

# Exact formula for two level system drive with fixed detunine and amplitude

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Original Hamiltonian

$$H_0 = \frac{1}{2} \begin{pmatrix} \Delta & \Omega e^{i(\delta t + \phi)} \\ \Omega e^{-i(\delta t + \phi)} & -\Delta \end{pmatrix}$$

Under a frame rotation

$$U_1 = \begin{pmatrix} e^{-i(\delta t + \phi)/2} & 0 \\ 0 & e^{i(\delta t + \phi)/2} \end{pmatrix}$$

The Hamiltonian becomes

$$\begin{aligned} H_1 &= U_1 H U_1^\dagger + i \frac{dU_1}{dt} U_1^\dagger \\ &= \frac{1}{2} \begin{pmatrix} \Delta' & \Omega \\ \Omega & -\Delta' \end{pmatrix} \end{aligned}$$

where  $\Delta' \equiv \Delta + \delta$ . We can diagonalize it with

$$U_2 = \begin{pmatrix} -\sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \\ \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \end{pmatrix}$$

where  $\Omega' \equiv \sqrt{\Delta'^2 + \Omega^2}$ , to,

$$\begin{aligned} H_2 &= U_2 H_1 U_2^\dagger \\ &= \begin{pmatrix} -\frac{\Omega'}{2} & 0 \\ 0 & \frac{\Omega'}{2} \end{pmatrix} \end{aligned}$$

Evolution under  $H_2$ ,

$$\begin{aligned} T_2 &= \exp(-iH_2 t) \\ &= \begin{pmatrix} \exp\left(i\frac{\Omega'}{2}t\right) & 0 \\ 0 & \exp\left(-i\frac{\Omega'}{2}t\right) \end{pmatrix} \end{aligned}$$

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Evolution under  $H_1$ ,

$$\begin{aligned}
& T_1 \\
&= U_2^\dagger T_2 U_2 \\
&= \begin{pmatrix} -\sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \exp\left(i\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \exp\left(-i\frac{\Omega'}{2}t\right) \\ \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \exp\left(i\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \exp\left(-i\frac{\Omega'}{2}t\right) \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \\ \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{1}{2} - \frac{\Delta'}{2\Omega'}\right) \exp\left(i\frac{\Omega'}{2}t\right) + \left(\frac{1}{2} + \frac{\Delta'}{2\Omega'}\right) \exp\left(-i\frac{\Omega'}{2}t\right) & \frac{\Omega}{2\Omega'} \left(\exp\left(-i\frac{\Omega'}{2}t\right) - \exp\left(i\frac{\Omega'}{2}t\right)\right) \\ \frac{\Omega}{2\Omega'} \left(\exp\left(-i\frac{\Omega'}{2}t\right) - \exp\left(i\frac{\Omega'}{2}t\right)\right) & \left(\frac{1}{2} + \frac{\Delta'}{2\Omega'}\right) \exp\left(i\frac{\Omega'}{2}t\right) + \left(\frac{1}{2} - \frac{\Delta'}{2\Omega'}\right) \exp\left(-i\frac{\Omega'}{2}t\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right) - i\frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) & -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) \\ -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) & \cos\left(\frac{\Omega'}{2}t\right) + i\frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix}
\end{aligned}$$

Original time evolution

$$\begin{aligned}
& T_0 \\
&= U_1^\dagger T_1 U_1(t=0) \\
&= \begin{pmatrix} \left(\cos\left(\frac{\Omega'}{2}t\right) - i\frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right)\right) e^{i\delta t/2} & -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) e^{i(\delta t/2+\phi)} \\ U - i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) e^{-i(\delta t/2+\phi)} & \left(\cos\left(\frac{\Omega'}{2}t\right) + i\frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right)\right) e^{-i\delta t/2} \end{pmatrix}
\end{aligned}$$

If  $\Delta = 0$ ,

$$T_0 = \begin{pmatrix} \left(\cos\left(\frac{\Omega'}{2}t\right) - i\frac{\delta}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right)\right) e^{i\delta t/2} & -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) e^{i(\delta t/2+\phi)} \\ -i\frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) e^{-i(\delta t/2+\phi)} & \left(\cos\left(\frac{\Omega'}{2}t\right) + i\frac{\delta}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right)\right) e^{-i\delta t/2} \end{pmatrix}$$

Now converting all the parameter to angles,

$$\theta \equiv \Omega t$$

$$\alpha \equiv \delta t$$

$$\theta' \equiv \sqrt{\theta^2 + \alpha^2}$$

$$\begin{aligned}
T_0 &= \begin{pmatrix} \left(\cos\left(\frac{\theta'}{2}\right) - i\frac{\alpha}{\theta'} \sin\left(\frac{\theta'}{2}\right)\right) e^{i\alpha/2} & -i\frac{\theta}{\theta'} \sin\left(\frac{\theta'}{2}\right) e^{i(\alpha/2+\phi)} \\ -i\frac{\theta}{\theta'} \sin\left(\frac{\theta'}{2}\right) e^{-i(\alpha/2+\phi)} & \left(\cos\left(\frac{\theta'}{2}\right) + i\frac{\alpha}{\theta'} \sin\left(\frac{\theta'}{2}\right)\right) e^{-i\alpha/2} \end{pmatrix} \\
&= \begin{pmatrix} \left(\cos\left(\frac{\theta'}{2}\right) - i\frac{\alpha}{2} \text{sinc}\left(\frac{\theta'}{2}\right)\right) e^{i\alpha/2} & -i\frac{\theta}{2} \text{sinc}\left(\frac{\theta'}{2}\right) e^{i(\alpha/2+\phi)} \\ -i\frac{\theta}{2} \text{sinc}\left(\frac{\theta'}{2}\right) e^{-i(\alpha/2+\phi)} & \left(\cos\left(\frac{\theta'}{2}\right) + i\frac{\alpha}{2} \text{sinc}\left(\frac{\theta'}{2}\right)\right) e^{-i\alpha/2} \end{pmatrix}
\end{aligned}$$

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## Check

Time derivative of  $T_1$

$$\begin{aligned} i \frac{dT_1}{dt} T_1^\dagger &= i \frac{\Omega'}{2} \begin{pmatrix} -\sin\left(\frac{\Omega'}{2}t\right) - i \frac{\Delta'}{\Omega'} \cos\left(\frac{\Omega'}{2}t\right) & -i \frac{\Omega}{\Omega'} \cos\left(\frac{\Omega'}{2}t\right) \\ -i \frac{\Omega}{\Omega'} \cos\left(\frac{\Omega'}{2}t\right) & -\sin\left(\frac{\Omega'}{2}t\right) + i \frac{\Delta'}{\Omega'} \cos\left(\frac{\Omega'}{2}t\right) \end{pmatrix} \\ &\quad \begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right) + i \frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) & i \frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) \\ i \frac{\Omega}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) & \cos\left(\frac{\Omega'}{2}t\right) - i \frac{\Delta'}{\Omega'} \sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix} \\ &= i \frac{\Omega'}{2} \begin{pmatrix} -i \frac{\Delta'}{\Omega'} & -i \frac{\Omega}{\Omega'} \\ -i \frac{\Omega}{\Omega'} & i \frac{\Delta'}{\Omega'} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \Delta' & \Omega \\ \Omega & -\Delta' \end{pmatrix} \\ &= H_1 \end{aligned}$$