Assignment

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CS21BTECH11052

Contents

***Abstract*—This manual provides solutions to the Assignment on Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.1.c
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The python code for the plot in Fig. 1.2 is given below,

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.2CDF.py
```

Download the above file and execute the command below to produce Fig.1.2

```
$ python3 1.2CDF.py
```

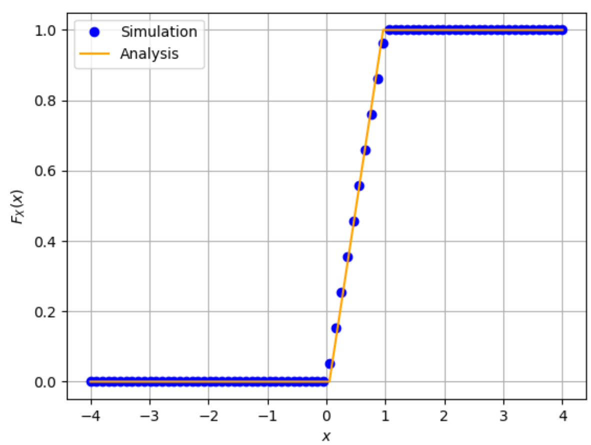

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable between 0 and 1,

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We have three parts:

i) For $x < 0$; $p_X(x) = 0$, So $F_U(x) = 0$.
ii) For $0 \leq x < 1$;

$$F_U(x) = \int_0^x (1)du = x \tag{1.3}$$

iii) For $x \geq 1$; CDF is 1 as all the random numbers are between 0 and 1.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\operatorname{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/1.4.c
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 1.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.500007 \tag{1.7}$$
$$\text{Variance} = 0.083301 \tag{1.8}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution: W.K.T,

$$\operatorname{var}[U] = E\left[U^2\right] - E[U]^2 \tag{1.10}$$

Where $E[U]$ is,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$
$$= \int_0^1 x \tag{1.12}$$
$$= \frac{1}{2} = 0.5 \tag{1.13}$$

And $E\left[U^2\right]$ is,

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.14}$$
$$= \int_0^1 x^2 dF_U(x) \tag{1.15}$$
$$= \frac{1}{3} \tag{1.16}$$
$$\tag{1.17}$$

Hence finally,

$$\therefore \boxed{\operatorname{var}[U] = \frac{1}{12} = 0.0833} \tag{1.18}$$

The theoretically calculated values are,

$$\text{Mean} = 0.5 \tag{1.19}$$
$$\text{Variance} = 0.0833 \tag{1.20}$$

These values matches with the actual analysis values from above,

$$\text{Mean} = 0.500007 \tag{1.21}$$
$$\text{Variance} = 0.083301 \tag{1.22}$$

Hence it verifies the result.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/2.1.c
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.2CDF.py
```

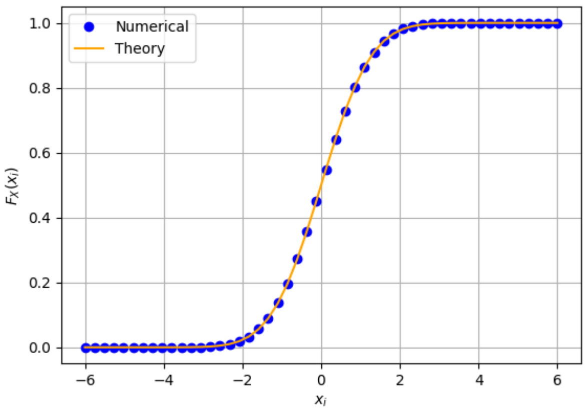

Fig. 2.2: The CDF of X

Download the above file and execute the following command to produce Fig.2.2

```
$ python3 2.2CDF.py
```

