Lec. VII. Random Variable

Random variable is a mathematical function

input → $\boxed{\text{function}}$ → output $\qquad y = f(x)$

y ↓ o/p, f ↓ funcₒ, x ↓ i/p

probₒ
- random expₒ
- outcome / sample point
- event / subset of sample space

sample point → $\boxed{\text{function}}$ → probability

i.e. tossing two coin

H < H, T
T < H, T

HH
HT
TH
TH

⇒ inputs

i/p = {HH, HT, TH, TT} → $\boxed{\text{function}}$ → unique probₒ

Input	HH	HT	TH	TT
random variable (X)	2	1	1	0

X : nos of head / nos of tails
↳ changing variable

$\longrightarrow$ $\boxed{\text{function}}$ $\longrightarrow$ $\{1/4, 2/4, 1/4\}$

i/p
$\{0, 1, 2\}$

prob. or
unique value

$P[X = 0H] = 1/4$ $\quad$ $P[X = 1H] = 2/4$ $\quad$ $P[X = 2H] = 1/4$

ie Tossing three coin

H H H	0T
H H T	1T
H T H	1T
H T T	2T
T H H	1T
T H T	2T
T T H	2T
T T T	3T

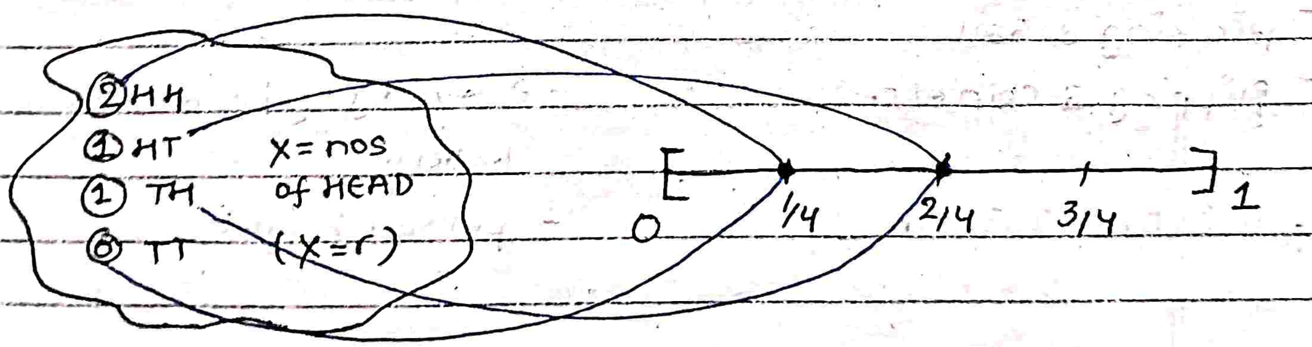
$X = 0, 1, 2, 3$

$P(X = r)$ random variable.

$P(X = 0T) \rightarrow 1/8$ $\quad$ $P(X = 1T) \rightarrow 3/8$

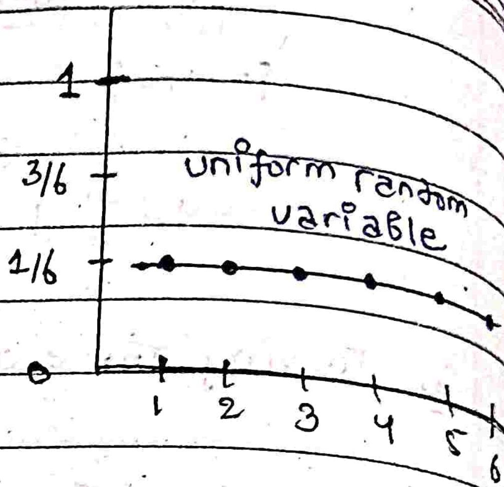
$P(X = 2T) \rightarrow 3/8$ $\quad$ $P(X = 3T) \rightarrow 1/8$

$\sum \text{all} \Rightarrow 1$

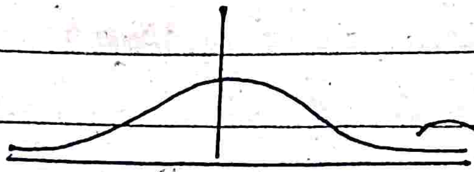


Rolling a balanced die

X	1	2	3	4	5	6
P(X=r)	1/6	1/6	1/6	1/6	1/6	1/6



for Large sample



gaussian distribution

Random Variable

Discrete random variable

countable

(infinite - countable set)

Continuous Random variable

Uncountable

(infinite - uncountable set).

Discrete,
- rolling a die,
- picking a ball
- flipping a coin etc

value : 0, 1, 2, 3, ...

always defined in interval

$(a \le x \le b)$.

- discharging / charging battery.
- person height

• Cumulative Distribution Function [C.d.f].

