

Mølmer-Sørensen gate simulation

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1 Goal

Derive the expression for simulating and optimizing a Mølmer-Sørensen gate pulse sequence.

2 Setup and scope

We'll discuss a simple two tone pulse sequence where the two tones are perfectly symmetric around the carrier. We'll ignore any error in the carrier frequency in this note. Crosstalk, coupling to carrier and other sideband orders are also ignored.

For a typical gate sequence, what we care about are

1. Ion motion:

The MS interaction will drive each of the motional mode in a spin-dependent way. For a proper MS gate, we'd like the final motional state to be identical to where we started. Any deviation from this results in a closure error.

2. Spin operation:

The enclosed area in phase-space from the driven motion results in a spin-dependent phase which is the main goal of the MS gate. Deviation in the control parameter could result in spin/angle error in the spin space.

3 Mølmer-Sørensen interaction

The effective Hamiltonian for a Mølmer-Sørensen gate sequence can be written as

$$H_{MS} = \frac{\Omega(t)}{2} \sum_{j=1,2} \sum_k \eta_{jk} \left(a_k e^{-i\theta_k(t)} + a_k^\dagger e^{i\theta_k(t)} \right) \sigma_x^j$$

where j is the ion index (simplified to 1 and 2) and k is the motional mode index. For the “fixed” parameters, η_{jk} is the Lamb-Dicke parameter for the j -th ion on the k -th mode. a_k and a_k^\dagger are the creation and annihilation operators for the k -th mode and the σ_x^j is the single qubit spin operator we are coupling to which we'll set as x in this note. (The error on the spin axis is ignored.) For the “variable” parameters in the pulse sequence, $\Omega(t)$ is the time dependent two-photon Rabi frequency (controlled by laser power) and $\theta_k(t)$ is the time-dependent phase offset between the laser and the k -th mode with,

$$\begin{aligned} \theta_k(t) &= \omega_k t - \theta(t) \\ &= \omega_k t - \int_0^t \delta(t') dt' \end{aligned}$$

where ω_k is the frequency of the k -th mode, $\theta(t)$ is the half the phase difference of the two lasers and $\delta(t')$ is the (symmetric) detuning of the lasers from the carrier. (If phase modulation is used, $\theta(t)$ and $\theta_k(t)$ may be discontinuous functions).

Using Magnus expansion, we can write down the unitary evolution of the system as

$$U_{MS}(\tau) = \exp \left[\sum_{j=1,2} \sum_k \frac{\eta_{kj}}{2} \left(\alpha_k(\tau) a_k^\dagger - \alpha_k^*(\tau) a_k \right) \sigma_x^j \right] \exp(i\Theta(\tau) \sigma_x^1 \sigma_x^2)$$

where

$$\alpha_k(\tau) = \int_0^\tau \Omega(t) e^{i\theta_k(t)} dt$$

describes the displacement of the k -th mode, and

$$\Theta(\tau) = \frac{1}{2} \sum_k \eta_{k1} \eta_{k2} \int_0^\tau dt \int_0^t dt' \Omega(t) \Omega(t') \sin(\theta_k(t) - \theta_k(t'))$$

is the angle of the two-qubit rotation. For a proper MS gate of length T , we need to have $\alpha_k(T) = 0$ for all k s and the rotation angle at the end of the pulse $\Theta(T)$ matching the angle we want.

For the purpose of optimization, the quantities we care about are.

1. Closure

$$\alpha_k(T) = \int_0^T \Omega(t) e^{i\theta_k(t)} dt$$

2. Enclosed area

$$\begin{aligned} A_k &= \int_0^T dt \int_0^t dt' \Omega(t) \Omega(t') \sin(\theta_k(t) - \theta_k(t')) \\ &= \int_0^T dt \int_0^t dt' \Omega(t) \Omega(t') (\sin(\theta_k(t)) \cos(\theta_k(t')) - \cos(\theta_k(t)) \sin(\theta_k(t'))) \\ &= \int_0^T dt \Omega(t) \sin(\theta_k(t)) \int_0^t dt' \Omega(t') \cos(\theta_k(t')) - \int_0^T dt \Omega(t) \cos(\theta_k(t)) \int_0^t dt' \Omega(t') \sin(\theta_k(t')) \end{aligned}$$

As well as their gradient w.r.t. mode frequencies which we'll calculate later.

The gradient of the closure w.r.t. mode frequencies does deserve a special mention, however.

$$\begin{aligned} \frac{\partial}{\partial \omega_k} \alpha_k(T) &= \int_0^T \frac{\partial}{\partial \omega_k} \Omega(t) e^{i\theta_k(t)} dt \\ &= i \int_0^T \Omega(t) e^{i\theta_k(t)} \frac{\partial \theta_k(t)}{\partial \omega_k} dt \\ &= i \int_0^T \Omega(t) e^{i\theta_k(t)} t dt \end{aligned}$$

When closure is assumed, this can be re-written as,

$$\begin{aligned}
\frac{\partial}{\partial \omega_k} \alpha_k(T) &= i \int_0^T \Omega(t) e^{i\theta_k(t)} t dt \\
&= i \left[t \int_0^t \Omega(t') e^{i\theta_k(t')} dt' \right]_0^T - i \int_0^T dt \int_0^t \Omega(t') e^{i\theta_k(t')} dt' \\
&= -i \int_0^T dt \int_0^t \Omega(t') e^{i\theta_k(t')} dt'
\end{aligned}$$

which is proportional to average displacement. Moreover, if the pulse is symmetric, zeroing this value will also automatically zero the final displacement thus remove the need to optimize two values at the same time.

4 Segmented pulse

When parameterizing the pulse shape used to drive MS gate, the pulse is usually decomposed into a few segments (in time) and the pulse shape within each segment is only sensitive to few parameters. We can take advantage of this structure to reduce the computational complexity when optimizing for a large number of free parameters by only dealing with one segment (and therefore few parameters) at a time.

More specifically, during the optimization process, we care not only about the values listed above (closure, area, and their gradient w.r.t. mode frequency) but also their gradient w.r.t. the pulse shape parameters. By dealing with the effect of each segment properly, we can compute all the gradient within $O(N_p + N_s)$ instead of $O(N_p N_s)$, where N_p is the number of parameters and N_s is the number of segments.

This is trivial to do for closure since it is a single integral and therefore is completely linear w.r.t. different segment of the integrand. The gradient of the integral on one segment is directly that on the final result and is independent of the functional form in any other segments.