Light shift and effective B field

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1 Goal

Derive and clarify some effects related to vector and tensor light shifts as well as a few different places they may appear in an experiment. Most, if not all of the discussion will be limited to E1 transitions. I'm not really looking for the most mathematically straight forward derivation, rather trying to see this from different angle for better understanding.

2 Some useful formulas

2.1 Spherical component of vector

Similar to the decomposition of light polarization into $\sigma^{\pm}$ and π, every 3D vector (operator) can be equivalently expressed as a rank-1 spherical tensor,

$$\begin{aligned} V_0 =& V_z && (1) \\ V_{\pm 1} =& \mp \frac{1}{\sqrt{2}}(V_x \pm \mathrm{i} V_y) && (2) \end{aligned}$$

In particular, when applied to the angular momentum operator,

$$\begin{aligned} J_0 =& J_z && (3) \\ J_{\pm 1} =& \mp \frac{1}{\sqrt{2}}(J_x \pm \mathrm{i} J_y) \\ =& \mp \frac{J_{\pm}}{\sqrt{2}} && (4) \end{aligned}$$

where $J_{\pm}$ are the angular momentum raising and lowering operators.

2.2 Wigner-Eckart theorem

This describes the relation between matrix elements of a vector/tensor operator in the angular momentum basis. The matrix element for different angular momentum states are related to each other with Clebsch-Gordan coefficients.

$$\langle j, m | T_q^{(k)} | j', m' \rangle = \langle j', k; m', q | j, m \rangle \langle j || T^{(k)} || j' \rangle \qquad (5)$$

where $T_q^{(k)}$ is the q-th component of the spherical tensor operator $T^{(k)}$ of rank k. This is the result of rotation symmetry between all the matrix elements.

Equivalently, this also means that no matter what the tensor operator is, it's matrix elements in this (between these) subspace differs from that of a different tensor operator only by a constant factor. (Note that this factor could depend on the j and j' (just not m and m') and it can of course be 0 as well), i.e.

$$\langle j,m|T_{1q}^{(k)}|j',m'\rangle \propto \langle j,m|T_{2q}^{(k)}|j',m'\rangle \tag{6}$$

2.3 When $j = j'$

A special case for the Wigner-Echart theorem is when $j = j'$. In this case we can plug in the angular momentum operator J (this would otherwise result in vanishing matrix elements if $j \neq j'$ since J conserves, well, j).

$$\begin{aligned}\langle j,m|J_q|j,m'\rangle =&\langle j,1;m',q|j,m\rangle\langle j||J||j\rangle \\ \propto&\langle j,1;m',q|j,m\rangle\end{aligned} \tag{7}$$

This allow us to replace the CG coefficents with the angular momentum operator, i.e.,

$$\langle j,m|V_q|j,m'\rangle \propto \langle j,m|J_q|j,m'\rangle \tag{8}$$

which could make some calculation/expression significantly simpler.

This relation basically states that within the subspace of a single j, we can treat any vector operator as proportional to the angular momentum. The proportionality factor can then be obtained from the dot product with angular momentum, i.e. the projection of the vector onto angular momentum.

2.3.1 $m = 0$ selection rule

The selection rule for $m = m' = 0$ transition directly follows from this relation since,

$$\begin{aligned}\langle j,m|V_0|j,m\rangle \propto&\langle j,m|J_0|j,m\rangle \\ =&\langle j,m|J_z|j,m\rangle \\ =&m\end{aligned} \tag{9}$$

which is 0 when $m = 0$.

2.3.2 Projection theorem

We can use this to derive the projection theorem. Explicitly writing down the proportionality factor in Eq. 8, we have,

$$\langle j,m|V_q|j,m'\rangle =c\langle j,m|J_q|j,m'\rangle \tag{10}$$

Multiply both sides with the angular momentum matrix element and sum over all m' and q

$$\sum_{m',q}\langle j,m|V_q|j,m'\rangle\langle j,m'|J_q^\dagger|j,m''\rangle =c\sum_{m',q}\langle j,m|J_q|j,m'\rangle\langle j,m'|J_q^\dagger|j,m''\rangle \tag{11}$$

$$\sum_q\langle j,m|V_qV_q^\dagger|j,m''\rangle =c\sum_q\langle j,m|J_qJ_q^\dagger|j,m''\rangle \tag{12}$$

$$\begin{aligned}\langle j,m|(\vec{V}\cdot\vec{J})|j,m''\rangle =&c\langle j,m|J^2|j,m''\rangle \\ =&cj(j+1)\end{aligned} \tag{13}$$

