Lec. VIII. Expected Value.!

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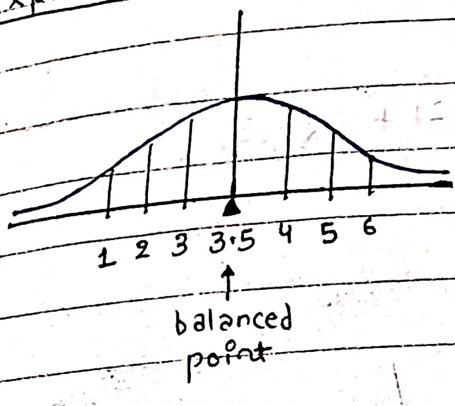
Expected Value or Average or Weighted Mean or COM →



Die roll → 1, 2, 3, 4, 5, 6

mean → 3.5

Large nos of trials

$\boxed{\text{median} = \text{mid-value}}$

- n-even → $\frac{n}{2}$, $\frac{(n+1)}{2}$
- n-odd → $\frac{(n+1)}{2}$ term

mode → most frequent values used

Central Measure of Tendency

$$\boxed{\text{Expected Value} = \sum_{i=1}^{n} P[X = x_i] \cdot x_i \qquad \mu = E[X].}$$

ie A coin is tossed thrice

$X = \{\text{nos of heads}\} = \{0, 1, 2, 3\}$

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

$$E[x] = \sum_{i=1}^{n} P[x = x_i]\cdot x_i$$

$\rightarrow \frac{1}{8}(0) + \frac{3}{8}(1) + \frac{3}{8}(2) + \frac{1}{8}(3)$

$\rightarrow 3/2.$

properties :

o If x & y are two independent random variables-

* $\boxed{E(x) + E(y) = E(x+y)}$

ie x: tossing a coin (H, T) y: tossing a die

x	H(1)	T(0)
$P(x=x_i)$	1/2	1/2

x	1	2	3	4	5	6
$P(x=x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

$E(x) = 1\times\frac{1}{2} + 1\times\frac{1}{2}$ $E(y) = 3.5$

$\rightarrow 0.5$

$E(x) + E(y) \Rightarrow 0.5 + 3.5 \rightarrow 4.0$

$x \rightarrow 0(H)$, $\rightarrow 1(T)$

y: 1, 2, 3, 4, 5, 6

$(x+y)$: $(x) \rightarrow 0 \rightarrow 1, 2, 3, 4, 5, 6$

$x+y$	
1	0n1
2	0n2
3	0n3
4	0n4
5	0n5
6	0n6

$P(0n1) \rightarrow P(0)\cdot P(1)$

$\hookrightarrow \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}$

$P(0n2) \rightarrow P(0)\cdot P(2)$

$\hookrightarrow \frac{1}{2}\times\frac{1}{6} \rightarrow \frac{1}{12}$

$(x) \rightarrow 1 \rightarrow 1, 2, 3, 4, 5, 6$

2	0n2	1n1
3	0n3	1n2
4	0n4	1n3
5	0n5	1n4
6	0n6	1n5
7	0n7	1n6

$(x+y)$	1	2	3	4	5	6	7
$P(x+y) = (x_i+y_i)$	1/12	2/12	2/12	2/12	2/12	2/12	2/12

$E(x+y) = \frac{48}{12} \rightarrow 4$

* $\boxed{E[Cx] = C\cdot E[x]}$

* $\boxed{E[ax+b] = a\cdot E[x] + b}$

* $E\left[\frac{1}{x}\right] \neq \frac{1}{E[x]}$ * $E\left[\frac{1}{x^2}\right] \neq \frac{1}{E[x^2]}.$

* $E[x-y] = E[x] - E[y]$

* $E[ax+by] = aE[x] + bE[y]$-

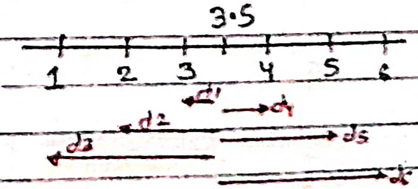
* $\boxed{\begin{array}{l}\text{Variance (deviation)} \rightarrow \\ E[x^2] - [E[x]]^2\end{array}}$

X	x_0	x_1	x_2	...	x_n
$P(X=x_i)$	P_0	P_1	P_2	...	P_n

1 $(1-3.5)^2$
2 $(2-3.5)^2$
3 $\Rightarrow \boxed{3.5}$ $(3-3.5)^2$
4 $(4-3.5)^2$
5 $(5-3.5)^2$
6 $(6-3.5)^2$

$E[x^2] =$

$x_0^2 P_0 + x_1^2 P_1 + \dots + x_n^2 P_n$



$E[x^2] =$

★ $\sum_{i=0}^{n} x_i^2 P_i$

$\sum f_i x_i^2 - [\sum f_i x_i]^2$

$\hookrightarrow f_i/N$

* variance = $var(x) = \sigma_x^2$ = E[...

