1 Matrix element

Wavefunction

$$\psi_n(x,z) = \sqrt{\frac{1}{2^n n! z\sqrt{\pi}}} \mathrm{e}^{-(x/z)^2/2} H_n(x/z)$$

Define

$$H'_n(x) \equiv \frac{H_n(x)}{2^n}$$
$$\psi_n(x,z) = \sqrt{\frac{2^n}{n! z\sqrt{\pi}}} \mathrm{e}^{-(x/z)^2/2} H'_n(x/z)$$

Matrix element

$$\begin{aligned}
\langle n1, m1 | n2, m2 \rangle =& \int_{-\infty}^{\infty} \psi_{n_1}(x,z_1)\psi_{n_2}(x,z_1)\psi_{m_1}(x,z_2)\psi_{m_2}(x,z_2)\mathrm{d}x \\
=& \int_{-\infty}^{\infty} \mathrm{d}x \sqrt{\frac{2_1^n}{n_1! z_1\sqrt{\pi}}} \mathrm{e}^{-(x/z_1)^2/2} H'_{n_1}(x/z_1) \sqrt{\frac{2_2^n}{n_2! z_1\sqrt{\pi}}} \mathrm{e}^{-(x/z_1)^2/2} H'_{n_2}(x/z_1) \\
& \sqrt{\frac{2_1^m}{m_1! z_2\sqrt{\pi}}} \mathrm{e}^{-(x/z_2)^2/2} H'_{m_1}(x/z_2) \sqrt{\frac{2_2^m}{m_2! z_2\sqrt{\pi}}} \mathrm{e}^{-(x/z_2)^2/2} H'_{m_2}(x/z_2) \\
=& \frac{1}{\pi z_1 z_2} \int_{-\infty}^{\infty} \mathrm{d}x \sqrt{\frac{2^{n_1+n_2}}{n_1! n_2!}} \mathrm{e}^{-(x/z_1)^2} H'_{n_1}(x/z_1) H'_{n_2}(x/z_1) \\
& \sqrt{\frac{2^{m_1+m_2}}{m_1! m_2!}} \mathrm{e}^{-(x/z_2)^2} H'_{m_1}(x/z_2) H'_{m_2}(x/z_2) \\
=& \frac{1}{\pi z_1 z_2} \sqrt{\frac{2^{n_1+n_2+m_1+m_2}}{n_1! n_2! m_1! m_2!}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-(x/z_1)^2} H'_{n_1}(x/z_1) H'_{n_2}(x/z_1) \\
& \mathrm{e}^{-(x/z_2)^2} H'_{m_1}(x/z_2) H'_{m_2}(x/z_2)
\end{aligned}$$