University of Burgundy

Applied Mathematics HomeWork 2

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1 Problem No 1

In a town called computer vision village the local newspaper the computer visionist has deter-mind that a citizen who purchases a copy of their paper one day has 70 % chance of buying the following day's edition. they have also determined that a person who does not purchase a copy of the computer visionist one day has 20% chance of purchasing in the next day. records show that of the 1000 citizens of computer vision village, exactly 750 purchased a copy of the newspapers on Days 0. to determine the appropriate amount of papers to press each day, the owner of the computer visionist, Mr Marr Rosenfeld, in interested the following types of questions:
1) If person purchased a paper today, how likely is he to purchase a paper on day 2 ? day 3? day n?
2) What sales figures can the computer visionist expect on day 2? day 3? day n?
3) Will the sales figures fluctuate a great deal from day to day or are they likely to become stable eventually?

Answer: Let's start with
Group 1: contain all citizen who purchases a copy of their paper on day n, guess **g1**.
Group 2: contain that citizen who dose not purchases a copy of their on day n, guess **g2**.

so vector is:

$$v_n = \begin{pmatrix} g1 \\ g2 \end{pmatrix} \quad \in R^2$$

where g_1 is the size of group 1 and g_2 the size of group 2.

so now construct a 2×2matrix M, which give the probability that citizen in one group one day will be in the same or another group next day.

given data is:

A citizen who purchases a copy of their paper one day has 70% chance of buying the following day's edition.

A person who does not purchases a copy of the paper one day has 20% chance of purchasing in the next day.

so according to Markov Rules(M):

$$V_n + 1 = \begin{bmatrix} 0.7g1 + 0.2g2 \\ 0.3g1 + 0.8g2 \end{bmatrix}$$

$$V_n + 1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} g1 \\ g2 \end{bmatrix}$$

$$V_n + 1 = M_v n$$
$$whereM = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

Now, find sales figures of the computer Visionist on Day 2,3 and n ? so, first find the power of M for getting M, we **diagonalization the Matrix M.**

$$Det(M - \lambda I) = 0$$

$$\begin{bmatrix} 0.7 - \lambda & 0.2 \\ 0.3 & 0.8 - \lambda \end{bmatrix} = 0$$
$$(0.7 - \lambda)(0.8 - \lambda) - 0.06 = 0$$
$$0.56 - 1.5\lambda + \lambda^2 = 0$$
$$\lambda^2 - 1.5\lambda + 0.5 = 0$$
$$\lambda^2 - \lambda - 0.5\lambda + 0.5 = 0$$
$$\lambda(\lambda - 1) - 0.5(\lambda - 1) = 0$$
$$(\lambda - 1)(\lambda - 0.5) = 0$$
$$\lambda = 1 \qquad \lambda = 0.5$$

Now find Eigen vectors using, $\lambda = 1$

$$N(M - \lambda_1 I)$$

$$= M - I$$
$$= \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}$$

Now we can Solve,

$$= \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$-0.3x_1 + 0.2x_2 = 0$$
$$0.3x_1 = 0.2x_2$$
$$x_2 = 1.5x_1$$

let, $x_1 = 1$

so eigen vector

$$x_1 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$$

check:

$$MX_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \lambda_1 X_1$$

Let, $\lambda_2 = 0.5$

So, same as before

$$M = \lambda_2 I = 0.5I = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}$$

So,

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$0.2x_1 + 0.2x_2 = 0$$

$$0.2x_1 = -0.2x_2$$

$$x_2 = -x_1$$

Let, $x_1 = 1$
So, eigen vecor

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

check:

$$Mx_2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = 0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_2 X_2$$

Now using diagonalization method we have,

$$M = S\Lambda S^{-1}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{-1}$$

So,

$$S = \begin{bmatrix} 1 & 1 \\ 1.5 & -1 \end{bmatrix}$$

Using simple formula of A^{-1} So, We get S^{-1}

$$S^{-1} = \frac{1}{-2.5}\begin{bmatrix} -1 & -1 \\ -1.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

So,

$$M = \begin{bmatrix} 1 & 1 \\ 1.5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

For day 2, we have $V_2 = MV_1 = M(MV_0) = M^2V_0$

So we have $M^k = S\Lambda^k S^{-1}$

So, $M^2 = S\Lambda^2 S^{-1}$

$$M^2 = \begin{bmatrix} 1 & 1 \\ 1.5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.25 \\ 1.5 & -0.25 \end{bmatrix}\begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{bmatrix}$$

Condition on day 0 is given as $v_0 = \begin{bmatrix} 750 \\ 250 \end{bmatrix}$ So,

$$v_2 = M^2 v_0$$

$$= \begin{bmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{bmatrix}\begin{bmatrix} 750 \\ 250 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 487.5 \\ 512.5 \end{bmatrix}$$

