Modeling, Analysis and Simulation on Searching for Global Optimum Region of Particle Swarm Optimization

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***Abstract*—In the case of particle swarm optimization, this paper mainly analyzes and discusses the mathematical model and the analysis on searching for the global optimum region. Firstly, the global optimum region Θ is defined and calculated in the convergence step and the divergence step. Furthermore, the rate μ of locating into the global optimum region is mathematically related to the number of particles, the number of generations, the fitness landscape, the ratio between exploration ability and exploitation ability, etc. Simulation results on Schaffer f6 function can help to understand the obtained results. Finally, those corresponding results and several remarks in the paper are helpful for the tradeoff between exploration ability and exploitation ability, together with the suitable searching strategy of particle swarm optimization algorithm.**

I. Introduction

Swarm optimization is one of the important and interesting topics in the fields of evolutionary computation, applied mathematics and computer science, etc. Its main task is to optimize the combinatorial parameters to minimize or maximize the given benchmark and practical optimization problems, while those swarm optimization methods mainly originate from the nature-inspired behaviors of swarm animals, the natural rules or the natural phenomenon, and so on. Tyically, several swarm optimization methods are genetic algorithm (GA) [1], particle swarm optimization (PSO) [2, 3], differential evolution (DE) [4], bacterial foraging optimization (BFO), and ant colony optimization (ACO) [5], etc. These swarm optimization methods utlize the cooperation and the competition among the swarm to find the global optimum. More importantly, the above-mentioned swarm optimization methods have been wildly and successfully solved in the continuous optimization problem [6], the discrete optimization problem [7], the multi-objectives optimization problem [8], the dynamical optimization problem [9] and the hybrid optimization problem. Additionally, those corresponding methods can be applied in the variety of some realistic applications, such as locating and tracking the optimums [10], intelligent control, intelligent power [11], trajectory planning [12, 13], pattern identification, etc.

There are several existing basic and fundamental theories on the convergence analysis of PSO algorithm. Ozcan and Mohan [14][15] analyze the trajectory of one particle and multiple particles without considering random variables. Trelea [16] provides the convergence analysis and the convergence condition and convergence speed under the deterministic transfer matrix of PSO algorithm. Clerc and Kennedy [17] analyze the stability of the simplified PSO algorithm and introduce the constriction factor method. Van den Bergh and Engelbrecht [18] discuss the fully detailed convergence analysis of the simplified PSO algorithm and the trajectory of single particle. Kadirkamanathan [19] considers PSO system as the nonlinear feedback system and discusses the stability of the random PSO algorithm. Jiang and Luo [20] provide the convergence condition and parameter selection of the random PSO algorithm. Poli [21] and Milan [22] consider the stability of the standard PSO algorithm under random variables and conclude convergence conditions from the perspective of mean and variance. Luis Fernandez and Esperanza Garcia [23] consider PSO algorithm as the stochastic damped mass-spring system, and analyze first and second order trajectories. Jun [24] introduces the joint spectral radius from the time-varying transfer matrix, and the joint spectral radius denotes the convergence speed of each particle, furthermore, the joint spectral radius can denote the tradeoff between exploration ability and exploitation ability. According to the existing theories on convergence analysis on the random PSO algorithm, there are few works on the basic and fundamental results of swarm optimization methods.

The motivation of this study is to discuss the mathematical model of the typical particle swarm optimization algorithm, together with the important results to find the better optimization result. Specifically speaking, we provide the basic and general model to analyze the important parameters and the rate of searching for the global optimum region. Main contributions of this paper can be highlighted as follows.

- The global optimum region is defined in the different dimensional space and the rate of searching for the global optimum region is calculated in different typical cases.
- The maximum solution space for each particle at next step by PSO algorithm is depicted and analyzed in this paper.
- Several important rules on the optimization strategy are

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concluded from particle swarm optimization.

- The rate of searching for the global optimum region is mathematically described and related to the number of particles, the number of generations, the ratio between exploration ability and exploitation ability, the objective landscape, etc.
- The corresponding results probably can be also used to other existing swarm optimization methods.

The reminder of this paper is organized as follows. Section II briefly gives the description of the particle swarm optimization and describes the maximum solution space for each particle at next step. The definition of global optimum region and several important results are described in Section III. The rate of locating into global optimum region is calculated in the divergence step or in the convergence step in Section IV. The mathematical model of evolutionary process from different swarm optimization methods is mathematically related to the number of particles, the number of generations, the ratio between exploration ability and exploitation ability, the landscape of objective function in Section V. Numerical results on the evolutionary process and the possible maximum space can demonstrate that the updating evolutionary procees and the next maximum space at next step are record in Section VI. Section VII concludes the main and interesting results, corresponding to the future related works.

II. Particle Swarm Optimization

Recently, there are several currently nature-inspired optimization methods in the areas of evolutionary computation. More specially, the current nature-inspired optimization methods consist of genetic algorithm, particle swarm optimization, differential evolution, ant colony optimization, and so on. Generally speaking, optimization searching strategy and parameter setting on those methods are influence the final results, therefore, it is of necessity to deeply analyze and discuss the mathematical model of searching for the global optimum region and the tradeoff between exploitation ability and exploration ability.

