Response of single bit rotation to amplitude noise

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1 Goal

Try to derive the pulse sequence to achieve robustness against (DC) amplitude noise in a brute force/generic way.

2 Pulse sequence with three rotations

1. $\sigma_1 \equiv \sigma_x \cos\theta_1 + \sigma_y \sin\theta_1$ by angle ψ_1
2. $\sigma_2 \equiv \sigma_x$ by angle ψ_2
3. $\sigma_3 \equiv \sigma_x \cos\theta_3 + \sigma_y \sin\theta_3$ by angle ψ_3

Full rotation.

$$U = \exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\exp\left(\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(\mathrm{i}\psi_3\sigma_3/2\right)$$

When ψ_1, ψ_2, ψ_3 are changed by a small fraction 2ε to $\psi_1(1+2\varepsilon)$, $\psi_2(1+2\varepsilon)$, $\psi_3(1+2\varepsilon)$ respectively.

The derivative of U w.r.t. ε is.

$$\begin{aligned}\frac{\partial U}{\partial\varepsilon} =& \mathrm{i}\psi_1\exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\sigma_1\exp\left(\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(\mathrm{i}\psi_3\sigma_3/2\right)+\\ &\mathrm{i}\psi_2\exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\sigma_2\exp\left(\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(\mathrm{i}\psi_3\sigma_3/2\right)+\\ &\mathrm{i}\psi_3\exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\exp\left(\mathrm{i}\psi_2\sigma_2/2\right)\sigma_3\exp\left(\mathrm{i}\psi_3\sigma_3/2\right)\end{aligned}$$

Simplifying it by changing coordinates

$$\begin{aligned}&\exp\left(-\mathrm{i}\psi_1\sigma_1/2\right)\frac{\partial U}{\partial\varepsilon}\exp\left(-\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(-\mathrm{i}\psi_3\sigma_3/2\right)\\ =&\mathrm{i}\psi_1\sigma_1 + \mathrm{i}\psi_2\sigma_2 + \mathrm{i}\psi_3\exp\left(\mathrm{i}\psi_2\sigma_2/2\right)\sigma_3\exp\left(-\mathrm{i}\psi_2\sigma_1/2\right)\\ =&\mathrm{i}\psi_1\sigma_1 + \mathrm{i}\psi_2\sigma_2 + \mathrm{i}\psi_3(\sigma_3\cos\psi_2 - \sigma_z\sin\theta_3\sin\psi_2 + \sigma_2\cos\theta_3(1-\cos\psi_2))\end{aligned}$$

For this to be 0, the coefficient for σ_z has to be 0. This means $\psi_3 = 0$ (two rotation only), $\sin\theta_3 = 0$ (third rotation on the same axis as the second one, so effectivel no third rotation), or $\sin\psi_2 = 0$. For a true three rotation sequence, the only option is $\sin\psi_2 = 0$, which leave us with two possibilities $\cos\psi_2 = \pm 1$.

For $\cos\psi_2 = 1$,

$$\begin{aligned}&\exp\left(-\mathrm{i}\psi_1\sigma_1/2\right)\frac{\partial U}{\partial\varepsilon}\exp\left(-\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(-\mathrm{i}\psi_3\sigma_3/2\right)\\ =&\mathrm{i}\psi_1\sigma_1 + \mathrm{i}\psi_2\sigma_2 + \mathrm{i}\psi_3\sigma_3\end{aligned}$$

For $\cos\psi_2 = -1$,

$$
\begin{aligned}
&\exp\left(-\mathrm{i}\psi_1\sigma_1/2\right)\frac{\partial U}{\partial\varepsilon}\exp\left(-\mathrm{i}\psi_2\sigma_2/2\right)\exp\left(-\mathrm{i}\psi_3\sigma_3/2\right)\\
=&\mathrm{i}\psi_1\sigma_1 + \mathrm{i}\psi_2\sigma_2 + \mathrm{i}\psi_3(-\sigma_3 + 2\sigma_2\cos\theta_3)\\
=&\mathrm{i}\psi_1\sigma_1 + \mathrm{i}\psi_2\sigma_2 + \mathrm{i}\psi_3(\sigma_x\cos\theta_3 - \sigma_y\sin\theta_3)
\end{aligned}
$$

