MATHEMATICS

STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is $\quad y - y_1 = m(x - x_1)$

Given two points: slope, $\quad m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $\quad m_1 = -1/m_2$

The distance between two points is

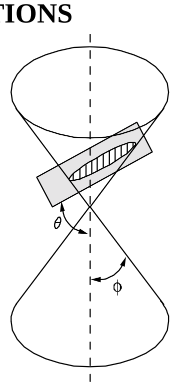
$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$Roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

CONIC SECTIONS



e = eccentricity = $\cos\theta/(\cos\phi)$

[Note: X' and Y', in the following cases, are translated axes.]

Case 1. Parabola $\quad e = 1$:

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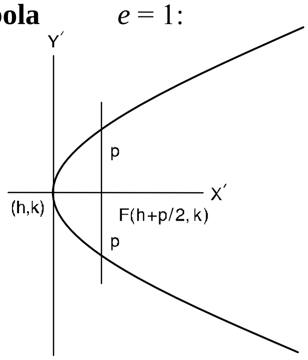

$$(y - k)^2 = 2p(x - h);\ \text{Center at } (h, k)$$

is the standard form of the equation. When $h = k = 0$,

Focus: $(p/2, 0)$; $\quad$ Directrix: $x = -p/2$

Case 2. Ellipse $\quad e < 1$:

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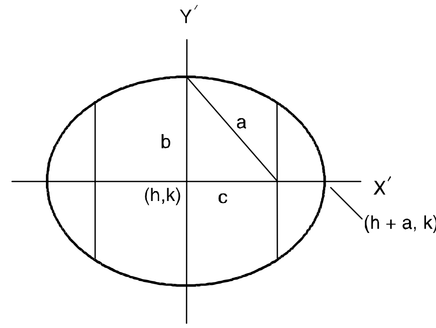

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \quad \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $\quad e = \sqrt{1 - \left(b^2/a^2\right)} = c/a$

$b = a\sqrt{1 - e^2};$

Focus: $(\pm ae, 0)$; $\quad$ Directrix: $x = \pm a/e$

Case 3. Hyperbola $\quad e > 1$:

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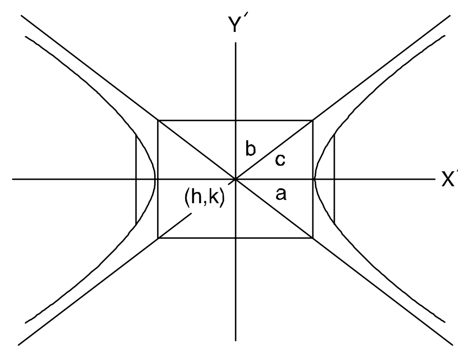


$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; \quad \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $\quad e = \sqrt{1 + \left(b^2/a^2\right)} = c/a$

$b = a\sqrt{e^2 - 1};$

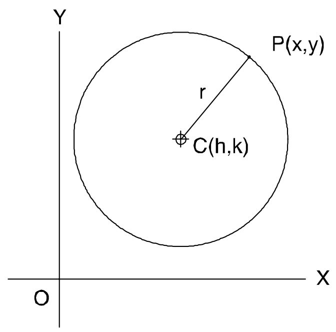
Focus: $(\pm ae, 0)$; $\quad$ Directrix: $x = \pm a/e$

Case 4. Circle $e = 0$:

$$(x - h)^2 + (y - k)^2 = r^2; \quad \text{Center at } (h, k)$$

is the general form of the equation with radius

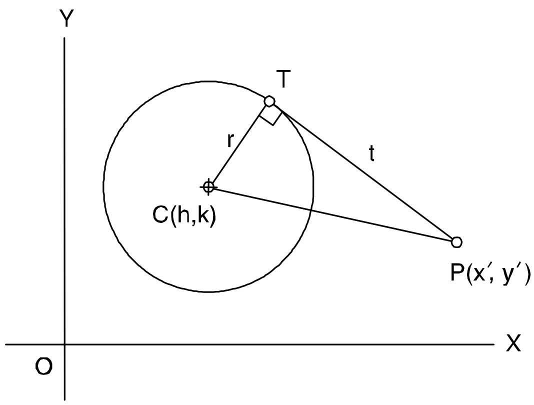
$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

• 

Length of the tangent from a point. Using the general form of the equation of a circle, the length of the tangent is found from

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$

by substituting the coordinates of a point $P(x',y')$ and the coordinates of the center of the circle into the equation and computing.

• 

Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

where not both A and C are zero.

If $B^2 - AC < 0$, an *ellipse* is defined.

If $B^2 - AC > 0$, a *hyperbola* is defined.

If $B^2 - AC = 0$, the conic is a *parabola*.

If $A = C$ and $B = 0$, a *circle* is defined.

If $A = B = C = 0$, a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a; \; k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a *point* at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

QUADRIC SURFACE (SPHERE)

The general form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

LOGARITHMS

The logarithm of x to the Base b is defined by

$$\log_b(x) = c, \text{ where } \quad b^c = x$$

Special definitions for $b = e$ or $b = 10$ are:

$\ln x$, Base $= e$

$\log x$, Base $= 10$

To change from one Base to another:

$$\log_b x = (\log_a x)/(\log_a b)$$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585\,(\log_{10} x)$

Identities

$\log_b b^n = n$

$\log x^c = c \log x$; $x^c = \text{antilog}\,(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$; $\log 1 = 0$

$\log x/y = \log x - \log y$

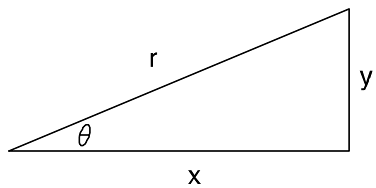
TRIGONOMETRY

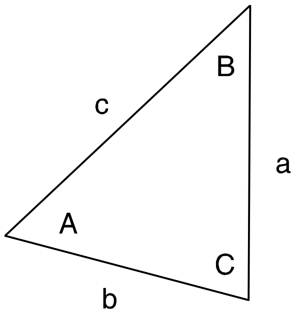
Trigonometric functions are defined using a right triangle.

