Phase of polarization components

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1 Introduction

After selecting a frame, we can always decompose the polarization into the σ^+, σ^- and π components. While the amplitude of this decomposition is relatively easy to figure out, their phases depends on the selection of basis. The basis, and therefore the phase, can be arbitrary as long as it's consistent. This consistency is trivial when only dealing with polarization vectors (as long as the same basis is used) but becomes more important when the light starts to interact with other stuff, e.g. atoms. It is therefore important to ensure that the selection of polarization basis is consistent with the basis for the angular momentum states of the atoms (and practically the phase of the CG-coefficient).

Although the mathematically correct way to do this is probably to start with a consistent definition and derive everything from there, I'd like to derive/check how to get a consistent definition of the phase based only on definitions/formulas that we use, to see how the definition can propagate from one part to another.

2 Derivation

2.1 Angular momentum eigenstate phase

The convention in quantum mechanics is to relate the angular momentum states using the ladder operators,

$$j_\pm = j_x \pm \mathrm{i} j_y \tag{1}$$

and we have

$$j_\pm |jm\rangle = C_\pm |j(m \pm 1)\rangle \tag{2}$$

where $C_\pm$ are non-negative constant factors. This uniquely defines the phase of the angular momentum eigenstates.

2.2 Tensor operator phase

When calculating the value of a tensor operator, we use Wigner-Eckart theorem to convert it to CG coefficients and a reduced matrix element. The derivation for Wigner-Eckart uses the definition/convention for the spherical basis,

$$V^0 = V_z \tag{3}$$

$$V^{\pm 1} = \mp \frac{1}{\sqrt{2}}(V_x \pm \mathrm{i} V_y) \tag{4}$$

2.3 Polarization phase

The light polarization can be expressed in the Cartesian and spherical basis as,

$$\sum_{i=x,y,z} A_i \hat{e}_i = \sum_{i=-1,0,1} A_i \hat{e}^i \tag{5}$$

Similarly, the inner product between the polarization and an operator can be expressed as,

$$\sum_{i=x,y,z} A_i V_i = \sum_{i=-1,0,1} A_i V^i \tag{6}$$

Comparing the two, we can see that the transformation between the spherical and Cartesian polarization basis should be the same as that for the operators, i.e.

$$\hat{e}^{\pi} = \hat{e}_z \tag{7}$$

$$\hat{e}^{\sigma^{\pm}} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm \mathrm{i}\hat{e}_y) \tag{8}$$

that satisfies

$$\vec{A} = \sum_{i=-1,0,1} A_i \hat{e}^i \tag{9}$$

and the conjugate basis,

$$\hat{e}_{\pi} = \hat{e}_z \tag{10}$$

$$\hat{e}_{\sigma^{\pm}} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \mp \mathrm{i}\hat{e}_y) \tag{11}$$

that satisfies,

$$\hat{e}_i \cdot \hat{e}^j = \delta_i^j \tag{12}$$

$$A_i = \hat{e}_i \cdot \vec{A} \tag{13}$$

3 Properties of polarization amplitude and phase

3.1 Amplitude transformation

From the relation between the basis, we can get the relation between the (complex) polarization amplitude in the two basis,

$$A_z = A_{\pi} \tag{14}$$

$$A_x = -\frac{1}{\sqrt{2}}(A_{\sigma^+} - A_{\sigma^-}) \tag{15}$$

$$A_y = -\frac{\mathrm{i}}{\sqrt{2}}(A_{\sigma^+} + A_{\sigma^-}) \tag{16}$$

and

$$A_{\pi} = A_z \tag{17}$$

$$A_{\sigma^{\pm}} = \frac{1}{\sqrt{2}}(\mathrm{i}A_y \mp A_x) \tag{18}$$

3.2 Condition for linear polarization

For linear polarization, we need $\vec{A} \times \vec{A}^* = 0$, or

$$\begin{aligned} A_x A_y^* =& A_x^* A_y \qquad (19)\\ A_y A_z^* =& A_y^* A_z \qquad (20)\\ A_z A_x^* =& A_z^* A_x \qquad (21)\end{aligned}$$

From $A_x A_y^* = A_x^* A_y$

$$\begin{aligned} -(A_{\sigma^+} - A_{\sigma^-})(A_{\sigma^+}^* + A_{\sigma^-}^*) =& (A_{\sigma^+}^* - A_{\sigma^-}^*)(A_{\sigma^+} + A_{\sigma^-}) \qquad (22)\\ |A_{\sigma^+}| =& |A_{\sigma^-}| \qquad (23)\end{aligned}$$

i.e. the $\sigma^\pm$ components must have the same amplitude.

