UMA201: Combinatorics

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Contents

1 Bijections

Definition 1.1 (Bijection)**.** Let A and B be sets. A function $f : A \to B$ is a *bijection* if it is both injective and surjective.

Notation. $A \sim B$ denotes that there exists a bijection from A to B.

Proposition 1.2 (Reflexivity)**.** For any set A, $A \sim A$.

Proof. Let $f : A \to A$ be the identity function. Then, f is a bijection. $\square$

Proposition 1.3 (Inverse)**.** Let $f : A \to B$ be a bijection. Then, there exists a unique function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proof. We define $g \subseteq B \times A$ as

$$g = \{P \in B \times A \mid \exists\, a \in A \exists\, b \in B (P = (b, a) \wedge (a, b) \in f)\}.$$

We claim that g is a function.

- Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $(a, b) \in f$. Thus, $(b, a) \in g$.

- Let $(b_0, a_0), (b_0, a_1) \in g$. Then, there exist $a, a' \in A$ and $b, b' \in B$ such that $(b_0, a_0) = (b, a)$, $(b_0, a_1) = (b', a')$, and $(a, b), (a', b') \in f$. But then $b = b' = b_0$. Since f is injective, $a = a'$, and so $a_0 = a_1$.

Let $a \in A$, and let $b = f(a)$. That is, $(a, b) \in f$. Then, $(b, a) \in g$. Thus $g(f(a)) = g(b) = a$ and so $g \circ f = \mathrm{id}_A$.

Let $b \in B$, and let $a = g(b)$. That is, $(b, a) \in g$. Then, $(a, b) \in f$. Thus $f(g(b)) = f(a) = b$ and so $f \circ g = \mathrm{id}_B$.

Suppose $g' : B \to A$ is a function such that $g' \circ f = \mathrm{id}_A$. Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Then, $g'(b) = g'(f(a)) = a = g(f(a)) = g(b)$. Thus, $g = g'$. This proves the uniqueness. □

Theorem 1.4. Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Then, f is a bijection.

Proof. Let $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. Then, $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Thus, f is injective.

Let $b \in B$. Then, $b = f(g(b))$. Thus, f is surjective. □

Proposition 1.5 (Symmetry)**.** If $A \sim B$, then $B \sim A$.

Proof. By proposition 1.3 and theorem 1.4. □

Proposition 1.6 (Composition)**.** If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Proof. Let $a_1, a_2 \in A$ with $g(f(a_1)) = g(f(a_2))$. Since g is injective, $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. Thus, $g \circ f$ is injective.

Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Thus, $g \circ f$ is surjective. □

Corollary 1.7 (Transitivity)**.** If $A \sim B$ and $B \sim C$, then $A \sim C$.

Theorem 1.8 (Equivalence)**.** $\sim$ is an equivalence relation on any set of sets.

Proof. By propositions 1.2 and 1.5 and corollary 1.7. □

Notation.

(i) ω denotes the minimal inductive set.

Theorem 1.9. Let $m, n \in \omega$. Then, $m \sim n$ if and only if $m = n$.

Proof. We prove this by induction on m. Let

$$U = \{m \in \omega \mid \forall n \in \omega(m \sim n \iff m = n)\}.$$

We first show that $\varnothing \in U$. Let $n \in \varnothing$ such that $\varnothing \sim n$. Then, there exists a bijection $f : \varnothing \to n$. Since f is surjective, $\forall n_0 \in n \exists m_0 \in \varnothing$ such that $(m_0, n_0) \in f$. But $f \subseteq \varnothing \times n = \varnothing$ and so $f = \varnothing$. Thus $n = \varnothing$.

Now suppose $m \in U$. Let $n \in \omega$ such that $m^+ \sim n$. We know that $m^+ \neq \varnothing$ and so $n \neq \varnothing$ (by the base case). Thus there exists $n^- \in n$ such that $n = (n^-)^+ = n^- \cup \{n^-\}$.

Let f be a bijection from m^+ to n. Then there exists an $m_0 \in m^+$ such that $f(m_0) = n^-$. We have two cases.

($m_0 = m$) Let $f' = f \setminus \{(m, n^-)\}$.

($m_0 \neq m$) Let $f' = f \setminus \{(m, f(m)), (m_0, n^-)\} \cup \{(m_0, f(m))\}$.

Then, $f' : m \to n^-$ is a bijection. Thus by the induction hypothesis, $m = n^-$ and so $m^+ = n$.

By induction, $U = \omega$. □

2 Cardinality

Definition 2.1 (Finite Cardinality)**.** Let A be a set. We say that A is finite if there exists an $n \in \omega$ such that $A \sim n$. We say that the *cardinality* of A is n and write $\#A = n$.

Remarks. n is guaranteed to be unique by theorems 1.8 and 1.9.

Lemma 2.2 (Disjoint Union)**.** Let A and B be disjoint finite sets. Then, $\#(A \cup B) = \#A + \#B$.

Proof. Let $n = \#A$ and $m = \#B$. Then, there exist bijections $f : A \to n$

and $g : B \to m$. Let $h : A \cup B \to n + m$ be defined as

$$h(x) = \begin{cases} f(x) & x \in A \\ n + g(x) & x \in B \end{cases}$$

Then, h is a bijection. □

Corollary 2.3 (Difference)**.** Let A and B be finite sets with $B \subseteq A$. Then $\#(A \setminus B) = \#A - \#B$.

Proof. $A \setminus B \cup B = A$ and $A \setminus B \cap B = \varnothing$. Thus □

Theorem 2.4 (Union)**.** Let A and B be finite sets. Then, $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.

