Simulation of time-bin remote entanglement generation

May 19, 2024

1 Full quantum treatment of the position/motion-dependent decoherence

1.1 From position-dependent phase to recoil

The interaction between an atom and the photon depends on the position of the atom. For excitation, the phase of the light affects the phase of the atomic excited state wavefunction[1]. For emission, the position of the atom affects the phase of the emitted photon.

This becomes a source of decoherence for time-bin entanglement on atomic systems. Classically, we can understand this since the position of the atom is not necessarily the same during the first and the second excitation/photon emission causing a different and random phase on the photon which averages out the coherence fringes. However, this predicts that there would be no decoherence if the motional state of the atom is stationary (i.e. if the atom is in the ground motional state or any of the Fock states $|n\rangle$). Since the thermal state can be expressed as a probabilistic mixture of pure Fock states, this conclusion cannot possibly be correct.

To handle this correctly, we need to construct the unitary operation that corresponds to the photon generation step. Without the position-dependent phase effect, this step maps the atomic (internal) wavefunction in the following way,

$$|0\rangle \rightarrow |0;\mathrm{ph}\rangle \tag{1}$$
$$|i\rangle \rightarrow |i\rangle \tag{2}$$

where the ph represent a photon being generated and $i \neq 0$ are the internal states of the atom that were not excited. When the effect of the motion is included, we can still use the classical picture to write out the new wavefunction. Instead of a fixed phase when the photon is generated, we acquire a phase from the absorbed and emitted photon,

$$|0,\vec{r}\rangle \rightarrow \mathrm{e}^{\mathrm{i}\Delta\vec{k}\cdot\vec{r}}|0,\vec{r};\mathrm{ph}\rangle \tag{3}$$
$$|i,\vec{r}\rangle \rightarrow |i,\vec{r}\rangle \tag{4}$$

where $\Delta\vec{k}$ is the difference between the wavevectors of the absorbed and emitted photon. Note that although we've stated the position dependent phase factor as being on the photon, it, being just a phase, can be treated as acting on any part of the wavefunction, including the motional wavefunction of the atom. Viewing this way, it is essentially the recoil on the motion of the atom and it creates entanglement between spin+photon state and the motional state of the atom.

[1] For a π-pulse, this is a global phase for a two-level system which can be ignored. However the time-bin entanglement scheme necessitate at least a three-level system. In such a system, this phase isn't global anymore and can be experimentally observed by comparing it to the third state

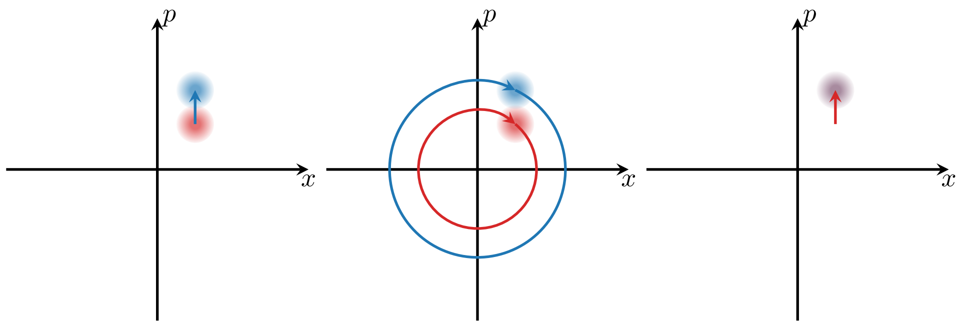

Figure 1:

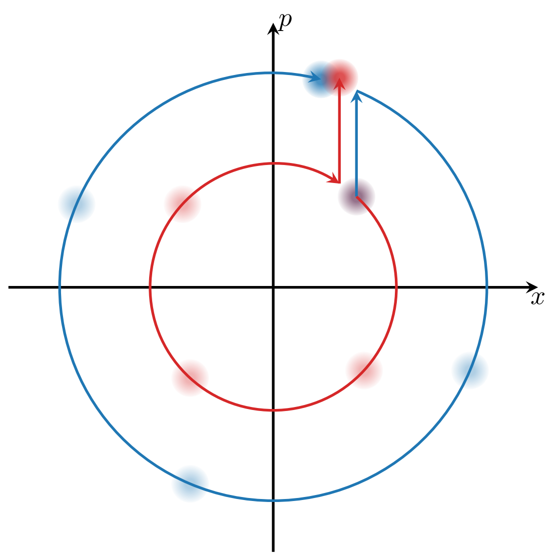

Figure 2:

Based on this understanding, the correct (or at least an equivalent) way to view the motion/position dependent decoherence is that (Fig. 2),

1. During the first photon generation step, the $|0\rangle$ spin state receives a recoil from the absorbed/emitted photon while the $|1\rangle$ spin state remains in the original motional state.

