signals

$$\phi_k(t) = e^{\mathrm{j}k[\frac{2\pi}{T}]t} = e^{\mathrm{j}k\omega t}$$

Discrete-time exponential signals

$$\phi_k[n] = e^{\mathrm{j}k[\frac{2\pi}{N}]n} = e^{\mathrm{j}k\frac{\omega}{m}n}$$

1.4 Unit Impulse and Step Functions

1.4.1 Discrete-Time Case

Unit Impulse

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

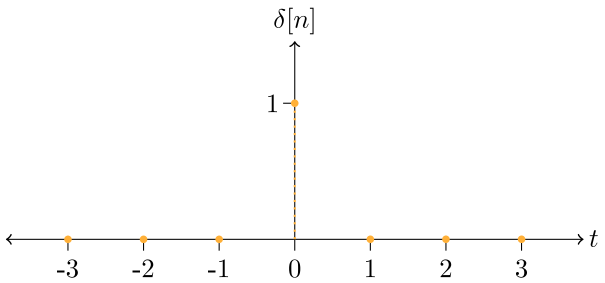

Figure 1.1: Unit impulse function in discrete-time

Unit Step

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

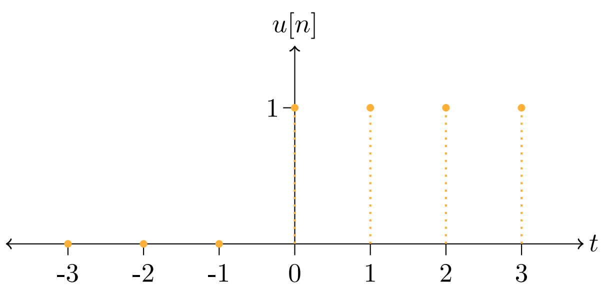

Figure 1.2: Unit step function in discrete-time

Properties

1. $\delta[n] = u[n] - u[n-1]$
2. $u[n] = \sum_{m=-\infty}^{n} \delta[n] = \sum_{k=0}^{\infty} \delta[n-k]$
3. **Sampling Property of unit impulse function**

$$x[n] \cdot \delta[n] = x[0] \cdot \delta[n]$$
$$x[n] \cdot \delta[n-n_0] = x[n_0] \cdot \delta[n-n_0]$$

1.4.2 Continuous-Time Case

Unit Step

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

This function is discontinuous at $t = 0$. It is modified a little bit to make it continuous.

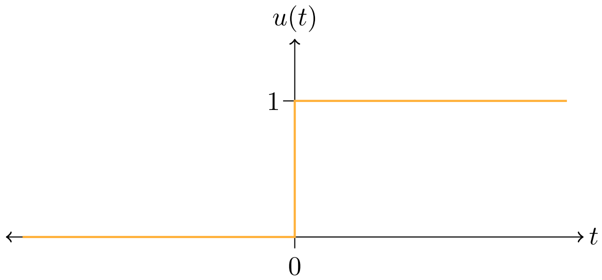

Figure 1.3: Unit step function in continuous-time

$$U_\Delta(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{\Delta} & 0 < t < \Delta \\ 1 & t \geq \Delta \end{cases}$$

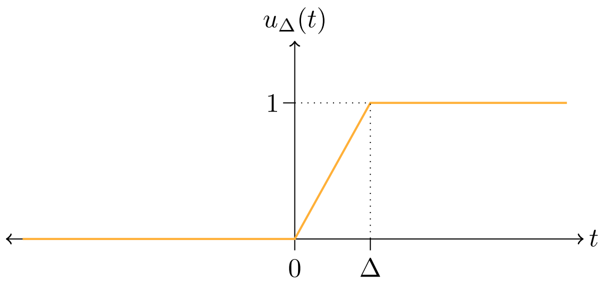

Figure 1.4: Continuous unit step function in continuous-time

Unit Impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$
$$\delta(t) = \frac{du(t)}{dt}$$

Since, $u(t)$ is not differentiable.

$$\delta(t) = \lim_{\Delta \to 0} \frac{dU_\Delta(t)}{dt}$$
$$\delta(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & t \geq \Delta \end{cases}$$

