Phase of polarization components

February 5, 2024

1 Introduction

After selecting a frame, we can always decompose the polarization into the σ^+, σ^- and π components. While the amplitude of this decomposition is relatively easy to figure out, their phases depends on the selection of basis. The basis, and therefore the phase, can be arbitrary as long as it's consistent. This consistency is trivial when only dealing with polarization vectors (as long as the same basis is used) but becomes more important when the light starts to interact with other stuff, e.g. atoms. It is therefore important to ensure that the selection of polarization basis is consistent with the basis for the angular momentum states of the atoms (and practically the phase of the CG-coefficient).

Although the mathematically correct way to do this is probably to start with a consistent definition and derive everything from there, I'd like to derive/check how to get a consistent definition of the phase based only on definitions/formulas that we use, to see how the definition can propagate from one part to another.

2 Calculation

The convention in quantum mechanics is to relate the angular momentum states using the ladder operators,

$$j_\pm = j_x \pm \mathrm{i} j_y \tag{1}$$

and we have

$$j_\pm |jm\rangle = C_\pm |j(m \pm 1)\rangle \tag{2}$$

where $C_\pm$ are non-negative constant factors. This uniquely defines the phase of the angular momentum eigenstates.

When calculating the value of a tensor operator, we use Wigner-Eckart theorem to convert it to CG coefficients and a reduced matrix element. The derivation for Wigner-Eckart uses the definition/convention for the spherical basis,

$$V_0 = V_z \tag{3}$$

$$V_{\pm 1} = \mp \frac{1}{\sqrt{2}}(V_x \pm \mathrm{i} V_y) \tag{4}$$

This should also be the polarization basis used for the π and $\sigma^\pm$ polarizations.

$$\hat{e}_\pi = \hat{e}_z \tag{5}$$

$$\hat{e}_{\sigma^\pm} = \mp \frac{1}{\sqrt{2}}(\hat{e}_x \pm \mathrm{i} \hat{e}_y) \tag{6}$$

Or the amplitude relation,

$$A_z = A_\pi \tag{7}$$

$$A_x = -\frac{1}{\sqrt{2}}(A_{\sigma^+} - A_{\sigma^-}) \tag{8}$$

$$A_y = -\frac{\mathrm{i}}{\sqrt{2}}(A_{\sigma^+} + A_{\sigma^-}) \tag{9}$$

and

$$A_\pi = A_z \tag{10}$$

$$A_{\sigma^+} = \frac{1}{\sqrt{2}}(\mathrm{i}A_y - A_x) \tag{11}$$

$$A_{\sigma^-} = \frac{1}{\sqrt{2}}(\mathrm{i}A_y + A_x) \tag{12}$$