2. During the time between the two excitations, the motion of the atom evolves freely (and differently due to the different initial motional state between the $|0\rangle$ and $|1\rangle$ spin states).

3. During the second photon generation step, the $|1\rangle$ spin state receives a recoil from the absorbed/emitted photon while the motion for the $|0\rangle$ spin state remains unchanged.

4. At this stage, the $|0\rangle$ motional state underwent recoil and then free evolution whereas the $|1\rangle$ motional state underwent free evolution and then recoil. If these two resulting motional states are not exactly the same, we've created a unwanted entanglement between the spin and the atom motion which reduces the fidelity of the spin+photon state that we would like to create[2].

[2]Much the same way motional closure error reduces the fidelity of a Mølmer-Sørensen gate

1.2 Calculation

To calculate the fidelity caused by this entanglement, we need to calculate the reduced density matrix of the spin and photon after the two photon generation steps. Here we'll make a few simplifications and ignore some independent error sources that can be taken into account separately,

1. We'll assume the excitation is complete and also ignore any branching into other spin states.
2. We'll assume the two qubit states are excited directly. In a real experiment the two photon excitation is more likely to be done on the same transition after swapping the two qubit state between the two excitations.
3. We'll only calculate up to the photon generated from the atom. After we've obtained the density matrix of an ion and the photon(s) it has generated, the effect of the photon collection/interference/detection etc., can be considered separately.
4. For simplicity, since full excitation is assumed, we'll omit the photon state in the notation below. It should be understood as $|0\rangle$ representing $|0;\text{early photon}\rangle$ after the first photon generation step, and $|1\rangle$ representing $|1;\text{late photon}\rangle$ after the second photon generation step.
5. We'll calculate for a single motional axis (and assume it's a harmonic oscillator). As long as there's no initial entanglment between the motions on different axis, the result of the calculation can just be applied on each axis independently.

With these simplifications, the initial density matrix of the system is,

$$\rho_0 =\frac{(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)}{2}\rho_{\mathrm{m}} \tag{5}$$

where ρ_{m} is the motional density matrix.

After first photon generation, the density matrix is,

$$\rho_1 =(P_0\mathcal{D}(\mathrm{i}\eta)+P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta)+P_1) \tag{6}$$

where P_0 and P_1 are the projection operators for the $|0\rangle$ and $|1\rangle$ states respectively, $\mathcal{D}(\mathrm{i}\eta)=\mathrm{e}^{\mathrm{i}kx}$ is the displacement operator representing the effect of the photon recoil and η is the Lamb-Dicke parameter.

After the free-evolution between the two generation steps,

$$\begin{aligned}\rho_2 =&\mathrm{e}^{-\mathrm{i}n\omega t}\rho_1\mathrm{e}^{\mathrm{i}n\omega t}\\ =&\mathrm{e}^{-\mathrm{i}n\omega t}(P_0\mathcal{D}(\mathrm{i}\eta)+P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta)+P_1)\mathrm{e}^{\mathrm{i}n\omega t}\end{aligned} \tag{7}$$

where ω is the trap frequency.

After second photon gneeration,

$$\begin{aligned}\rho_3 =&(P_0+P_1\mathcal{D}(\mathrm{i}\eta))\rho_2(P_0+P_1\mathcal{D}(-\mathrm{i}\eta))\\ =&(P_0+P_1\mathcal{D}(\mathrm{i}\eta))\mathrm{e}^{-\mathrm{i}n\omega t}(P_0\mathcal{D}(\mathrm{i}\eta)+P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta)+P_1)\mathrm{e}^{\mathrm{i}n\omega t}(P_0+P_1\mathcal{D}(-\mathrm{i}\eta))\\ =&\big(P_0\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)+P_1\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\big)\rho_0\big(P_0\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}+P_1\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)\end{aligned} \tag{8}$$

This result is currently generic and can be applied to any initial motional state. For the experiment, however, we are more interested in using a thermal state as the initial state. It is useful to use a representation of the density matrix using coherent states (as will be obvious from the derivation below),

$$\rho_{\mathrm{m}} = \frac{1}{\pi\bar{n}} \int \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha| \mathrm{e}^{-|\alpha|^2/\bar{n}} \tag{9}$$

where $\bar{n} \equiv \dfrac{1}{\mathrm{e}^{\beta\omega} - 1}$ is the average n.

