Exact formula for two level system drive with fixed detunine and amplitude

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Original Hamiltonian

$$H_0 = \frac{1}{2}\begin{pmatrix} \Delta & \Omega e^{i(\delta t+\phi)} \\ \Omega e^{-i(\delta t+\phi)} & -\Delta \end{pmatrix}$$

Under a frame rotation

$$U_1 = \begin{pmatrix} e^{-i(\delta t+\phi)/2} & 0 \\ 0 & e^{i(\delta t+\phi)/2} \end{pmatrix}$$

The Hamiltonian becomes

$$\begin{aligned} H_1 =& U_1 H U_1^\dagger + i\frac{dU_1}{dt}U_1^\dagger \\ =& \frac{1}{2}\begin{pmatrix} \Delta' & \Omega \\ \Omega & -\Delta' \end{pmatrix} \end{aligned}$$

where $\Delta' \equiv \Delta + \delta$. We can diagonalize it with

$$U_2 = \begin{pmatrix} -\sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \\ \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \end{pmatrix}$$

where $\Omega' \equiv \sqrt{\Delta'^2 + \Omega^2}$, to,

$$\begin{aligned} H_2 =& U_2 H_1 U_2^\dagger \\ =& \begin{pmatrix} -\frac{\Omega'}{2} & 0 \\ 0 & \frac{\Omega'}{2} \end{pmatrix} \end{aligned}$$

Evolution under H_2,

$$\begin{aligned} T_2 =& \exp\left(-iH_2 t\right) \\ =& \begin{pmatrix} \exp\left(i\frac{\Omega'}{2}t\right) & 0 \\ 0 & \exp\left(-i\frac{\Omega'}{2}t\right) \end{pmatrix} \end{aligned}$$

Evolution under H_1,

$$\begin{aligned}
&T_1\\
=&U_2^\dagger T_2 U_2\\
=&\begin{pmatrix} -\sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}}\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}}\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)\\ \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}}\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}}\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)\end{pmatrix}\begin{pmatrix} -\sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}}\\ \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}}\end{pmatrix}\\
=&\begin{pmatrix} \left(\frac{1}{2}-\frac{\Delta'}{2\Omega'}\right)\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)+\left(\frac{1}{2}+\frac{\Delta'}{2\Omega'}\right)\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) & \frac{\Omega}{2\Omega'}\left(\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)-\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)\right)\\ \frac{\Omega}{2\Omega'}\left(\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)-\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)\right) & \left(\frac{1}{2}+\frac{\Delta'}{2\Omega'}\right)\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)+\left(\frac{1}{2}-\frac{\Delta'}{2\Omega'}\right)\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)\end{pmatrix}\\
=&\begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & \cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\end{pmatrix}
\end{aligned}$$

Original time evolution

$$\begin{aligned}
&T_0\\
=&U_1^\dagger T_1 U_1(t=0)\\
=&\begin{pmatrix} \left(\cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\right)\mathrm{e}^{\mathrm{i}\delta t/2} & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{\mathrm{i}(\delta t/2+\phi)}\\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{-\mathrm{i}(\delta t/2+\phi)} & \left(\cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\right)\mathrm{e}^{-\mathrm{i}\delta t/2}\end{pmatrix}
\end{aligned}$$

If $\Delta = 0$,

$$T_0 = \begin{pmatrix} \left(\cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\delta}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\right)\mathrm{e}^{\mathrm{i}\delta t/2} & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{\mathrm{i}(\delta t/2+\phi)}\\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{-\mathrm{i}(\delta t/2+\phi)} & \left(\cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\delta}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\right)\mathrm{e}^{-\mathrm{i}\delta t/2}\end{pmatrix}$$

Now converting all the parameter to angles,

$$\begin{aligned}
\theta \equiv&\Omega t\\
\alpha \equiv&\delta t\\
\theta' \equiv&\sqrt{\theta^2+\alpha^2}
\end{aligned}$$

$$\begin{aligned}
T_0 =&\begin{pmatrix} \left(\cos\left(\frac{\theta'}{2}\right)-\mathrm{i}\frac{\alpha}{\theta'}\sin\left(\frac{\theta'}{2}\right)\right)\mathrm{e}^{\mathrm{i}\alpha/2} & -\mathrm{i}\frac{\theta}{\theta'}\sin\left(\frac{\theta'}{2}\right)\mathrm{e}^{\mathrm{i}(\alpha/2+\phi)}\\ -\mathrm{i}\frac{\theta}{\theta'}\sin\left(\frac{\theta'}{2}\right)\mathrm{e}^{-\mathrm{i}(\alpha/2+\phi)} & \left(\cos\left(\frac{\theta'}{2}\right)+\mathrm{i}\frac{\alpha}{\theta'}\sin\left(\frac{\theta'}{2}\right)\right)\mathrm{e}^{-\mathrm{i}\alpha/2}\end{pmatrix}\\
=&\begin{pmatrix} \left(\cos\left(\frac{\theta'}{2}\right)-\mathrm{i}\frac{\alpha}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\right)\mathrm{e}^{\mathrm{i}\alpha/2} & -\mathrm{i}\frac{\theta}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\mathrm{e}^{\mathrm{i}(\alpha/2+\phi)}\\ -\mathrm{i}\frac{\theta}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\mathrm{e}^{-\mathrm{i}(\alpha/2+\phi)} & \left(\cos\left(\frac{\theta'}{2}\right)+\mathrm{i}\frac{\alpha}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\right)\mathrm{e}^{-\mathrm{i}\alpha/2}\end{pmatrix}
\end{aligned}$$

Check

Time derivative of T_1

$$
\begin{aligned}
\mathrm{i}\frac{\mathrm{d}T_1}{\mathrm{d}t}T_1^\dagger =& \mathrm{i}\frac{\Omega'}{2}\begin{pmatrix} -\sin\left(\frac{\Omega'}{2}t\right) - \mathrm{i}\frac{\Delta'}{\Omega'}\cos\left(\frac{\Omega'}{2}t\right) & -\mathrm{i}\frac{\Omega}{\Omega'}\cos\left(\frac{\Omega'}{2}t\right) \\ -\mathrm{i}\frac{\Omega}{\Omega'}\cos\left(\frac{\Omega'}{2}t\right) & -\sin\left(\frac{\Omega'}{2}t\right) + \mathrm{i}\frac{\Delta'}{\Omega'}\cos\left(\frac{\Omega'}{2}t\right) \end{pmatrix} \\
& \begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right) + \mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & \mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \\ \mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & \cos\left(\frac{\Omega'}{2}t\right) - \mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix} \\
=& \mathrm{i}\frac{\Omega'}{2}\begin{pmatrix} -\mathrm{i}\frac{\Delta'}{\Omega'} & -\mathrm{i}\frac{\Omega}{\Omega'} \\ -\mathrm{i}\frac{\Omega}{\Omega'} & \mathrm{i}\frac{\Delta'}{\Omega'} \end{pmatrix} \\
=& \frac{1}{2}\begin{pmatrix} \Delta' & \Omega \\ \Omega & -\Delta' \end{pmatrix} \\
=& H_1
\end{aligned}
$$