$$\boxed{F_X(x_i) = P(X \le x_i)}$$

$F_X(0) = P[X \le 0] = P_0$

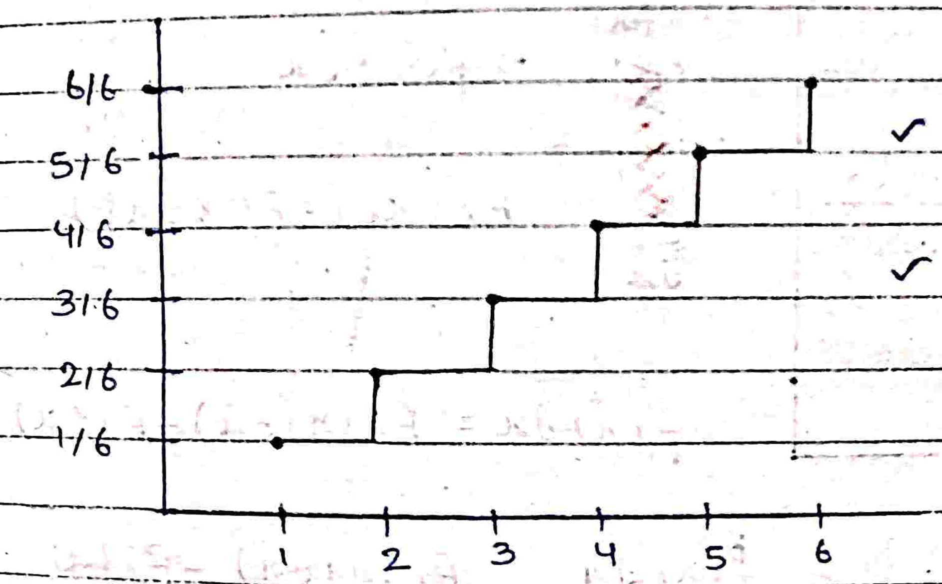
$F_X(1) = P[X \le 1] = P[X=0] + P[X=1]$

$F_X(2) = P[X \le 2] = P[X=0] + P[X=1] + P[X=2]$

ooooooo

$F_X(x_i) = P(X \le x_i) = P_0 + P_1 + P_2 + \dots + P_n$

X	1	2	3	4	5	6
$F_X(x_i)$	$F_X(1) = 1/6$	$F_X(2) = 2/6$	$F_X(3) = 3/6$	$F_X(4) = 4/6$	$F_X(5) = 5/6$	$F_X(6) = 6/6$



✓ monotonically increasing

✓ always +ve

↓

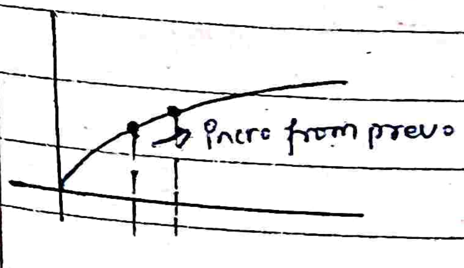
$F(x_i) \ge 0$

$p_i \ge 0$

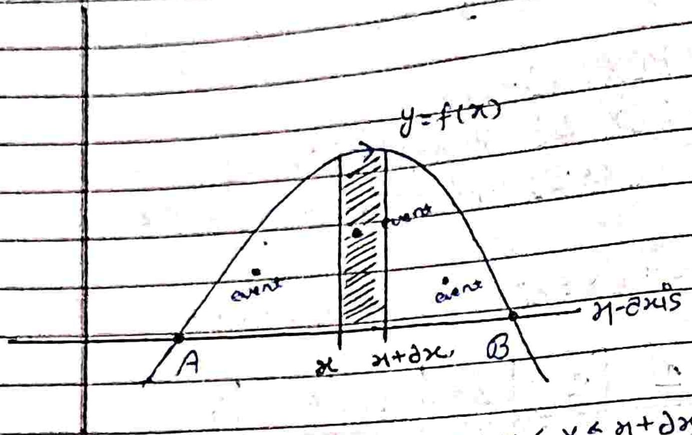
✓ $F_X(\infty) = 1$

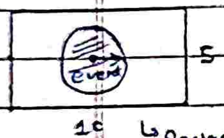
$F_X(-\infty) = 0$

monotonically



- Continuous Probability Distribution :





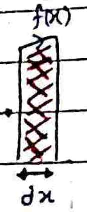
↳ Rectangle.

p(event) ⇒ $\frac{\text{favourable region}}{\text{total region}}$

$\frac{\text{Ar. of circle}}{\text{Ar. of rect.}}$

$P(x < X \le x+dx)$

= Ar. of rectangle



$\Rightarrow f(x) \cdot dx$

$F_X(x_i) = P(X \le x_i)$

↓

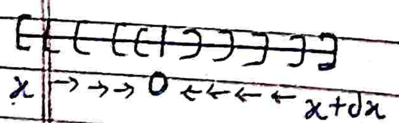
$f(x)dx = F_X(x+dx) - F_X(x)$

$$\frac{f(x)\,dx}{dx} = \frac{F_X(x+dx)}{dx} - \frac{F_X(x)}{dx}$$

$$\lim_{dx \to 0} \frac{F_X(x+dx) - F_X(x)}{dx} = f(x)$$

$dx \to 0$

↳



~~RC~~ $P(x \le X \le x+dx) = f(x)\,dx$

$$\Rightarrow \lim_{dx \to 0} \frac{F_X(x+dx) - F_X(x)}{dx} = f(x)$$

$\Rightarrow F'_X(x) = f(x)$

In a continuous probo ~~density~~ distribution

$F'_X(x) = f(x)$

$$\boxed{F_X(x) = cdf = \int_R f(x)\,dx}$$

$$F_X(b) - F_X(a) = \int_a^b f(x)\,dx$$

$$P_X(X \ge a) \to \int_a^{\infty} f(x)\,dx$$

$$P_X(X \le a) \to \int_{-\infty}^{a} f(x)\,dx$$

\# $f(x)$ is a $\boxed{\text{propability - density - function}}$ & valid probo density $\int_{-\infty}^{\infty} f(x)\,dx = 1$

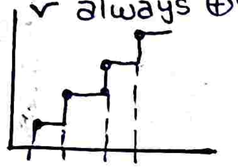
random variable → Discrete random variable → Cummulative distribution func. {cdf} → probability mass function

random variable → Continuous random variable → Continuous Probability Distribution → probability density function

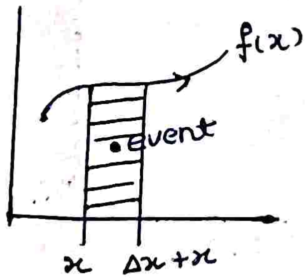
$$\sum_{i=1}^{n} P[X=x_i]=1$$

✓ monotonically increasing
✓ always +ve

$F(x_i) \geq 0$
$P_i \geq 0$
$F_X(\infty)=1$
$F_X(-\infty)=0$



$$\int_{region} f(x)\,dx = 1$$



i.e. rolling a die

sample point(x)	1	2	3	4	5	6
P(X = r)	1/6	1/6	1/6	1/6	1/6	1/6
$F_X(x_i) = P(X \leq x_i)$	1/6	2/6	3/6	4/6	5/6	6/6

$$F_X(x_i) = \int_{-\infty}^{\infty} \text{ or } \int_{region} f(x)\,dx$$

= 1 (valid pdf).

$$\frac{\partial F_X(x_i)}{\partial x} = f(x)$$

↳ pdf

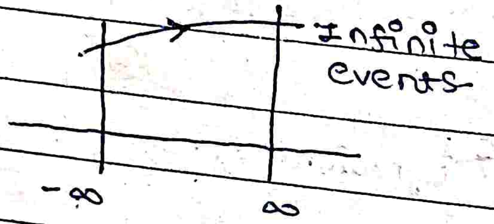
P) If x is a continuous random variable whose probability density function is given by —
(34)

$$f(x) = \begin{cases} k(5x - 2x^2), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Then $P(x \ge 1)$ is —

$$P(x \ge 1) = \int_1^2 k(5x - 2x^2)\,dx = ?$$

Note: In continuous random variable,



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

valid probo function

Here,
$$\int_0^2 k(5x - 2x^2)\,dx$$

$$\rightarrow k \int_0^2 (5x - 2x^2)\,dx$$

$$\rightarrow k \left[\frac{5x^2}{2} - \frac{2x^3}{3} \right]_0^2 \Rightarrow 1$$

$$\rightarrow k \left[10 - \frac{16}{3} \right] = 1 \rightarrow k = 3/14$$

$$P(X\geq 1) = \int_1^2 k(5x-2x^2)\,dx$$

$$= \int_1^2 \frac{3}{14}(5x-2x^2)\,dx$$

$$\Rightarrow \frac{3}{14}\left[\left(\frac{5\times(2)^2}{2} - \frac{2\times(2)^3}{3}\right) - \left(\frac{5\times(1)^2}{2} - \frac{2\times(1)^3}{3}\right)\right]$$

$$\Rightarrow \frac{3}{14}\left[\left(10-\frac{16}{3}\right) - \left(\frac{5}{2}-\frac{2}{3}\right)\right] \Rightarrow 17/28.$$