$\sin \theta = y/r$, $\cos \theta = x/r$

$\tan \theta = y/x$, $\cot \theta = x/y$

$\csc \theta = r/y$, $\sec \theta = r/x$





Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Identities

$\csc \theta = 1/\sin \theta$

$\sec \theta = 1/\cos \theta$

$\tan \theta = \sin \theta/\cos \theta$

$\cot \theta = 1/\tan \theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2 \sin^2\alpha = 2 \cos^2\alpha - 1$

$\tan 2\alpha = (2 \tan \alpha)/(1 - \tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2 \cot \alpha)$

$\tan (\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$

$\cot (\alpha + \beta) = (\cot \alpha \cot \beta - 1)/(\cot \alpha + \cot \beta)$

$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\tan (\alpha - \beta) = (\tan \alpha - \tan \beta)/(1 + \tan \alpha \tan \beta)$

$\cot (\alpha - \beta) = (\cot \alpha \cot \beta + 1)/(\cot \beta - \cot \alpha)$

$\sin (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/2}$

$\cos (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/2}$

$\tan (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/(1 + \cos \alpha)}$

$\cot (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/(1 - \cos \alpha)}$

$\sin \alpha \sin \beta = (1/2)[\cos (\alpha - \beta) - \cos (\alpha + \beta)]$

$\cos \alpha \cos \beta = (1/2)[\cos (\alpha - \beta) + \cos (\alpha + \beta)]$

$\sin \alpha \cos \beta = (1/2)[\sin (\alpha + \beta) + \sin (\alpha - \beta)]$

$\sin \alpha + \sin \beta = 2 \sin (1/2)(\alpha + \beta) \cos (1/2)(\alpha - \beta)$

$\sin \alpha - \sin \beta = 2 \cos (1/2)(\alpha + \beta) \sin (1/2)(\alpha - \beta)$

$\cos \alpha + \cos \beta = 2 \cos (1/2)(\alpha + \beta) \cos (1/2)(\alpha - \beta)$

$\cos \alpha - \cos \beta = -2 \sin (1/2)(\alpha + \beta) \sin (1/2)(\alpha - \beta)$

COMPLEX NUMBERS

Definition $i = \sqrt{-1}$

$(a + ib) + (c + id) = (a + c) + i (b + d)$

$(a + ib) - (c + id) = (a - c) + i (b - d)$

$(a + ib)(c + id) = (ac - bd) + i (ad + bc)$

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

$(a + ib) + (a - ib) = 2a$

$(a + ib) - (a - ib) = 2ib$

$(a + ib)(a - ib) = a^2 + b^2$

Polar Coordinates

$x = r \cos \theta$; $y = r \sin \theta$; $\theta = \arctan (y/x)$

$r = |x + iy| = \sqrt{x^2 + y^2}$

$x + iy = r (\cos \theta + i \sin \theta) = re^{i\theta}$

$$[r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)] = r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

$$(x + iy)^n = [r (\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Euler's Identity

$e^{i\theta} = \cos \theta + i \sin \theta$

$e^{-i\theta} = \cos \theta - i \sin \theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Roots

If k is any positive integer, any complex number (other than zero) has k distinct roots. The k roots of $r (\cos \theta + i \sin \theta)$ can be found by substituting successively $n = 0, 1, 2, \ldots, (k - 1)$ in the formula

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360^\circ}{k}\right) + i \sin\left(\frac{\theta}{k} + n\frac{360^\circ}{k}\right)\right]$$

MATRICES

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication

If $\boldsymbol{A} = (a_{ik})$ is an $m \times n$ matrix and $\boldsymbol{B} = (b_{kj})$ is an $n \times s$ matrix, the matrix product AB is an $m \times s$ matrix

$$\boldsymbol{C} = \left(c_{i\,j}\right) = \left(\sum_{l=1}^{n} a_{il} b_{lj}\right)$$

where n is the common integer representing the number of columns of $\boldsymbol{A}$ and the number of rows of $\boldsymbol{B}$ (l and $k = 1, 2, \ldots, n$).

Addition

If $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ are two matrices of the same size $m \times n$, the sum $\boldsymbol{A} + \boldsymbol{B}$ is the $m \times n$ matrix $\boldsymbol{C} = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$.

Identity

The matrix $\mathbf{I} = (a_{ij})$ is a square $n \times n$ identity matrix where $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$.

Transpose

The matrix $\boldsymbol{B}$ is the transpose of the matrix $\boldsymbol{A}$ if each entry b_{ji} in $\boldsymbol{B}$ is the same as the entry a_{ij} in $\boldsymbol{A}$ and conversely. In equation form, the transpose is $\boldsymbol{B} = \boldsymbol{A}^T$.

Inverse

The inverse $\boldsymbol{B}$ of a square $n \times n$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}, \text{ where}$$

adj($\boldsymbol{A}$) = adjoint of $\boldsymbol{A}$ (obtained by replacing $\boldsymbol{A}^T$ elements with their cofactors, see **DETERMINANTS**) and

$|\boldsymbol{A}|$ = determinant of $\boldsymbol{A}$.

DETERMINANTS

A *determinant of order n* consists of n^2 numbers, called the *elements* of the determinant, arranged in n rows and n columns and enclosed by two vertical lines. In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the hth column and the kth row. The *cofactor* of this element is the value of the minor of the element (if $h + k$ is *even*), and it is the negative of the value of the minor of the element (if $h + k$ is *odd*).

