GEARS

CONTENTS:

Introduction

Definitions:

A means of transmitting power between two shafts. In this way the angular velocity and direction of rotation can also be changed.

Gears are not the only method of achieving this belt drives could also be used, however, gears have the advantages of providing positive drive without slip and permit high torques to be transmitted. Gears can also be used between shafts, which can be parallel or inclined to one another.

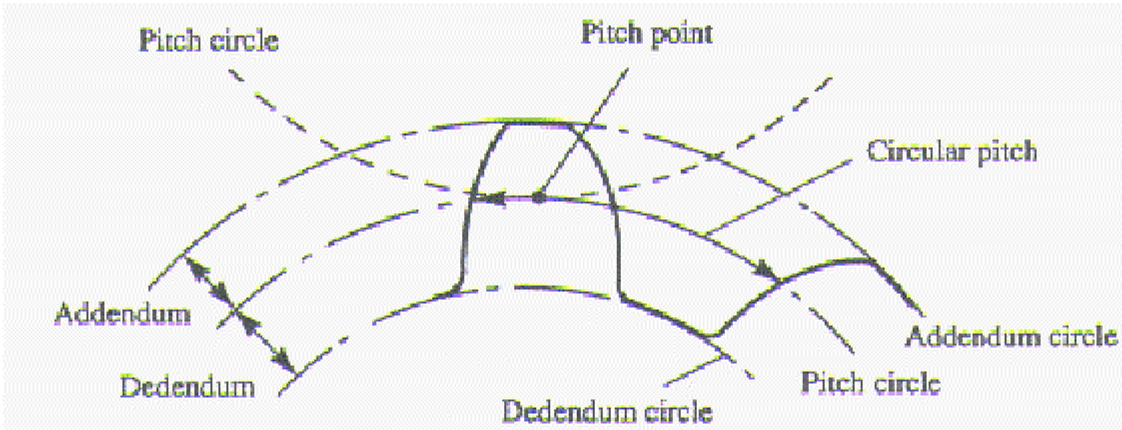
Gears for use with parallel shaft axes that have axial teeth with the teeth cut along the axial lines parallel to the shaft are called SPUR gears. Alternately they can have helical teeth with the teeth being cut on the helix, these are called HELICAL gears. Helical gears are less noisy than spur gears because with helical gears there is a gradual engagement of any individual tooth. This also gives a smoother drive and longer life. Unfortunately, because of the inclination of the teeth helical gears will create an axial force component on the shaft bearing. This can be overcome by using double helical gears with two sets of teeth cut back to back and cut with opposite inclinations.

Gears for use with shafts inclined to one another are called BEVEL gears when the lines of the shafts intersect. The teeth may be cut straight or curved. The curved teeth give the same benefits as helical gears.

The terms SKEW or SPIRAL gears are used for gears when the shafts are non-intersecting. A special form of such a gear is called the WORM and WHEEL.

When two gears are in mesh together the larger gear is called the WHEEL or CROWN with bevel gears and the smaller wheel is the PINION (or WORM in the case of worm and wheel) For bevel gears, when the ratio is 1:1 they are termed MITER gears.

The figure shows some of the terms used with spur gears. We will consider a gear as a disc with teeth.



The *pitch circle diameters* (PCD) are the diameter of the discs, without teeth, which would transmit the same velocities by friction as the gears if the PCD for the wheels in a pair are d_A and d_B, then since the teeth will be the same size

$$d_A / d_B = t_A / t_B$$

If their are 50 teeth on wheel A and 100 teeth on wheel B then, wheel A must rotate through two revolutions in same time as B rotates through one.

The *pitch point* is the point of contact between the two pitch circles. While the *circular pitch, p* is the distance measured along the pitch circle between a point on tooth and the corresponding point on the next tooth

The diametral pitch, P is the number of teeth per millimetre of pitch circle diameter.

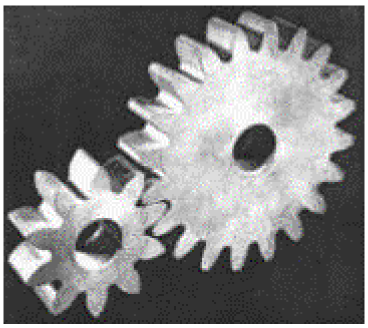
$$P = \pi / p$$

The *module, m* is the number of millimetres of pitch circle diameter per tooth.

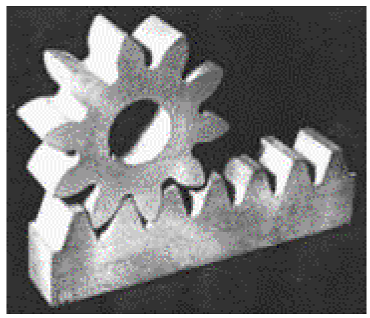
$$m = 1/P = p/\pi$$

Spur Gears

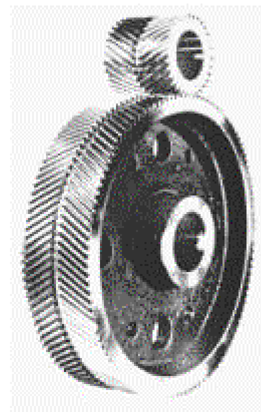
Gears are used to transmit power between shafts rotating usually at different speeds. Some of the many types of gears are illustrated below.



A pair of ***spur*** gears for mounting on parallel shafts. The 10 teeth of the smaller ***pinion*** and the 20 teeth of the ***wheel*** lie parallel to the shaft axes

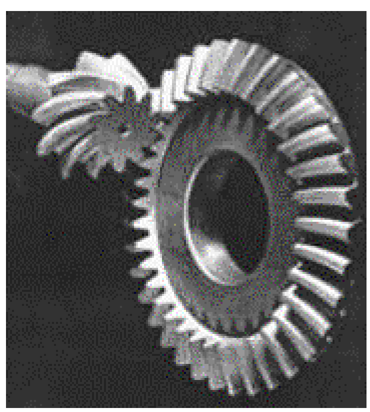


A ***rack*** and pinion. The straight rack translates rectilinearly and may be regarded as part of a wheel of infinite diameter



Like spur gears ***helical*** gears connect parallel shafts, however the teeth are not parallel to the shaft axes but lie along helices about the axes

Straight ***bevel*** gears for shafts whose axes intersect

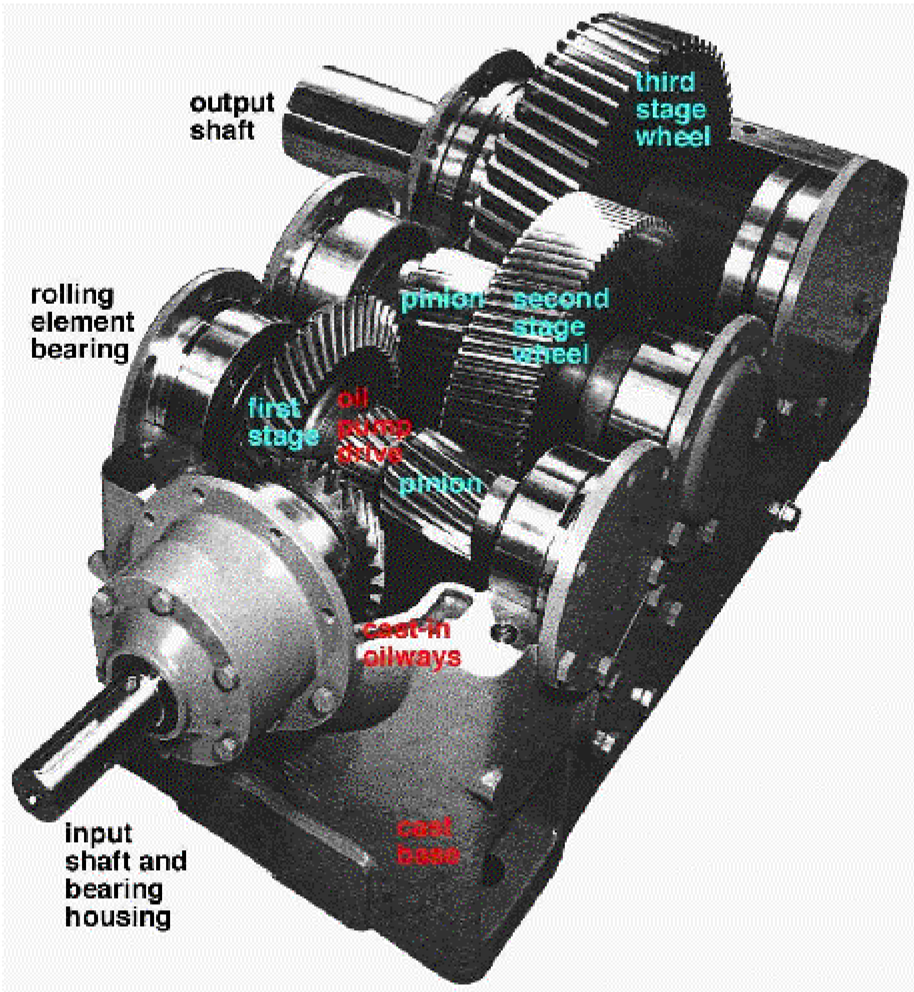


Hypoid gears - one of a number of gear types for offset shafts

A ***worm*** and wormwheel gives a large speed ratio but with significant sliding

In order to demonstrate briefly the development of gear drives, from first principles through to safety implications, we consider here only spur gears.

Knowledge of these is fundamental to understanding the behavior of geometrically more complex types, including helical gears that are generally preferred to spurs since they are more compact and smoother in operation, thus permitting higher speeds.



A typical commercial gearbox is shown with its cover removed. It demonstrates that it is usually more attractive economically to split a larger speed ratio into a number of *stages* (pairs of gears) rather than to effect it with a single pair. There are three stages here - the first spiral bevel pair is followed by two helical pairs.
A couple of features of the box are immediately apparent :

compactness - shafts are short and simply supported where practicable, with gears located as close as possible to bearings in order to minimize shaft bending

sturdiness increasing from input through output - the sizes of input & output shafts, and second & third stage gear teeth, should be compared.

A pair of meshing gears is a power transformer, a coupler or interface which marries the ***speed and torque*** characteristics of a power ***source*** and a power ***sink*** (load). A single pair may be inadequate for certain sources and loads, in which case more complex combinations such as the above gearbox, known as ***gear trains,*** are necessary. In the vast majority of applications such a device acts as a **speed reducer** in which the power source drives the device through the high speed low torque input shaft, while power is fed from the device to the load through the low speed high torque output shaft.

Speed reducers are much more common than speed -up drives not so much because they reduce speed, but rather because they amplify torque. Thus gears are used to accelerate a car from rest, not to provide the initial low speeds (which could be accomplished by easing up on the accelerator pedal) but to increase the torque at the wheels which is necessary to accelerate the vehicle. Torque amplification is the reason for the gearbox's increasing sturdiness mentioned above.

These notes will consider the following aspects of spur gearing :

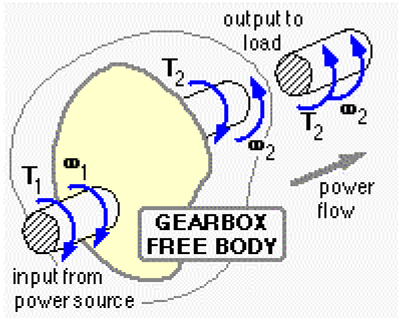
- overall kinetics of a gear pair (for cases only of steady speeds and loads)
- tooth geometry requirements for a constant velocity ratio
- detailed geometry of the involute tooth and meshing gears
- the consequences of power transfer on the fatigue life of the components, and hence
- the essentials of gear design.

Some of the main features of spur gear teeth are illustrated. The teeth extend from the root, or ***dedendum*** cylinder (or colloquially, *"circle"*) to the tip, or ***addendum*** circle: both these circles can be measured. The useful portion of the tooth is the ***flank*** (or face), it is this surface which contacts the mating gear. The ***fillet*** in the root region is cinematically irrelevant since there is no contact there, but it is important insofar as fatigue is concerned.

Overall kinetics of a gear pair

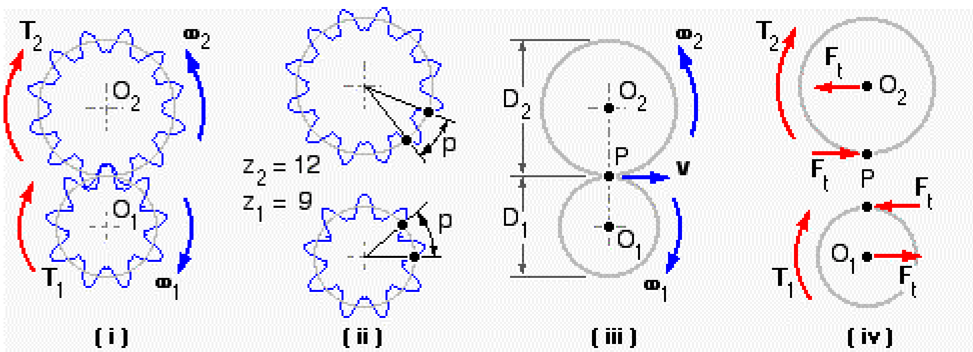
Analysis of gears follows along familiar lines in that we consider kinetics of the overall assembly first, before examining internal details such as individual gear teeth.