We can now write the final density matrix as,

$$\begin{aligned}\rho_3 = &\int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \big(|0\rangle \mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta) + |1\rangle\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\big)|\alpha\rangle \\ &\langle\alpha|\big(\langle 0|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t} + \langle 1|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)\end{aligned} \tag{10}$$

After tracing out the motion part, the spin reduced density matrix is,

$$\begin{aligned}\rho_{3,\mathrm{s}} =& \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \begin{pmatrix} |0\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \\ +|1\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \end{pmatrix} \\ =& \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \begin{pmatrix} |0\rangle\langle 0| + |1\rangle\langle 1| \\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \end{pmatrix}\end{aligned} \tag{11}$$

The coefficient on the off-diagonal term can be calculated as (thanks to using the coherent states as the basis),

$$\begin{aligned} &\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \\ =&\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\mathrm{e}^{-\mathrm{i}\omega t}\alpha\rangle \\ =&\mathrm{e}^{(\mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t})/2}\mathrm{e}^{(-\mathrm{i}\eta\alpha^*-\mathrm{i}\eta\alpha)/2}\langle\alpha+\mathrm{i}\eta|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\rangle \\ =&\exp\left(\frac{1}{2}\begin{pmatrix} \mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{i}\eta\alpha^* - \mathrm{i}\eta\alpha \\ -|\alpha+\mathrm{i}\eta|^2 - \left|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\right|^2 + 2(\alpha^*-\mathrm{i}\eta)\big(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\big)\end{pmatrix}\right) \\ =&\exp\Big(\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{i}\eta\alpha^* - |\alpha|^2 - \eta^2 + (\alpha^*-\mathrm{i}\eta)\big(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\big)\Big) \\ =&\exp\Big(2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) + \eta^2\big(\mathrm{e}^{\mathrm{i}\omega t}-1\big)\Big)\end{aligned} \tag{12}$$

which gives us

$$\begin{aligned}\rho_{3,\mathrm{s}} =& \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \begin{pmatrix} |0\rangle\langle 0| + |1\rangle\langle 1| \\ +|0\rangle\langle 1|\exp\big(-2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) + \eta^2\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) \\ +|1\rangle\langle 0|\exp\big(2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) + \eta^2\big(\mathrm{e}^{\mathrm{i}\omega t}-1\big)\big)\end{pmatrix} \\ =& \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + |0\rangle\langle 1| \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \exp\big(-2\mathrm{i}\eta\mathrm{Re}\big(\alpha\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) + \eta^2\big(\mathrm{e}^{-\mathrm{i}\omega t}-1\big)\big) + h.c.\end{aligned} \tag{13}$$

And we can calculate the integral,

$$\begin{aligned}
&\int \mathrm{d}^2\alpha \mathrm{e}^{-|\alpha|^2/\bar{n}} \exp\left(2\mathrm{i}\eta \mathrm{Re}\left(\alpha\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)\right)+\eta^2\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)\right)\\
=&\mathrm{e}^{\eta^2\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)}\int \mathrm{d}^2\alpha \mathrm{e}^{-|\alpha|^2/\bar{n}} \exp\left(2\mathrm{i}\eta \mathrm{Re}((\alpha_x+\mathrm{i}\alpha_y)(\cos\omega t-1-\mathrm{i}\sin\omega t))\right)\\
=&\mathrm{e}^{\eta^2\left(\mathrm{e}^{-\mathrm{i}\omega t}-1\right)}\int \mathrm{d}^2\alpha \exp\left(-\alpha_x^2/\bar{n}+2\mathrm{i}\alpha_x\eta(\cos\omega t-1)\right)\exp\left(-\alpha_y^2/\bar{n}+2\mathrm{i}\alpha_y\eta\sin\omega t\right)\\
=&\pi\bar{n}\exp\left(-\eta^2\left(1-\mathrm{e}^{-\mathrm{i}\omega t}+2\bar{n}(1-\cos\omega t)\right)\right)\\
=&\pi\bar{n}\exp\left(-\mathrm{i}\eta^2\sin\omega t\right)\exp\left(-\eta^2(1-\cos\omega t)(2\bar{n}+1)\right)
\end{aligned} \tag{14}$$

And the final reduced density matrix,

$$\rho_{3,\mathrm{s}} = \frac{|0\rangle\langle 0|+|1\rangle\langle 1|+|0\rangle\langle 1|\exp\left(\mathrm{i}\eta^2\sin\omega t\right)\exp\left(-\eta^2(1-\cos\omega t)(2\bar{n}+1)\right)+h.c.}{2} \tag{15}$$

The off-diagonal element of the reduced density matrix is the important part that specifies the final spin (+ photon) state. This is the number that can directly be multiplied when multiple motional axis is considered.

Its amplitude quatifies the purity of the spin state (i.e. parity scan contrast) which is

$$\exp\left(-\eta^2(1-\cos\omega t)(2\bar{n}+1)\right) \tag{16}$$

It's also worth noting that the is a small (and temperature independent) spin phase factor as well (i.e. parity scan phase) that is

$$\exp\left(\mathrm{i}\eta^2\sin\omega t\right) \tag{17}$$

The maximum phase is η^2 which can be completely ignored if η is small enough.

