

# Sideway circular polarization in tightly focused beam

February 23, 2023

## 1 Goal

Trying to accurately understand the origin of the sideway circular polarization on the side of the focus of a tightly focused beam.

## 2 Qualitative description

For a collimated light beam, the polarization vector is generally within the plane perpendicular to the wave propagation direction (since light is a transverse wave). However, for a focused beam, the “wave propagation direction” isn’t very well defined anymore, which allows polarization parallel to the optical axis to occur. This “axial” polarization component could even lead to circular polarization that rotates in a plane parallel to the optical axis, especially near the focus.

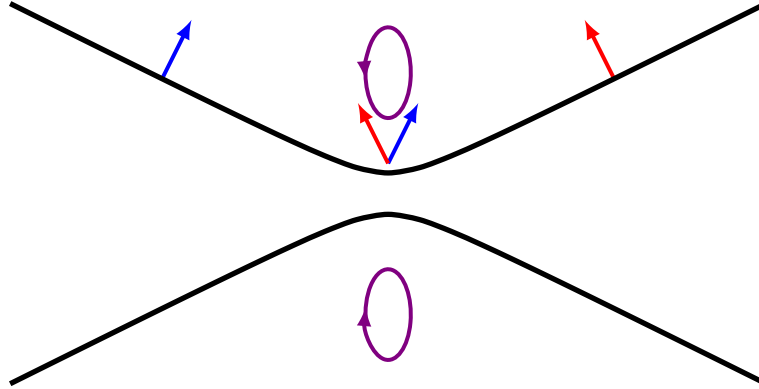


Figure 1: Sideway circular polarization near the focus of a tightly focused beam. The red and blue error shows the polarization vector on the same edge of a tightly focused beam before and after the focus.

We can qualitatively see this happening by looking at the field on the edge of the beam. The two edges have significantly different  $k$  vectors and therefore different polarization vectors as well. As shown in Fig. 1, the polarization on the two edges of the beam acquires an axial component due to the large angle between the  $k$  vector and the optical axis. While the two sides of the beam are generally far away from each other and their different polarization directions cause little problem, this is not the case anymore near the focus as the edge of the beam changes direction from converging to diverging and the polarization in that area (next to the focus in the focal plane) would have a polarization somewhere in between. This of course doesn’t guarantee that there are any circular

polarization or even axial polarization component, which would be the topic of the next section.

### 3 Semi-quantitative explanation

Full quantitative understanding of the sideways circular polarization near the focus of the beam requires a full calculation of the vector field. It is possible, however, to understand why such a polarization exists based on some continuity and symmetry considerations.

#### 3.1 General idea

We can see from the far field (away from the focus) that the axial polarization does exist (Fig. 1). We can therefore break down the proof of the existence of the sideways circular polarization in two steps.

1. Show that the axial polarization also exist on the focal plane.
2. Prove that the axial polarization has to be out-of-phase with the transverse polarization.

These two would guarantee a sideways circular polarization.

#### 3.2 Existence of axial polarization on the focal plane

Even though the three components of the electric field are related to one another, they do all satisfy the scalar wave equation on their own (Appendix A). This allows us to consider the propagation of each components independent of other ones.

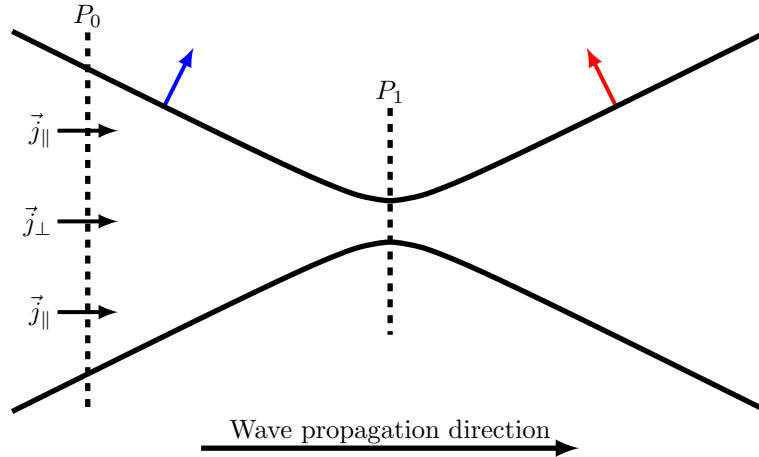


Figure 2: Scalar wave flux (Appendix B) for the transverse  $\vec{j}_{\perp}$  and the longitudinal  $\vec{j}_{\parallel}$  component of the field. In a plane far from the focus,  $P_0$  we can verify that the flux for both components points to the right creating a net positive flux from left to right. For this reason, the net flux for both components should be non-null at all other parallel planes, including the focal plane  $P_1$ .

