Magnus expansion with linearly changing Hamiltonian

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With a Hamiltonian

$$H(t) = H_0 + H_1 t \tag{1}$$

The commutators for the the leading order of Magnus expansion,

$$\begin{aligned}
[H(t_1), H(t_2)] &= [H_0 + H_1 t_1, H_0 + H_1 t_2] \\
&= [H_0, H_1] t_2 + [H_1, H_0] t_1 \\
&= [H_0, H_1](t_2 - t_1)
\end{aligned} \tag{2}$$

$$\begin{aligned}
[[H(t_1), H(t_2)], H(t_3)] &= [[H_0, H_1](t_2 - t_1), H_0 + H_1 t_3] \\
&= ([[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3)(t_2 - t_1)
\end{aligned} \tag{3}$$

$$\begin{aligned}
&[[[H(t_1), H(t_2)], H(t_3)], H(t_4)] \\
=&[([[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3), H_0 + H_1 t_4](t_2 - t_1) \\
=&[[[H_0, H_1], H_0], H_0](t_2 - t_1) + [[[H_0, H_1], H_0], H_1](t_2 - t_1) t_4 \\
&+ [[[H_0, H_1], H_1], H_0] t_3 (t_2 - t_1) + [[[H_0, H_1], H_1], H_1] t_3 t_4 (t_2 - t_1)
\end{aligned} \tag{4}$$

The terms in the time integral,

$$\begin{aligned}
&[H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \\
=&[[H_0, H_1], H_0](t_2 - t_3) + [[H_0, H_1], H_1] t_1 (t_2 - t_3) \\
&+ [[H_0, H_1], H_0](t_2 - t_1) + [[H_0, H_1], H_1] t_3 (t_2 - t_1) \\
=&[[H_0, H_1], H_0](2t_2 - t_1 - t_3) + [[H_0, H_1], H_1](t_1 t_2 + t_2 t_3 - 2 t_1 t_3)
\end{aligned} \tag{5}$$

$$\begin{aligned}
&[[[H(t_1), H(t_2)], H(t_3)], H(t_4)] + [[[H(t_3), H(t_2)], H(t_4)], H(t_1)] \\
&+ [[[H(t_3), H(t_4)], H(t_2)], H(t_1)] + [[[H(t_4), H(t_1)], H(t_3)], H(t_2)] \\
=&[[[H_0, H_1], H_0], H_0](t_2 - t_1 + t_2 - t_3 + t_4 - t_3 + t_1 - t_4) \\
&+ [[[H_0, H_1], H_0], H_1](t_2 t_4 - t_1 t_4 + t_2 t_1 - t_3 t_1 + t_4 t_1 - t_3 t_1 + t_1 t_2 - t_4 t_2) \\
&+ [[[H_0, H_1], H_1], H_0](t_2 t_3 - t_1 t_3 + t_2 t_4 - t_3 t_4 + t_4 t_2 - t_3 t_2 + t_1 t_3 - t_4 t_3) \\
&+ [[[H_0, H_1], H_1], H_1](t_2 t_3 t_4 - t_1 t_3 t_4 + t_2 t_1 t_4 - t_3 t_1 t_4 + t_4 t_1 t_2 - t_3 t_1 t_2 + t_1 t_2 t_3 - t_4 t_2 t_3) \\
=&[[[H_0, H_1], H_0], H_0] 2(t_2 - t_3) + [[[H_0, H_1], H_0], H_1] 2 t_1 (t_2 - t_3) \\
&+ [[[H_0, H_1], H_1], H_0] 2 t_4 (t_2 - t_3) + [[[H_0, H_1], H_1], H_1] 2 t_1 t_4 (t_2 - t_3)
\end{aligned} \tag{6}$$

Integrals,

$$\begin{aligned}
\Omega_1 =& \int_{\tau_1}^{\tau_2} H(t_1)\mathrm{d}t_1 \\
=& \int_{\tau_1}^{\tau_2} H_0 + H_1 t_1 \mathrm{d}t_1 \\
=& (\tau_2 - \tau_1)\left(H_0 + \frac{\tau_2 + \tau_1}{2} H_1\right)
\end{aligned} \tag{7}$$

$$\begin{aligned}
\Omega_2 =& \frac{1}{2}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 \int_{\tau_1}^{t_1} \mathrm{d}t_2 [H(t_1), H(t_2)] \\
=& [H_0, H_1]\frac{1}{2}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 \int_{\tau_1}^{t_1} \mathrm{d}t_2 (t_2 - t_1) \\
=& [H_0, H_1]\frac{1}{4}\int_{\tau_1}^{\tau_2} \mathrm{d}t_1 (t_1 - \tau_1)^2 \\
=& -\frac{(\tau_2 - \tau_1)^3}{12}[H_0, H_1]
\end{aligned} \tag{8}$$

$$\begin{aligned}
\Omega_3 =& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 [H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \\
=& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 [[H_0, H_1], H_0](2t_2 - t_1 - t_3) + [[H_0, H_1], H_1](t_1 t_2 + t_2 t_3 - 2t_1 t_3) \\
=& \frac{(\tau_2 - \tau_1)^5}{240}[[H_0, H_1], H_1]
\end{aligned} \tag{9}$$

