

Assignment

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Abstract—This manual provides solutions to the Assignment on Random Numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 1.1.c
$ ./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The python code for the plot in Fig. 1.2 is given below,

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.2CDF.py>

Download the above file and execute the command below to produce Fig.1.2

```
$ python3 1.2CDF.py
```

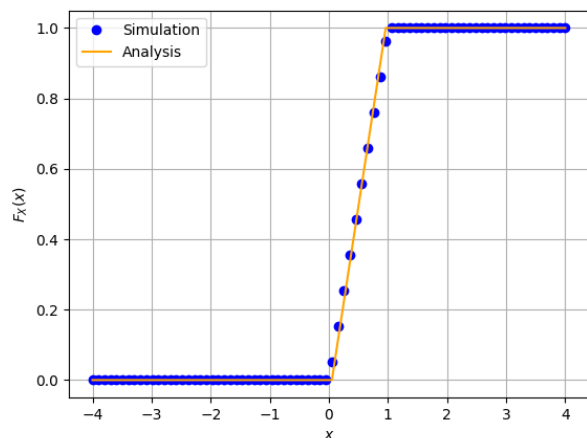


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given U is a uniform Random Variable between 0 and 1,

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (1.2)$$

We have three parts:

- i) For $x < 0$; $p_X(x) = 0$, So $F_U(x) = 0$.
- ii) For $0 \leq x < 1$;

