Light shift and effective B field

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1 Goal

Derive and clarify some effects related to vector and tensor light shifts as well as a few different places they may appear in an experiment. Most, if not all of the discussion will be limited to E1 transitions. I'm not really looking for the most mathematically straight forward derivation, rather trying to see this from different angle for better understanding.

2 Some useful formulas

2.1 Spherical component of vector

Similar to the decomposition of light polarization into $\sigma^{\pm}$ and π, every 3D vector (operator) can be equivalently expressed as a rank-1 spherical tensor,

$$V_0 = V_z \tag{1}$$

$$V_{\pm 1} = \mp \frac{1}{\sqrt{2}}(V_x \pm \mathrm{i} V_y) \tag{2}$$

In particular, when applied to the angular momentum operator,

$$J_0 = J_z \tag{3}$$

$$\begin{aligned} J_{\pm 1} &= \mp \frac{1}{\sqrt{2}}(J_x \pm \mathrm{i} J_y) \\ &= \mp \frac{J_{\pm}}{\sqrt{2}} \end{aligned} \tag{4}$$

where $J_{\pm}$ are the angular momentum raising and lowering operators.

2.2 Wigner-Eckart theorem

This describes the relation between matrix elements of a vector/tensor operator in the angular momentum basis. The matrix element for different angular momentum states are related to each other with Clebsch-Gordan coefficients.

$$\langle j, m|T_q^{(k)}|j', m'\rangle = \langle j', k; m', q|j, m\rangle \langle j||T^{(k)}||j'\rangle \tag{5}$$

where $T_q^{(k)}$ is the q-th component of the spherical tensor operator $T^{(k)}$ of rank k. This is the result of rotation symmetry between all the matrix elements.

Equivalently, this also means that no matter what the tensor operator is, it's matrix elements in this (between these) subspace differs from that of a different tensor operator only by a constant factor. (Note that this factor could depend on the j and j' (just not m and m') and it can of course be 0 as well), i.e.

$$\langle j, m|{T_1}_q^{(k)}|j', m'\rangle \propto \langle j, m|{T_2}_q^{(k)}|j', m'\rangle \tag{6}$$

2.3 When $j = j'$

A special case for the Wigner-Echart theorem is when $j = j'$. In this case we can plug in the angular momentum operator J (this would otherwise result in vanishing matrix elements if $j \neq j'$ since J conserves, well, j).

$$\begin{aligned}\langle j, m|J_q|j, m'\rangle =&\langle j, 1; m', q|j, m\rangle\langle j||J||j\rangle \\ \propto&\langle j, 1; m', q|j, m\rangle\end{aligned} \tag{7}$$

This allow us to replace the CG coefficents with the angular momentum operator, i.e.,

$$\langle j, m|V_q|j, m'\rangle \propto \langle j, m|J_q|j, m'\rangle \tag{8}$$

which could make some calculation/expression significantly simpler.

This relation basically states that within the subspace of a single j, we can treat any vector operator as proportional to the angular momentum. The proportionality factor can then be obtained from the dot product with angular momentum, i.e. the projection of the vector onto angular momentum.

2.3.1 $m = 0$ selection rule

The selection rule for $m = m' = 0$ transition directly follows from this relation since,

$$\begin{aligned}\langle j, m|V_0|j, m\rangle \propto&\langle j, m|J_0|j, m\rangle \\ =&\langle j, m|J_z|j, m\rangle \\ =&m\end{aligned} \tag{9}$$

which is 0 when $m = 0$.

2.3.2 Projection theorem

We can use this to derive the projection theorem. Explicitly writing down the proportionality factor in Eq. 8, we have,

$$\langle j, m|V_q|j, m'\rangle = c\langle j, m|J_q|j, m'\rangle \tag{10}$$

Multiply both sides with the angular momentum matrix element and sum over all m' and q

$$\sum_{m',q}\langle j, m|V_q|j, m'\rangle\langle j, m'|J_q^\dagger|j, m''\rangle = c\sum_{m',q}\langle j, m|J_q|j, m'\rangle\langle j, m'|J_q^\dagger|j, m''\rangle \tag{11}$$

$$\sum_q\langle j, m|V_qV_q^\dagger|j, m''\rangle = c\sum_q\langle j, m|J_qJ_q^\dagger|j, m''\rangle \tag{12}$$

$$\begin{aligned}\langle j, m|(\vec{V}\cdot\vec{J})|j, m''\rangle =&c\langle j, m|J^2|j, m''\rangle \\ =&cj(j+1)\end{aligned} \tag{13}$$