From $A_y A_z^* = A_y^* A_z$ and $A_z A_x^* = A_z^* A_x$,

$$\begin{aligned} A_\pi^*(A_{\sigma^+} + A_{\sigma^-}) =& -A_\pi(A_{\sigma^+}^* + A_{\sigma^-}^*) \qquad (24)\\ A_\pi^*(A_{\sigma^+} - A_{\sigma^-}) =& A_\pi(A_{\sigma^+}^* - A_{\sigma^-}^*) \qquad (25)\\ A_\pi^* A_{\sigma^+} =& -A_\pi A_{\sigma^-}^* \qquad (26)\end{aligned}$$

Let ϕ_π and $\phi_{\sigma^\pm}$ be the phase for the three polarization components, i.e. $A_\pi = |A_\pi|\mathrm{e}^{\mathrm{i}\phi_\pi}$ and $A_{\sigma^\pm} = |A_{\sigma^\pm}|\mathrm{e}^{\mathrm{i}\phi_{\sigma^\pm}}$, also given $|A_{\sigma^+}| = |A_{\sigma^-}| = |A_\sigma|$

$$|A_\pi||A_\sigma|\left(\mathrm{e}^{\mathrm{i}\left(\phi_{\sigma^+} + \phi_{\sigma^-} - 2\phi_\pi\right)} - 1\right) = 0 \qquad (27)$$

i.e. either one of $|A_\pi|$ and $|A_\sigma|$ is zero, or $\phi_\sigma - \phi_\pi = n\pi + \frac{\pi}{2}$, where $\phi_\sigma \equiv \frac{\phi_{\sigma^+} + \phi_{\sigma^-}}{2}$ is the average phase of the σ polarizations. In another word, unless the polarization is purely z or purely in the $x-y$ plane, linear polarization is achieved when the phase between π polarization and the average σ polarizations is off by $\frac{\pi}{2}$ (90°).

3.3 ϕ_{HV}

We can generalize the result above by introducing a phase difference between horizontal ($x-y$ plane) and vertical (z) polarization.

$$\begin{aligned} \vec{A} =& A_\pi \vec{e}^\pi + A_{\sigma^+}\vec{e}^{\sigma^+} + A_{\sigma^-}\vec{e}^{\sigma^-}\\ =& A_\pi \vec{e}_z - \frac{1}{\sqrt{2}}|A_{\sigma^+}|\mathrm{e}^{\mathrm{i}\phi_{\sigma^+}}(\hat{e}_x + \mathrm{i}\hat{e}_y) + \frac{1}{\sqrt{2}}|A_{\sigma^-}|\mathrm{e}^{\mathrm{i}\phi_{\sigma^-}}(\hat{e}_x - \mathrm{i}\hat{e}_y)\end{aligned} \qquad (28)$$

Define $A_H \equiv \frac{|A_{\sigma^+}| + |A_{\sigma^-}|}{\sqrt{2}}$, $A_\Delta \equiv \frac{|A_{\sigma^+}| - |A_{\sigma^-}|}{\sqrt{2}}$, $\theta \equiv \frac{\phi_{\sigma^+} - \phi_{\sigma^-}}{2}$

$$\begin{aligned} \vec{A} =& A_\pi \vec{e}_z - \frac{\mathrm{e}^{\mathrm{i}\phi_\sigma}}{2}\left((A_H + A_\Delta)\mathrm{e}^{\mathrm{i}\theta}(\hat{e}_x + \mathrm{i}\hat{e}_y) - (A_H - A_\Delta)\mathrm{e}^{-\mathrm{i}\theta}(\hat{e}_x - \mathrm{i}\hat{e}_y)\right)\\ =& A_\pi \vec{e}_z - \frac{\mathrm{e}^{\mathrm{i}\phi_\sigma}}{2}\left(A_H\left(\mathrm{e}^{\mathrm{i}\theta}(\hat{e}_x + \mathrm{i}\hat{e}_y) - \mathrm{e}^{-\mathrm{i}\theta}(\hat{e}_x - \mathrm{i}\hat{e}_y)\right) + A_\Delta\left(\mathrm{e}^{\mathrm{i}\theta}(\hat{e}_x + \mathrm{i}\hat{e}_y) + \mathrm{e}^{-\mathrm{i}\theta}(\hat{e}_x - \mathrm{i}\hat{e}_y)\right)\right)\\ =& |A_\pi|\mathrm{e}^{\mathrm{i}\phi_\pi}\vec{e}_z - \mathrm{e}^{\mathrm{i}\phi_\sigma}(\mathrm{i}A_H(\sin\theta\hat{e}_x + \cos\theta\hat{e}_y) + A_\Delta(\cos\theta\hat{e}_x - \sin\theta\hat{e}_y))\\ =& \mathrm{e}^{\mathrm{i}\phi_\pi}\left(|A_\pi|\vec{e}_z + A_H\mathrm{e}^{\mathrm{i}\phi_{HV}}(\sin\theta\hat{e}_x + \cos\theta\hat{e}_y) + A_\Delta\mathrm{e}^{\mathrm{i}(\phi_{HV}+\pi/2)}(\cos\theta\hat{e}_x - \sin\theta\hat{e}_y)\right)\end{aligned} \qquad (29)$$

where $\phi_{HV} \equiv \phi_\sigma - \phi_\pi - \frac{\pi}{2}$.

This describes a polarization with an $x - y$ plane major axis along $\sin\theta\hat{e}_x + \cos\theta\hat{e}_y$ with amplitude A_H and minor axis along $\cos\theta\hat{e}_x - \sin\theta\hat{e}_y$ with amplitude A_Δ. The phase difference between the $x - y$ plane major axis and the z polarization is ϕ_{HV} which, again, is $\frac{\pi}{2}$ (90°) off from the phase difference between average σ polarization and π.