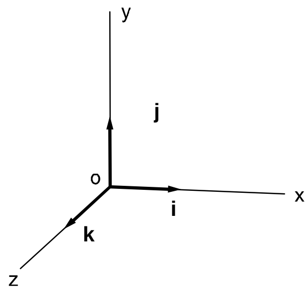
If n is greater than 1, the *value* of a determinant of order n is the sum of the n products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

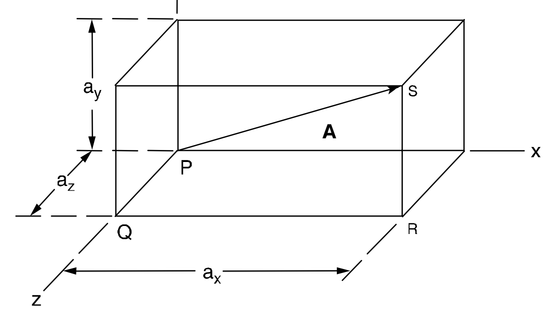
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

VECTORS





$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Addition and *subtraction:*

$$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$$

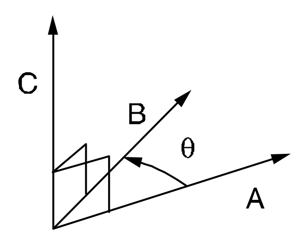
$$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$$

The *dot product* is a *scalar product* and represents the projection of $\mathbf{B}$ onto $\mathbf{A}$ times $|\mathbf{A}|$. It is given by

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= a_x b_x + a_y b_y + a_z b_z \\ &= |\mathbf{A}|\,|\mathbf{B}| \cos\theta = \mathbf{B}\cdot\mathbf{A} \end{aligned}$$

The *cross product* is a *vector product* of magnitude $|\mathbf{B}|\,|\mathbf{A}|$ sin θ which is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. The product is

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B} \times \mathbf{A}$$



The sense of $\mathbf{A} \times \mathbf{B}$ is determined by the right-hand rule.

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\mathbf{n} \sin\theta, \text{ where}$$

$\mathbf{n}$ = unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

Gradient, Divergence, and Curl

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi$$

$$\nabla \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times \left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

The Laplacian of a scalar function ϕ is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Identities

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$; $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$

If $\mathbf{A}\cdot\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$

If $\mathbf{A} \times \mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is parallel to $\mathbf{B}$.

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = (\nabla \cdot \nabla)\phi$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

PROGRESSIONS AND SERIES

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a.
2. The common difference is d.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = a + (n - 1)d$$

$$S = n(a + l)/2 = n\,[2a + (n - 1)\,d]/2$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression (G.P.), divide each number after the first by the preceding number. If the quotients are equal, the series is geometric.

1. The first term is a.
2. The common ratio is r.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = ar^{n-1}$$

$$S = a\,(1 - r^n)/(1 - r);\ r \neq 1$$

$$S = (a - rl)/(1 - r);\ r \neq 1$$

$$\lim_{n\to\infty} S_n = a/(1 - r);\quad r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| \geq 1$.

Properties of Series

$$\sum_{i=1}^{n} c = nc; \quad c = \text{constant}$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} (x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$

$$\sum_{x=1}^{n} x = (n + n^2)/2$$

1. A power series in x, or in $x - a$, which is convergent in the interval $-1 < x < 1$ (or $-1 < x - a < 1$), defines a function of x which is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term, and the resulting series has the same interval of convergence as the original series (except possibly at the end points of the interval).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series*.

PROBABILITY AND STATISTICS

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is

   $$P(n,r) = \frac{n!}{(n-r)!}$$

2. The number of different *combinations* of n distinct objects *taken r at a time* is

   w $$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{[r!(n-r)!]}$$

3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i,

   where $i = 1, 2, \ldots, k$ and $\Sigma n_i = n$, is

   $$P(n; n_1, n_2, \ldots n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Laws of Probability

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$P(A + B) = P(A) + P(B) - P(A, B)$, where

$P(A + B)$ = the probability that either A or B occur alone or that both occur together,

$P(A)$ = the probability that A occurs,

$P(B)$ = the probability that B occurs, and

$P(A, B)$ = the probability that both A and B occur simultaneously.

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$P(A, B) = P(A)P(B \mid A) = P(B)P(A \mid B)$, where

$P(B \mid A)$ = the probability that B occurs given the fact that A has occurred, and

$P(A \mid B)$ = the probability that A occurs given the fact that B has occurred.

If either $P(A)$ or $P(B)$ is zero, then $P(A, B) = 0$.

Probability Functions

A random variable x has a probability associated with each of its values. The probability is termed a discrete probability if x can assume only the discrete values

$x = X_1, X_2, \ldots, X_i, \ldots, X_N$

The *discrete probability* of the event $X = x_i$ occurring is defined as $P(X_i)$.

Probability Density Functions

If x is continuous, then the *probability density function* $f(x)$ is defined so that

$\int_{x_1}^{x_2} f(x)dx$ = the probability that x lies between x_1 and x_2.

The probability is determined by defining the equation for $f(x)$ and integrating between the values of x required.

Probability Distribution Functions

The *probability distribution function* $F(X_n)$ of the discrete probability function $P(X_i)$ is defined by

$$F(X_n) = \sum_{k=1}^{n} P(X_k) = P(X_i \le X_n)$$

When x is continuous, the *probability distribution function* $F(x)$ is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

which implies that $F(a)$ is the probability that $x \le a$.

The *expected value* $g(x)$ of any function is defined as

$$E\{g(x)\} = \int_{-\infty}^{x} g(t)f(t)dt$$

BINOMIAL DISTRIBUTION

$P(x)$ is the probability that x will occur in n trials. If p = probability of success and q = probability of failure = $1 - p$, then

$$P(x) = C(n,x)p^x q^{n-x} = \frac{n!}{x!(n-x)!}p^x q^{n-x}$$

where

x = 0, 1, 2, …, n,

$C(n, x)$ = the number of combinations, and

n, p = parameters.

NORMAL DISTRIBUTION (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of *n* observations tend to become normally distributed as *n* increases. The variate *x* is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)/2\sigma^2}, \quad \text{where}$$

μ = the population mean,

σ = the standard deviation of the population, and

$-\infty \le x \le \infty$

When $\mu = 0$ and $\sigma^2 = \sigma = 1$, the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{where } -\infty \le x \le \infty.$$

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

$F(x)$ = the area under the curve from $-\infty$ to x,

$R(x)$ = the area under the curve from x to ∞, and

$W(x)$ = the area under the curve between $-x$ and x.

