Simulation of time-bin remote entanglement generation

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1 Full quantum treatment of the position/motion-dependent decoherence

1.1 From position-dependent phase to recoil

The interaction between an atom and the photon depends on the position of the atom. For excitation, the phase of the light affects the phase of the atomic excited state wavefunction[1]. For emission, the position of the atom affects the phase of the emitted photon.

This becomes a source of decoherence for time-bin entanglement on atomic systems. Classically, we can understand this since the position of the atom is not necessarily the same during the first and the second excitation/photon emission causing a different and random phase on the photon which averages out the coherence fringes. However, this predicts that there would be no decoherence if the motional state of the atom is stationary (i.e. if the atom is in the ground motional state or any of the Fock states $|n\rangle$). Since the thermal state can be expressed as a probabilistic mixture of pure Fock states, this conclusion cannot possibly be correct.

To handle this correctly, we need to construct the unitary operation that corresponds to the photon generation step. Without the position-dependent phase effect, this step maps the atomic (internal) wavefunction in the following way,

$$|0\rangle \to |0; \mathrm{ph}\rangle \tag{1}$$
$$|i\rangle \to |i\rangle \tag{2}$$

where the ph represent a photon being generated and $i \neq 0$ are the internal states of the atom that were not excited. When the effect of the motion is included, we can still use the classical picture to write out the new wavefunction. Instead of a fixed phase when the photon is generated, we acquire a phase from the absorbed and emitted photon,

$$|0, \vec{r}\rangle \to e^{i\Delta\vec{k}\cdot\vec{r}}|0, \vec{r}; \mathrm{ph}\rangle \tag{3}$$
$$|i, \vec{r}\rangle \to |i, \vec{r}\rangle \tag{4}$$

where $\Delta\vec{k}$ is the difference between the wavevectors of the absorbed and emitted photon. Note that although we've stated the position dependent phase factor as being on the photon, it, being just a phase, can be treated as acting on any part of the wavefunction, including the motional wavefunction of the atom. Viewing this way, it is essentially the recoil on the motion of the atom and it creates entanglement between spin+photon state and the motional state of the atom.

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[1]For a π-pulse, this is a global phase for a two-level system which can be ignored. However the time-bin entanglement scheme necessitate at least a three-level system. In such a system, this phase isn't global anymore and can be experimentally observed by comparing it to the third state

Based on this understanding, the correct (or at least an equivalent way) to view the motion/position dependent decoherence is that,

1. During the first photon generation step, the $|0\rangle$ spin state received a recoil from the absorbed/emitted photon while the $|1\rangle$ spin state remains in the original motional state.

2. During the free evolution time between the two excitations, the motion of the atom evolves freely (and differently due to the different initial motional state between the $|0\rangle$ and $|1\rangle$ spin states).

3. During the second photon generation step, the $|1\rangle$ spin state received a recoil from the absorbed/emitted photon while the motion for the $|0\rangle$ spin state remains unchanged.

4. At this state, the $|0\rangle$ motional state underwent recoil-free evolution whereas the $|1\rangle$ motional state underwent free evolution-recoil. If these two motional states are not exactly the same, we've created a unwanted entanglement between the spin and the atom motion which reduces the fidelity of the spin+photon state that we would like to create[2].

1.2 Calculation

To calculate the fidelity caused by this entanglement, we need to calculate the reduced density matrix of the spin and photon after the two photon generation steps. Here we'll make a few simplifications and ignore some independently error sources that can be taken into account separately,

1. We'll assume the excitation is complete
   and also ignore any branching into other states.

2. We'll assume the two qubit state are excited directly. In a real experiment the two photon excitation is more likely to be done on the same transition after swapping the two qubit state between the two excitations.

3. We'll only calculate up to the photon generated from the atom. After we've obtained the density matrix of an ion and the photon(s) it generated, the effect of the photon collection/interference/detection etc., can be considered separately.

4. For simplicity, since full excitation is assumed, we'll omit the photon state in the notation below. It should be understood as $|0\rangle$ representing $|0;\mathrm{earlyphoton}\rangle$ after the first photon generation step, and $|1\rangle$ representing $|1;\mathrm{latephoton}\rangle$ after the second photon generation step.

