**HOMEWORK 1**

UMA 201 (2023)

**Color codes:**

- Problems are for submission. Solve them on your own.
- Problems are for practise - they are crucial to get a grip on the subject. Feel free to work on them in small groups.
- Problems are somewhat more challenging. Try them only if you are comfortable with the more basic ones.
- Food for thought is just for fun.

**Problem 1.** A fair coin is tossed $n$ times. Show that the probability that there is exactly one run of heads and two runs of tails is $\frac{(n-1)(n-2)}{2^{n+1}}$. [*Note:* A run is a consecutive sequence of tosses giving the same result. For example, $001101000$ has three runs of tails and two runs of heads]

**Problem 2.** A well-shuffled deck of cards is dealt among four players (so 13 cards each). What is the chance that no player has two cards of the same kind (i.e., two aces or two ones, or two queens, etc). Find an exact answer, and also give an approximate numerical value.

**Problem 3.** A box contains $n$ coupons labelled $1, 2, \ldots, n$. A total of $k$ coupons are drawn from the box at random. There are two schemes, write the probability space in each case and calculate the probability of the event that $1$ is not in the chosen sample.

(1) With replacement sampling: This means that a coupon is drawn uniformly at random, the number noted and the coupon is returned to the box, then a coupon is drawn again at random, and so on.

(2) Without replacement sampling: Similar to above, but the drawn coupon is not returned to the box (equivalently, we may say that $k$ coupons are drawn simultaneously from the box).

**Problem 4.** A die is thrown repeatedly till the face $6$ comes up for the first time. Write the probability space and calculate the probability of the event that the face $1$ never showed up.

**Problem 5.** A fair die is thrown, and if the number $k$ shows up, a fair coin is tossed $k$ times. Write down the probability space fully and calculate the probability of the event that no heads show up.

**Problem 6.** Below, $n, m, r$ etc are positive integers. Here $\binom{p}{q}$ is interpreted as $0$ if $q > p$ or $q < 0$.

(1) Show that the number of ways to place $r$ distinguishable balls in $m$ labelled bins is $m^r$.

(2) Show that the number of ways to place $r$ indistinguishable balls in $m$ labelled bins is $\binom{m+r-1}{m-1}$.

(3) Show that the number of ways to place $r$ indistinguishable balls in $m$ labelled bins in such a way that each bin can contain at most one ball, is $\binom{m}{r}$.

(4) Show that the number of ways to place $r$ distinguishable balls in $m$ distinguishable bins in such a way that there are $k_1$ balls in the first bin, $k_2$ in the second and so on (where $k_1 + \ldots + k_m = r$) is $\frac{r!}{k_1!k_2!\ldots k_m!}$ (this is usually written $\binom{r}{k_1,\ldots,k_m}$ and has the name "multinomial coefficient").

(5) Show that the number of ways to place $r$ indistinguishable balls in $m$ labelled bins in such a way that no bin is empty, is $\binom{r-1}{m-1}$.

**Problem 7.** Prove the following facts about binomial coefficients.

(1) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1}$.

(2) $\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \ldots + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k}\binom{m}{0} = \binom{m+n}{k}$.

(3) $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \ldots + (-1)^{n-1}n\binom{n}{n} = 0$.

(4) As $k$ ranges from $1$ to $n$, the numbers $\binom{n}{k}$ increase and then decrease. At what value of $k$ is the maximum attained?

**Problem 8.** Generalize the first problem to compute the probability that in a sequence of $n$ tosses of a fair coin, there are a total of $k$ runs (note that if there are $j$ runs of heads, the number of runs of tails is either $j-1$ or $j$ or $j+1$).

**Problem 9.** There are 4 billion women in the world. What is the largest number $k$ for which you can assert with a probability of $0.99$ or more that there are two (unrelated) women who share the same birthday, whose mothers share the same birthday, whose grandmothers share the same birthday $\ldots$ and so on for $k$ generations?

**Food for thought:** A frustrated professor of statistics has the habit of throwing up pieces of chalk during lecture. Directly above is a fan, which the piece of chalk may hit or not. Does the chance of hitting it increase or decrease if the fan is turned on? Think before reading further.

If you say that it increases (or that it decreases), here is my doubt. Imagine the vacant space between the wings of the fan as three "empty wings". These empty wings also rotate when the fan is turned on, so the chance of hitting them must also increase (or decrease) by your logic. That means that the chance of not hitting the fan must also increase (or decrease). How is that possible?