The free body of a typical single stage gearbox is shown. The power source



applies the torque T_1 to the input shaft, driving it at speed ω_1 in the sense of the torque (clockwise here). For a single pair of gears the output shaft rotates at speed ω_2 in the opposite sense to the input shaft, and the torque T_2 supplied by the gearbox drives the load in the sense of ω_2. The reaction to this latter torque is shown on the free body of the gearbox - apparently the output torque T_2 must act on the gearbox ***in the same sense*** as that of the input torque T_1.

The gears appear in more detail in the figure (i) below. O_1 and O_2 are the centers of the pinion and wheel respectively. We may regard the gears as equivalent ***pitch cylinders*** which roll together without slip - the requirements for preventing slip due to the ***positive drive*** provided by the meshing teeth is examined below. Unlike the addendum and dedendum cylinders, pitch cylinders cannot be measured directly; they are notional and must be inferred from other measurements.



One essential for correct meshing of the gears is that the ***size*** of the teeth on the pinion is the same as the size of teeth on the wheel. One measure of size is the ***circular pitch,*** p, the distance between adjacent teeth around the pitch circle (**ii**); thus $p = \pi D/z$ where z is the number of teeth on a gear of pitch diameter D. The IS measure of size is the ***module,*** $m = p/\pi$ which should not be confused with the IS abbreviation for meter. So the geometry of pinion 1 and wheel 2 must be such that :

$$D_1 / z_1 = D_2 / z_2 = p / \pi = m$$

. . . . that is the module must be common to both gears. For the rack, both the diameter and tooth number tend to infinity, but their quotient remains the finite module.

The pitch circles contact one another at the ***pitch point,*** P Fig (**iii**), which is also notional. Since the positive drive precludes slip between the pitch cylinders, the pinion's pitch line velocity, v, must be identical to the wheel's pitch line velocity :

$v = \omega_1 R_1 = \omega_2 R_2$; where pitch circle radius

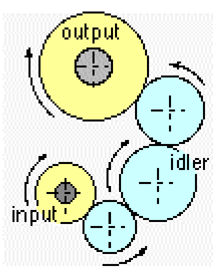
$$R = D/2$$

Separate free bodies of pinion and wheel appear in (**iv**).
F_t is the tangential component of action -reaction at the pitch point due to contact between the gears. The corresponding radial component plays no part in power transfer and is therefore not shown on the bodies. Ideal gears only are considered initially, so the friction force due to sliding contact is omitted also. The free bodies show that the magnitude of the shaft reactions must be F_t, and that for equilibrium :

$F_t = T_1 / R_1 = T_2 / R_2$ in the absence of friction.

The preceding concepts may be combined conveniently into :

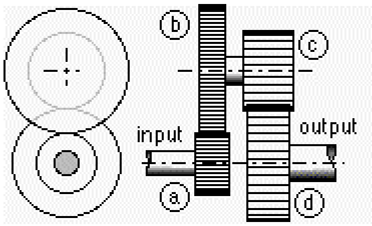
(**1**) $\omega_1 / \omega_2 = T_2 / T_1 = D_2 / D_1 = z_2 / z_1$; $D = mz$



That is, gears reduce speed and amplify torque in proportion to their teeth numbers.
In practice, rotational speed is described by N (rev/min or Hz) rather than by ω (rad/s).

The only way that the input and output shafts of a gear pair can be made to rotate in the same sense is by interposition of an odd number of intermediate gears as shown - these do not affect the speed ratio between input and output shafts. Such a gear train is called a ***simple*** train. If there is no power flow through the shaft of an intermediate gear then it is an ***idler*** gear.

A gear train comprising two or more pairs is termed ***compound*** when the wheel of one stage is mounted on the same shaft as the pinion of the next stage.



A compound train as in the above gearbox is used when the desired speed ratio cannot be achieved economically by a single pair. Applying (**1**) to each stage in turn, the overall speed ratio for a compound train is found to be the product of the speed ratios for the individual stages.

Selecting suitable integral tooth numbers to provide a specified speed ratio can be awkward if the speed tolerance is tight and the range of available tooth numbers is limited. Until the advent of computers allowed such problems to be solved by iterative trials, techniques based on ***continued fractions*** were used.

Unlike the above gearbox, the input and output shafts are coaxial in the train illustrated here; this is rather an unusual feature, but necessary in certain change speed boxes and the like.

Gear Trains

Definition:

A gear train is a series of gear wheels, which are used to transmit rotational motion from an input shaft to an output shaft.

For any pair of meshing wheels, whatever type of gear they are, the ratio of the angular speeds is proportional to the inverse ratio of the number of teeth on the wheels. The *gear ratio, G* is the ratio of the angular speeds.

$$G = \omega_A / \omega_B = - t_B / t_A$$

The minus sign is often included in an equation to indicate that the angular velocity of wheel B is in an opposite direction to that of wheel A.

Simple Trains

Definition:

Gears train in which each shaft carries a single gear.

For a simple gear train involving three gear wheels, the overall *gear ratio G* is the ratio of the angular velocities at the input and output shafts are thus ω_A / ω_C. Therefore,

$$G = \omega_A / \omega_C = \omega_A / \omega_B \times \omega_B / \omega_C$$

The gear ratio is independent of the intermediate wheel (termed an *idler*) however it does result in a change of angular direction in the output wheel.

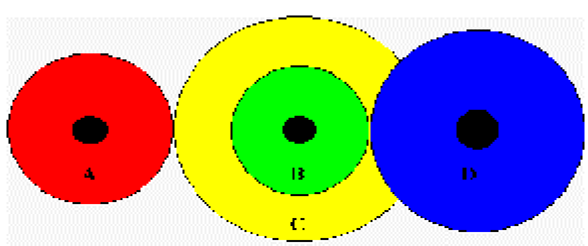
Compound Trains

Definition:

A gear train in which two or more wheels are mounted on a common shaft.

The figure shows an example of a compound gear train. The two wheels mounted on the same shaft rotate at the same angular velocity. Thus $\omega_B = \omega_C$. The overall gear ratio G which is ω_A / ω_D, is given by:

$$G = \omega_A / \omega_D = \omega_A / \omega_C \times \omega_C / \omega_B \times \omega_B / \omega_D$$



Epicycle Gear Trains

Definition:

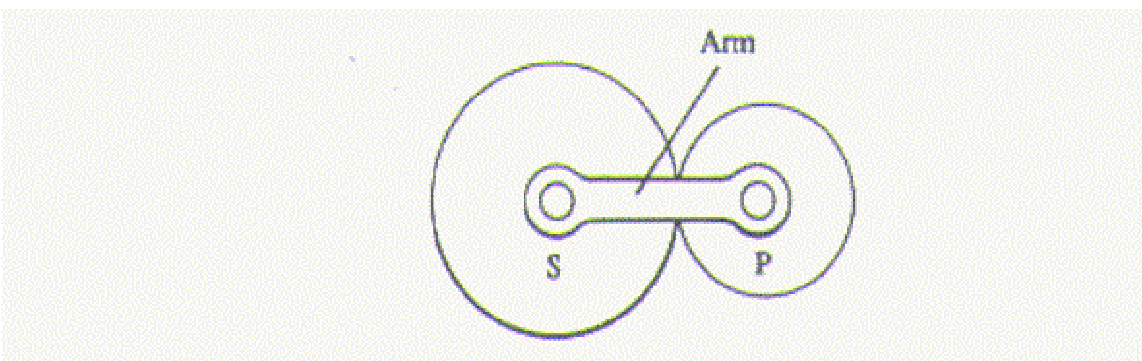
> In an epicycle gear train one or more wheels is carried on an arm, which can rotate about the main axes of the train. Such wheels are called planets and the wheel around which the planets rotate is called the sun.

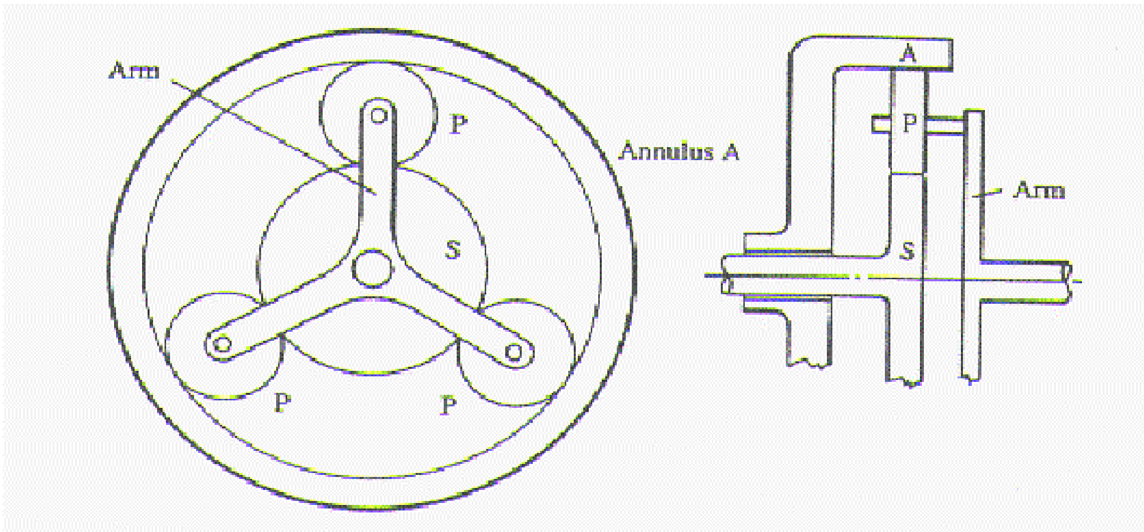
Next figure shows a simple epicycle gear train. An arm links the centres of the planet wheel P and the sun, S. If the sun wheel is fixed then rotation of the arm around the sun causes the planet wheel, P to rotate. In order to determine the amount of rotation a simple technique can be used.

1. Imagine the link (arm) to be fixed while wheel S rotates through +1 revolutions. This causes wheel P to rotate through t_S / t_P revolutions.

2. Imagine the gears to be locked solid and give a rotation of -1 revolutions to all the wheels and the arm about the axis through S.

3. Imagine the sun to be fixed and the arm rotating through -1 revolutions. This has the same effect as adding together the two previous steps.





The figure shows an epicycle gear train in which A is an *annulus* having internal teeth and there are three planets that can rotate about pins through the arms. The planets mesh externally with the sun wheel, S and internally with the annulus. There are usually three or more planets in order to improve dynamic characteristics. The same technique as above can be used to establish the relative motion of the wheels and arm.

Torques on Gear Trains

Consider a gear system which has an input torque of T_i at an angular velocity ω_I and an output torque (i.e. resisting torque on the output shaft) of T_O at an angular velocity of ω_O.

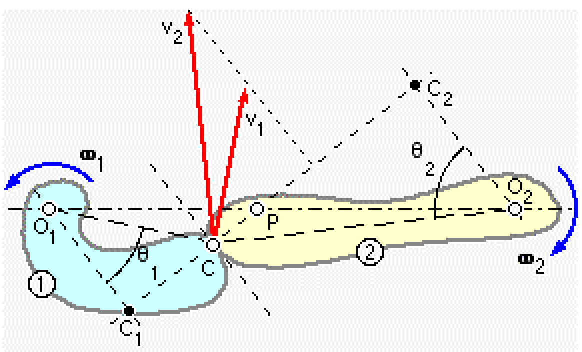
The input torque is in the same direction as the angular rotation the output resistive torque, however, is in the opposite direction to the output angular rotation. This will be true if the input and output shafts rotate in the same direction or in opposite directions. If there is no angular acceleration of the system then the net torque on the gear system must be zero. Therefore, we must have a *holding torque T_h* that prevents the gear system as a whole rotating. The direction of this holding torque will depend on the size and directions of the input and output torques i.e.

$$\mathbf{T_i + T_O + T_h = 0}$$

The input power is $T_i\ \omega_I$ and the output power is - $T_O\ \omega_O$. The minus sign shows that the output reaction torque acts in the opposite sense to the angular velocity ω_O. The transmission efficiency, ξ is the ratio of the output power to input power.

$$\mathbf{T_O\ \omega_O + \xi\ T_i\ \omega_i = 0}$$

Conjugate tooth action



We have seen that one essential for correctly meshing gears is that the size of the teeth (the module) must be the same for the two gears. We now examine another requirement - the shape of teeth necessary for the speed ratio to remain constant during an increment of rotation; this behavior of the contacting surfaces (i.e.. the teeth flanks) is known as ***conjugate action.***

Consider the two rigid bodies 1 and 2 which rotate about fixed centers, O, with angular velocities ω. The bodies touch at the contact point, C, through which the common tangent and normal are drawn.
The absolute velocity v of the contact point reckoned as a point on either body, is perpendicular to the radius from that body's center O to the contact point. For the bodies to remain in contact, there must be no component of relative motion along the common normal, so that from the velocity triangles :-

$$v_2 \cos\theta_2 = v_1 \cos\theta_1 \quad \text{where} \quad v_1 = \omega_1 . O_1C \ ; \quad v_2 = \omega_2 . O_2C$$

Note that the tangential components of velocity are generally different, so sliding must occur. For the speed ratio to be constant therefore, from the above and similar triangles :-

(**3**) $\omega_2/\omega_1 = v_2 . O_1C/v_1 . O_2C = O_1C.\cos\theta_1/O_2C.\cos\theta_2$
$= O_1C_1/O_2C_2 = O_1P / O_2P$ i.e.. this ratio also must be constant.