Therefore we have

$$c =\frac{\langle j,m|(\vec{V}\cdot\vec{J})|j,m''\rangle}{j(j+1)} \tag{14}$$

$$\langle j,m|V_q|j,m'\rangle =\frac{\langle j,m|(\vec{V}\cdot\vec{J})|j,m''\rangle}{j(j+1)}\langle j,m|J_q|j,m'\rangle \tag{15}$$

2.3.3 Explicit calculation

Just for completeness, we can verify this relation between angular momentum and CG coefficients explicitly. This part can be ignored without affecting the understanding of the rest.
First the expression using angular momentum operators,

$$\begin{aligned}\langle j,m|J_0|j,m'\rangle =&\langle j,m|m'|j,m'\rangle \\ =&m'\delta_{mm'}\end{aligned} \tag{16}$$

$$\begin{aligned}\langle j,m|J_{\pm 1}|j,m'\rangle =& \mp\frac{1}{\sqrt{2}}\langle j,m|J_\pm|j,m'\rangle \\ =& \mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\langle j,m|j,m'\pm 1\rangle \\ =& \mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\delta_{m,m'\pm 1}\end{aligned} \tag{17}$$

Using the explicit formula for the CG coefficients,

$$\begin{aligned}\langle j,1;m',q|j,m\rangle =&\delta_{m,m'+q}\sqrt{\frac{(2j+1)(j+j-1)!(j-j+1)!(j+1-j)!}{(j+1+j+1)!}} \\ &\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!} \\ &\sum_k\frac{(-1)^k}{k!(j+1-j-k)!(j-m'-k)!(1+q-k)!(j-1+m+k)!(j-j-q+k)!} \\ =&\delta_{m,m'+q}\frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!}}{2\sqrt{(j+1)j}} \\ &\sum_k\frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+q-k)!(j-1+m'+k)!(-q+k)!}\end{aligned} \tag{18}$$

For $q=0$

$$\begin{aligned}\langle j,1;m',0|j,m\rangle =&\delta_{mm'}\frac{\sqrt{(j+m)!(j-m)!(j-m)!(j+m)!}}{2\sqrt{(j+1)j}} \\ &\sum_{k=0,1}\frac{(-1)^k}{k!(1-k)!(j-m-k)!(1-k)!(j-1+m+k)!k!} \\ =&\delta_{mm'}\frac{(j-m)!(j+m)!}{2\sqrt{(j+1)j}}\left(\frac{1}{(j-m)!(j-1+m)!}-\frac{1}{(j-m-1)!(j+m)!}\right) \\ =&m\frac{\delta_{mm'}}{\sqrt{j(j+1)}}\end{aligned} \tag{19}$$

For $q = \pm 1$

$$\begin{aligned}
\langle j, 1; m', \pm 1 | j, m\rangle =& \delta_{m,m'\pm 1} \frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1\mp 1)!(1\pm 1)!}}{2\sqrt{(j+1)j}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m+k)!(\mp 1+k)!} \\
=& \frac{\delta_{m,m'\pm 1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'\pm 1)!(j-m'\mp 1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m'+k)!(\mp 1+k)!}
\end{aligned} \tag{20}$$

For $q = 1$

$$\begin{aligned}
\langle j, 1; m', 1 | j, m\rangle =& \frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+1-k)!(j-1+m'+k)!(-1+k)!} \\
=& -\frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2(j-m'-1)!(j-m'-1)!(j+m')!(j+m')!}} \\
=& -\sqrt{\frac{(j+m'+1)(j-m')}{2}} \frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}}
\end{aligned} \tag{21}$$

For $q = -1$

$$\begin{aligned}
\langle j, 1; m', -1 | j, m\rangle =& \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2}} \\
& \sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(-k)!(j-1+m'+k)!(1+k)!} \\
=& \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}} \sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2(j-m')!(j-m')!(j+m'-1)!(j+m'-1)!}} \\
=& \sqrt{\frac{(j-m'+1)(j+m')}{2}} \frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}}
\end{aligned} \tag{22}$$