Some of the properties of CDF

a) $F_X(x)$ is non decreasing function.
b) As $x \to -\infty$, $F_X(x) \to 0$ and when $x \to \infty$, $F_X(x) \to 1$
c) Graph is linear upto some region around $x = 0$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.3PDF.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3PDF.py
```

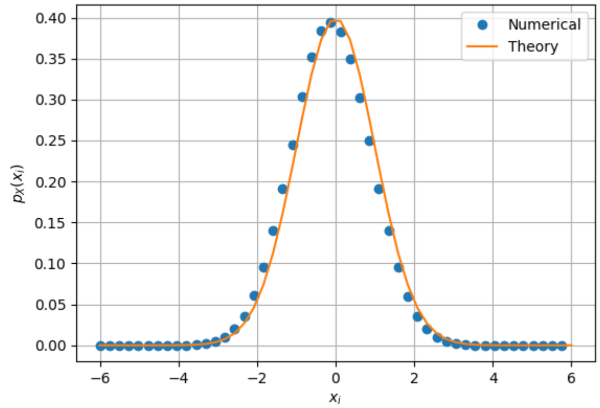

Fig. 2.3: The PDF of X

Some of the properties of the PDF:

a) Symmetric about $x = \mu$
b) Area under the PDF graph is unity.
c) Increasing function for $x < \mu$ and decreasing for $x > \mu$ and attains maximum at $x = \mu$.

Let $x \sim \aleph(0, 1)$. The Q-function $Q(x)$ is defined as:

$$Q(x) = Pr(X > x) \tag{2.3}$$
$$= 1 - Pr(X \le x) \tag{2.4}$$
$$= 1 - F_X(x) \tag{2.5}$$

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.4.c
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 2.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.000294 \tag{2.6}$$
$$\text{Variance} = 0.999561 \tag{2.7}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.8)$$

repeat the above exercise theoretically.

Solution: CDF is defined as

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \quad (2.9)$$

$$\text{W.K.T,} \quad \boxed{F_X(x) = 1} \quad (2.10)$$

Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x)dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

It is a odd function function, So its value is 0.

$$\boxed{E(x) = 0} \quad (2.13)$$

$E(x^2)$ is given by,

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right)dx \quad (2.14)$$

On applying by-parts for the above integral we get,

$$\int_{-\infty}^{\infty} x \times x \exp\left(-\frac{x^2}{2}\right)dx = -x \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \quad (2.15)$$

$$+ \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right)dx \quad (2.16)$$

Where, $x \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{\infty} = 0$; because it is converges to zero as x tends to infinity.

$$\therefore E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \quad (2.17)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.18)$$

$$= 1 \quad (2.19)$$

Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.20)$$

$$\therefore \boxed{\text{var}[U] = 1} \quad (2.21)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/3.1.c
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 3.1.c −lm
$ ./a.out
```

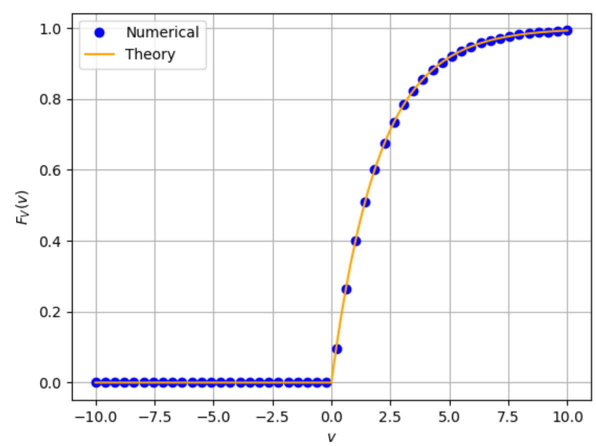

Fig. 3.1: The PDF of X

The CDF of V is plotted in Fig. 3.1 using the code below

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/3.1CDF.py
```

Download the above files and execute the following commands to produce plot Fig.3.1

```
$ python3 3.1CDF.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If Y = g(X),