$$\begin{aligned}
\Omega_4 =& \frac{1}{12}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 \int_{\tau_1}^{t_3}\mathrm{d}t_4 [[[H(t_1), H(t_2)], H(t_3)], H(t_4)] + [[[H(t_3), H(t_2)], H(t_4)], H(t_1)] \\
&+ [[[H(t_3), H(t_4)], H(t_2)], H(t_1)] + [[[H(t_4), H(t_1)], H(t_3)], H(t_2)] \\
=& \frac{1}{6}\int_{\tau_1}^{\tau_2}\mathrm{d}t_1 \int_{\tau_1}^{t_1}\mathrm{d}t_2 \int_{\tau_1}^{t_2}\mathrm{d}t_3 \int_{\tau_1}^{t_3}\mathrm{d}t_4 [[[H_0, H_1], H_0], H_0](t_2 - t_3) + [[[H_0, H_1], H_0], H_1] t_1 (t_2 - t_3) \\
&+ [[[H_0, H_1], H_1], H_0] t_4 (t_2 - t_3) + [[[H_0, H_1], H_1], H_1] t_1 t_4 (t_2 - t_3) \\
=& \frac{(\tau_2 - \tau_1)^5}{720}\Bigg([[[H_0, H_1], H_0], H_0] + \frac{\tau_1 + 5\tau_2}{6}[[[H_0, H_1], H_0], H_1] \\
&+ \frac{5\tau_1 + \tau_2}{6}[[[H_0, H_1], H_1], H_0] + \frac{\tau_1^2 + 5\tau_1\tau_2 + \tau_2^2}{7}[[[H_0, H_1], H_1], H_1]\Bigg)
\end{aligned} \tag{10}$$

For Pauli,

$$H_0 = \vec{h}_0 \cdot \vec{\sigma} \tag{11}$$

$$H_1 = \vec{h}_1 \cdot \vec{\sigma} \tag{12}$$

Commutators,

$$\begin{aligned}
[H_0, H_1] =& \left[\vec{h}_0 \cdot \vec{\sigma}, \vec{h}_1 \cdot \vec{\sigma}\right] \\
=& 2\mathrm{i}\left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma
\end{aligned} \tag{13}$$

$$\begin{aligned}
[[H_0, H_1], H_0] =& -4\left(\left(\vec{h}_0 \times \vec{h}_1\right) \times \vec{h}_0\right) \cdot \sigma \\
=& -4\left(\vec{h}_1 \left|\vec{h}_0\right|^2 - \vec{h}_0\left(\vec{h}_0 \cdot \vec{h}_1\right)\right) \cdot \sigma
\end{aligned} \tag{14}$$

$$\begin{aligned}
[[H_0, H_1], H_1] =& -4\left(\left(\vec{h}_0 \times \vec{h}_1\right) \times \vec{h}_1\right) \cdot \sigma \\
=& -4\left(\vec{h}_1\left(\vec{h}_0 \cdot \vec{h}_1\right) - \vec{h}_0 \left|\vec{h}_1\right|^2\right) \cdot \sigma
\end{aligned} \tag{15}$$

$$\begin{aligned}
[[[H_0, H_1], H_0], H_0] =& -8\mathrm{i}\left(\left(\vec{h}_1 \left|\vec{h}_0\right|^2 - \vec{h}_0\left(\vec{h}_0 \cdot \vec{h}_1\right)\right) \times \vec{h}_0\right) \cdot \sigma \\
=& 8\mathrm{i}\left|\vec{h}_0\right|^2 \left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma
\end{aligned} \tag{16}$$

$$\begin{aligned}
[[[H_0, H_1], H_0], H_1] =& -8\mathrm{i}\left(\left(\vec{h}_1 \left|\vec{h}_0\right|^2 - \vec{h}_0\left(\vec{h}_0 \cdot \vec{h}_1\right)\right) \times \vec{h}_1\right) \cdot \sigma \\
=& 8\mathrm{i}\left(\vec{h}_0 \cdot \vec{h}_1\right)\left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma
\end{aligned} \tag{17}$$

$$[[[H_0, H_1], H_1], H_0] = 8\mathrm{i}\left(\vec{h}_0 \cdot \vec{h}_1\right)\left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma \tag{18}$$

$$[[[H_0, H_1], H_1], H_1] = 8\mathrm{i}\left|\vec{h}_1\right|^2 \left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma \tag{19}$$

Integrals,

$$\Omega_1 = (\tau_2 - \tau_1)\left(\vec{h}_0 + \frac{\tau_2 + \tau_1}{2}\vec{h}_1\right) \cdot \sigma \tag{20}$$

$$\Omega_2 = -\mathrm{i}\frac{(\tau_2 - \tau_1)^3}{6}\left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma \tag{21}$$

$$\Omega_3 = -\frac{(\tau_2 - \tau_1)^5}{60}\left(\left(\vec{h}_0 \cdot \vec{h}_1\right)\vec{h}_1 - \left|\vec{h}_1\right|^2 \vec{h}_0\right) \cdot \sigma \tag{22}$$

$$\Omega_4 = \mathrm{i}\frac{(\tau_2 - \tau_1)^5}{90}\left(\left|\vec{h}_0\right|^2 + (\tau_1 + \tau_2)\left(\vec{h}_0 \cdot \vec{h}_1\right) + \frac{\tau_1^2 + 5\tau_1\tau_2 + \tau_2^2}{7}\left|\vec{h}_1\right|^2\right)\left(\vec{h}_0 \times \vec{h}_1\right) \cdot \sigma \tag{23}$$