Q) 35 Given that x is a random variable in the range $[0,\infty)$ with a probability density function $\frac{e^{-x/2}}{k}$, the value of the constant k is –

$$\int_0^\infty f(x)\,dx \quad \text{(valid pdf)}$$

$$\int e^x\,dx = e^x$$

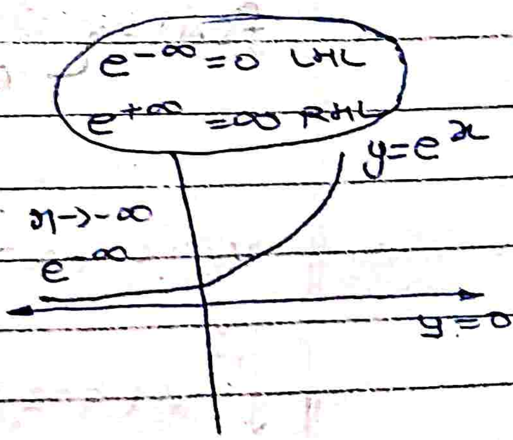
$$\int_0^\infty \frac{e^{-x/2}}{k}\,dx = 1$$

put $\frac{x}{2} = t$

$dx = 2\,dt$

$$\Rightarrow \frac{1}{k}\int_0^\infty e^{-t}\cdot 2\,dt = 1$$

$$\Rightarrow \frac{2}{k}\int_0^\infty e^{-t}\,dt = 1$$



$$\Rightarrow \frac{2}{k}\left[-e^{-t}\right]\Big|_0^\infty = 1$$

$$\Rightarrow \frac{2}{k}\left[-e^{-\infty} - (-e^{-0})\right] = 1$$

$$\Rightarrow \frac{2}{k}\left[0+1\right] = 1 \Rightarrow k = 2$$

The random variable X takes on the values 1, 2 or 3 with probabilities $\frac{2+5P}{3}$, $\frac{1+3P}{5}$ and $\frac{1.5+2P}{5}$ respectively. The value of P is—

$X = 1, 2, 3$ (discrete random variable)

X	1	2	3
$P(X=x)$	$\frac{2+5P}{3}$	$\frac{1+3P}{5}$	$\frac{1.5+2p}{5}$

in discrete random variable,

$$\sum_{x=0}^{n} P(X=x) = 1$$

$$\frac{2+5P}{5} + \frac{1+3P}{5} + \frac{1.5+2P}{5} = 1 \rightarrow p = 1/20$$

find the value of λ such that the function $f(x)$ is a valid probability density function

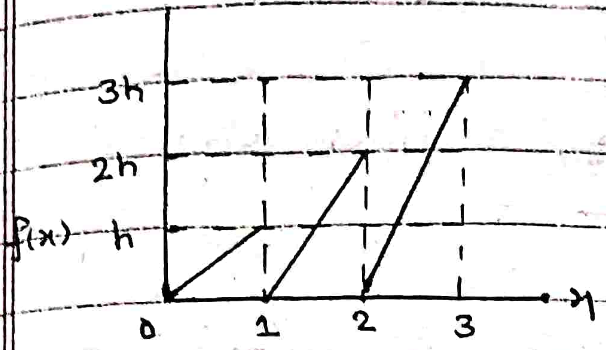
$$f(x) = \begin{cases} \lambda(x-1)(2-x) & \text{for } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

valid pdf

$$\int_1^2 \lambda(x-1)(2-x)\,dx = 1$$

$$\lambda \int_1^2 -x^2 + 3x - 2 = 1$$

Q) (38) The graph of a function $f(x)$ is shown in fig —



for $f(x)$ ~~to~~ be a valid pdf, the value of h is —

for valid pdf

$$\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{\text{total Area}} = 1$$

$$\frac{h \times 1}{2} + \frac{2h \times 1}{2} + \frac{3h \times 1}{2} = 1$$

$$\frac{h}{2} + h + \frac{3h}{2} \rightarrow \frac{6h}{2} = 1 \longrightarrow h = \frac{1}{3}$$

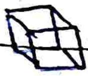
Q) (39) Lifetime of a electric-Bulb is a random variable with density $f(x) = kx^2$ where x is measured in years. If the minimum & maximum lifetimes of Bulb are 1 & 2 years respectively, then the value of k is ___

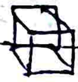
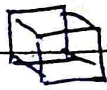
$f(x)$: x is a continuous random variable.

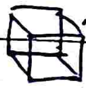
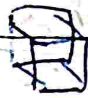
$$f(x) = \begin{cases} kx^2 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

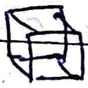
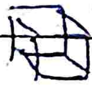
if valid pdf $\int_1^2 kx^2\,dx = 1$

Consider a die with the property that the probability of a face with 'n' dots showing up is proportional to 'n'. The probability of face with three dots showing-up is —

 → 1, 2, 3, 4, 5, 6 → discrete random variable

roll  → $1 \propto 1 \to 1 \cdot k$  → $4 \propto 4 \to 4 \cdot k$

 → $2 \propto 2 \to 2 \cdot k$  → $5 \propto 5 \to 5 \cdot k$

 → $3 \propto 3 \to 3 \cdot k$  → $6 \propto 6 \to 6 \cdot k$

pmf (probability mass function)

