Exact formula for quantum jump method in two level system with decay and coupling including detuning

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1 The problem

The derivation in the previous note does not include detuning caused by trap anharmonicity. The anharmonicity should be small in the small range of states that is driven by any given orders but we would still like to include it in the simulation to see how big the effect actually is.
Another effect that can be included is the decoherence. We should be able to simulate the fast component with the decay term and the slow component (e.g. B field drift that's slower than the experimental cycles) by a random overall detuning.
Since we are still only dealing with a single drive, the Hamiltonian will still be time independent. We are also still ignoring the off-resonant scattering so the system is still two-level. The main difference is that the diagnal part of the Hamiltonian will now have a real part and the resulting state may be imaginary.

2 Modified Hamiltonian

With the detuning included, the effective Hamiltonian is now,

$$H' = -\frac{\mathrm{i}}{2}\begin{pmatrix}\Gamma_1 + \mathrm{i}\delta & \Omega \\ -\Omega & \Gamma_2 - \mathrm{i}\delta\end{pmatrix}$$

Define

$$\begin{aligned}\Gamma_1 \equiv& \Gamma + \gamma\\ \Gamma_2 \equiv& \Gamma - \gamma\end{aligned}$$

We have

$$H' = -\frac{\mathrm{i}}{2}\begin{pmatrix}\Gamma + \gamma + \mathrm{i}\delta & \Omega \\ -\Omega & \Gamma - \gamma - \mathrm{i}\delta\end{pmatrix}$$

3 Time evolution

Formally the time evolution is

$$\begin{aligned}&\exp\left(-\mathrm{i}H't\right)\\ =&\exp\left(-\frac{t}{2}\begin{pmatrix}\Gamma + \gamma + \mathrm{i}\delta & \Omega \\ -\Omega & \Gamma - \gamma - \mathrm{i}\delta\end{pmatrix}\right)\end{aligned}$$

Define $\Omega' = \sqrt{\Omega^2 - (\gamma + \mathrm{i}\delta)^2}$ (complex)

$$
\begin{aligned}
&\exp\left(-\mathrm{i}H't\right)\\
=&\frac{\mathrm{e}^{-\Gamma t/2}}{2\mathrm{i}\Omega'}\begin{pmatrix}\gamma\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} - \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right) + \mathrm{i}\Omega'\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} + \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right) & \Omega\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} - \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right)\\ -\Omega\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} - \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right) & -\gamma\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} - \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right) + \mathrm{i}\Omega'\left(\mathrm{e}^{-\frac{\mathrm{i}\Omega' t}{2}} + \mathrm{e}^{\frac{\mathrm{i}\Omega' t}{2}}\right)\end{pmatrix}\\
=&\frac{\mathrm{e}^{-\Gamma t/2}}{2\mathrm{i}\Omega'}\begin{pmatrix}-2\mathrm{i}\gamma\sin\dfrac{\Omega' t}{2} + 2\mathrm{i}\Omega'\cos\dfrac{\Omega' t}{2} & -2\mathrm{i}\Omega\sin\dfrac{\Omega' t}{2}\\ 2\mathrm{i}\Omega\sin\dfrac{\Omega' t}{2} & 2\mathrm{i}\gamma\sin\dfrac{\Omega' t}{2} + 2\mathrm{i}\Omega'\cos\dfrac{\Omega' t}{2}\end{pmatrix}\\
=&\frac{\mathrm{e}^{-\Gamma t/2}}{\Omega'}\begin{pmatrix}-\gamma\sin\dfrac{\Omega' t}{2} + \Omega'\cos\dfrac{\Omega' t}{2} & -\Omega\sin\dfrac{\Omega' t}{2}\\ \Omega\sin\dfrac{\Omega' t}{2} & \gamma\sin\dfrac{\Omega' t}{2} + \Omega'\cos\dfrac{\Omega' t}{2}\end{pmatrix}
\end{aligned}
$$

Starting from the atom in state 1, the wave functions are

$$
\begin{aligned}
\psi =&\frac{\mathrm{e}^{-\Gamma t/2}}{\Omega'}\begin{pmatrix}-\gamma\sin\dfrac{\Omega' t}{2} + \Omega'\cos\dfrac{\Omega' t}{2} & -\Omega\sin\dfrac{\Omega' t}{2}\\ \Omega\sin\dfrac{\Omega' t}{2} & \gamma\sin\dfrac{\Omega' t}{2} + \Omega'\cos\dfrac{\Omega' t}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
=&\frac{\mathrm{e}^{-\Gamma t/2}}{\Omega'}\begin{pmatrix}-\gamma\sin\dfrac{\Omega' t}{2} + \Omega'\cos\dfrac{\Omega' t}{2}\\ \Omega\sin\dfrac{\Omega' t}{2}\end{pmatrix}
\end{aligned}
$$

The quantities we care about are:

- Total decay probability ($1 - \langle\psi|\psi\rangle$)
- Instantaneous decay probability for each channel ($\langle\psi|C_m^\dagger C_m|\psi\rangle$ for $m = 1,\ 2$)

These results will be different from the on-resonance case since we are taking absolute value (probabilities).

