Response of dressing rotation to amplitude noise

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Dressing Hamiltonian

$$H = \Omega\sigma_x + \delta\sigma_z$$

Large Ω for the target ion and small Ω for the crosstalk ion.

The rotation should robustly turn the $|0\rangle$ and $|1\rangle$ states into the eigen state of the dressing Hamiltonian for both small and large Ω.

The two eigen states are

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm\sqrt{1 \pm 1/\sqrt{1+(\Omega/\delta)^2}} \\ \sqrt{1 \mp 1/\sqrt{1+(\Omega/\delta)^2}} \end{pmatrix}$$

The transformation needed is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1 + 1/\sqrt{1+(\Omega/\delta)^2}} & -\sqrt{1 - 1/\sqrt{1+(\Omega/\delta)^2}} \\ \sqrt{1 - 1/\sqrt{1+(\Omega/\delta)^2}} & \sqrt{1 + 1/\sqrt{1+(\Omega/\delta)^2}} \end{pmatrix}$$

For small Ω,

$$\begin{aligned} U \approx& \begin{pmatrix} 1 & -\Omega/\delta/2 \\ \Omega/\delta/2 & 1 \end{pmatrix} \\ =& 1 - \frac{\mathrm{i}\Omega}{2\delta}\sigma_y \end{aligned}$$

For large Ω, if $\Omega = \Omega_0(1+\varepsilon)$

$$\begin{aligned} &\sqrt{1 \pm 1/\sqrt{1+(\Omega/\delta)^2}} \\ \approx& \sqrt{1 \pm 1/\sqrt{1+(\Omega_0/\delta)^2}} - \frac{(\Omega_0/\delta)^2}{2(1+(\Omega_0/\delta)^2)^{3/2}\sqrt{1 \pm 1/\sqrt{1+(\Omega_0/\delta)^2}}}\varepsilon \\ \approx& \sqrt{1 \pm 1/\sqrt{1+(\Omega_0/\delta)^2}} - \frac{(\Omega_0/\delta)\sqrt{1 \mp 1/\sqrt{1+(\Omega_0/\delta)^2}}}{2(1+(\Omega_0/\delta)^2)}\varepsilon \end{aligned}$$

$$
\begin{aligned}
U =& \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+1/\sqrt{1+(\Omega/\delta)^2}} & -\sqrt{1-1/\sqrt{1+(\Omega/\delta)^2}} \\ \sqrt{1-1/\sqrt{1+(\Omega/\delta)^2}} & \sqrt{1+1/\sqrt{1+(\Omega/\delta)^2}} \end{pmatrix} \\
\approx& \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+1/\sqrt{1+(\Omega_0/\delta)^2}} & -\sqrt{1-1/\sqrt{1+(\Omega_0/\delta)^2}} \\ \sqrt{1-1/\sqrt{1+(\Omega_0/\delta)^2}} & \sqrt{1+1/\sqrt{1+(\Omega_0/\delta)^2}} \end{pmatrix} \\
&+ \frac{(\Omega_0/\delta)\varepsilon}{2\sqrt{2}(1+(\Omega_0/\delta)^2)} \begin{pmatrix} \sqrt{1-1/\sqrt{1+(\Omega_0/\delta)^2}} & -\sqrt{1+1/\sqrt{1+(\Omega_0/\delta)^2}} \\ \sqrt{1+1/\sqrt{1+(\Omega_0/\delta)^2}} & \sqrt{1-1/\sqrt{1+(\Omega_0/\delta)^2}} \end{pmatrix} \\
=& \sqrt{\frac{1+1/\sqrt{1+(\Omega_0/\delta)^2}}{2}} - \sqrt{\frac{1-1/\sqrt{1+(\Omega_0/\delta)^2}}{2}} \mathrm{i}\sigma_y \\
&+ \frac{(\Omega_0/\delta)\sqrt{1-1/\sqrt{1+(\Omega_0/\delta)^2}}}{2\sqrt{2}(1+(\Omega_0/\delta)^2)}\varepsilon + \frac{(\Omega_0/\delta)\sqrt{1+1/\sqrt{1+(\Omega_0/\delta)^2}}}{2\sqrt{2}(1+(\Omega_0/\delta)^2)} \mathrm{i}\sigma_y\varepsilon
\end{aligned}
$$

The pulses should consists of rotations with the same Rabi frequency but may have variable detuning, phase and time. Note that only the xy component of the unitary matters. The z component and the global phase can be perfectly cancelled out by using the time reversal as the undress pulse sequence.

