1 Simplifies Optical Bloch Equation for Sideband Cooling Simulation.

Rabi frequency between state m and n (assume to be real since the phase is not important for sidband cooling.): Ω_{mn}
Pumping rate from state n to m: Γ_{mn}

Diagonal terms,

$$\frac{\mathrm{d}\rho_{mm}}{\mathrm{d}t} = -\rho_{mm}\sum_k \Gamma_{km} + \sum_k \rho_{kk}\Gamma_{mk} + \mathrm{i}\sum_k \left(\rho_{mk}\Omega_{km} - \Omega_{mk}\rho_{km}\right)$$

Off-diagnal terms,

$$\frac{\mathrm{d}\rho_{mn}}{\mathrm{d}t} = -\frac{\rho_{mn}}{2}\sum_k \left(\Gamma_{km} + \Gamma_{kn}\right) + \mathrm{i}\sum_k \left(\rho_{mk}\Omega_{kn} - \Omega_{mk}\rho_{kn}\right)$$

When only one sideband is driven,

$$\Omega_{mn} = \Omega_m\delta_{m,n-\Delta} + \Omega_n\delta_{n,m-\Delta}$$

where $Delta$ include both the change in vibrational level and internal level.
Define $p_n = \rho_{nn}$, the equations becomes,

$$\frac{\mathrm{d}p_m}{\mathrm{d}t} = \sum_k \left(p_k\Gamma_{mk} - p_m\Gamma_{km}\right) + \mathrm{i}(\rho_{m,m-\Delta}\Omega_{m-\Delta} - \Omega_m\rho_{m+\Delta,m} + \rho_{m,m+\Delta}\Omega_m - \Omega_{m-\Delta}\rho_{m-\Delta,m})$$

$$\frac{\mathrm{d}\rho_{m,m+\Delta}}{\mathrm{d}t} = -\frac{\rho_{m,m+\Delta}}{2}\sum_k \left(\Gamma_{km} + \Gamma_{k,m+\Delta}\right) + \mathrm{i}\Omega_m(p_m - p_{m+\Delta})$$

ρ_{mn}'s with $|m-n| \neq 0, \Delta$ are ignored since they are 0. In particular, since Δ includes change of internal levels, elements with $|m-n| \geqslant 2\Delta$ does not exist.

Define $q_n = \mathrm{i}\rho_{n,n+\Delta}$,

$$\rho_{n,n+\Delta} = -\mathrm{i}q_n$$
$$\rho_{n+\Delta,n} = \mathrm{i}q_n^*$$

$$\frac{\mathrm{d}p_m}{\mathrm{d}t} = \sum_k \left(p_k\Gamma_{mk} - p_m\Gamma_{km}\right) + \mathrm{i}\left(\mathrm{i}q_{m-\Delta}^*\Omega_{m-\Delta} - \mathrm{i}\Omega_m q_m^* - \mathrm{i}q_m\Omega_m + \mathrm{i}\Omega_{m-\Delta}q_{m-\Delta}\right)$$

$$-\mathrm{i}\frac{\mathrm{d}q_m}{\mathrm{d}t} = \mathrm{i}\frac{q_m}{2}\sum_k \left(\Gamma_{km} + \Gamma_{k,m+\Delta}\right) + \mathrm{i}\Omega_m(p_m - p_{m+\Delta})$$

$$\frac{\mathrm{d}p_m}{\mathrm{d}t} = \sum_k \left(p_k\Gamma_{mk} - p_m\Gamma_{km}\right) + \Omega_m(q_m^* + q_m) - \Omega_{m-\Delta}\left(q_{m-\Delta}^* + q_{m-\Delta}\right)$$

$$\frac{\mathrm{d}q_m}{\mathrm{d}t} = -\frac{q_m}{2}\sum_k \left(\Gamma_{km} + \Gamma_{k,m+\Delta}\right) + \Omega_m(p_{m+\Delta} - p_m)$$

For a process starting with $q_m = 0$, q_m will always remain real.

$$\frac{\mathrm{d}p_m}{\mathrm{d}t} = \sum_k p_k\Gamma_{mk} - p_m\Gamma_m + 2\Omega_m q_m - 2\Omega_{m-\Delta}q_{m-\Delta}$$

$$\frac{\mathrm{d}q_m}{\mathrm{d}t} = -\frac{q_m}{2}(\Gamma_m + \Gamma_{m+\Delta}) + \Omega_m(p_{m+\Delta} - p_m)$$

where $\Gamma_m \equiv \sum_k \Gamma_{km}$ is the decay rate of state m.

