## THEORY QUESTIONS

**Q2** Consider a set of *N* vectors ***X*** = {$x_1, x_2, ..., x_N$ } each in $R^d$, with average vector ***X***.We have seen in class that the direction e such that

$$\sum_{i=1}^{N} \|x_i - \bar{x} - (e \cdot (x_i - \bar{x}))e\|^2$$

is minimized, is obtained by maximizing $\mathbf{e^T Ce}$, where C is the covariance matrix of the vectors in ***X*** . This vector e is the eigenvector of matrix C with the highest eigenvalue. Prove that the direction f perpendicular to e for which $f^T Cf$ is maximized, is the eigenvector of C with the second highest eigenvalue. For simplicity, assume that all non-zero eigenvalues of C are distinct and that rank(C) > 2.

**Sol:** Using Lagrange multipliers $\lambda_1$ and $\lambda_2$, we have to minimize

$$G(f) = f^TCf - \lambda_1(f^Tf - 1) - \lambda_2(f^Te)$$

$\Rightarrow \Delta_f\, G(f) = 2Cf - 2\lambda_1 f - \lambda_2 e = 0 \quad \Rightarrow 2e^TC f - 2\lambda_1 e^T f - \lambda_2 e^T e = 0 \quad \Rightarrow 2e^TC f - 2\lambda_1 e^T f - \lambda_2 = 0$ **-----(1)**

[$\because$ e is a unit vector]

$\Rightarrow 2(C^Te)^T f - 2\lambda_1 e^T f - \lambda_2 = 0 \Rightarrow 2(Ce)^T f - 2\lambda_1 e^T f - \lambda_2 = 0$ . Since e is an eigenvector of C,

$Ce = \lambda e$ where $\lambda$ is the highest eigen value of C. Also, f is perpendicular to e. $\Rightarrow e^Tf = 0$

$$\Rightarrow 2\lambda\, e^T f - 2\lambda_1 e^T f - \lambda_2 = 0$$

$$\Rightarrow (2\lambda - 2\lambda_1)e^T f - \lambda_2 = 0$$

$$\Rightarrow \lambda_2 = 0$$

Now, using original eq., $\quad 2Cf - 2\lambda_1 f = 0$

$$\Rightarrow Cf = \lambda_1 f$$

So, f is an eigenvector of C with eigen value $\lambda_1$. Also, all the non-zero eigenvalues of C are distinct, C has atleast 2 non zero eigenvalues (rank(C)>2) and the highest eigen value is $\lambda$ corresponding to eigenvector e, and, $f^TCf = \lambda_1$ had to be maximized. So, $\lambda_1$ must be the second largest eigen value of C.