5. We'll calculate for a single motional axis (and assume it's a harmonic oscillator). As long as there's no initial entanglment between the motions on different axis, the result of the calculation can just be applied on each axis independently.

With these simplifications, the initial density matrix of the system is,

$$\rho_0 = \frac{(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)}{2}\rho_{\mathrm{m}} \tag{5}$$

where ρ_{m} is the motional density matrix.

After first photon generation, the density matrix is,

$$\rho_1 = (P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1) \tag{6}$$

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[2]Much the same way motional closure error reduces the fidelity of a Mølmer-Sørensen gate

where P_0 and P_1 are projection operation for the $|0\rangle$ and $|1\rangle$ states, $\mathcal{D}(\mathrm{i}\eta) = \mathrm{e}^{\mathrm{i}kx}$ is the displacement operator representing the effect of the photon recoil and η is the Lamb-Dicke parameter.

After the free-evolution between the two generation steps,

$$\rho_2 = \mathrm{e}^{-\mathrm{i}n\omega t}(P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1)\mathrm{e}^{\mathrm{i}n\omega t} \tag{7}$$

where ω is the trap frequency.

After second photon gneeration,

$$\begin{aligned}
\rho_3 &= (P_0 + P_1\mathcal{D}(\mathrm{i}\eta))\mathrm{e}^{-\mathrm{i}n\omega t}(P_0\mathcal{D}(\mathrm{i}\eta) + P_1)\rho_0(P_0\mathcal{D}(-\mathrm{i}\eta) + P_1)\mathrm{e}^{\mathrm{i}n\omega t}(P_0 + P_1\mathcal{D}(-\mathrm{i}\eta)) \\
&= \big(P_0\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta) + P_1\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\big)\rho_0\big(P_0\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t} + P_1\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)
\end{aligned} \tag{8}$$

This result is currently generic and can by applied to any initial motional state. For the experiment, however, we are more interested in using a thermal state as the initial state. It is useful to use a representation of the density matrix using coherent states (as will be obvious from the derivation below),

$$\rho_m = \frac{1}{\pi\bar{n}} \int \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha|\mathrm{e}^{-|\alpha|^2/\bar{n}} \tag{9}$$

where $\bar{n} \equiv \dfrac{1}{\mathrm{e}^{\beta\omega} - 1}$ is the average n.

We can now write the final density matrix as,

$$\begin{aligned}
\rho_3 = \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \big(|0\rangle\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta) + |1\rangle\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\big)|\alpha\rangle \\
\langle\alpha|\big(\langle 0|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t} + \langle 1|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\big)
\end{aligned} \tag{10}$$

After tracing out the motion part, the spin reduced density matrix is,

$$\begin{aligned}
\rho_{3,\mathrm{s}} &= \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \begin{pmatrix} |0\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \\ +|1\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \end{pmatrix} \\
&= \int \mathrm{d}^2\alpha \frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}} \begin{pmatrix} |0\rangle\langle 0| + |1\rangle\langle 1| \\ +|0\rangle\langle 1|\langle\alpha|\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\alpha\rangle \\ +|1\rangle\langle 0|\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle \end{pmatrix}
\end{aligned} \tag{11}$$

The coefficient on the off-diagonal term can be calculated as (thanks to using the coherent states

as the basis),

$$
\begin{aligned}
&\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)\mathrm{e}^{-\mathrm{i}n\omega t}|\alpha\rangle\\
=&\langle\alpha|\mathcal{D}(-\mathrm{i}\eta)\mathrm{e}^{\mathrm{i}n\omega t}\mathcal{D}(\mathrm{i}\eta)|\mathrm{e}^{-\mathrm{i}\omega t}\alpha\rangle\\
=&\mathrm{e}^{(\mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t})/2}\mathrm{e}^{(-\mathrm{i}\eta\alpha^*-\mathrm{i}\eta\alpha)/2}\langle\alpha+\mathrm{i}\eta|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\rangle\\
=&\exp\left(\frac{1}{2}\left(\begin{array}{c}\mathrm{i}\eta\alpha^*\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t}-\mathrm{i}\eta\alpha^*-\mathrm{i}\eta\alpha\\-|\alpha+\mathrm{i}\eta|^2-\left|\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t}\right|^2+2(\alpha^*-\mathrm{i}\eta)(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t})\end{array}\right)\right)\\
=&\exp\left(\mathrm{i}\eta\alpha\mathrm{e}^{-\mathrm{i}\omega t}-\mathrm{i}\eta\alpha^*-|\alpha|^2-\eta^2+(\alpha^*-\mathrm{i}\eta)(\alpha+\mathrm{i}\eta\mathrm{e}^{\mathrm{i}\omega t})\right)\\
=&\exp\left(2\mathrm{i}\eta\mathrm{Re}(\alpha(\mathrm{e}^{-\mathrm{i}\omega t}-1))+\eta^2(\mathrm{e}^{\mathrm{i}\omega t}-1)\right)
\end{aligned}
\tag{12}
$$