1 Single rotation

$$
\begin{aligned}
&\exp\left(\mathrm{i}(X(\cos\varphi\sigma_x + \sin\varphi\sigma_y) + Z\sigma_z)\right) \\
=& \cos\left(\sqrt{X^2+Z^2}\right) + \mathrm{i}\sin\left(\sqrt{X^2+Z^2}\right)\frac{X\cos\varphi}{\sqrt{X^2+Z^2}}\sigma_x \\
&+ \mathrm{i}\sin\left(\sqrt{X^2+Z^2}\right)\frac{X\sin\varphi}{\sqrt{X^2+Z^2}}\sigma_y + \mathrm{i}\sin\left(\sqrt{X^2+Z^2}\right)\frac{Z}{\sqrt{X^2+Z^2}}\sigma_z
\end{aligned}
$$

xy component

$$
\begin{aligned}
a_1 =& \sqrt{\frac{X^2\cos^2\left(\sqrt{X^2+Z^2}\right)+Z^2}{X^2+Z^2}} \\
x_1 =& -\mathrm{i}\frac{\sin\left(\sqrt{X^2+Z^2}\right)X}{\sqrt{X^2\cos^2\left(\sqrt{X^2+Z^2}\right)+Z^2}}\left(\cos\left(\sqrt{X^2+Z^2}\right)\cos\varphi + \frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{\sqrt{X^2+Z^2}}\right) \\
y_1 =& \mathrm{i}\frac{\sin\left(\sqrt{X^2+Z^2}\right)X}{\sqrt{X^2\cos^2\left(\sqrt{X^2+Z^2}\right)+Z^2}}\left(\cos\left(\sqrt{X^2+Z^2}\right)\sin\varphi + \frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\cos\varphi}{\sqrt{X^2+Z^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cos\left(\sqrt{X^2+Z^2}\right)\cos\varphi+\frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{\sqrt{X^2+Z^2}}=0\\
&-\sin\left(\sqrt{X^2+Z^2}\right)\cos\varphi+\frac{Z\cos\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{\sqrt{X^2+Z^2}}-\frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{X^2+Z^2}=0\\
&\cos\left(\sqrt{X^2+Z^2}\right)\cos\varphi=-\frac{Z\sin\left(\sqrt{X^2+Z^2}\right)}{\sqrt{X^2+Z^2}}\sin\varphi\\
&\sin\left(\sqrt{X^2+Z^2}\right)\cos\varphi=\frac{Z\cos\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{\sqrt{X^2+Z^2}}-\frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\sin\varphi}{X^2+Z^2}\\
&-\frac{Z\sin^2\left(\sqrt{X^2+Z^2}\right)}{\sqrt{X^2+Z^2}}=\frac{Z\cos^2\left(\sqrt{X^2+Z^2}\right)}{\sqrt{X^2+Z^2}}-\frac{Z\sin\left(\sqrt{X^2+Z^2}\right)\cos\left(\sqrt{X^2+Z^2}\right)}{X^2+Z^2}\\
&\sin\left(\sqrt{X^2+Z^2}\right)\cos\left(\sqrt{X^2+Z^2}\right)=\sqrt{X^2+Z^2}
\end{aligned}
$$

Can't satisfy.

1.1 Small power limit

$$
\begin{aligned}
&\exp\left(\mathrm{i}(X(\cos\varphi\sigma_x+\sin\varphi\sigma_y)+Z\sigma_z)\right)\\
\approx&\cos Z+\mathrm{i}\sin Z\sigma_z+\mathrm{i}X\frac{\sin Z}{Z}(\cos\varphi\sigma_x+\sin\varphi\sigma_y)
\end{aligned}
$$