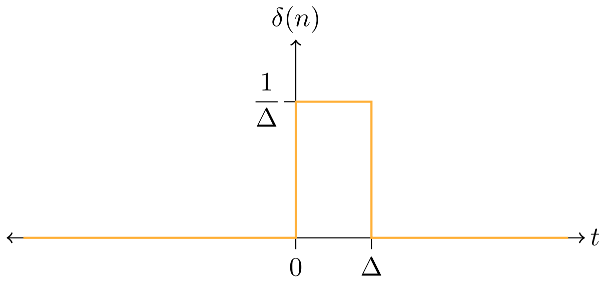

Figure 1.5: Unit impulse function in continuous-time

Properties

1. $\displaystyle\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$

2. **Sampling Property of unit impulse function**

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$
$$x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$$

1.5 Systems

System is the interconnection of components/devices/subsystems. A *continuous-time system* is represented by *differential equation* and a *discrete-time system* is represented by *difference equation*.

1.5.1 Interconnection of Systems

1. **Series (Cascade) Connection**

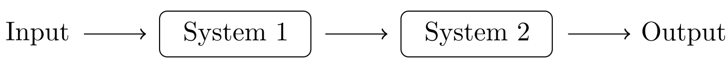

Figure 1.6: Example of series interconnection

2. **Parallel Connection**

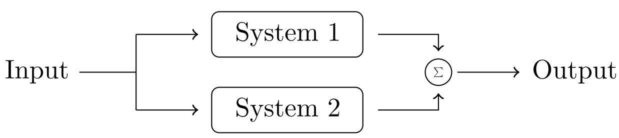

Figure 1.7: Example of parallel interconnection

3. **Series-Parallel Connection**
   It is combination of series and parallel connections.

4. **Feedback Interconnection**
   In this kind of interconnection, some part of output is again fed as input to the system.

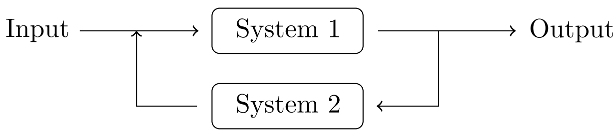

Figure 1.8: Example of feedback interconnection

1.6 Basic Properties of a System

Some basic properties of a system are

1. Memory
2. Invertibility
3. Causality
4. Stability
5. Time-invariance
6. Linearity

In all examples below, $x(t)$(or $x[n]$) is the input to the system and the output is $y(t)$(or $y[n]$).

1.6.1 Memory

Systems can be divided into two classes on basis of memory:

1. **Memoryless:** System whose output depends only on present input.
   Examples
   - $y[n] = (2x[n] - x^2[n])^2$
   - $y(t) = Rx(t)$

2. **With memory:** System whose output depends on past/future value also.
   Examples
   - $y[n] = x[n-1]$
   - $y[n] = \sum\limits_{k=-\infty}^{n} x[k]$
   - $y(t) = \frac{1}{c} \int\limits_{-\infty}^{\infty} x(\tau) d\tau$

1.6.2 Invertibility

Systems can be divided into two classes on basis of invertibility:

1. **Invertible:** Systems for which distinct input leads to distinct outputs (i.e. they are one-one and onto).
   Examples
   - $y(t) = 2x(t)$

When a system and its inverse system are cascaded then output is same as input and such group of system is called *identity systems* ($x(t) = t$).

2. **Non-invertible:** Systems which are not invertible.
   EXAMPLES
   - $y(t) = c$

1.6.3 Causality

Systems can be divided into two classes on basis of causality:

1. **Causal/Non-anticipative:** Systems whose output is dependent only on present and past (but not future) values of the input.
   EXAMPLES
   - All memoryless systems(they use only present input)

   All practical systems are causal, unless they use recorded signals as future values.

   Note: *Causal signals* are signals which start after $t = 0$, *non-causal signals* are signals that start before $t = 0$ and *anti-causal signals* are signals that end after $t = 0$.

2. **Non-causal system:** Systems which are not causal.
   EXAMPLES
   - $y(t) = x(t + 1)$

1.6.4 Stablility

Systems which produce bounded output for bounded input are called *stable system.* Such systems are are called BIBO(bounded input, bounded output) stable.
EXAMPLES

- Charging/discharging capacitor

Note: A signal is called bounded if $\exists\ B > 0$ such that the signal magnitude never exceeds B. [2]

1.6.5 Time Invariance

Systems can be divided into two classes on basis of time-invariance:

1. **Time invariant:** System whose output does not depend on the instant or time of applying the input. Delay in input produces same delay in output.
   EXAMPLES
   - $y(t) = x(t)$