This indicates that, since the centers are fixed, ***the point P is fixed*** too.

In general therefore, whatever the shapes of the bodies, the contact point C will move along some locus as rotation proceeds; but if the action is to be conjugate then the body geometry must be such that the common normal at the contact point passes always through one unique point lying on the line of centers - this point is the pitch point referred to above, and the pitch circles' radii are O_1P and O_2P.

There exists a host of shapes which ensure conjugacy - indeed it is possible, within certain restrictions, to arbitrarily choose the shape of one body then determine the shape of the second necessary for conjugacy. But by far the most common gear geometry which satisfies conjugacy is based on ***the*** involute, in which case both gears are similar in form, and the contact point's locus is a simple straight line - the ***line of action.***

The involute tooth

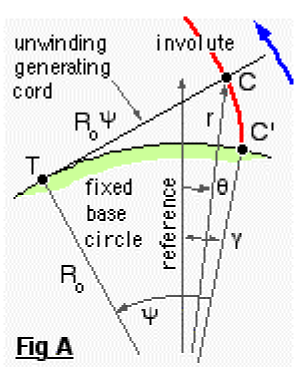
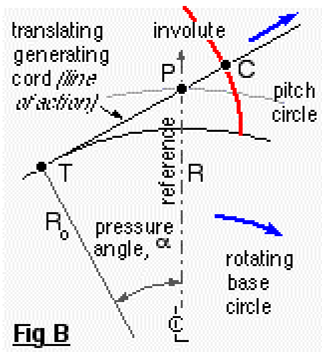


Fig A

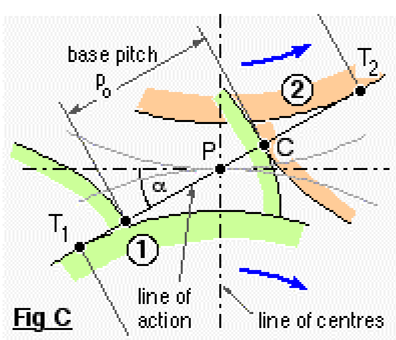
One method of generating an involute is shown in Fig A. A generating cord, in which there is a knot C, is wrapped around a fixed cylinder - the ***base*** cylinder (idiomatically circle) of radius R_o. When the taut cord is subsequently unwound, the knot traces out an involute whose polar coordinates may be expressed implicitly in terms of the variable generating angle ψ, reckoned from the radius through the initial knot position, C'. The coordinate origin is taken at the circle center, O, with a fixed reference direction defined at some constant angle γ, also from the initial radius. The tangent, TC, is normal to the involute at C, and since the tangent length TC is equal to the arc length TC', the polar coordinates of C (r, θ) are :-

(**4**) $r = R_o\sqrt{(1 + \psi^2)}$; $\theta = \gamma - \psi + \arctan \psi$



In order to see how the involute leads to gear teeth and conjugate action, we place a slightly different interpretation on the above model. The cord is wrapped around the base cylinder which in Fig B is now free to rotate about its center as the cord is pulled off in a fixed direction. This fixed cord direction forms the line of action, tangent to the base cylinder at the fixed point T, and clearly satisfies conjugacy by cutting the fixed reference at the fixed pitch point P through which the pitch cylinder passes. The line of action is inclined to the pitch point tangent at the ***pressure angle,*** α. The knot C always moves along the line of action, tracing out an involute with respect to the rotating cylinder. The relation between the base and pitch circle radii is evidently :

(**5**) $R_o = R \cos \alpha$



Extending this to two cylinders - representing meshing gears, 1 & 2 Fig C - the taut cord winds off one base cylinder and onto the other to form the line of action inclined at the pressure angle α. The knot, C, on the mating involutes coincides with the contact point and moves along the line of action as the gears and base cylinders rotate. The pitch cylinders extend to the pitch point P situated at the intersection of the lines of action and of centers. Evidently the distance between the cylinders does not affect the speed ratio since the base cylinder diameters are fixed.

The distance between knots - i.e.. between tooth flanks along the line of action, Fig C - is the ***base*** pitch, po, given by :-

(**6**) $p_o = \pi D_o / z = p \cos\alpha = \pi m \cos\alpha$ . . . . . from (**1**)

For continuous motion transfer, at least two pairs of teeth must be in contact as one of the pairs comes into or leaves mesh. The teeth in Fig C are truncated in practice to permit rotation.

Involute generation by knotted cord is all very well conceptually, but hardly practicable as a basis for manufacturing. Only one of the many methods of gear manufacture is considered here - the **rack generation** technique is fundamental to the understanding of gear behavior.

Gear tooth generation

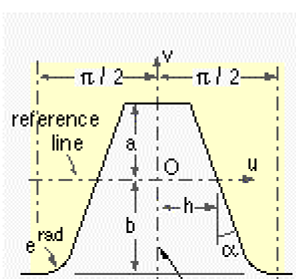


Fig D

A ***tooth*** system is defined by a unique pressure angle and set of tooth proportions which characterize the system's ***basic*** rack of Fig D. Various systems find application in particular specialized sectors of industry, but by far the most widespread - and the only system considered here - is the ***20° full depth*** system which incorporates a 20° pressure angle (α) and the proportions :-

half-width	$h = \pi/4$	addendum	$a = 1$
dedendum	$b = 1.25$	fillet radius	$e = 0.38$

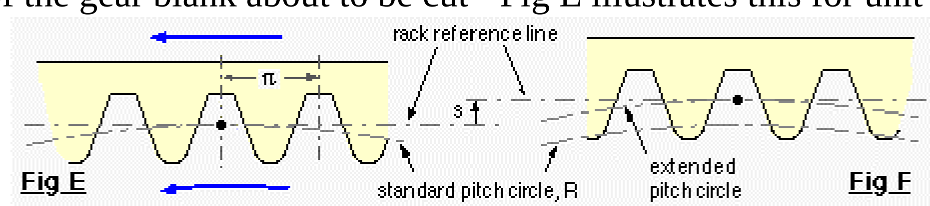
that is the tooth occupies half the pitch (π) measured along the reference line.

A circular gear of specified module is generated by a ***rack cutter*** which takes the form of the shaded mask. The rack's profile is a cutting edge whose straight sides and fillets machine respectively the active flanks and the roots of the gear's teeth. The cutter is set out about the reference and center lines, with **absolute dimensions derived from the system's proportions multiplied by the specified module**. Cutters are available commercially to the following modules :-

m = 1 1.25 1.5 2 2.5 3 4 5 6 8 10 12 16 20 25 32 40 50 mm

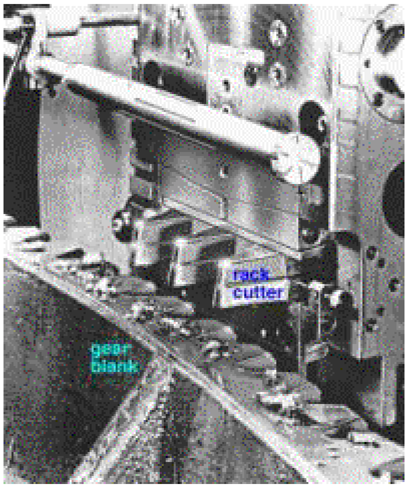
Gear proportions and absolute dimensions are often used interchangeably - this should not cause confusion provided the absence or presence of units is noted carefully. Thus the dedendum of a 4 mm module gear may appear alternately as 1.25 (no units) or as 5 mm (i.e.. 1.25 multiplied by 4 mm). If this gear possessed 20 teeth then its pitch radius could be quoted as 40 mm (**1**) or as 10 (no units).

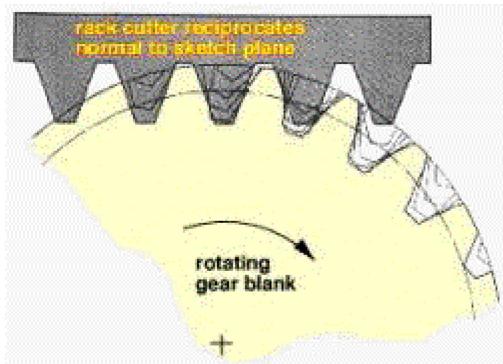
Generated gears (of the same system) which have the same number of teeth are geometrically similar to one another. One cutter only is necessary for each module, no matter how many teeth are in the gear being cut. Once the cutter has been prepared, generation commences by aligning the rack's reference line with the pitch circle of the gear blank about to be cut - Fig E illustrates this for unit module.

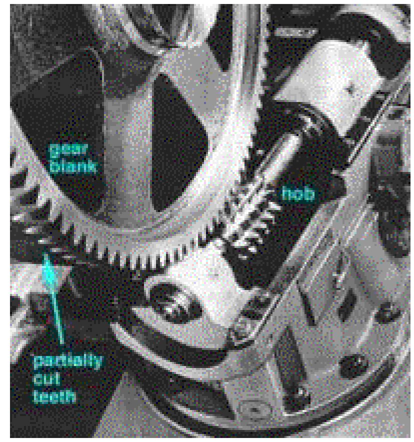


During machining of the gear blank by the rack, the cutter undergoes two distinct simple motions :

it reciprocates perpendicularly to the sketch plane Fig E, shaving material off the blank as shown in the photograph of a partially cut gear at left below, and it translates parallel to its reference line while the circular gear blank rotates, an external mechanism being provided so that there is no slip between these feed movements.







Generation may be carried out by a hob in the form of a rotating worm-wheel rather than by a reciprocating rack.

Before cutting commences, the rack may be displaced transverse to its reference line by an amount **s** (or m^*s dimensionally) where s is the so-called ***profile shift coefficient*** (also known as the *addendum modification coefficient*),Fig F. Profile shift is reckoned positive if the rack is moved away from the blank as shown above; negative if the cutter is moved towards the gear blank.

A gear's ***pitch circle*** represents the notional cylinder which rolls without slip on the pitch circle of its mating gear. The introduction of profile shift makes it necessary to distinguish between three different pitch circles :

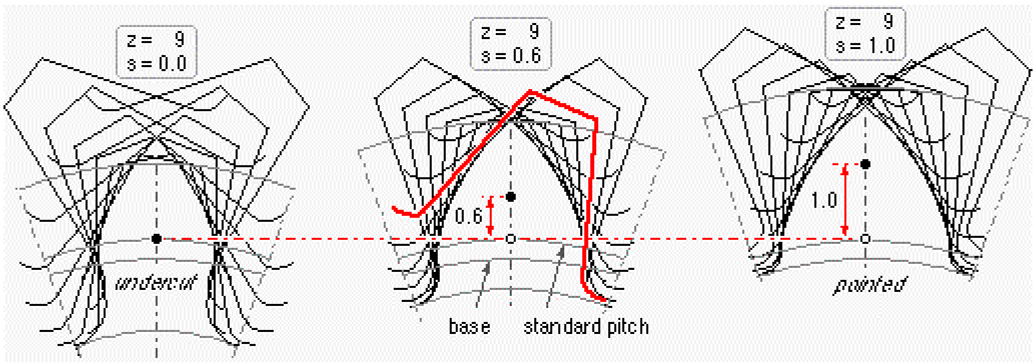
The **standard pitch circle** (Fig E) is defined by the gear's tooth number only, thus from (**1**) this circle's proportional radius is $R = z/2$, or in dimensional terms $R = m.z/2$.

When profile shift s is present then the pitch circle is referred to as the **extended pitch circle**; its radius is $R + s$ proportionally as may be seen from Fig F.