Same as for day 3,

$$M^3 = \begin{bmatrix} 1 & 1 \\ 1.5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.125 \end{bmatrix}\begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.125 \\ 1.5 & -0.125 \end{bmatrix} \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{bmatrix}$$

As we know $V_0 = \begin{bmatrix} 750 \\ 250 \end{bmatrix}$

$$v_3 = M^3 v_0$$

$$= \begin{bmatrix} 0.475 & 0.35 \\ 0.525 & 0.65 \end{bmatrix} \begin{bmatrix} 750 \\ 250 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 443.75 \\ 556.25 \end{bmatrix}$$

For day n:

$$= M^n \begin{bmatrix} 1 & 1 \\ 1.5 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.5^n \end{bmatrix} \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5^n \\ 1.5 & -0.5^n \end{bmatrix} \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & -0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 + (0.6)(0.5)^n & (0.4)(1 - 0.5^n) \\ (0.6)(1 - 0.5^n) & (0.6) + (0.4)(0.5)^n \end{bmatrix}$$

$$v_n = M^n v_0$$

$$= \begin{bmatrix} 0.4 + (0.6)(0.5)^n & (0.4)(1 - 0.5^n) \\ (0.6)(1 - 0.5^n) & 0.6 + (0.4)(0.5)^n \end{bmatrix} \begin{bmatrix} 750 \\ 250 \end{bmatrix}$$

$$v_n = \begin{bmatrix} 400 + (350)(0.5)^n \\ 600 + (350)(0.5)^n \end{bmatrix}$$

1) If person purchased a paper today, how likely is he to purchase a paper day 2,3 and n?

As per above calculation, and according to Markov matrix, for day 2,3 and n it would be 0.55, 0.475 and $0.4 + (0.6)(0.5)^n$ respectively.

2) what sales figures can the computer visionist expect on day 2,3 and n ?

the sales figures expected for day 2,3 and n it would be 488(round figure), 444(round figure) and $400 + (350)(0.5)^n$ respectively, that much person buy the news paper and 512, 556 and $600 - (350)(0.5)^n$ that much person don't.

3) will the sales figures fluctuate a great deal from day to day, or are they likely to become stable eventually?

the sales figure will become stable eventually as days progress. as M in a Markov matrix, from the property of Markov matrix the corresponding eigen vector of eigen value one is steady state of the system that every column of M^n will approach as $m \to \infty$, so here for eigen vector which is given $\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$ and for large values of n the sales figure tends to this eigen vector. so we can say that the sales figure will become stable at 400 persons buying the newspaper and 600 persons don't buy.

2 Problem No 2

This second problem is about another application of eigenvalues / eigenvectors. It deals with the Google page-Rank algorithm.

1) Read the paper How Google ranks Web Pages? which is available from Edmodo and Write a summary of the ranking method.

2) Explain how the eigenvalue problem is solved. you don't need to review what is the paper, just put down what you understand.

3) Starting with a 5×5 Markov matrix, apply the algorithm described in section 5 of the paper to find the eigenvectors associated with eigenvalue 1.

Answer: Now a days, there are thousands of the people are using Google web page. when Google was launched by Sergey Brin and Larry page while they were graduate students at Stanford. At that time there are many internet search engines like Google, yahoo and MSN receive over 81% of all search requests despite claims that the quality of search provided by yahoo and MSN now equals that of Google. all search engines using Web Pages content to the rank the Web pages but this is not good way because web page develops could easily manipulate the ordering of search results by placing concealed information on Web pages so brin and page developed algorithm that gives ranking named page rank. page rank has connection to numerous areas of mathematics and computer science such as matrix theory, numerical analysis, information retrieval and graph theory, as a result much research continues to be developed to the explaining and improving page rank. In the page rank method of ranking of pages is based on how pages are liked and not on the content of the pages or how often the pages are visited. web is constantly changing so the ranking changes too.

The ranking form the entries of vector $v \in \mathbb{R}^n$

Where N - total number of pages on the web

v - non negative numbers

v_i - measure of the value of pages i

Ranking vector V followed by several improvement leading to the page rank method.

The Very Simple Method

if page j has a link to page i, imagine that page j is recommending page i. and let, N be the total number of web pages, let A be the N by N matrix whose ij entry is

$a_{ij} = 1$ if page j has a link to page j

O if not

Then the number of recommendation that page i gets i

$$a_{i1} + a_{i2} + \cdots + a_{in}$$

We can think of this as a measure of the value v_i of page i

$$v_i = a_{i1} + a_{i2} + \cdots + a_{in}$$

This equation can be written very concisely using matrix multiplication

$$v = Au$$

where $u \in \mathbb{R}^N$ is the vector each of whose component is 1.