DISPERSION, MEAN, MEDIAN, AND MODE VALUES

If X_1, X_2, …, X_n represent the values of *n* items or observations, the *arithmetic mean* of these items or observations, denoted $\bar{X}$, is defined as

$$\bar{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n)\sum_{i=1}^{n} X_i$$

$\bar{X} \to \mu$ for sufficiently large values of *n*. Therefore, for the purposes of this handbook, the following is accepted:

μ = population mean = $\bar{X}$

The *weighted arithmetic mean* is

$$\bar{X}_w = \frac{\sum w_i X_i}{\sum w_i}, \quad \text{where}$$

$\bar{X}_w$ = the weighted arithmetic mean,

X_i = the values of the observations to be averaged, and

w_i = the weight applied to the X_i value.

The *variance* of the observations is the *arithmetic mean* of the *squared deviations from the population mean*. In symbols, X_1, X_2, …, X_n represent the values of the *n sample* observations of a *population of size N*. If μ is the arithmetic mean of the population, the *population variance* is defined by

$$\sigma^2 = (1/N)[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \ldots + (X_N - \mu)^2]$$
$$= (1/N)\sum_{i=1}^{N} (X_i - \mu)^2$$

The *standard deviation* of a population is

$$\sigma = \sqrt{(1/N)\sum (X_i - \mu)^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

The *sample standard deviation* is

$$s = \sqrt{\left[\frac{1}{n-1}\right]\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}$$

The *coefficient of variation* = $CV = s/\bar{X}$

The *geometric mean* = $\sqrt[n]{X_1 X_2 X_3 \ldots X_n}$

The *root-mean-square value* = $\sqrt{(1/n)\sum X_i^2}$

The *median* is defined as the *value of the middle item* when the data are *rank-ordered* and the number of items is *odd*. The *median* is the *average of the middle two items* when the rank-ordered data consists of an *even* number of items.

The *mode* of a set of data is the *value that occurs with greatest frequency*.

t-DISTRIBUTION

The variate *t* is defined as the quotient of two independent variates *x* and *r* where *x is unit normal* and *r is the root mean square* of *n* other independent *unit normal variates*; that is, $t = x/r$. The following is the *t*-distribution with *n* degrees of freedom:

$$f(t) = \frac{\Gamma[(n+1)]/2}{\Gamma(n/2)\sqrt{n\pi}} \frac{1}{\left(1 + t^2/n\right)^{(n+1)/2}}$$

where $-\infty \le t \le \infty$.

A table at the end of this section gives the values of $t_{\alpha n}$ for values of α and *n*. Note that in view of the symmetry of the *t*-distribution,

$t_{1-\alpha,n} = -t_{\alpha,n}$. The function for α follows:

$$\alpha = \int_{t_{\alpha,n}}^{\infty} f(t)\,dt$$

A table showing probability and density functions is included on page 121 in the **INDUSTRIAL ENGINEERING SECTION** of this handbook.

GAMMA FUNCTION

$$\Gamma(n) = \int_o^\infty t^{n-1} e^{-t}\, dt, \quad n > 0$$

CONFIDENCE INTERVALS

Confidence Interval for the Mean μ of a Normal Distribution

(a) Standard deviation σ is known

$$\bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

(b) Standard deviation σ is not known

$$\bar{X} - t_{\alpha/2} \frac{s}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ corresponds to n – 1 degrees of freedom.

Confidence Interval for the Difference Between Two Means
μ_1 and μ_2

(a) Standard deviations σ_1 and σ_2 known

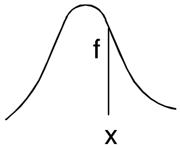
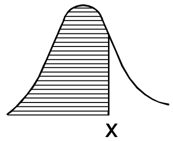
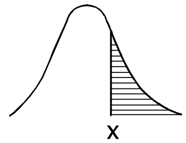
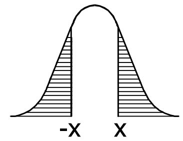
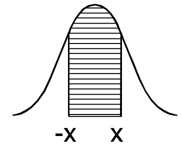
$$\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(b) Standard deviations σ_1 and σ_2 are not known

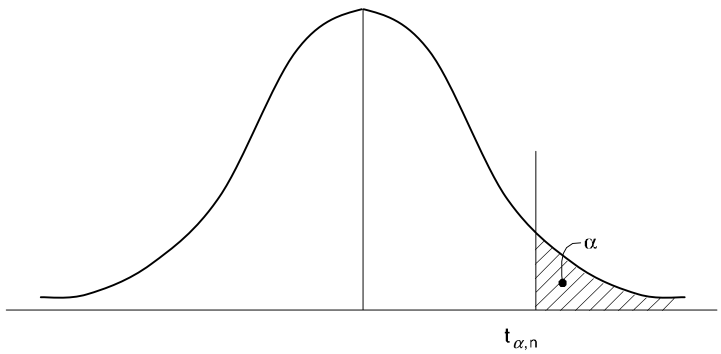
$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[(n-1)S_1^2 + (n_2 - 1)S_2^2\right]}{n_1 + n_2 - 2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 - t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[(n-1)S_1^2 + (n_2 - 1)S_2^2\right]}{n_1 + n_2 - 2}}$$

where $t_{\alpha/2}$ corresponds to $n_1 + n_2 - 2$ degrees of freedom.

UNIT NORMAL DISTRIBUTION TABLE

					
x	***f*(*x*)**	***F*(*x*)**	***R*(*x*)**	**2*R*(*x*)**	***W*(*x*)**
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

t-DISTRIBUTION TABLE

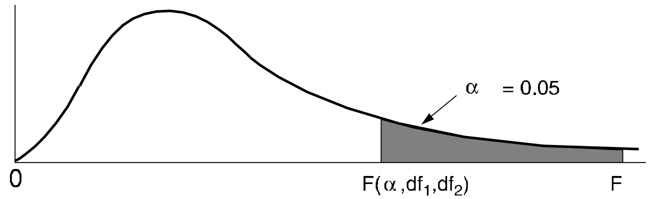


VALUES OF $t_{\alpha,n}$

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	n
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
inf.	1.282	1.645	1.960	2.326	2.576	inf.