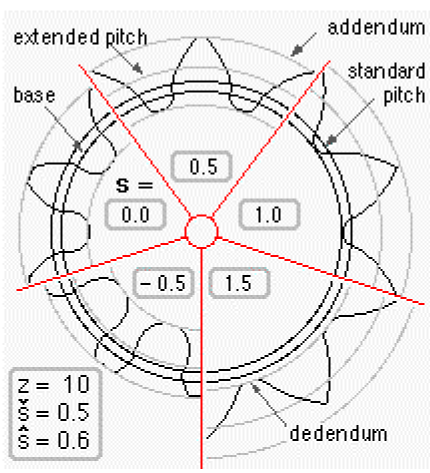
The **(actual) pitch circles**, R'_1 and R'_2 , at which two involute gears operate depend on the distance between their centers, as will be explained in the **gear meshing** page below.

The Macintosh program ***"Gear Tooth Generator"*** simulates the rack generation process, duplicating the relative motion of a unit-module cutting rack with respect to a gear blank whilst holding the latter stationary. The program illustrates how the cutter shaves a portion of the tooth during each reciprocation. Program output for a

20° full depth 9-tooth gear appears below and illustrates the advantages to be gained by using profile shift.



The base, standard pitch, addendum and dedendum circles appear in each plot - the first two remain fixed while the latter vary with profile shift. The rack reference origin is highlighted. It is apparent that an involute exists only outside its base circle; inside that circle the tooth root fillet is trochoidal.



- With no profile shift as shown on the left sketch, the rack origin lies on the pitch circle and the tooth is severely undercut, seriously weakening the cantilevered tooth in the region of highest bending stress.
- A profile shift of 0.6 in the center sketch removes this undercut tendency but the reduced tip width would be dangerous in practice as the tip is liable to be easily broken off, especially if the teeth are hardened.
- A larger shift in the rightmost tooth leads to extreme pointing; the tooth does not reach the theoretical addendum circle.

The effect of profile shift is further exemplified by this montage of five different gears. They all have 10 teeth so they share common base and standard pitch circles - their teeth are therefore all part of the same involute. However the gears' profile shifts differ and so the teeth are based on different portions of that involute.
For a particular number of teeth there is a critical lower profile shift below which the teeth become undercut, and a critical upper profile shift above which the teeth tips become pointed. It is unnecessary to consider tooth geometry further here.

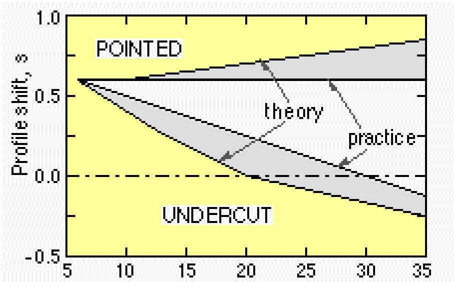


Fig G

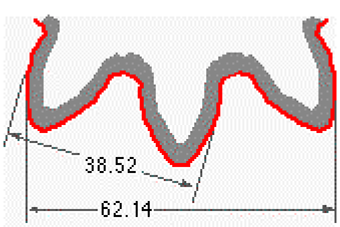
Fig G illustrates the profile shift limits for the 20° full depth system revealed by that theory, together with the usual practical limits suggested by ISO TR 4467, namely :

(**7**) $\max [-0.5, (30-z)/40] \leq s \leq 0.6$

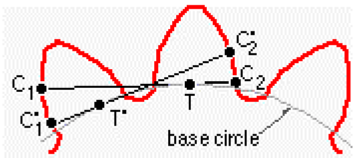
Desirable compactness necessitates the smallest tooth numbers possible, but these tooth form limits become more constraining as numbers decrease - eg. 20 is the minimum which theoretically avoids undercutting in the absence of profile shift. Depending upon the drive as a whole, spur gear tooth numbers of 12 are possible with suitable profile shift. Despite the indications of Fig G, numbers less than 12 are seldom used - except for lightly-loaded essentially cinematic applications - due to other considerations such as the weakness of non-pointed but thin tooth tips and the need for two pairs of teeth to be in contact simultaneously as the drive is transferred from one pair to the next, manufacturing and/or mounting inaccuracies notwithstanding *(refer to **contact ratio** below).*

The specialized business of gear manufacture and inspection is complex. The following example gives some insight into the necessity for the involute geometry of Appendix B. Not infrequently the standardized 20° full depth system is altered subtlety to improve operation.

EXAMPLE

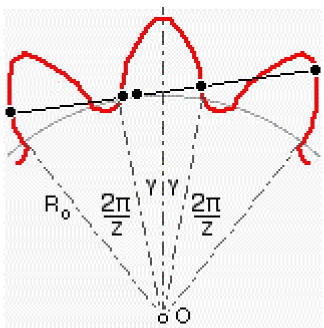


Show that a spur gear is a body of constant width when measured as shown across tooth flanks, the difference between two such measurements giving the module of the gear.
With what standard module rack would this gear



have been generated?
If there are 13 teeth on the gear, what is the profile shift ?

The measurement C_1C_2 is normal to the teeth flanks and so coincides with the tangent to the base circle at T. In a similar manner, the measurement $C'_1C'_2$ coincides with the tangent at T'.
But the arc length TT' must equal the differences in generating cord lengths $TC_1 - T'C'_1$ etc. so the measurements are equal.



Radii are drawn to the starts of the flank involutes on the base circle; these radii are inclined at γ to the tooth center line, and corresponding radii on adjacent teeth subtend $2\pi/z$ at the center, O.
Equality of the generating cord and arc lengths requires that the measurement across n teeth is $L_n = R_o(2\gamma + (n-1)\ 2\pi/z)$, so that the difference in measurements between n and n+1 teeth is evidently $D_o\pi/z$, or, using (**1**, **5**), π m cos α. Assuming a 20° pressure angle here, then $m = (62.14 - 38.52)/\pi \cos 20° = 8$ mm.
A 13-tooth gear therefore has a pitch diameter of $D = mz = 104$ mm and from the L_n equation above :-

$$\gamma = L_2/ D \cos\alpha - \pi/z = 38.52/104 \cos 20° - \pi/13 = 0.1525 \text{ rad}$$

We now need to evaluate γ as a function of s.
Figures D and F indicate that the rack space half-width on the gear rolling (pitch) circle is $(h + s\tan\alpha)$, and this must equal the arc subtended by the gear half-tooth around the pitch circle, i.e.. $r\theta$, where r and θ are given by (**4**) at r = R. From this we have :-

$$\gamma = (h + s\tan\alpha)/R + \tan\alpha - \alpha \quad \text{or, inserting known values}$$

$$0.1525 = (\pi/4 + s\tan 20°)\ 8/52 + \tan 20° - \pi/9$$

whence $s = 0.30$

One of the major advantages of involute gears is their ability to work successfully when the distance between the centers of meshing gears is altered.

Gear meshing

Involute gears have the invaluable ability of providing conjugate action when the gears' center distance is varied either deliberately or involuntarily due to manufacturing and/or mounting errors.

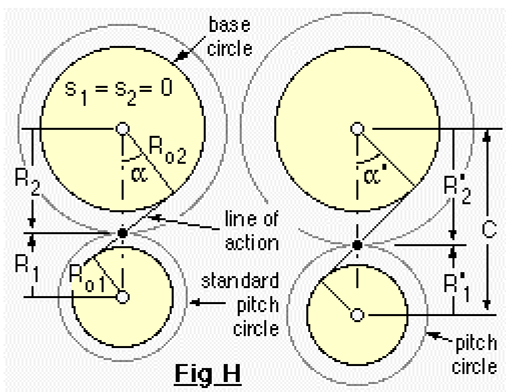


Fig H

The exaggerated effect of this is shown in Fig H for gears manufactured to pressure angle α without profile shift. On the left, the gears are mounted at the standard center distance (the sum of the two standard radii given by (**1**)) with the "cord" line of action wrapped around the base circles whose radii are given by (**5**).

On the right of Fig H the gears' center distance is increased - but since the same generating cord is wrapped around the same base circles then it follows that the speed ratio is unaltered and the same involutes (and hence teeth) are involved. Clearly there are practical limits to center distance variations - eg. the gears may lose contact completely - however provided these limits are not reached then the pitch point and resulting pitch circles are defined by intersection of the lines of action and of centers, exactly as occurred in Fig C above. From the similar triangles of Fig H :-

(**8**) $R'_i = D'_i/2 = C.z_i/\Sigma z = R_{oi} \sec \alpha'$ ($i = 1,2$) using (**5**)

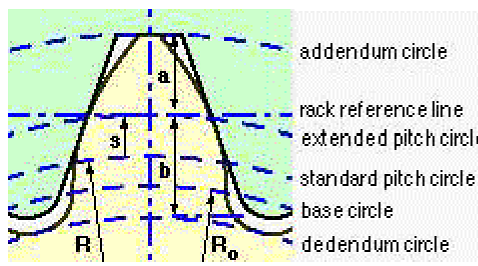
There is an infinite number of possible center distances for a given pair of profile shifted gears, however we consider only the particular case known as the ***extended center distance*** which corresponds to mutual tangency of the two extended pitch circles mentioned above, that is :-

(**9**) $C = (R_1 + s_1) + (R_2 + s_2) = {}^1/_2\, \Sigma z + \Sigma s$ proportionally, using (**1**), or

$= m\, ({}^1/_2\, \Sigma z + \Sigma s)$ dimensionally.

EXAMPLE

A pair of 20° full depth gears mesh at extended centers - the 12 tooth pinion's profile shift is 0.5, the 18 tooth wheel's is 0.4. What are the pitch and addendum diameters of the gears and what is the pressure angle ?



No absolute dimensions are given so it is proportions which are relevant in this example.

From (**1**) the diameters of the standard pitch circles are $D_1 = 12$, $D_2 = 18$. Corresponding base circle diameters are $D_{o1} = 12 * \cos 20° = 11.28$, $D_{o2} = 18 * \cos 20° = 16.92$

The pinion's extended pitch radius is $R_1 + s_1 = 12/2 + 0.5 = 6.5$ and similarly the wheel's is $18/2 + 0.4 = 9.4$

The extended center distance is thus (**9**) 6.5 +9.4 = 15.9

From (**8**) the actual pitch diameters are $D'_1 = 2 * 15.9 * 12/(12+18) = 12.72$ and $D'_2 = 2 * 15.9 * 18/(12+18) = 19.08$

Note that the actual pitch circles preserve the velocity ratio :
19.08/12.72 = 16.92/11.28 = 18/12 = 1.5

Also from the geometry and (**8**) the addendum diameters are D_{a1} = 2(6.5 +1.0) = 15.0, D_{a2} = 2(9.4 +1.0) = 20.8, while the actual pressure angle is $\alpha' = \arccos(D_{oi} / D'_i) = \arccos(16.92/19.08) = 27.5^o$

Without profile shift, the choice of center distance is limited by the modules available from the standard list and by integral tooth numbers, however the profile shift sum in (**9**) provides flexibility in the choice of (extended) center distance - invaluable in a coaxial reduction for example.

The meshing of typical gears at extended centers is further detailed in Figs I and J which are particularized for the 12:18 drive of the foregoing example.
Fig I shows the initial no-load situation. The gears are mounted so that the extended pitch circles (of radii 6.5 and 9.4 in the example) are mutually tangential with the pinion tooth symmetrically disposed with respect to the wheel inter-tooth space. Tangency of the extended pitch circles implies that the generating rack (shown dashed) is simultaneously tangent to the tooth profiles of both gears, and leads to a gap - an absence of contact - between the pinion tooth and both adjacent wheel teeth.