For a traveling wave from left to right (Fig. 2), we can consider the flux of this scalar wave (Appendix B) through a vertical plane. Since the flux is always in the direction of the beam propagation

(i.e. phase gradient), the net flux for both components through a plane in the far field  $P_0$  point to the right. This means that the flux for both fields, and therefore the value of both fields are non-null somewhere on the focal plane  $P_1$ . We can therefore reasonably expect that there are points on the focal plane where both fields are non-zero.

### 3.3 Phase between the transverse and longitudinal polarization on the focal plane

In this subsection we'll discuss a beam that's perfectly symmetric around the focal plane.<sup>1</sup> More specifically, this means that the mirror reflection of the beam about the focal plane should result in the same beam propagating backward (i.e. mirror reflection plus time reversal should result in the same field).

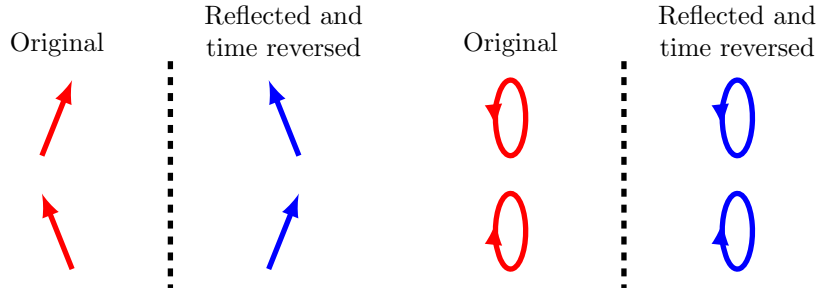


Figure 3: Time reversal and spacial reflection changes linear polarization that is not aligned with or perpendicular to the reflection plane (left). However, it will maintain an elliptical polarization if the axis of ellipse is straight (right).

For a point on the focal plane with both longitudinal and transverse polarization components, this require the polarization itself to be symmetric under this transformation. Any in phase component between the two polarization directions would break such symmetry. In fact, the symmetry is only preserved if the two components are off by a phase  $\pm\frac{\pi}{2}$ , or when the polarization is a straight ellipse (Fig. 3).

The result will of course not fully applicable to physical beam without the perfect symmetry but the general result should still apply. In another word, the point next to the focus on the focal plane may not have a straight elliptical polarization (i.e. a perfectly out-of-phase axial compoment) but the polarization should still be elliptical and the straight ellipse may be achieved on a nearby plane/point.

### 3.4 Approximate the axial polarization using Hermite-Gaussian modes

In the far field, the wavefront of the axial polarization is approximately  $x\tilde{A}(x, y)$  where  $\tilde{A}(x, y)$  is the wavefront for the transverse polarization and we are assuming that the polarization is along  $\hat{x}$ . Therefore, when the beam (i.e. the normal, transversely polarized portion) is a Gaussian beam, the longitudinal polarization is approximately a  $\text{TEM}_{10}$  Gaussian mode<sup>2</sup>, and we should be able

<sup>1</sup>I believe the existance of such field is a reasonably assumption to make, given the time reversal symmetry of EM field.

<sup>2</sup>[https://en.wikipedia.org/wiki/Gaussian\\_beam#Hermite-Gaussian\\_modes](https://en.wikipedia.org/wiki/Gaussian_beam#Hermite-Gaussian_modes)

---

to use it to check the properties we've predicted above (Fig. 4).