CRITICAL VALUES OF THE *F* DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of *F* corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1																		
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	**∞**
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

DIFFERENTIAL CALCULUS

The Derivative

For any function $y = f(x)$,

the derivative $= D_x y = dy/dx = y'$

$$y' = \underset{\Delta x \to 0}{\text{limit}} \left[(\Delta y)/(\Delta x)\right]$$
$$= \underset{\Delta x \to 0}{\text{limit}} \left\{\left[f(x+\Delta x) - f(x)\right]/(\Delta x)\right\}$$

y' = the slope of the curve $f(x)$.

TEST FOR A MAXIMUM

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

TEST FOR A MINIMUM

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

TEST FOR A POINT OF INFLECTION

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If *y is kept fixed*, the function

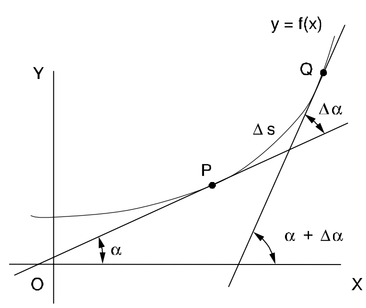
$z = f(x, y)$

becomes a function of the *single variable x*, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

The Curvature of Any Curve

♦



The curvature K of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \underset{\Delta s \to 0}{\text{limit}} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

CURVATURE IN RECTANGULAR COORDINATES

$$K = \frac{y''}{\left[1 + (y')^2\right]^{3/2}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = dx/dy$$
$$K = \frac{-x''}{\left[1 + (x')^2\right]^{3/2}}$$

THE RADIUS OF CURVATURE

The *radius of curvature R* at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \left|\frac{\left[1 + (y')^2\right]^{3/2}}{|y''|}\right| \qquad (y'' \neq 0)$$

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms 0/0 or ∞/∞ (where α is finite or infinite), then

$$\underset{x \to \alpha}{\text{limit}}\, f(x)/g(x)$$

is equal to the first of the expressions

$$\underset{x \to \alpha}{\text{limit}} \frac{f'(x)}{g'(x)}, \underset{x \to \alpha}{\text{limit}} \frac{f''(x)}{g''(x)}, \underset{x \to \alpha}{\text{limit}} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

INTEGRAL CALCULUS

The definite integral is defined as:

$$\underset{n \to \infty}{\text{limit}} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is available on page 15. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6),

B. Integration by Substitution, and

C. Separation of Rational Fractions into Partial Fractions.

♦ Wade, Thomas L., *Calculus*, Copyright © 1953 by Ginn & Company. Diagram reprinted by permission of Simon & Schuster Publishers.

DERIVATIVES AND INDEFINITE INTEGRALS

In these formulas, u, v, and w represent functions of x. Also, a, c, and n represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed: arcsin $u = \sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\, du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\, dv/dx + v\, du/dx$
6. $d(uvw)/dx = uv\, dw/dx + uw\, dv/dx + vw\, du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\, du/dx - u\, dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\, du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\, du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)a^u \dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\, du/dx$
15. $d(u^v)/dx = vu^{v-1}\, du/dx + (\ln u)\, u^v\, dv/dx$
16. $d(\sin u)/dx = \cos u\, du/dx$
17. $d(\cos u)/dx = -\sin u\, du/dx$
18. $d(\tan u)/dx = \sec^2 u\, du/dx$
19. $d(\cot u)/dx = -\csc^2 u\, du/dx$
20. $d(\sec u)/dx = \sec u \tan u\, du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\, du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad(-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad(0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad(-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad(0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$
    $(0 \le \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$
    $(0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

1. $\int d\, f(x) = f(x)$
2. $\int dx = x$
3. $\int a\, f(x)\, dx = a \int f(x)\, dx$
4. $\int [u(x) \pm v(x)]\, dx = \int u(x)\, dx \pm \int v(x)\, dx$
5. $\int x^m dx = \dfrac{x^{m+1}}{m+1}$ $\quad(m \ne -1)$
6. $\int u(x)\, dv(x) = u(x)\, v(x) - \int v(x)\, du(x)$
7. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b|$
8. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x\, dx = \dfrac{a^x}{\ln a}$
10. $\int \sin x\, dx = -\cos x$
11. $\int \cos x\, dx = \sin x$
12. $\int \sin^2 x dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
13. $\int \cos^2 x dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$
14. $\int x \sin x\, dx = \sin x - x\cos x$
15. $\int x \cos x\, dx = \cos x + x \sin x$
16. $\int \sin x \cos x\, dx = (\sin^2 x)/2$
17. $\int \sin ax \cos bx\, dx = -\dfrac{\cos(a-b)x}{2(a-b)} - \dfrac{\cos(a+b)x}{2(a+b)}$ $\quad(a^2 \ne b^2)$
18. $\int \tan x\, dx = -\ln|\cos x| = \ln|\sec x|$
19. $\int \cot x\, dx = -\ln|\csc x| = \ln|\sin x|$
20. $\int \tan^2 x\, dx = \tan x - x$
21. $\int \cot^2 x\, dx = -\cot x - x$
22. $\int e^{ax}\, dx = (1/a)\, e^{ax}$
23. $\int xe^{ax}\, dx = (e^{ax}/a^2)(ax - 1)$
24. $\int \ln x\, dx = x\,[\ln(x) - 1]$ $\quad(x > 0)$
25. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ $\quad(a \ne 0)$
26. $\int \dfrac{dx}{ax^2+c} = \dfrac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\dfrac{a}{c}}\right),$ $\quad(a > 0, c > 0)$

27a. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}}$ $\quad(4ac - b^2 > 0)$

27b. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|$ $\quad(b^2 - 4ac > 0)$

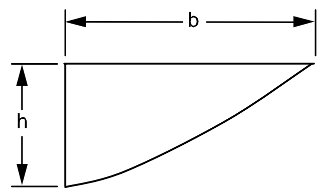
27c. $\int \dfrac{dx}{ax^2+bx+c} = -\dfrac{2}{2ax+b},$ $\quad(b^2 - 4ac = 0)$