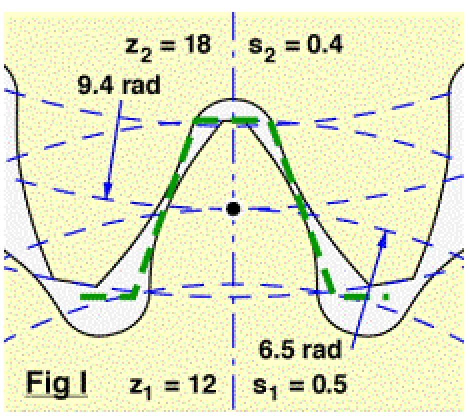


Fig I

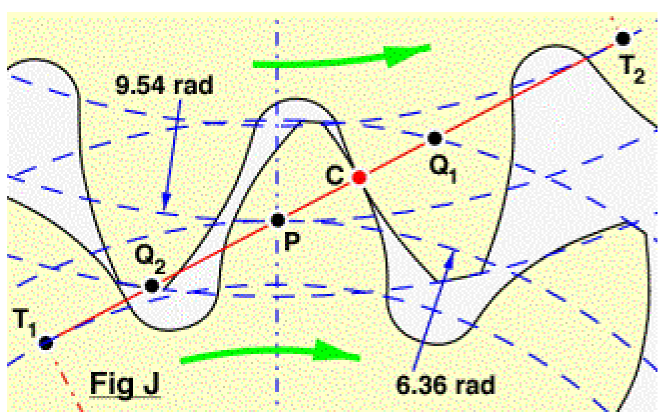


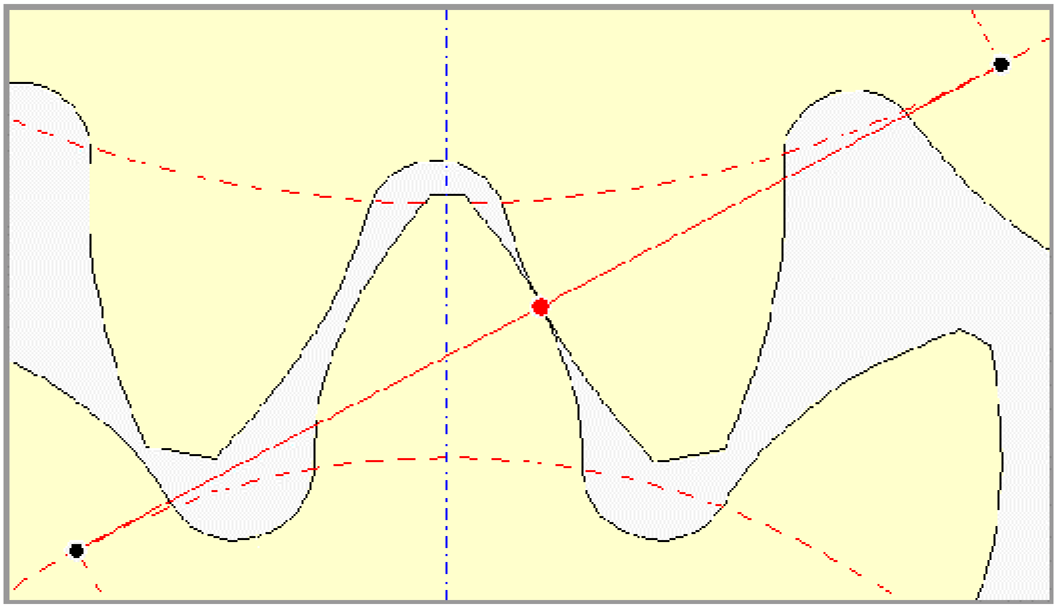
Fig J

A subsequent on-load state of affairs when the pinion rotates clockwise is shown in Fig J. The taut generating cord (line of action) extends between the tangent points T_1 and T_2 on the base circles. The cord's intersection with the line of centers defines the actual pitch point P - exactly as in Fig H - and in turn the actual pitch circles whose diameters in the particular case are 12.72 and 19.08.

A pinion tooth touches a wheel tooth at the contact point C (the knot) which moves up the line of action and along the teeth faces as rotation proceeds. Since contact cannot occur outside the teeth, it takes place along the line of action only between the points Q_2 and Q_1 on the line of action and inside both addendum circles. The line segment Q_2Q_1 is named the ***path of contact.***

The animation shows clearly :

- the contact point marching along the line of action
- the path of contact bounded by the two addenda
- the orthogonality between line of action and involute tooth flanks at the contact point
- how load is transferred from one pair of contacting teeth to the next as rotation proceeds
- relative sliding between the teeth - particularly noticeable at the beginning and end of contact
- guaranteed tooth tip clearance due to the dedendum exceeding the addendum
- a significant gap between the non-drive face of a pinion tooth and the adjacent wheel tooth



Gears formed by a milling cutter instead of being generated by a rack cutter as above, may exhibit undercutting or interference (which prevents complete rotation of the two gears due to teeth binding) - these faults result from Q_2 for example lying to the left of T_1 in Fig J.

The gap between the non-drive face of the pinion tooth and the adjacent wheel tooth is known as ***backlash.*** If the rotational sense of the pinion were to reverse, then a period of unrestrained pinion motion would take place until the backlash gap closed and contact with the wheel tooth re-established impulsively. Shock in a torsionally vibrating drive is exacerbated by significant backlash, though a small amount of backlash is provided in all drives to prevent binding due to manufacturing or mounting inaccuracies and to facilitate lubrication. Backlash may be reduced by subtle alterations to tooth profile or by shortening the center distance from the extended value, however we consider gears meshing only at the extended center distance.

Continuous motion transfer requires two pairs of teeth in contact at the ends of the path of contact, though there is only one pair in contact in the middle of the path, as in Fig J. The average number of teeth in contact is an important parameter - if it is too low due to the use of inappropriate profile shifts or to an excessive center distance for example, then manufacturing inaccuracies may lead to loss of cinematic continuity - that is to impact, vibration and noise. The average number of teeth in contact is also a guide to load sharing between teeth; it is termed the ***contact ratio,*** ε_γ, given by :-

ε_γ = length of path of contact / distance between teeth along the line of action

= Q_2PQ_1 / base pitch, p_o and for extended centers with (**6**) for the 20° system :

(**10**) $(2\pi \cos\alpha)\, \varepsilon_\gamma \;=\; \Sigma_{i=1,2} \sqrt{[(z_i + 2(1+s_i))^2 - (z_i \cos\alpha)^2]} - \sqrt{[(\Sigma z + 2\, \Sigma s)^2 - (\Sigma z \cos\alpha)^2]}$

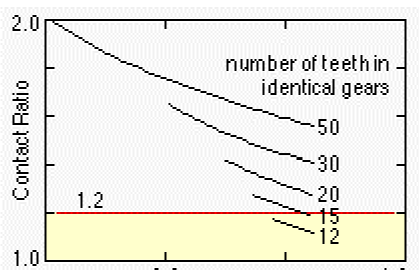


FIG K

Gears having a contact ratio below about 1.2 are not normally recommended as the gears themselves, their shafts and bearings would all require especial care in design and manufacture to preserve conjugacy. The effect of tooth number on contact ratio is shown in Fig K for two identical gears with a profile shift which varies between the practical limits of Fig G. Evidently it becomes more difficult to achieve an acceptable contact ratio as tooth numbers decrease - hence the statement above that 12 teeth are the usual minimum. The plot further indicates that contact ratio increases as profile shift decreases.

In design, Σs is often taken as zero; or, if the center distance is closely specified, by the selection of Σs for use in (**9**) - the individual profile shifts may be estimated from a specified Σs as follows :-

(**11**) $s_1 = [\lambda z_2 + (\Sigma s - \lambda) z_1] / \Sigma z$; $s_2 = \Sigma s - s_1$;
where $0.5 \leq \lambda \leq 0.75$
which tends to balance the strengths of the gears.

EXAMPLE

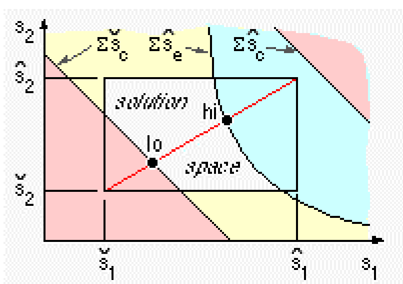
Outline a procedure for selecting a spur gear pair on cinematic/geometric grounds, for the front end of a general design program whose later segments could examine fatigue and reliability.

Input data are :

minimum and maximum bounds of speed ratio (≥ 1) and of extended center distance (mm)

while output is :

module (mm), with corresponding tooth numbers, z_1 & z_2 (say 12 ≤ z_1 ≤ 25), and profile shifts, s_1 & s_2, to give a speed ratio and extended center distance falling within the specified bounds, and a contact ratio no less than some stipulated critical value (eg. 1.2).



The procedure might take the form of a number of nested loops, each assigning a trial value to one of the design variables. For example the outermost loop would select a module from the standard list and the next would assume a trial pinion tooth number between the limits suggested above. The wheel tooth number would be assigned in the next inner loop, corresponding to the pinion teeth and a speed ratio between the specified limits.

The innermost loop would consider both profile shifts simultaneously and is best understood through a graph of s_2 versus s_1, realizing that the module and tooth numbers have been settled in outer loops prior to this aspect being examined. The tooth numbers will define practical limits to the profile shifts (**7**), hence solutions must lie within the known rectangular region of the graph.

In addition to these limitations, there are further bounds, $\Sigma\, s_c^{min}$ and $\Sigma\, s_c^{max}$, corresponding to the center distance limits from (**9**) - these appear as the straight boundaries superimposed on the graph; and a high bound, $\Sigma\, s_e^{max}$, corresponding to the critical contact ratio via (**10**), which appears as the curved boundary. The actual disposition of all these boundaries on the graph - which may or may not enclose a viable solution space - depends of course on the problem in hand and the values of module and tooth numbers assigned in outer loops. Having defined any such solution space, the procedure could output any solution lying within it - any point on the line between the two extreme points 'lo' and 'hi' would appear to be a suitable compromise - or alternatively (**11**) might be implemented.

Gear failure – reliability

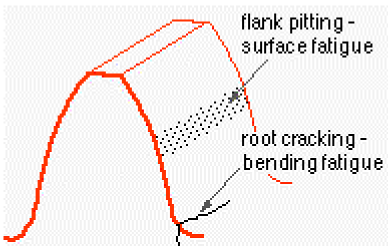
Photo elasticity is used to estimate the stresses in a loaded element. Polarized light is transmitted through the transparent model of a slice of the element. The model deforms under load, causing the transmitted beam to interfere with a reference beam. The resulting interference pattern consists of a series of bands, each representing an area of constant prototype stress whose value can be predicted by calibration and by counting bands from unloaded areas. Qualitatively, the closer these bands are bunched up in the model, the higher are the prototypical stresses.

The photo elastic results of contacting gear teeth shown here have important implications for tooth safety :

- high stresses occur where the teeth contact one another;
- bending stresses occur at the root of the lower cantilevered tooth whose fillet radius is large
- much higher bending stresses occur at the root of the upper tooth whose fillet radius is small: this is yet another manifestation of stress concentration in way of geometric singularities.

These stresses are alternating since a particular tooth is loaded only briefly during one rotation of the gear. Gear failure is therefore very much a case of fatigue, though a one-off static overload obviously may cause failure if sufficiently large.

The following treatment of gear reliability is a gross simplification of the American Gear Manufacturers Association (AGMA) code of practice, AGMA 2001. The treatment aims to demonstrate one approach to fatigue design rather than to transform the reader into a gearing expert.



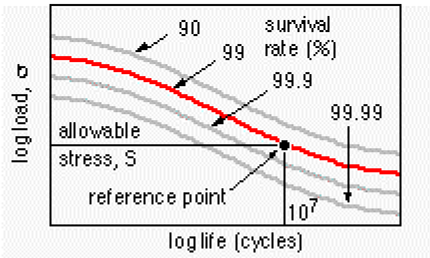
Apart from one-off overloads, there are three common modes of tooth failure

- ***bending*** fatigue leading to root cracking,
- surface contact fatigue leading to flank ***pitting,*** and
- lubrication breakdown leading to ***scuffing.***

We shall not examine the last of these - it involves the relative velocity and pressure between the teeth together with the lubricant viscosity which is affected by the complex interaction between the heat generated due to the gears' inefficiency, the thermal inertia and dissipative properties of the complete gearbox, and the temperature of the surrounds.

Since fatigue is prevalent in the other two failure modes, the AGMA employs a ***reliability*** approach rather than a safety factor approach in assessing a tooth's tolerance of damage - its ***bending strength*** and its ***pitting resistance.*** In a reliability analysis, knowledge of a component's load-life (S-N) relationship enables a load to be considered from the point of view of its effect on component **life** rather than whether it leads to total failure or to total non-failure. A smaller load increases life, a larger load reduces life; whether the safety factor is greater or less than one is irrelevant - indeed the whole concept of safety factor is inappropriate in the context of reliability. In the succeeding sections therefore, we first determine the effective damaging tooth force F* due to the power transmitted and the shock / vibration intrinsic to the gears' manufacture and operational environment. The stress, , due to this force is then deduced from bending theory, or from Hertzian contact theory in the case of pitting, and the resulting life follows from the load-life curve appropriate to bending or to contact fatigue as the case may be.

It will be recalled that the load-life diagram for a particular material and given type of loading is generally of the form illustrated, with curves corresponding to different survival rates being approximately parallel to one another - the 99% survival curve eg. implies that one sample in a hundred fails to reach the life given by the curve for a particular load. We shall consider only steel materials (carbon steels induction- or through-hardened, nitride alloy steels etc). Rather than provide the complete load-life diagram for each steel, the AGMA chooses a reference point on the curve corresponding to 99% survival rate after 10^7 **unidirectional** loading cycles, and cites the corresponding allowable stress, S, as a representative property of the steel.



So, for any given load ( ) the life and survival rate (reliability) may be correlated through :

(**12**) $\sigma = S \times$ life factor (K_L or C_L) / reliability factor (K_R) where :

-The ***reliability factor***, $\mathbf{K_R}$, caters for survival rates other than 99%. Since the survival contours are essentially parallel to one another on a logarithmic scale, then simple multiplying factors enable load correlation, as tabled :

- The ***life factor***(K_L for bending, C_L for pitting) caters for lives other than 10^7 cycles. Since the load-life diagrams for all the steels considered are of the same shape essentially, normalizing by the allowable stress will result in a unique K_L (or C_L) -versus- life curve for all steels which will be presented later.