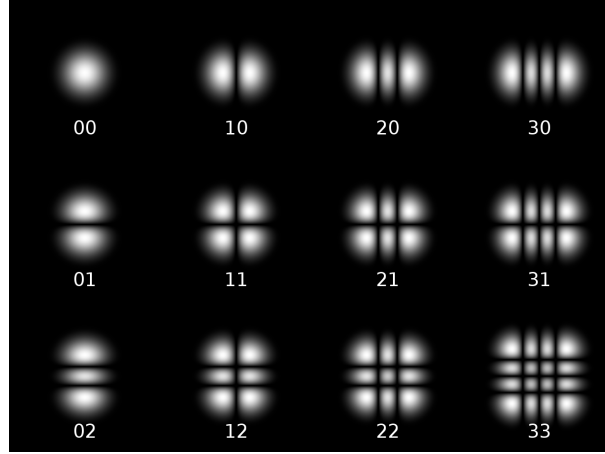


Figure 4: Power distribution of some Hermite Gaussian modes. The wavefront of the longitudinal polarization for a focused  $\text{TEM}_{00}$  can be approximated by the  $\text{TEM}_{10}$  mode.

Compared to the normal Gaussian beam (i.e. the  $\text{TEM}_{00}$  mode) there are two main differences that we care about.

1. The amplitude profile of its wavefront is proportional to  $xe^{-(x^2+y^2)/w^2}$  instead of  $e^{-(x^2+y^2)/w^2}$  for the normal ( $\text{TEM}_{00}$ ) mode. Note that the amplitude profile remains this shape and non-zero, which is consistent with our conclusion in section 3.2.
2. The phase of the beam is different. Compared to a plain wave, which has a phase  $e^{ikz}$  in the propagation direction, a focused beam picks up an additional phase, the so-called “Gouy phase”, in the form  $N \arctan(z/z_R)$  where  $z_R$  is the Rayleigh range of the beam. For the normal  $\text{TEM}_{00}$  mode,  $N$  is 1 and for the  $\text{TEM}_{10}$  mode,  $N$  is 2. The longitudinal and transverse field are in phase in the far field and so they acquire a relative  $\frac{\pi}{2}$  phase after propagating to the focus. This phase is exactly the same as the one we found in section 3.3, which explains the sideways elliptical polarization.

---

## A Wave equation for electric field

From the Maxwell equation,

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= -\nabla \times \frac{\partial \vec{B}}{\partial t} \\ &= -\frac{\partial \nabla \times \vec{B}}{\partial t}\end{aligned}\tag{1}$$

$$\begin{aligned}&= -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla \times (\nabla \times \vec{E}) &= \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \\ &= -\nabla^2 \vec{E}\end{aligned}\tag{2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}\tag{3}$$

Equation 3 contains no mixing between the different components of the electric field which shows that each compoments of the electric field all satisfy the scalar wave equation.

## B Continuity equation for scalar wave

Consider a complex scalar wave  $u(\vec{r}, t)$ , it satisfy the wave equation,

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}\tag{4}$$

If the wave is monotonic<sup>3</sup>, we can write it out as  $u(\vec{r}, t) = u_0(\vec{r})e^{i\omega t}$ , in which case the wave equation becomes,

$$\nabla^2 u_0 = k^2 u_0\tag{5}$$

where the wave vector  $k \equiv \frac{\omega}{c}$ . We can now define the flux of the wave  $\vec{j} \equiv u_0^* \nabla u_0 - u_0 \nabla u_0^*$ , we have,

$$\begin{aligned}\nabla \cdot \vec{j} &= \nabla \cdot (u_0^* \nabla u_0 - u_0 \nabla u_0^*) \\ &= \nabla u_0^* \cdot \nabla u_0 - \nabla u_0 \cdot \nabla u_0^* + u_0^* \nabla^2 u_0 - u_0 \nabla^2 u_0^* \\ &= u_0^* k^2 u_0 - u_0 k^2 u_0^* \\ &= 0\end{aligned}\tag{6}$$

i.e.  $\vec{j}$  is a continuous flow that cannot terminate in free space.

For a plane wave, this vector  $\vec{j}$  points in the direction of the wave propagation. Generically, this points in the direction of the phase gradient (i.e. “local” propagation direction).

---

<sup>3</sup>We'll consider only a monotonic field to simplify the math but a time-dependenty continuity equation for a generic scalar wave also exists.