2 Motion of the atom during emission

The discussion in the previous section assumes the emission of the photon to be instantaneous. However, in reality, this happens over the lifetime of the excited state. For photon that's emitted at different time, the recoil on the atom would also happen at a different time, resulting in a different final atom motional wavefunction. This resulting entanglement between the photon wavefunction and the motion wavefunction could affect the fidelity of the final spin-entanglement.

For this discussion, we will ignore the effect of imperfect timing mismatch between the two excitations since it's already covered above. We can then simply assume that the two excitations happened at exactly the same time and label the two photon states $|A\rangle$ and $|B\rangle$ instead. (This two photon states can be early vs late for time-bin qubit, horizontal/vertical for polarization qubit or different frequencies for frequency qubit.)

For an initial state of

$$\rho_0 = \frac{(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)}{2}\rho_\mathrm{m} \tag{18}$$

After excitation the state is

$$\rho_1 = \frac{(|e_0\rangle+|e_1\rangle)(\langle e_0|+\langle e_1|)}{2}\mathcal{D}(\mathrm{i}\eta_\mathrm{d})\rho_\mathrm{m}\mathcal{D}(-\mathrm{i}\eta_\mathrm{d}) \tag{19}$$

where e_0 and e_1 are the two excited states and, η_d is the Lamb-Dicke parameter for the drive (excitation) photon.

Since the decay happen over a variable length of time, it's difficult to put a bound on when the decay "finished". Therefore, the post-decay wavefunction we'll use is the one that is back-time-propagated to the start of the decay using the free-evolution Hamiltonian. This is of course not physical, but it allows us to treat the time evolution after the excitation to be simply the free evolution.

$$\begin{aligned}\rho_2 =&\Gamma^2\int_0^\infty \mathrm{d}\tau_1 \mathrm{e}^{-\Gamma\tau_1/2}\int_0^\infty \mathrm{d}\tau_2 \mathrm{e}^{-\Gamma\tau_2/2}\frac{(|0,A(\tau_1)\rangle+|1,B(\tau_1)\rangle)(\langle 0,A(\tau_2)|+\langle 1,B(\tau_2)|)}{2}\\ &\mathrm{e}^{-\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{d})\rho_\mathrm{m}\mathcal{D}(-\mathrm{i}\eta_\mathrm{d})\mathrm{e}^{-\mathrm{i}H\tau_2}\mathcal{D}(-\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_2}\end{aligned}\tag{20}$$

where τ_1 and τ_2 are "emission time" for the photons. Substitude in the thermal motional density matrix.

$$\begin{aligned}\rho_2 =&\frac{\Gamma^2}{\pi\bar{n}}\int_0^\infty \mathrm{d}\tau_1 \mathrm{e}^{-\Gamma\tau_1/2}\int_0^\infty \mathrm{d}\tau_2 \mathrm{e}^{-\Gamma\tau_2/2}\frac{(|0,A(\tau_1)\rangle+|1,B(\tau_1)\rangle)(\langle 0,A(\tau_2)|+\langle 1,B(\tau_2)|)}{2}\\ &\mathrm{e}^{-\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{d})\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha|\mathrm{e}^{-|\alpha|^2/\bar{n}}\mathcal{D}(-\mathrm{i}\eta_\mathrm{d})\mathrm{e}^{-\mathrm{i}H\tau_2}\mathcal{D}(-\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_2}\end{aligned}\tag{21}$$

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{d})|\alpha\rangle\\ =&\exp\left(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)\right)\mathrm{e}^{-\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{e})\mathrm{e}^{\mathrm{i}H\tau_1}|\alpha+\mathrm{i}\eta_\mathrm{d}\rangle\\ =&\exp\left(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)\right)\mathrm{e}^{-\mathrm{i}H\tau_1}\mathcal{D}(\mathrm{i}\eta_\mathrm{e})|\mathrm{e}^{\mathrm{i}\omega\tau_1}(\alpha+\mathrm{i}\eta_\mathrm{d})\rangle\\ =&\exp\left(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)+\mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^*-\mathrm{i}\eta_\mathrm{d}))\right)\mathrm{e}^{-\mathrm{i}H\tau_1}|\mathrm{e}^{\mathrm{i}\omega\tau_1}(\alpha+\mathrm{i}\eta_\mathrm{d})+\mathrm{i}\eta_\mathrm{e}\rangle\\ =&\exp\left(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)+\mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^*-\mathrm{i}\eta_\mathrm{d}))\right)|\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_1}\rangle\end{aligned}\tag{22}$$