These aspects when combined will result in gear tooth design equations, one for bending strength and one for pitting resistance, which embrace load (transmitted power), dimensions (primarily size characterized by module), material (allowable stress) and life rather than safety.

RELIABILITY FACTOR	% Survival	K_R
Fewer than one failure in 10,000	99.99	1.50
Fewer than one failure in 1,000	99.9	1.25
Fewer than one failure in 100	99	1.00
Fewer than one failure in 10	90	0.85

Tooth forces

It will be recalled that the free body of a gear pair necessitates identical senses of the torques on pinion and wheel - Fig L below repeats the essentials of Fig J with clockwise torques. Two teeth are in contact at the point C on the line of action, which is tangential to the two base circles and inclined at the pressure angle, a. (Strictly, parameters corresponding to extended centers should be considered here, but this sophistication is not necessary for the present introduction to gear reliability.)

On the right of Fig L, separate incomplete free bodies of the gears are shown to highlight the equal-and-opposite contact forces, F, acting on the gears at the contact point. Presuming negligible friction, these forces are normal to the involutes i.e.. along the line of action. Using the principle of transmissibility, the contact force on the lower gear has been shown moved along the line of action to act at the pitch point, P, where it is resolved into a tangential component F_t and a radial component F_r. The same can be done with the upper gear - in both cases the torque is equilibrated by the contact force, so using (**5**) :

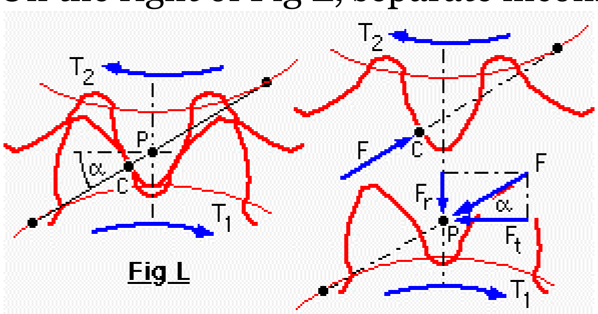


Fig L

$$(\mathbf{13})\quad T_i = F R_{oi} = F (R_i \cos\alpha) = (F \cos\alpha) R_i = F_t R_i \ ; \quad i = 1{,}2$$

where F_t is the useful tangential component of the contact force at the pitch point. The radial component, F_r, is useless and tends only to separate the gears (or pitch cylinders) - note that (**13**) tallies with (**1**) which neglected F_r.

The pressure angle in Fig L should more correctly be the **extended** value, and correspondingly (**13**) should really be cast in terms of the

actual pitch radius. We neglect such niceties in the present general development.

The steady power transferred, P, is :-

(**14**) $P = \omega_i T_i$; $i = 1, 2$

the power transfer in rotational terms equals

$$= \omega_i F_t R_i = F_t v$$

the power transfer in translation terms, from (**1**)

Although the positive drive ensures that there is no speed loss, there will be torque and power losses due to sliding. Spur gear efficiencies exceeding 99% have been reported, however a value of around 96% is more appropriate for run-of-the-mill design - this allows for bearing and aerodynamic losses in addition to tooth friction.

Gear analysis or design usually starts with a known time-averaged (i.e.. steady) power transfer - these last equations enable the uniform transmitted load F_t to be found. The corresponding failure-producing (i.e.. damaging) load F^* will be greater than F_t because of shock, vibration caused by less-than-perfect tooth profiles and mounting rigidity, and so on. We write :

(**15**) $F^* = K_a K_v K_m F_t$

in which the various empirical K-factors, all ≥ 1, each reflects the extra damage caused by a particular, separately identifiable practical non-uniformity, as follows :

$\mathbf{K_a}$ is an ***application factor*** to allow for the non-uniformity of input and/or output torque inherent in the machinery connected to the gears.

Typical values are :

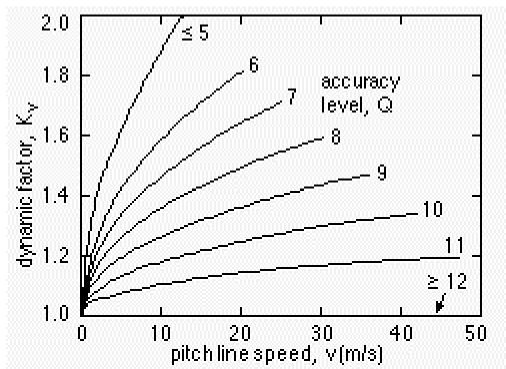
APPLICATION FACTOR K_a FOR REDUCTION GEARS	*Driving Machinery*	*uniform* electric motor steam turbine gas turbine	*light shocks* multicylinder combustion engine	*heavy shocks* single cylinder combustion engine
uniform	generator, belt conveyor, light elevator, electric hoist, machine tool feed drive, ventilator, turbo-blower, turbo compressor, mixer (constant density)	1	1.25	1.5
medium shocks	machine tool main drive, heavy elevator, crane turning gears, mine ventilator, mixer (variable density), multi cylinder piston pump, feed pump	1.25	1.5	1.75
heavy shocks	press, shear, rolling mill drive, heavy centrifuge, heavy feed pump, pug mill, power shovel, rotary drilling apparatus, briquette press	1.75	2 or higher	2.25 or higher

K_v is a ***dynamic factor*** which accounts for internally generated tooth loads induced by non-conjugate action of the teeth, by gear mesh stiffness variation, by gear imbalance and the like. The AGMA code defines a transmission accuracy level number, Q, which reflects these latter (high Q implies low excitation, a low Q refers to low accuracy and high vibration) and suggests the K_v factors shown below, expressing them empirically as follows (with velocity v in m/s) :

$Q \leq 5$ — **$K_v = 1 + \sqrt{v}/3.6$ gears of this class are limited to speeds under 13m/s.**

$6 \leq Q \leq 11$ — **$K_v = (1 + \sqrt{v} / (7.6 - 4 B))^B$ where $B = 0.25 \square (12 - Q)^{2/3}$**

$Q \geq 12$ — **$K_v = 1$ - this corresponds to ultra precision gearing, or to situations where dynamic loads have been accurately forecast and separately allowed for.**



The selection of a suitably realistic accuracy level requires considerable experience.
K_v does not account for vibration induced by running near shaft critical speeds or by other resonance's - these should be avoided or separately catered for.
Note that the AGMA defines K_v as the reciprocal of the K_v used here.

K_m is a ***load distribution factor*** which reflects the non-uniformity of tooth loading over the face width of the teeth, arising from gear and mounting inaccuracy, elastic deformation of shafting and the like.
For a face width f (mm) the AGMA proposes the empiricisms :

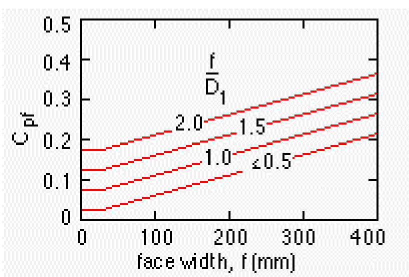
$K_m = 1 + C_{pf} + C_{ma}$ where

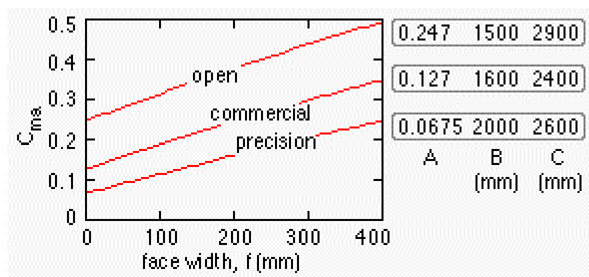
C_{pf} is a pinion proportion factor which reflects misalignment due to load induced elastic deformations of the pinion; it may be evaluated (with dimensions in mm) from :

$C_{pf} = 0.1 \max(0.5, f/D_1) + \max (0, f - 25)/2000 - 0.025$ graphed below.

C_{ma} is a mesh alignment factor, which accounts for misalignment due to causes other than elastic deformation such as inaccurate location of shaft bearings. It is approximated :

$C_{ma} = A + f/B - (f/C)^2$ in which the constants A, B & C for various classes of gears - open, commercial and precision enclosed - are shown below.





The face width should be reasonably proportioned to other gear dimensions. If a tooth is too wide it may bend excessively across its width, if it is too narrow then an uneconomically large diameter must be provided to compensate for lack of width. Proportions may be expressed as :

(**16**) $f = \beta m$ where, usually, $9 \le \beta \le 15$ for economic gears

These limits should not be regarded as inviolable, but costs should be expected to escalate if they are exceeded.

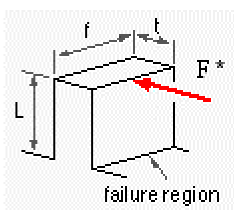
We have seen how to evaluate (**15**) the damaging fatigue load on a gear tooth, F^*, due to the torque transmitted combined with the quality of the gear's manufacture and its operating environment.

To estimate the tooth's life as a result of each failure mechanism, this load is applied to a geometrically simplified model of the tooth, viz. :

- a rectangular cantilever in **bending**, or
- one of a pair of cylinders in Hertzian **contact**

to determine the corresponding stresses in the simplified model. Stresses in the geometrically complex tooth are projected from these model stresses via a ***geometry factor***- obtainable by comparison with exhaustive testing by the AGMA and others - thus enabling the life to be estimated from the S-N curve.

Bending strength

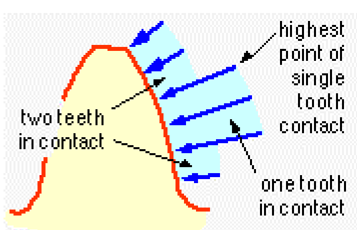


As a first approximation the tooth is modeled as a cantilever of rectangular cross-section and length L and subjected to the damaging load, F*. The maximum bending stress at the criticalfailureregionis:

$$\sigma = M\,y / I = (F^* L)(t/2) / (f\,t^3/12)$$

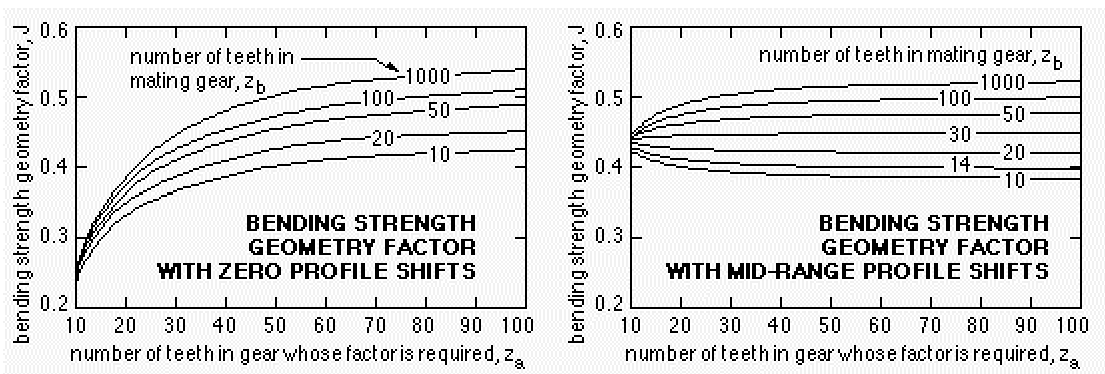
So :

(**17**) $\sigma = F^*/mfJ$



in which the AGMA proportionality constant, J, is known as the ***bending strength geometry factor*** and is evaluated by photo-elastic or finite element methods. It correlates the stress in the simple cantilever model with the actual maximum stress in a gear tooth, which is influenced by stress concentration, by variation in load direction and moment arm length, and by load sharing between teeth as the contact point moves along the tooth as suggested by the sketch - though the two statically indeterminate teeth pairs prevent the sudden steps sketched.

The factor's value depends upon the number of teeth and profile shift of both gears and is obtainable from AGMA 2001 or calculated by the Macintosh program "***Steel Spur Gears***" is based on AS 2938. Its variation is graphed below when both profile shifts are zero and when they are both at the middle of the practical range (Fig G).