MENSURATION OF AREAS AND VOLUMES

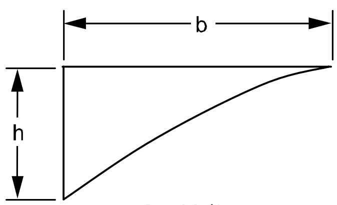
Nomenclature

A = total surface area
P = perimeter
V = volume

Parabola



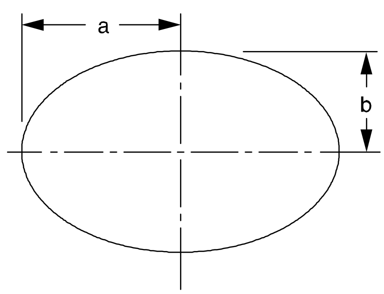
A = 2bh/3



A = bh/3

Ellipse

♦



A = πab

$$P_{approx} = 2\pi\sqrt{\left(a^2 + b^2\right)/2}$$

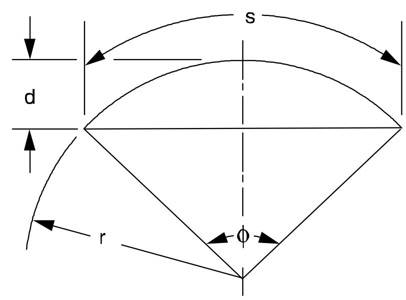
$$P = \pi(a+b)\begin{bmatrix} 1 + (1/2)^2\lambda^2 + (1/2 \times 1/4)^2\lambda^4 \\ + (1/2 \times 1/4 \times 3/6)^2\lambda^6 + (1/2 \times 1/4 \times 3/6 \times 5/8)^2\lambda^8 \\ + (1/2 \times 1/4 \times 3/6 \times 5/8 \times 7/10)^2\lambda^{10} + \ldots \end{bmatrix},$$

where

$\lambda = (a - b)/(a + b)$

Circular Segment

♦

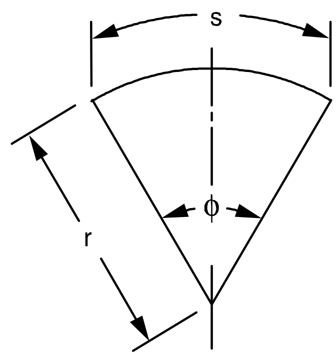


$A = [r^2 (\phi - \sin \phi)]/2$

$\phi = s/r = 2\{\arccos [(r - d)/r]\}$

Circular Sector

♦

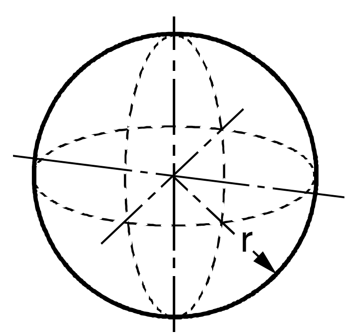


$A = \phi r^2/2 = sr/2$

$\phi = s/r$

Sphere

♦



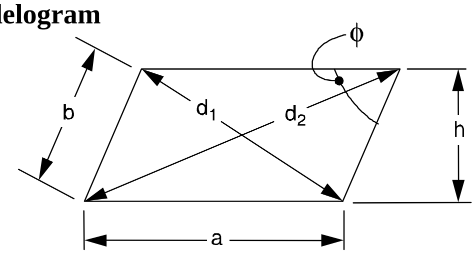
$V = 4\pi r^3/3 = \pi d^3/6$

$A = 4\pi r^2 = \pi d^2$

♦ Gieck, K. & Gieck R., *Engineering Formulas*, 6th Ed., Copyright © 1967 by Gieck Publishing. Diagrams reprinted by permission of Kurt Gieck.

MENSURATION OF AREAS AND VOLUMES

Parallelogram



$P = 2(a + b)$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$

$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$

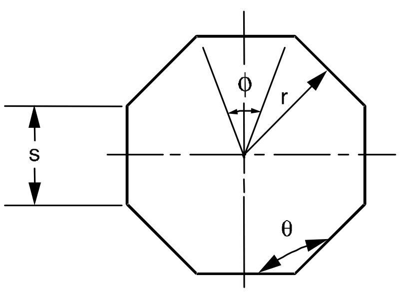
$$d_1^2 + d_2^2 = 2\left(a^2 + b^2\right)$$

$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (*n* equal sides)

♦



$\phi = 2\pi/n$

$$\theta = \left[\frac{\pi(n-2)}{n}\right] = \pi - 2$$

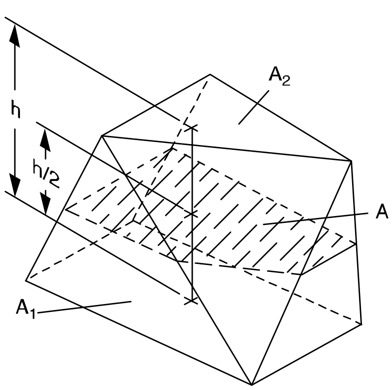
$P = ns$

$s = 2r\ [\tan(\phi/2)]$

$A = (nsr)/2$

Prismoid

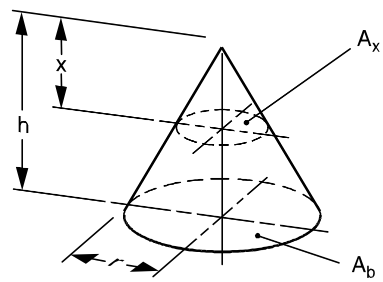
♦



$V = (h/6)(A_1 + A_2 + 4A)$

Right Circular Cone

♦



$V = (\pi r^2 h)/3$

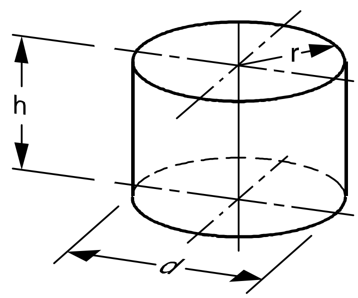
A = side area + base area

$$= \pi r\left(r + \sqrt{r^2 + h^2}\right)$$

$A_x : A_b = x^2 : h^2$

Right Circular Cylinder

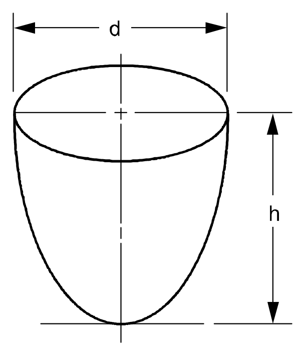
♦



$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$

$$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$$

Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

♦ Gieck, K. & R. Gieck, *Engineering Formulas*, 6th Ed., Copyright 8 1967 by Gieck Publishing. Diagrams reprinted by permission of Kurt Gieck.

