

Assignment

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Abstract—This manual provides solutions to the Assignment on Random Numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 1.1.c
$ ./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The python code for the plot in Fig. 1.2 is given below,

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.2CDF.py>

Download the above file and execute the command below to produce Fig.1.2

```
$ python3 1.2CDF.py
```

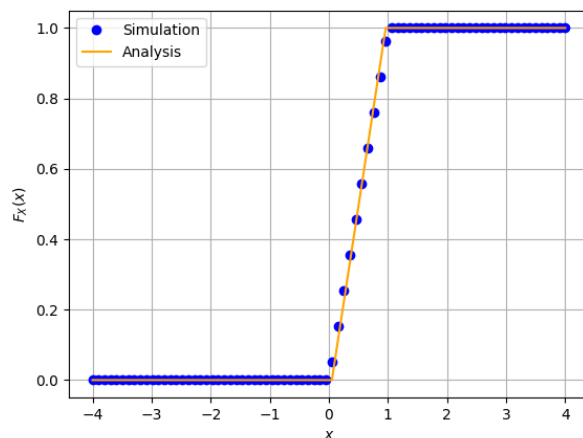


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given U is a uniform Random Variable between 0 and 1,

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (1.2)$$

We have three parts:

- i) For $x < 0$; $p_X(x) = 0$, So $F_U(x) = 0$.
- ii) For $0 \leq x < 1$;

$$F_U(x) = \int_0^x (1) du = x \quad (1.3)$$

- iii) For $x \geq 1$; CDF is 1 as all the random numbers are between 0 and 1.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.4.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 1.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.500007 \quad (1.7)$$

$$\text{Variance} = 0.083301 \quad (1.8)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution: W.K.T,

$$\text{var}[U] = E[U^2] - E[U]^2 \quad (1.10)$$

Where $E[U]$ is,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.11)$$

$$= \int_0^1 x \quad (1.12)$$

$$= \frac{1}{2} = 0.5 \quad (1.13)$$

And $E[U^2]$ is,

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.14)$$

$$= \int_0^1 x^2 dF_U(x) \quad (1.15)$$

$$= \frac{1}{3} \quad (1.16)$$

$$(1.17)$$

Hence finally,

$$\therefore \text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.18)$$

The theoretically calculated values are,

$$\text{Mean} = 0.5 \quad (1.19)$$

$$\text{Variance} = 0.0833 \quad (1.20)$$

These values matches with the actual analysis values from above,

$$\text{Mean} = 0.500007 \quad (1.21)$$

$$\text{Variance} = 0.083301 \quad (1.22)$$

Hence it verifies the result.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.2CDF.py>

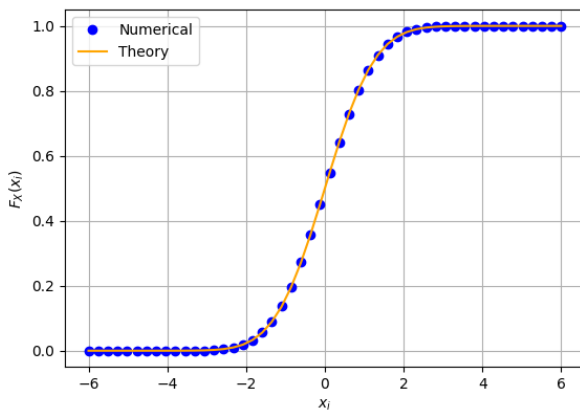


Fig. 2.2: The CDF of X

Download the above file and execute the following command to produce Fig.2.2

\$ python3 2.2CDF.py

Some of the properties of CDF

- $F_X(x)$ is non decreasing function.
- As $x \rightarrow -\infty$, $F_X(x) \rightarrow 0$ and when $x \rightarrow \infty$, $F_X(x) \rightarrow 1$
- Graph is linear upto some region around $x = 0$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.3PDF.py>

Download the above files and execute the following commands to produce Fig.2.3

\$ python3 2.3PDF.py

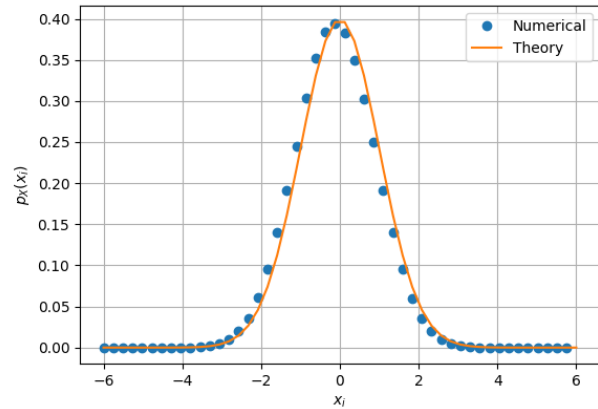


Fig. 2.3: The PDF of X

Some of the properties of the PDF:

- Symmetric about $x = \mu$
- Area under the PDF graph is unity.
- Increasing function for $x < \mu$ and decreasing for $x > \mu$ and attains maximum at $x = \mu$.