W.K.T,

$$F_Y(y) = F_X(g^{-1}(y)) \quad (3.2)$$

$$V = -2\ln(1-U) \tag{3.3}$$
$$1-U = e^{\frac{-V}{2}} \tag{3.4}$$
$$U = 1-e^{\frac{-V}{2}} \tag{3.5}$$
$$F_V(x) = F_U(1-e^{\frac{-x}{2}}) \tag{3.6}$$

$$\implies F_V(x) = \begin{cases} 0 & x < 0 \\ 1-e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.1.c
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2 using the code below

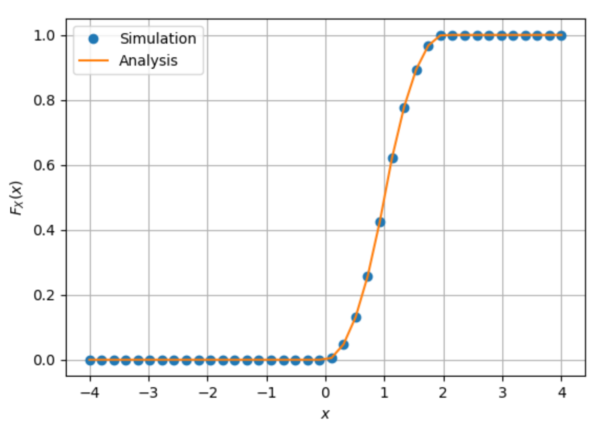

Fig. 4.2: The CDF of T

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.2CDF.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.2CDF.py
```

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.3PDF.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.3PDF.py
```

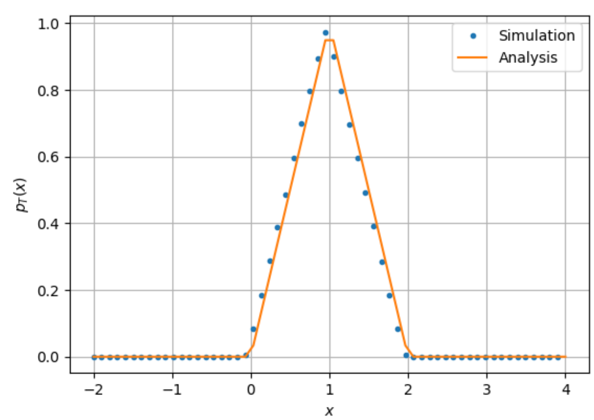

Fig. 4.3: The PDF of T

4.4 Find the Theoreotical Expression for the PDF and CDF of T
Solution:

$$T = U_1 + U_2 \tag{4.2}$$
$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x)p_{U2}(y)dx \tag{4.3}$$
$$\text{As,} p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{4.4}$$
$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u)p_U(t-u)du \tag{4.5}$$

a) Theoretical PDF

i) For $0 < t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \tag{4.6}$$

$$\implies p_T(t) = \int_0^t du = t \tag{4.7}$$

ii) For $1 < t \leq 2$

$$p_T(t) = \int_0^1 p_U(t-u)du \tag{4.8}$$

$$\implies p_T(t) = \int_{t-1}^1 du = 2-t \tag{4.9}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \tag{4.10}$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot
Solution: The above theoretical results are verified in the plots Fig 4.2 and Fig 4.3

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−Assignments/blob/main/RandomNumbers/codes/5.1.c
https://github.com/bhargav0383/AI1110−Assignments/blob/main/RandomNumbers/codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 5.1.c
$ ./a.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \aleph(0, 1)$.
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−Assignments/blob/main/RandomNumbers/codes/5.2.c
https://github.com/bhargav0383/AI1110−Assignments/blob/main/RandomNumbers/codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 5.2.c
$ ./a.out
```

5.3 Plot Y using a scatter plot.
Solution: The CDF of V is plotted in Fig. 5.3 using the code below

```
https://github.com/bhargav0383/AI1110−Assignments/blob/main/RandomNumbers/codes/5.3scatt.py
```

Download the above files and execute the following commands to produce Fig.5.3

