Q3. Consider a matrix A of size $m \times n$, $m \le n$. Define $P = A^T A$ and $Q = AA^T$. (Note: all matrices, vectors and scalars involved in this question are real-valued).

(a) Prove that for any vector y with appropriate number of elements, we have $\mathbf{y^t P y} \ge 0$. Similarly show that $\mathbf{z^T Q z} \ge 0$ for a vector z with appropriate number of elements. Why are the eigenvalues of P and Q non-negative?

Sol: We have, $y^T P y = y^T(A^T A)y = (y^T A^T)(Ay) = (Ay)^T(Ay) = \| Ay \|_2^2 \ge 0$ ------- **(1)**

$[\because (AB)^T = B^T A^T]$

Also, $z^T Q z = z^T(AA^T)z = (z^T A)(A^T z) = (A^T z)^T(A^T z) = \| A^T z \|_2^2 \ge 0$

Now, Let λ be the eigen value of P, with eigen vector x $\Rightarrow Px = \lambda x$

Multiplying both the sides by x^T, we have $x^T Px = x^T \lambda x = \lambda x^T x = \lambda \| x\|_2^2$

From (1), we have, $x^T P x \ge 0 \Rightarrow \lambda \| x\|_2^2 \ge 0, \Rightarrow \lambda \ge 0$

Similarly, let z be an eigen vector of Q with eigen value ω.

$\Rightarrow Qz = \omega z \Rightarrow z^T Q z = \omega \| z\|_2^2 \ge 0 \Rightarrow \omega \ge 0$. Hence, the eigen values of P and Q are non- negative. P and Q are positive semi-definite matrices.

(b) If **u** is an eigenvector of P with eigenvalue λ, show that **Au** is an eigenvector of **Q** with eigenvalue λ. If **v** is an eigenvector of **Q** with eigenvalue μ, show that $\mathbf{A^T v}$ is an eigenvector of **P** with eigenvalue $\boldsymbol{\mu}$. What will be the number of elements in u and v?

Sol: We have, $Pu = \lambda u \Rightarrow (A^T A)u = \lambda u$

Now , $Q(Au) = (AA^T)(Au) = A(A^T Au) = A(\lambda u) = \lambda (Au)$ $[\because \lambda$ is ascalar]. $\Rightarrow \mathbf{Q(Au) = \lambda (Au)}$. Hence, Au is an eigen vector of Q with eigen value of Q with eigen value λ

Also, v is an eigen vector of Q with eigen value μ. $\Rightarrow Qv = \mu v \Rightarrow (AA^T)v = \mu v$

Now, $P(A^T v) = (A^T A)(A^T v) = A^T (A A^T v) = A^T \mu v \Rightarrow \mathbf{P(A^T v) = \mu(A^T v)}$

Thus $A^T v$ is an eigen vector of P with eigen value μ .

(c) If v_i is an eigenvector of Q and we define $u_i \triangleq A^T v_i / \| A^T v_i \|_2$. Then prove that there will exist some real, non-negative γ_i such that $Au_i = \gamma_i v_i$

Sol: We have, $Au_i = A(A^Tv_i / \| A^Tv_i \|_2) = (A A^Tv_i)/\| A^Tv_i \|_2 = Qv_i / \| A^Tv_i \|_2 = \mu_i v_i / \| A^Tv_i \|_2$.

Let $y_i = (\mu_i / \| A^Tv_i \|_2)$, -----------(2)

$\Rightarrow y_i \geq 0$ [From (a) and (b), and $\| a \|_2 \geq 0$]. Hence, $\exists$ a real non-negative y_i such that $\mathbf{Au_i = \gamma_i v_i}$

(d)It can be shown that $u^T_i u_j = 0$ for $i \neq j$ and likewise $v^T_i v_j = 0$ for $i \neq j$ for correspondingly distinct eigenvalues[1]. Now, define $U = [v_1|v_2|v_3|...|v_m]$ and $V = [u_1|u_2|u_3|...|u_m]$. Now show that $A = U \Gamma V^T$ where Γ is a diagonal matrix

containing the non-negative values $\gamma_1, \gamma_2, ..., \gamma_m$. With this, you have just established the existence of the singular value decomposition of any matrix A. This is a key result in linear algebra and it is widely used in image processing, computer vision, computer graphics, statistics, machine learning, numerical analysis, natural language processing and data mining.

Sol: We have, $u^T_i u_j = 0$ for $i \neq j$.

For $i = j$, $u^T_i u_i = (A^T v_i)^T (A^T v_i) / \| A^T v_i \|_2^2 = 1$ [$\because u^T u = \| u \|_2^2$].

Now, $Au_i = \gamma_i v_i$ (from (c), and γ_i is non-negative) .

Note that P is n*n matrix with at least n-m linearly independent eigenvectors corresponding to zero eigenvalues. Let those eigenvectors be u_i, i=m+1,m+2,...,n

Let $V' = [u_1|u_2|u_3|...|u_m|u_{m+1}| ....|u_n]$, where V' is a square, orthogonal (n*n) matrix. (Columns of V' are all the eigenvectors of P, and since P is real symmetric, V' is orthogonal).

Also, $U \Gamma V^T = \sum_{i=1}^{m} (\gamma_i v_i u^T_i) = \sum_{i=1}^{m} (Au_i u^T_i)) = A * \sum_{i=1}^{n} (u_i u^T_i) = AV'V'^T$ [$\because \sum_{i=m+1}^{n}$

$(A_i u_i) = 0$]

Hence, $U \Gamma V^T = AV'V'^T = A$