$$\begin{aligned}\rho_2 =&\frac{\Gamma^2}{\pi\bar{n}}\int_0^\infty \mathrm{d}\tau_1 \mathrm{e}^{-\Gamma\tau_1/2}\int_0^\infty \mathrm{d}\tau_2 \mathrm{e}^{-\Gamma\tau_2/2}\frac{(|0,A(\tau_1)\rangle+|1,B(\tau_1)\rangle)(\langle 0,A(\tau_2)|+\langle 1,B(\tau_2)|)}{2}\\ &\int \mathrm{d}^2\alpha\exp\left(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)+\mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^*-\mathrm{i}\eta_\mathrm{d}))\right)\exp\left(-\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha)-\mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_2}(\alpha^*-\mathrm{i}\eta_\mathrm{d}))\right)\\ &\mathrm{e}^{-|\alpha|^2/\bar{n}}|\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_1}\rangle\langle\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_2}|\end{aligned}\tag{23}$$

Note that unlike the previous case, the unwanted entanglement create is entirely between two degrees of freedoms outside the qubit space and does not affect the fidelity of an atom-photon entanglement fidelity. To see the effect of this, we need to look at the state of the two atoms combined.

$$\begin{aligned}\rho_2' =&\frac{\Gamma^4}{\pi^2\bar{n}_a\bar{n}_b}\int_0^\infty \mathrm{d}\tau_{a1}\mathrm{e}^{-\Gamma\tau_{a1}/2}\int_0^\infty \mathrm{d}\tau_{a2}\mathrm{e}^{-\Gamma\tau_{a2}/2}\int_0^\infty \mathrm{d}\tau_{b1}\mathrm{e}^{-\Gamma\tau_{b1}/2}\int_0^\infty \mathrm{d}\tau_{b2}\mathrm{e}^{-\Gamma\tau_{b2}/2}\\ &\frac{(|0_a,A_a(\tau_{a1})\rangle+|1_a,B_a(\tau_{a1})\rangle)(\langle 0_a,A_a(\tau_{a2})|+\langle 1_a,B_a(\tau_{a2})|)}{2}\\ &\frac{(|0_b,A_b(\tau_{b1})\rangle+|1_b,B_b(\tau_{b1})\rangle)(\langle 0_b,A_b(\tau_{b2})|+\langle 1_b,B_b(\tau_{b2})|)}{2}\\ &\int \mathrm{d}^2\alpha_a\exp\left(\mathrm{i}\eta_{a\mathrm{d}}\mathrm{Re}(\alpha_a)+\mathrm{i}\eta_{a\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a1}}(\alpha_a^*-\mathrm{i}\eta_{a\mathrm{d}}))\right)\exp\left(-\mathrm{i}\eta_{a\mathrm{d}}\mathrm{Re}(\alpha_a)-\mathrm{i}\eta_{a\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a2}}(\alpha_a^*-\mathrm{i}\eta_{a\mathrm{d}}))\right)\\ &\mathrm{e}^{-|\alpha_a|^2/\bar{n}_a}|\alpha_a+\mathrm{i}\eta_{a\mathrm{d}}+\mathrm{i}\eta_{a\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a1}}\rangle\langle\alpha_a+\mathrm{i}\eta_{a\mathrm{d}}+\mathrm{i}\eta_{a\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a2}}|\\ &\int \mathrm{d}^2\alpha_b\exp\left(\mathrm{i}\eta_{b\mathrm{d}}\mathrm{Re}(\alpha_b)+\mathrm{i}\eta_{b\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b1}}(\alpha_b^*-\mathrm{i}\eta_{b\mathrm{d}}))\right)\exp\left(-\mathrm{i}\eta_{b\mathrm{d}}\mathrm{Re}(\alpha_b)-\mathrm{i}\eta_{b\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b2}}(\alpha_b^*-\mathrm{i}\eta_{b\mathrm{d}}))\right)\\ &\mathrm{e}^{-|\alpha_b|^2/\bar{n}_b}|\alpha_b+\mathrm{i}\eta_{b\mathrm{d}}+\mathrm{i}\eta_{b\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b1}}\rangle\langle\alpha_b+\mathrm{i}\eta_{b\mathrm{d}}+\mathrm{i}\eta_{b\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b2}}|\end{aligned}\tag{24}$$