* $\boxed{\text{standard deviation} = \sqrt{\text{variance}}}$ σ_x

var > 0

var cann't be negative

continuous random variable.

- $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$ $\quad -\infty \le x \le \infty$ $\downarrow$ continuous random variable

- $E[X] = \int_a^b x \cdot f(x)\, dx$

- $var[X] = E[X^2] - [E[X]]^2$

$\int_a^b x^2 f(x)\, dx$

- $E[x^n] = \int_a^b x^n f(x)\, dx$

- $var(x) = \int_a^b x^2 f(x)\, dx - \left[\int_a^b x \cdot f(x)\, dx\right]^2$

Examples,

Q) 41) A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3, 1/6, respectively. Then mean value & the variance of the no. of defective pieces produced by

$X = \{0, 1, 2\}$ → Discrete random variable

X	0	1	2
$P(X=x_i)$	1/6	2/3	1/6

$E[X] = (0)\left(\frac{1}{6}\right) + (1)\left(\frac{2}{3}\right) + (2)\left(\frac{1}{6}\right) \Rightarrow \frac{2}{3} + \frac{1}{3} \Rightarrow 1$

$var(x) = E[X^2] - [E(X)]^2$

$E[X^2] \to (0)^2\left(\frac{1}{6}\right) + (1)^2\left(\frac{2}{3}\right) + (2)^2\left(\frac{1}{6}\right) \to \frac{2}{3}$

$V[X] \to \frac{2}{3} - (1)^2 \to \frac{1}{3}$

Q) 42) Consider the following probability mass function (p.m.f) of a random variable X.

$$p(x, q) = \begin{cases} q & \text{if } x = 0 \\ 1-q & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

if $q = 0.4$, then variance of X is ——

X	0	1
$P(X=x_i)$	q	$1-q$

$Var(X) = E[X^2] - (E[X])^2$

$$(0)^2(q) + (1)^2(1-q) - \left[(0)(q) + (1)(1-q)\right]^2$$

$$(1-q) - [(1-q)]^2$$

$$(1-0.4) - [(1-0.4)]^2$$

$$0.6 - 0.36 \Rightarrow 0.24$$

Q. 43 Each of the nine words in the sentence "The quick brown fox jumps over the lazy dog." is written on a seprate piece of paper. These 9 pieces of paper are kept in a Box.

One of the piece at random is drawn from the Box. The expected length of the word drawn is —

The $\to$ 3	quick $\to$ 5	brown $\to$ 5
fox $\to$ 3	jumps $\to$ 5	over $\to$ 4
the $\to$ 3	lazy $\to$ 4	dog. $\to$ 3

expected length $\to$ 3 letter word

$\to$ 4 letter word

$\to$ 5 letter word

X	3	4	5
$P(X=x_i)$	4/9	2/9	3/9

$$E[X] = 3 \times \frac{4}{9} + 4 \times \frac{2}{9} + 5 \times \frac{3}{9} \Rightarrow 3.89$$

Q. 44 A random variable X has probability density function $f(x)$ as given Below :

$$f(x) = \begin{cases} a + bx & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If expected value $E[X] = 2/3$ then $P[X < 0.5]$ is —

If $f(x)$ is valid pdf,

$$\int_0^1 f(x)\,dx = 1 \to \int_0^1 (a+bx)\,dx = 1 \quad \text{— (1)}$$

$$a + \frac{b}{2} = 1$$

If X is a continuous random variable

$$\int_0^1 x \cdot f(x)\,dx = 2/3$$

$$\int_0^1 x \cdot (a+bx)\,dx = 2/3 \quad \text{— (2)} \to \frac{a}{2} + \frac{b}{3} = 1$$

to find $\int_{0}^{0.5} (a+bx)\,dx$

Q. (45) Two cards are drawn successively with replacement for a well shuffled pack of 52 cards. Find the expected value for the number of Aces?

4 Aces (52 cards) — Two cards — Aces — ~~Expected value~~

→ No Ace : 0 Ace

→ (Ace ∧ other) + (other ∧ Ace) : 1 Ace

→ 2 Aces : 2 Ace.