X	1	2	3	4	5	6
P(X=n)	1K	2K	3K	4K	5K	6K

$\sum P(X=n) = 1$

$K + 2K + 3K + 4K + 5K + 6K = 1$

$K = 1/21$

$P(X=n) \to P(X=3K) \to 3 \times \frac{1}{21}$

$\to \frac{1}{7}$

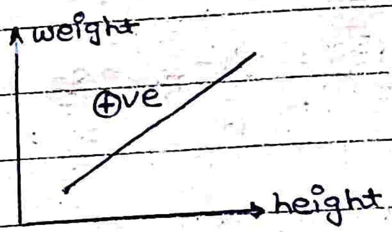
Lec. IX. Bivariate R.V

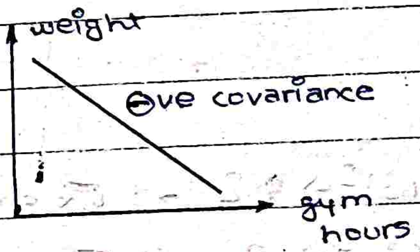
Date :

Page :

Covariance	Correlation
$Cov(x,y) = E(xy) - E(x)E(y)$	corr. coeff. $= r = \dfrac{Cov(x,y)}{\sigma_x \sigma_y}$
or $\dfrac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{N}$	
If x & y are independent $cov(x,y) = 0$	If x & y are independent $r = 0$

Covariance (continued)





$-\infty \leq cov < \infty$

Correlation (continued)

$-1 \leq r \leq 1$ strength

(gives magn. & dirn. Both)

(i) $r = 1$, perfectly correlated in positive sense

(ii) $r = -1$, -"- in negative sense (one feature increase other decrease).

o Regression

if supply is constant then

(y) Demand $\propto \dfrac{1}{\text{price}(x)} \Rightarrow y \propto \dfrac{1}{x}$ (customer) — y dependent

price $\propto$ Demand $\Rightarrow x \propto y$ (shopkeeper) — x dependent

two diff. P.O.V not possible for correlation

R Line y on $x \rightarrow$ $y - \bar{y} = b_{yx}(x - \bar{x})$, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

R line x on $y \rightarrow$ $x - \bar{x} = b_{xy}(y - \bar{y})$, $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

b_{yx} & $b_{xy} \rightarrow$ regression coefficient

Q)

X	1	2	3	4	5
Y	3	5	7	9	11

evaluate correlation coeff., regression coeff., regression line

	X	Y	XY	X^2	Y^2
	1	3	3	1	9
	2	5	10	4	25
	3	7	21	9	49
	4	9	36	16	81
	5	11	55	25	121
Σ	15	35	125	55	285

$E(X) = \frac{\Sigma x}{N} \rightarrow \frac{15}{5} \rightarrow 3$

$E(X^2) = \frac{\Sigma X^2}{N} \rightarrow \frac{55}{5} \rightarrow 11$

$E(Y) = \frac{\Sigma Y}{N} \rightarrow \frac{35}{5} \rightarrow 7$

$E[Y^2] = \frac{\Sigma Y^2}{N} \rightarrow \frac{285}{5} \rightarrow 57$

$E[XY] = \frac{\Sigma XY}{N} \rightarrow \frac{125}{5} \rightarrow 25$

$Var(X) = E[X^2] - [E[X]]^2 \rightarrow 11-(3)^2 \rightarrow 2 \rightarrow \sigma_x = \sqrt{2}$

$Var(Y) = E[Y^2] - [E[Y]]^2 \rightarrow 57-(7)^2 \rightarrow 8 \rightarrow \sigma_y = \sqrt{8} = 2\sqrt{2}$

① $r = \frac{Cov(X,Y)}{\sigma_x \sigma_y} \rightarrow \frac{4}{(\sqrt{2})(2\sqrt{2})} \Rightarrow 1$ perfect correlation

② $b_{yx} \rightarrow 1 \cdot \frac{2\sqrt{2}}{\sqrt{2}} = 2$ or $\frac{\sigma_y}{\sigma_x}$

$b_{xy} \rightarrow 1 \cdot \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$

③ reg line y on x

$y - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x}) \rightarrow y - 7 = 2(x-3)$

$\rightarrow y = 2x + 1$ —①

reg x on y

$$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \rightarrow (x-3) = \frac{1}{2}(y-7)$$

$$\rightarrow x = \frac{y}{2} - \frac{1}{2} \quad — (2)$$

Q. (5) If $\bar{x}=10$, $\bar{y}=90$, $\sigma x=3$, $\sigma y=12$, $r=0.8$ then find regression line.

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \rightarrow (0.8)\left(\frac{12}{3}\right) \rightarrow 3.2$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} \rightarrow (0.8)\left(\frac{3}{12}\right) \rightarrow 0.2$$

rego line y on x $\rightarrow$ $y-\bar{y} = b_{yx}(x-\bar{x})$

$$y-90 = 3.2(x-10)$$

$$y = 3.2x + 58$$

rego line x on y $\rightarrow$ $x-\bar{x} = b_{xy}(y-\bar{y})$

$$x-10 = 0.2(y-90)$$

$$x = 0.2y - 8$$

Note: Coefficient of correlation is geometric mean of coefficient of regression

$$\sqrt{b_{yx} \times b_{xy}} \Rightarrow \sqrt{r\frac{\sigma_y}{\sigma_x} \times \frac{\sigma_x}{\sigma_y}\cdot r} \Rightarrow \sqrt{r^2} \Rightarrow r$$

Double Integral working rule :

① (51) $I = \int_0^3 \int_0^2 x^2 y \, dx \, dy = ?$

$$\Rightarrow \int_{y=0}^{y=3} \int_{x=0}^{x=2} (x^2 y)\, dx \cdot dy$$

$$\Rightarrow \int_{x=0}^{2} x^2 \, dx \cdot \int_{y=0}^{3} y \, dy \Rightarrow \left(\frac{x^3}{3}\right)\Big|_0^2 \left(\frac{y^2}{2}\right)\Big|_0^3$$

$$\Rightarrow \frac{8}{3} \times \frac{9}{2} \Rightarrow 4 \times 3 \Rightarrow 12$$

② (52) $\int_0^1 \int_0^2 e^{x+y} \, dy \, dx = ?$

$$\Rightarrow \int_{x=0}^{x=1} \int_{y=0}^{y=2} (e^{x+y} \, dy)\, dx$$

$$\Rightarrow \int_{x=0}^{x=1} e^x \, dx \int_{y=0}^{y=2} e^y \, dy \Rightarrow (e-1)(e^2-1)$$

③ $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y) \, dx \, dy = ?$

$$\Rightarrow \int_{y=0}^{y=\pi/2} \int_{x=0}^{x=\pi/2} \sin(x+y) \, dx \, dy$$

method (I),

$$I = \int_{y=0}^{y=\pi/2} \left[\int_{x=0}^{x=\pi/2} \sin(x+\underset{\uparrow}{y})\, dx \right] dy$$

treat as 'constant'.