2 Two rotations

$$
\begin{aligned}
&\exp\left(\mathrm{i}(X_1(\cos\varphi_1\sigma_x + \sin\varphi_1\sigma_y) + Z_1\sigma_z)\right)\exp\left(\mathrm{i}(X_2(\cos\varphi_2\sigma_x + \sin\varphi_2\sigma_y) + Z_2\sigma_z)\right)\\
=&\left(\cos\left(\sqrt{X_1^2+Z_1^2}\right) + \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\cos\varphi_1}{\sqrt{X_1^2+Z_1^2}}\sigma_x\right.\\
&\left. + \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\sin\varphi_1}{\sqrt{X_1^2+Z_1^2}}\sigma_y + \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{Z_1}{\sqrt{X_1^2+Z_1^2}}\sigma_z\right)\\
&\left(\cos\left(\sqrt{X_2^2+Z_2^2}\right) + \mathrm{i}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\cos\varphi_2}{\sqrt{X_2^2+Z_2^2}}\sigma_x\right.\\
&\left. + \mathrm{i}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\sin\varphi_2}{\sqrt{X_2^2+Z_2^2}}\sigma_y + \mathrm{i}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{Z_2}{\sqrt{X_2^2+Z_2^2}}\sigma_z\right)\\
=&\cos\left(\sqrt{X_1^2+Z_1^2}\right)\cos\left(\sqrt{X_2^2+Z_2^2}\right) - \frac{\sin\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)Z_1Z_2}{\sqrt{X_1^2+Z_1^2}\sqrt{X_2^2+Z_2^2}}\\
&-\sin\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_1X_2\cos\left(\varphi_1-\varphi_2\right)}{\sqrt{X_1^2+Z_1^2}\sqrt{X_2^2+Z_2^2}}\\
&+\mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\cos\varphi_1}{\sqrt{X_1^2+Z_1^2}}\cos\left(\sqrt{X_2^2+Z_2^2}\right)\sigma_x + \mathrm{i}\cos\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\cos\varphi_2}{\sqrt{X_2^2+Z_2^2}}\sigma_x\\
&+\mathrm{i}\frac{\sin\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)(Z_1X_2\sin\varphi_2 - X_1Z_2\sin\varphi_1)}{\sqrt{X_1^2+Z_1^2}\sqrt{X_2^2+Z_2^2}}\sigma_x\\
&+\mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\sin\varphi_1}{\sqrt{X_1^2+Z_1^2}}\cos\left(\sqrt{X_2^2+Z_2^2}\right)\sigma_y - \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{Z_1}{\sqrt{X_1^2+Z_1^2}}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\cos}{\sqrt{X_2^2 -}}\\
&+\mathrm{i}\cos\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\sin\varphi_2}{\sqrt{X_2^2+Z_2^2}}\sigma_y + \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\cos\varphi_1}{\sqrt{X_1^2+Z_1^2}}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{Z_2}{\sqrt{X_2^2 -}}\\
&+\mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{Z_1}{\sqrt{X_1^2+Z_1^2}}\cos\left(\sqrt{X_2^2+Z_2^2}\right)\sigma_z + \mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\sin\varphi_1}{\sqrt{X_1^2+Z_1^2}}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\cos}{\sqrt{X_2^2 -}}\\
&-\mathrm{i}\sin\left(\sqrt{X_1^2+Z_1^2}\right)\frac{X_1\cos\varphi_1}{\sqrt{X_1^2+Z_1^2}}\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{X_2\sin\varphi_2}{\sqrt{X_2^2+Z_2^2}}\sigma_z + \mathrm{i}\cos\left(\sqrt{X_1^2+Z_1^2}\right)\sin\left(\sqrt{X_2^2+Z_2^2}\right)\frac{Z_2}{\sqrt{X_2^2 -}}
\end{aligned}
$$

