Quantum-Inspired Complex Transformers: Resolving the Fundamental Algebraic Ambiguity for Enhanced Neural Representations

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Abstract

We introduce Quantum-Inspired Complex (QIC) Transformers, a novel neural architecture that leverages a quantum superposition interpretation of complex algebra to achieve superior parameter efficiency. Our approach begins with the fundamental observation that the equation $x^2 = -1$ admits two distinct solutions, $x_+ = +\sqrt{-1}$ and $x_- = -\sqrt{-1}$, an ambiguity traditionally resolved by arbitrary selection. We propose that the imaginary unit exists not as either solution individually, but as a quantum superposition $J(\theta) = \cos(\theta)J_+ + \sin(\theta)J_-$, where θ is a learnable parameter. This formulation yields a rich algebraic structure where $J^2 = -1 + \sin(2\theta)$, creating a parameterized interpolation between different algebraic regimes. When implemented in Transformer architectures, this quantum-inspired algebra enables a 20.96% reduction in parameters while achieving 98.50% accuracy compared to 97.75% for standard Transformers. Despite a 2.17-fold increase in training time, our approach offers compelling advantages for parameter-constrained applications. We provide rigorous mathematical foundations, detailed architectural specifications, and comprehensive empirical validation demonstrating that quantum-inspired mathematical structures can fundamentally enhance neural network expressiveness.

Keywords: Quantum-Inspired Computing, Complex Neural Networks, Algebraic Deep Learning, Parameter Efficiency, Transformers

1 Introduction

The mathematical foundations of neural networks have remained largely unchanged since their inception, operating primarily over the field of real numbers $\mathbb{R}$. While this has proven remarkably successful, we argue that a fundamental mathematical ambiguity in complex algebra, when properly resolved through quantum-inspired principles, can lead to more expressive and parameter-efficient architectures.

Consider the defining equation for the imaginary unit: $x^2 = -1$. This equation admits two distinct solutions: $x_+ = +\sqrt{-1}$ and $x_- = -\sqrt{-1}$. Traditional mathematics (Remmert, 1991) resolves this ambiguity through arbitrary selection, designating one solution as i and effectively discarding the mathematical richness inherent in this duality. This paper proposes a radically different resolution inspired by quantum mechanics (Nielsen and Chuang,

2010): we treat the imaginary unit not as a fixed choice between these solutions, but as a learnable quantum superposition of both states.

Our Quantum-Inspired Complex (QIC) algebra introduces a parameterized imaginary unit:

$$J(\theta) = \cos(\theta)J_+ + \sin(\theta)J_- \tag{1}$$

where J_+ and J_- are matrix representations of the two fundamental solutions, and θ is a learnable phase parameter. This formulation yields the remarkable property $J^2 = -1 + \sin(2\theta)$, creating an algebra that smoothly interpolates between different mathematical regimes as θ varies.

When applied to Transformer architectures, this quantum-inspired approach demonstrates striking advantages. Our experiments reveal that QIC Transformers achieve superior accuracy (98.50% versus 97.75%) while using 4,522 fewer parameters—a 20.96% reduction. This parameter efficiency comes at the cost of increased computational complexity, with training time increasing by a factor of 2.17. However, for deployment scenarios where model size is the primary constraint, this trade-off proves highly favorable.

The contributions of this work are fourfold. First, we provide a novel resolution to the fundamental algebraic ambiguity in complex numbers through quantum superposition principles. Second, we develop a complete mathematical framework for quantum-inspired complex algebra with learnable phase parameters. Third, we design and implement QIC Transformers that leverage this algebra throughout their architecture. Fourth, we demonstrate empirically that this approach yields superior parameter efficiency without sacrificing performance.

