Student: Bhargavi Gundla **Date:** 04/20/22	**Instructor:** Plamen Iliev **Course:** Math 2550G, Spring 2022	**Assignment:** H10

Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 9$. Then find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.

To find the center of mass bounded below by $z = x^2 + y^2$ and above by the plane $z = 9$, first find the mass $M = \iiint_D \delta \, dz \, dy \, dx$. Next find first moments M_{yz}, M_{xz}, and M_{xy}. Lastly use the formulas $\bar{x} = \frac{M_{yz}}{M}$, $\bar{y} = \frac{M_{xz}}{M}$, and $\bar{z} = \frac{M_{xy}}{M}$ to find the coordinates of the center of mass.

Since the density is constant, the mass of the solid is the solid's volume times its density. Note part (b) says to find the plane $z = c$ that divides the solid's volume in half. Thus, to save time and effort, find the mass of the solid bounded below by $z = x^2 + y^2$ and above by an arbitrary plane $z = h$. Then to find the mass of the given solid, substitute 9 in for h and to find the volume of the solid, divide the solid's mass by its density.

The bounds on z in terms of h, x, and y are $x^2 + y^2 \le z \le h$.

Find the mass M of the solid with the constant density δ.

$$\iint_R \int_{x^2+y^2}^{h} \delta \, dz \, dy \, dx = \delta \iint_R \int_{x^2+y^2}^{h} dz \, dy \, dx$$

Pull the constant δ out of the integral.

Begin by integrating M with respect to z.

$$\delta \iint_R \int_{x^2+y^2}^{h} dz \, dy \, dx = \delta \iint_R [z]_{x^2+y^2}^{h} \, dy \, dx$$

Substitute the upper and lower limits of z into the obtained result.

$$\delta \iint_R [z]_{x^2+y^2}^{h} \, dy \, dx = \delta \iint_R \left(h - \left(x^2 + y^2\right)\right) dy \, dx$$

Notice that $\left(x^2 + y^2\right)$ in the Cartesian system is r^2 in the polar coordinate system. The integration in the polar coordinate system gives $\iint_R \left(h - r^2\right) r \, dr \, d\theta$.

The x-y region will be the shadow cast by the region. The edge of the shadow is directly below the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = h$. Setting the equations equal to find this intersection we see the x-y shadow is bounded by $x^2 + y^2 \le h$.

The limits of integration polar coordinates are given below.

$0 \le r \le$	$\sqrt{h}$	The limits of integration for r.
$0 \le \theta \le$	2π	The limits of integration for the angle θ.

Simplifying $\iint_R \left(h - r^2\right) r \, dr \, d\theta$ and entering in the limits of integration gives $\int_0^{2\pi} \int_0^{\sqrt{h}} \left(hr - r^3\right) dr \, d\theta$.

Now take the integral with respect to r.

$$\delta\int_0^{2\pi}\int_0^{\sqrt{h}}\left(hr-r^3\right)\,dr\,d\theta=\delta\int_0^{2\pi}\left[\frac{hr^2}{2}-\frac{r^4}{4}\right]_0^{\sqrt{h}}\,d\theta$$

Substitute the upper and lower limits of h into the obtained result.

$$\begin{aligned}\delta\int_0^{2\pi}\left[\frac{hr^2}{2}-\frac{r^4}{4}\right]_0^{\sqrt{h}}\,d\theta &= \delta\int_0^{2\pi}\left(\frac{h\left(\sqrt{h}\right)^2}{2}-\frac{\left(\sqrt{h}\right)^4}{4}\right)-0\\ &= \delta\int_0^{2\pi}\left(\frac{h^2}{4}\right)\,d\theta\end{aligned}$$

Lastly take the integral with respect to $d\theta$.

$$\delta\int_0^{2\pi}\left(\frac{h^2}{4}\right)\,d\theta=\delta\left[\frac{h^2}{4}\theta\right]_0^{2\pi}$$

Substitute the upper and lower limits of θ into the obtained result.

$$\begin{aligned}\delta\left[\frac{h^2}{4}\theta\right]_0^{2\pi} &= \delta\left(\frac{h^2(2\pi)}{4}-\frac{h^2(0)}{4}\right)\\ &= \delta\frac{\pi h^2}{2}\end{aligned}$$

The mass of the object is $M=\delta\frac{\pi h^2}{2}$.

Next find the coordinates of the center of mass $\bar{x}$, $\bar{y}$, and $\bar{z}$. Since the solid is symmetrical about the z-axis, its center of mass is located on the z axis $\bar{x}=\bar{y}=0$.