For both of these cases, the robust condition is for the sum of the three Pauli vectors to be 0 (in the first case these are the Pauli vector that corresponds to the rotation and in the second case the last one is replaced with its reflection about the x axis). The robust condition for the first one is also the same as the cross talk calculation condition which allows the pulse sequence to cancel cross talk and overrotation error to the first order at the same time.

As for the actual rotation, for the first case, the second pulse is a full rotation so the equivalent final operation is simply the combination of the first and the third one. Unless one of them is a π or 2π rotation, the combined result will have a component that is a z rotation. This operation cannot be removed by changing the coordinate system (i.e. by making the second rotation along another axis rather than the x axis). This could be cancelled out for single-qubit gate by adjusting the phase but may not be as easy for two-qubit gates.

For the second case,

$$
\begin{aligned}
U =& \exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\sigma_x\exp\left(\mathrm{i}\psi_3\sigma_3/2\right)\\
=&\left(\cos\frac{\psi_1}{2} + \mathrm{i}\sigma_x\cos\theta_1\sin\frac{\psi_1}{2} + \mathrm{i}\sigma_y\sin\theta_1\sin\frac{\psi_1}{2}\right)\sigma_x\left(\cos\frac{\psi_3}{2} + \mathrm{i}\sigma_x\cos\theta_3\sin\frac{\psi_3}{2} + \mathrm{i}\sigma_y\sin\theta_3\sin\frac{\psi_3}{2}\right)\\
=&\sigma_x\cos\frac{\psi_1}{2}\cos\frac{\psi_3}{2} + \mathrm{i}\cos\frac{\psi_1}{2}\cos\theta_3\sin\frac{\psi_3}{2} - \sigma_z\cos\frac{\psi_1}{2}\sin\theta_3\sin\frac{\psi_3}{2}\\
&+ \mathrm{i}\cos\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2} - \sigma_x\cos\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} - \sigma_y\cos\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2}\\
&+ \sigma_z\sin\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2} - \sigma_y\sin\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} + \sigma_x\sin\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2}\\
=&\mathrm{i}\cos\theta_3\cos\frac{\psi_1}{2}\sin\frac{\psi_3}{2} + \mathrm{i}\cos\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2}\\
&+ \sigma_x\cos\frac{\psi_1}{2}\cos\frac{\psi_3}{2} - \sigma_x\cos\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} + \sigma_x\sin\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2}\\
&- \sigma_y\cos\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} - \sigma_y\sin\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2}\\
&- \sigma_z\sin\theta_3\cos\frac{\psi_1}{2}\sin\frac{\psi_3}{2} + \sigma_z\sin\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2}
\end{aligned}
$$

(Seems possible to achieve pure $x-y$ plane rotation though not sure if there's a way to express the condition in a concise way.)

2.1 Maximizing the final rotation angle

Given the same total pulse lens (i.e. total rotation angle for all the pulses), we'd like to maximize the total effective rotation angle to make the pulse sequence the most efficient.