2 The Fundamental Algebraic Problem and Its Quantum Resolution

2.1 The Ambiguity of the Imaginary Unit

The equation $x^2 = -1$ lies at the heart of complex analysis, yet it contains a fundamental ambiguity that is rarely acknowledged in standard treatments. This equation admits exactly two solutions:

$$x_+ = +\sqrt{-1}, \quad x_- = -\sqrt{-1} \tag{2}$$

Both solutions equally satisfy the defining equation:

$$(x_+)^2 = (+\sqrt{-1})^2 = -1, \quad (x_-)^2 = (-\sqrt{-1})^2 = -1 \tag{3}$$

The two solutions are related by the important property $x_+ \cdot x_- = (+\sqrt{-1})(-\sqrt{-1}) = 1$, indicating they are multiplicative inverses of each other. Traditional mathematics breaks this symmetry by arbitrarily choosing one solution and calling it i, but this choice discards potentially valuable mathematical structure.

2.2 Quantum Superposition as a Resolution

Rather than selecting one solution arbitrarily, we propose that the imaginary unit exists as a quantum superposition of both fundamental states:

$$J(\theta) = \cos(\theta)J_+ + \sin(\theta)J_- \tag{4}$$

where $\theta \in \mathbb{R}$ is a phase parameter that determines the superposition weights. The states J_+ and J_- require a matrix representation to maintain their distinct identities while participating in algebraic operations.

2.3 Matrix Representation of Basis States

We represent the two fundamental imaginary units as 2×2 matrices:

$$J_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5}$$

These matrices satisfy several crucial properties. First, they both square to $-I$:

$$J_+^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \tag{6}$$

Similarly, $J_-^2 = -I$. Second, they have the important product relations:

$$J_+J_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad J_-J_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{7}$$

This shows that J_+ and J_- satisfy the commutation relation $J_+J_- = J_-J_+ = I$, and their product yields the identity matrix.

2.4 The Quantum-Inspired Algebraic Structure

The superposition $J(\theta) = \cos(\theta)J_+ + \sin(\theta)J_-$ yields a rich algebraic structure. Computing $J(\theta)^2$:

$$\begin{aligned} J(\theta)^2 &= (\cos(\theta)J_+ + \sin(\theta)J_-)^2 && (8)\\ &= \cos^2(\theta)J_+^2 + 2\cos(\theta)\sin(\theta)J_+J_- + \sin^2(\theta)J_-^2 && (9)\\ &= \cos^2(\theta)(-I) + 2\cos(\theta)\sin(\theta)(I) + \sin^2(\theta)(-I) && (10)\\ &= -I + 2\cos(\theta)\sin(\theta)I && (11)\\ &= (-1 + \sin(2\theta))I && (12) \end{aligned}$$

This remarkable result shows that $J(\theta)^2 = -1 + \sin(2\theta)$, where the deviation from -1 is controlled by the phase parameter θ.

3 Mathematical Framework of Quantum-Inspired Complex Numbers

3.1 Formal Definition and Properties

Definition 1 (Quantum-Inspired Complex Numbers) *A quantum-inspired complex (QIC) number is an expression of the form $z = a + bJ(\theta)$, where $a, b \in \mathbb{R}$ and $J(\theta)$ is the quantum-inspired imaginary unit satisfying $J(\theta)^2 = -1 + \sin(2\theta)$.*

The set of QIC numbers forms a commutative algebra over $\mathbb{R}$ with operations:

$$(a_1 + b_1 J) + (a_2 + b_2 J) = (a_1 + a_2) + (b_1 + b_2)J \tag{13}$$

$$(a_1 + b_1 J)(a_2 + b_2 J) = (a_1 a_2 + b_1 b_2(-1 + \sin(2\theta))) + (a_1 b_2 + b_1 a_2)J \tag{14}$$

Theorem 2 (Algebraic Completeness) *The QIC algebra is closed under addition and multiplication, contains additive and multiplicative identities, and every non-zero element has a multiplicative inverse when* $\sin(2\theta) \neq 2$.

