Adiabatic trap lowering

September 20, 2017

Try to compute the lowering trajectory that has the minimum probability of changing the motional state of the atom in the trap.
Let the time varying trapping frequency be ω_t. The wave function is $|\psi_t\rangle$. The time dependent Hamiltonian

$$H_t = \frac{p^2}{2m} + \frac{m\omega_t^2 x^2}{2}$$

which have (time dependent) eigen vectors and eigen values

$$\begin{aligned}H_t|\phi_{nt}\rangle =& E_{nt}|\phi_{nt}\rangle\\ =& \hbar\omega_t\left(n+\frac{1}{2}\right)|\phi_{nt}\rangle\end{aligned}$$

We can expand the wavefunction in this basis

$$|\psi_t\rangle = \sum_n c_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle$$

where the phase

$$\begin{aligned}\varphi_{nt} =& \int_0^t \frac{E_{nt'}\mathrm{d}t'}{\hbar}\\ =& \left(n+\frac{1}{2}\right)\int_0^t \omega_{t'}\mathrm{d}t'\\ =& \left(n+\frac{1}{2}\right)\theta_t\end{aligned}$$

Subtitute these into the Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle = H_t|\psi_t\rangle$$

We have

$$\begin{aligned}0 =& \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle - H_t|\psi_t\rangle\\ =& \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\sum_n c_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle - H_t\sum_n c_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle\\ =& \mathrm{i}\hbar\sum_n\frac{\mathrm{d}c_{nt}}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle + \mathrm{i}\hbar\sum_n c_{nt}\frac{\mathrm{d}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}}{\mathrm{d}t}|\phi_{nt}\rangle + \mathrm{i}\hbar\sum_n c_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}\frac{\mathrm{d}|\phi_{nt}\rangle}{\mathrm{d}t} - \sum_n c_{nt}E_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle\\ =& \mathrm{i}\hbar\sum_n\frac{\mathrm{d}c_{nt}}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}|\phi_{nt}\rangle + \mathrm{i}\hbar\sum_n c_{nt}\mathrm{e}^{-\mathrm{i}\varphi_{nt}}\frac{\mathrm{d}|\phi_{nt}\rangle}{\mathrm{d}t}\end{aligned}$$

Left multiply by $\frac{\langle \phi_{lt}|}{\mathrm{i}\hbar}$

$$
\begin{aligned}
0 =& \langle \phi_{lt}| \sum_n \frac{\mathrm{d}c_{nt}}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}\varphi_{nt}} |\phi_{nt}\rangle + \langle \phi_{lt}| \sum_n c_{nt} \mathrm{e}^{-\mathrm{i}\varphi_{nt}} \frac{\mathrm{d}|\phi_{nt}\rangle}{\mathrm{d}t} \\
\frac{\mathrm{d}c_{lt}}{\mathrm{d}t} =& -\langle \phi_{lt}| \sum_n c_{nt} \mathrm{e}^{\mathrm{i}(l-n)\theta_t} \frac{\mathrm{d}|\phi_{nt}\rangle}{\mathrm{d}t}
\end{aligned}
$$

The wavefunction of a harmonic oscillator

$$
\phi_n(\omega, x) = \sqrt[4]{\omega} f_n(\sqrt{\omega} x)
$$

where f_n is independent of ω. Therefore

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega} \phi_n(\omega, x) =& \frac{\mathrm{d}}{\mathrm{d}\omega} \sqrt[4]{\omega} f_n(\sqrt{\omega} x) \\
=& \frac{\mathrm{d}\sqrt[4]{\omega}}{\mathrm{d}\omega} f_n(\sqrt{\omega} x) + \sqrt[4]{\omega} \frac{\mathrm{d}}{\mathrm{d}\omega} f_n(\sqrt{\omega} x) \\
=& \frac{\omega^{-3/4}}{4} f_n(\sqrt{\omega} x) + \sqrt[4]{\omega} \frac{\mathrm{d}\sqrt{\omega}}{\mathrm{d}\omega} \frac{\mathrm{d}}{\mathrm{d}\sqrt{\omega}} f_n(\sqrt{\omega} x) \\
=& \frac{\omega^{-3/4}}{4} f_n(\sqrt{\omega} x) + \frac{\omega^{-1/4}}{2} \frac{x}{\sqrt{\omega}} \frac{\mathrm{d}}{\mathrm{d}x} f_n(\sqrt{\omega} x) \\
=& \frac{\omega^{-3/4}}{4} \left(1 + \frac{2\mathrm{i}xp}{\hbar}\right) f_n(\sqrt{\omega} x) \\
=& \frac{1}{4\omega} \left(1 + \frac{2\mathrm{i}xp}{\hbar}\right) \phi_n(\omega, x)
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega} |\phi_{nt}\rangle =& \frac{1}{4\omega} \left(1 + \frac{2\mathrm{i}xp}{\hbar}\right) |\phi_{nt}\rangle \\
=& \frac{1}{4\omega} \left(1 + \frac{2\mathrm{i}}{\hbar} \mathrm{i} \frac{\hbar}{2} \left(a^\dagger + a\right)\left(a^\dagger - a\right)\right) |\phi_{nt}\rangle \\
=& \frac{1}{4\omega} \left(1 - \left(a^\dagger + a\right)\left(a^\dagger - a\right)\right) |\phi_{nt}\rangle \\
=& \frac{1}{4\omega} \left(a^2 + a^\dagger a - a a^\dagger + 1 - a^{\dagger 2}\right) |\phi_{nt}\rangle \\
=& \frac{1}{4\omega} \left(a^2 - a^{\dagger 2}\right) |\phi_{nt}\rangle \\
=& \frac{1}{4\omega} \left(\sqrt{n(n-1)} |\phi_{(n-2)t}\rangle - \sqrt{(n+1)(n+2)} |\phi_{(n+2)t}\rangle\right)
\end{aligned}
$$

Substitute into the equation for c_{lt}

$$
\begin{aligned}
\frac{\mathrm{d}c_{lt}}{\mathrm{d}t} =& -\langle \phi_{lt}| \sum_n c_{nt} \mathrm{e}^{\mathrm{i}(l-n)\theta_t} \frac{\mathrm{d}\ln \omega_t}{\mathrm{d}t} \frac{1}{4} \left(\sqrt{n(n-1)} |\phi_{(n-2)t}\rangle - \sqrt{(n+1)(n+2)} |\phi_{(n+2)t}\rangle\right) \\
=& -c_{(l+2)t} \mathrm{e}^{-2\mathrm{i}\theta_t} \frac{\mathrm{d}\ln \omega_t}{\mathrm{d}t} \frac{1}{4} \sqrt{(l+2)(l+1)} + c_{(l-2)t} \mathrm{e}^{2\mathrm{i}\theta_t} \frac{\mathrm{d}\ln \omega_t}{\mathrm{d}t} \frac{1}{4} \sqrt{(l-1)l} \\
=& \frac{1}{4} \frac{\mathrm{d}\ln \omega_t}{\mathrm{d}t} \left(\sqrt{(l-1)l} \mathrm{e}^{2\mathrm{i}\theta_t} c_{(l-2)t} - \sqrt{(l+2)(l+1)} \mathrm{e}^{-2\mathrm{i}\theta_t} c_{(l+2)t}\right)
\end{aligned}
$$