After the beam splitter we have

$$A_a(\tau) \to \frac{1}{\sqrt{2}}\big(A^1(\tau) + A^2(\tau)\big) \tag{25}$$

$$B_a(\tau) \to \frac{1}{\sqrt{2}}\big(B^1(\tau) + B^2(\tau)\big) \tag{26}$$

$$A_b(\tau) \to \frac{1}{\sqrt{2}}\big(A^1(\tau) - A^2(\tau)\big) \tag{27}$$

$$B_b(\tau) \to \frac{1}{\sqrt{2}}\big(B^1(\tau) - B^2(\tau)\big) \tag{28}$$

where A^1, B^1, A^2, B^2 are the four potential detections at the output of the beam splitter. For simplicity, we'll only consider one detection possiblity $A^1(\tau_A)B^1(\tau_B)$. We now just need to compute the conditional density matrix when this event happens. The unnormalized reduced density matrix (trace out motion),

$$\begin{aligned}
\rho_3' = &\int_0^\infty \mathrm{d}\tau_{a1}\mathrm{e}^{-\Gamma\tau_{a1}/2}\int_0^\infty \mathrm{d}\tau_{a2}\mathrm{e}^{-\Gamma\tau_{a2}/2}\int_0^\infty \mathrm{d}\tau_{b1}\mathrm{e}^{-\Gamma\tau_{b1}/2}\int_0^\infty \mathrm{d}\tau_{b2}\mathrm{e}^{-\Gamma\tau_{b2}/2}\\
&\big(\delta(\tau_{b1}-\tau_A)\delta(\tau_{a1}-\tau_B)|1_a,0_b\rangle + \delta(\tau_{a1}-\tau_A)\delta(\tau_{b1}-\tau_B)|0_a,1_b\rangle\big)\\
&\big(\langle 1_a,0_b|\delta(\tau_{b2}-\tau_A)\delta(\tau_{a2}-\tau_B) + \langle 0_a,1_b|\delta(\tau_{a2}-\tau_A)\delta(\tau_{b2}-\tau_B)\big)\\
&\int \mathrm{d}^2\alpha_a \exp\big(\mathrm{i}\eta_{ad}\mathrm{Re}(\alpha_a) + \mathrm{i}\eta_{ae}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a1}}(\alpha_a^* - \mathrm{i}\eta_{a\mathrm{d}}))\big)\exp\big(-\mathrm{i}\eta_{ad}\mathrm{Re}(\alpha_a) - \mathrm{i}\eta_{ae}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a2}}(\alpha_a^* - \mathrm{i}\eta_{a\mathrm{d}}))\big)\\
&\mathrm{e}^{-|\alpha_a|^2/\bar{n}_a}\langle \alpha_a + \mathrm{i}\eta_{a\mathrm{d}} + \mathrm{i}\eta_{ae}\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a2}}|\alpha_a + \mathrm{i}\eta_{a\mathrm{d}} + \mathrm{i}\eta_{ae}\mathrm{e}^{-\mathrm{i}\omega_a\tau_{a1}}\rangle\\
&\int \mathrm{d}^2\alpha_b \exp\big(\mathrm{i}\eta_{b\mathrm{d}}\mathrm{Re}(\alpha_b) + \mathrm{i}\eta_{b\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b1}}(\alpha_b^* - \mathrm{i}\eta_{b\mathrm{d}}))\big)\exp\big(-\mathrm{i}\eta_{b\mathrm{d}}\mathrm{Re}(\alpha_b) - \mathrm{i}\eta_{b\mathrm{e}}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b2}}(\alpha_b^* - \mathrm{i}\eta_{b\mathrm{d}}))\big)\\
&\mathrm{e}^{-|\alpha_b|^2/\bar{n}_b}\langle \alpha_b + \mathrm{i}\eta_{b\mathrm{d}} + \mathrm{i}\eta_{b\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b2}}|\alpha_b + \mathrm{i}\eta_{b\mathrm{d}} + \mathrm{i}\eta_{b\mathrm{e}}\mathrm{e}^{-\mathrm{i}\omega_b\tau_{b1}}\rangle
\end{aligned} \tag{29}$$