Particle swarm optimization (PSO), which is one of the typical swarm optimization algorithms and is mainly inspired by searching for the food of fish and birds, can be mathematically formulated as

$$v_{ij}(t+1) = \omega(t)v_{ij}(t)+c_1r_1(pbest_{ij}(t)-x_{ij}(t))+c_2r_2(gbest_{ij}(t)-x_{ij}(t)) \tag{1}$$

$$x_{ij}(t+1) = v_{ij}(t+1) + x_{ij}(t) \tag{2}$$

where $v_{ij}(t)$ and $x_{ij}(t)$ denote the jth velocity and position of ith particle at tth step, respectively. $\omega(t)$ is the time-varying inertia weight in the whole evolutionary process, while c_1 and c_2 setting to 2.0 denote the acceleration coefficients. r_1 and r_2 are the uniformly random variables in the range from 0 to 1. $pbest_{ij}(t)$ and $gbest_{ij}(t)$ denote the jth best position of ith particle and the jth best position of best particle, respectively.

According to the PSO algorithm, one particle at each step has its possible solution space, which is closely related to the number of random vectors in the updating equation. Taking the general PSO algorithm for example, the particle in the general PSO algorithm is updated by

$$\begin{aligned} x_{ij}(t+1) = x_{ij}(t) + \omega(t)v_{ij}(t) + c_1r_1(pbest_{ij}(t) - x_{ij}(t)) \\ +c_2r_2(gbest_{ij}(t) - x_{ij}(t)) \end{aligned} \tag{3}$$

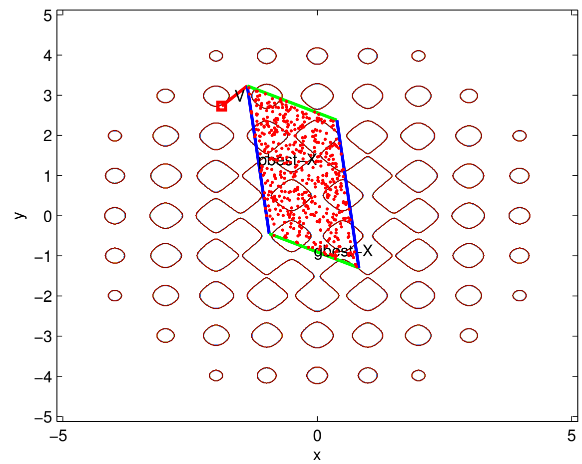

Fig. 1. The largest solution space of one particle in the general PSO algorithm

As depicted in Fig. 2, the possible solution space at the next step is the projection of the convex hull space in the two-dimensional space, while the convex hull space is composed of $\{x+v, x+v+v_1, x+v+v_2, x+v+v_1+v_2\}$, where v denotes $\omega(t)v_{ij}(t)$ and v_1 denotes $c_1r_1(pbest_{ij}(t) - x_{ij}(t))$, while v_2 is $c_2r_2(gbest_{ij}(t) - x_{ij}(t))$.

Additionally, each particle in the improved PSO algorithm, which adds $mbest - X$ into the general PSO algorithm, can be mathematically expressed by

$$\begin{aligned} x_{ij}(t+1) = x_{ij}(t) + \omega(t)v_{ij}(t) + c_1r_1(pbest_{ij}(t) - x_{ij}(t)) \\ + c_2r_2(gbest_{ij}(t) - x_{ij}(t)) + c_3r_3(mbest_{ij}(t) - x_{ij}(t)) \end{aligned} \tag{4}$$

where $mbest_{ij}(t)$ denotes the jth mean position of all particles at tth step.

As shown in Fig. 2, the possible maximum solution space at the next step is the projection of the **convex hull** space in the two-dimensional space, while the convex hull space is composed of $\{x+v, x+v+v_1, x+v+v_2, x+v+v_3, x+v+v_1+v_2, x+v+v_1+v_3, x+v+v_2+v_3, x+v+v_1+v_2+v_3\}$, where v_3 denotes $c_3r_3(mbest_{ij}(t) - x_{ij}(t))$.

According to the existing particle swarm optimization methods, the general mathematical model of updating each particle probably can be mathematically expressed as

$$x_{ij}(t+1) = x_{ij}(t) + \triangle x_{ij}(t+1) \tag{5}$$

where $x_{ij}(t+1)$ and $\triangle x_{ij}(t+1)$ denote the jth current and the modified value of ith particle at $t+1$th step, respectively. The major difference of those swarm optimization methods is to differently update the $\triangle x_{ij}(t+1)$ on the basis of different natural phenomenon and swarm animals in the nature.

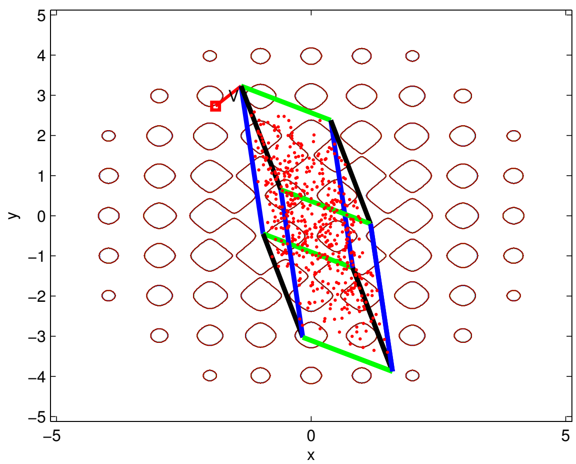

Fig. 2. The largest solution space of one particle in the improved PSO algorithm

Specifically speaking, the general model of particle swarm optimization can be briefly described by

$$x_{ij}(t+1) = x_{ij}(t) + k_1 v_{ij}(t) + k_2 \sum_{i=1}^{m} (C_i(q_{ij}(t) - x_{ij}(t))) + k_3 r \quad (6)$$

where k_1 denotes the inertia weight of previous velocity and k_2 denotes the weight of the distance between its position and its attractor. In addition, k_3 is the weight of random variable r, whose value reduces in the evolutionary process. $v_{ij}(t)$ denotes the jth previous velocity of ith particle at tth step and $q_{ij}(t)$ denotes the jth attractor of ith particle at tth step.

