Assignment

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CS21BTECH11052

Contents

***Abstract*—This manual provides solutions to the Assignment on Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.1.c
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The python code for the plot in Fig. 1.2 is given below,

```
https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.2CDF.py
```

Download the above file and execute the command below to produce Fig.1.2

```
$ python3 1.2CDF.py
```

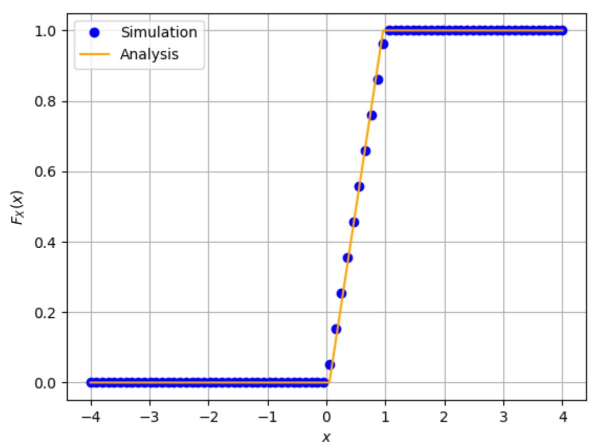

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable between 0 and 1,

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We have three parts:

i) For $x < 0$; $p_X(x) = 0$, So $F_U(x) = 0$.
ii) For $0 \leq x < 1$;

$$F_U(x) = \int_0^x (1)du = x \tag{1.3}$$

iii) For $x \geq 1$; CDF is 1 as all the random numbers are between 0 and 1.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110-
    Assignments/blob/main/RandomNumbers/
    codes/1.4.c
https://github.com/bhargav0383/AI1110-
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 1.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.500007 \tag{1.7}$$
$$\text{Variance} = 0.083301 \tag{1.8}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution: W.K.T,

$$\text{var}[U] = E\left[U^2\right] - E[U]^2 \tag{1.10}$$

Where $E[U]$ is,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$
$$= \int_{0}^{1} x \tag{1.12}$$
$$= \frac{1}{2} = 0.5 \tag{1.13}$$

And $E\left[U^2\right]$ is,

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.14}$$
$$= \int_{0}^{1} x^2 dF_U(x) \tag{1.15}$$
$$= \frac{1}{3} \tag{1.16}$$
$$\tag{1.17}$$

Hence finally,

$$\therefore \boxed{\text{var}[U] = \frac{1}{12} = 0.0833} \tag{1.18}$$

The theoretically calculated values are,

$$\text{Mean} = 0.5 \tag{1.19}$$
$$\text{Variance} = 0.0833 \tag{1.20}$$

These values matches with the actual analysis values from above,

$$\text{Mean} = 0.500007 \tag{1.21}$$
$$\text{Variance} = 0.083301 \tag{1.22}$$

Hence it verifies the result.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110-
    Assignments/blob/main/RandomNumbers/
    codes/2.1.c
https://github.com/bhargav0383/AI1110-
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.2CDF.py
```

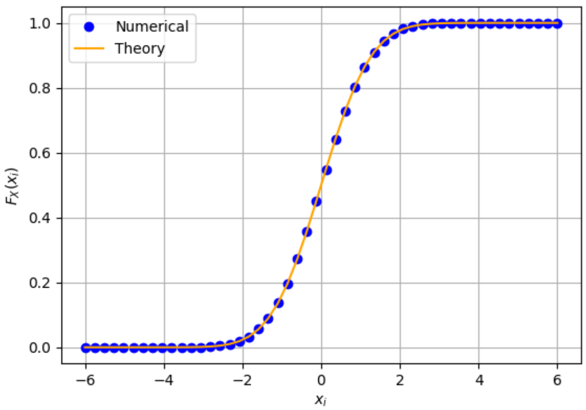

Fig. 2.2: The CDF of X

Download the above file and execute the following command to produce Fig.2.2

```
$ python3 2.2CDF.py
```

Some of the properties of CDF

a) $F_X(x)$ is non decreasing function.
b) As $x \to -\infty$, $F_X(x) \to 0$ and when $x \to \infty$, $F_X(x) \to 1$
c) Graph is linear upto some region around $x = 0$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.3PDF.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3PDF.py
```

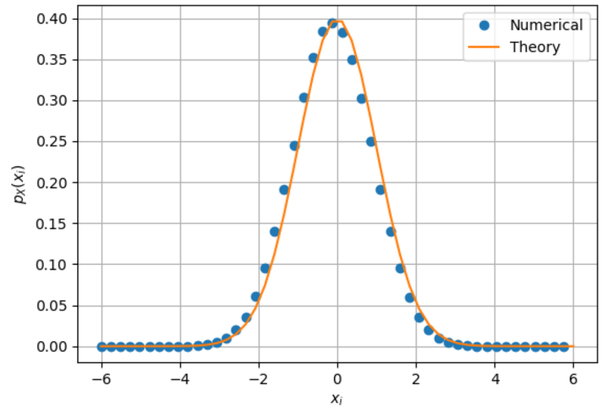

Fig. 2.3: The PDF of X

Some of the properties of the PDF:

a) Symmetric about $x = \mu$
b) Area under the PDF graph is unity.
c) Increasing function for $x < \mu$ and decreasing for $x > \mu$ and attains maximum at $x = \mu$.