Proof. We have $A \cup B = (A \setminus (A \cap B)) \cup B$ where $A \setminus (A \cap B)$ and B are disoint. Thus

$$\begin{aligned} \#(A \cup B) &= \#(A \setminus (A \cap B)) + \#B \\ &= \#A + \#B - \#(A \cap B). \end{aligned}$$

□

Theorem 2.5 (Product)**.** Let A and B be finite sets with cardinalities n and m respectively. Then, $\#(A \times B) = n \times m$.

Proof. We prove this by induction over m. The case $m = 0$ is trivial.

Let $m = 1$. Then, $B = \{b\}$ for some b. Let $f = a \in A \mapsto (a, b) \in A \times B$. f is a bijection and so $\#(A \times B) = \#A = n = n \times 1$.

Suppose $m \in \omega \setminus \{0, 1\}$ and the theorem holds for m. Let A be a set with cardinality n and B be a set with Cardinality m^+. Then $B \sim m \cup \{m\}$.

Let $f : m \cup \{m\} \to B$ be a bijection. Consider $B' = B \setminus f(m)$. Then, $B' \sim m$ and so $\#(A \times B') = n \times m$.

We have

$$A \times B = \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) \mid \exists\, a \in A \exists\, b \in B (x = (a, b))\}$$

and

$$\begin{aligned} A \times B' &= \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) \mid \exists\, a \in A \exists\, b \in B' (x = (a, b))\}. \\ A \times \{f(m)\} &= \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) \mid \exists\, a \in A (x = (a, f(m)))\}. \end{aligned}$$

We first note that $A \times B'$ and $A \times \{f(m)\}$ are disjoint. This is since $x \in A \times \{f(m)\}$ implies $x = (a, f(m))$ for some $a \in A$, but $f(m) \notin B'$ and so $x \notin A \times B'$.

We now prove that $A \times B = (A \times B') \cup (A \times \{f(m)\})$.

- Let $x \in A \times B$. Then there exist $a \in A$ and $b \in B$ such that $x = (a, b)$. If $b \in B'$, then $x \in A \times B'$. Else $b = f(m)$ and so $x \in A \times \{f(m)\}$.
- Let $x \in (A \times B') \cup (A \times \{f(m)\})$. Then $x \in A \times B'$ or $x \in A \times \{f(m)\}$. In either case, $x \in A \times B$.

Thus

$$\begin{aligned} A \times B &= (A \times B') \cup (A \times \{f(m)\}) \\ \#(A \times B) &= \#(A \times B') + \#(A \times \{f(m)\}) \\ &= n \times m + n \\ &= n \times m^+. \end{aligned}$$

By induction, the theorem holds for all $m \in \omega$. □

Theorem 2.6 (Power Set)**.** Let A be a finite set with cardinality n. Then, $\#\mathscr{P}(A) = 2^n$.

3 Binomial Coefficients

Definition 3.1. The number of subsets of n with cardinality k is denoted by $\binom{n}{k}$ and is called "n choose k".

Lemma 3.2. For any $n \in \mathbb{N}$, $k \in [0..n]$,

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof. Let A_j be the set of subsets of n with cardinality j. Let B_j be the set of subsets of $n + 1$ with cardinality j.

Note that for $i \neq j$, $A_i \cap A_j = \varnothing$. Let $f : A_k \cup A_{k+1} \to B_{k+1}$ be defined as

$$f(S) = \begin{cases} S \cup \{n\} & S \in A_k \\ S & S \in A_{k+1} \end{cases}$$

and let $g : B_{k+1} \to A_k \cup A_{k+1}$ be defined as

$$g(T) = T \setminus \{n\}.$$

Let $h = g \circ f$. Then

$$\begin{aligned} h(S) &= \left.\begin{cases} S \cup \{n\} & S \in A_k \\ S & S \in A_{k+1} \end{cases}\right\} \setminus \{n+1\} \\ &= \begin{cases} S & S \in A_k \\ S & S \in A_{k+1} \end{cases} \\ &= S. \end{aligned}$$

Thus f is an injection.

Let $T \in B_{k+1}$. If $n \notin T$, then $T \in A_{k+1}$ and so $f(T) = T$. If $n \in T$, then $T \setminus \{n\} \in A_k$ and so $f(T \setminus \{n\}) = T$. Thus f is a surjection.

Therefore f is a bijection and so $\#(A_k \cup A_{k+1}) = \#B_{k+1}$. Since $A_k \cap A_{k+1} = \varnothing$,

$$\begin{aligned} \#(A_k \cup A_{k+1}) &= \#A_k + \#A_{k+1} \\ \#B_{k+1} &= \#A_k + \#A_{k+1} \\ \binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1}. \end{aligned}$$

□

Theorem 3.3 (Binomial Expansion)**.** For any $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

Proof. Let $P(n)$ be the given statement for some n. For $n = 0$, the statement is trivial.

Suppose $P(k)$ is true. Then

$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&= (1+x)\sum_{j=0}^{k}\binom{k}{j}x^j \\
&= \sum_{j=0}^{k}\binom{k}{j}x^j + \sum_{j=0}^{k}\binom{k}{j}x^{j+1} \\
&= \binom{k}{0}x^0 + \sum_{j=1}^{k}\binom{k}{j}x^j + \sum_{j=1}^{k}\binom{k}{j-1}x^j + \binom{k}{k}x^{k+1} \\
&= \binom{k+1}{0}x^0 + \sum_{j=1}^{k}\left[\binom{k}{j} + \binom{k}{j-1}\right]x^j + \binom{k+1}{k+1}x^{k+1} \\
&= \sum_{j=0}^{k+1}\binom{k+1}{j}x^j.
\end{aligned}
$$

Thus, $P(k+1)$ is true.

By induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$