Example: The daily consumption of milk in a city in excess of 20000 galllons is approximately distributed as a Gamma distribution with parameters $\beta = \frac{1}{10000}$ and $\alpha = 2$. The city has a daily stock of 30000 gallons. What is the probability that the stock is insufficient on a particular day.

Solution: Let X denotes the daily consumption of milk (in litres) in a city

Then the random variable $Y = X - 20000$ has a Gamma distribution with pdf

$$f(y) = \frac{1}{(10000)^2\Gamma(2)} y^{2-1} e^{-\frac{y}{10000}} \ ; \quad y > 0$$

Since the city has a daily stock of 30000 gallons, the probability that the city is insufficient for a (single) day is

$$P(X > 30000) \ = P(Y > 10000) = \int_{10000}^{\infty} f(y)\,dy.$$

$$P(X > 30000) = P(Y > 10000)$$

$$= \int_{10000}^{\infty} \frac{y e^{-\frac{y}{10000}}}{(10000)^2}\,dy$$

Put $z = \frac{y}{10000}$

$\Rightarrow \ dz = \frac{dy}{10000}$

$$= \int_1^{\infty} z e^{-z}\,dz$$

$$= (-z e^{-z})_1^{\infty} - (e^{-z})_1^{\infty}$$

$$= \frac{2}{e}$$

Note: Here $\alpha = 2$ so the integration is easily done.

However, for general values of α, β the integral is evaluated by using table of incomplete Gamma integral of the form $\int_0^{\infty} \frac{x^{n-1}e^{-x}}{\Gamma(n)}\,dx$, which has been tabulated for different values of α, β .

Example: Let X be a random variable with the probability density function (p.d.f.)

$$f(x) = \begin{cases} a\,e^{-\frac{x}{3}}; & x > 0 \\ 0 & ; otherwise \end{cases}$$

Find (i) Value of a (ii) $P(X > 3)$ (iii) $P(1 < X < 4)$

Solution: 1st method

$$f(x) = \begin{cases} a\,e^{-x/3}; & x>0 \\ 0 & ; else \end{cases}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0 \end{cases}$$

$$\Rightarrow \quad \boxed{a = \frac{1}{3}}$$

(i) For p.d.f, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} a\, e^{-x/3}\,dx = 1$$

$$\Rightarrow a\left(\frac{e^{-\frac{x}{3}}}{\frac{-1}{3}}\right)_{0}^{\infty} = 1$$

$$\Rightarrow\ -3a(0-1) = 1$$

$$\Rightarrow a = \frac{1}{3}$$

$$f(x) = \begin{cases} a\, e^{-\frac{x}{3}}; & x > 0 \\ 0 & ; otherwise \end{cases}$$

(ii) $P(X > 3)$

1st method by P.D.F.

$$= \int_{3}^{\infty} \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3}\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right)_{3}^{\infty}$$

$$= -1(0 - e^{-1})$$

$$= \frac{1}{e}$$

2nd method by using CDF

$P(X > 3)$

$$= 1 - P(X \le 3)$$

$$= 1 - F(3)$$

$$= 1 - \left[1 - e^{-\lambda(3)}\right]$$

$$= e^{-1}$$

(iii) $P(1 < X < 4)$

1st method by using P.D.F.

$$= \int_1^4 \frac{1}{3} e^{-x/3} dx$$

$$= \frac{1}{3}\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right)_1^4$$

$$= -1\left(e^{-4/3} - e^{-1/3}\right)$$

$$= e^{-1/3} - e^{-4/3}$$

2nd method by using CDF

$P(1 < X < 4)$

$$= F(4) - F(1)$$

$$= \left[1 - e^{-\lambda(4)}\right] - \left[1 - e^{-\lambda(1)}\right]$$

$$= e^{-1/3} - e^{-4/3}$$ Ans

Example: The time (in hours) required to repair a machine is exponential distributed with parameter 1/3. What is the probability that the repair time exceeds 3 hours?

Solution: Required probability $= P(X > 3)$

$$= \int_3^\infty \frac{1}{3} e^{-x/3} dx$$

$$= \frac{1}{3}\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right)_3^\infty$$

$$= -1(0 - e^{-1})$$

$$= \frac{1}{e}$$

2nd method by using CDF

$P(X > 3)$

$$= 1 - P(X \le 3)$$

$$= 1 - F(3)$$

$$= 1 - \left[1 - e^{-\lambda(3)}\right]$$

$$= e^{-1}$$