UMA201: Probability and Statistics

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August 2023

Contents

Lecture 07

Tue 22 Aug '23

0.1 Principle of Inclusion & Exclusion

Let $A_1, A_2, \dots, A_n$ be events in a probability space. Then

$$\Pr(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{j=1}^{n} (-1)^{j+1} S_j$$

where $S_j = \sum_{1 \le i_1 < \dots < i_j \le n} \Pr(A_{i_1} \cap \dots \cap A_{i_j})$.

Proof. Let $A_1, \dots, A_n$ be events. Let $\omega \in \Omega$. If $\omega \notin A_1 \cup \dots \cup A_n$, Then ω is not counted on either the left or the right side.

If $\omega \in A_1 \cup \dots \cup A_n$, Then ω is counted once on the left side. Let $l = \#\{i \in [1..n] \mid \omega \in A_i\}$.

Then p_ω is counted $\binom{l}{j}$ times in S_j. Thus p_ω is counted $\sum_{j=1}^{n} (-1)^{j+1} \binom{l}{j}$

times on the right side.

$$\begin{aligned}
\sum_{j=1}^{l}(-1)^{j+1}\binom{l}{j} &= \sum_{j=0}^{l}(-1)^{j+1}\binom{l}{j} + 1 \\
&= 1 - \sum_{j=0}^{l}(-1)^{j}\binom{l}{j} \\
&= 1 \qquad \square
\end{aligned}$$

Example. Prof. Trelawny guesses a deck of n cards.

$$\Omega = S_n \times S_n$$

where S_n is the set of permutations of n. Assuming Prof. Trelawny is a lying bitch,

$$p(\pi, \sigma) = \left(\frac{1}{n!}\right)^2.$$

What is the probability that all guesses are wrong? That is, $\pi(j) \neq \sigma(j)$ for all $j \in n$.

$$E = \{(\pi, \sigma) \in \Omega \mid \pi(j) \neq \sigma(j) \,\forall\, j \in n\}.$$

We have that

$$E^c = \{(\pi, \sigma) \in \Omega \mid \exists\, j \in n(\pi(j) = \sigma(j))\}$$

is a union of events

$$E_j^c = \{(\pi, \sigma) \in \Omega \mid \pi(j) = \sigma(j)\}.$$

Thus

$$\Pr(E^c) = S_1 - S_2 + \ldots$$

where

$$\begin{aligned}
S_k &= \sum_{0 \leq i_1 < \cdots < i_k < n} \Pr(A_{i_1} \cap \cdots \cap A_{i_k}) \\
&= \binom{n}{k} \cdot \frac{n!(n-k)!}{n!^2} \\
&= \frac{1}{k!}
\end{aligned}$$

Thus

$$\begin{aligned}
\Pr(E^c) &= \frac{1}{1!} - \frac{1}{2!} + \cdots - (-1)^n \frac{1}{n!} \\
\implies \Pr(E) &= 1 - \Pr(E^c) \\
&= \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \\
&\approx \frac{1}{e} \text{ for large } n \\
&\approx 0.37
\end{aligned}$$

Problem 0.1. Show that the probability of exactly k correct guesses is approximately $\frac{1}{e}\frac{1}{k!}$ for small k and large n.

Example. The probability of fraudster Trelawney getting at least 10 cards right is approx

$$1 - \frac{1}{e}\left(1 + \frac{1}{2!} + \cdots + \frac{1}{10!}\right) \approx 1 - \frac{1}{e} \cdot e = 0$$

Problem 0.2. What is the probability that Trelawney gets at least k cards right?

0.2 Bonferroni's Inequalities

Theorem 0.1 (Bonferroni's Inequalities)**.** Let $A_1, \ldots, A_n$ be events in a probability space. Then

$$S_1 - S_2 \le \Pr(A_1 \cup \cdots \cup A_n) \le S_1.$$

The right inequality is sometimes called the *union bound.*

Proof. Let $l = \#\{i \in [1..n] \mid \omega \in A_i\}$. Then p_ω is counted $\binom{l}{j}$ times in S_j. Thus ω is counted $\binom{l}{1} - \binom{l}{2}$ times on the left, $\mathbb{1}_\Omega$ in the middle, and $\binom{l}{1}$ times on the right.

$\binom{l}{1} - \binom{l}{2} = 1$ for $l = 1, 2$ and ≤ 0 otherwise. □

Example. We throw r distinguishable balls into m labelled bins. What is the probability that some bin is empty?

We let $\Omega = [1..m]^r$ with uniform probability. Let A_i be the event that bin i is empty. Then $P(A_k) = \frac{(m-1)^r}{m^r}$ for all k. $P(A_k \cap A_l) = \frac{(m-2)^r}{m^r}$ for all

$k \neq l$.

