Exact solutions for 2x2 master equation

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1 Master equation

Master equation for a driven two level system

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} =& -\mathrm{i}[H,\rho] + \sum_m C_m \rho C_m^\dagger - \frac{1}{2}\sum_m \left(C_m^\dagger C_m \rho + \rho C_m^\dagger C_m\right)\\
H =& \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i}\Omega \\ \mathrm{i}\Omega & 0 \end{pmatrix}\\
\rho =& \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\end{aligned}$$

2 Non-state changing decay

Jump operators

$$\begin{aligned}C_1 =& \sqrt{\Gamma}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\\
C_2 =& \sqrt{\Gamma}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\\
C_1^\dagger =& \sqrt{\Gamma}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\\
C_2^\dagger =& \sqrt{\Gamma}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\\
C_1^\dagger C_1 =& \Gamma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\\
C_2^\dagger C_2 =& \Gamma\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

Decay terms

$$\begin{aligned}&\sum_m C_m \rho C_m^\dagger\\
=& \begin{pmatrix} \Gamma\rho_{11} & 0 \\ 0 & \Gamma\rho_{22} \end{pmatrix}\\
&-\frac{1}{2}\sum_m \left(C_m^\dagger C_m \rho + \rho C_m^\dagger C_m\right)\\
=& -\frac{1}{2}\Big(C_1^\dagger C_1 \rho + \rho C_1^\dagger C_1 + C_2^\dagger C_2 \rho + \rho C_2^\dagger C_2\Big)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\Gamma}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\rho + \rho\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) - \frac{\Gamma}{2}\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\rho + \rho\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right)\\
&= -\begin{pmatrix}\Gamma\rho_{11} & \Gamma\rho_{12}\\ \Gamma\rho_{21} & \Gamma\rho_{22}\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\mathrm{i}[H,\rho] + \begin{pmatrix}\Gamma\rho_{11} & 0\\ 0 & \Gamma\rho_{22}\end{pmatrix} - \begin{pmatrix}\Gamma\rho_{11} & \Gamma\rho_{12}\\ \Gamma\rho_{21} & \Gamma\rho_{22}\end{pmatrix}\\
&= \frac{1}{2}\left(\begin{pmatrix}0 & -\Omega\\ \Omega & 0\end{pmatrix}\rho - \rho\begin{pmatrix}0 & -\Omega\\ \Omega & 0\end{pmatrix}\right) - \begin{pmatrix}0 & \Gamma\rho_{12}\\ \Gamma\rho_{21} & 0\end{pmatrix}\\
&= \frac{1}{2}\left(\begin{pmatrix}\Omega\rho_{12} & -\Omega\rho_{11}\\ \Omega\rho_{22} & -\Omega\rho_{21}\end{pmatrix} - \begin{pmatrix}-\Omega\rho_{21} & -\Omega\rho_{22}\\ \Omega\rho_{11} & \Omega\rho_{12}\end{pmatrix}\right) - \begin{pmatrix}0 & \Gamma\rho_{12}\\ \Gamma\rho_{21} & 0\end{pmatrix}\\
&= \begin{pmatrix}\frac{\Omega}{2}(\rho_{12}+\rho_{21}) & \frac{\Omega}{2}(\rho_{22}-\rho_{11}) - \Gamma\rho_{12}\\ \frac{\Omega}{2}(\rho_{22}-\rho_{11}) - \Gamma\rho_{21} & -\frac{\Omega}{2}(\rho_{12}+\rho_{21})\end{pmatrix}
\end{aligned}
$$

Let $x = \rho_{12} + \rho_{21}$, $y = \rho_{22} - \rho_{11}$

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \Omega y - \Gamma x\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\Omega x\\
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\Omega\frac{\mathrm{d}x}{\mathrm{d}t}\\
&= -\Omega(\Omega y - \Gamma x)\\
&= -\Omega^2 y + \Gamma\Omega x\\
&= -\Omega^2 y - \Gamma\frac{\mathrm{d}y}{\mathrm{d}t}\\
0 &= \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \Gamma\frac{\mathrm{d}y}{\mathrm{d}t} + \Omega^2 y
\end{aligned}
$$

Assume underdamped ($4\Omega^2 > \Gamma^2$). Let $\Omega' = \sqrt{4\Omega^2 - \Gamma^2}/2$

$$
y = \exp\left(-\frac{\Gamma t}{2}\right)(c_1 \sin\Omega' t + c_2 \cos\Omega' t)
$$

Initial condition

$$
\begin{aligned}
x(0) &= 0\\
y(0) &= -1
\end{aligned}
$$

So $c_2 = -1$

$$
\begin{aligned}
y &= \exp\left(-\frac{\Gamma t}{2}\right)(c_1 \sin\Omega' t - \cos\Omega' t)\\
\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} &= \frac{\Gamma}{2} + c_1\Omega'\\
y &= -\exp\left(-\frac{\Gamma t}{2}\right)\left(\frac{\Gamma}{2\Omega'}\sin\Omega' t + \cos\Omega' t\right)
\end{aligned}
$$