For the $\cos\psi_2 = 1$ case, the final rotation is

$$\begin{aligned}
U =& \exp\left(\mathrm{i}\psi_1\sigma_1/2\right)\exp\left(\mathrm{i}\psi_3\sigma_3/2\right) \\
=& \left(\cos\frac{\psi_1}{2} + \mathrm{i}\sigma_x\cos\theta_1\sin\frac{\psi_1}{2} + \mathrm{i}\sigma_y\sin\theta_1\sin\frac{\psi_1}{2}\right)\left(\cos\frac{\psi_3}{2} + \mathrm{i}\sigma_x\cos\theta_3\sin\frac{\psi_3}{2} + \mathrm{i}\sigma_y\sin\theta_3\sin\frac{\psi_3}{2}\right) \\
=& \cos\frac{\psi_1}{2}\cos\frac{\psi_3}{2} + \mathrm{i}\sigma_x\cos\theta_3\cos\frac{\psi_1}{2}\sin\frac{\psi_3}{2} + \mathrm{i}\sigma_y\sin\theta_3\cos\frac{\psi_1}{2}\sin\frac{\psi_3}{2} \\
&+ \mathrm{i}\sigma_x\cos\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2} - \cos\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} - \mathrm{i}\sigma_z\cos\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} \\
&+ \mathrm{i}\sigma_y\sin\theta_1\sin\frac{\psi_1}{2}\cos\frac{\psi_3}{2} + \mathrm{i}\sigma_z\sin\theta_1\cos\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2} - \sin\theta_1\sin\theta_3\sin\frac{\psi_1}{2}\sin\frac{\psi_3}{2}
\end{aligned}$$

A Decompose an arbitray single qubit rotation into an xy rotation and a z rotation

Arbitrary single qubit rotation,

$$U = a_0 + x_0\sigma_x + y_0\sigma_y + z_0\sigma_z$$

xy rotation followed by z rotation

$$\begin{aligned} U =& U_{xy}U_z \\ =& (a_1 + x_1\sigma_x + y_1\sigma_y)(\cos\theta + \mathrm{i}\sin\theta\sigma_z) \\ =& a_1\cos\theta + \mathrm{i}a_1\sin\theta\sigma_z + \cos\theta x_1\sigma_x + \mathrm{i}\sin\theta x_1\sigma_x\sigma_z + \cos\theta y_1\sigma_y + \mathrm{i}\sin\theta y_1\sigma_y\sigma_z \\ =& a_1\cos\theta + (\cos\theta x_1 - \sin\theta y_1)\sigma_x + (\cos\theta y_1 + \sin\theta x_1)\sigma_y + \mathrm{i}a_1\sin\theta\sigma_z \end{aligned}$$

$$\begin{aligned} a_0 =& a_1\cos\theta \\ x_0 =& x_1\cos\theta - y_1\sin\theta \\ y_0 =& y_1\cos\theta + x_1\sin\theta \\ z_0 =& \mathrm{i}a_1\sin\theta \end{aligned}$$

so we have

$$\begin{aligned} a_1 =& \sqrt{a_0^2 - z_0^2} \\ \cos\theta =& \frac{a_0}{\sqrt{a_0^2 - z_0^2}} \\ \sin\theta =& -\frac{\mathrm{i}z_0}{\sqrt{a_0^2 - z_0^2}} \\ x_1 =& \frac{a_0x_0 - \mathrm{i}z_0y_0}{\sqrt{a_0^2 - z_0^2}} \\ y_1 =& \frac{a_0y_0 + \mathrm{i}z_0x_0}{\sqrt{a_0^2 - z_0^2}} \end{aligned}$$

For the special case of $a_1 = 0$ (i.e. $a_0^2 = z_0^2$), we can prove that in this case we must have $a_0 = z_0 = 0$ for U to remain unitary (Appendix B) [1]. In this case, U already contains no σ_z term so we can simply return U and I as the decomposition.

[1] A non-unitary U may have a similar decomposition though the z part may also not be unitary anymore.

