Simulation of time-bin remote entanglement generation

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1 Full quantum treatment of the position/motion-dependent decoherence

The interaction between an atom and the photon depends on the position of the atom. For excitation, the phase of the light affects the phase of the atomic excited state wavefunction[1]. For emission, the position of the atom affects the phase of the emitted photon.

This becomes a source of decoherence for time-bin entanglement on atomic systems. Classically, we can understand this since the position of the atom is not necessarily the same during the first and the second excitation/photon emission causing a different and random phase on the photon which averages out the coherence fringes. However, this predicts that there would be no decoherence if the motional state of the atom is stationary (i.e. if the atom is in the ground motional state or any of the Fock states $|n\rangle$). Since the thermal state can be expressed as a probabilistic mixture of pure Fock states, this conclusion cannot possibly be correct.

To handle this correctly, we need to construct the unitary operation that corresponds to the photon generation step. Without the position-dependent phase effect, this step maps the atomic (internal) wavefunction in the following way,

$$|0\rangle \rightarrow |0; \mathrm{ph}\rangle \tag{1}$$
$$|i\rangle \rightarrow |i\rangle \tag{2}$$

where the ph represent a photon being generated and $i \neq 0$ are the internal states of the atom that were not excited. When the effect of the motion is included, we can still use the classical picture to write out the new wavefunction. Instead of a fixed phase when the photon is generated, we acquire a phase from the absorbed and emitted photon,

$$|0, \vec{r}\rangle \rightarrow \mathrm{e}^{\mathrm{i}\Delta\vec{k}\cdot\vec{r}}|0, \vec{r}; \mathrm{ph}\rangle \tag{3}$$
$$|i, \vec{r}\rangle \rightarrow |i, \vec{r}\rangle \tag{4}$$

where $\Delta\vec{k}$ is the difference between the wavevectors of the absorbed and emitted photon. Note that although we've stated the position dependent phase factor as being on the photon, it, being just a phase, can be treated as acting on any part of the wavefunction, including the motional wavefunction of the atom. Viewing this way, it is essentially a recoil on the motion of the atom and it creates entanglement between spin+photon state and the motional state of the atom.

Based on this understanding, the correct (or at least an equivalent way) to view the motion/position dependent decoherence is that,

[1]For a π-pulse, this is a global phase for a two-level system which can be ignored. However the time-bin entanglement scheme necessitate at least a three-level system. In such a system, this phase isn't global anymore and can be experimentally observed by comparing it to the third state

1. During the first photon generation step, the $|0\rangle$ spin state received a recoil from the absorbed/emitted photon while the $|1\rangle$ spin state remains in the original motional state.
2. During the free evolution time between the two excitations, the motion of the atom evolves freely (and differently due to the different initial motional state between the $|0\rangle$ and $|1\rangle$ spin states).
3. During the second photon generation step, the $|1\rangle$ spin state received a recoil from the absorbed/emitted photon while the motion for the $|0\rangle$ spin state remains unchanged.
4. At this state, the $|0\rangle$ motional state underwent recoil-free evolution whereas the $|1\rangle$ motional state underwent free evolution-recoil. If these two motional states are not exactly the same, we've created a unwanted entanglement between the spin and the atom motion which reduces the fidelity of the spin+photon state that we would like to create[2].

Initial density matrix

$$\rho_0 = \frac{(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)}{2}\rho_\mathrm{m} \tag{5}$$

After first kick

$$\rho_1 = (P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1) \tag{6}$$

After time evolution

$$\rho_2 = \mathrm{e}^{-in\omega t}(P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1)\mathrm{e}^{in\omega t} \tag{7}$$

After second kick

$$\begin{aligned}\rho_3 =& (P_0 + P_1\mathcal{D}(\mathrm{i}\eta))\mathrm{e}^{-in\omega t}(P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1)\mathrm{e}^{in\omega t}(P_0 + P_1\mathcal{D}(-\mathrm{i}\eta))\\ =& \big(P_0\mathrm{e}^{-in\omega t}\mathcal{D}(\mathrm{i}\eta) + P_1\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-in\omega t}\big)\rho_0\big(P_0\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{in\omega t} + P_1\mathrm{e}^{in\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)\end{aligned} \tag{8}$$

For a thermal initial state

$$\rho_m = \frac{1}{\pi n_B}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha|\mathrm{e}^{-|\alpha|^2/n_B} \tag{9}$$

where $n_B \equiv \dfrac{1}{\mathrm{e}^{\beta\omega} - 1}$

$$\begin{aligned}\rho_3 =& \int \mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/n_B}}{2\pi n_B}\big(|0\rangle\mathrm{e}^{-in\omega t}\mathcal{D}(\mathrm{i}\eta) + |1\rangle\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-in\omega t}\big)|\alpha\rangle\\ &\langle\alpha|\big(\langle 0|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{in\omega t} + \langle 1|\mathrm{e}^{in\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)\end{aligned} \tag{10}$$