To find $\bar{z}$, find M_{xy} and then use the formula $\bar{z}=\frac{M_{xy}}{M}$ where $M_{xy}=\delta\iiint z\,dz\,dy\,dx$.

Begin by intergrating M_{xy} with respect to z.

$$\delta\iint\int_{x^2+y^2}^{h} z\,dz\,dy\,dx=\delta\iint\left[\frac{z^2}{2}\right]_{x^2+y^2}^{h}\,dy\,dx$$

Substitute the upper and lower limits of z into the obtained result.

$$\delta\iint\left[\frac{z^2}{2}\right]_{x^2+y^2}^{h}\,dy\,dx = \delta\iint\left(\frac{h^2}{2}-\frac{\left(x^2+y^2\right)^2}{2}\right)\,dy\,dx$$

Similarly to the previous part, transfering the integration into the polar coordinates system gives $\delta\int_0^{2\pi}\int_0^{\sqrt{h}}\left(\frac{h^2}{2}-\frac{r^4}{2}\right)r\,dr\,d\theta$.

Simplifying the integral gives $\delta\int_0^{2\pi}\int_0^{\sqrt{h}}\left(\frac{h^2r}{2}-\frac{r^5}{2}\right)\,dr\,d\theta$.

Now take the integral with respect to dr.

$$\delta\int_0^{2\pi}\int_0^{\sqrt{h}}\left(\frac{h^2 r}{2}-\frac{r^5}{2}\right)dr\,d\theta=\delta\int_0^{2\pi}\left[\frac{h^2r^2}{4}-\frac{r^6}{12}\right]_0^{\sqrt{h}}d\theta$$

Substitute the upper and lower limits of r into the obtained result.

$$\delta\int_0^{2\pi}\left[\frac{h^2r^2}{4}-\frac{r^6}{12}\right]_0^{\sqrt{h}}d\theta=\delta\left(\frac{h^2\left(\sqrt{h}\right)^2}{4}-\frac{\left(\sqrt{h}\right)^6}{12}\right)$$

$$=\delta\int_0^{2\pi}\left(\frac{h^3}{6}\right)d\theta$$

Lastly integrate with respect to θ.

$$\delta\int_0^{2\pi}\left(\frac{h^3}{6}\right)d\theta=\delta\left[\frac{h^3}{6}\theta\right]_0^{2\pi}$$

Substitute the upper and lower limits of θ into the obtained result.

$$\delta\left[\frac{h^3}{6}\theta\right]_0^{2\pi}=\delta\left(\frac{h^3(2\pi)}{6}-\frac{h^3(0)}{6}\right)$$

$$=\delta\frac{h^3\pi}{3}$$

Therefore the first moment M_{xy} is $\delta\frac{\pi h^3}{3}$. To find the z-coordinate of the center of mass, use $\bar{z}=\frac{M_{xy}}{M}$. Thus $\bar{z}=\frac{\frac{\pi h^3}{3}\delta}{\frac{\pi h^2}{2}\delta}=\frac{2h}{3}$.

To find the z-coordinate of the center of mass of the object bounded below by $z=x^2+y^2$ and above by the plane $z=9$, substitute 9 for h into $\bar{z}=\frac{2h}{3}$. The center of mass when $z=9$ is (0,0,6).

To find the plane $z=c$ that cuts the object into two parts of equal volume, first find the volume of the whole object, and then find the volume of the bottom half of which the limits of z range from zero to c. Set the volume of the whole object to be equal to two times the volume of the bottom half, and solve for c.

Recall that the mass of an object is the object's volume times its density. Thus the volume of an object is $V=\frac{M}{\delta}$, where M is the mass and δ is the density of the object. From the previous part, the mass M was found to be $\delta\frac{\pi h^2}{2}$. Therefore the volume of the given solid is $V=\frac{\frac{\pi h^2\delta}{2}}{\delta}=\frac{\pi h^2}{2}$.

To find the volume of the object bounded by $z=x^2+y^2$ and $z=9$, substitute 9 for h into $V=\frac{\pi h^2}{2}$. The volume of the object if $z=h=9$ is $\frac{81}{2}\pi$. The volume of the bottom half of the object at $z=h=c$ is $\frac{\pi c^2}{2}$.

Find the value of c. Set the volume of the whole object $V = \frac{81}{2}\pi$ to be two times the volume of the bottom half of the object, $V = \frac{\pi c^2}{2}$.

$$2\frac{\pi c^2}{2} = \frac{81}{2}\pi$$

$$c = \frac{9}{2}\sqrt{2}$$

The plane $z = \frac{9}{2}\sqrt{2}$ cuts the object bounded by $z = x^2 + y^2$ and $z = 9$ into two parts of equal volumes.