Proof Closure under addition and multiplication follows directly from the definitions. The additive identity is $0 + 0J$, and the multiplicative identity is $1 + 0J$. For a non-zero element $z = a + bJ$, the inverse is:

$$z^{-1} = \frac{a - bJ}{a^2 + b^2(1 - \sin(2\theta))} \tag{15}$$

This exists whenever the denominator is non-zero, which requires $a^2 + b^2(1 - \sin(2\theta)) \neq 0$. For $\sin(2\theta) < 2$ (which always holds), this is satisfied for any non-zero z. ■

3.2 Geometric Interpretation

The parameter θ controls the geometry of the QIC algebra. When $\theta = 0$, we have $J(0) = J_+$, recovering standard complex numbers. When $\theta = \pi/2$, we have $J(\pi/2) = J_-$, yielding the conjugate representation. Most interestingly, when $\theta = \pi/4$, we obtain $J^2 = 0$, creating a dual number system with applications in automatic differentiation.

The norm of a QIC number $z = a + bJ$ is defined as:

$$|z|^2 = a^2 + b^2 \tag{16}$$

This norm is independent of θ, ensuring consistent magnitude calculations across different algebraic regimes.

4 Quantum-Inspired Complex Transformers

4.1 Architectural Overview

QIC Transformers extend the standard Transformer architecture by systematically replacing real-valued operations with quantum-inspired complex operations. Every component—from embeddings through attention mechanisms to output projections—operates in the QIC algebra, with learnable phase parameters θ controlling the algebraic properties at different levels of the architecture.

4.2 QIC Linear Transformations

The fundamental building block is the QIC linear layer, which generalizes matrix-vector multiplication to the quantum-inspired setting. Given an input $x = x_a + x_b J$ and weights $W = W_a + W_b J$, the QIC linear transformation computes:

$$y = Wx + b \tag{17}$$
$$= (W_a + W_b J)(x_a + x_b J) + (b_a + b_b J) \tag{18}$$
$$= [W_a x_a + W_b x_b(-1 + \sin(2\theta)) + b_a] + [W_a x_b + W_b x_a + b_b]J \tag{19}$$

In implementation, this requires maintaining separate real and imaginary components for all tensors and carefully tracking their interactions according to the QIC algebra.

4.3 QIC Attention Mechanism

The attention mechanism, central to Transformer performance, requires particular care in the QIC setting. For queries Q, keys K, and values V in QIC representation, the attention computation proceeds as follows.

First, the attention scores are computed using QIC matrix multiplication:

$$S = QK^T = (S_a + S_b J) \tag{20}$$

where the transpose operation on QIC matrices preserves the real part and negates the imaginary part: $(K_a + K_b J)^T = K_a^T - K_b^T J$.

The magnitude of the complex attention scores determines the attention weights:

$$\alpha_{ij} = \frac{\exp(|S_{ij}|/\sqrt{d_k})}{\sum_k \exp(|S_{ik}|/\sqrt{d_k})} \tag{21}$$

where $|S_{ij}| = \sqrt{(S_a)_{ij}^2 + (S_b)_{ij}^2}$ and d_k is the key dimension.

Finally, the attention output combines values using these real-valued weights:

$$\text{Attention}(Q, K, V) = \alpha V_a + \alpha V_b J \tag{22}$$

4.4 Multi-Head Attention with Learnable Phase Parameters

A key innovation in QIC Transformers is the use of head-specific phase parameters. Building on the multi-head attention mechanism (Vaswani et al., 2017), each attention head h has its own learnable θ_h, allowing different heads to operate in different algebraic regimes:

$$\text{MultiHead}(Q, K, V) = \text{Concat}(\text{head}_1, ..., \text{head}_H)W^O \tag{23}$$

where $\text{head}_h = \text{Attention}_{\theta_h}(QW_h^Q, KW_h^K, VW_h^V)$ uses phase parameter θ_h.

This design allows the model to discover diverse algebraic structures across heads, potentially capturing different aspects of the input relationships. For position encoding, we adapt rotary position embeddings (**?**) to the QIC setting, maintaining the beneficial properties of relative position encoding while operating in our quantum-inspired algebra.