which gives us

$$
\begin{aligned}
\rho_{3,\mathrm{s}}=&\int\mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}}\left(\begin{array}{c}|0\rangle\langle0|+|1\rangle\langle1|\\+|0\rangle\langle1|\exp\left(-2\mathrm{i}\eta\mathrm{Re}(\alpha(\mathrm{e}^{-\mathrm{i}\omega t}-1))+\eta^2(\mathrm{e}^{-\mathrm{i}\omega t}-1)\right)\\+|1\rangle\langle0|\exp\left(2\mathrm{i}\eta\mathrm{Re}(\alpha(\mathrm{e}^{-\mathrm{i}\omega t}-1))+\eta^2(\mathrm{e}^{\mathrm{i}\omega t}-1)\right)\end{array}\right)\\
=&\frac{|0\rangle\langle0|+|1\rangle\langle1|}{2}+|0\rangle\langle1|\int\mathrm{d}^2\alpha\frac{\mathrm{e}^{-|\alpha|^2/\bar{n}}}{2\pi\bar{n}}\exp\left(-2\mathrm{i}\eta\mathrm{Re}(\alpha(\mathrm{e}^{-\mathrm{i}\omega t}-1))+\eta^2(\mathrm{e}^{-\mathrm{i}\omega t}-1)\right)+h.c.
\end{aligned}
\tag{13}
$$

And we can calculate the integral,

$$
\begin{aligned}
&\int\mathrm{d}^2\alpha\mathrm{e}^{-|\alpha|^2/\bar{n}}\exp\left(2\mathrm{i}\eta\mathrm{Re}(\alpha(\mathrm{e}^{-\mathrm{i}\omega t}-1))+\eta^2(\mathrm{e}^{-\mathrm{i}\omega t}-1)\right)\\
=&\mathrm{e}^{\eta^2(\mathrm{e}^{-\mathrm{i}\omega t}-1)}\int\mathrm{d}^2\alpha\mathrm{e}^{-|\alpha|^2/\bar{n}}\exp\left(2\mathrm{i}\eta\mathrm{Re}((\alpha_x+\mathrm{i}\alpha_y)(\cos\omega t-1-\mathrm{i}\sin\omega t))\right)\\
=&\mathrm{e}^{\eta^2(\mathrm{e}^{-\mathrm{i}\omega t}-1)}\int\mathrm{d}^2\alpha\exp\left(-\alpha_x^2/\bar{n}+2\mathrm{i}\alpha_x\eta(\cos\omega t-1)\right)\exp\left(-\alpha_y^2/\bar{n}+2\mathrm{i}\alpha_y\eta\sin\omega t\right)\\
=&\pi\bar{n}\exp\left(-\eta^2(1-\mathrm{e}^{-\mathrm{i}\omega t}+2\bar{n}(1-\cos\omega t))\right)\\
=&\pi\bar{n}\exp\left(-\mathrm{i}\eta^2\sin\omega t\right)\exp\left(-\eta^2(1-\cos\omega t)(2\bar{n}+1)\right)
\end{aligned}
\tag{14}
$$

And the final reduced density matrix,

$$
\rho_{3,\mathrm{s}}=\frac{|0\rangle\langle0|+|1\rangle\langle1|+|0\rangle\langle1|\exp\left(\mathrm{i}\eta^2\sin\omega t\right)\exp\left(-\eta^2(1-\cos\omega t)(2\bar{n}+1)\right)+h.c.}{2}
\tag{15}
$$