$$\Rightarrow \int_{y=0}^{\pi/2} \left\{ -\cos(x+y) \right\}_{x=0}^{\pi/2} \cdot dy \Rightarrow \int_{y=0}^{\pi/2} -\left\{ \cos\left(\frac{\pi}{2}+y\right) - \cos y \right\} dy$$

$$\Rightarrow \int_{0}^{\pi/2} -\left\{ -\sin y - \cos y \right\} dy \Rightarrow \int_{0}^{\pi/2} (\sin y + \cos y)\, dy \Rightarrow (-\cos y + \sin y)\Big|_{0}^{\pi/2}$$

method (2),

$$I = \int_{x=0}^{\pi/2} \left[\int_{y=0}^{y=\pi/2} \sin(\underset{\uparrow}{x}+y)\, dy \right] dx$$

treat as 'constant'.

$$\Rightarrow \int_{x=0}^{\pi/2} \left\{ -\cos(x+y) \right\}_{y=0}^{\pi/2} \cdot dx \Rightarrow \int_{x=0}^{\pi/2} -\left\{ \cos\left(\frac{\pi}{2}+x\right) - \cos x \right\} dx$$

$$\Rightarrow \int_{x=0}^{\pi/2} \left\{ \sin x + \cos x \right\} dx \Rightarrow (-\cos x + \sin x)\Big|_{0}^{\pi/2}$$

- case I: Both the variable have constant limits and integrals is an explicit function of x and y we can integrate seperately.

- case III: If both the variables have constant limits & integral is an implicit function of x and y then we can integrate one by one in any order.

• **Case III:** for vertical strip → if y have variable limits then we should first integrate dy keeping x constant.

Q. 53 $I = \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx = ?$

$\Rightarrow \int_{x=0}^{x=2} \int_{y=0}^{y=x/2} e^{x^2}\, dy\, dx \rightarrow \int_{x=0}^{2} \left[\int_{y=0}^{x/2} 1 \cdot dy \right] (e^{x^2}\, dx)$

$\Rightarrow \int_{x=0}^{2} \left(\frac{x}{2}\right) \cdot e^{x^2}\, dx$ $\left\{ \begin{array}{l} \text{put } x^2 = t \\ x\, dx = dt/2 \\ \text{at } x = 0 \rightarrow t = 0 \\ x = 2 \rightarrow t = 4 \end{array} \right\}$

$\Rightarrow \int_0^4 \frac{1}{2} e^t \cdot \frac{dt}{2}$

$\Rightarrow \frac{1}{4} (e^t) \Big|_0^4 \Rightarrow \frac{e^4 - 1}{4}$

Q. 54 $I = \int_0^1 \int_{y^2}^{5y} (1)\, dx\, dy = ?$ $x = f(y)$ ✓ $y = f(y)$ ✗

$\Rightarrow \int_{y=0}^{1} \int_{x=y^2}^{5y} (1)\, dx\, dy$

$\Rightarrow \int_{y=0}^{1} \left[\int_{x=y^2}^{5y} (1)\, dx \right] dy \Rightarrow \int_{y=0}^{1} (5y - y^2)\, dy$

$\Rightarrow \left(\frac{y^{3/2}}{3/2} - \frac{y^3}{3} \right) \Big|_0^1 \Rightarrow \frac{1}{3}$

• **Case IV:** If 'x' has variable limits then we should first integrate 'dx' keeping 'y' constant.

Q. 55 $I = \int_0^2 \int_0^{x^2} (xy^2)\, dx\, dy = ?$

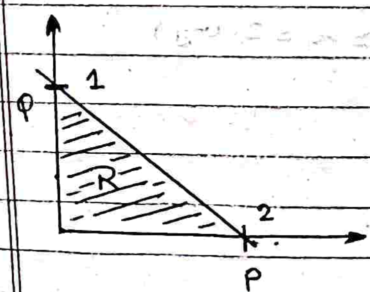
$\Rightarrow \int_{x=0}^{2} \int_{y=0}^{y=x^2} (xy^2)\, dy\, dx \Rightarrow \int_{x=0}^{2} \left[\int_{y=0}^{x^2} y^2\, dy \right] (x\, dx)$

$\Rightarrow \int_{x=0}^{2} \left(\frac{x^6}{3} \right) x\, dx \Rightarrow \frac{1}{3} \int_{x=0}^{2} \cdot x^7\, dx \Rightarrow \frac{1}{3} \left(\frac{x^8}{8} \right) \Big|_0^2$

$\Rightarrow \frac{2^8}{3 \times 8} \Rightarrow \frac{2^5}{3} \Rightarrow \frac{32}{3}$

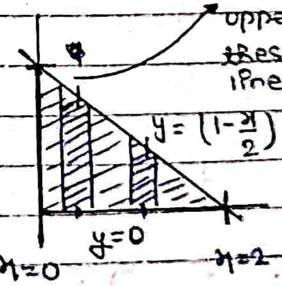
Note: The concept of explicit function is applicable only when both the limits are constant.

Q. 56 $\iint_R xy\, dx\, dy = ?$ over the region R shown in fig.?



PQ: $\frac{x}{2} + \frac{y}{1} = 1$

$y = \left(1 - \frac{x}{2}\right)$



upper portion of all these strips lie on this line.

$0 \le y \le \left(1 - \frac{x}{2}\right)$

$0 \le x \le 2$

$$\int_{x=0}^{2}\int_{y=0}^{y=(1-\frac{x}{2})} (xy)\,dy\,dx$$

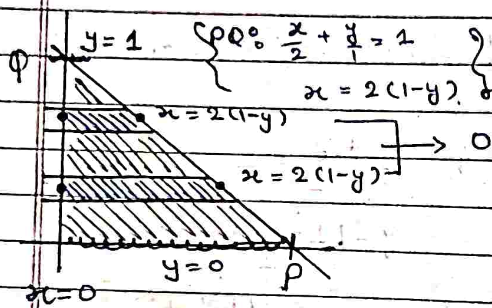
$$\int_{x=0}^{2}\left[\int_{y=0}^{y=(1-x/2)} y\,dy\right] x\,dx$$

$$\int_{x=0}^{2}\left[\frac{\left(1-\frac{x}{2}\right)^2}{2}\right](x\,dx)$$

$$\frac{1}{8}\int_{0}^{2}(4+x^3-4x)\,dx$$

$$\frac{1}{8}\left[2x^2+\frac{x^4}{4}-\frac{4x^3}{3}\right]_0^2$$

*method II) using 'horizontal strip'.