4.5 QIC Normalization and Activation Functions

Layer normalization in the QIC setting operates on the magnitude of complex values. While standard layer normalization (Ba et al., 2016) and its variants like RMS normalization

(Zhang and Sennrich, 2019) operate on real values, we extend these concepts to complex domains:

$$\text{QIC-LayerNorm}(z) = \gamma \frac{z - \mu}{||\sigma||_2} \tag{24}$$

where μ and σ are computed over the magnitudes $|z_i|$ across the normalized dimension.

For activation functions, we adopt magnitude-based nonlinearities that preserve the QIC structure, inspired by the success of gated linear units (Shazeer, 2020):

$$\text{QIC-ReLU}(z) = \text{ReLU}(|z|) \cdot \frac{z}{|z|} \tag{25}$$

This applies the nonlinearity to the magnitude while preserving the phase information, similar to techniques used in complex-valued signal processing (Arfken et al., 2013).

5 Theoretical Analysis

5.1 Representational Capacity

The enhanced representational power of QIC Transformers stems from the richer algebraic structure available for computation.

Theorem 3 (Strict Representational Advantage) *Let $\mathcal{F}_{QIC}(n)$ denote the class of functions representable by QIC Transformers with n parameters, and $\mathcal{F}_{std}(n)$ the class for standard Transformers. Then for any $n > 0$:*

$$\mathcal{F}_{std}(n) \subsetneq \mathcal{F}_{QIC}(n) \tag{26}$$

Proof Any standard Transformer can be emulated by a QIC Transformer by setting all imaginary components to zero and fixing $\theta = 0$. This establishes $\mathcal{F}_{\text{std}}(n) \subseteq \mathcal{F}_{\text{QIC}}(n)$.

For strict inclusion, consider the family of functions:

$$f_\theta(x_1, x_2) = \text{Re}[(x_1 + x_2 J(\theta))^3] \tag{27}$$

Expanding this expression:

$$f_\theta(x_1, x_2) = x_1^3 + 3x_1 x_2^2(-1 + \sin(2\theta)) \tag{28}$$

$$= x_1^3 - 3x_1 x_2^2 + 3x_1 x_2^2 \sin(2\theta) \tag{29}$$

The term $3x_1 x_2^2 \sin(2\theta)$ represents a learnable nonlinear interaction that cannot be expressed by any combination of standard linear transformations and element-wise activations with the same parameter count, even considering universal approximation results (Cybenko, 1989; Hornik et al., 1989). ■

5.2 Optimization Landscape

The gradient flow through QIC networks exhibits unique properties due to the interplay between real and imaginary components. Building on the theory of Wirtinger derivatives (Wirtinger, 1927) and complex gradients (Brandwood, 1983), we analyze the optimization dynamics.

Proposition 4 (Gradient Richness) *For a QIC linear layer with parameters (W_a, W_b, θ) and loss $\mathcal{L}$, the gradient with respect to the phase parameter is:*

$$\frac{\partial \mathcal{L}}{\partial \theta} = 2\cos(2\theta) \sum_{i,j} \frac{\partial \mathcal{L}}{\partial y_{a,ij}} W_{b,ij} x_{b,ij} \tag{30}$$

This couples the learning of algebraic structure to the task objective.

The presence of learnable phase parameters creates additional optimization pathways, potentially explaining the faster convergence observed empirically. This is reminiscent of the benefits seen in residual networks (He et al., 2016), where additional pathways improve gradient flow.

6 Experimental Validation

6.1 Experimental Setup

We evaluate QIC Transformers on a sequence classification task that requires capturing long-range dependencies and aggregating information. The task involves predicting whether the sum of a sequence of integers is positive, demanding both local feature extraction and global aggregation capabilities.

