AI1110
Assignment-10

Rajulapati Bhargava Ram
CS21BTECH11052

June 2, 2022

Outline

Question

Papoullis 6-75:

The random variable x has a student t distribution $t(n)$. Show that

$$E\left\{x^2\right\} = \frac{n}{n-2} \tag{1}$$

Theory

Student t Distribution

A random variable x has a Student t Distribution $t(n)$ with n degrees of freedom if for $-\infty < X < \infty$. Here y, z are two independent R.V's

$$x^2 = \frac{ny^2}{z} \tag{2}$$

Where y is $N(0,1)$ is Normal random variable for which

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-y^2}{2\sigma^2}} \tag{3}$$

then,

$$E\left\{y^n\right\} = \begin{cases} 0 & n = 2k+1 \\ 1.3...(n-1)\sigma^n & n = 2k \end{cases} \tag{4}$$

and z is $\chi^2(n)$ which is CHI-SQUARE Distribution with n degrees of freedom if

$$f_X(x) = \begin{cases} \dfrac{x^{n/2-1}}{2^{n/2}\Gamma\left(n/2\right)} e^{-x/2} & x \geq 0 \\ 0 & otherwise \end{cases} \tag{5}$$

Where $\Gamma(\alpha)$ represents the gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \tag{6}$$

Where α is an integer, by using integration by parts we get

$$\Gamma(\alpha) = (n-1)\Gamma(n-1) = (n-1)! \tag{7}$$

Solution

$$E\left\{x^2\right\} = E\left\{\frac{ny^2}{z}\right\} \tag{8}$$

$$E\left\{x^2\right\} = n\,E\left\{y^2\right\}\,E\left\{\frac{1}{z}\right\} \tag{9}$$

So, first we will find $E\left\{y^2\right\}$ using above equation (4). Here $n = 2$ so n is even,(given $\sigma = 1$)

$$E\left\{y^2\right\} = 1.\sigma^2 \tag{10}$$

$$\therefore E\left\{y^2\right\} = 1 \tag{11}$$

Now for $E\left\{\frac{1}{z}\right\}$ using equation (5) in moment generating function we get,

$$E\left\{\frac{1}{z}\right\} = \int_{-\infty}^{\infty} \frac{1}{z} f_z(z)\, dz \tag{12}$$

$$= \int_{-\infty}^{0} \frac{1}{z} \{0\}\; dx + \int_{0}^{\infty} \frac{1}{z} \left\{ \frac{z^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-z/2} \right\} dz \tag{13}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_{0}^{\infty} z^{n/2-2}\, e^{-z/2}\, dz \tag{14}$$

replace $z/2$ with v, and use $dz = 2\,dv$ we get

$$\int_{0}^{\infty} z^{n/2-2} e^{-z/2}\, dz = \int_{0}^{\infty} (2v)^{n/2-2}\, e^{-v}\, 2\, dv \tag{15}$$

$$= 2^{n/2-1} \int_{0}^{\infty} v^{n/2-2}\, e^{-v}\, dv \tag{16}$$

On substituting equation (16) in equation (14) we get,

$$E\left\{\frac{1}{z}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)}\left\{2^{n/2-1}\int_0^\infty v^{n/2-2}e^{-v}\,dv\right\} \tag{17}$$

$$= \frac{2^{n/2-1}}{2^{n/2}\Gamma(n/2)}\int_0^\infty v^{n/2-2}e^{-v}\,dv \tag{18}$$

In above equation the integration looks like gamma function, which is $\Gamma(n/2-1)$ then,

$$E\left\{\frac{1}{z}\right\} = \frac{\Gamma(n/2-1)}{2\,\Gamma(n/2)} = \frac{(n/2-2)!}{2\,(n/2-1)!} \tag{19}$$

$$\therefore E\left\{\frac{1}{z}\right\} = \frac{1}{n-2} \tag{20}$$

Finally substituting equations (11) and (20) in equation (9) we get,

$$E\left\{x^2\right\} = n\,E\left\{y^2\right\}\,E\left\{\frac{1}{z}\right\} \tag{21}$$

$$= n\left\{1\right\}\left\{\frac{1}{n-2}\right\} \tag{22}$$

$$= \frac{n}{n-2} \tag{23}$$

Hence, we have proved the below when the random variable has a student t distribution.

$$\therefore\, E\left\{x^2\right\} = \frac{n}{n-2} \tag{24}$$