Exact formula for two level system drive with fixed detunine and amplitude

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Original Hamiltonian

$$H_0 = \frac{1}{2}\begin{pmatrix} \Delta & \Omega \mathrm{e}^{\mathrm{i}(\delta t+\phi)} \\ \Omega \mathrm{e}^{-\mathrm{i}(\delta t+\phi)} & -\Delta \end{pmatrix}$$

Under a frame rotation

$$U_1 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\delta t+\phi)/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\delta t+\phi)/2} \end{pmatrix}$$

The Hamiltonian becomes

$$\begin{aligned} H_1 =& U_1 H U_1^\dagger + \mathrm{i}\frac{\mathrm{d}U_1}{\mathrm{d}t} U_1^\dagger \\ =& \frac{1}{2}\begin{pmatrix} \Delta' & \Omega \\ \Omega & -\Delta' \end{pmatrix} \end{aligned}$$

where $\Delta' \equiv \Delta + \delta$. We can diagonalize it with

$$U_2 = \begin{pmatrix} -\sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} \\ \sqrt{\frac{1}{2} + \frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2} - \frac{\Delta'}{2\Omega'}} \end{pmatrix}$$

where $\Omega' \equiv \sqrt{\Delta'^2 + \Omega^2}$, to,

$$\begin{aligned} H_2 =& U_2 H_1 U_2^\dagger \\ =& \begin{pmatrix} -\frac{\Omega'}{2} & 0 \\ 0 & \frac{\Omega'}{2} \end{pmatrix} \end{aligned}$$

Evolution under H_2,

$$\begin{aligned} T_2 =& \exp\left(-\mathrm{i}H_2 t\right) \\ =& \begin{pmatrix} \exp\left(\mathrm{i}\frac{\Omega'}{2}t\right) & 0 \\ 0 & \exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) \end{pmatrix} \end{aligned}$$

Evolution under H_1,

$$
\begin{aligned}
&T_1\\
=&U_2^\dagger T_2 U_2\\
=&\begin{pmatrix} -\sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}}\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}}\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) \\ \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}}\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right) & \sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}}\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) \end{pmatrix}\begin{pmatrix} -\sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}} \\ \sqrt{\frac{1}{2}+\frac{\Delta'}{2\Omega'}} & \sqrt{\frac{1}{2}-\frac{\Delta'}{2\Omega'}} \end{pmatrix}\\
=&\begin{pmatrix} \left(\frac{1}{2}-\frac{\Delta'}{2\Omega'}\right)\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)+\left(\frac{1}{2}+\frac{\Delta'}{2\Omega'}\right)\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) & \frac{\Omega}{2\Omega'}\left(\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)-\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)\right) \\ \frac{\Omega}{2\Omega'}\left(\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right)-\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)\right) & \left(\frac{1}{2}+\frac{\Delta'}{2\Omega'}\right)\exp\left(\mathrm{i}\frac{\Omega'}{2}t\right)+\left(\frac{1}{2}-\frac{\Delta'}{2\Omega'}\right)\exp\left(-\mathrm{i}\frac{\Omega'}{2}t\right) \end{pmatrix}\\
=&\begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & \cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix}
\end{aligned}
$$

Original time evolution

$$
\begin{aligned}
&T_0\\
=&U_1^\dagger T_1 U_1\\
=&\begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{\mathrm{i}(\delta t+\phi)} \\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{-\mathrm{i}(\delta t+\phi)} & \cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\Delta'}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix}
\end{aligned}
$$

If $\Delta=0$,

$$
T_0=\begin{pmatrix} \cos\left(\frac{\Omega'}{2}t\right)-\mathrm{i}\frac{\delta}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) & -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{\mathrm{i}(\delta t+\phi)} \\ -\mathrm{i}\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right)\mathrm{e}^{-\mathrm{i}(\delta t+\phi)} & \cos\left(\frac{\Omega'}{2}t\right)+\mathrm{i}\frac{\delta}{\Omega'}\sin\left(\frac{\Omega'}{2}t\right) \end{pmatrix}
$$

Now converting all the parameter to angles,

$$
\begin{aligned}
\theta \equiv&\Omega t\\
\alpha \equiv&\delta t\\
\theta' \equiv&\sqrt{\theta^2+\alpha^2}
\end{aligned}
$$

$$
\begin{aligned}
T_0=&\begin{pmatrix} \cos\left(\frac{\theta'}{2}\right)-\mathrm{i}\frac{\alpha}{\theta'}\sin\left(\frac{\theta'}{2}\right) & -\mathrm{i}\frac{\theta}{\theta'}\sin\left(\frac{\theta'}{2}\right)\mathrm{e}^{\mathrm{i}(\alpha+\phi)} \\ -\mathrm{i}\frac{\theta}{\theta'}\sin\left(\frac{\theta'}{2}\right)\mathrm{e}^{-\mathrm{i}(\alpha+\phi)} & \cos\left(\frac{\theta'}{2}\right)+\mathrm{i}\frac{\alpha}{\theta'}\sin\left(\frac{\theta'}{2}\right) \end{pmatrix}\\
=&\begin{pmatrix} \cos\left(\frac{\theta'}{2}\right)-\mathrm{i}\frac{\alpha}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right) & -\mathrm{i}\frac{\theta}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\mathrm{e}^{\mathrm{i}(\alpha+\phi)} \\ -\mathrm{i}\frac{\theta}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right)\mathrm{e}^{-\mathrm{i}(\alpha+\phi)} & \cos\left(\frac{\theta'}{2}\right)+\mathrm{i}\frac{\alpha}{2}\mathrm{sinc}\left(\frac{\theta'}{2}\right) \end{pmatrix}
\end{aligned}
$$