B Direct prove that $a_0 = z_0 = 0$ if $a_0^2 = z_0^2$

For U to be unitary, we must have $UU^\dagger = 1$

$$
\begin{aligned}
UU^\dagger =& (a_0 + x_0\sigma_x + y_0\sigma_y + z_0\sigma_z)(a_0^* + x_0^*\sigma_x + y_0^*\sigma_y + z_0^*\sigma_z) \\
=& a_0a_0^* + a_0x_0^*\sigma_x + a_0y_0^*\sigma_y + a_0z_0^*\sigma_z + a_0^*x_0\sigma_x + x_0^*x_0\sigma_x\sigma_x + y_0^*x_0\sigma_x\sigma_y + z_0^*x_0\sigma_x\sigma_z + \\
& a_0^*y_0\sigma_y + x_0^*y_0\sigma_y\sigma_x + y_0^*y_0\sigma_y\sigma_y + z_0^*y_0\sigma_y\sigma_z + a_0^*z_0\sigma_z + x_0^*z_0\sigma_z\sigma_x + y_0^*z_0\sigma_z\sigma_y + z_0^*z_0\sigma_z\sigma_z \\
=& |a_0|^2 + |x_0|^2 + |y_0|^2 + |z_0|^2 \\
& + (a_0x_0^* + a_0^*x_0 + \mathrm{i}z_0^*y_0 - \mathrm{i}y_0^*z_0)\sigma_x \\
& + (a_0y_0^* + a_0^*y_0 + \mathrm{i}x_0^*z_0 - \mathrm{i}z_0^*x_0)\sigma_y \\
& + (a_0z_0^* + a_0^*z_0 + \mathrm{i}y_0^*x_0 - \mathrm{i}x_0^*y_0)\sigma_z \\
=& |a_0|^2 + |x_0|^2 + |y_0|^2 + |z_0|^2 \\
& + 2(\mathrm{Re}(a_0x_0^* + \mathrm{i}y_0z_0^*))\sigma_x \\
& + 2(\mathrm{Re}(a_0y_0^* + \mathrm{i}z_0x_0^*))\sigma_y \\
& + 2(\mathrm{Re}(a_0z_0^* + \mathrm{i}x_0y_0^*))\sigma_z
\end{aligned}
$$

So we have

$$
\begin{aligned}
|a_0|^2 + |x_0|^2 + |y_0|^2 + |z_0|^2 =& 1 \\
\mathrm{Re}(a_0x_0^* + \mathrm{i}y_0z_0^*) =& 0 \\
\mathrm{Re}(a_0y_0^* + \mathrm{i}z_0x_0^*) =& 0 \\
\mathrm{Re}(a_0z_0^* + \mathrm{i}x_0y_0^*) =& 0
\end{aligned}
$$

For $a_0^2 = z_0^2$, we have either $a_0 = z_0$ or $a_0 = -z_0$. If $a_0 = z_0$, the second and the third constaints turns into,

$$
\begin{aligned}
\mathrm{Re}(a_0x_0^* - \mathrm{i}a_0y_0^*) =& 0 \\
\mathrm{Im}(a_0x_0^* - \mathrm{i}a_0y_0^*) =& 0
\end{aligned}
$$

or

$$
a_0(x_0^* - \mathrm{i}y_0^*) = 0
$$

the last constraint turns into,

$$
\mathrm{Re}\Big(|a_0|^2 + \mathrm{i}x_0y_0^*\Big) = 0
$$

If $a_0 \neq 0$, we have $x_0^* = \mathrm{i}y_0^*$,

$$
\mathrm{Re}\Big(|a_0|^2 + |x_0|^2\Big) = 0
$$

which requires $a_0 = 0$ and $x_0 = 0$ contradicting with $a_0 \neq 0$.
If $a_0 = z_0$, the second and the third constaints turns into,

$$
\begin{aligned}
\mathrm{Re}(a_0x_0^* + \mathrm{i}a_0y_0^*) =& 0 \\
\mathrm{Im}(a_0x_0^* + \mathrm{i}a_0y_0^*) =& 0
\end{aligned}
$$

or

$$
a_0(x_0^* + \mathrm{i}y_0^*) = 0
$$

the last constraint turns into,

$$\mathrm{Re}\Big(-|a_0|^2 + \mathrm{i}x_0 y_0^*\Big) = 0$$

If $a_0 \neq 0$, we have $x_0^* = -\mathrm{i}y_0^*$,

$$\mathrm{Re}\Big(-|a_0|^2 - |x_0|^2\Big) = 0$$

which requires $a_0 = 0$ and $x_0 = 0$ contradicting with $a_0 \neq 0$. So in both cases we must have $a_0 = z_0 = 0$.