Assignment 01

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Notation.

$$[a..b] \coloneqq [a,b] \cap \mathbb{Z}$$
$$[P] \coloneqq \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$$
$$S_n \coloneqq \{\text{permutations of } [1..n]\}$$

Problem 1.1. A fair coin is tossed n times. Show that the probability that there is exactly one run of heads and two runs of tails is $\frac{(n-1)(n-2)}{2^{n+1}}$. [*Note*: A run is a consecutive sequence of tosses giving the same result. For example, 001101000 has three runs of tails and two runs of heads]

Solution. We model the probability space as $\Omega = \{0,1\}^n$ with p_ω being $\frac{1}{\#\Omega} = \frac{1}{2^n}$ for all ω. This is because the coin is fair, and each toss is assumed to be independent of the others.

Let A be the event that there is exactly one run of heads and two runs of tails. This implies that the first head is at an index greater than 1 and the last head at an index less than n (1-indexed). Thus we can write A as

$$A = \{\omega \in \Omega \mid \exists\, x < y \in [2..n] \text{ such that } \omega_j = [x \le j < y]\}.$$

Let B be the set of subsets of $[2..n]$ of size 2. Let $f : B \to A$ be given as $f(b) = a$ where

$$a_j = [\min b \le j < \max b] \text{ for all } j.$$

Then f is a bijection.

Thus we have $\#A = \#B = \binom{n-1}{2}$ and so the probability of A is

$$\begin{aligned}\Pr(A) &= \sum_{\omega \in A} p_\omega \\ &= \frac{\#A}{2^n} \\ &= \frac{(n-1)(n-2)}{2^{n+1}}\end{aligned}$$

as required.

Problem 1.2. A well-shuffled deck of cards is dealt among four players (so 13 cards each). What is the chance that no player has two cards of the same kind (*i.e.*, two aces or two ones, or two queens, etc)? Find an exact answer, and also give an approximate numerical value.

Solution. We assign a number from 1 to 4 to each player, a number from 1 to 4 to each suit, and a number from 1 to 13 to each face.

We can thus model each card as $(f, s) \in C = [1..13] \times [1..4]$, where f is the face and s is the suit.

We model the sample space as

$$\Omega = \{Q \in M_{13\times 4}([1..4]) \mid f(p, Q) = 13 \text{ for all } p \in [1..4]\},$$

where $f(x, A)$ is the number of occurences of x in A, and Q_{ij} represents the player to whom the card (i, j) is dealt.

We can find $\#\Omega$ as follows:
First we choose 13 out of the 52 positions in the matrix to be 1.
Then we choose 13 out of the remaining 39 positions to be 2.
Then we choose 13 out of the remaining 26 positions to be 3.
Finally, we choose 13 out of the remaining 13 positions to be 4.

This gives a total of

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

possible ways to form an element of Ω.

Thus $\#\Omega = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{13!^4}$.

Since the deck is well-shuffled, it is fair to assume that each element of Ω is equally likely. Thus $p_\omega = \frac{1}{\#\omega} = \frac{13!^4}{52!}$ for all $\omega \in \Omega$.

We can model the event as

$$E = \{\omega \in \Omega \mid \forall\, i \in [1..13](\forall\, j \in [1..4] \forall\, k \in [1..4](j \neq k \implies \omega_{ij} \neq \omega_{ik}))\}$$

and so by the pigeonhole principle,

$$E = \left\{\omega \in \Omega \mid \forall\, i \in [1..13] \left(\{\omega_{ij}\}_{j \in [1..4]} = [1..4] \right) \right\}$$

and thus

$$E = \left\{\omega \in \Omega \mid \forall\, i \in [1..13] \left((\omega_{ij})_{j \in [1..4]} \in S_4 \right) \right\}.$$

We can define a bijection $\phi : S_4^{13} \to E$ by

$$\phi(\varsigma)_{ij} = (\varsigma_i)_j$$

and so $\#E = \#S_4^{13} = 4!^{13}$.

Therefore,

$$\Pr(E) = \sum_{\omega \in E} p_\omega = \frac{4!^{13} 13!^4}{52!}.$$

This is approximately equal to $1.63 \cdot 10^{-11}$.

If interpreted as seconds, it is roughly the time it takes light to travel a distance of 5 mm.