

Ion-photon state fidelity

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April 9, 2025

1 Goal

Entangling two qudits by pair-wise interference of the photons emitted from each qudit states produces entangled qubit states in which the two qubit states are probabilistically distributed between different qudit state pairs. However, the state produced this way, a qubit Bell state, contains less entanglement than the original atom-photon states. While making full use of the atom-photon state may require complex optical setup, it would still be interesting to put a bound on the atom-photon state fidelity based on the pairwise entanglement result.

2 Definition of fidelity

Given an target state $|\psi\rangle$, the fidelity of a density matrix ρ is defined as its overlap with the state,

$$f = \langle \psi | \rho | \psi \rangle \quad (1)$$

The d -dimensional target state for the atom-photon pair is,

$$|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_n |n_A, n_P\rangle \quad (2)$$

where the A and P subscripts are for the atom and photon state. The target state for the pair-wise atom-atom state between the m and n -th qudit is,

$$|\psi_0\rangle_{mn} = \frac{|mn\rangle + |nm\rangle}{\sqrt{2}} \quad (3)$$

where m and n are the two qudit state entangled. However, both of these states contain phase degrees of freedoms that does not affect their utility as long as the phase is known/can be measured. It is therefore more useful to define the fidelity as the maximum overlap between the density matrix and the target state while varying the phase of the target state.

For the d -dimensional atom-photon state, the final fidelity is,

$$\begin{aligned} f^{AP} &= \frac{1}{d} \max_{\{\phi\}} \sum_{m,n} e^{i(\phi_n - \phi_m)} \langle m_A, m_P | \rho | n_A, n_P \rangle \\ &= \frac{1}{d} + \frac{2}{d} \max_{\{\phi\}} \sum_{m < n} \text{Re} \left(e^{i(\phi_n - \phi_m)} \langle m_A, m_P | \rho | n_A, n_P \rangle \right) \end{aligned} \quad (4)$$

For the atom-atom Bell state, the final fidelity is,

$$\begin{aligned} f_{mn}^{AA} &= \frac{1}{2} \max_{\phi} \left(\langle mn | + e^{-i\phi} \langle nm | \right) \rho \left(|mn\rangle + e^{i\phi} |nm\rangle \right) \\ &= \frac{1}{2} + |\langle mn | \rho | nm \rangle| \end{aligned} \quad (5)$$

The second step in the derivation for both assumes that the density matrix is normalized to the subspace span by the states involved. If the probability of being in this subspace is not 1, this probability should be multiplied with the number here to obtain the full fidelity. It is worth noting that in both cases, the fidelity does not depend explicitly on the diagonal elements of the density matrix.

3 Relation between the fidelity/density matrix of the atom-atom and atom-photon state

3.1 Correlation

First we discuss the probability of the system being in the correct Hilbert space, the one where the state of the photon to be perfectly correlated with the atom that emits the photon. If the probability of atom-photon pair $i = 1, 2$ being outside the correlated Hilbert space is ε_C^i , the probability of the ion-ion state being outside the correlated Hilbert space is,

$$\varepsilon'_C \geq \varepsilon_{C1} + \varepsilon_{C2} \quad (6)$$

so we have a bound for the atom-photon correlation $\varepsilon_C^i \leq \varepsilon'_C$. This could potentially also be bound directly by measuring the single chamber atom-photon correlation.

3.2 Fidelity within perfectly correlated subspace

From now on, we can assume both atom-photon pairs are in the perfectly correlated Hilbert space $|n_A, n_P\rangle^i$. With the ion-photon density matrix $\rho^i \equiv \sum_{m,n} |m_A, m_P\rangle^i \rho_{mn}^i \langle n_A, n_P|^i$ for the i -th pair.

The (unnormalized) density matrix after detecting two photons in the n and m time bins is,

$$P_{mn}(\rho^1 \rho^2) = \begin{pmatrix} \rho_{nn}^1 \rho_{mm}^2 & \rho_{nm}^1 \rho_{mn}^2 \\ \rho_{mn}^1 \rho_{nn}^2 & \rho_{mm}^1 \rho_{nn}^2 \end{pmatrix} \quad (7)$$

which gives a rate of

$$r_{mn} = \rho_{nn}^1 \rho_{mm}^2 + \rho_{mm}^1 \rho_{nn}^2 \quad (8)$$

and a fidelity of (eq. 5 extended for unnormalized density matrix)

$$f_{mn} = \frac{1}{2} + \frac{|\rho_{nm}^1 \rho_{mn}^2|}{r_{mn}} \quad (9)$$

The atom-atom entanglement measurement allows both r_{mn} and f_{mn} to be bounded, which in turns provides a bound for the atom-photon state fidelity. In particular, the balance between r_{mn} 's constraint the diagonal elements of the atom-photon density matrix and the atom-atom fidelities f_{mn} constraint ratio of the absolute values of the off-diagonal elements and the diagonal elements. Although the diagonal elements doesn't directly affect the atom-photon fidelity (eq. 4), they do involve in the constraints on the off-diagonal elements (eq 9) and also in the constraints from the positive-semidefiniteness of the density matrix (see below).