2.1 Small power limit

$$
\begin{aligned}
&\exp\left(\mathrm{i}(X_1(\cos\varphi_1\sigma_x+\sin\varphi_1\sigma_y)+Z_1\sigma_z)\right)\exp\left(\mathrm{i}(X_2(\cos\varphi_2\sigma_x+\sin\varphi_2\sigma_y)+Z_2\sigma_z)\right)\\
\approx&\left(\cos Z_1+\mathrm{i}\sin Z_1\sigma_z+\mathrm{i}X_1\frac{\sin Z_1}{Z_1}(\cos\varphi_1\sigma_x+\sin\varphi_1\sigma_y)\right)\\
&\left(\cos Z_2+\mathrm{i}\sin Z_2\sigma_z+\mathrm{i}X_2\frac{\sin Z_2}{Z_2}(\cos\varphi_2\sigma_x+\sin\varphi_2\sigma_y)\right)\\
\approx&\cos(Z_1+Z_2)+\mathrm{i}\sin(Z_1+Z_2)\sigma_z\\
&+\mathrm{i}X_2\frac{\sin Z_2}{Z_2}(\cos(\varphi_2-Z_1)\sigma_x+\sin(\varphi_2-Z_1)\sigma_y)\\
&+\mathrm{i}X_1\frac{\sin Z_1}{Z_1}(\cos(\varphi_1+Z_2)\sigma_x+\sin(\varphi_1+Z_2)\sigma_y)
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(\mathrm{i}(X_1(\cos\varphi_1\sigma_x+\sin\varphi_1\sigma_y)+Z_1\sigma_z)\right)\exp\left(\mathrm{i}(X_2(\cos\varphi_2\sigma_x+\sin\varphi_2\sigma_y)+Z_2\sigma_z)\right)\\
&\exp\left(\mathrm{i}(X_3(\cos\varphi_3\sigma_x+\sin\varphi_3\sigma_y)+Z_3\sigma_z)\right)\\
\approx&\left(\cos Z_1+\mathrm{i}\sin Z_1\sigma_z+\mathrm{i}X_1\frac{\sin Z_1}{Z_1}(\cos\varphi_1\sigma_x+\sin\varphi_1\sigma_y)\right)\\
&\left(\cos Z_2+\mathrm{i}\sin Z_2\sigma_z+\mathrm{i}X_2\frac{\sin Z_2}{Z_2}(\cos\varphi_2\sigma_x+\sin\varphi_2\sigma_y)\right)\\
&\left(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z+\mathrm{i}X_3\frac{\sin Z_3}{Z_3}(\cos\varphi_3\sigma_x+\sin\varphi_3\sigma_y)\right)\\
\approx&(\cos(Z_1+Z_2)+\mathrm{i}\sin(Z_1+Z_2)\sigma_z)(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z)\\
&+\mathrm{i}X_3\frac{\sin Z_3}{Z_3}(\cos(Z_1+Z_2)+\mathrm{i}\sin(Z_1+Z_2)\sigma_z)(\cos\varphi_3\sigma_x+\sin\varphi_3\sigma_y)\\
&+\mathrm{i}X_2\frac{\sin Z_2}{Z_2}(\cos(\varphi_2-Z_1)\sigma_x+\sin(\varphi_2-Z_1)\sigma_y)(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z)\\
&+\mathrm{i}X_1\frac{\sin Z_1}{Z_1}(\cos(\varphi_1+Z_2)\sigma_x+\sin(\varphi_1+Z_2)\sigma_y)(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z)\\
\approx&\cos(Z_1+Z_2+Z_3)+\mathrm{i}\sin(Z_1+Z_2+Z_3)\sigma_z\\
&+\mathrm{i}X_3\frac{\sin Z_3}{Z_3}(\cos(Z_1+Z_2)+\mathrm{i}\sin(Z_1+Z_2)\sigma_z)(\cos\varphi_3\sigma_x+\sin\varphi_3\sigma_y)\\
&+\mathrm{i}X_2\frac{\sin Z_2}{Z_2}(\cos(\varphi_2-Z_1)\sigma_x+\sin(\varphi_2-Z_1)\sigma_y)(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z)\\
&+\mathrm{i}X_1\frac{\sin Z_1}{Z_1}(\cos(\varphi_1+Z_2)\sigma_x+\sin(\varphi_1+Z_2)\sigma_y)(\cos Z_3+\mathrm{i}\sin Z_3\sigma_z)
\end{aligned}
$$

3 Derivation of eigenstates/values

Hamiltonian

$$
H=\frac{\delta}{2}(|0\rangle\langle 0|-|1\rangle\langle 1|)+\frac{\Omega}{2}(|0\rangle\langle 1|+|1\rangle\langle 0|)
$$

Eigenvalues

$$E_{\pm} = \pm\frac{\sqrt{\Omega^2+\delta^2}}{2}$$

Eigenstate equations ($d \equiv \dfrac{\delta}{\Omega}$)

$$0 = \Big(d(|0\rangle\langle 0| - |1\rangle\langle 1|) + (|0\rangle\langle 1| + |1\rangle\langle 0|) \mp \sqrt{1+d^2}\Big)|\psi_{\pm}\rangle$$

With $|\psi_{\pm}\rangle \equiv b^{\pm}|0\rangle + c^{\pm}|1\rangle$

$$\begin{aligned}
0 =& \Big(d(|0\rangle\langle 0| - |1\rangle\langle 1|) + (|0\rangle\langle 1| + |1\rangle\langle 0|) \mp \sqrt{1+d^2}\Big)\big(b^{\pm}|0\rangle + c^{\pm}|1\rangle\big) \\
0 =& \Big(d \mp \sqrt{1+d^2}\Big)b^{\pm}|0\rangle + c^{\pm}|0\rangle - \Big(d \pm \sqrt{1+d^2}\Big)c^{\pm}|1\rangle + b^{\pm}|1\rangle \\
c^{\pm} =& \pm\left(\sqrt{1+d^2} \mp d\right)b^{\pm} \\
b^{\pm} =& \pm\left(\sqrt{1+d^2} \pm d\right)c^{\pm} \\
c^{\pm} =& \sqrt{\frac{1 \mp d/\sqrt{1+d^2}}{2}} \\
b^{\pm} =& \pm\sqrt{\frac{1 \pm d/\sqrt{1+d^2}}{2}} \\
|\psi_{\pm}\rangle =& \frac{\pm\sqrt{1 \pm d/\sqrt{1+d^2}}|0\rangle + \sqrt{1 \mp d/\sqrt{1+d^2}}|1\rangle}{\sqrt{2}}
\end{aligned}$$