UMA201: Probability and Statistics

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Contents

Lecture 01: Introduction

Tue 01 Aug '23

1 The Course

Half or a little more of the course will cover probability. We will have a quiz every alternate week, and a homework every alternate week. Tentatively, one quiz and one homework will be dropped.

1.1 Professor

Name: Manjunath Krishnapur
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1.2 References

- Probability and Statistics for Engineers and Scientists by Sheldon M. Ross.
- Statistics by Freedman, Pisani and Purves. This is very light on mathematics.

1.3 Grading

(i) **Final:** 50%

(ii) **Midterm:** 20%

(iii) **Quizzes/Assignments:** 30%

Statistics is about making the best use of data to make reasonable decisions. Statistics makes racism mathematical.

2 Probability

Definition 2.1 (Discrete Probability Space)**.** A *discrete probability space* is an ordered pair (Ω, p) where:

(i) Ω is a finite or countable set.

(ii) $p : \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

We define three more notions:

(i) **Event:** any subset of Ω.

(ii) **Probability of an event:** for an event $A \subseteq \Omega$, $P(A) = \sum_{\omega \in A} p(\omega)$ is the probability of A.

(iii) **Random variable:** a function $X : \Omega \to \mathbb{R}$.

Notation. $p(\omega)$ is often denoted as p_ω.

Lecture 02

Thu 03 Aug '23

(i) $\Omega = \{1, 2, \ldots, 6\}$ with $p(\omega) = \frac{1}{6}$ for all $\omega \in \Omega$.

$A = \{1, 6\}$ is an event, and $P(A) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

This is a potential "model" for a fair d6.

(ii) "Toss n coins." We propose a probability space to describe this situation. Suppose $n = 2$. We label the coins and represent the set of all possible outcomes as

$$\Omega = \{HH, HT, TH, TT\} \text{ or as } \Omega = \{2H, 1H1T, 2T\}.$$

We choose the first representation since it carries the most information.

$\Omega = \{0, 1\}^n$. We propose $p(\omega)$ to be $\frac{1}{2^n}$ for all ω. Indeed, $\sum_{\omega \in \Omega} p_\omega = 1$.

Let A be the event that we get $k \in \{0, 1, \ldots, n\}$ or more heads. That is,

$$A = \left\{ x = (x_1, \ldots, x_n) \in \Omega \mid \sum x \geq k \right\}$$

and $P(A) = \sum_{a \in A} p_a = \frac{1}{2^n} |A|$.

To compute $|A|$, we note that $A = B_k \cup B_{k+1} \cup \cdots \cup B_n$ where $B_j = \{\omega \in \Omega \mid \sum \omega = k\}$ are pairwise disjoint. Thus, $|A| = \sum_{j=k}^{n} |B_j|$ and so

$$\begin{aligned} P(A) &= \frac{|B_k| + \cdots + |B_n|}{2^n} \\ &= \frac{1}{2^n} \left[\binom{n}{k} + \binom{n}{k+1} + \cdots + \binom{n}{n} \right] \end{aligned}$$

(iii) "Toss a coin n times." $\Omega = \{0, 1\}^n$ again. The index i in an element of Ω is whether the ith toss landed heads or tails. In the previous example, the index i was whether the ith coin landed heads or tails.

Is p the same as before? If the coin has no "memory", then yes. But suppose the coin is made of clay. The shape of the coin changes as we toss it. Then, the probability of getting heads on the ith toss depends on the previous tosses. The coin has "memory".

Of course, we don't usually bother with such details. It is still important to note that we are assuming that the coin has no memory.

(iv) "Throw r balls into m bins at random." This situation describes several others.

   (i) Toss r coins ($m = 2$).

   (ii) Roll an m-sided dice r times.

   (iii) Record the birthdays of r people ($m = 365$ or 366).

   (iv) Study the occurence of letters in an English text. Here $m = 26$ and r is the number of letters in the text.

We propose $\Omega = \{1, \ldots, m\}^r$. "Random" is not descriptive enough.

We have $|\Omega| = m^r$ and so a reasonable p is $\omega \mapsto \frac{1}{m^r}$.

An alternative is $\tilde{\Omega} = \{\omega \in \{0, \ldots, r\}^m \mid \sum \omega = r\}$. Note that $\left|\tilde{\Omega}\right| = \binom{m+r-1}{m-1}$ A $\tilde{p}$ corresponding to the above p is complicated. In choosing Ω, we made an implicit assumption that the r balls are distinguishable, as are the m bins.

Suppose we have r indistinguishable balls and m distinguishable bins. Is $p = \omega \mapsto \frac{1}{|\tilde{\Omega}|}$ reasonable?