Therefore, the possible maximum solution space is the projection of the corresponding convex hull, and the convex hull is mainly determined by the linear combination of all vectors.

III. Basic Concepts and Characteristics of Searching for the Global Optimum Region

In order to deeply investigate the general mechanism of particle swarm optimization, this section is to introduce the basic and important concept so-called the global optimum region Θ and several important remarks. More importantly, some assumptions and characteristics of particle swarm optimization are also provided to discuss the whole evolutionary process.

Definition 3.1: (The global optimum region Θ): Given the objective function $f(\mathbf{x})$ and the second global optimum, the global optimum region is the intersection set between the hyperplane of the second global optimum and the hypersurface landscape of objective function.

- In the one dimensional space, the global optimum region Θ, as depicted in Fig. 3, is the segment line $\overline{bc}$ between the curve line of objective function $f(x)$ and the parallel line of x-axis on the second global optimum a.
- In the two dimensional space, the global optimum region Θ is the irregular plane between three-dimensional surface of objective function $f(x_1, x_2)$ and the parallel plane of xy-plane on the second global optimum.
- In the three or high dimensional space, the global optimum region Θ is the irregular hyper surface between the hyper surface of objective function $f(\mathbf{x})$ and the parallel hyper plane on the second global optimum.

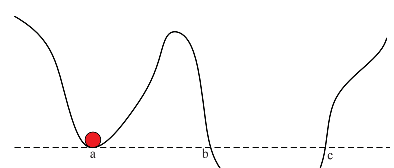

Fig. 3. The global optimum segment line $\overline{bc}$ (b $\leq$ c)

Taking Rastrigin function for example, the global optimum region in the three dimensional space can be depicted in Fig. 4. In Fig. 4, the first subgraph is the landscape of calculating objective functions and the second subgraph is the landscape in xy plane, while the third subgraph is the global optimum region and the final subgraph is the global optimum region in xy plane.

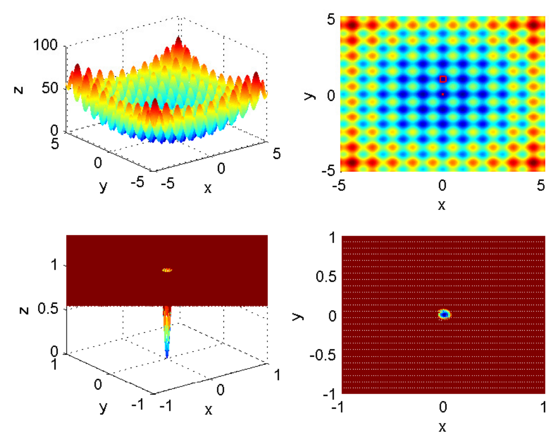

Fig. 4. The global optimum region of Rastrigin function

Considering Schaffer f6 function, the global optimum region in the three dimensional space can be depicted in Fig. 5.

Definition 3.2: (The local optimum region Λ): Given the objective function $f(\mathbf{x})$ and the constant objective fitness ($f^* > f_{min}$), the local optimum region Λ is the intersection set between the hyperplane on the constant objective fitness and the hypersurface of local optimum in the objective landscape.

Assumption 3.1: More importantly, it highlights that the adjacent space of the global optimum region is strictly larger than 0 for any optimization problems.

$$\Theta > 0. \quad (7)$$

Assumption 3.2: For the existing swarm optimization methods, the adjacent region of the current best position has better

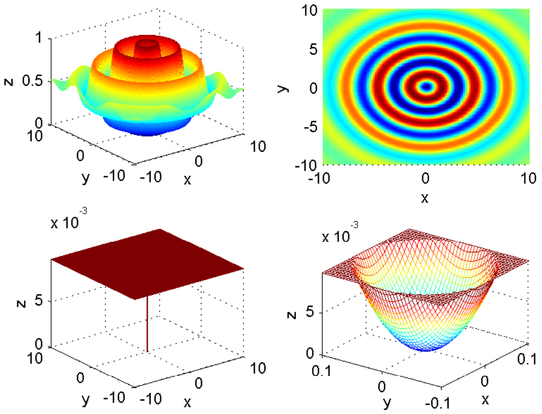

Fig. 5. The global optimum region of Rastrigin function

position in the solution space, except for the local optimum or the global optimum.

Remark 3.1: As to the minimization problem, the fitness of the current global optimum $X_g(t)$ at tth step by swarm optimization methods becomes smaller and smaller.

$$f(X_g(1)) \geq \cdots \geq f(X_g(t)) \geq f(X_g(t+1)) \geq \cdots \geq f(X_g(T_{max})) \tag{8}$$

where T_{max} denotes the maximum number of generations in the whole evolutionary process.