After writing the two internal states (a and b) explicitly and adding a third state (c) to take into account m_F pumping,

$$\begin{aligned}
\frac{\mathrm{d}p_m^a}{\mathrm{d}t} &= \sum_{\alpha=a,b,c;k} p_k^\alpha \Gamma_{mk}^{a\alpha} - p_m^a \Gamma_m^a + 2\Omega_m q_m \\
\frac{\mathrm{d}p_m^b}{\mathrm{d}t} &= \sum_{\alpha=a,b,c;k} p_k^\alpha \Gamma_{mk}^{b\alpha} - p_m^b \Gamma_m^b - 2\Omega_{m-\delta} q_{m-\delta} \\
\frac{\mathrm{d}p_m^c}{\mathrm{d}t} &= \sum_{\alpha=a,b,c;k} p_k^\alpha \Gamma_{mk}^{c\alpha} - p_m^c \Gamma_m^c \\
\frac{\mathrm{d}q_m}{\mathrm{d}t} &= -\frac{q_m}{2}\left(\Gamma_m^a + \Gamma_{m+\delta}^b\right) + \Omega_m\left(p_{m+\delta}^b - p_m^a\right)
\end{aligned}$$

For sodium, states a, b and c corresponds to $|F=1, m_F=1\rangle$, $|F=2, m_F=2\rangle$ and $|F=2, m_F=1\rangle$.

When the trapping frequency is much smaller than the line width, Γ_m^α is independent with m and is proportional to the corresponding pumping power. $\Gamma_{mn}^{\alpha\beta}$ can be written as,

$$\begin{aligned}
\Gamma_{mn}^{\alpha\beta} =& \Gamma^{\alpha\beta}\gamma_{mn} \\
=& \begin{pmatrix} \Gamma^{aa} & \Gamma^{ab} & \Gamma^{ac} \\ \Gamma^{ba} & \Gamma^{bb} & \Gamma^{bc} \\ \Gamma^{ca} & \Gamma^{cb} & \Gamma^{cc} \end{pmatrix}\gamma_{mn} \\
=& \begin{pmatrix} \Gamma^a B^{aa} & \Gamma^b B^{ab} & \Gamma^c B^{ac} \\ \Gamma^a B^{ba} & \Gamma^b B^{bb} & \Gamma^c B^{bc} \\ \Gamma^a B^{ca} & \Gamma^b B^{cb} & \Gamma^c B^{cc} \end{pmatrix}\gamma_{mn} \\
=& \begin{pmatrix} \Gamma_1 B^{aa} & \varepsilon\Gamma_2 B^{ab} & \Gamma_2 B^{ac} \\ \Gamma_1 B^{ba} & \varepsilon\Gamma_2 B^{bb} & \Gamma_2 B^{bc} \\ \Gamma_1 B^{ca} & \varepsilon\Gamma_2 B^{cb} & \Gamma_2 B^{cc} \end{pmatrix}\gamma_{mn}
\end{aligned}$$

where $B^{\alpha\beta}$ is the branching ratio, Γ_1 is the F pumping power ($(1,1) \to (2,2)'$ light), Γ_2 is the m_F pumping power ($(2,1) \to (2,2)'$ light), ε is the off-resonance coupling of the m_F pumping light to the $(2,2) \to (2,3)'$ transition and γ_{mn} is the state independent direction averaged recoil coupling between state m and n.

$$\begin{aligned}
\gamma_{mn} =& \gamma_{nm} \\
\sum_m \gamma_{mn} =& 1
\end{aligned}$$

With 0.6% polarization misalignment and for Γ_1 and Γ_2 normalized to the pumping rate of the $(1,1)$ and $(2,1)$ states.

$$\begin{aligned}
\Gamma^{\alpha\beta} =& \begin{pmatrix} 0.500\Gamma_1 & 0.006\Gamma_2 & 0.500\Gamma_2 \\ 0.333\Gamma_1 & 0.171\Gamma_2 & 0.333\Gamma_2 \\ 0.167\Gamma_1 & 0.002\Gamma_2 & 0.167\Gamma_2 \end{pmatrix} \\
\Gamma_m^a =& \Gamma_1 \\
\Gamma_m^b =& 0.179\Gamma_2 \\
\Gamma_m^c =& \Gamma_2
\end{aligned}$$

If we use the $D1$ transition to do the pumping instead the matrix becomes

$$\begin{aligned}
\Gamma^{\alpha\beta} =& \begin{pmatrix} 0.500\Gamma_1 & 0.006\Gamma_2 & 0.500\Gamma_2 \\ 0.333\Gamma_1 & 0.004\Gamma_2 & 0.333\Gamma_2 \\ 0.167\Gamma_1 & 0.002\Gamma_2 & 0.167\Gamma_2 \end{pmatrix} \\
\Gamma_m^a =& \Gamma_1 \\
\Gamma_m^b =& 0.012\Gamma_2 \\
\Gamma_m^c =& \Gamma_2
\end{aligned}$$