Magnus expansion with linearly changing Hamiltonian

April 17, 2024

With a Hamiltonian

$$H(t) = H_0 + H_1 t \tag{1}$$

The commutators for the the leading order of Magnus expansion,

$$\begin{aligned}[H(t_1), H(t_2)] =& [H_0 + H_1 t_1, H_0 + H_1 t_2] \\ =& [H_0, H_1] t_2 + [H_1, H_0] t_1 \\ =& [H_0, H_1](t_2 - t_1)\end{aligned} \tag{2}$$

$$\begin{aligned}[[H(t_1), H(t_2)], H(t_3)] =& [[H_0, H_1](t_2 - t_1), H_0 + H_1 t_3] \\ =& ([[[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3)(t_2 - t_1)\end{aligned} \tag{3}$$

$$\begin{aligned}& [[[H(t_1), H(t_2)], H(t_3)], H(t_4)] \\ =& [([[[H_0, H_1], H_0] + [[H_0, H_1], H_1] t_3), H_0 + H_1 t_4](t_2 - t_1) \\ =& [[[[H_0, H_1], H_0], H_0](t_2 - t_1) + [[[H_0, H_1], H_0], H_1](t_2 - t_1) t_4 \\ & + [[[[H_0, H_1], H_1], H_0] t_3 (t_2 - t_1) + [[[[H_0, H_1], H_1], H_1] t_3 t_4 (t_2 - t_1)\end{aligned} \tag{4}$$

The terms in the time integral,

$$\begin{aligned}& [H(t_1), [H(t_2), H(t_3)]] + [H(t_3), [H(t_2), H(t_1)]] \\ =& [[H_0, H_1], H_0](t_2 - t_3) + [[H_0, H_1], H_1] t_1 (t_2 - t_3) \\ & + [[H_0, H_1], H_0](t_2 - t_1) + [[H_0, H_1], H_1] t_3 (t_2 - t_1) \\ =& [[H_0, H_1], H_0](2t_2 - t_1 - t_3) + [[H_0, H_1], H_1](t_1 t_2 + t_2 t_3 - 2 t_1 t_3)\end{aligned} \tag{5}$$

$$\begin{aligned}& [[[H(t_1), H(t_2)], H(t_3)], H(t_4)] + [[[H(t_3), H(t_2)], H(t_4)], H(t_1)] \\ & + [[[H(t_3), H(t_4)], H(t_2)], H(t_1)] + [[[H(t_4), H(t_1)], H(t_3)], H(t_2)] \\ =& [[[H_0, H_1], H_0], H_0](t_2 - t_1 + t_2 - t_3 + t_4 - t_3 + t_1 - t_4) \\ & + [[[H_0, H_1], H_0], H_1](t_2 t_4 - t_1 t_4 + t_2 t_1 - t_3 t_1 + t_4 t_1 - t_3 t_1 + t_1 t_2 - t_4 t_2) \\ & + [[[H_0, H_1], H_1], H_0](t_2 t_3 - t_1 t_3 + t_2 t_4 - t_3 t_4 + t_4 t_2 - t_3 t_2 + t_1 t_3 - t_4 t_3) \\ & + [[[H_0, H_1], H_1], H_1](t_2 t_3 t_4 - t_1 t_3 t_4 + t_2 t_1 t_4 - t_3 t_1 t_4 + t_4 t_1 t_2 - t_3 t_1 t_2 + t_1 t_2 t_3 - t_4 t_2 t_3) \\ =& [[[H_0, H_1], H_0], H_0] 2(t_2 - t_3) + [[[H_0, H_1], H_0], H_1] 2 t_1 (t_2 - t_3) \\ & + [[[H_0, H_1], H_1], H_0] 2 t_4 (t_2 - t_3) + [[[H_0, H_1], H_1], H_1] 2 t_1 t_4 (t_2 - t_3)\end{aligned} \tag{6}$$