Remark 3.2: For the maximization problem, the current fitness of global optimum $X_g(t)$ in different swarm optimizations becomes larger and larger.

$$f(X_g(1)) \leq \cdots \leq f(X_g(t)) \leq f(X_g(t+1)) \leq \cdots \leq f(X_g(T_{max})). \tag{9}$$

Remark 3.3: Suppose that there are N suboptimal points in the whole solution space. The global optimum region is the subset in one of the adjacency convergence regions Γ_i $(1 \leq i \leq N)$ of the ith local optimum. Therefore, the global optimum region Θ is the subset of the adjacency convergence region Γ_i.

$$\Theta \subseteq \Gamma_i \tag{10}$$

where the objective of those optimized algorithms in the early search stage is to search for the ith local optimum.

Remark 3.4: Given one constant objective fitness f_*, the rate $\xi_k(t)$ denotes the intersection region between the hyperplane f_* and the convergence area in the kth local optimum region at tth step.

As for the minimization problem in the general evolutionary process, each particle searches for the current best solution $X_g(t)$ of all particles, so it is necessary to discuss the mathematical model between the rate $\xi_k(t)$ and the best solution $X_g(t)$.

$$\begin{cases} \sum_{k=1}^{Q(t)} \xi_k(t) \geq \sum_{k=1}^{Q(t+1)} \xi_k(t+1) & f(X_g(t)) > f(X_g(t+1)) \\ \sum_{k=1}^{Q(t)} \xi_k(t) = \sum_{k=1}^{Q(t+1)} \xi_k(t+1) & f(X_g(t)) \leq f(X_g(t+1)) \end{cases} \tag{11}$$

where the parameter $Q(t)$ denotes the local optimum number between the hyper plane $X_g(t)$ and the hyper surface of objective function, while the parameter $Q(t+1)$ denotes the local optimum number between the hyper plane $X_g(t+1)$ and the hyper surface of objective function. It is noticed that the number $Q(t)$ is generally larger than the number $Q(t+1)$.

Concerning the minimization problem, it is also important to discuss the rate $\xi_k(t)$ for each particle in the evolutionary process.

$$\begin{cases} \sum_{k=1}^{Q(t)} \xi_k(t) \geq \sum_{k=1}^{Q(t+1)} \xi_k(t+1) & f(X_i(t)) > f(X_i(t+1)) \\ \sum_{k=1}^{Q(t)} \xi_k(t) = \sum_{k=1}^{Q(t+1)} \xi_k(t+1) & f(X_i(t)) \leq f(X_i(t+1)) \end{cases} \tag{12}$$

where $X_i(t)$ denotes the current value of ith particle at tth step.

For each adjacency region of the kth suboptimal optimum, the rate $\xi_k(t)$ on the basis of the best position $X_g(t)$ or the current position $X_i(t)$ reduces from the maximum region Γ_k to 0.

$$\xi_k(t) \geq \xi_k(t+1) \quad 0 \leq \xi_k(t) < \Gamma_k. \tag{13}$$

IV. Rate μ in Convergence Behavior and Divergence Behavior

The objective of this section is to calculate the rate μ of searching for the global optimum region in the different typical cases. In the whole evolutionary process, the PSO algorithm has the convergence behavior and the divergence behavior to search for better position, therefore, it is necessary to discuss and calculate the rate μ in the convergence behavior and the divergence behavior at one step.

A. Rate μ under Uniformly Random Solution Variation

Definition 4.1: The rate $\mu(t)$ of searching the global optimum region is mathematically defined as the rate that one particle locates into the global optimum region at one step.

Let the current solution be x_0 and the random variation of the solution v_0. In the one dimensional space, if the particle at next step updating by x_0 and v_0 has no opportunity to locate into the global optimum region, the rate μ_c can be mathematically expressed by

$$\mu_c = 0. \tag{14}$$

As depicted in Fig. 3, if the next particle has the partial intersection between the searching region and the global optimum region, the rate μ_c, in the presence of $b \leq x_0 + v_0 \leq c$ and $x_0 < b$, can be mathematically described by

$$\mu_c = \frac{x_0 + v_0 - b}{v_0} \tag{15}$$

or the rate μ_c, under the conditions of $b \leq x_0 + v_0 \leq c$ and $x_0 > c$, is

$$\mu_c = \frac{c - x_0 - v_0}{|v_0|}. \tag{16}$$

If the next solution can cover the whole global optimum region in one dimensional space, the rate μ_c can be mathematically calculated by

$$\mu_c = \frac{c - b}{|v_0|}. \tag{17}$$

Therefore, as for the one dimensional space, the rate of locating into the global optimum region can be expressed by

$$\mu_c = \begin{cases} 0 & x_0 + v_0 < d_{min} \\ \frac{x_0+v_0-d_{min}}{|v_0|} & d_{min} \leq x_0 + v_0 \leq d_{max} \\ \frac{d_{max}-d_{min}}{|v_0|} & x_0 + v_0 > d_{max} \end{cases} \tag{18}$$

where d_{min} and d_{max} denote the minimum and maximum distance from x_0 to the global optimum region, respectively.