Therefore we have

$$c = \frac{\langle j, m|(\vec{V}\cdot\vec{J})|j, m''\rangle}{j(j+1)} \tag{14}$$

$$\langle j, m|V_q|j, m'\rangle = \frac{\langle j, m|(\vec{V}\cdot\vec{J})|j, m''\rangle}{j(j+1)}\langle j, m|J_q|j, m'\rangle \tag{15}$$

2.3.3 Explicit calculation

Just for completeness, we can verify this relation between angular momentum and CG coefficients explicitly. This part can be ignored without affecting the understanding of the rest.
First the expression using angular momentum operators,

$$
\begin{aligned}
\langle j,m|J_0|j,m'\rangle =&\langle j,m|m'|j,m'\rangle \\
=&m'\delta_{mm'}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\langle j,m|J_{\pm 1}|j,m'\rangle =& \mp\frac{1}{\sqrt{2}}\langle j,m|J_{\pm}|j,m'\rangle \\
=& \mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\langle j,m|j,m'\pm 1\rangle \\
=& \mp\sqrt{\frac{(j\mp m')(j\pm m'+1)}{2}}\delta_{m,m'\pm 1}
\end{aligned}
\tag{17}
$$

Using the explicit formula for the CG coefficients,

$$
\begin{aligned}
\langle j,1;m',q|j,m\rangle =&\delta_{m,m'+q}\sqrt{\frac{(2j+1)(j+j-1)!(j-j+1)!(j+1-j)!}{(j+1+j+1)!}} \\
&\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!} \\
&\sum_k \frac{(-1)^k}{k!(j+1-j-k)!(j-m'-k)!(1+q-k)!(j-1+m+k)!(j-j-q+k)!} \\
=&\delta_{m,m'+q}\frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1-q)!(1+q)!}}{2\sqrt{(j+1)j}} \\
&\sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+q-k)!(j-1+m'+k)!(-q+k)!}
\end{aligned}
\tag{18}
$$

For $q=0$

$$
\begin{aligned}
\langle j,1;m',0|j,m\rangle =&\delta_{mm'}\frac{\sqrt{(j+m)!(j-m)!(j-m)!(j+m)!}}{2\sqrt{(j+1)j}} \\
&\sum_{k=0,1}\frac{(-1)^k}{k!(1-k)!(j-m-k)!(1-k)!(j-1+m+k)!k!} \\
=&\delta_{mm'}\frac{(j-m)!(j+m)!}{2\sqrt{(j+1)j}}\left(\frac{1}{(j-m)!(j-1+m)!}-\frac{1}{(j-m-1)!(j+m)!}\right) \\
=&m\frac{\delta_{mm'}}{\sqrt{j(j+1)}}
\end{aligned}
\tag{19}
$$

For $q = \pm 1$

$$
\begin{aligned}
\langle j,1;m',\pm 1|j,m\rangle =&\delta_{m,m'\pm 1}\frac{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!(1\mp 1)!(1\pm 1)!}}{2\sqrt{(j+1)j}} \\
&\sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m+k)!(\mp 1+k)!} \\
=&\frac{\delta_{m,m'\pm 1}}{\sqrt{(j+1)j}}\sqrt{\frac{(j+m'\pm 1)!(j-m'\mp 1)!(j-m')!(j+m')!}{2}} \\
&\sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1\pm 1-k)!(j-1+m'+k)!(\mp 1+k)!}
\end{aligned}
\tag{20}
$$

For $q = 1$

$$
\begin{aligned}
\langle j,1;m',1|j,m\rangle =&\frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}}\sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2}} \\
&\sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(1+1-k)!(j-1+m'+k)!(-1+k)!} \\
=&-\frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}}\sqrt{\frac{(j+m'+1)!(j-m'-1)!(j-m')!(j+m')!}{2(j-m'-1)!(j-m'-1)!(j+m')!(j+m')!}} \\
=&-\sqrt{\frac{(j+m'+1)(j-m')}{2}}\frac{\delta_{m,m'+1}}{\sqrt{(j+1)j}}
\end{aligned}
\tag{21}
$$