Then from the Bonferroni inequalities,

$$m \cdot \frac{(m-1)^r}{m^r} - \binom{m}{2} \cdot \frac{(m-2)^r}{m^r} \leq \Pr(A_1 \cup \cdots \cup A_m) \leq m \cdot \frac{(m-1)^r}{m^r}$$

Lecture 08

Tue 29 Aug '23

Lecture 09: Conditional Probability

Tue 29 Aug '23

1 Conditional Probability

Definition 1.1 (Conditional Probability)**.** Let (Ω, p) be a probability space and let B be an event with $\Pr(B) > 0$. For any event A, we define

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Remarks.

- $\Pr(B \mid B) = 1$.
- $\Pr(B^c \mid B) = 0$.
- $\Pr(\,\cdot \mid B)$ has the properties of probability.
  - If A_1, A_2 are disjoint, then
  $$\Pr(A_1 \cup A_2 \mid B) = \Pr(A_1 \mid B) + \Pr(A_2 \mid B).$$
  - $\Pr(A^c \mid B) = 1 - \Pr(A \mid B)$.

Remarks. Events A and B are independent if and only if $\Pr(B) = 0$ or $\Pr(A \mid B) = \Pr(A)$.

Examples.

- A shuffled deck of cards. $\Omega = S_{52}$, $p : \omega \mapsto \frac{1}{52!}$.

  Let B be the event that the first card is a spade, let A be the event that the second card is a diamond, and let C be the event that the second card is a queen. We have

$$\begin{aligned}
\Pr(A) &= \frac{1}{4}, \\
\Pr(B) &= \frac{1}{4}, \\
\Pr(C) &= \frac{1}{13}.
\end{aligned}$$

For the intersections we have,

$$\begin{aligned}
\Pr(A \cap B) &= \frac{13 \cdot 13 \cdot 50!}{52!} \\
&= \frac{13}{4 \cdot 51}, \\
\Pr(B \cap C) &= \frac{(12 \cdot 4 + 1 \cdot 3) \cdot 50!}{52!} \\
&= \frac{1}{52}, \\
\Pr(C \cap A) &= \frac{1}{52}.
\end{aligned}$$

Now for the conditional probabilities,

$$\begin{aligned}
\Pr(A \mid B) &= \frac{13}{51} && > \Pr(A) \\
\Pr(C \mid B) &= \frac{1}{4} && = \Pr(C) \\
\Pr(A \mid C) &= \frac{1}{13} && = \Pr(A) \\
\Pr(B \mid C) &= \frac{1}{13} && = \Pr(B) \\
\Pr(B \mid A) &= \frac{13}{51} && > \Pr(B) \\
\Pr(C \mid A) &= \frac{1}{4} && = \Pr(C).
\end{aligned}$$

Notation. We notate $\Pr(A \mid B \cap C)$ as $\Pr(A \mid B, C)$.

Proposition 1.2 (Intersections)**.** For any events $A_1, \ldots, A_n$ provided $\Pr(A_1 \cap \ldots \cap A_{n-1}) \neq 0$, we have

$$\Pr(A_1 \cap \ldots \cap A_n) = \Pr(A_1) \Pr(A_2 \mid A_1) \ldots \Pr(A_n \mid A_1, \ldots, A_{n-1}).$$

Proposition 1.3 (Law of Total Probability)**.** Let $B_1, \ldots, B_n$ be a partition of Ω. Then for any event A, we have

$$\Pr(A) = \sum_{i=1}^{n} \Pr(B_i) \Pr(A \mid B_i)$$

where multiplication short-circuits to 0.

Theorem 1.4 (Bayes' Theorem)**.** Let $B_1, \ldots, B_n$ be a partition of Ω. Then for any event A with $\Pr(A) > 0$, we have

$$\Pr(B_i \mid A) = \frac{\Pr(B_i) \Pr(A \mid B_i)}{\sum_{j=1}^{n} \Pr(B_j) \Pr(A \mid B_j)},$$

where multiplication short-circuits to 0.

Example. Suppose there is a disease X that affects 1 in 10000 people. There is a test T for X that is 99% accurate. That is, $\Pr(T \mid X) = 0.99 = \Pr(T^c \mid X^c)$.

A person gets tested and the test is positive. What is the chance they have the disease?

$$\begin{aligned}
\Pr(X \mid T) &= \frac{\Pr(X) \Pr(T \mid X)}{\Pr(X) \Pr(T \mid X) + \Pr(X^c) \Pr(T \mid X^c)} \\
&= \frac{10^{-4} \cdot 0.99}{10^{-4} \cdot 0.99 + (1 - 10^{-4}) \cdot 0.01} \\
&= \frac{0.99}{0.99 + 9999 \cdot 0.01} \\
&= \frac{0.99}{100.98} \\
&\approx 0.98\%.
\end{aligned}$$