CENTROIDS AND MOMENTS OF INERTIA

The *location of the centroid of an area*, bounded by the axes and the function $y = f(x)$, can be found by integration.

$$x_c = \frac{\int x dA}{A}$$

$$y_c = \frac{\int y dA}{A}$$

$$A = \int f(x) dx$$

$$dA = f(x) dx = g(y) dy$$

The *first moment of area* with respect to the y-axis and the x-axis, respectively, are:

$$M_y = \int x \, dA = x_c A$$

$$M_x = \int y \, dA = y_c A$$

when either $\bar{x}$ or $\bar{y}$ is of finite dimensions then $\int xdA$ or $\int ydA$ refer to the centroid x or y of dA in these integrals. The *moment of inertia* (*second moment of area*) with respect to the y-axis and the x-axis, respectively, are:

$$I_y = \int x^2 \, dA$$

$$I_x = \int y^2 \, dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located d units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

In a plane, $\tau = \int r^2 dA = I_x + I_y$

Values for standard shapes are presented in a table in the **DYNAMICS** section.

DIFFERENTIAL EQUATIONS

A common class of ordinary linear differential equations is

$$b_n \frac{d^n y(x)}{dx^n} + \ldots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where $b_n, \ldots, b_i, \ldots, b_1, b_0$ are constants.

When the equation is a homogeneous differential equation, $f(x) = 0$, the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_i e^{r_i x} + \ldots + C_n e^{r_n x}$$

where r_n is the nth distinct root of the characteristic polynomial $P(x)$ with

$$P(r) = b_n r^n + b_{n-1} r^{n-1} + \ldots + b_1 r + b_0$$

If the root $r_1 = r_2$, then $C_2 e^{r_2 x}$ is replaced with $C_2 x e^{r_1 x}$. Higher orders of multiplicity imply higher powers of x. The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is any solution with $f(x)$ present. If $f(x)$ has $e^{r_n x}$ terms, then resonance is manifested. Furthermore, specific $f(x)$ forms result in specific $y_p(x)$ forms, some of which are:

$f(x)$	$y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}$, $\alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time t, then transient dynamic solutions are implied.

First-Order Linear Homogeneous Differential Equations With Constant Coefficients

$y' + ay = 0$, where a is a real constant:

Solution, $y = Ce^{-at}$, where

C = a constant that satisfies the initial conditions.

First-Order Linear Nonhomogeneous Differential Equations

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad x(t) = \begin{Bmatrix} A & t < 0 \\ B & t > 0 \end{Bmatrix}$$

$$y(0) = KA$$

τ is the time constant

K is the gain

The solution is

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \quad \text{or}$$

$$\frac{t}{\tau} = \ln\left[\frac{KB - KA}{KB - y}\right]$$

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

An equation of the form

$$y'' + 2ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + 2ar + b)\, Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + 2ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = -a \pm \sqrt{a^2 - b}$$

and can be real and distinct for $a^2 > b$, real and equal for $a^2 = b$, and complex for $a^2 < b$.

If $a^2 > b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x) e^{r_1 x}$$

If $a^2 < b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

where

$$\alpha = -a$$

$$\beta = \sqrt{b - a^2}$$

FOURIER SERIES

Every function $F(t)$ which has the period $\tau = 2\pi/\omega$ and satisfies certain continuity conditions can be represented by a series plus a constant.

$$F(t) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

The above equation holds if $F(t)$ has a continuous derivative $F'(t)$ for all t. Multiply both sides of the equation by $\cos m\omega t$ and integrate from 0 to τ.

$$\int_0^\tau F(t)\cos m\omega t dt = \int_0^\tau (a_0/2)\cos m\omega t dt$$

$$\int_0^\tau F(t)\cos m\omega t dt = \int_0^\tau (a_0/2)\cos m\omega t dt$$
$$+ \sum_{n=1}^{\infty}[a_n \int_0^\tau \cos n\omega t \cos m\omega t dt$$
$$+ b_n \int_0^\tau \sin n\omega t \cos m\omega dt]$$

Term-by-term integration of the series can be justified if $F(t)$ is continuous. The *coefficients* are

$$a_n = (2/\tau)\int_0^\tau F(t)\cos n\omega t dt \quad \text{and}$$

$$b_n = (2/\tau)\int_0^\tau F(t)\sin n\omega t dt, \quad \text{where}$$

$\tau = 2\pi/\omega$. The constants a_n, b_n are the *Fourier coefficients* of $F(t)$ for the interval 0 to τ, and the corresponding series is called the *Fourier series* of $F(t)$ over the same interval. The integrals have the same value over any interval of length τ.

If a Fourier series representing a periodic function is truncated after term $n = N$, the mean square value F_N^2 of the truncated series is given by the Parseval relation. This relation says that the mean square value is the sum of the mean square values of the Fourier components, or

$$F_N^2 = (a_0/2)^2 + (1/2)\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)$$

and the RMS value is then defined to be the square root of this quantity or F_N.

