AI1110
Assignment-12

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June 19, 2022

Outline

Question

Papoullis 10-8:

The input to a real system $H(\omega)$ is a WSS process $x(t)$ and the output equals $y(t)$. Show that if

$$R_{xx}(\tau) = R_{yy}(\tau) \qquad R_{xy}(-\tau) = -R_{xy}(\tau)$$

as in (10-130), then $H(\omega) = jB(\omega)$ where $B(\omega)$ is a function taking only the values $+1$ and -1.
Special case: If $y(t) = \widehat{x}(t)$, then $B(\omega) = -sgn\ \omega$

Theory

CORRELATION

The **Auto – correlation** of a process $x(t)$, real or complex, is by definition the mean of the product $x(t_1)x^*(t_2)$. Which is

$$R_{xx}(t_1, t_2) = E\{x(t_1)x^*(t_2)\} = R^*_{xx}(t_2, t_1) \tag{1}$$

The **Cross – correlation** of two process $x(t)$ and $y(t)$ is the function,

$$R_{xy}(t_1, t_2) = E\{x(t_1)y^*(t_2)\} = R^*_{yx}(t_2, t_1) \tag{2}$$

WIDE SENSE STATIONARY

A stochastic process $x(t)$ is called wide-sense stationary (abbreviated WSS) if its mean is constant

$$E\{x(t)\} = \eta \tag{3}$$

and its correlation depends only on $\tau = t_1 - t_2$:

$$E\{x(t+\tau)x^*(t)\} = R_{xx}(\tau) \tag{4}$$

Two processes $x(t)$ and $y(t)$ are called jointly WSS if each is WSS and their cross-correlation depends only on $\tau = t_1 - t_2$:

$$R_{xy}(\tau) = E\{x(t+\tau)y^*(t)\} \tag{5}$$

THE POWER SPECTRUM

The power spectrum (or spectral density) of a WSS process $x(t)$, real or complex, is the Fourier transform $S(\omega)$ of its autocorrelation, $R_{xx}(\tau) = E\{x(t+\tau)x^*(t)\}$:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{6}$$

If $x(t)$ is a real process, then $R(\tau)$ is real and even; hence $S(\omega)$ is also real and even. In this case,

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, \cos\omega\tau\, d\tau \tag{7}$$

From above we get, $S_{xx}(\omega) = S_{xx}(-\omega)$

The cross-power spectrum of two processes $x(t)$ and $y(t)$ is the Fourier transform $S_{xy}(\omega)$ of their cross-correlation $R_{xy}(\tau) = E\{x(t+\tau)y^*(t)\}$:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{8}$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \qquad R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \tag{9}$$

$$S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega) \qquad S_{yy}(\omega) = S_{xy}(\omega)H(\omega) \tag{10}$$

Combining above equations we get,

$$R_{yy} = R_{xx}(\tau) * h^*(-\tau) * h(\tau) = R_{xx}(\tau) * \rho(\tau) \tag{11}$$

$$S_{yy}(\omega) = S_{xx}(\omega)H^*(\omega)H(\omega) = S_{xx}(\omega)|H(\omega)|^2 \tag{12}$$

Where $H(\omega)$ is called Transfer function.

Solution

In question given is,

$$R_{xx}(\tau) = R_{yy}(\tau) \tag{13}$$

$$R_{xy}(-\tau) = -R_{xy}(\tau) \tag{14}$$

Begin with cross-power spectrum $S_{yy}(\omega)$,

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{15}$$

$$S_{xy}(-\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{j\omega\tau}\, d\tau \tag{16}$$

Replace τ with $-\tau$,

$$S_{xy}(-\omega) = \int_{-\infty}^{\infty} R_{xy}(-\tau)\, e^{-j\omega\tau}\, d\tau \tag{17}$$

Using equation (14),

$$S_{xy}(-\omega) = \int_{-\infty}^{\infty} -R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{18}$$

$$S_{xy}(-\omega) = -\int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{19}$$

$$S_{xy}(-\omega) = -S_{xy}(\omega) \tag{20}$$

Similarly using equation (13) we can get,

$$S_{xx}(\omega) = S_{yy}(\omega) \tag{21}$$

Comparing above equation with equation (12) we get,

$$|H(\omega)|^2 = 1 \tag{22}$$

Using equation (10),

$$S_{xy}(-\omega) = S_{xx}(-\omega)H^*(-\omega) \tag{23}$$
$$-S_{xy}(\omega) = S_{xx}(-\omega)H^*(-\omega) \tag{24}$$

From equation (7) $S_{xx}(-\omega) = S_{xx}(\omega)$, So

$$-S_{xy}(\omega) = S_{xx}(\omega)H^*(-\omega) \tag{25}$$
$$\Rightarrow -H^*(\omega) = H^*(-\omega) \tag{26}$$
$$\Rightarrow H(-\omega) = -H(\omega) \tag{27}$$

From above, the real part in $H(\omega)$ is zero, odd function and its in the form,

$$\therefore H(\omega) = j\,B(\omega) \tag{28}$$

Where $B(\omega)$ is a real function from equation (22),

$$\therefore |B(\omega)| = 1 \tag{29}$$

The graph of function $B(\omega)$,

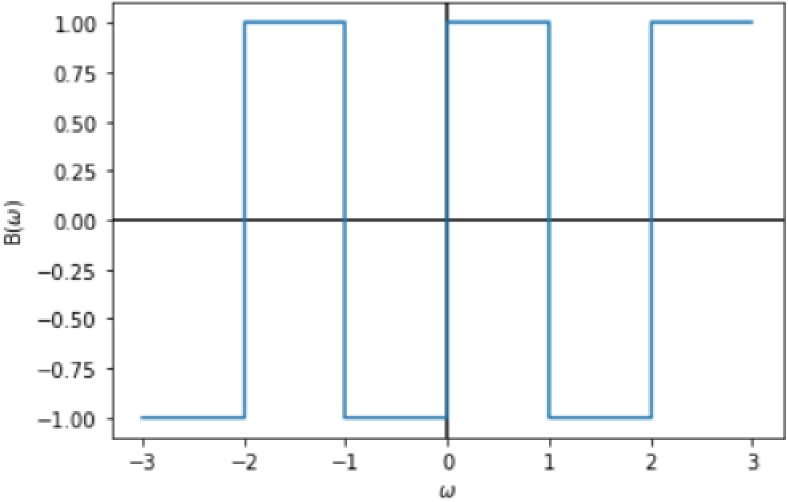

Figure 0