```
$ python3 5.3scatt.py
```

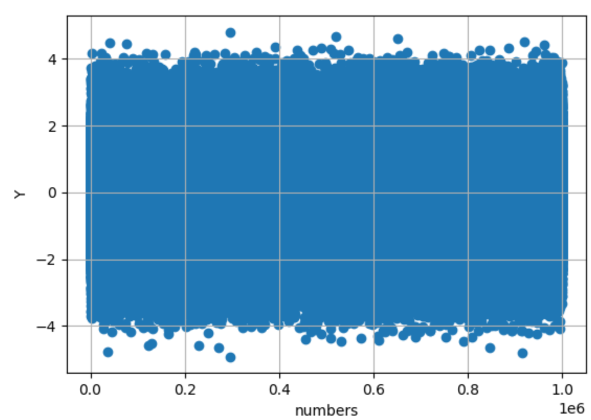

Fig. 5.3: The Scatter Plot

5.4 Guess how to estimate X from Y.
Solution:

a) If $Y < 0$ then probably $X = -1$
b) If $Y > 0$ then probably $X = 1$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110–Assignments/blob/main/RandomNumbers/codes/5.5.c
https://github.com/bhargav0383/AI1110–Assignments/blob/main/RandomNumbers/codes/functions.h
```

Download the above files and execute the following commands to get the result

```
$ gcc 5.5.c
$ ./a.out
```

$$P_{e|0} = 0.499035 \tag{5.4}$$

$$P_{e|1} = 0.500176 \tag{5.5}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = P(X = 1)P_{e|0} + P(X = -1)P_{e|1} \tag{5.6}$$

Since X is equiprobable

$$P(X = 1) = P(X = -1) = 0.5 \tag{5.7}$$

$$\implies P_e = \frac{P_{e|0} + P_{e|1}}{2} \tag{5.8}$$

$$\implies \boxed{P_e = Q_N(A)} \tag{5.9}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.10}$$

$$P_{e|0} = \Pr\left(AX + N < 0|X = 1\right) \tag{5.11}$$

$$P_{e|0} = \Pr\left(N < -A\right) \tag{5.12}$$

$$P_{e|0} = \int_{-\infty}^{-A} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{5.13}$$

$$P_{e|0} = \int_{A}^{\infty} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \tag{5.14}$$

$$P_{e|0} = Q_N(A) \tag{5.15}$$

$$\text{Similarly, } P_{e|1} = Q_N(A) \tag{5.16}$$

```
https://github.com/bhargav0383/AI1110–Assignments/blob/main/RandomNumbers/codes/5.7.py
```

Download the above files and execute the following commands to produce Fig.5.7

```
$ python3 5.7.py
```

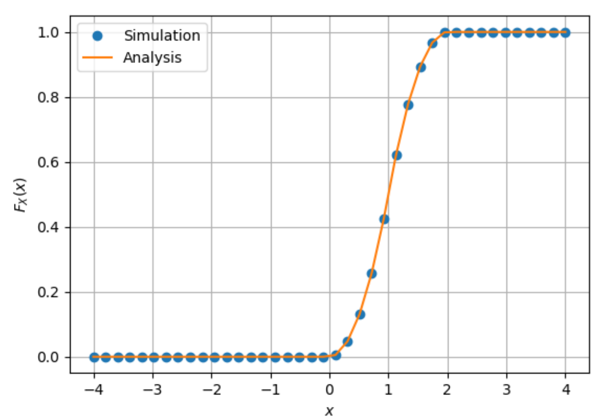

Fig. 5.7: $P_e(A)$ with semilog-y axis

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: To estimate X from Y, we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.17}$$