Concerning the two dimensional space, the rate μ_c can be mathematically described by

$$\mu_c = \begin{cases} 0 & x_0 + v_0 < d_{min} \\ \pi v_0^2 \bigcap \Theta & d_{min} \leq x_0 + v_0 \leq d_{max} \\ \frac{\Theta}{\pi v_0^2} & x_0 + v_0 > d_{max}. \end{cases} \tag{19}$$

In the case of the three dimensional space, the rate μ_c on the basis of the above-mentioned results is mathematically computed by

$$\mu_c = \begin{cases} 0 & x_0 + v_0 < d_{min} \\ \frac{4}{3}\pi |v_0|^3 \bigcap \Theta & d_{min} \leq x_0 + v_0 \leq d_{max} \\ \frac{3\Theta}{4\pi |v_0|^3} & x_0 + v_0 > d_{max}. \end{cases} \tag{20}$$

B. Rate μ_c under Normally Random Solution Variation

Several improved PSO algorithms include the normal random vectors in the velocity and position's equation. Assume the probability density function of solution variation as

$$f_p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}} \tag{21}$$

where x_0 and σ denote the mean and variance of the probability function $f_p(x)$, respectively.

In the one dimensional space, the rate of searching for the global optimum region can be mathematically calculated by

$$\mu_c = \frac{\int_{d_{min}}^{d_{max}} f_p(x)dx}{\int_{x_0}^{x_0+v_0} f_p(x)dx}. \tag{22}$$

In the two dimensional space, the rate μ_c can be mathematically described by

$$\mu_c = \frac{\int_{d_{min}}^{d_{max}} \int_{y(d_{min})}^{y(d_{max})} f_p(x)dxdy}{\int_{x_0}^{x_0+v_0} \int_{x_1}^{x_1+v_1} f_p(x)dxdy}. \tag{23}$$

In the three or high dimensional space, the rate μ_c can be mathematically formulated as

$$\mu_c = \frac{\Theta}{\Psi} \tag{24}$$

where Ψ denotes the whole solution space under the normally distributed solution variation.

C. Rate μ_d under the Divergence Behavior

In order to achieve the good tradeoff between exploration ability and exploitation ability, particle swarm optimization is composed of the convergence behavior and the divergence behavior. The above-mentioned subsections mainly discuss the rate μ_c in the convergence behavior, and this subsection is to calculate and analyze the rate μ_d in the divergence behavior.

In the whole evolutionary process, the PSO algorithm has the divergence operator to avoid easily getting into the local optimum, so it is necessary to analyze and calculate the rate of locating into the global optimum region in the divergence behavior. The additional variation of solution is set to be R ($R \geq 0$). When the particle's variation submits to the uniformly random, the rate μ_d under the one dimensional space is also computed by

$$\mu_d = \begin{cases} 0 & x_0 + v_0 + R < d_{min} \\ \frac{x_0+v_0+R-d_{min}}{|v_0|+R} & d_{min} \leq x_0 + v_0 + R \leq d_{max} \\ \frac{d_{max}-d_{min}}{|v_0|+R} & x_0 + v_0 + R > d_{max}. \end{cases} \tag{25}$$

In the two dimensional space, the rate μ_d in the divergence behavior at one step is

$$\mu_d = \begin{cases} 0 & x_0 + v_0 + R < d_{min} \\ \pi(|v_0| + R)^2 \bigcap \Theta & d_{min} \leq x_0 + v_0 + R \leq d_{max} \\ \frac{\Theta}{\pi(|v_0|+R)^2} & x_0 + v_0 + R > d_{max}. \end{cases} \tag{26}$$

In the three dimensional space, the rate μ_d in the divergence behavior can be mathematically described by

$$\mu_d = \begin{cases} 0 & x_0 + v_0 + R < d_{min} \\ \frac{4}{3}\pi(|v_0| + R)^3 \bigcap \Theta & d_{min} \leq x_0 + v_0 + R \leq d_{max} \\ \frac{3\Theta}{4\pi(|v_0|+R)^3} & x_0 + v_0 + R > d_{max}. \end{cases} \tag{27}$$

In the high dimensional space, the corresponding rate μ_d is mathematically calculated as

$$\mu_d = \begin{cases} 0 & x_0 + v_0 + R < d_{min} \\ \Psi \bigcap \Theta & d_{min} \leq x_0 + v_0 + R \leq d_{max} \\ \frac{\Theta}{\Psi} & x_0 + v_0 + R > d_{max}. \end{cases} \tag{28}$$

When the particle's variation submits to the normally probability function, the rate μ_d in the one dimensional space is

$$\mu_d = \frac{\int_{d_{min}}^{d_{max}} f_p(x)dx}{\int_{x_0}^{x_0+v_0+R} f_p(x)dx}. \tag{29}$$

In the two dimensional space, the corresponding rate μ_d in the convergence behavior is

$$\mu_d = \frac{\int_{d_{min}}^{d_{max}} \int_{y(d_{min})}^{y(d_{max})} f_p(x)dxdy}{\int_{x_0}^{x_0+v_0+R} \int_{x_1}^{x_1+v_1+R} f_p(x)dxdy}. \tag{30}$$