2. **Time variant:** Systems whose output depend on the time of application of input.
   EXAMPLES
   - $y(t) = tx(t)$,

1.6.6 Linearity

Systems can be divided into two classes on basis of linearity:

1. **Linear:** Systems that can be superimposed i.e. they are additive and homogeneous.
   EXAMPLES
   - $y(t) = x(t_0)$

Additivity: If $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then

$$x_1(t) + x_2(t) \to y_1(t) + y_2(t)$$

Homogenity: If $x(t) \to y(t)$, then

$$ax(t) \to ay(t) \quad (a \in \mathbb{C})$$

2. **Non-linear:** Systems which do not follow additivity or homogenity or both.
   EXAMPLES

   - $y(t) = x^2(t) + c$

1.7 Linear Time Invariant(LTI) Systems

Unit impulse signal will be used as a basis for creating other signals. In other words, we will be writing signals as a linear combination of shifted unit-impulse signals.

1.7.1 Discrete-Time LTI Systems

Consider an arbitrary signal $x[n]$, it can be represented as linear combination of shifted unit-impulse signals.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{1.3}$$

This is called *sifting property* of impulse in discrete-time case.

Let $h_k[n]$ be the impulse response (system response to the unit-impulse $\delta[n-k]$). ($h_0 = h$)

$\because$ System is linear
$\therefore$ From eq. (1.3), combining the system response to different unit impulses will give the output

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \tag{1.4}$$

$\because$ System is time-invariant
$\therefore h_k[n] = h[n-k]$
The eq. (1.4), reduces to

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{1.5}$$

Using convolution sum[3], eq. (1.5) reduces to

$$y[n] = x[n] * h[n]$$

1.7.2 Continuous-Time LTI Systems

Like discrete-time case, we can write any arbitrary signal $x(t)$ as a linear combination of shifted unit-impulse signals.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{1.6}$$

This is called *sifting property* of impulse in continuous-time case.

Let $h_\tau(t)$ be the impulse response (system response to the unit-impulse $\delta(t-\tau)$). ($h_0 = h$)

$\because$ System is linear
$\therefore$ From eq. (1.6), combining the system response to different unit impulses will give the output

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau \tag{1.7}$$

$\because$ System is time-invariant
$\therefore h_\tau(t) = h(t-\tau)$
The eq. (1.7), reduces to

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1.8}$$

Using convolution integral[3], eq. (1.8) reduces to

$$y(t) = x(t) * h(t)$$

Corollary: Impulse response is complete characterization of LTI system.

1.7.3 Properties of LTI Systems

The properties given below are for LTI systems only.

1. **Commutative Property**

$$y(t) = x(t) * h(t) = h(t) * x(t)$$
$$y[n] = x[n] * h[n] = h[n] * x[n]$$

2. **Distributive Property**

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$
$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

3. **Associative Property**

$$x[n] * [h_1[n] * h_2[n]] = [x[n] * h_1[n]] * h_2[n]$$
$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

4. When LTI system is memoryless
$h[n] = 0$ for $n \neq 0$ and $h(t) = 0$ for $t \neq 0$

$$h(t) = k\delta(t) \qquad y(t) = kx(t)$$
$$h[n] = k\delta[n] \qquad y[n] = kx[n]$$

Note: $x(t) * \delta(t) = x(t)$ and $x[n] * \delta[n] = x[n]$

5. LTI system $h_1(t)$(or $h_1[n]$) is invertible if $\exists$ $h_2(t)$(or $h_2[n]$), such that

$$h_1(t) * h_2(t) = \delta(t)$$
$$\text{or } h_1[n] * h_2[n] = \delta[n]$$

6. When LTI system is causal

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{n} x[k]h[n-k] \quad (\because\ x[k]=0, k>n) \\ &= \sum_{k=0}^{\infty} x[n-k]h[k] \end{aligned}$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \quad (\because\ x(\tau)=0, \tau>t) \\ &= \int_{0}^{\infty} x(t-\tau)h(\tau) \end{aligned}$$

So, we can also say that for a causal **system**

$$\begin{aligned} h(t) &= 0\ \forall\ t<0 \\ h[n] &= 0\ \forall\ n<0 \end{aligned}$$

Similarly, for a causal **signal**

$$\begin{aligned} x(t) &= 0\ \forall\ t<0 \\ x[n] &= 0\ \forall\ n<0 \end{aligned}$$

Initial test: A causal system gives no output unless an input is given.

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