For $q = -1$

$$
\begin{aligned}
\langle j,1;m',-1|j,m\rangle =&\frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}}\sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2}} \\
&\sum_k \frac{(-1)^k}{k!(1-k)!(j-m'-k)!(-k)!(j-1+m'+k)!(1+k)!} \\
=&\frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}}\sqrt{\frac{(j+m'-1)!(j-m'+1)!(j-m')!(j+m')!}{2(j-m')!(j-m')!(j+m'-1)!(j+m'-1)!}} \\
=&\sqrt{\frac{(j-m'+1)(j+m')}{2}}\frac{\delta_{m,m'-1}}{\sqrt{(j+1)j}}
\end{aligned}
\tag{22}
$$

Comparing the result from the two methods, we can see that the proportionality factor is $\sqrt{(j+1)j}$, or

$$
\langle j,m|J_q|j,m'\rangle =\sqrt{(j+1)j}\langle j,1;m',q|j,m\rangle \tag{23}
$$

3 Vector and tensor light shift

4 Vector light shift as effective magnetic field

5 Source of circular polarization

As we saw, a laser beam with circular polarization could create a non-zero vector light shift. In an experiment, this circular polarization could be generated for an otherwise linearly polarized light by birefringence of material as the beam passes through them, and from reflection off of optical coating that introduces non-trivial phase shift between different polarization components. [1]

The circular polarization may also happen, however, for an otherwise perfectly linearly polarized light, near the focus of a beam with large NA, paricularly for optical tweezers or other focused beams for individual addressing. This is because a beam with a tight focus breaks the paraxial approximation such that the beam cannot be treated as a scalar field anymore. The field on the edge of the beam have significantly different k vectors and therefore different polarization vectors as well. As shown in Fig. 1, the polarization on the two edges of the beam acquires an axial component due to the large angle between the k vector and the optical axis. While the two sides of the beam are generally far away from each other and their different polarization directions cause little problem, this is not the case anymore near the focus as the edge of the beam changes direction from converging to diverging.

A point on the side of the focus is therefore affected by the combination of the two edge polarizations and varies rapidly within roughly a Rayleigh length. Although not a perfect analog, the effect this may cause can be understood by looking at the interference between two plane waves at an angle with in-plane polarization. Depending on the position (and therefore relative phase) the polarization will vary between in-plane linear and in-plane elliptical. For a normal focused beam, it turns out that the point next to the focus in the focal point would have an in-plane elliptical polarization. From symmetry, in the ideal case, the polarization in the center of the focus is still exactly linear and the rotation direction of the circular polarization on the two sides of the focus are opposite.

Base on the previous discussions, this corresponds to having an effective magnetic field that is out of the plane, i.e. perpendicular to both the optical axis and the polarization.

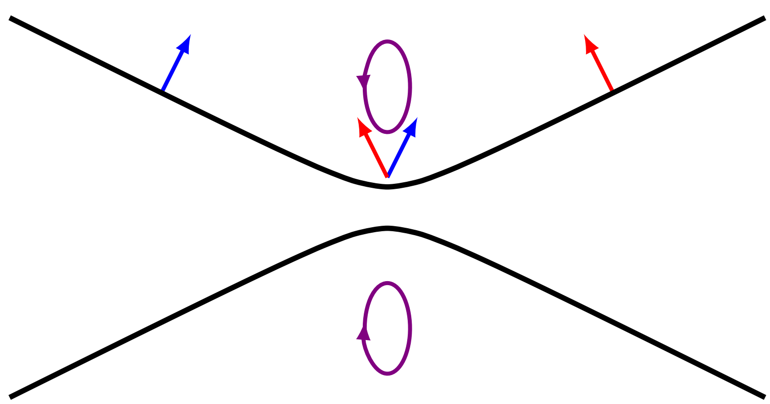

Figure 1: Transverse circular polarization near the focus of a tighly focused beam. The red and blue error shows the polarization vector on the same edge of a tightly focused beam before and after the focus. Near the focus though, the polarization becomes a superposition of the two creating the in-plane elliptical polarization next to the focus.

[1] Note that while Fresnel equations generally do not introduce non-trivial phase shift between S and P polarizations when the index of refraction is completely real (i.e. no absorption), except maybe for total internal reflection, this does not hold true anymore when interference between multiple reflected beams is involved, which is generally the case for optical coatings. As such, unless the incident beam is (almost) normal to the surface, in which case the symmetry prevents any polarization change from occuring, a dielectric mirror will generally change the polarization of a linear light if the polarization isn't purely S or P initially.

6 Mitigating the effect of transverse circular polarization in optical tweezers