In the three or high dimensional space, the corresponding rate μ_d in the convergence behavior is

$$\mu_d = \frac{\Theta}{\Psi}. \tag{31}$$

V. The Mathematical Model of Searching the Global Optimum Region

The above-mentioned sections mainly analyze the rate μ of locating into the global optimum region in the presence of convergence behavior and divergence behavior at one step. In order to discuss the rate of converging into the global optimum region in the whole evolutionary process, we assume several additional notations including the maximum number T_s of generations, the number n of all particles, the ratio τ of convergence behavior in the whole process. So, the rate μ_s of converging into the global optimum region in the whole process can be mathematically described by

$$\mu_s = \sum_{t_1=1}^{T_s\tau} \sum_{i=1}^{n} \mu_{ci}(t_1) + \sum_{t_2=1}^{T_s(1-\tau)} \sum_{i=1}^{n} \mu_{di}(t_2) \tag{32}$$

where $\mu_{ci}(t_1)$ and $\mu_{di}(t_2)$ denote the rate of locating the global optimum region under the convergence behavior at t_1th step and the divergence behavior at t_2th step, respectively. Then, the number T of generations locating into the global optimum region can be computed by

$$T = \frac{1}{\mu_s} = \frac{1}{\sum_{t_1=1}^{T\tau} \sum_{i=1}^{n} \mu_{ci}(t_1) + \sum_{t_2=1}^{T(1-\tau)} \sum_{i=1}^{n} \mu_{di}(t_2)}. \tag{33}$$

Generally speaking, premature convergence is one of the typical and existing problems in the area of swarm optimization. When the particles converge into one the local optimum and the solution variation in the convergence behavior is so small that the solutions do not easily escape from the local optimum in the convergence behavior, therefore, the rate μ_s of locating into the global optimum region utilizing by the divergence behavior, under the uniformly random solution variation case, can be roughly calculated by

$$\mu_s = \sum_{t_2=1}^{T_s(1-\tau)} \sum_{i=1}^{n} \mu_{di}(t_2) \tag{34}$$

so the number T of generations locating into the global optimum region is

$$T = \frac{1}{\sum_{i=1}^{n} \mu_{di}(t)} \times \frac{1}{1-\tau} \tag{35}$$

where the ratio τ is the ratio between the number of convergence behavior and the number of generations.

According to the important mathematical model (32) of locating into the global optimum region, several crucial rules can be concluded as follows.

- The rate of locating into the global optimum region is closely related to the hyper surface and the second global optimum of objective function. Generally speaking, the global optimum region is large or the objective fitness between the global optimum and the second global optimum is so large, the particles easily locate into the global optimum region, probably giving rise to the large rate μ.
- The rate μ is also related to the ratio of convergence behavior in the whole process, the number of particles, the number of generations, the fitness landscape of objective function, the initial solution and its variation at each step. Therefore, in order to achieve high rate μ, we not only increase the number of particles and the number of generations, but also reduce the corresponding solution space and adjust the tradeoff between convergence behavior and divergence behavior.
- In order to get the good solution in the optimization method, it is better to utilize the professional knowledge to reduce the whole solution space, resulting in the relatively large rate μ at each step.
- More importantly, the crucial parameter τ can control the ratio of convergence behavior in the whole process and indirectly determine the tradeoff between exploration ability and exploitation ability.
- According to the definition of the global optimum region, when the objective fitness quickly decreases to lead to the low possibility in the global optimum region and the low rate of locating into the global optimum region.

VI. Numerical Results

In order to show the effectiveness of the obtained results and easily understand the remarks in this paper, the objective of this section is to discuss the evolutionary process on Schaffer's F6 function by particle swarm optimization algorithm, together with the corresponding maximum solution space. Particularly, the rate μ of locating the global optimum region is discussed to better balance the tradeoff the exploration ability and exploitation ability.

A. Schaffer's F6 Function

Schaffer's f6 function has many local optima around the global optimum in the solution space $[-10, 10]^2$. Once getting into the local optimum, the particles hardly get out of local optima in the solution space. The formula of Schaffer's f6 function can be mathematically described by

$$f_1(x, y) = 0.5 + \frac{(\sin\sqrt{x^2 + y^2}) - 0.5}{(1 + 0.001(x^2 + y^2))^2} \tag{36}$$

where $|x| \leq 100$, $|y| \leq 100$. As depicted in Fig. 6, the global optimum of Schaffer's f6 function is (0,0).

And the number of particles or agents is set to 20, and the maximum number of generations is set to 200. It is so important to discuss the possible space at the next step of DE algorithm and PSO algorithm.

B. Evolutionary Process

In order to show the effectiveness of the obtained results in this paper, it is key to analyze and discuss the evolutionary process by using PSO algorithm and DE algorithm, etc. Specifically speaking, it is of importance to discuss the global optimum region, the rate of finding better solution, the possible solution space by the PSO algorithm and the DE algorithm at next step.

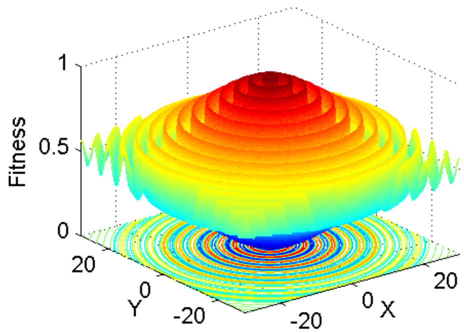

Fig. 6. The landscape of Schaffer's f6 function in the two-dimensional space

It is key to analyze that the next points are the continuous region by the PSO algorithm. Taking Schaffer f6 function by the PSO algorithm or DE algorithm for example, the objective landscape of Schaffer f6 function in $[-10, 10]^2$ can be depicted in Fig. 6. As for the PSO algorithm, the updating equations on velocity and position in one PSO algorithm are referred to as $pbest - X$ and $gbest - X$, and another updating equations are referred to as $pbest - X$, $gbest - X$ and $mbest - X$. The DE algorithm also consists of $DE/best/1/bin$, $DE/rand-to-best/1/bin$, $DE/best/2/bin$ and $DE/rand/2/bin$.