Comparing the result from the two methods, we can see that the proportionality factor is $\sqrt{(j+1)j}$, or

$$\langle j, m | J_q | j, m'\rangle = \sqrt{(j+1)j} \langle j, 1; m', q | j, m\rangle \tag{23}$$

3 Vector and tensor light shift

3.1 Direct derivation

The coupling between state $|F, m_F\rangle$ and $|F', m'_F\rangle$

$$\langle F, m_F | d_q | F', m'_F\rangle = \langle F || \mathbf{d} || F'\rangle \langle F, m_F | F', 1; m'_F, q\rangle \tag{24}$$

$$= \langle F || \mathbf{d} || F'\rangle (-1)^{F'-1+m_F} \sqrt{2F+1} \begin{pmatrix} F' & 1 & F \\ m'_F & q & m_F \end{pmatrix} \tag{25}$$

where F and m_F (F' and m'_F) are the total angular momentum and its projection for the initial (final) state. d is the dipole operator and q is the label for the spherical harmonic component (-1, 0, or 1). $q = \pm 1$ corresponds to the $\sigma^\pm$ polarization/transition and $q = 0$ corresponds to the π polarization/transition.

From this, we can first calculate the stark shift for a pure (σ^+, π or σ^-) polarization. We only care about a single polarization and a single pair of state for each shift so there's no summation. We'll also ignore the detuning and all of the other pre-factors since we only care about the ratio between different states/polaritions.

$$\Delta E \propto \langle F, m_F | d_q | F', m'_F \rangle \langle F', m'_F | d_q | F, m_F \rangle \tag{26}$$

$$= \langle F || \mathbf{d} || F' \rangle^2 \left| \langle F, m_F | F', 1; m'_F, q \rangle \right|^2 \tag{27}$$

$$= \langle F || \mathbf{d} || F' \rangle^2 (2F+1) \begin{pmatrix} F' & 1 & F \\ m'_F & q & m_F \end{pmatrix}^2 \tag{28}$$

Amplitude of the light A_q

$$\langle F, m_F | H | F', m'_F \rangle \tag{29}$$

$$\propto \sum_{F'', m''_F, q, q'} \langle F, m_F | A_q d_q | F'', m''_F \rangle \langle F'', m''_F | A^*_{q'} d_{q'} | F', m'_F \rangle \tag{30}$$

$$\propto \sum_{F'', m''_F, q, q'} A_q A^*_{q'} \langle F, m_F | F'', 1; m''_F, q \rangle \langle F'', 1; m''_F, q' | F', m'_F \rangle \tag{31}$$

$$= \sum_{k,p} T^k_p \sum_{F'', m''_F, q, q'} \langle k, p | 1, 1; q, q' \rangle \langle F, m_F | F'', 1; m''_F, q \rangle \langle F'', 1; m''_F, q' | F', m'_F \rangle \tag{32}$$

$$= T^0_0 \sum_{F'', m''_F, q, q'} \langle 0, 0 | 1, 1; q, q' \rangle \langle F, m_F | F'', 1; m''_F, q \rangle \langle F'', 1; m''_F, q' | F', m'_F \rangle \tag{33}$$

$$+ \sum_p T^1_p \sum_{F'', m''_F, q, q'} \langle 1, p | 1, 1; q, q' \rangle \langle F, m_F | F'', 1; m''_F, q \rangle \langle F'', 1; m''_F, q' | F', m'_F \rangle \tag{34}$$

$$+ \sum_p T^2_p \sum_{F'', m''_F, q, q'} \langle 2, p | 1, 1; q, q' \rangle \langle F, m_F | F'', 1; m''_F, q \rangle \langle F'', 1; m''_F, q' | F', m'_F \rangle \tag{35}$$

$$\langle F, m_F | V_q | F', m'_F \rangle = \langle F || V || F' \rangle \langle F, 1; m_F, q | F', m'_F \rangle \tag{36}$$

$$\langle F, m_F | F_q | F, m'_F \rangle = \langle F, m_F | F_q | F, m'_F \rangle \tag{37}$$

4 Vector light shift as effective magnetic field

5 Mitigating the effect of transverse circular polarization in optical tweezers