The motion integral,

$$\begin{aligned}
&\int \mathrm{d}^2\alpha \exp\big(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) + \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\exp\big(-\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) - \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_2}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\\
&\mathrm{e}^{-|\alpha|^2/\bar{n}}\langle \alpha + \mathrm{i}\eta_\mathrm{d} + \mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_2}|\alpha + \mathrm{i}\eta_\mathrm{d} + \mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_1}\rangle\\
=&\int \mathrm{d}^2\alpha \exp\big(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) + \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\exp\big(-\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) - \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_2}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\\
&\mathrm{e}^{-|\alpha|^2/\bar{n}}\mathrm{e}^{-\left(\left|\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_2}\right|^2+\left|\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_1}\right|^2-2\left(\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_2}\right)^*\left(\alpha+\mathrm{i}\eta_\mathrm{d}+\mathrm{i}\eta_\mathrm{e}\mathrm{e}^{-\mathrm{i}\omega\tau_1}\right)\right)/2}\\
=&\int \mathrm{d}^2\alpha \exp\big(\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) + \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_1}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\exp\big(-\mathrm{i}\eta_\mathrm{d}\mathrm{Re}(\alpha) - \mathrm{i}\eta_\mathrm{e}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega\tau_2}(\alpha^* - \mathrm{i}\eta_\mathrm{d}))\big)\\
&\mathrm{e}^{-|\alpha|^2/\bar{n}}\exp\Big(\mathrm{i}\eta_\mathrm{e}\mathrm{Re}\big((\alpha+\mathrm{i}\eta_\mathrm{d})(\mathrm{e}^{\mathrm{i}\omega\tau_1} - \mathrm{e}^{\mathrm{i}\omega\tau_2})\big) - \eta_\mathrm{e}^2\Big(1 - \mathrm{e}^{\mathrm{i}\omega(\tau_2-\tau_1)}\Big)\Big)\\
=&\exp\Big(-2\mathrm{i}\eta_\mathrm{e}\eta_\mathrm{d}(\sin\omega\tau_1 - \sin\omega\tau_2) - \eta_\mathrm{e}^2\Big(1 - \mathrm{e}^{\mathrm{i}\omega(\tau_2-\tau_1)}\Big)\Big)\\
&\int \mathrm{d}^2\alpha \exp\big(-\alpha_x^2/\bar{n} + 2\mathrm{i}\eta_\mathrm{e}(\cos\omega\tau_1 - \cos\omega\tau_2)\alpha_x - \alpha_y^2/\bar{n} - 2\mathrm{i}\eta_\mathrm{e}(\sin\omega\tau_1 - \sin\omega\tau_2)\alpha_y\big)\\
=&\exp\Big(-2\mathrm{i}\eta_\mathrm{e}\eta_\mathrm{d}(\sin\omega\tau_1 - \sin\omega\tau_2) - \eta_\mathrm{e}^2\Big(1 - \mathrm{e}^{\mathrm{i}\omega(\tau_2-\tau_1)}\Big)\Big)\\
&\pi\bar{n}\exp\Big(-\bar{n}\eta_\mathrm{e}^2(\cos\omega\tau_1 - \cos\omega\tau_2)^2\Big)\exp\Big(-\bar{n}\eta_\mathrm{e}^2(\sin\omega\tau_1 - \sin\omega\tau_2)^2\Big)\\
=&\pi\bar{n}\exp\Big(-2\mathrm{i}\eta_\mathrm{e}\eta_\mathrm{d}(\sin\omega\tau_1 - \sin\omega\tau_2) - \eta_\mathrm{e}^2\Big(1 - \mathrm{e}^{\mathrm{i}\omega(\tau_2-\tau_1)}\Big) - 2\bar{n}\eta_\mathrm{e}^2(1 - \cos\omega(\tau_1-\tau_2))\Big)
\end{aligned} \tag{30}$$

Ignoring the constant factors the unnormalized reduced density matrix,

$$\begin{aligned}\rho_3' =& \int_0^\infty \mathrm{d}\tau_{a1} \mathrm{e}^{-\Gamma\tau_{a1}/2} \int_0^\infty \mathrm{d}\tau_{a2} \mathrm{e}^{-\Gamma\tau_{a2}/2} \int_0^\infty \mathrm{d}\tau_{b1} \mathrm{e}^{-\Gamma\tau_{b1}/2} \int_0^\infty \mathrm{d}\tau_{b2} \mathrm{e}^{-\Gamma\tau_{b2}/2}\\
&(\delta(\tau_{b1}-\tau_A)\delta(\tau_{a1}-\tau_B)|1_a,0_b\rangle + \delta(\tau_{a1}-\tau_A)\delta(\tau_{b1}-\tau_B)|0_a,1_b\rangle)\\
&(\langle 1_a,0_b|\delta(\tau_{b2}-\tau_A)\delta(\tau_{a2}-\tau_B) + \langle 0_a,1_b|\delta(\tau_{a2}-\tau_A)\delta(\tau_{b2}-\tau_B))\\
&\exp\left(-2\mathrm{i}\eta_{ae}\eta_{\mathrm{ad}}(\sin\omega_a\tau_{a1}-\sin\omega_a\tau_{a2}) - \eta_{ae}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_a(\tau_{a2}-\tau_{a1})}\right) - 2\bar{n}_a\eta_{ae}^2(1-\cos\omega_a(\tau_{a1}-\tau_{a2}))\right)\\
&\exp\left(-2\mathrm{i}\eta_{\mathrm{be}}\eta_{\mathrm{bd}}(\sin\omega_b\tau_{b1}-\sin\omega_b\tau_{b2}) - \eta_{\mathrm{be}}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_b(\tau_{b2}-\tau_{b1})}\right) - 2\bar{n}_b\eta_{\mathrm{be}}^2(1-\cos\omega_b(\tau_{b1}-\tau_{b2}))\right)\end{aligned} \tag{31}$$