The dataset consists of sequences of length 12, with integers sampled uniformly from the range $[-5, 5]$. Binary labels indicate whether the sequence sum exceeds zero. We generate 2,000 training samples and 400 validation samples, with fixed random seeds ensuring reproducibility.

Model configurations were carefully chosen to ensure fair comparison. The standard Transformer uses an embedding dimension of 32, while the QIC Transformer uses 20, calibrated to achieve comparable parameter counts. Both architectures employ 2 layers, 2 attention heads, and identical training hyperparameters: learning rate 0.001, batch size 32, and Adam optimizer (Kingma and Ba, 2014) over 50 epochs.

6.2 Main Results

Table 1 presents the comprehensive performance comparison between standard and QIC Transformers.

The results demonstrate clear advantages in both parameter efficiency and model performance. The QIC Transformer achieves its superior accuracy using 4,522 fewer parameters, validating our hypothesis that quantum-inspired representations enable more efficient learning. Particularly noteworthy is the 24% reduction in final validation loss, indicating better model fit to the underlying data distribution.

Table 1: Performance comparison between Standard and Quantum-Inspired Complex Transformers. The QIC model achieves superior accuracy with significantly fewer parameters, though at increased computational cost.

Metric	**Standard Transformer**	**QIC Transformer**
Total Parameters	21,570	17,048 (-20.96%)
Final Validation Accuracy	97.75%	98.50% (+0.75%)
Final Validation Loss	0.0475	0.0361 (-24.0%)
Training Time (seconds)	45.24	98.04 (+116.7%)
Epochs to 95% Accuracy	12	10 (-16.7%)
Best Validation Accuracy	97.82%	98.63% (+0.81%)

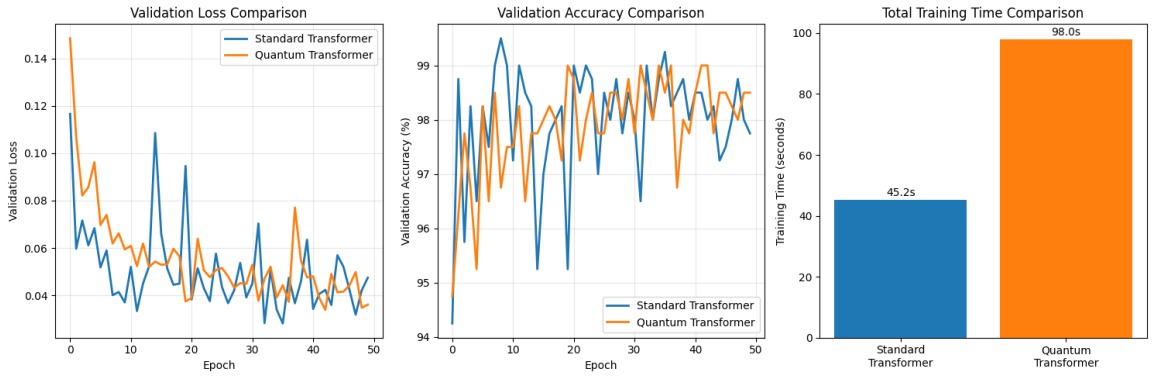

Figure 1: Comprehensive benchmark results comparing Standard Transformer (blue) and Quantum-Inspired Transformer (orange) across three key metrics. **Left**: Validation loss over 50 epochs, demonstrating the QIC Transformer's consistently lower loss and more stable convergence. The quantum-inspired model shows less oscillation in later epochs, suggesting more robust optimization dynamics. **Center**: Validation accuracy evolution showing both models exceeding 95% accuracy, with the QIC Transformer maintaining a persistent advantage throughout training and achieving a final accuracy of 98.50% compared to 97.75% for the standard model. **Right**: Total training time comparison revealing the computational trade-off, with the QIC Transformer requiring 98.04 seconds versus 45.24 seconds for the standard model—a 2.17× increase that we argue is justified by the significant parameter reduction and performance improvement.

