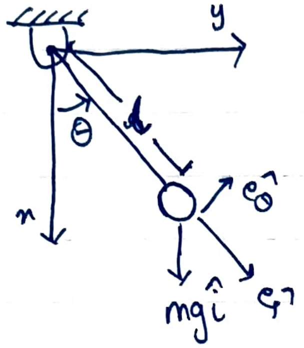
Q22. Simple Pendulum 7 ways

Equation to derive : $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

~~FBD~~ Sketch



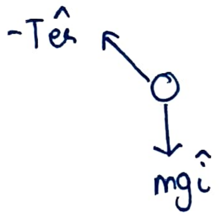
Apply LMB,

$$\sum \vec{F} = m\vec{a}$$

$$mg\hat{i} - T\hat{e_r} = m(l\ddot{\theta}\hat{e_\theta} - l\dot{\theta}^2\hat{e_r}) \quad \text{—} ①$$

(tension T is the constraint force)

FBD



a) Resolve eq^n ① into $\hat{e_r}$, $\hat{e_\theta}$

$$mg\cos\theta\hat{e_r} - mg\sin\theta\hat{e_\theta} - T\hat{e_r} = m(l\ddot{\theta}\hat{e_\theta} - l\dot{\theta}^2\hat{e_r})$$

Taking dot product with $\hat{e_r}$, $\hat{e_\theta}$,

$\underline{T = mg\cos\theta - ml\dot{\theta}^2}$

$$-mg\sin\theta = ml\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

b) $\{mg\cos\theta\hat{e_r} - mg\sin\theta\hat{e_\theta} - T\hat{e_r} = m(l\ddot{\theta}\hat{e_\theta} - l\dot{\theta}^2\hat{e_r})\}\cdot\hat{e_\theta}$

$\Rightarrow \quad -mg\sin\theta\hat{e_\theta} = ml\ddot{\theta}\hat{e_\theta}$

$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$

c) $\{mg\cos\theta\hat{e_r} - mg\sin\theta\hat{e_\theta} - T\hat{e_r} = m(l\ddot{\theta}\hat{e_\theta} - l\dot{\theta}^2\hat{e_r})\}\times\hat{e_r}$

$\Rightarrow \{-mg\sin\theta(-\hat{k}) = ml\ddot{\theta}(-\hat{k})\}\cdot\hat{k}$

$\Rightarrow \quad mg\sin\theta = -ml\ddot{\theta}$

$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$

d) Apply AMB.

$$\sum M_{/O} = \vec{r} \times m\vec{a}$$

$$l\hat{e_r} \times (mg\cos\theta\,\hat{e_r} - mg\sin\theta\,\hat{e_\theta} - T\hat{e_r}) = l\hat{e_r} \times [m(l\ddot{\theta}\hat{e_\theta} - l\dot{\theta}^2\hat{e_r})]$$

$$\{-mgl\sin\theta\,\hat{k} = ml^2\ddot{\theta}\,\hat{k}\}\cdot\hat{k}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g\sin\theta}{l} = 0}$$

e) Conservation of Energy

$$E_K + E_P = \text{constant}$$

$$\Rightarrow \frac{d}{dt}\left(E_K + E_P\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\,\vec{v}\cdot\vec{v} - mgl\cos\theta\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m(l\dot{\theta})^2 - mgl\cos\theta\right) = 0$$

$$\Rightarrow ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\,\dot{\theta} = 0$$

$$\Rightarrow ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right) = 0$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$ for $\dot{\theta} \neq 0$ (assume 0 amt of time spent at ends where $\dot{\theta} = 0$)

f) Power Balance.

$$P = \dot{E_K} = \vec{F}\cdot\vec{v}$$

$$\Rightarrow ml^2\dot{\theta}\ddot{\theta} = (mgl\cos\theta\,\hat{e_r} - mgl\sin\theta\,\hat{e_\theta} - T\hat{e_r})\cdot(l\dot{\theta}\hat{e_\theta})$$

$$\Rightarrow ml^2\dot{\theta}\ddot{\theta} = -mgl^2\sin\theta\,\dot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$ for $\dot{\theta} \neq 0$.

g) Lagrangian

$$\mathcal{L} = E_K - E_P = \frac{1}{2}m(l\dot{\theta})^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0 \quad \Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$