The phase of the atom-atom state is also measured. However, the phase is the sum of the phase from the two atom-photon state and therefore is a bit difficult to use as constraints on the phase of the individual atom-photon state.

4 Numerical calculation

I was unable to derive an analytical expression for the atom-photon fidelity f^{AP} and therefore had to use numerical optimization to bound the fidelity.

4.1 Physical constraints on the density matrix

Other than the constraints listed above (mostly on the absolute values of the matrix elements), the density matrix is further constrained since it must be a positive semidefinite matrix. This constraint is important since allowing arbitrary phase on the off-diagonal matrix element of the density matrix could produce non-physical density matrix with very small, or even negative fidelities. Numerically, this constraint can be expressed using the Sylvester's criterion, which stated that the leading principal minors must be non-negative. However, during optimization, it was found that this constraint is not numerically robust, but can be improved by requiring the determinant of all of the sub-density matrixes to be non-negative.

4.2 All constraints

In all, the constraints used for optimization are.

1. Lower and upper bound on the atom-atom entangling rates

$$r_{mn}^{\text{lo}} \leq \rho_{nn}^1 \rho_{mm}^2 + \rho_{mm}^1 \rho_{nn}^2 \leq r_{mn}^{\text{hi}} \quad (10)$$

2. Lower and upper bound on the atom-atom entangling fidelities

$$f_{mn}^{\text{lo}} \leq \frac{1}{2} + \frac{|\rho_{nm}^1 \rho_{mn}^2|}{\rho_{nn}^1 \rho_{mm}^2 + \rho_{mm}^1 \rho_{nn}^2} \leq f_{mn}^{\text{hi}} \quad (11)$$

3. Both of the atom-photon density matrices are positive-semidefinite.

4.3 Implementation notes

The degrees of freedoms of the problem can be slightly reduced by removing redundant phase variables. In eq. 4, we can set the phase for the first state ϕ_1 to 0. This corresponds to a global phase that does not affect the result. Similarly, we can also require the off-diagonal matrix elements for the first row and column of the atom-photon density matrix to be real numbers, this is always possible to achieve by varying the phases on the basis, which does not affect the fidelity.

The optimization for both problems (computing the maximum fidelity given a qudit density matrix and computing the minimum atom-photon fidelity given constraints from atom-atom rate and fidelity) are implemented with gradient based local optimizations¹. For finding the atom-photon fidelity bound given the atom-atom result constraints, I find that it is more reliable to do multiple rounds of optimizations and use random sampling to find an off-diagonal phase that could minimize the fidelities. This seems to help the optimizer to get out of any local minima. OTOH, the gradient based optimizer seems to have less trouble finding the distribution of diagonal matrix elements that minimizes the fidelity.

¹I settled on *NLOPT_LD_TNEWTON_PRECOND_RESTART* for finding the max fidelity and *LD_SLSQP* for finding the minimum fidelity bound

5 Short version

The atom-atom entanglement measurements done in the experiment only contains information about the coherence between each pair of the qudit states. This is therefore not a direct measurement of the fidelity of the atom-photon entangled state, which is higher dimensional than the atom-atom entangled states. Fortunately, under the assumption that the preparation of the two atom-photon states are independent, we can express each of the atom-photon state as a density matrix. By expressing the atom-atom entanglement fidelity as a function of these density matrices, we can use the experimental result to constraint the values of matrix elements as well as the atom-photon state fidelity.

The experimental data provides three different types of constraints on the atom-photon density matrix.

1. The population data measures the correlation between the state of the atom and the photon it emits. This bounds the probability for the atom-photon to be in the perfectly-correlated subspace. Since the desired atom-photon state is within this subspace, this probability is directly proportional to the atom-photon fidelity.
2. The rate for a particular pair of atom-atom states to be entangled is proportional to the product of the diagonal density matrix elements from the two atom-photon pairs. The balance between the rates for each atom-atom state pair therefore provides a bound for the uniformity of the atom-photon state distribution.
3. The fidelity of an atom-atom entangled states is proportional to the product of the off-diagonal density matrix elements from the two atom-photon pairs. This therefore measures the coherence of the atom-photon state. Note that since the phase of the atom-atom state is the sum of that for the two atom-photon states, the measurement on the atom-atom state does not provide a good phase measure for the atom-photon state. This bound is therefore only on the amplitude of the off-diagonal matrix element while leaving their phase free to change.

We combine these constraints from experimental results with the requirement for a physical density matrix (that they have to be positive-semidefinite) to compute a lower bound for the atom-photon state. This is done numerically and the result can be seen in table 1. The fidelity shown is the minimum fidelity at least one of the atom-photon state must have in order to reproduce the atom-atom entanglement result.

d	f_{AP}
$d = 4$	91.3 %
$d = 3$	93.1 %
$d = 2$	95.9 %

Table 1: Lower bound on the atom-photon fidelity different dimensionalities $d = \{2, 3, 4\}$.