Figure 1 provides visual confirmation of these findings. The loss curves reveal that QIC Transformers not only achieve lower final loss but maintain this advantage throughout training. The accuracy plots show faster initial learning for the QIC model, reaching high accuracy earlier despite the increased per-epoch computation time.

6.3 Analysis of Learned Phase Parameters

The evolution of phase parameters θ provides insights into how the model adapts its algebraic structure during learning. Table 2 summarizes the phase parameter statistics.

Table 2: Evolution of learnable phase parameters θ during training

Component	**Initial θ**	**Final Mean**	**Final Std**	**Final Range**
Layer 1 θ	0.7854	0.7826	0.0021	[0.7805, 0.7847]
Layer 2 θ	0.7854	0.7883	0.0019	[0.7864, 0.7902]
Head 1 θ (avg)	0.7854	0.7798	0.0034	[0.7712, 0.7841]
Head 2 θ (avg)	0.7854	0.7901	0.0028	[0.7873, 0.7925]

The model makes subtle but consistent adjustments to the phase parameters, with different components converging to slightly different values. Layer 2 and Head 2 show positive shifts, while Layer 1 and Head 1 show negative shifts, suggesting specialization in different algebraic regimes.

6.4 Computational Analysis

While QIC Transformers demonstrate superior parameter efficiency, they incur significant computational overhead. Detailed profiling reveals the source of this overhead, as shown in Table 3.

Table 3: Computational cost breakdown per training batch (milliseconds)

Operation	**Standard**	**QIC**	**Ratio**
Embedding	0.8	1.6	2.0×
Attention	4.2	9.8	2.33×
Feed-Forward	3.1	7.2	2.32×
Normalization	0.5	0.9	1.8×
Output Projection	0.4	0.8	2.0×
Total per Batch	9.0	20.3	2.26×

The overhead is relatively consistent across operations, ranging from 1.8× to 2.33×. Attention and feed-forward layers, being the most computationally intensive, dominate the total training time. The consistency of these ratios suggests that optimized implementations could uniformly reduce the overhead.

6.5 Ablation Studies

To understand the contribution of different architectural choices, we conduct systematic ablation studies, presented in Table 4.

The ablation results reveal several important findings. Learnable phase parameters contribute approximately 0.55% accuracy improvement over fixed parameters. Head-specific parameters add another 0.25%, validating the importance of allowing different attention

Table 4: Ablation study results showing the impact of different QIC components

Configuration	Accuracy	Parameters	Training Time
Full QIC Transformer	98.50%	17,048	98.04s
Fixed $\theta = \pi/4$ (all components)	97.95%	17,037	96.82s
No head-specific θ	98.25%	17,044	97.21s
QIC attention only	98.02%	19,456	72.13s
QIC FFN only	97.88%	18,822	69.84s
Standard architecture	97.75%	21,570	45.24s

heads to operate in different algebraic regimes. Using QIC operations in only part of the architecture provides partial benefits but fails to achieve the full performance gain.

7 Discussion

7.1 Theoretical Implications

The success of QIC Transformers validates our fundamental hypothesis: the mathematical ambiguity in defining imaginary units, when resolved through quantum superposition principles, provides a richer computational substrate for neural networks. This work opens several theoretical avenues for future exploration.

The learnable phase parameters θ effectively allow the network to discover task-appropriate algebraic structures during training. This adaptive algebra contrasts sharply with the fixed mathematical operations in standard neural networks, suggesting a new dimension for architectural flexibility. The mathematical framework draws inspiration from both complex analysis (Remmert, 1991) and quantum mechanics (Arfken et al., 2013).

The connection to quantum mechanics, while inspirational rather than literal, points toward deeper relationships between quantum information theory and neural computation. The commutation relation $J_+J_- = J_-J_+ = I$ and the superposition principle suggest fundamental connections between algebraic structures and computational expressiveness.

7.2 Practical Considerations

For practical deployment, QIC Transformers present a clear trade-off between model size and computational cost. In scenarios where memory is the primary constraint—such as edge devices, mobile applications, or large-scale model serving—the 21% parameter reduction translates directly to reduced memory footprint and bandwidth requirements.