$$F_U(x) = \int_0^x (1) du = x \quad (1.3)$$

- iii) For $x \geq 1$; CDF is 1 as all the random numbers are between 0 and 1.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/1.4.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 1.4.c
$ ./a.out
```

The actual analysis values,

$$\text{Mean} = 0.500007 \quad (1.7)$$

$$\text{Variance} = 0.083301 \quad (1.8)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution: W.K.T,

$$\text{var}[U] = E[U^2] - E[U]^2 \quad (1.10)$$

Where $E[U]$ is,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.11)$$

$$= \int_0^1 x \quad (1.12)$$

$$= \frac{1}{2} = 0.5 \quad (1.13)$$

And $E[U^2]$ is,

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.14)$$

$$= \int_0^1 x^2 dF_U(x) \quad (1.15)$$

$$= \frac{1}{3} \quad (1.16)$$

$$(1.17)$$

Hence finally,

$$\therefore \text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.18)$$

The theoretically calculated values are,

$$\text{Mean} = 0.5 \quad (1.19)$$

$$\text{Variance} = 0.0833 \quad (1.20)$$

These values matches with the actual analysis values from above,

$$\text{Mean} = 0.500007 \quad (1.21)$$

$$\text{Variance} = 0.083301 \quad (1.22)$$

Hence it verifies the result.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.2CDF.py>

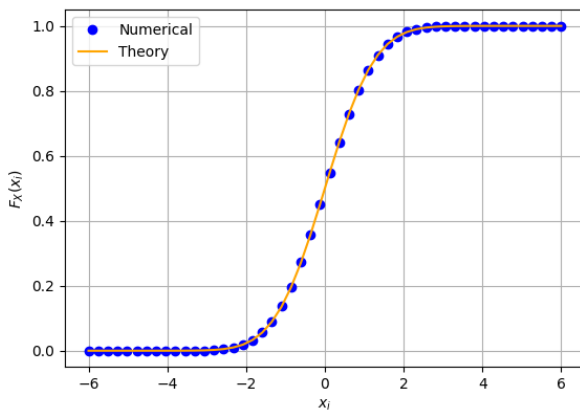


Fig. 2.2: The CDF of X

Download the above file and execute the following command to produce Fig.2.2

\$ python3 2.2CDF.py

Some of the properties of CDF

- $F_X(x)$ is non decreasing function.
- As $x \rightarrow -\infty$, $F_X(x) \rightarrow 0$ and when $x \rightarrow \infty$, $F_X(x) \rightarrow 1$
- Graph is linear upto some region around $x = 0$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.3PDF.py>

Download the above files and execute the following commands to produce Fig.2.3

\$ python3 2.3PDF.py

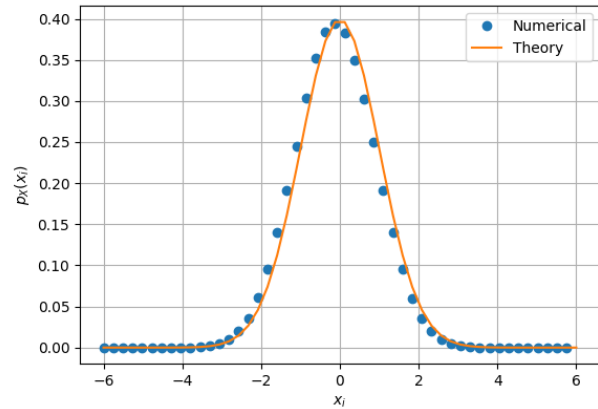


Fig. 2.3: The PDF of X

Some of the properties of the PDF:

- Symmetric about $x = \mu$
- Area under the PDF graph is unity.
- Increasing function for $x < \mu$ and decreasing for $x > \mu$ and attains maximum at $x = \mu$.

Let $x \sim \mathcal{N}(0, 1)$. The Q-function $Q(x)$ is defined as:

$$Q(x) = Pr(X > x) \quad (2.3)$$

$$= 1 - Pr(X \leq x) \quad (2.4)$$

$$= 1 - F_X(x) \quad (2.5)$$

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/2.4.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

\$ gcc 2.4.c
\$./a.out

The actual analysis values,

$$\text{Mean} = 0.000294 \quad (2.6)$$

$$\text{Variance} = 0.999561 \quad (2.7)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.8)$$

repeat the above exercise theoretically.

Solution: CDF is defined as

$$F_X(x) = \int_{-\infty}^x p_X(x) dx \quad (2.9)$$

$$\text{W.K.T, } \boxed{F_X(x) = 1} \quad (2.10)$$

Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

It is an odd function, So its value is 0.

$$\boxed{E(x) = 0} \quad (2.13)$$

$E(x^2)$ is given by,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.14)$$

$$= \int_{-\infty}^{\infty} x \times x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.15)$$

$$= x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.16)$$

$$+ \int_{-\infty}^{\infty} 1 \times \int \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.17)$$

$$= x \times 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.18)$$

$$= 1 \quad (2.19)$$

$$(2.20)$$

Where, $x \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} = 0$; because it is an odd function.

Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.21)$$

$$\therefore \boxed{\text{var}[U] = 1} \quad (2.22)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following files and execute the C program.

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/3.1.c>
<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/functions.h>

Execute the above C program files using the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

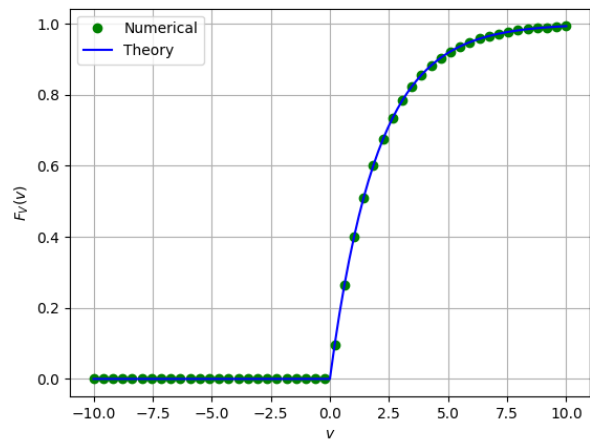


Fig. 3.1: The PDF of X

The CDF of V is plotted in Fig. 3.1 using the code below

<https://github.com/bhargav0383/AI1110-Assignments/blob/main/RandomNumbers/codes/3.1CDF.py>

Download the above files and execute the following commands to produce plot Fig.3.1

```
$ python3 3.1CDF.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If $Y = g(X)$,

W.K.T,

$$F_Y(y) = F_X(g^{-1}(y)) \quad (3.2)$$

$$V = -2 \ln(1 - U) \quad (3.3)$$

$$1 - U = e^{\frac{-V}{2}} \quad (3.4)$$

$$U = 1 - e^{\frac{-V}{2}} \quad (3.5)$$

$$F_V(x) = F_U(1 - e^{\frac{-x}{2}}) \quad (3.6)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \quad (3.7)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

Execute the above C program files using the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. ?? using the code below

Download the above files and execute the following commands to produce Fig.??

4.3 Find the PDF of T .

Solution: The PDF of T is plotted in Fig. ?? using the code below

Download the above files and execute the following commands to produce Fig.??

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

4.5 Verify your results through a plot

Solution: The Results are verified in the plots Fig ?? and Fig ??