Tracing out the motion

$$\begin{aligned}\rho_{3,\mathrm{s}} =& \int \mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/n_B}}{2\pi n_B}\begin{pmatrix} |0\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{in\omega t}\mathrm{e}^{-in\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle\\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{in\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-in\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle\\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{in\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-in\omega t}|\alpha\rangle\\ +|1\rangle\langle 1|\langle\alpha|\mathrm{e}^{in\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-in\omega t}|\alpha\rangle\end{pmatrix}\\ =& \int \mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/n_B}}{2\pi n_B}\begin{pmatrix} |0\rangle\langle 0| + |1\rangle\langle 1|\\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{in\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-in\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle\\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{in\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-in\omega t}|\alpha\rangle\end{pmatrix}\end{aligned} \tag{11}$$

[2]Much the same way motional closure error reduces the fidelity of a Mølmer-Sørensen gate

$$
\begin{aligned}
&\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle\\
=&\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\mathrm{e}^{-\mathrm{i}\omega t}\alpha\rangle\\
=&\mathrm{e}^{(\mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t})/2}\mathrm{e}^{(-\mathrm{i}\eta\alpha^*-\mathrm{i}\eta\alpha)/2}\langle\alpha+\mathrm{i}\eta|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\rangle\\
=&\exp\left(\frac{1}{2}\left(\begin{array}{c}\mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t}-\mathrm{i}\eta\alpha^*-\mathrm{i}\eta\alpha\\ -|\alpha+\mathrm{i}\eta|^2-\left|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\right|^2+2(\alpha^*-\mathrm{i}\eta)\big(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\big)\end{array}\right)\right)\\
=&\exp\Big(\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t}-\mathrm{i}\eta\alpha^*-|\alpha|^2-\eta^2+(\alpha^*-\mathrm{i}\eta)\big(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\big)\Big)\\
=&\exp\Big(2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+\eta^2\big(\mathrm{e}^{\mathrm{i}\omega t}-1\big)\Big)
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\rho_{3,\mathrm{s}}=&\int\mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/n_B}}{2\pi n_B}\left(\begin{array}{c}|0\rangle\langle0|+|1\rangle\langle1|\\ +|0\rangle\langle1|\exp\big(-2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+\eta^2\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)\\ +|1\rangle\langle0|\exp\big(2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+\eta^2\big(\mathrm{e}^{\mathrm{i}\omega t}-1\big)\big)\end{array}\right)\\
=&\frac{|0\rangle\langle0|+|1\rangle\langle1|}{2}\\
&+|0\rangle\langle1|\int\mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/n_B}}{2\pi n_B}\exp\big(-2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+\eta^2\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+h.c.
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\int\mathrm{d}^2\alpha\mathrm{e}^{-|\alpha|^2/n_B}\exp\big(2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)+\eta^2\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big)\\
=&\mathrm{e}^{\eta^2\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)}\int\mathrm{d}^2\alpha\mathrm{e}^{-|\alpha|^2/n_B}\exp\left(2\mathrm{i}\eta\mathrm{Re}((\alpha_x+\mathrm{i}\alpha_y)(\cos\omega t-1-\mathrm{i}\sin\omega t))\right)\\
=&\mathrm{e}^{\eta^2\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)}\int\mathrm{d}^2\alpha\exp\big(-\alpha_x^2/n_B+2\mathrm{i}\alpha_x\eta(\cos\omega t-1)\big)\exp\big(-\alpha_y^2/n_B+2\mathrm{i}\alpha_y\eta\sin\omega t\big)\\
=&\pi n_B\exp\big(-\eta^2\big(1-\mathrm{e}^{-\mathrm{i}\omega t}+2n_B(1-\cos\omega t)\big)\big)\\
=&\pi n_B\exp\big(-\mathrm{i}\eta^2\sin\omega t\big)\exp\big(-\eta^2(1-\cos\omega t)(2n_B+1)\big)
\end{aligned}
\tag{14}
$$

$$
\rho_{3,\mathrm{s}}=\frac{|0\rangle\langle0|+|1\rangle\langle1|+|0\rangle\langle1|\exp\big(\mathrm{i}\eta^2\sin\omega t\big)\exp\big(-\eta^2(1-\cos\omega t)(2n_B+1)\big)+h.c.}{2}
\tag{15}
$$