1 Quantum model

The initial state of the system (thermal) and the time evolution of it in the harmonic trap as well as free space can both be easily described by the Wigner function. After obtaining the Wigner function of the final state after recapturing, we will assume that the escape probability is the total distribution in the phase space that has too high energy. We also need to consider the 3D case, where the atom can escape as long as the total energy in all dimension exceeds the trap depth.

2 Initial state

Define dimensionless quantities,

$$\begin{aligned} x =& \sqrt{\frac{m\omega}{\hbar}} x_0 \\ p =& \frac{p_0}{\sqrt{m\hbar\omega}} \\ t =& \omega t_0 \\ \beta =& \beta_0 \hbar\omega \end{aligned}$$

where x_0, p_0, t_0, etc. are the original dimensional quantities and x, p, t, etc. are dimensionless. In this unit, the energy is

$$\begin{aligned} E =& \frac{m\omega^2 x_0^2}{2} + \frac{p_0^2}{2m} \\ =& \frac{\hbar\omega}{2}\left(x^2 + p^2\right) \\ \equiv& \hbar\omega\varepsilon \end{aligned}$$

For the initial thermal state at temperature T, the (dimensionless) Wigner function is

$$W(x,p) = \frac{\tanh(f)}{\pi} \exp\left(-\tanh(f)\left(x^2 + p^2\right)\right)$$

where $f \equiv \frac{1}{2}\beta$.

3 Evolution

The evolution of the Wigner function in free space is a shearing along x. After time t, the distribution $W_0(x,p)$ becomes,

$$W_t(x,p) = W_t(x - pt, p)$$

For the thermal initial state, this is

$$\begin{aligned} W_t(x,p) =& \frac{\tanh(f)}{\pi} \exp\left(-\tanh(f)\left((x-pt)^2 + p^2\right)\right) \\ =& \frac{\tanh(f)}{\pi} \exp\left(-\tanh(f)\left(x^2 - 2txp + (1+t^2)p^2\right)\right) \end{aligned}$$

4 Energy distribution

To get the energy distribution after the recapture, we can integrate the Wigner function over the polar angle.

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} =& \int_0^{2\pi} \mathrm{d}\theta r W_t(r\cos\theta, r\sin\theta) \\
=& r\frac{\tanh(f)}{\pi}\int_0^{2\pi} \mathrm{d}\theta \exp\left(-\tanh(f) r^2 \left(\cos^2\theta - 2t\cos\theta\sin\theta + (1+t^2)\sin^2\theta\right)\right) \\
=& r\frac{\tanh(f)}{\pi}\mathrm{e}^{-\tanh(f)\left(2+t^2\right)\varepsilon}\int_0^{2\pi} \mathrm{d}\theta \exp\left(\tanh(f)\varepsilon\left(t^2\cos 2\theta + 2t\sin 2\theta\right)\right)
\end{aligned}
$$

Energy distribution

$$
\frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \frac{\tanh(f)}{\pi}\mathrm{e}^{-\tanh(f)\left(2+t^2\right)\varepsilon}\int_0^{2\pi} \mathrm{d}\theta \exp\left(\tanh(f)\varepsilon\left(t^2\cos 2\theta + 2t\sin 2\theta\right)\right)
$$

Shift θ

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}\varepsilon} =& \frac{\tanh(f)}{\pi}\mathrm{e}^{-\tanh(f)\left(2+t^2\right)\varepsilon}\int_0^{2\pi} \mathrm{d}\theta \exp\left(\tanh(f) t\sqrt{4+t^2}\varepsilon\cos 2\theta\right) \\
=& 2\tanh(f)\mathrm{e}^{-\tanh(f)\left(2+t^2\right)\varepsilon} I_0\left(\tanh(f) t\sqrt{4+t^2}\varepsilon\right)
\end{aligned}
$$

where $I_n(x)$ is the modified Bessel function of the first kind. Define

$$
\begin{aligned}
G(x,t) \equiv& 2\mathrm{e}^{-\left(2+t^2\right)x} I_0\left(t\sqrt{4+t^2}x\right) \\
\frac{\mathrm{d}P}{\mathrm{d}\varepsilon} =& \tanh\left(\frac{\beta}{2}\right) G\left(\varepsilon\tanh\left(\frac{\beta}{2}\right), t\right)
\end{aligned}
$$