Let $x \sim \mathcal{N}(0, 1)$. The Q-function $Q(x)$ is defined as:

$$Q(x) = Pr(X > x) \quad (2.3)$$

$$= 1 - Pr(X \leq x) \quad (2.4)$$

$$= 1 - F_X(x) \quad (2.5)$$

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.4.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

\$ gcc 2.4.c
\$./a.out

The actual analysis values,

$$\text{Mean} = 0.000294 \quad (2.6)$$

$$\text{Variance} = 0.999561 \quad (2.7)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.8)$$

repeat the above exercise theoretically.

Solution: CDF is defined as

$$F_X(x) = \int_{-\infty}^x p_X(x) dx \quad (2.9)$$

$$\text{W.K.T, } \boxed{F_X(x) = 1} \quad (2.10)$$

Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

It is a odd function function, So its value is 0.

$$\boxed{E(x) = 0} \quad (2.13)$$

$E(x^2)$ is given by,

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.14)$$

On applying by-parts for the above integral we get,

$$\int_{-\infty}^{\infty} x \times x \exp\left(-\frac{x^2}{2}\right) dx = -x \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.15)$$

$$+ \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.16)$$

Where, $x \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} = 0$; because it is a odd function.

$$\therefore E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.17)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.18)$$

$$= 1 \quad (2.19)$$

Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.20)$$

$$\therefore \boxed{\text{var}[U] = 1} \quad (2.21)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/3.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

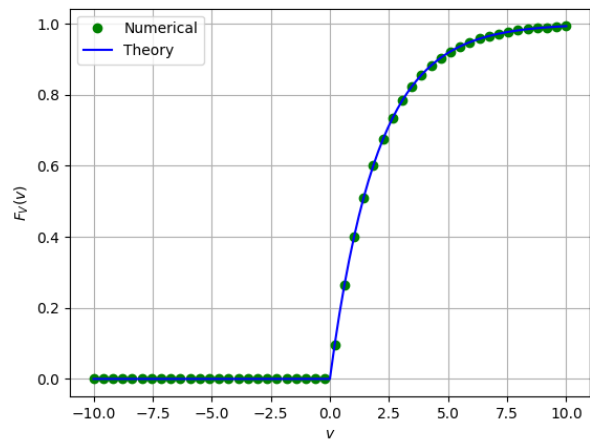


Fig. 3.1: The PDF of X

The CDF of V is plotted in Fig. 3.1 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/3.1CDF.py>

Download the above files and execute the following commands to produce plot Fig.3.1

```
$ python3 3.1CDF.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If $Y = g(X)$,

W.K.T,

$$F_Y(y) = F_X(g^{-1}(y)) \quad (3.2)$$

$$V = -2 \ln(1 - U) \quad (3.3)$$

$$1 - U = e^{\frac{-V}{2}} \quad (3.4)$$

$$U = 1 - e^{\frac{-V}{2}} \quad (3.5)$$

$$F_V(x) = F_U(1 - e^{\frac{-x}{2}}) \quad (3.6)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \quad (3.7)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. 4.2 using the code below

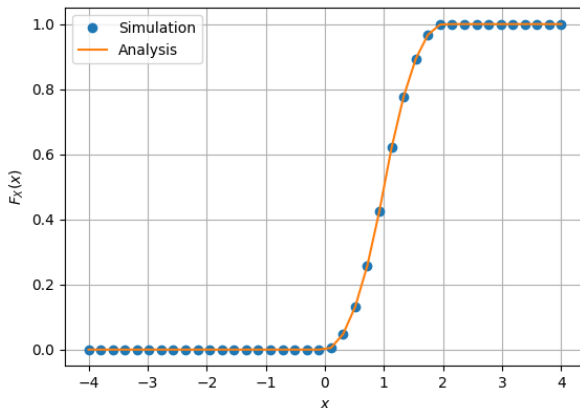


Fig. 4.2: The CDF of T

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.2CDF.py>

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.2CDF.py
```

4.3 Find the PDF of T .

Solution: The PDF of T is plotted in Fig. 4.2 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/4.3PDF.py>

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.3PDF.py
```

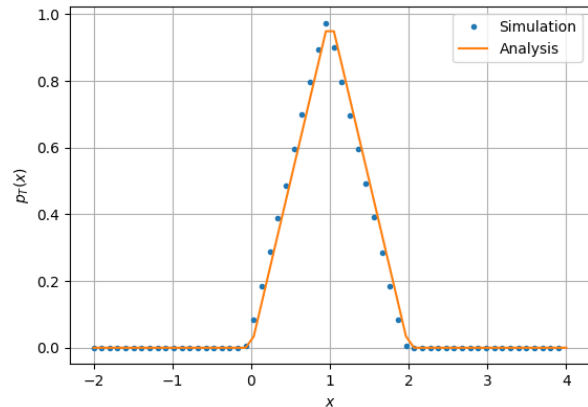


Fig. 4.3: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U1}(x)p_{U2}(y)dx \quad (4.3)$$

$$\text{As, } p_{U1}(x) = p_{U1}(y) = p_U(u) \quad (4.4)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

a) Theoretical PDF

i) For $0 < t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (4.7)$$

ii) For $1 < t \leq 2$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2 - t \quad (4.9)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (4.10)$$

$$\therefore F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

Solution: The above theoretical results are verified in the plots Fig 4.2 and Fig 4.3