Some equations useful for computing these values,

$$
\begin{aligned}
\Omega'_r =& \Re(\Omega') \\
\Omega'_i =& \Im(\Omega') \\
\Omega' =& \Omega'_r + \mathrm{i}\Omega'_i \\
& \sin(a + \mathrm{i}b) \\
=& \sin a \cos \mathrm{i}b + \cos a \sin \mathrm{i}b \\
=& \sin a \cosh b + \mathrm{i} \cos a \sinh b \\
& \cos(a + \mathrm{i}b) \\
=& \cos a \cos \mathrm{i}b + \sin a \sin \mathrm{i}b \\
=& \cos a \cosh b + \mathrm{i} \sin a \sinh b \\
& \sin^2 \frac{a}{2} \sinh^2 \frac{b}{2} + \cos^2 \frac{a}{2} \cosh^2 \frac{b}{2} \\
=& \frac{(1 - \cos a)(\cosh b - 1) + (1 + \cos a)(\cosh b + 1)}{4} \\
=& \frac{\cosh b + \cos a}{2} \\
& \sin^2 \frac{a}{2} \cosh^2 \frac{b}{2} + \cos^2 \frac{a}{2} \sinh^2 \frac{b}{2} \\
=& \frac{(1 - \cos a)(\cosh b + 1) + (1 + \cos a)(\cosh b - 1)}{4} \\
=& \frac{\cosh b - \cos a}{2}
\end{aligned}
$$

$$
\begin{aligned}
& -\gamma \sin \frac{\Omega' t}{2} + \Omega' \cos \frac{\Omega' t}{2} \\
=& -\gamma \left(\sin \frac{\Omega'_r t}{2} \cosh \frac{\Omega'_i t}{2} - \mathrm{i} \sinh \frac{\Omega'_i t}{2} \cos \frac{\Omega'_r t}{2} \right) \\
& + (\Omega'_r + \mathrm{i}\Omega'_i) \left(\cos \frac{\Omega'_r t}{2} \cosh \frac{\Omega'_i t}{2} + \mathrm{i} \sin \frac{\Omega'_r t}{2} \sinh \frac{\Omega'_i t}{2} \right) \\
=& \left(\Omega'_r \cos \frac{\Omega'_r t}{2} - \gamma \sin \frac{\Omega'_r t}{2} \right) \cosh \frac{\Omega'_i t}{2} - \Omega'_i \sin \frac{\Omega'_r t}{2} \sinh \frac{\Omega'_i t}{2} \\
& + \mathrm{i} \left(\left(\gamma \cos \frac{\Omega'_r t}{2} + \Omega'_r \sin \frac{\Omega'_r t}{2} \right) \sinh \frac{\Omega'_i t}{2} + \Omega'_i \cos \frac{\Omega'_r t}{2} \cosh \frac{\Omega'_i t}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\gamma\sin\frac{\Omega' t}{2}+\Omega'\cos\frac{\Omega' t}{2}\right|^2\\
=&{\Omega'_r}^2\cos^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}+\gamma^2\sin^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}+{\Omega'_i}^2\sin^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}\\
&-2\gamma\Omega'_r\sin\frac{\Omega'_r t}{2}\cos\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}-2\Omega'_i\Omega'_r\sin\frac{\Omega'_r t}{2}\cos\frac{\Omega'_r t}{2}\sinh\frac{\Omega'_i t}{2}\cosh\frac{\Omega'_i t}{2}\\
&+2\gamma\Omega'_i\sin^2\frac{\Omega'_r t}{2}\sinh\frac{\Omega'_i t}{2}\cosh\frac{\Omega'_i t}{2}\\
&+\gamma^2\cos^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}+{\Omega'_r}^2\sin^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}+{\Omega'_i}^2\cos^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}\\
&+2\gamma\Omega'_r\sin\frac{\Omega'_r t}{2}\cos\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}+2\gamma\Omega'_i\cos^2\frac{\Omega'_r t}{2}\sinh\frac{\Omega'_i t}{2}\cosh\frac{\Omega'_i t}{2}\\
&+2\Omega'_i\Omega'_r\sin\frac{\Omega'_r t}{2}\cos\frac{\Omega'_r t}{2}\sinh\frac{\Omega'_i t}{2}\cosh\frac{\Omega'_i t}{2}\\
=&|\Omega'|^2\left(\sin^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}+\cos^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}\right)\\
&+2\gamma\Omega'_i\left(\sinh\frac{\Omega'_i t}{2}\cosh\frac{\Omega'_i t}{2}-\sin\frac{\Omega'_r t}{2}\cos\frac{\Omega'_r t}{2}\right)\\
&+\gamma^2\left(\sin^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}+\cos^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}\right)\\
=&\frac{|\Omega'|^2-\gamma^2}{2}\cos\Omega'_r t+\frac{|\Omega'|^2+\gamma^2}{2}\cosh\Omega'_i t+\gamma\Omega'_i(\sinh\Omega'_i t-\sin\Omega'_r t)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\Omega\sin\frac{\Omega' t}{2}\right|^2\\
=&\Omega^2\left|\sin\frac{\Omega'_r t}{2}\cosh\frac{\Omega'_i t}{2}-\mathrm{i}\sinh\frac{\Omega'_i t}{2}\cos\frac{\Omega'_r t}{2}\right|^2\\
=&\Omega^2\left(\sin^2\frac{\Omega'_r t}{2}\cosh^2\frac{\Omega'_i t}{2}+\cos^2\frac{\Omega'_r t}{2}\sinh^2\frac{\Omega'_i t}{2}\right)\\
=&\frac{\Omega^2}{2}(\cosh\Omega'_i t-\cos\Omega'_r t)
\end{aligned}
$$