Diagonal element of the reduced density matrix element (trace out motion),

$$\begin{aligned}&\langle 0_a,1_b|\rho_3'|0_a,1_b\rangle\\
=&\mathrm{e}^{-\Gamma(\tau_A+\tau_B)}\\
&\exp\left(-2\mathrm{i}\eta_{ae}\eta_{\mathrm{ad}}(\sin\omega_a\tau_A-\sin\omega_a\tau_A) - \eta_{ae}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_a(\tau_A-\tau_A)}\right) - 2\bar{n}_a\eta_{ae}^2(1-\cos\omega_a(\tau_A-\tau_A))\right)\\
&\exp\left(-2\mathrm{i}\eta_{\mathrm{be}}\eta_{\mathrm{bd}}(\sin\omega_b\tau_B-\sin\omega_b\tau_B) - \eta_{\mathrm{be}}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_b(\tau_B-\tau_B)}\right) - 2\bar{n}_b\eta_{\mathrm{be}}^2(1-\cos\omega_b(\tau_B-\tau_B))\right)\\
=&\mathrm{e}^{-\Gamma\tau_A}\mathrm{e}^{-\Gamma\tau_B}\end{aligned} \tag{32}$$

$$\begin{aligned}&\langle 1_a,0_b|\rho_3'|1_a,0_b\rangle\\
=&\mathrm{e}^{-\Gamma(\tau_A+\tau_B)}\end{aligned} \tag{33}$$

These suggests that the population isn't affected by this entanglment and the probability of detect goes as the lifetime suggests. This result isn't surprising or interesting.

Now the more interesting term is the off-diagonal element.

$$\begin{aligned}&\langle 0_a,1_b|\rho_3'|1_a,0_b\rangle\\
=&\mathrm{e}^{-\Gamma(\tau_A+\tau_B)}\\
&\exp\left(-2\mathrm{i}\eta_{ae}\eta_{\mathrm{ad}}(\sin\omega_a\tau_A-\sin\omega_a\tau_B) - \eta_{ae}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_a(\tau_B-\tau_A)}\right) - 2\bar{n}_a\eta_{ae}^2(1-\cos\omega_a(\tau_A-\tau_B))\right)\\
&\exp\left(-2\mathrm{i}\eta_{\mathrm{be}}\eta_{\mathrm{bd}}(\sin\omega_b\tau_B-\sin\omega_b\tau_A) - \eta_{\mathrm{be}}^2\left(1-\mathrm{e}^{\mathrm{i}\omega_b(\tau_A-\tau_B)}\right) - 2\bar{n}_b\eta_{\mathrm{be}}^2(1-\cos\omega_b(\tau_B-\tau_A))\right)\\
=&\mathrm{e}^{-\Gamma(\tau_A+\tau_B)}\\
&\exp\left(-2\mathrm{i}(\eta_{\mathrm{ae}}\eta_{\mathrm{ad}}+\eta_{\mathrm{be}}\eta_{\mathrm{bd}})(\sin\omega_a\tau_A-\sin\omega_a\tau_B) - \mathrm{i}\eta_{\mathrm{ae}}^2(\sin\omega_a(\tau_A-\tau_B)) + \mathrm{i}\eta_{\mathrm{be}}^2(\sin\omega_b(\tau_A-\tau_B))\right)\\
&\exp\left(-(2\bar{n}_a+1)\eta_{\mathrm{ae}}^2(1-\cos\omega_a(\tau_A-\tau_B))\right)\exp\left(-(2\bar{n}_b+1)\eta_{\mathrm{be}}^2(1-\cos\omega_b(\tau_A-\tau_B))\right)\end{aligned} \tag{34}$$

We can compare this with the normalization factor to get the real diagonal element. Similar to the previous case, there is a phase factor that's proportional to the square of the LD parameters

$$\exp\left(-2\mathrm{i}(\eta_{\mathrm{ae}}\eta_{\mathrm{ad}}+\eta_{\mathrm{be}}\eta_{\mathrm{bd}})(\sin\omega_a\tau_A-\sin\omega_a\tau_B) - \mathrm{i}\eta_{\mathrm{ae}}^2(\sin\omega_a(\tau_A-\tau_B)) + \mathrm{i}\eta_{\mathrm{be}}^2(\sin\omega_b(\tau_A-\tau_B))\right) \tag{35}$$

This factor can be completely ignored for ion experiments.

The fidelity factor is,

$$\exp\left(-(2\bar{n}_a+1)\eta_{\mathrm{ae}}^2(1-\cos\omega_a(\tau_A-\tau_B))\right)\exp\left(-(2\bar{n}_b+1)\eta_{\mathrm{be}}^2(1-\cos\omega_b(\tau_A-\tau_B))\right) \tag{36}$$

This is 1 when $\tau_A = \tau_B$, which makes sense since there's no leakage of information from the motional state of the atoms.