Magnus expansion with linearly changing Hamiltonian

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With a Hamiltonian

$$H(t) = H_0 + H_1 t \tag{1}$$

The commutators for the the leading order of Magnus expansion,

$$\begin{aligned}
[H(t_1), H(t_2)] =& [H_0 + H_1 t_1, H_0 + H_1 t_2] \\
=& [H_0, H_1] t_2 + [H_1, H_0] t_1 \\
=& [H_0, H_1](t_2 - t_1)
\end{aligned} \tag{2}$$

$$\begin{aligned}
[[H(t_1), H(t_2)], H(t_3)] =& [[H_0, H_1](t_2 - t_1), H_0 + H_1 t_3] \\
=& ([[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3)(t_2 - t_1)
\end{aligned} \tag{3}$$

$$\begin{aligned}
&[[[H(t_1), H(t_2)], H(t_3)], H(t_4)] \\
=& [([[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3), H_0 + H_1 t_4](t_2 - t_1) \\
=& [[[H_0, H_1], H_0], H_0](t_2 - t_1) + [[[H_0, H_1], H_0], H_1](t_2 - t_1) t_4 \\
&+ [[[H_0, H_1], H_1], H_0] t_3 (t_2 - t_1) + [[[H_0, H_1], H_1], H_1] t_3 t_4 (t_2 - t_1)
\end{aligned} \tag{4}$$

The terms in the time integral,

$$\begin{aligned}
&[H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \\
=& [[H_0, H_1], H_0](t_2 - t_3) + [[H_0, H_1], H_1] t_1 (t_2 - t_3) \\
&+ [[H_0, H_1], H_0](t_2 - t_1) + [[H_0, H_1], H_1] t_3 (t_2 - t_1) \\
=& [[H_0, H_1], H_0](2t_2 - t_1 - t_3) + [[H_0, H_1], H_1](t_1 t_2 + t_2 t_3 - 2 t_1 t_3)
\end{aligned} \tag{5}$$

$$\begin{aligned}
&[[[H(t_1), H(t_2)], H(t_3)], H(t_4)] + [[[H(t_3), H(t_2)], H(t_4)], H(t_1)] \\
&+ [[[H(t_3), H(t_4)], H(t_2)], H(t_1)] + [[[H(t_4), H(t_1)], H(t_3)], H(t_2)] \\
=& [[[H_0, H_1], H_0], H_0](t_2 - t_1 + t_2 - t_3 + t_4 - t_3 + t_1 - t_4) \\
&+ [[[H_0, H_1], H_0], H_1](t_2 t_4 - t_1 t_4 + t_2 t_1 - t_3 t_1 + t_4 t_1 - t_3 t_1 + t_1 t_2 - t_4 t_2) \\
&+ [[[H_0, H_1], H_1], H_0](t_2 t_3 - t_1 t_3 + t_2 t_4 - t_3 t_4 + t_4 t_2 - t_3 t_2 + t_1 t_3 - t_4 t_3) \\
&+ [[[H_0, H_1], H_1], H_1](t_2 t_3 t_4 - t_1 t_3 t_4 + t_2 t_1 t_4 - t_3 t_1 t_4 + t_4 t_1 t_2 - t_3 t_1 t_2 + t_1 t_2 t_3 - t_4 t_2 t_3) \\
=& [[[H_0, H_1], H_0], H_0] 2(t_2 - t_3) + [[[H_0, H_1], H_0], H_1] 2 t_1 (t_2 - t_3) \\
&+ [[[H_0, H_1], H_1], H_0] 2 t_4 (t_2 - t_3) + [[[H_0, H_1], H_1], H_1] 2 t_1 t_4 (t_2 - t_3)
\end{aligned} \tag{6}$$

Integrals,

$$
\begin{aligned}
\Omega_1 =& \int_{\tau_1}^{\tau_2} H(t_1)\mathrm{d}t_1 \\
=& \int_{\tau_1}^{\tau_2} H_0 + H_1 t_1 \mathrm{d}t_1 \\
=& (\tau_2 - \tau_1)\left(H_0 + \frac{\tau_2 + \tau_1}{2} H_1\right)
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\Omega_2 =& \frac{1}{2}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 \int_{\tau_1}^{t_1} \mathrm{d}t_2 [H(t_1), H(t_2)] \\
=& [H_0, H_1]\frac{1}{2}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 \int_{\tau_1}^{t_1} \mathrm{d}t_2 (t_2 - t_1) \\
=& [H_0, H_1]\frac{1}{4}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 (t_1 - \tau_1)^2 \\
=& -\frac{(\tau_2 - \tau_1)^3}{12}[H_0, H_1]
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\Omega_3 =& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 [H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \\
=& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 [[H_0, H_1], H_0](2t_2 - t_1 - t_3) + [[H_0, H_1], H_1](t_1 t_2 + t_2 t_3 - 2t_1 t_3) \\
=& \frac{(\tau_2 - \tau_1)^5}{240}[[H_0, H_1], H_1]
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\Omega_4 =& \frac{1}{12}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 \int_{\tau_1}^{t_3}\mathrm{d}t_4 [[[H(t_1), H(t_2)], H(t_3)], H(t_4)] + [[[H(t_3), H(t_2)], H(t_4)], H(t_1)] \\
&+ [[[H(t_3), H(t_4)], H(t_2)], H(t_1)] + [[[H(t_4), H(t_1)], H(t_3)], H(t_2)] \\
=& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 \int_{\tau_1}^{t_3}\mathrm{d}t_4 [[[H_0, H_1], H_0], H_0](t_2 - t_3) + [[[H_0, H_1], H_0], H_1] t_1 (t_2 - t_3) \\
&+ [[[H_0, H_1], H_1], H_0] t_4 (t_2 - t_3) + [[[H_0, H_1], H_1], H_1] t_1 t_4 (t_2 - t_3) \\
=& \frac{(\tau_2 - \tau_1)^5}{720}\Bigg([[[H_0, H_1], H_0], H_0] + \frac{\tau_1 + 5\tau_2}{6}[[[H_0, H_1], H_0], H_1] \\
&+ \frac{5\tau_1 + \tau_2}{6}[[[H_0, H_1], H_1], H_0] + \frac{\tau_1^2 + 5\tau_1\tau_2 + \tau_2^2}{7}[[[H_0, H_1], H_1], H_1]\Bigg)
\end{aligned} \tag{10}
$$