eq: $\left\{\frac{x}{2}+\frac{y}{1}=1 \quad x = 2(1-y)\right\}$

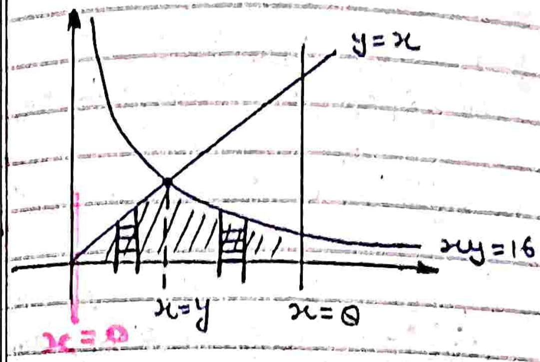
$\longrightarrow 0 \le x \le 2(1-y)$

$$\int_{y=0}^{1}\int_{x=0}^{2(1-y)} xy\,dx\,dy$$

$$\int_{y=0}^{1}\left[\int_{x=0}^{2(1-y)} x\,dx\right] y\,dy$$

$$\int_{y=0}^{1}\left[2(1-y)^2\right] y\,dy \Rightarrow 1/6$$

Q. (5) find $\iint x^2\,dy\,dx = ?$ over the region bounded by $xy=16$, $y=x$, $x=0$, $x=8$?



$$I = \int_{x=0}^{y}\int_{y=0}^{x} x^2\,dy\,dx + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$$

Lec. X. Joint P.M.F & P.D.F

Joint P.M.F

Q). (58) A fair coin is tossed thrice. If X denotes the number of heads on first toss and Y denotes the ~~total number of heads~~ then write it's joint pmf.

$X = \{0, 1\}$ (0 → head, 1 → 1 head) $\xrightarrow{\text{then}}$ $Y = \{0, 1, 2, 3\}$

$S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$

$X = \{\text{nos of heads in } I^{st} \text{ toss}\} = \{0, 1\}$

$p_1 = P(X = 0H \text{ in } I^{st} \text{ toss}) = (4/8)$

$p_2 = P(X = 1H \text{ in } I^{st} \text{ toss}) = (4/8)$

X	0	1
P(X)	4/8	4/8

$Y = (\text{Total nos. head}) = \{0, 1, 2, 3\}$

$p_3 = P(Y=0H) = 1/8$, $p_4 = P(Y=3H) = 1/8$

$p_5 = P(Y=1H) = 3/8$, $p_6 = P(Y=2H) = 3/8$

Y	0	1	2	3
P(Y)	1/8	3/8	3/8	1/8

Now, joint probability distribution,

X \ Y	0	1	2	3	P(X)
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{4}{8}$
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
P(Y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

(i) marginal distribution of X & Y.

X	0	1
P(X)	4/8	4/8

Y	0	1	2	3
P(Y)	1/8	3/8	3/8	1/8

(ii) $P(X=1, Y \geq 1) = P(X=1 \cap Y = 1 \text{ or } 2 \text{ or } 3)$

$= P[(1,1) \text{ or } (1,2) \text{ or } (1,3)]$

$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} \Rightarrow \frac{1}{2}$

(iii) $P(X \geq 0, Y \geq 2) = P\{(X = 0 \text{ or } 1) \cap (Y = 2 \text{ or } 3)\}$

$= \{(0,2) \text{ or } (0,3) \text{ or } (1,2) \text{ or } (1,3)\}$

$\Rightarrow \frac{1}{8} + 0 + \frac{2}{8} + \frac{1}{8} \Rightarrow \frac{1}{2}$

(iv) $P(0 \leq X \leq 1, Y = 3) = P[(X = 0 \text{ or } 1) \cap (Y=3)]$

$= P[(0,3) \text{ or } (1,3)]$

$\Rightarrow 0 + \frac{1}{8} \Rightarrow \frac{1}{8}$

(v) $P(X=4) \Rightarrow P[(0,0) \text{ or } (1,1)]$

$\Rightarrow \frac{1}{8} + \frac{1}{8} \Rightarrow \frac{1}{4}$

(vi) $P[1 \le Y \le 3] \Rightarrow P[(X=0,1) \cap (Y=1,2,3)]$.

$\Rightarrow P[(0,1) \text{ or } (0,2) \text{ or } (0,3) \text{ or } (1,1) \text{ or } (1,2) \text{ or } (1,3)]$

$\Rightarrow 7/8$

"or"

$P[Y = \text{"1" or "2" or "3"}]$

$\Rightarrow \frac{3}{8} + \frac{3}{8} + \frac{1}{8} \Rightarrow \frac{7}{8}$

Note: * If x_i & y_j are Discrete Random Variable then this joint p.m.f satisfies,

① $P[X=x_i, Y=y_j] = P[x_i, y_j] \ge 0$

② $\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) = 1$

* Individual probability distribution are called marginal distribution.

* if $P[x_i, y_j] = P(x_i) \cdot P(y_j) \ \forall i \,\&\, j$ then X & Y are independent random variables.

* conditional distribution of X for given Y.

$$P\left[\frac{X=x}{Y=y}\right] = \frac{P[x,y]}{P[Y=y]}$$

Conditional distribution of Y for given X:

$$P\left[\frac{Y=y}{X=x}\right] = \frac{P(x,y)}{P(X=x)}.$$

(59) four coins are tossed, Let x denotes nos of heads and y denotes (number of Heads - number of tails) Then write joint pmf for such type of random experiment -

S ≡ { HHHH
HHHT
HHTH
HHTT
HTHH
HTHT
HTTH
HTTT
THHH
THHT
THTH
THTT
TTHH
TTHT
TTTH
TTTT }

X ≡ {nos of heads}

≡ {0, 1, 2, 3, 4}

(marginal distr of x)

X	0	1	2	3	4
P(X)	1/16	4/16	6/16	4/16	1/16

Y ≡ {nos of head - nos of tail}

{0-4, 1H-3T, 2H-2T, 3H-1T ...}

⇓

{-4, -2, 0, 2, 4}

(marginal distr of Y)

Y:	-4	-2	0	2	4
P(Y)	1/16	4/16	6/16	4/16	1/16

$P(Y=-4) \rightarrow 0H-4T \rightarrow 1/16$

$P(Y=-2) = P(1H \,\&\, 3T) \rightarrow 4/16$

$P(Y=0) \Rightarrow P(2H \,\&\, 2T) \rightarrow 6/16$

$P(Y=2) \Rightarrow P(3H \,\&\, 1T) \rightarrow 4/16$

$P(Y=4) \Rightarrow P(4H \,\&\, 0T) \rightarrow 1/16$

X \ Y	-4	-2	0	2	4	P(X)
0	1/16	0	0	0	0	1/16
1	0	4/16	0	0	0	4/16
2	0	0	6/16	0	0	6/16
3	0	0	0	4/16	0	4/16
4	0	0	0	0	1/16	1/16.
P(Y)	1/16	4/16	6/16	4/16	1/16	1

(i) $P(-2 \le y \le 2) = P[y = -2, 0, 2)$

$\Rightarrow \frac{4}{16} + \frac{6}{16} + \frac{4}{16} \Rightarrow \frac{14}{16}$

(ii) $P(x \le 2, y \ge 2) = P[x = 0,1,2 \cap y = 2,4]$.

