AI1110
Assignment-9

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Outline

Question

Papoullis 5-32:

(a) Show that if m is the median of X, then

$$E\{|x-a|\} = E\{|x-m|\} + 2\int_a^m (x-a)f(x)\,\mathrm{d}x \tag{1}$$

for any a.

(b) Find c such that $E\{|x-c|\}$ is minimum.

Definitions

Cumulative Distribution Function

If $f_X(x)$ is probability density function of a random variable X, then it's corresponding Probability (cumulative) distribution function $F_X(x)$ is

$$F_X(x) = P\{X \leq x\} = \int_{-\infty}^{x} f_X(x)\,dx \tag{2}$$

Expected Value

The expected value or mean of a random variable X is by definition integral

$$E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\,dx \tag{3}$$

Solution

From the given information;

$$\frac{\partial|x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x \geq a \end{cases} \tag{4}$$

From definition we know that,

$$E\{|X-a|\} = \int_{-\infty}^{\infty} |x-a| f_X(x)\, dx \tag{5}$$

Let $I(a) = E\{|X - a|\}$ then, the partial derivative of $I(a)$ w.r.t to a $(I'(a))$ is

$$
\begin{aligned}
\frac{\partial I(a)}{\partial a} &= \frac{\partial\{\int_{-\infty}^{\infty} |x - a| f_X(x)\, dx\}}{\partial a} && (6)\\
&= \int_{-\infty}^{\infty} \frac{\partial |x - a|}{\partial a} f_X(x)\, dx && (7)\\
&= \int_{-\infty}^{a} \frac{\partial |x - a|}{\partial a} f_X(x)\, dx + \int_{a}^{\infty} \frac{\partial |x - a|}{\partial a} f_X(x)\, dx && (8)\\
&= \int_{-\infty}^{a} (1) f_X(x)\, dx + \int_{a}^{\infty} (-1) f_X(x)\, dx && (9)\\
&= P\{X \le a\} - P\{X > a\} && (10)\\
\therefore I'(a) &= 2F_X(a) - 1 && (11)
\end{aligned}
$$

Part (a)

Start with

$$\int_m^a I'(\alpha)\, d\alpha = I(a) - I(m) \tag{12}$$

$$I(a) = I(m) + \int_m^a [2F_X(\alpha) - 1]\, d\alpha \tag{13}$$

$$= I(m) + 2\int_m^a F_X(\alpha)\, d\alpha - \int_m^a 1\, d\alpha \tag{14}$$

$$= I(m) + 2\int_m^a F_X(\alpha)\, d\alpha - (a - m) \tag{15}$$

Where, replace α with x in the integral below

$$\int_m^a F_X(\alpha)\, d\alpha = aF_X(a) - mF_X(m) - \int_m^a xf_X(x)\, dx \tag{16}$$

As m is the median of X, C.M.F $F_X(m) = \frac{1}{2}$
On substituting equation (16) in equation (15) we get,

$$I(a) = I(m) + 2\left\{aF_X(a) - mF_X(m) - \int_m^a xf_X(x)\,dx\right\} - (a - m) \tag{17}$$

$$= I(m) + 2\int_a^m xf_X(x)\,dx + 2\left\{aF_X(a) - m(\frac{1}{2})\right\} + m - a \tag{18}$$

$$= I(m) + 2\int_a^m xf_X(x)\,dx + a(2F_X(a) - 1) \tag{19}$$

$$= I(m) + 2\int_a^m xf_X(x)\,dx + 2a(F_X(a) - F_X(m)) \tag{20}$$

$$= I(m) + 2\int_a^m xf_X(x)\,dx + 2a\int_m^a f_X(x)\,dx \tag{21}$$

$$= I(m) + 2\int_a^m (x - a)f_X(x)\,dx \tag{22}$$

We have used this in above equation (21),

$$\int_{m}^{a} f_X(x)\,dx = \int_{-\infty}^{a} f_X(x)\,dx - \int_{-\infty}^{m} f_X(x)\,dx \tag{23}$$

$$= F_X(a) - F_X(m) \tag{24}$$

$$\therefore I(a) = I(m) + 2\int_{a}^{m} (x-a) f_X(x)\,dx \tag{25}$$

$$E\{|X-a|\} = E\{|X-m|\} + 2\int_{a}^{m} (x-a) f_X(x)\,dx \tag{26}$$

Hence proved, If m is the median of X then the above equation is true for any a.

Part (b)

The value of c for which $E\{|X - c|\}$ is minimum can be found by,
$I(c) = E\{|X - c|\}$ is minimum if,

$$I'(c) = 0 \tag{27}$$

$$2F_X(c) - 1 = 0 \tag{28}$$

$$\therefore F_X(c) = \frac{1}{2} \tag{29}$$

$$\therefore c = m \tag{30}$$

Hence, the value of c is m which is the median of X.