At the 2th step in the whole process, the possible maximum space by the PSO algorithm with $pbest - X$ and $gbest - X$ can be shown in Fig. 7, while the corresponding maximum space by the PSO algorithm with three vectors also can be shown in Fig. 8. The possible maximum solution space by the PSO algorithm is one continuous region, which not only is governed by the number of random vectors, but also it is influenced by the best position of personal particle and all particles. If the number of random vectors in PSO algorithm is smaller than the dimension of solution space, the possible maximum space is the projection region of random vectors.

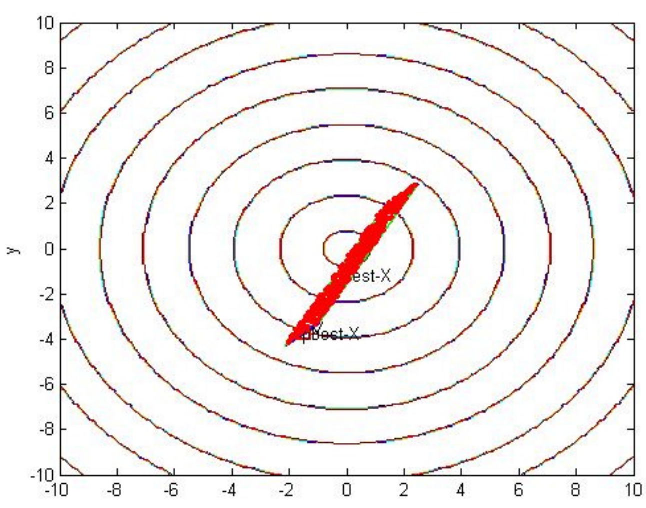

Fig. 7. The 2th step

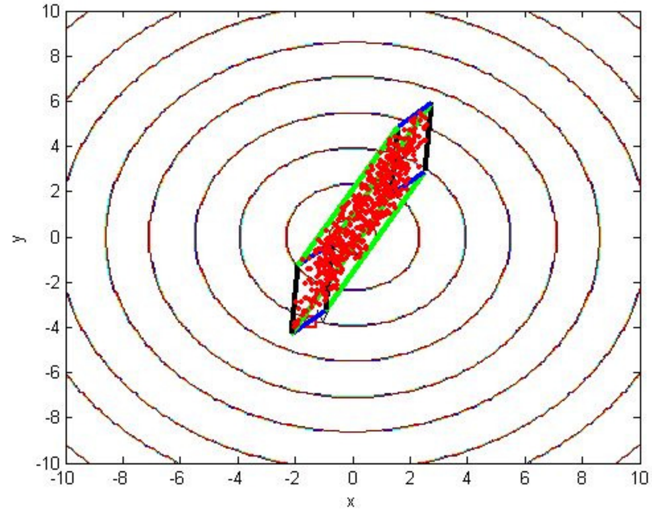

Fig. 8. The 2th step

However, the possible maximum solution space by the DE algorithm is the discrete point set, which is mainly dependent of the number of random variables in the updating equation of DE algorithm. The possible maximum space by the DE algorithm can be depicted in Fig. 9. Except for the convergence behavior of DE algorithm, the possible maximum space is so large and its exploration ability is larger than that of PSO algorithm.

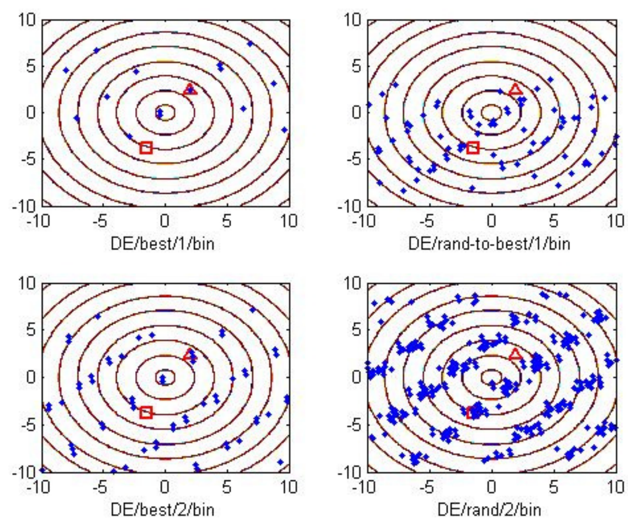

Fig. 9. The 2th step

At the 22th step, the maximum space by the PSO algorithm with two vectors can be shown in Fig. 10, while the maximum space by the PSO algorithm with three vectors can be shown in Fig. 11. Additionally, the maximum soultion space by the DE algorithm can be depicted in Fig. 12.