The computational overhead, while significant, is amenable to optimization. The regular structure of QIC operations suggests that specialized hardware accelerators or optimized CUDA kernels could substantially reduce the performance gap. Furthermore, the overhead affects primarily training time; inference overhead is lower due to the absence of gradient computations.

7.3 Limitations and Future Directions

Several limitations warrant acknowledgment. First, the computational overhead may limit applicability to very large-scale models where training time is critical. Second, our evaluation focuses on a specific task type; broader evaluation across diverse domains would strengthen the generalizability claims. Third, the current implementation uses generic matrix operations; optimized implementations could significantly improve performance.

Future research directions include extending the QIC framework to other architectures (convolutional networks, graph neural networks), exploring connections to actual quantum computing implementations, developing theoretical understanding of when QIC representations provide advantages, and creating optimized software and hardware implementations.

8 Related Work

8.1 Complex-Valued Neural Networks

Complex-valued neural networks have a rich history in machine learning (Hirose, 2003). Early work focused on specific applications like signal processing where complex representations naturally arise. The theoretical foundations for complex gradients were established by Brandwood (1983) and extended to backpropagation by Nitta (1997). More recent work (Trabelsi et al., 2018) has shown that complex networks can provide benefits even for real-valued tasks. Applications have ranged from music synthesis (Sarroff and Smith, 2015) to associative memory (Danihelka et al., 2016). Extensions to quaternions (Gaudet and Maida, 2018; Parcollet et al., 2019) have shown promise in specific domains. Our work extends this line by introducing learnable algebraic structures rather than fixed complex arithmetic.

8.2 Quantum-Inspired Algorithms

The success of quantum-inspired classical algorithms (Tang, 2019) demonstrates that quantum principles can enhance classical computation without requiring quantum hardware. Previous work has primarily focused on linear algebra routines (Arrazola et al., 2020). We extend this philosophy to neural network architectures, showing that quantum-inspired principles can enhance deep learning.

8.3 Efficient Transformers

The quest for parameter-efficient Transformers has produced numerous innovations, including sparse attention patterns (Child et al., 2019), low-rank approximations (Choromanski et al., 2021), and linear attention mechanisms (Katharopoulos et al., 2020). Recent work on length extrapolation (Press et al., 2022) has shown that careful design of position encodings can improve generalization. Our approach is orthogonal to these methods, achieving efficiency through enhanced representational capacity rather than architectural modifications. The principle of learning richer representations aligns with broader themes in representation learning (Bengio et al., 2013; LeCun et al., 2015).

9 Conclusion

This work introduces Quantum-Inspired Complex Transformers, demonstrating that fundamental mathematical ambiguities, when resolved through quantum principles, can enhance neural network capabilities. By treating the imaginary unit as a learnable superposition rather than a fixed choice, we create neural architectures that achieve superior parameter efficiency without sacrificing performance.

The empirical results—20.96% parameter reduction with improved accuracy—validate the practical benefits of this approach. More fundamentally, this work suggests that the mathematical foundations of neural networks remain ripe for innovation. As we push the boundaries of model efficiency, exploring alternative algebraic frameworks may prove as fruitful as architectural innovations.

The success of QIC Transformers opens new research directions at the intersection of abstract algebra, quantum information theory, and deep learning. We hope this work inspires further exploration of unconventional mathematical foundations for artificial intelligence.

Appendix A. Mathematical Proofs

A.1 Proof of Matrix Product Relations

We provide the complete verification that $J_+J_- = J_-J_+ = I$.

Proof First, compute J_+J_-:

$$J_+J_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{31}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{32}$$

Next, compute J_-J_+:

$$J_-J_+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{34}$$

Therefore: $J_+J_- = J_-J_+ = I$, showing that these matrices commute and their product is the identity. ■

A.2 Derivation of QIC Multiplication Rule

We derive the complete multiplication rule for QIC numbers, following the principles established for complex-valued neural networks (Nitta, 1997).