4 Decay rates

The decay rate for state 1 is

$$
\begin{aligned}
&\langle\psi|C_1^\dagger C_1|\psi\rangle\\
=&\frac{\Gamma_1\mathrm{e}^{-\Gamma t}}{|\Omega'|^2}\left|-\gamma\sin\frac{\Omega' t}{2}+\Omega'\cos\frac{\Omega' t}{2}\right|^2\\
=&\frac{\Gamma_1\mathrm{e}^{-\Gamma t}}{2|\Omega'|^2}\left(\left(|\Omega'|^2-\gamma^2\right)\cos\Omega'_r t+\left(|\Omega'|^2+\gamma^2\right)\cosh\Omega'_i t+2\gamma\Omega'_i(\sinh\Omega'_i t-\sin\Omega'_r t)\right)
\end{aligned}
$$

The decay rate for state 2 is

$$
\begin{aligned}
&\langle\psi|C_2^\dagger C_2|\psi\rangle \\
=&\frac{\Gamma_2 \mathrm{e}^{-\Gamma t}}{|\Omega'|^2}\left|\Omega\sin\frac{\Omega' t}{2}\right|^2 \\
=&\frac{\Gamma_2\Omega^2\mathrm{e}^{-\Gamma t}}{2|\Omega'|^2}(\cosh\Omega'_i t-\cos\Omega'_r t)
\end{aligned}
$$

4.1 Total decay probability

The total probability of not decaying,

$$
\begin{aligned}
\langle\psi|\psi\rangle =&\frac{\mathrm{e}^{-\Gamma t}}{|\Omega'|^2}\left(\left|-\gamma\sin\frac{\Omega' t}{2}+\Omega'\cos\frac{\Omega' t}{2}\right|^2+\left|\Omega\sin\frac{\Omega' t}{2}\right|^2\right) \\
=&\frac{\mathrm{e}^{-\Gamma t}}{|\Omega'|^2}\left(\frac{|\Omega'|^2-\gamma^2-\Omega^2}{2}\cos\Omega'_r t+\frac{|\Omega'|^2+\gamma^2+\Omega^2}{2}\cosh\Omega'_i t+\gamma\Omega'_i(\sinh\Omega'_i t-\sin\Omega'_r t)\right)
\end{aligned}
$$

4.2 Total rate

Calculate the derivative of $\langle\psi|\psi\rangle$ to help root finding with Newton's method.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\psi\rangle \\
=&-\langle\psi|C_1^\dagger C_1|\psi\rangle-\langle\psi|C_2^\dagger C_2|\psi\rangle \\
=&-\frac{\mathrm{e}^{-\Gamma t}}{2|\Omega'|^2}\left(\Gamma_1\left(|\Omega'|^2-\gamma^2\right)\cos\Omega'_r t+\Gamma_1\left(|\Omega'|^2+\gamma^2\right)\cosh\Omega'_i t+2\Gamma_1\gamma\Omega'_i(\sinh\Omega'_i t-\sin\Omega'_r t)\right) \\
&-\frac{\mathrm{e}^{-\Gamma t}}{2|\Omega'|^2}\left(\Gamma_2\Omega^2\cosh\Omega'_i t-\Gamma_2\Omega^2\cos\Omega'_r t\right) \\
=&-\frac{\mathrm{e}^{-\Gamma t}}{2|\Omega'|^2}\Bigg(\left(\Gamma_1|\Omega'|^2-\Gamma_1\gamma^2-\Gamma_2\Omega^2\right)\cos\Omega'_r t+\left(\Gamma_1|\Omega'|^2+\Gamma_1\gamma^2+\Gamma_2\Omega^2\right)\cosh\Omega'_i t \\
&+2\Gamma_1\gamma\Omega'_i(\sinh\Omega'_i t-\sin\Omega'_r t)\Bigg)
\end{aligned}
$$