Therefore,

$$\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (5.18)\\
&= \Pr\left(AX + N < \delta | X = 1\right) && (5.19)\\
\implies P_{e|0} &= \Pr\left(N < \delta - A\right) && (5.20)\\
&= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx && (5.21)\\
&= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx && (5.22)\\
\implies P_{e|0} &= Q_N(A - \delta) && (5.23)
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_{e|1} &= Q_N(A + \delta) && (5.24)\\
P_e &= P_{e|0} \Pr\left(X = 1\right) + P_{e|1} \Pr\left(X = -1\right) && (5.25)\\
&= \frac{Q_N(A - \delta) + Q_N(A + \delta)}{2} && (5.26)
\end{aligned}$$

Differentiating the above equation wrt δ:

$$\begin{aligned}
0 &= \frac{d}{d\delta}\left(\frac{Q_N(A - \delta) + Q_N(A + \delta)}{2}\right) && (5.27)\\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}\right) && (5.28)\\
\implies (\delta - A)^2 &= (\delta + A)^2 && (5.29)\\
\implies & \boxed{\delta = 0} && (5.30)
\end{aligned}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.31)$$

Solution: Using Eq. (5.25), we have:

$$\begin{aligned}
P_e &= P_{e|0} p + P_{e|1}(1 - p) && (5.32)\\
&= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta) && (5.33)
\end{aligned}$$

Differentiating as before, we get:

$$0 = p\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - (1 - p)\frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \quad (5.34)$$

$$\begin{aligned}
e^{\frac{(\delta + A)^2 - ((\delta - A))^2}{2}} &= \frac{1 - p}{p} && (5.35)\\
\implies & \boxed{\delta = 0} && (5.36)
\end{aligned}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: Assume that

$$\begin{aligned}
\Pr\left(X = -1\right) &= p && (5.37)\\
\Pr\left(X = 1\right) &= (1 - p) && (5.38)
\end{aligned}$$

From Total Probability Theorem, we have:

$$\begin{aligned}
p_Y(y) &= p_{Y|X=-1}(y| - 1) \Pr\left(X = -1\right)\\
&\quad + p_{Y|X=1}(y|1) \Pr\left(X = 1\right) && (5.39)\\
p_Y(y) &= p \times p_{(-A+N)}(y) && (5.40)\\
&\quad + (1 - p) \times p_{(A+N)}(y) && (5.41)
\end{aligned}$$

Now, $p_{(-A+N)}$ is just the pdf of a shifted normal distribution, and therefore:

$$p_Y(y) = p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \quad (5.42)$$

To use the MAP criterion, we must find $p_{X|Y}(x|y)$. To do this, we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \quad (5.43)$$

When $X = 1$, we have:

$$\begin{aligned}
p_{X|Y}(1|y) &= \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} && (5.44)\\
&= \frac{(1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} && (5.45)\\
&= \frac{(1 - p)\, e^{2yA}}{p + (1 - p)\, e^{2yA}} && (5.46)
\end{aligned}$$

Similarly, when $X = -1$, we get:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1 - p)\, e^{2yA}} \quad (5.47)$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\begin{aligned}
\frac{(1 - p)\, e^{2yA}}{p + (1 - p)\, e^{2yA}} &> \frac{p}{p + (1 - p)\, e^{2yA}} && (5.48)\\
e^{2yA} &> \frac{p}{(1 - p)} && (5.49)\\
y &> \frac{1}{2A} \ln \frac{p}{(1 - p)} && (5.50)
\end{aligned}$$

Therefore, when Eq. (5.50), we can assert that $X = 1$, and $X = -1$ otherwise. Now, consider when $p = \frac{1}{2}$. We have:

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)} \tag{5.51}$$

$$= \frac{1}{2A} \ln 1 \tag{5.52}$$

$$= 0 \tag{5.53}$$

Therefore, when $y > 0$, we choose $X = 1$, and we choose $X = -1$ otherwise.

6 Gaussian to Other

6.1 Let $X_1 \sim \aleph(0,1)$ $X_2 \sim \aleph(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$