Homework 05

Naman Mishra
22223
Section A

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Problem 3.1. Suppose r distinguishable balls are thrown at random into m labelled bins. Let A_k be the event that the kth bin is empty.

(a) Are A_1, A_2, A_3 independent?

(b) What happens (regarding independence) if $r = 3m$ and $m \to \infty$.

Solution. We model the sample space as

$$\Omega = [m]^r \qquad p : \omega \mapsto \frac{1}{m^r}.$$

We have

$$A_k = ([m] \setminus \{k\})^r$$

and more generally,

$$\bigcap_{j \in J} A_j = ([m] \setminus J)^r$$

for any $J \subseteq [m]$.

Since the probability distribution is uniform, we have

$$\begin{aligned}\Pr\left(\bigcap_{j \in J} A_j\right) &= \frac{\#([m] \setminus J)^r}{m^r} \\ &= \frac{(m - \#J)^r}{m^r} \\ &= \left(1 - \frac{\#J}{m}\right)^r\end{aligned}$$

Thus

$$\begin{aligned}
\Pr(A_1) = \Pr(A_2) = \Pr(A_3) &= \left(1 - \frac{1}{m}\right)^r \\
\Pr(A_1 \cap A_2) = \Pr(A_1 \cap A_3) = \Pr(A_2 \cap A_3) &= \left(1 - \frac{2}{m}\right)^r \\
\Pr(A_1 \cap A_2 \cap A_3) &= \left(1 - \frac{3}{m}\right)^r
\end{aligned}$$

Note that

$$\begin{aligned}
& \Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2) \\
\iff & \frac{(m-2)^r}{m^r} = \frac{(m-1)^{2r}}{m^{2r}} \\
\iff & (m-2)m = (m-1)^2 \\
\iff & 0 = 1
\end{aligned}$$

for $r > 0$, which is a contradiction. Thus A_1, A_2, and A_3 are dependent. More strongly, by symmetry, they are pairwise dependent.

Now suppose $r = 3m$. Then

$$\begin{aligned}
\Pr\left(\bigcap_{j \in J} A_j\right) &= \left(1 - \frac{\#J}{m}\right)^{3m} \\
\lim_{m\to\infty} \Pr\left(\bigcap_{j \in J} A_j\right) &= e^{-3\#J}
\end{aligned}$$

Thus $\lim_{m\to\infty} \Pr(A_k) = e^{-3}$. Notice that

$$\begin{aligned}
\lim_{m\to\infty} \prod_{j\in J} \Pr(A_j) &= \prod_{j \in J} \lim_{m\to\infty} \Pr(A_j) && \text{(all limits exist)} \\
&= \prod_{j\in J} e^{-3} \\
&= e^{-3\#J} \\
&= \lim_{m\to\infty} \Pr\left(\bigcap_{j\in J} A_j\right)
\end{aligned}$$

Thus A_1, A_2, A_3 are independent in the limit. In fact, any finite set of A_k is independent in the limit. $\square$

Problem 3.2. Let A and B be events of positive probability in a probability space, such that the probability of their intersection is also positive.

(1) Show that $\Pr(A \mid B) = \Pr(B \mid A)$ if and only if $\Pr(A) = \Pr(B)$.

(2) Show that $\Pr(A \mid B) > \Pr(A)$ if and only if $\Pr(B \mid A) > \Pr(B)$.

Solution. We simply manipulate equations and inequalities using the fact that $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$ are all positive.

(1) We have

$$\begin{aligned} \Pr(A \mid B) = \Pr(B \mid A) &\iff \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(A)} \\ &\iff \frac{1}{\Pr(B)} = \frac{1}{\Pr(A)} \\ &\iff \Pr(A) = \Pr(B) \end{aligned}$$

(2) We have

$$\begin{aligned} \Pr(A \mid B) > \Pr(A) &\iff \frac{\Pr(A \cap B)}{\Pr(B)} > \Pr(A) \\ &\iff \Pr(A \cap B) > \Pr(A)\Pr(B) \\ &\iff \frac{\Pr(B \cap A)}{\Pr(A)} > \Pr(B) \\ &\iff \Pr(B \mid A) > \Pr(B) \end{aligned}$$

□