According to the obtained results, some interesting and useful remarks can be concluded to deeply study the swarm

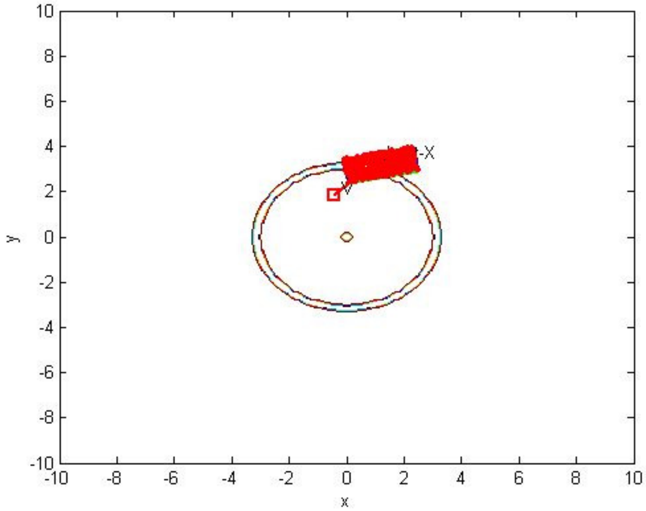

Fig. 10. The 22th step

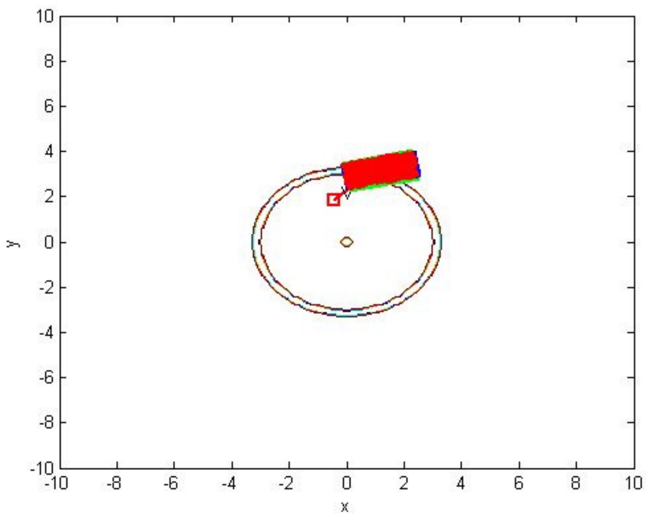

Fig. 11. The 22th step

optimization methods and get good solution in different fitness landscapes.

- In the case of the PSO algorithm, the maximum solution space is mainly dependent of the updated rule and the number of random vectors or random variables.
- According to the figure considering finding better solution in the landscape, the rate of finding better solution or updating better solution is relatively large at the beginning of search stage, while the corresponding rate is so small at the end of search stage. It is better to slowly update the current best position of all particles.
- The parameters $\xi_k(t)$ denotes the intersection between the hyperplane $f_*(t)$ and the kth local optimum region. If the parameter $\xi_k(t)$ decreases from large value to small

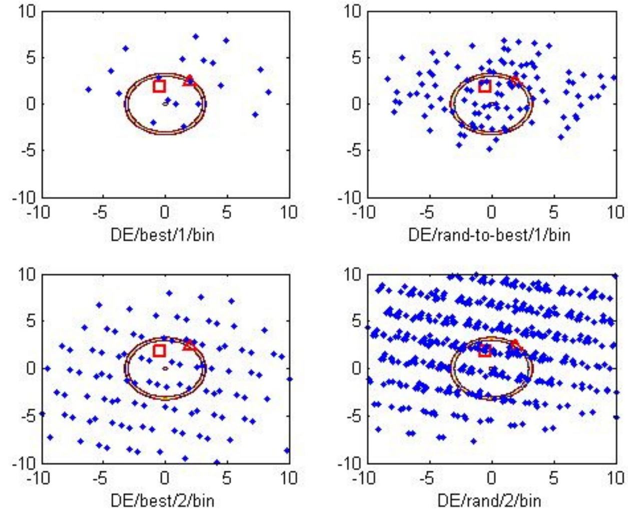

Fig. 12. The 22th step

value, so it is better to slowly update the current best solution. If the parameter $\xi_k(t)$ does not change in the whole evolutionary process, it can update the current best position as soon as possible.
- According to the obtained results from the two-dimensional landscape, the global optimum region Θ is typical few solution space, therefore, it is one challenging problem to search for the global optimum region in the presence of the unknown global optimum in the high dimensional space. In order to solve this problem, it is of importance to change the considered problem into another optimization problem with few parameters with the aide of the professional knowledge.
- In order to find the global optimum, the global optimum region is one of key factors to searching for the global optimum, so it is key to increase the global optimum region by utilizing stretching technology on the fitness landscape.

VII. Conclusion

In order to deeply discuss and analyze the key characteristics of the PSO algorithm, this paper mainly introduces the basic and fundamental concepts, which are mainly composed of the global optimum region, the local optimum region and the maximum solution space at next step, and it calculates the rate of locating into the global optimum region by PSO algorithm and DE algorithm at each step. More importantly, the mathematical model of searching for the global optimum region is closely related to the corresponding rate μ and the number of particles, the number of generations, the fitness landscape, the ratio between exploration ability and exploitation ability, and the updating equations on position and velocity. Additionally, in the early search stage, swarm optimization methods with high rate update the better solution or the current global optimum, while those swarm optimization methods with low rate update the current global optimum in the latter search stage. Finally, several important results and remarks are concluded according to the corresponding mathematical model. According to the mathematical model in (32), the best way of improving the searching efficiency is to reduce the whole solution space by the professional knowledge and enlarge the global optimum region by the new technology.

The future works can be mainly concluded in the paper. The first work in the future is mainly to discuss the rate of finding the global optimum region by other swarm optimization methods. The second work in the future is to develop the new swarm optimization inspired from new swarm animals or swarm behavior and design the updating equations according to the obtained results. The third work in the future is to design and improve the effectiveness of several typical optimization methods according to the model (32).

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