First Improvement

The weight of a recommendation made by page j should be inversely proportional to the total number of recommendation that j makes. thus replace the formula

$$v_i = \frac{a_{i1}}{n_1} + \frac{a_{i2}}{n_2} + \cdots + \frac{a_{iN}}{n_N}$$

Where n_j the total number of recommendation is that page j makes:

$$N_j = \sum_i a_{ij}$$

But here is a problem if page j is a dead-end. if there are no links from page j because then formula involves dividing by 0. Google gets around this problem by pretending that a page will no links out in fact has a link to every page, thus we redefine the matrix A as follow:

$a_{ij} = 1$ if page j has a link to page i or if page j is a dead end.

0 if there is a link from page j to some page, but not to page i.

Now $n_j = \Sigma_i a_{ij}$ will always be positive so formula makes sense.

$$V = Pu$$

Where P is the N by N matrix whose ij entry is

$$P_{ij} = \frac{a_{ij}}{n_j}$$

The weight of a recommendation should depend on the importance of the recommender since v_j is supposed to reflect the value or importance of page j we thus multiply the j^{th} term in 2 by v_j to get the formula

$$V_i = \frac{a_{i1}}{n_1} v_1 + \frac{a_{i2}}{n_2} v_2 + \cdots + \frac{a_{iN}}{n_N} u_N$$

in matrix notation:

$$V = Pv$$

Thus we are led to conclude that the ranking vector v should be an eigen vector of p with eigenvalue 1. For this to give a ranking of the Web Pages, 1 had better be an eigenvalue of P.fortunately it must be because P is a Markov matrix. That is, the entries of P are all>= 0 and the sum of the entries in each column is 1.

But if there are several clusters of Web Pages not connected to each other, then P with have a set of two or more linearly eigen vector with eigenvalue 1. In that case,

The equation $V = Pv$ will not determine a unique ranking.

In mathematical notation, we let

$$v = Qvor(q-1)v = 0$$

By Gaussian elimination takes too long

We repeatedly multiply by Q to get Q_w, Q^2w, Q^3w, and so on. By theorem ? these vectors coverge to v. so we keep going until the sequence settles down i.e until $Q_k + 1w$ is nearly equal to Q^kw. Then QKw is our eigen vector V By method we can find the eigen vector corresponding to eigen value 1.

2. Explain how the eigen value problem is solved ?

We have Markov matrix the equation to get the rank vector is $v = Qv$ so here it is in the form of $Ax = \lambda x$, Where λ are called eigenvalues of A and here $\lambda = 1$. As this is a linear system it would be generally solved by Gaussian elimination method. but here the problem is we have millions of Web Pages and so to rank them by the process of Gauss elimination would be very hectic process. So, we go for other methods to solve such type of problems. Here Q is Markov matrix so as $k \to \infty$ then $Q^kw(w$ is vector whose sum of entries is 1), converges to v, the eigen vector which corresponds to eigen value 1. Here practically we check until $Q^{k+1}w$ is equal to Q^w so finally Q^k is our eigen vector v which is the rank vector in our problem.

3. starting with 5×5 Markov matrix, apply the algorithm described in section 5 of the paper to find the eigen vector associated with eigen value 1.

So, let assume a matrix which has same value in the starting now there are 5 web pages for in the link chain.

$$Q = \begin{bmatrix} 0.6 & 0.3 & 0.2 & 0.1 & 0.5 \\ 0.2 & 0.3 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.4 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.2 & 0 & 0 \end{bmatrix}$$

Let us begin with nonzero vector whose elements are all non negative and divide this vector by sum of its elements to get vector w:

$$W = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}'$$

Compute QW, Q^2W, Q^3W, $\cdots$, until Q^nW.

$$QW = \begin{bmatrix} 0.3400 & 0.2600 & 0.2600 & 0.800 & 0.0600 \end{bmatrix}'$$

$$Q^2W = \begin{bmatrix} 0.3720 & 0.2220 & 0.2360 & 0.0920 & 0.0780 \end{bmatrix}'$$

$$Q^3W = \begin{bmatrix}0.3852 & 0.2248 & 0.2298 & 0.0908 & 0.0694\end{bmatrix}'$$

$$Q^4W = \begin{bmatrix}0.3883 & 0.2246 & 0.2277 & 0.0909 & 0.0684\end{bmatrix}'$$

$$Q^5W = \begin{bmatrix}0.3892 & 0.2247 & 0.2272 & 0.0909 & 0.0680\end{bmatrix}'$$

$$Q^6W = \begin{bmatrix}0.3895 & 0.2247 & 0.2270 & 0.0909 & 0.0679\end{bmatrix}'$$

$$Q^7W = \begin{bmatrix}0.3895 & 0.2247 & 0.2270 & 0.0909 & 0.0679\end{bmatrix}'$$

So, above show to the solution we can see that Q^6W and Q^7W are same. so we can say that property of Markov matrix, that Q^6W has converged to vector V.

MATLAB code attached in .zip file

In comment, I would like to say that. this page rank algorithm is more effective, great and simple.