$\Rightarrow P[(0,2) \text{ or } (0,4) \text{ or } (1,2) \text{ or } (1,4) \text{ or } (2,2) \text{ or } (2,4)]$

$\Rightarrow 0+0+0+0+0+0 \Rightarrow 0$.

(iii) $P(x \ge 3, -2 \le y \le 4] \Rightarrow P[x = 3,4 \cap y = 0,2,4]$

$\Rightarrow P[(3,0), (3,2), (3,4), (4,0), (4,2), (4,4)]$

$\Rightarrow 0 + \frac{4}{16} + 0 + 0 + 0 + \frac{1}{16} \Rightarrow \frac{5}{16}$

$P(X = Y) \Rightarrow P[(0,0), (2,2), (4,4)] \Rightarrow 1/16$

Conditional distribution of x given that $y=0$.

$$P\left[\frac{X = x}{y = 0}\right] \Rightarrow \frac{P[X = x \cap y = 0]}{P[y = 0]} \quad \text{—①} \quad \left\{ P(A|B) = P(A \cap B)/P(B) \right\}$$

Here $P(y=0) \rightarrow 6/16$

$P(x = 0 | y = 0) \Rightarrow \frac{P(0,0)}{P(y=0)} \Rightarrow \frac{0}{6/16}$

$P(x = 1 | y = 0) \Rightarrow \frac{P(1,0)}{P(y=0)} \Rightarrow \frac{0}{6/16}$

$P(x = 2 | y = 0) \Rightarrow \frac{P(2,0)}{P(y=0)} \Rightarrow \frac{6/16}{6/16}$

$P(x = 3 | y = 0) \Rightarrow \frac{P(3,0)}{P(y=0)} \Rightarrow \frac{0}{6/16}$

$P(x = 4 | y = 0) \Rightarrow \frac{P(4,0)}{P(y=0)} \Rightarrow \frac{0}{6/16}$.

X = x	0	1	2	3	4
P(X = x | y = 0)	0	0	1	0	0

Conditional distribution of y for given x = 1

$P(Y=0|X=1) \Rightarrow \frac{P(1,0)}{P(X=1)} \Rightarrow \frac{0}{4/8} \Rightarrow 0$

$P(Y=1|X=1) \Rightarrow \frac{P(1,1)}{P(X=1)} \Rightarrow \frac{1/8}{4/8} \Rightarrow 1/4$

$P(Y=2|X=1) \Rightarrow \frac{P(1,2)}{P(X=1)} \Rightarrow \frac{2/8}{4/8} \Rightarrow 2/4$

$P(Y=3|X=1) \Rightarrow \frac{P(1,3)}{P(X=1)} \Rightarrow \frac{1/8}{4/8} \Rightarrow 1/4$

$Y=y$	0	1	2	3
$P\left[\frac{Y=y}{X=1}\right]$	0	1/4	2/4	1/4

ec. XI. Joint P.M.F & P.D.F

Date: Page:

If x and y are continuous random variable then their joint PDF is denoted by $f(x, y)$ and we have

① $f(x,y) \geq 0$; ② $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

③ $P(a \leq x \leq b \cap c \leq y \leq d) = \int_{c}^{d} \int_{a}^{b} f(x,y)\, dx\, dy$

④ if 'x' & 'y' are independent random variables =
$f(x,y) = f(x) \cdot f(y)$

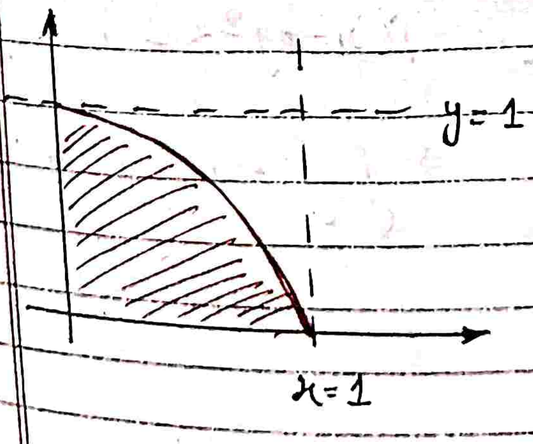
⑤ marginal density function ~~for~~ is $f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

⑥ marginal density function of $f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

Q). Two point are randomly selected at random in the interval $[0,1]$. Find the prob. that sum of their squares is less than one.

$0 \leq x \leq 1$, $0 \leq y \leq 1$

selected point 1   selected point 2



Total Area $\Rightarrow (1)(1) \Rightarrow 1$

favourable case : $x^2 + y^2 < 1$

fav° area $\Rightarrow \frac{1}{4}\pi(1)^2 \Rightarrow \frac{\pi}{4}$

req° probability $\Rightarrow \frac{fA}{TA} \Rightarrow \frac{\pi/4}{1} \Rightarrow \frac{\pi}{4}$

"or"

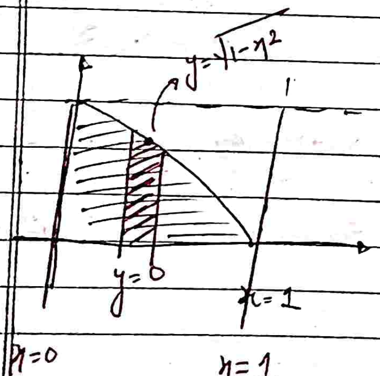
$x \in [0,1]$ & x is a uniform random variable then pdf, $f(x) = \begin{cases} \frac{1}{1-0} ; & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$

$y \in [0,1]$ & y is a uniform random variable then pdf, ~~pdf~~ $f(y) = \begin{cases} \frac{1}{1-0} ; & 0 \le y \le 1 \\ 0 ; & \text{otherwise} \end{cases}$

$\because$ x & y are also independent so joint pdf -- →

$f(x) \cdot f(y) = f(x,y)$ (property ①)

$f(x,y) = \begin{cases} 1 ; & 0 \le x \le 1, 0 \le y \le 1 \\ 0 ; & \text{otherwise} \end{cases}$