$X = \{0, 1, 2\}$ discrete random variable

X	0	1	2
$P(X=x_i)$	144/169	24/169	16/169.

4 Aces, 52 cards → $\frac{48}{52} \times \frac{48}{52} \Rightarrow \frac{144}{169}$

$P[X=0] = P[X = \text{No Aces}]$

$P[X=1] \rightarrow P[\text{one Ace \& one another card}]$.

$P(A \wedge O) + P(O \wedge A)$

$\frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} \rightarrow \frac{24}{169}$

$P[X=2] \rightarrow P[\text{both Aces}]$.

$\rightarrow \frac{4}{52} \times \frac{4}{52} \rightarrow \frac{16}{\ }$

Q. (46) The distribution of continuous random variable x is defined by —

$$f(x) = \begin{cases} x^3 & 0 < x \leq 1 \\ (2-x)^3 & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

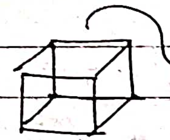
obtain expected value of x.

$E(x) = \int_0^1 x(x^3)\,dx + \int_1^2 x(2-x)^3\,dx$

$(A-B)^3 \rightarrow A^3 - B^3 - 3AB(A-B)$

Q. (47) A fair die with faces $\{1,2,3,4,5,6\}$ is thrown repeatedly till '3' is observed for the first time. Let, X denote the nos of time the dice is thrown. The expected value of X is —

X = nos of die thrown



→ 1,2,3,4,5,6

X = 1 → 3 (1/6)

X = 2 → $\bar{3}$ (Not) 3 (Yes) $\left(\frac{5}{6} \times \frac{1}{6}\right)$

X = 3 → $\bar{3}$ $\bar{3}$ 3 (Not) (Not) (Yes) $\left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$

X = 4 → $\bar{3}$ $\bar{3}$ $\bar{3}$ 3 $\left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$

X is a Discrete random variable.

X	1	2	3	4	5
$P(X=x_i)$	$1/6$	$5/6 \times 1/6$	$(5/6)^2 \times 1/6$	$(5/6)^3 \times 1/6$	$(5/6)^4 \times 1/6$

↑ infinite countable

$$E[X] = \underset{A}{\boxed{(1)}}\underset{GP}{\left(\frac{1}{6}\right)} + \underset{A}{\boxed{(2)}}\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \underset{A}{\boxed{3}}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \dots$$

whole series multiply by common ratio,

$$\frac{T2}{T1} \Rightarrow \frac{(5/6)^2}{(5/6)} \rightarrow 5/6.$$

$$\cancel{\frac{5}{6}E(x) = (5)\frac{1}{6} + (2)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)}$$

$$\frac{5}{6}E(x) = \quad + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + 3\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \dots$$

$$E(x) = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 3\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + 4\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \dots$$

(-)

$$\frac{1}{6}E(x) \rightarrow (1)\left(\frac{1}{6}\right) + (1)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \dots$$

$$\frac{1}{6}E(x) = \frac{1}{6} + \frac{1}{6}\underbrace{\left[\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \dots\right]}_{\text{infinite terms of G.P.}}$$

$$\frac{1}{6}E[x] = \frac{1}{6} + \frac{1}{6}\left[\frac{5/6}{1-(5/6)}\right] \quad \left\{ \begin{array}{l} a + ar + ar^2 + ar^3 + \dots \\ S_\infty = \dfrac{a}{(1-r)} \end{array} \right.$$

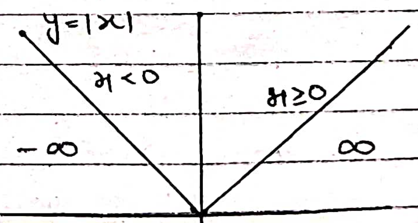
$$\Rightarrow \frac{1}{6}E(x) = \frac{1}{6} + \frac{1}{6}\left[\frac{5/6}{1/6}\right]$$

$$E[X] = 6$$

Q. (48) $P_x(x) = Me^{(-2|x|)} + Ne^{(-3|x|)}$ is the p.d.f for real random variable x over the entire x-axis, M & N are both positive real nos, eqn relating M & N = ?

$$f(x) = P_x(x) = Me^{(-2|x|)} + Ne^{(-3|x|)}$$

if func is valid p.d.f $\int_{-\infty}^{\infty} f(x)\,dx = 1$



$$|x| \begin{cases} x & ; x>0 \\ -x & ; x<0 \end{cases}$$

x = 0 (Not Differentiable)
↓
∞ tangent can be drawn