The graphs demonstrate that without profile shifts, J-factors decrease and stresses (**17**) increase markedly with decreasing tooth numbers - however with small numbers of teeth, significant increases of J-factor and reductions of stress result from incorporating suitable profile shifts. When both profile shifts are mid-range the J-factor may be approximated by :

(**18**) $J \approx \{ 0.535 + (- 3.030 + 14.5 / z_b) / z_b \} + \{ - 0.904 + (28.43 - 147.0 / z_b) / z_b \} / z_a$

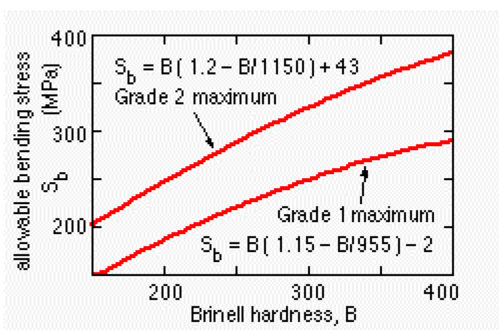
in which z_a is the number of teeth in the gear whose J-factor is required and z_b is the number of teeth in the mating gear.

Combining (**12**) (**15**) (**16**) and (**17**) leads to the final design equation for tooth bending strength which is applicable to each of two mating gears :

(**19**) $\quad (\pi m^3 \beta N_1 z_1 / K_a K_v K_m K_R P) . (J S_b K_L)_i \geq 1; i = 1,2$

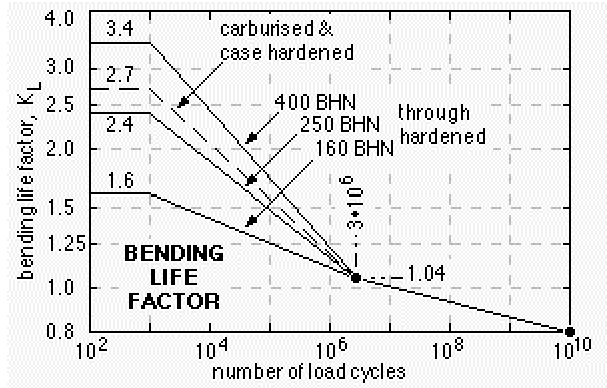
where S_b is the ***allowable bending stress*** for the gear tooth material.

The first bracketed term of (**19**) is common to both gears, noting that $N_1 z_1 = N_2 z_2$. In the analysis situation, the two bending life factors K_L and hence the lives of the two gears may be evaluated individually from the equations (**19**). In design, a trial module or face width is often selected which must satisfy the inequalities (**19**) for both gears, whose minimum life would be specified - the weaker gear, that with the lesser $J.S_b.K_L$ product, dictates the minimum module acceptable.



As an indication of the information available in AGMA 2001, the maximum allowable bending stresses for two grades of through -hardened carbon steel gears are shown here as functions of Brinell hardness - the differences between the grades are metallurgical as explained in AGMA 2001, which gives also allowable stresses for other materials. These stresses are the result of extensive gear testing; attempts to deduce them from other material properties such as the reversed bending endurance limit are ill advised.

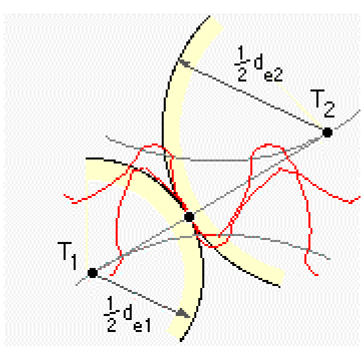
AGMA proposals for the bending life factor, K_L, of steel gears (except nitrided) are as shown - however values are influenced by factors other than the Brinell Hardness.



We shall consider application of this theory once the theory for the other failure mechanism, pitting, has been developed.

Pitting resistance

Flank pitting is caused by alternating normal pressure on the contact surfaces of the teeth. It is found to occur most frequently at the pitch circle - where relative sliding of the teeth is zero and the hydrodynamic lubricant film tends to break down - so attention is focused on this location. From previous work on Hertzian contact stresses, the maximum normal pressure, p, between contacting cylinders of equivalent diameters d_{e1} and d_{e2} and length L, and on which the normal force is F, is :



(20) $$p^2 = (2 E^* F / \pi L) (1/d_{e1} + 1/d_{e2})$$

where $1/E^* = (1 - \nu_1^2) / E_1 + (1 - \nu_2^2) / E_2$

i.e.. $E^* = E / 2 (1 - \nu^2)$

if the materials are identical, and
= 113 GPa

if the materials are both steel.

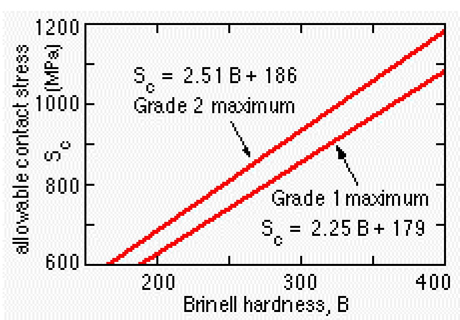
This simplified model is adapted to involute teeth contacting at the pitch point as follows :

- The radii of the equivalent cylinders are the radii of curvature of the contacting teeth at the pitch point - the distances T_1P and T_2P of Fig J (in which the contact point C has moved to the pitch point). So $d_e = 2\,T\,P = 2\,R \sin\alpha = D \sin\alpha = m\,z \sin\alpha$

- The model cylinders' common contact length is the gears' face width, i.e.. $L = f = \beta\,m$ using (**16**)

- The maximum surface stress, σ, in the gear materials at the contact is proportional to the maximum pressure on the model cylinder's surface, i.e.. $\sigma \propto p$

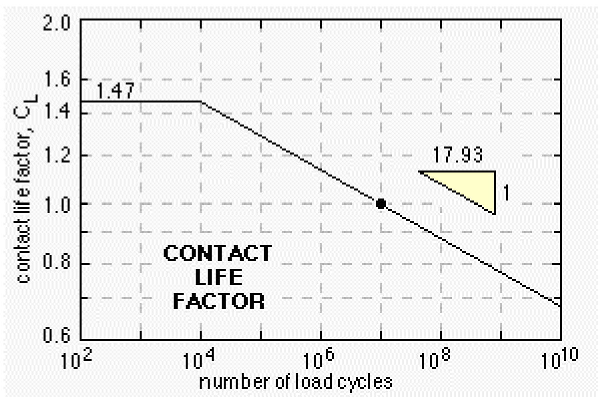
- The contact force on the cylinders, F, corresponds to the normal force between the teeth, Fig L. Since it is the effective damaging tangential force, F*, which is relevant here, the corresponding damaging normal force is $F = F^* \sec\alpha$ from the triangle of forces, Fig L.



Making these adjustments to (**20**), invoking (**12**) (**15**) (**16**) and introducing a constant of proportionality - to correlate simple model with actual tooth - leads to the final design equation for pitting resistance :

(**21**) $\left(\pi^2 m^3 \beta N_1 z_1^2 I / K_a K_v K_m K_R^2 E^* P \right) . \left(S_c C_L \right)_i^2 \geq 1$; $i = 1,2$

in which S_c is the ***allowable contact*** stress for the gear tooth material. Values are cited in AGMA 2001 - maximum values for two grades of through-hardened materials are shown here :



C_L is a gear's contact life factor - the normalized load-life curve under repeated contact loading as explained above. The AGMA relationship for through-hardened steels is given here. The implications of the slope of the (log -log) load-life plot must be appreciated - a load increase of only 2% leads to a life reduction of 30%.

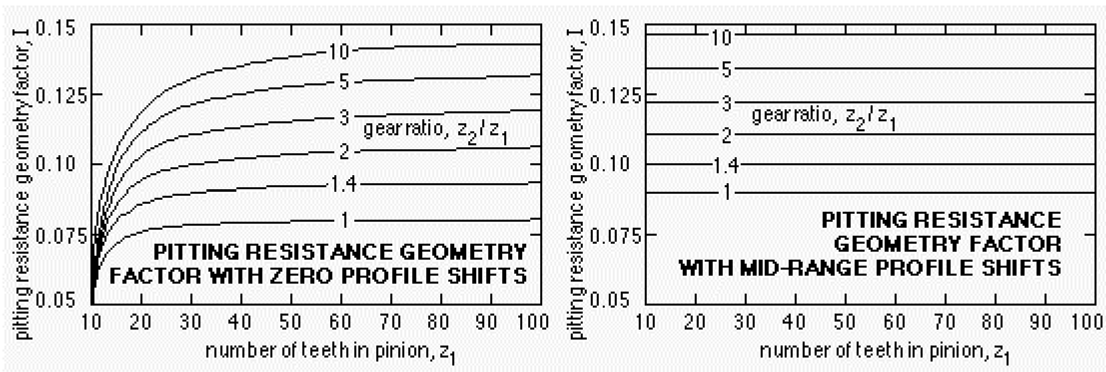
The contact design equation (**21**) is applied to the two gears individually in the same manner as is the bending design equation

(**19**) - again, the first bracketed term is common to both gears. In applying (**21**) to design therefore, the gear set will be limited by the weaker gear - that which has the lesser $S_c C_L$ product.

The remaining parameter in (**21**) as yet unexplained is the ***pitting resistance geometry factor,*** I, which correlates the stresses between the simple Hertzian model and actuality. It is analogous to J, and like J its value depends upon tooth numbers and profile shifts, and is obtainable from AGMA 2001 or calculated by the Macintosh program "***Steel Spur Gears***".
The value is the same for both gears and tends to $\sin 2\alpha / 4(1+z_1/z_2)$ as the number of pinion teeth z_1 becomes large. It is plotted below corresponding to the profile shifts of both gears being either zero or in the middle of the practical range (Fig G). When profile shifts are mid -

range the I-factor is essentially dependent only upon the gear ratio ρ, and may be approximated by :

(**22**) $I \approx (0.0404 + 0.1127\,\rho) / (1 + 0.70\,\rho)$ in which $\rho = z_2 / z_1 \geq 1$



Application of fatigue theory

Both fatigue design equations (**19**), (**21**) indicate that the capacity (the maximum continuous power transferable) is proportional to the cube of the module - i.e.. a slight increase in module enables a large increase in capacity. The following examples illustrate application of the equations to analysis and to design.

EXAMPLE (analysis)

A crane traversing gear set is driven at 425 rev/min by an electric motor through primary gearing. The open gear set transmits 75 kW. The gears are manufactured to a transmission accuracy level of 5, with 8 mm module and 110 mm face width from through -hardened grade 1 steel, the 20 tooth pinion to 250 BHN and the 53 tooth wheel to 180

BHN. The profile shift of each gear is mid-range.
What life can be expected of the gear set at 99.9% reliability ?

The life of the pair will be the minimum life for the pinion and for the wheel due to both potential failure mechanisms - bending and contact.

Evaluating the various common factors first :

K_a

Referring to the table of Application Factors, the input is uniform and the output rather less damaging than medium shock, so assume a value of $K_a = 1.15$

K_v

The velocity is needed to evaluate this . . . . $D_1 = m.z_1 = 8^*20 = 160$ mm, so the pitch line speed is $v = \pi DN = \pi\ 0.160^*425/60 = 3.56$ m/s. Since Q =5 the corresponding dynamic factor is $K_v = 1 + \sqrt{v}/3.6 = 1 + \sqrt{3.56}/3.6 = 1.52$ - which tallies with the K_v graph.

K_m

The geometry is known, so the parts of K_m may be evaluated - $f/D_1 = 110/160 = 0.688$ which gives a pinion proportion factor $C_{pf} = 0.1 \max(0.5, 0.688) + \max(0, 110 - 25)/2000 - 0.025 = 0.086$.
The mesh alignment factor for open gearing of this face width is $C_{ma} = 0.247 + 110/1500 - (110/2900)^2 = 0.319$.
So the load distribution factor is $K_m = 1 + 0.086 + 0.319 = 1.41$

K_R

corresponding to a reliability of 99.9% is 1.25 I
Since both gears' profile shifts are mid-range, then for a ratio of 53/20 = 2.65, I = 0.119 approximately, from (**22**)

Apply the bending design equation (**19**) to find the life factors and hence the lives of the two gears. Calculations are shown for pinion only.

		pinion	*wheel*
tooth number		**20**	**53**
bending strength geometry factor, J, for mid-range shifts, from (18)		**0.462**	**0.423**
Brinell hardness number		**250**	**180**
allowable bending stress, S_b, from diagram (grade 1)	**MPa**	**220**	**171**
$K_L = K_a K_v K_m K_R P / p m^2 f N_1 z_1 J S_b$			
1.15*1.52*1.41*1.25*75 E3 / p*0.008^{2}*0.11*(425/60)*20*0.462*220E6	**-**	**0.726**	**1.020**
Nm/s $1/m^2$ 1/m s *check units !* **m^2/N**			
corresponding life from K_L plot	**Mc**	**205000**	**5.4**
dividing by rotational speed for life in units of time	**khr**	**8020!!**	**0.56**

Apply contact design equation (21) in a similar manner to obtain life factors, C_L, and hence the lives of the two gears in units of time.

allowable contact stress, S_c, from diagram (grade 1)	**MPa**	**740**	**585**
$C_L = \sqrt{(K_a K_v K_m E^* P / m^2 f N_1 I)} K_R / p z_1 S_c$			
√(1.15*1.52*1.41*113E9*75E3/0.008^{2}*0.11*(425/60)*0.119)*1.25 / p*20*740E6	**-**	**1.60**	**2.02**
√(N/m^2 Nm/s $1/m^2$ 1/m s) m^2/N			
corresponding life from C_L plot	**Mc**	**??**	**??**

The maximum possible contact life factor is 1.47 (refer to the C_L graph); since the calculated values of C_L for both pinion and wheel exceed this, there is no way that the desired reliability can be achieved with a reasonable life. Note that C_L K_R above, so a substantially reduced probability of survival would result in gears with non -zero lives. But the probability of the gears surviving 75 kW with an economic life is low.