The Q-function $Q(x)$ is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \tag{2.3}$$

$$= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{2}}^\infty e^{-t^2} dt\right) \tag{2.4}$$

$$= \frac{1}{2}\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-t^2} dt\right) \tag{2.5}$$

$$= \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right) \tag{2.6}$$

Where erf() is error function defined as,

$$\text{erf(z)} = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{2.7}$$

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/2.4.c
https://github.com/bhargav0383/AI1110–
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 2.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.000294 \tag{2.8}$$

$$\text{Variance} = 0.999561 \tag{2.9}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.10}$$

repeat the above exercise theoretically.

Solution: CDF is defined as

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \tag{2.11}$$

$$\text{W.K.T,} \quad \boxed{F_X(x) = 1} \tag{2.12}$$

Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x)dx \tag{2.13}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.14}$$

It is a odd function function, So its value is 0.

$$\boxed{E(x) = 0} \tag{2.15}$$

$E(x^2)$ is given by,

$$E(x^2) = \int_{-\infty}^{\infty} x^2 p_X(x)dx \tag{2.16}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 exp\left(-\frac{x^2}{2}\right) dx \tag{2.17}$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int xexp\left(-\frac{x^2}{2}\right) dx\right) \tag{2.18}$$

$$- \frac{1}{\sqrt{2\pi}} \int \int \left(xexp\left(-\frac{x^2}{2}\right)\right) dx.dx \tag{2.19}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2}\right) dx \tag{2.20}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \tag{2.21}$$

$$= 1 \tag{2.22}$$

Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \tag{2.23}$$

$$\therefore \boxed{\text{var}[U] = 1} \tag{2.24}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program.

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/3.1.c
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/functions.h
```

Execute the above C program files using the following commands

```
$ gcc 3.1.c −lm
$ ./a.out
```

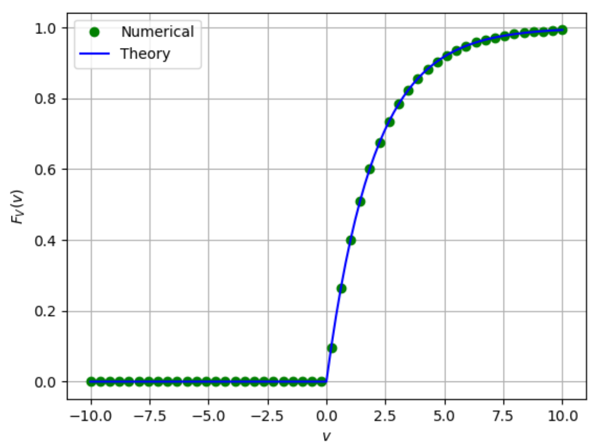

Fig. 3.1: The PDF of X

The CDF of V is plotted in Fig. 3.1 using the code below

```
https://github.com/bhargav0383/AI1110−
    Assignments/blob/main/RandomNumbers/
    codes/3.1CDF.py
```

Download the above files and execute the following commands to produce plot Fig.3.1

```
$ python3 3.1CDF.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: If Y = g(X),
W.K.T,

$$F_Y(y) = F_X(g^{-1}(y)) \tag{3.2}$$

$$V = -2\ln(1-U) \tag{3.3}$$

$$1-U = e^{\frac{-V}{2}} \tag{3.4}$$

$$U = 1-e^{\frac{-V}{2}} \tag{3.5}$$

$$F_V(x) = F_U(1-e^{\frac{-x}{2}}) \tag{3.6}$$

$$\implies F_V(x) = \begin{cases} 0 & x < 0 \\ 1-e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{3.7}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

Download the above files and execute the following commands

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. **??** using the code below

Download the above files and execute the following commands to produce Fig.**??**

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. **??** using the code below

Download the above files and execute the following commands to produce Fig.**??**

4.4 Find the Theoreotical Expression for the PDF and CDF of T
Solution:

4.5 Verify your results through a plot
Solution: The Results are verfied in the plots Fig **??** and Fig **??**