

Exact solutions for 2x2 master equation

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1 Master equation

Master equation for a driven two level system

$$\begin{aligned}\frac{d\rho}{dt} &= -i[H, \rho] + \sum_m C_m \rho C_m^\dagger - \frac{1}{2} \sum_m (C_m^\dagger C_m \rho + \rho C_m^\dagger C_m) \\ H &= \frac{1}{2} \begin{pmatrix} 0 & -i\Omega \\ i\Omega & 0 \end{pmatrix} \\ \rho &= \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\end{aligned}$$

2 Non-state changing decay

Jump operators

$$\begin{aligned}C_1 &= \sqrt{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C_2 &= \sqrt{\Gamma} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ C_1^\dagger &= \sqrt{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C_2^\dagger &= \sqrt{\Gamma} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ C_1^\dagger C_1 &= \Gamma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C_2^\dagger C_2 &= \Gamma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

Decay terms

$$\begin{aligned}& \sum_m C_m \rho C_m^\dagger \\ &= \begin{pmatrix} \Gamma \rho_{11} & 0 \\ 0 & \Gamma \rho_{22} \end{pmatrix} \\ & \quad - \frac{1}{2} \sum_m (C_m^\dagger C_m \rho + \rho C_m^\dagger C_m) \\ &= -\frac{1}{2} (C_1^\dagger C_1 \rho + \rho C_1^\dagger C_1 + C_2^\dagger C_2 \rho + \rho C_2^\dagger C_2)\end{aligned}$$

$$\begin{aligned}
&= -\frac{\Gamma}{2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho + \rho \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) - \frac{\Gamma}{2} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rho + \rho \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= - \begin{pmatrix} \Gamma \rho_{11} & \Gamma \rho_{12} \\ \Gamma \rho_{21} & \Gamma \rho_{22} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\frac{d\rho}{dt} &= -i[H, \rho] + \begin{pmatrix} \Gamma \rho_{11} & 0 \\ 0 & \Gamma \rho_{22} \end{pmatrix} - \begin{pmatrix} \Gamma \rho_{11} & \Gamma \rho_{12} \\ \Gamma \rho_{21} & \Gamma \rho_{22} \end{pmatrix} \\
&= \frac{1}{2} \left(\begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix} \rho - \rho \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix} \right) - \begin{pmatrix} 0 & \Gamma \rho_{12} \\ \Gamma \rho_{21} & 0 \end{pmatrix} \\
&= \frac{1}{2} \left(\begin{pmatrix} \Omega \rho_{12} & -\Omega \rho_{11} \\ \Omega \rho_{22} & -\Omega \rho_{21} \end{pmatrix} - \begin{pmatrix} -\Omega \rho_{21} & -\Omega \rho_{22} \\ \Omega \rho_{11} & \Omega \rho_{12} \end{pmatrix} \right) - \begin{pmatrix} 0 & \Gamma \rho_{12} \\ \Gamma \rho_{21} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\Omega}{2}(\rho_{12} + \rho_{21}) & \frac{\Omega}{2}(\rho_{22} - \rho_{11}) - \Gamma \rho_{12} \\ \frac{\Omega}{2}(\rho_{22} - \rho_{11}) - \Gamma \rho_{21} & -\frac{\Omega}{2}(\rho_{12} + \rho_{21}) \end{pmatrix}
\end{aligned}$$

Let $x = \rho_{12} + \rho_{21}$, $y = \rho_{22} - \rho_{11}$

$$\begin{aligned}
\frac{dx}{dt} &= \Omega y - \Gamma x \\
\frac{dy}{dt} &= -\Omega x \\
\frac{d^2 y}{dt^2} &= -\Omega \frac{dx}{dt} \\
&= -\Omega(\Omega y - \Gamma x) \\
&= -\Omega^2 y + \Gamma \Omega x \\
&= -\Omega^2 y - \Gamma \frac{dy}{dt} \\
0 &= \frac{d^2 y}{dt^2} + \Gamma \frac{dy}{dt} + \Omega^2 y
\end{aligned}$$

Assume underdamped ($4\Omega^2 > \Gamma^2$). Let $\Omega' = \sqrt{4\Omega^2 - \Gamma^2}/2$

$$y = \exp\left(-\frac{\Gamma t}{2}\right) (c_1 \sin \Omega' t + c_2 \cos \Omega' t)$$

Initial condition

$$\begin{aligned}
x(0) &= 0 \\
y(0) &= -1
\end{aligned}$$

So $c_2 = -1$

$$\begin{aligned}
y &= \exp\left(-\frac{\Gamma t}{2}\right) (c_1 \sin \Omega' t - \cos \Omega' t) \\
\left. \frac{dy}{dt} \right|_{t=0} &= \frac{\Gamma}{2} + c_1 \Omega' \\
y &= -\exp\left(-\frac{\Gamma t}{2}\right) \left(\frac{\Gamma}{2\Omega'} \sin \Omega' t + \cos \Omega' t \right)
\end{aligned}$$