This example reveals the general truism - pitting of steel gears is much more damaging than bending, i.e.. the life of the gear set is more often than not limited by contact fatigue. A gear's capacity is proportional to the **square** of its allowable surface contact stress (**21**) so that surface hardening of the tooth flanks (by case- or flame- hardening, by nitriding or carburising etc.) leads to a better balanced solution in which the contact -limited fatigue life may be increased to approach the bending -limited fatigue life. AGMA 2001 cites properties for surface -hardened steel gears - allowable contact stresses exceeding 2 GPa are possible.

EXAMPLE (synthesis)

Select suitable gears for the final stage of a commercial quality 100 kW gearbox whose input and output shafts rotate at 200 and at about 55 rev/min respectively. The box will be interposed between an electric motor and a variable density mixer, and should have a life of 16 khr with 99% reliability.

The general design approach, once materials are tentatively selected, is trial -and -error choice of tooth numbers which must satisfy the speed ratio limits and which should not be too large otherwise compactness suffers. For given tooth numbers, the necessary module and face width are obtainable from design equations (**19**) and (**21**).
We shall exemplify design calculations for one particular candidate $z_1 = 18$ ($z_2 = 65$) and mid-range profile shifts.

An application factor of 1.25 seems appropriate, though in practice further details of the mixer's charge would enable more certainty in this choice.

From the lessons learned in the previous example, both gears will be made from grade 2 steel - the pinion will be flame hardened to 52 HRC while the wheel will be through hardened to 360 BHN. The pinion's allowable stresses must be **obtained directly** from AGMA 2001 -

they are S_c = 1320 MPa, S_b = 380 MPa.
As contact rather than bending is expected to be critical, the gears will be designed for contact, then checked in bending.

		pinion	*wheel*
tooth number	-	18	65
allowable contact stress, S_c (grade 2)	MPa	1320	1090
life (16 khr*rotational speed)	Mc	190	53
life factor corresponding to this life, from graph, C_L	-	0.849	0.912
product $S_c C_L$ to deduce the weaker gear	MPa	1120	994

The wheel is weaker than the pinion in contact fatigue - it might pay to use a somewhat less sophisticated pinion material / heat treatment, but any decision will be delayed until all results (eg. bending too) are to hand.
The contact design equation (**21**) will now be solved to obtain an approximate idea of the module - and to this end, middle-of-the-road values will be assumed for those parameters which depend upon module for evaluation. Thus take :

- β = 12, midway between the limits of (**16**)

- K_v = 1.1 since the pinion rotates fairly slowly at 200 rev/min and so the pitch speed will be low
- K_m = 1.2 since the mesh alignment factor cannot be less than 0.13 for commercial enclosed gears.

It is known that K_R = 1, corresponding to a reliability of 99%, and that the contact geometry factor I = 0.127 from (**22**) corresponding to a ratio of 65/18. So applying (**21**) :

$$m^3 = K_a K_v K_m K_R^2 \; E^* \; P / \pi^2 \beta \; N_1 \; z_1^2 \; I \; (S_c C_L)^2$$
$$1.25 * 1.1 * 1.2 * 1^2 * 113E9 * 1E5 / \pi^2 * 12 * (200/60) * 18^2 * 0.127 * (994E6)^2$$

whence $m = 10.5$ mm
N/m^2 Nm/s s m^4/N^2

A standard module of either 10 or 12 mm is indicated.
A modules of $m = 12$ mm will be tried - this will enable factors to be tightened up when using the design equation once again, this time to evaluate the face width. Thus :

K_v
$D_1 = m.z_1 = 12*18 = 216$ mm; $v = \pi D_1 N_1 = \pi*0.216*(200/60) = 2.26$ m/s (i.e.. very low as expected)
Assuming a transmission accuracy level of $Q = 8$, then, from the graph, $K_v = 1.26$
K_m
Assuming an average value of $\beta = 9$ (on the low side to offset the choice of module on the high side of the range around 10.5 mm).
Then the face width $f = 12*9 = 108$, say 110 mm which leads to $C_{pf} = 0.068$; $C_{ma} = 0.194$ and so $K_m = 1 + 0.068 + 0.194 = 1.26$

So, solving (**21**) for the face width ratio, β, and for the weaker gear :

$$\beta = K_a K_v K_m K_R^2 E^* P / \pi^2 m^3 N_1 z_1^2 I (S_c C_L)^2$$

$$1.25*1.26*1.26*1.0^2*113E9*1E5/\pi^2*0.012^3*(200/60)*18^2*0.127*(994E6)^2$$

whence $\beta = 9.7$

This is within the usual economic range $9 \leq \beta \leq 15$ so looks like a practical candidate. The face width is $f = 9.7*12 = 116$ mm. Take $f = 120$ mm - sufficiently close to 110 mm to obviate need to update K_m for refined calculations.
The chosen parameters so far (m = 12 mm, f = 120 mm) will now be checked via the bending equation (**19**), from which the bending lives of the two gears will be deduced, thus :

$(J\ S_b\ K_L)_i\ =\ K_a\ K_v\ K_m\ K_R P / \pi\ m^2\ f\ \ N_1\ z_1$

$1.25*1.26*1.26*1.0*1E5/\pi*0.012^2*0.120*(200/60)*18\ =\ 60.9$ Mpa

Nm/s 1/m^2 1/m s

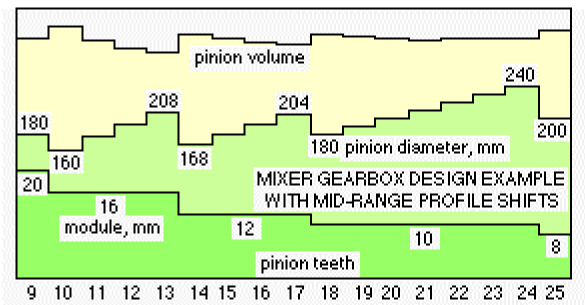
		pinion	*wheel*
product $J\ S_b\ K_L$	MPa	60.9	60.9
bending strength - S_b, pinion from AGMA 2001; wheel corr. to 360 BHN	MPa	380	360
bending strength geometry factor, J, from (**18**)	-	0.464	0.415
$K_L = (J\ S_b\ K_L) / J\ S_b$	-	0.345	0.408

It may be seen from the graph that the lives corresponding to these K_L factors are very much greater than those specified, i.e.. bending again is less critical than contact.

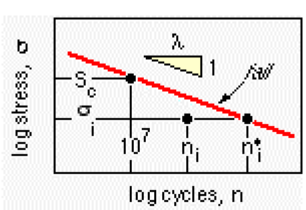
Summarizing, the candidate considered utilizes a module of 12 mm, a face width of 120 mm, gears of 18 and 65 teeth with mid-range profile shifts and materials as specified above. In practice, further design candidates with various steels and heat treatments etc. would be carried out to aid an economic assessment. The program **Steel Spur Gears** is based on the foregoing theory and may be used to assist in the design task - a **dialogue with the program** for these examples is provided.

The design example adopts flame hardened teeth on the pinion, however the life of the chosen candidate is also dictated by contact. Clearly different materials and surface treatments are design variables which, together with different tooth numbers, characterize solution candidates in a real problem - and the optimum choice is mainly a question of economics. To put lives in perspective, a life of 1000 khr represents more than a century of continuous running - all day, every day!

The graph shows the results of repeating the design example in an effort to obtain the most compact box. Trial pinion tooth numbers increasing from 9 to 25 lead to modules which decrease monotonically from 20 to 8 mm, pinion diameters (D = m*z) which generally increase and face widths which generally decrease. The surprising outcome of this study is that pinion (and hence gearbox) volume is essentially constant over the range studied - that is small pinion tooth numbers are not necessary for compactness. However the analysis recognizes only two fatigue failure modes; it ignores eg. the possibility of tooth yield due to a single large overload - a possibility which would be reduced by the larger teeth associated with small tooth numbers. AGMA 2001 is much more comprehensive than the fatigue overview given above; it caters for overloads and other eventualities, and introduces a number of additional factors.



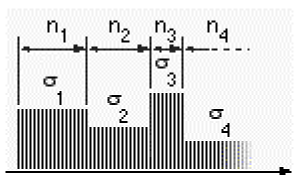
Periodic duty



If a component is subjected to a repeating stress of constant magnitude, σ_i , then it will fail after n^*_i application cycles - that is its life is n^*_i cycles. At some stage prior to failure it will have been subjected to n_i cycles, so the cumulative damage at that stage (the fraction of its life used up) is n_i/n^*_i. These lives are shown here on a stress-life diagram which, in the case of steel gears and when normalized by the allowable contact stress S_c, is just the contact life factor C_L examined above. It follows that for steel gears which fail by pitting :

(23) $n^*_i = 10 \, (S_c/\sigma_i)^{\lambda=17.93}$ Mc ; $\sigma_i \le 1.47 \, S_c$

The concept may be extended to a component under different stresses according to the spectrum shown below which comprises a block of n_1 cycles at stress magnitude σ_1, a block of n_2 cycles at σ_2, and so on. The life of the component if σ_1 were applied alone is n^*_1, its life



under σ_2 alone is n^*_2 etc. It follows that the fraction of life used up in the first block is n_1/n^*_1, the fraction used up in the second is n_2/n^*_2 etc. Assuming that the stress blocks may be treated independently, failure of the component occurs when the sum of the life-fractions used up by the various blocks (the cumulative damage) reaches 100%. That is, at failure :

$\Sigma_{i=1} \, n_i/n^*_i = 1$

Miner's Rule (strictly the *'Palmgren-Miner Rule'*)

This convenient but simplistic view implies that the damage caused by a stress block is constant irrespective of whether block occurred at the start of loading or just before failure. This is not the case in practice so Miner's Rule can be rather inexact, but is preferred in many non-critical applications to more complex / expensive techniques for handling variable load fatigue such as the *'Rainflow Method'* popular in aeronautical circles.

EXAMPLE

Estimate the life at 99% reliability of a steel gear u

under the given stress spectrum.

Failure is through pitting - the allowable contact

stress S_c being 1.55 GPa.

If the gear's life is L (khr) then the total number

of cycles in each stress block over that life is :

$n_i = (t_i / \Sigma t_i)\, L * N_i$

The life n^*_i corresponds to the stress σ_i in each

stress block acting alone, and follows from (**23**) :

σ_i	GPa	1.63	1.54	1.45	1.37
N_i	rev /min	65	85	125	14
t_i	hr	1	2	3	4
n_i	Mc	0.39L	1.02L	2.25L	0.34L
n^*_i	Mc	4.1	11.2	33.1	91.5

Applying Miner's Rule : $L (0.39/4.1 + 1.02/11.2 + 2.25/33.1 + 0.34/91.5) = 1$

whence $L = 3.9$ khr.

Many components other than gears follow approximate logarithmically linear stress-life relationships : $\sigma^{\lambda}_i n^*_i = \text{constant}$. Inserting this into Miner's Rule leads to :

$$\Sigma_{i=1} \sigma^{\lambda}_i . n_i = \text{constant}, \quad \text{and} \quad = \sigma^{\lambda}_e \Sigma_{i=1} n_i$$

in which σ_e is the equivalent stress - the single stress level which causes the same damage as the actual stress spectrum over the period

of interest. Stress is usually proportional to the load raised to some index - thus for gears which fail by pitting, the contact stresses are proportional to the square root of the tangential force and hence of the load (power or torque). Load may then be substituted for stress in the foregoing, leading to the ubiquitous relationship : load$^{\lambda}$ * life = constant.
The life of a gear n^*_i due to the power in the i'th block of a periodic load spectrum can be computed using (**21**) as if the power were constant and not part of a spectrum. All potential failure modes must be considered when applying Miner's Rule.

In conclusion, it should be reiterated that the foregoing treatment does not pretend to be a recipe for instant gear design - rather it illustrates one approach to the application of fatigue theory, and hopefully will enable the reader to specify, apply and adapt gears intelligently. Gear design is an art - gear geometry and metallurgy in particular can be subtle.
As AGMA 2001 puts it *"This Standard is intended for use by the experienced gear designer capable of selecting reasonable values for the factors. It is not intended for use by the engineering public at large."*