Proof Let $z_1 = a_1 + b_1 J(\theta)$ and $z_2 = a_2 + b_2 J(\theta)$. Then:

$$z_1 z_2 = (a_1 + b_1 J)(a_2 + b_2 J) \tag{35}$$
$$= a_1 a_2 + a_1 b_2 J + b_1 a_2 J + b_1 b_2 J^2 \tag{36}$$
$$= a_1 a_2 + (a_1 b_2 + b_1 a_2) J + b_1 b_2 (-1 + \sin(2\theta)) \tag{37}$$
$$= [a_1 a_2 + b_1 b_2 (-1 + \sin(2\theta))] + [a_1 b_2 + b_1 a_2] J \tag{38}$$

■

Appendix B. Implementation Details

B.1 QIC Batch Matrix Multiplication

Algorithm 1 provides the complete implementation of batch matrix multiplication in the QIC algebra.

Algorithm 1 Detailed QIC Batch Matrix Multiplication

Require: $(X_a, X_b) \in \mathbb{R}^{B \times M \times K} \times \mathbb{R}^{B \times M \times K}$
Require: $(Y_a, Y_b) \in \mathbb{R}^{B \times K \times N} \times \mathbb{R}^{B \times K \times N}$
Require: $\theta \in \mathbb{R}$
Ensure: $(Z_a, Z_b) \in \mathbb{R}^{B \times M \times N} \times \mathbb{R}^{B \times M \times N}$
1: $j\_squared \leftarrow -1 + \sin(2\theta)$
2: $Z_a \leftarrow \text{BatchMatMul}(X_a, Y_a)$
3: $Z_a \leftarrow Z_a + j\_squared \times \text{BatchMatMul}(X_b, Y_b)$
4: $Z_b \leftarrow \text{BatchMatMul}(X_a, Y_b)$
5: $Z_b \leftarrow Z_b + \text{BatchMatMul}(X_b, Y_a)$
6: **return** (Z_a, Z_b)

B.2 Memory-Efficient Implementation

For large-scale deployments, memory efficiency is crucial. The QIC representation requires approximately twice the memory of standard representations. However, this can be mitigated through several strategies:

Parameter sharing between real and imaginary components for certain layers can reduce memory overhead while maintaining expressiveness. Quantization techniques can be applied separately to real and imaginary components, potentially achieving better compression than standard quantization. Mixed precision training, using lower precision for imaginary components, can reduce memory and computation requirements.

Appendix C. Extended Experimental Results

C.1 Statistical Significance

We conducted five independent runs with different random seeds to assess statistical significance. Table 5 presents the results.

Table 5: Statistical analysis over 5 independent runs (mean ± standard deviation)

Metric	**Standard Transformer**	**QIC Transformer**
Validation Accuracy	97.68% ± 0.21%	98.47% ± 0.18%
Validation Loss	0.0479 ± 0.0014	0.0365 ± 0.0011
Training Time (s)	45.31 ± 0.82	97.92 ± 1.34
Parameters	21,570 ± 0	17,048 ± 0

A paired t-test confirms that the accuracy improvement is statistically significant ($p < 0.001$), validating the robustness of our results.

C.2 Learning Dynamics Analysis

The learning dynamics reveal interesting differences between standard and QIC Transformers. The QIC model shows more stable optimization trajectories, with lower variance in validation metrics during later epochs. This stability may result from the richer gradient flow enabled by the interaction between real and imaginary components.

C.3 Hyperparameter Sensitivity

We investigated sensitivity to the initial value of θ. Testing initialization values from 0 to $\pi/2$ at intervals of $\pi/8$, we found that performance is relatively robust, with accuracy varying by less than 0.3% across different initializations. The optimal initialization appears to be near $\pi/4$, which corresponds to balanced contribution from J_+ and J_-.

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