3 State changing decay

Jump operators

$$
\begin{aligned}
C_1 =& \sqrt{\Gamma}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
C_2 =& \sqrt{\Gamma}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
C_1^\dagger =& \sqrt{\Gamma}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
C_2^\dagger =& \sqrt{\Gamma}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
C_1^\dagger C_1 =& \Gamma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
C_2^\dagger C_2 =& \Gamma\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Decay terms

$$
\begin{aligned}
&\sum_m C_m \rho C_m^\dagger \\
=& \begin{pmatrix} \Gamma\rho_{22} & 0 \\ 0 & \Gamma\rho_{11} \end{pmatrix} \\
&- \frac{1}{2}\sum_m \left(C_m^\dagger C_m \rho + \rho C_m^\dagger C_m\right) \\
=& -\frac{1}{2}\Big(C_1^\dagger C_1 \rho + \rho C_1^\dagger C_1 + C_2^\dagger C_2 \rho + \rho C_2^\dagger C_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
=& -\frac{\Gamma}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\rho + \rho\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) - \frac{\Gamma}{2}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\rho + \rho\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
=& -\begin{pmatrix} \Gamma\rho_{11} & \Gamma\rho_{12} \\ \Gamma\rho_{21} & \Gamma\rho_{22} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} =& -\mathrm{i}[H,\rho] + \begin{pmatrix} \Gamma\rho_{22} & 0 \\ 0 & \Gamma\rho_{11} \end{pmatrix} - \begin{pmatrix} \Gamma\rho_{11} & \Gamma\rho_{12} \\ \Gamma\rho_{21} & \Gamma\rho_{22} \end{pmatrix} \\
=& \frac{1}{2}\left(\begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}\rho - \rho\begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}\right) + \begin{pmatrix} \Gamma(\rho_{22}-\rho_{11}) & -\Gamma\rho_{12} \\ -\Gamma\rho_{21} & -\Gamma(\rho_{22}-\rho_{11}) \end{pmatrix} \\
=& \frac{1}{2}\left(\begin{pmatrix} \Omega\rho_{12} & -\Omega\rho_{11} \\ \Omega\rho_{22} & -\Omega\rho_{21} \end{pmatrix} - \begin{pmatrix} -\Omega\rho_{21} & -\Omega\rho_{22} \\ \Omega\rho_{11} & \Omega\rho_{12} \end{pmatrix}\right) + \begin{pmatrix} \Gamma(\rho_{22}-\rho_{11}) & -\Gamma\rho_{12} \\ -\Gamma\rho_{21} & -\Gamma(\rho_{22}-\rho_{11}) \end{pmatrix} \\
=& \begin{pmatrix} \frac{\Omega}{2}(\rho_{12}+\rho_{21}) + \Gamma(\rho_{22}-\rho_{11}) & \frac{\Omega}{2}(\rho_{22}-\rho_{11}) - \Gamma\rho_{12} \\ \frac{\Omega}{2}(\rho_{22}-\rho_{11}) - \Gamma\rho_{21} & -\frac{\Omega}{2}(\rho_{12}+\rho_{21}) - \Gamma(\rho_{22}-\rho_{11}) \end{pmatrix}
\end{aligned}
$$

Let $x = \rho_{12} + \rho_{21}$, $y = \rho_{22} - \rho_{11}$

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} =& \Omega y - \Gamma x \\
\frac{\mathrm{d}y}{\mathrm{d}t} =& - \Omega x - 2\Gamma y \\
\Omega x =& - \frac{\mathrm{d}y}{\mathrm{d}t} - 2\Gamma y \\
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} =& - \Omega \frac{\mathrm{d}x}{\mathrm{d}t} - 2\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} \\
=& - \Omega(\Omega y - \Gamma x) - 2\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} \\
=& - \Omega^2 y + \Gamma \Omega x - 2\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} \\
=& - \Omega^2 y - \Gamma \left(\frac{\mathrm{d}y}{\mathrm{d}t} + 2\Gamma y \right) - 2\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} \\
=& - 3\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} - \left(\Omega^2 + 2\Gamma^2\right) y \\
0 =& \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\Gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \left(\Omega^2 + 2\Gamma^2\right) y
\end{aligned}
$$

Assume underdamped ($4\Omega^2 > \Gamma^2$). Let $\Omega' = \sqrt{4\Omega^2 - \Gamma^2}/2$

$$
y = \exp\left(-\frac{\Gamma t}{2}\right)(c_1 \sin \Omega' t + c_2 \cos \Omega' t)
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Initial condition

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\begin{aligned}
y =& \exp\left(-\frac{\Gamma t}{2}\right)(c_1 \sin \Omega' t - \cos \Omega' t) \\
\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} =& \frac{\Gamma}{2} + c_1 \Omega' \\
y =& - \exp\left(-\frac{\Gamma t}{2}\right)\left(\frac{\Gamma}{2\Omega'} \sin \Omega' t + \cos \Omega' t\right)
\end{aligned}
$$