$P(x^2+y^2 \le 1)$

$\Rightarrow \int\int f(x,y)\,dy\,dx$

$P[x^2+y^2 \le 1] = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

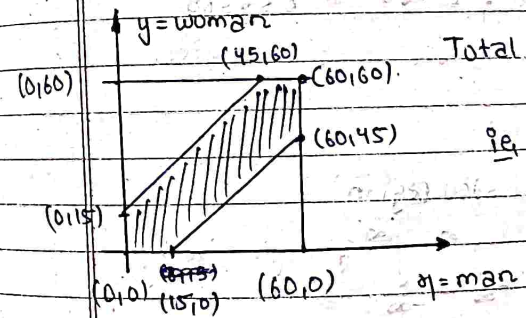
$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} (1)\,dy\,dx \Rightarrow \int_{x=0}^{1} (\sqrt{1-x^2})\,dx$

$\Rightarrow \frac{\pi}{4}$

Q61). A couple wants to meet in park between 1 PM and 2PM with the understanding that each will wait no longer than 15 minutes for other. what is the probability that they will meet?

x = time of arrival of man

y = —"— woman



Total Area = (60)(60)

ie, $x=10, y=2 \to$ ✓ meet

$x=20, y=2 \to$ ✗ never meet

Unfavourable area = 2(ar. of Δ) $\Rightarrow 2\left[\frac{1}{2} \times 45 \times 45\right]$

$b = 60-15$, $h = 45-0$

$\Rightarrow$ P(favourable area) = 1 - P(unfavo area).

$$1 - \left(\frac{45 \times 45}{60 \times 60}\right) \Rightarrow \frac{7}{16}$$

"Or"

$x \in (0,60)$ & x is uniform r.v

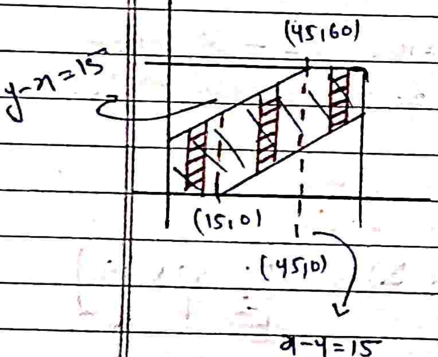
$$f(x) = \begin{cases} \frac{1}{60} & ; 0 < x < 60 \\ 0 & \text{otherwise} \end{cases}$$

$y \in (0,60)$ & y is uniform r.v

$$f(y) = \begin{cases} \frac{1}{60} & ; 0 \le y \le 60 \\ 0 & ; \text{otherwise.} \end{cases}$$

$\therefore$ x & y are independent so joint pdf is—

$$f(x,y) = f(x) \cdot f(y) \Rightarrow \begin{cases} \frac{1}{60 \times 60} & ; 0 \le x \le 60,\ 0 \le y \le 60 \\ 0 & ; \text{otherwise.} \end{cases}$$



$$P(\text{shaded region}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$\int_{x=0}^{15} \int_{y=0}^{x+15} \left(\frac{1}{3600}\right) dy\, dx + \int_{x=15}^{45} \int_{y=x-15}^{x+15} \left(\frac{1}{3600}\right) dy\, dx$$

$$+ \int_{x=45}^{60} \int_{y=x-15}^{60} \left(\frac{1}{3600}\right) dy\, dx$$

[NAT] **Ques. 62**

The joint pdf of 2D random variable (x,y) is given as—

$$f(x,y) = \begin{cases} 2, & 0 < x < 1,\ 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

(i) verify that it's valid pdf

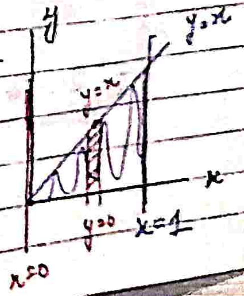
$$\int_{x=0}^{1} \int_{y=0}^{x} (2)\, dy\, dx \Rightarrow \int_{x=0}^{1} x\, dx \Rightarrow 2\left(\frac{x^2}{2}\right)\Big|_0^1$$

$$\Rightarrow 1$$

Hence, verified.

(ii) find marginal density function of x & y.

Marginal density func of x is—

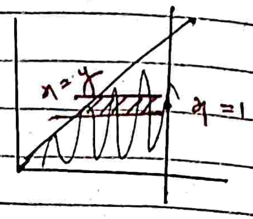


$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$f(x) = 2x, \; 0<x<1$

Marginal density funcⁿ of y is –

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$



$f(y) = \int_{x=y}^{1} 2\,dx$

$\Rightarrow 2(x)\Big|_{x=y}^{1} \Rightarrow 2(1-y); \; 0<y<1$

Note:

(7)

Conditional density funcⁿ of 'x' for given 'y' is $f(x|y) = \dfrac{f(x,y)}{f(y)}$

Conditional density funcⁿ of 'y' for given 'x' is $f(y|x) = \dfrac{f(x,y)}{f(x)}$

(iii) Find ~~given~~ conditional density funcⁿ of y for given X = x

$f(y|x) = \dfrac{f(x,y)}{f(x)} \Rightarrow \dfrac{2}{2x} \Rightarrow \dfrac{1}{x}, \; 0<x<1$

(iv) Find the conditional density funcⁿ of X for given Y = y

$f(x|y) = \dfrac{f(x,y)}{f(y)} \Rightarrow \dfrac{2}{2(1-y)} \Rightarrow \dfrac{1}{1-y}, \; 0<y<1$

(Q63) Joint pdf of x and y is given as →

$f(x,y) = 4xy\,e^{-(x^2+y^2)}, \; x\ge 0, \; y\ge 0$

(i) Verify that it's valid pdf

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\int_0^{\infty}\int_0^{\infty} 4xy\,e^{-(x^2+y^2)}\,dx\,dy$

$\Rightarrow \int_0^{\infty} 2x\,e^{-x^2}\,dx \times \int_0^{\infty} 2y\,e^{-y^2}\,dy$

$\Rightarrow 2x\int_0^{\infty} e^{-x^2}\,dx \times 2y\int_0^{\infty} e^{-y^2}\,dy$

$\Rightarrow \int_0^{\infty} e^{-t}\,dt \times \int_0^{\infty} e^{-t}\,dt$ $\left\{ \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ x = \sqrt{t} \\ x=0 \to t=0 \\ x=\infty \to t=\infty \end{array} \right.$

$\Rightarrow -e^{-t}\Big|_0^{\infty} \times -e^{-t}\Big|_0^{\infty}$

$\Rightarrow (-e^{-\infty})-(-e^{-0}) \times (-e^{-\infty})-(-e^{-0})$

$\Rightarrow -1 \times -1 \Rightarrow 1$