FOURIER TRANSFORM

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$f(t) = [1/(2\pi)]\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

f(t)	***F(ω)***
$\delta(t)$	1
$u(t)$	$(1/2)\delta(\omega) + 1/j\omega$
$u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right) = r_{rect}\,\frac{t}{\tau}$	$\tau\frac{\sin(\omega\tau/2)}{\varpi\tau/2}$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing s with $j\omega$ provided

$$f(t) = 0,\ t < 0$$
$$\int_0^\infty \left|f(t)\right| dt < \infty$$

LAPLACE TRANSFORMS

The unilateral Laplace transform pair

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st} dt$$

represents a powerful tool for the transient and frequency response of linear time invariant systems. Some useful Laplace transform pairs are [Note: The last two transforms represent the Final Value Theorem (F.V.T.) and Initial Value Theorem (I.V.T.) respectively. It is assumed that the limits exist.]:

f(t)	***F(s)***
$\delta(t)$, Impulse at $t = 0$	1
$u(t)$, Step at $t = 0$	$1/s$
$t[u(t)]$, Ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s + \alpha)$
$te^{-\alpha t}$	$1/(s + \alpha)^2$
$e^{-\alpha t} \sin \beta t$	$\beta/[(s + \alpha)^2 + \beta^2]$
$e^{-\alpha t} \cos \beta t$	$(s + \alpha)/[(s + \alpha)^2 + \beta^2]$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1} \frac{d^m f(0)}{d^m t}$
$\int_0^t f(\tau) d\tau$	$(1/s)F(s)$
$\int_0^t x(t-\tau) h(t) d\tau$	$H(s)X(s)$
$f(t - \tau)$	$e^{-\tau s}F(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$

DIFFERENCE EQUATIONS

Difference equations are used to model discrete systems. Systems which can be described by difference equations include computer program variables iteratively evaluated in a loop, sequential circuits, cash flows, recursive processes, systems with time-delay components, etc. Any system whose input $v(t)$ and output $y(t)$ are defined only at the equally spaced intervals $t = kT$ can be described by a difference equation.

First-Order Linear Difference Equation

The difference equation

$$P_k = P_{k-1}(1 + i) - A$$

represents the balance P of a loan after the kth payment A. If P_k is defined as $y(k)$, the model becomes

$$y(k) - (1 + i)\, y(k - 1) = - A$$

Second-Order Linear Difference Equation

The Fibonacci number sequence can be generated by

$$y(k) = y(k - 1) + y(k - 2)$$

where $y(-1) = 1$ and $y(-2) = 1$. An alternate form for this model is $f(k + 2) = f(k + 1) + f(k)$

with $f(0) = 1$ and $f(1) = 1$.

z-Transforms

The transform definition is

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

The inverse transform is given by the contour integral

$$f(k) = \frac{1}{2\pi i} \oint_{\Gamma} F(z) z^{k-1} dz$$

and it represents a powerful tool for solving linear shift invariant difference equations. A limited unilateral list of *z*-transform pairs follows [Note: The last two transform pairs represent the Initial Value Theorem (I.V.T.) and the Final Value Theorem (F.V.T.) respectively.]:

***f*(*k*)**	***F*(*z*)**
$\delta(k)$, Impulse at $k = 0$	1
$u(k)$, Step at $k = 0$	$1/(1 - z^{-1})$
β^k	$1/(1 - \beta z^{-1})$
$y(k - 1)$	$z^{-1}Y(z) + y(-1)$
$y(k - 2)$	$z^{-2}Y(z) + y(-2) + y(-1)z^{-1}$
$y(k + 1)$	$zY(z) - zy(0)$
$y(k + 2)$	$z^2Y(z) - z^2y(0) - zy(1)$
$\sum_{m=0}^{\infty} X(k - m)h(m)$	$H(z)X(z)$
$\lim_{k\to 0} f(k)$	$\lim_{z\to\infty} F(z)$
$\lim_{k\to\infty} f(k)$	$\lim_{z\to 1} (1 - z^{-1}) F(z)$

EULER'S APPROXIMATION

$$x_{i+1} = x_i + \Delta t\, (dx_i/dt)$$

NUMERICAL METHODS

Newton's Method of Root Extraction

Given a polynomial $P(x)$ with n simple roots, $a_1, a_2, \ldots, a_n$ where

$$P(x) = \prod_{m=1}^{n} (x - a_m)$$
$$= x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \ldots + \alpha_n$$

and $P(a_i) = 0$. A root a_i can be computed by the iterative algorithm

$$a_i^{j+1} = a_i^{j} - \left| \frac{P(x)}{\partial P(x)/\partial x} \right| \qquad x = a_i^j$$

with $\left| P\left(a_i^{j+1}\right) \right| \le \left| P\left(a_i^{j}\right) \right|$ Convergence is quadratic.

Newton's Method of Minimization

Given a scalar value function

$$h(\boldsymbol{x}) = h(x_1, x_2, \ldots, x_n)$$

find a vector $\boldsymbol{x}^* \in R_n$ such that

$$h(\boldsymbol{x}^*) \le h(\boldsymbol{x}) \text{ for all } \boldsymbol{x}$$

Newton's algorithm is

$$\boldsymbol{x}_{K+1} = \boldsymbol{x}_K - \left(\left. \frac{\partial^2 h}{\partial x^2} \right|_{\boldsymbol{x} = \boldsymbol{x}_K} \right)^{-1} \left. \frac{\partial h}{\partial x} \right|_{\boldsymbol{x} = \boldsymbol{x}_K}$$

where

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \ldots \\ \ldots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)dx$$

are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)dx \approx \Delta x \left[\frac{f(a) + f(b)}{2}\right]$$

for $n > 1$

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)

for $n = 2$

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \geq 4$

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[\begin{array}{l} f(a) + 2\sum\limits_{k=2,4,6,\ldots}^{n-2} f(a + k\Delta x) \\ +4\sum\limits_{k=1,3,5,\ldots}^{n-1} f(a + k\Delta x) + f(b) \end{array}\right]$$

with $\Delta x = (b - a)/n$

Numerical Solution of Ordinary Differential Equations

Given a differential equation

$$dy/dt = f(y, t) \text{ with } y(0) = y_o$$

At some general time $k\Delta t$

$$y[(k + 1)\Delta t] \cong y(k\Delta t) + \Delta t\, f\,[\, y(k\Delta t), k\Delta t]$$

which can be used with starting condition y_o to solve recursively for $y(\Delta t)$, $y(2\Delta t)$, …, $y(n\Delta t)$.

The method can be extended to nth order differential equations by recasting them as n first-order equations.