Mølmer-Sørensen detuning and red-blue asymmetry error

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In the laser frame (rotating at the average frequency of the red and blue tones),

$$
\begin{aligned}
H(t) = &\sum_{j=1,2} \frac{\Omega(t)}{2} \sum_k \eta_{jk} \Big((1+\varepsilon_j) \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t)} \sigma_+^j + a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t)} \sigma_-^j \Big) \\
&+ (1-\varepsilon_j) \Big(a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t)} \sigma_+^j + a_k \mathrm{e}^{-\mathrm{i}\theta_k(t)} \sigma_-^j \Big) \Big) - \frac{\Delta_j}{2} \sigma_z^j \\
= &\sum_{j=1,2} \frac{\Omega(t)}{2} \sum_k \eta_{jk} \Big(\sigma_x^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t)} + a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t)} \Big) + \mathrm{i}\varepsilon_j \sigma_y^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t)} - a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t)} \Big) \Big) - \frac{\Delta_j}{2} \sigma_z^j
\end{aligned}
$$

First order in Magnus expansion,

$$
\begin{aligned}
M_1(\tau) = &\int_0^\tau H(t) \mathrm{d}t \\
= &\frac{1}{2} \sum_{j=1,2} \sum_k \eta_{jk} \Big(\sigma_x^j \Big(a_k \alpha_k(t) + a_k^\dagger \alpha_k^*(t) \Big) + \mathrm{i}\varepsilon_j \sigma_y^j \Big(a_k \alpha_k(t) - a_k^\dagger \alpha_k^*(t) \Big) \Big) - \frac{\Delta_j t}{2} \sigma_z^j
\end{aligned}
$$

where

$$
\alpha_k(t) \equiv \Omega(t) \mathrm{e}^{-\mathrm{i}\theta_k(t)}
$$

similar to the ideal case. Assuming closure of all α_k, the none trivial part of this term would also vanish other than the detuning.
Second order in Magnus expansion,

$$
\begin{aligned}
&[H(t_1), H(t_2)] \\
= &\Bigg[\sum_{j=1,2} \frac{\Omega(t_1)}{2} \sum_k \eta_{jk} \Big(\sigma_x^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t_1)} + a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t_1)} \Big) + \mathrm{i}\varepsilon_j \sigma_y^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t_1)} - a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t_1)} \Big) \Big) - \frac{\Delta_j}{2} \sigma_z^j, \\
&\sum_{j=1,2} \frac{\Omega(t_2)}{2} \sum_k \eta_{jk} \Big(\sigma_x^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t_2)} + a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t_2)} \Big) + \mathrm{i}\varepsilon_j \sigma_y^j \Big(a_k \mathrm{e}^{-\mathrm{i}\theta_k(t_2)} - a_k^\dagger \mathrm{e}^{\mathrm{i}\theta_k(t_2)} \Big) \Big) - \frac{\Delta_j}{2} \sigma_z^j \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=&\Bigg[\sum_{j=1,2}\frac{\Omega(t_1)}{2}\sum_k\eta_{jk}\left(\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)+\mathrm{i}\varepsilon_j\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)\right),\\
&\sum_{j=1,2}\frac{\Omega(t_2)}{2}\sum_k\eta_{jk}\left(\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)+\mathrm{i}\varepsilon_j\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)\right)\Bigg]\\
&-\left[\sum_{j=1,2}\frac{\Omega(t_1)}{2}\sum_k\eta_{jk}\left(\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)+\mathrm{i}\varepsilon_j\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)\right),\sum_{j=1,2}\frac{\Delta_j}{2}\sigma_z^j\right]\\
&-\left[\sum_{j=1,2}\frac{\Delta_j}{2}\sigma_z^j,\sum_{j=1,2}\frac{\Omega(t_2)}{2}\sum_k\eta_{jk}\left(\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)+\mathrm{i}\varepsilon_j\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)\right)\right]\\
=&\frac{\Omega(t_1)\Omega(t_2)}{4}\sum_k\left(\sum_{j=1,2}\sigma_x^j\eta_{jk}\right)^2\left[a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)},a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right]\\
&+\mathrm{i}\frac{\Omega(t_1)\Omega(t_2)}{4}\left[\sum_{j=1,2}\sigma_x^j\sum_k\eta_{jk}\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right),\sum_{j=1,2}\varepsilon_j\sigma_y^j\sum_k\eta_{jk}\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)\right]\\
&+\mathrm{i}\frac{\Omega(t_1)\Omega(t_2)}{4}\left[\sum_{j=1,2}\varepsilon_j\sigma_y^j\sum_k\eta_{jk}\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right),\sum_{j=1,2}\sigma_x^j\sum_k\eta_{jk}\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)\right]\\
&-\frac{\Omega(t_1)\Omega(t_2)}{4}\sum_k\left(\sum_{j=1,2}\varepsilon_j\sigma_y^j\eta_{jk}\right)^2\left[a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)},a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right]\\
&+\sum_{j=1,2}\frac{\Omega(t_1)\Delta_j}{2}\sum_k\eta_{jk}\left(\mathrm{i}\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)+\varepsilon_j\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_1)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_1)}\right)\right)\\
&-\sum_{j=1,2}\frac{\Omega(t_2)\Delta_j}{2}\sum_k\eta_{jk}\left(\mathrm{i}\sigma_y^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}+a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)+\varepsilon_j\sigma_x^j\left(a_k\mathrm{e}^{-\mathrm{i}\theta_k(t_2)}-a_k^\dagger\mathrm{e}^{\mathrm{i}\theta_k(t_2)}\right)\right)
\end{aligned}
$$